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Courses of Mathematical Analysis 3

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Introduction

These courses contain the official program for the subject Analysis 3 intended primarily for students in the second year of a mathematics degree. The content of this subject is the basis of any introduction to mathematical analysis. It is considered a direct extension of the two subjects Analysis 1 and Analysis 2 seen in the first year of a mathematics degree. For this reason, I recommend that all students who take the subject Analysis 3 also take the two subjects Analysis 1 and Analysis 2, which essentially aims to consolidate the knowledge acquired in secondary school in order to be used in the second year of a mathematics degree.

These courses contain six main chapters, where the concepts of numerical series, sequences or series of functions, power series, Fourier series, generalized integrals and functions defined by integrals are presented. Each chapter of these courses ends with uncorrected exercises allowing the sudent to go further in understanding and assimilation of the mathematical concepts introduced.

Chapter 1

Numerical series

1.1 Series with real or complex terms

Definition 1.1.1. *Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of real or complex numbers. We call a numerical series (respectively complex series) of general term u_n, any expression of the form:*

$$u_0 + u_1 + ... + u_n + ... = \sum_{n\geq 0} u_n. \tag{1.1}$$

The real numbers (respectively the complexe numbers) $u_0, u_1, ...\ u_n, ...$ are called terms of the series.
Let us now consider the following partial sums:

$$S_n = u_0 + u_1 + ... + u_n = \sum_{k=0}^{n} u_k. \tag{1.2}$$

The number S_n is called partial sum of order n of the series $\sum_{n\geq 0} u_n$, and the sequence (S_n) is called sequence of partial sums of the series $\sum_{n\geq 0} u_n$.

Definition 1.1.2. *Let $\sum_{n\geq 0} u_n$ be a series with real terms or complex. We say that the series $\sum_{n\geq 0} u_n$ converges if the sequence of partial sums (S_n) converges, and it diverges if the sequence of partial sums diverges.*

Definition 1.1.3. *When the series* $\sum_{n\geq 0} u_n$ *converges, we call sum of the series, the limit* S *of the sequence of partial sums and we write:*

$$S = \lim_{n\to+\infty} S_n = \sum_{n\geq 0} u_n. \tag{1.3}$$

Definition 1.1.4. *Let* $\sum_{n\geq 0} u_n$ *be a convergent series of sum* S*. We call rest of order* n *of the series* $\sum_{n\geq 0} u_n$*, the number* R_n *which defined by:*

$$R_n = S - S_n = \sum_{k\geq n+1} u_k. \tag{1.4}$$

We then have the following equivalence:

$$\sum_{n\geq 0} u_n \text{ converge } \Leftrightarrow \lim_{n\to+\infty} S_n = S \Leftrightarrow \lim_{n\to+\infty} R_n = 0. \tag{1.5}$$

Remak 1.1. *We also deduce that the nature of a series does not change, in removing a finite number of its terms. On the other hand, if the series converges, the value of its sum depends on all the terms of the series.*

Example 1.1.1. *(Geometric series)*
The geometric series $\sum_{n\geq 0} R^n$ *is:*

1. *convergent if and only if* $|R| < 1$*, in this case* $S = \dfrac{1}{1-R}$.
2. *divergent if and only if* $|R| \geq 1$.

Proposition 1.1.1. *(* ***Telescopic process****) Let* (u_n) *and* (v_n) *be two sequences of real or complex numbers, such that* $u_n = v_{n+1} - v_n$*. Then, the series* $\sum_{n\geq 0} u_n$ *converges if and only if the sequence* (v_n) *converges, and in this case:*

$$\sum_{n\geq 0} u_n = \lim_{n\to+\infty} v_n - v_0. \tag{1.6}$$

Proof. Indeed, the proof is made of the following equality:

$$\sum_{k=0}^{n} u_k = \sum_{k=0}^{n} (v_{k+1} - v_k) = v_{k+1} - v_0.$$

□

Example 1.1.2. *(Case of convergence) Let us consider the series of general term*

$$u_n = \frac{1}{n(n+1)},\ n \geq 1.$$

The term u_n can be rewritten as:

$$u_n = \frac{1}{n} - \frac{1}{n+1}, n \geq 1.$$

The term u_n can be rewritten as:

$$\begin{aligned} S_n &= \sum_{k=1}^{n} u_k = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \ldots + \left(\frac{1}{n} - \frac{1}{n+1}\right) \\ &= 1 - \frac{1}{n+1}. \end{aligned}$$

Since $\lim_{n\to+\infty} S_n = 1$, it follows that the series $\sum\limits_{n\geq 1} u_n$ is convergent of sum $S = 1$.

Example 1.1.3. *(Case of divergence) Let us consider the series of general term*

$$u_n = \ln(1 + \frac{1}{n}),\ n \geq 1.$$

The term u_n can be rewritten as:

$$u_n = \ln(n+1) - \ln(n),\ n \geq 1. \tag{1.7}$$

Hence:

$$\begin{aligned} S_n &= \sum_{k=1}^{n} u_k = (\ln 2 - \ln 1) + (\ln 3 - \ln 2) + \ldots + (\ln(n+1) - \ln(n)) \\ &= \ln(n+1). \end{aligned}$$

Since $\lim_{n\to+\infty} S_n = +\infty$, it follows that the series $\sum\limits_{n\geq 1} u_n$ diverges.

Proposition 1.1.2. *(Necessary condition of convergence**)** For a numerical series $\sum\limits_{n\geq 0} u_n$ to be convergent, it is necessary that its general term u_n tends towards zero.*

Proof. Suppose that $\sum\limits_{n\geq 0} u_n$ converges to $S = \lim_{n\to+\infty} S_n$. We then have:

$$\lim_{n\to+\infty} u_n = \lim_{n\to+\infty} S_n - S_{n-1} = S - S = 0. \tag{1.8}$$

Corollary 1.1.1. *(Sufficient condition of divergence) A sufficient condition for a serie is divergent, is that its general term does not tend towards zero.*

Corollary 1.1.2. *The converse of Proposition 1.1.2 is false in general. Indeed, the series* ***harmonic*** $\sum_{n\geq 1} \frac{1}{n}$ *diverges, while its general term tends towards zero.*

□

1.1.1 Algebraic structure of the set of convergent series

Proposition 1.1.3. *Let* $\sum_{n\geq 0} u_n$ *and* $\sum_{n\geq 0} v_n$ *be two numerical or complex series. We then have the following properties:*

1. If $\sum_{n\geq 0} u_n$ *is convergent with sum* S_1 *and if* $\sum_{n\geq 0} v_n$ *is convergent with sum* S_2*, then* $\sum_{n\geq 0} (u_n + v_n)$ *is convergent with sum* $S_1 + S_2$.

2. If $\sum_{n\geq 0} u_n$ *is convergent with sum* S_1 *and if* $\alpha \in \mathbb{R}$ *(where* $\mathbb{C}$ *), then* $\sum_{n\geq 0} (\alpha u_n)$ *is convergent with sum* αS_1.

Proof. The proof of this proposition follows immediately from the properties of the limits of sequences. □

Remak 1.2. *With these two previous operations, we can easily demonstrate that the set of convergent series is a subspace vector.*

1.1.2 Other algebraic operations

Proposition 1.1.4. *Let* $\sum_{n\geq 0} u_n$ *and* $\sum_{n\geq 0} v_n$ *be two numerical or complex series. We then have the following supplimentary properties:*

1. If $\sum_{n\geq 0} u_n$ *diverges and if* $\alpha \in \mathbb{R}^*$*, then* $\sum_{n\geq 0} (\alpha u_n)$ *diverges.*

2. If $\sum_{n\geq 0} u_n$ *converges and if* $\sum_{n\geq 0} v_n$ *diverges, then* $\sum_{n\geq 0} (u_n + v_n)$ *diverges.*

3. If the two series $\sum_{n\geq 0} u_n$ *and* $\sum_{n\geq 0} v_n$ *are divergent, we cannot conclude anything about the nature of the series* $\sum_{n\geq 0} (u_n + v_n)$*, it can be convergent, as it can be divergent.*

Proof. The proof of this proposition follows immediately from the properties of the limits of sequences. □

1.1.3 Cauchy criterion

Theorem 1.1.1. *Let $\sum_{n\geq 0} u_n$ be a series with real terms or complex. This series is convergent if and only if:*

$$\forall \epsilon > 0,\ \exists n_0 \in \mathbb{N},\ \forall p, q \in \mathbb{N},\ p > q \geq n_0,\ \text{we have} \left|\sum_{k=q+1}^{p} u_k\right| < \epsilon. \tag{1.9}$$

Proof. The proof of this theorem is done using the Cauchy criterion following the partial sums $S_n = \sum_{k=0}^{n} u_k$, and the fact that $S_p - S_q = \sum_{k=q+1}^{p} u_k$. □

1.2 Positive terms series

Definition 1.2.1. *We call a series with a positive terms any series whose general term u_n verifies:*

$$u_n \geq 0, \text{ for all } n \geq 0. \tag{1.10}$$

Proposition 1.2.1. *Let $\sum_{n\geq 0} u_n$ be a series with real terms positive. Then this series converges towards S, if and only if the sequence (S_n) of its partial sums is majorized. In this case, we have:*

$$S_n \leq S, \text{ for all } n \geq 0. \tag{1.11}$$

Proof. Since $S_{n+1} - S_n = u_{n+1} \geq 0$, for all $n \geq 0$, it follows that (S_n) is increased, so for it to be convergent, it is necessary and sufficient that it be majorized. In this case, the limit of the sequence of partial sums (S_n) majore all the terms of the sequence. □

Remak 1.3. *If (S_n) is not majorized, $\lim_{n\to+\infty} S_n = +\infty$, and the series $\sum_{n\geq 0} u_n$ diverges.*

1.2.1 Comparison theorems

Theorem 1.2.1. *Given two series with positive terms $\sum_{n\geq 0} u_n$ and $\sum_{n\geq 0} v_n$ verifying :*

$$\exists n_0 \in \mathbb{N}, \text{ such that for all } n \geq n_0, u_n \leq v_n, \tag{1.12}$$

we then have:

1. $\sum_{n\geq 0} v_n$ *converge implies* $\sum_{n\geq 0} u_n$ *converges.*
2. $\sum_{n\geq 0} u_n$ *diverges implies* $\sum_{n\geq 0} v_n$ *diverges.*

Proof. 1. Let's pose for all $n \in \mathbb{N}$, $S_n = \sum_{k=0}^{n} u_k$ and $T_n = \sum_{k=0}^{n} v_k$. Since $u_n \leq v_n$, for all $n \geq n_0$, we then obtain :

$$S_n \leq T_n, \text{ for all } n \geq n_0. \tag{1.13}$$

If the series $\sum_{n\geq 0} v_n$ converges, the sequence (T_n) is therefore majorized, then the sequence (S_n) is also majorizd, and thus the series $\sum_{n\geq 0} u_n$ converges.
2. The second property is the contrapositive of the first, so is also true. □

Corollary 1.2.1. *Let* $\sum_{n\geq 0} u_n$ *and* $\sum_{n\geq 0} v_n$ *be two series with positive terms. Suppose that there exist two strictly positive real numbers* α *and* β *verifying:*

$$\alpha u_n \leq v_n \leq \beta u_n, \tag{1.14}$$

then $\sum_{n\geq 0} u_n$ *and* $\sum_{n\geq 0} v_n$ *are of the same nature.*

Proof. Applying the Theorem 1.2.1 twice, gives us: if the series with general term v_n converges, the series with general term u_n converges and if the series with general term u_n converges, the series with general term v_n converges. Which shows that the two series are of the same nature. □

Example 1.2.1. *Consider the general term number series:*

$$u_n = \frac{\theta^n}{\sqrt{n}}, \ \theta \geq 0 \text{ and } n > 0. \tag{1.15}$$

* *If* $\theta \geq 1$, $u_n \geq \frac{1}{\sqrt{n}} \geq \frac{1}{n}$. *Since* $\sum_{n\geq 1} \frac{1}{n}$ *diverges, the considered series also diverges.*
* *If* $0 \leq \theta < 1$, $u_n \leq \theta^n$. *Since* $\sum_{n\geq 1} \theta^n$ *converges (geometric series), the considered series also converges.*

Theorem 1.2.2. *Let* $\sum_{n\geq 0} u_n$ *and* $\sum_{n\geq 0} v_n$ *be two series with positive terms. Suppose there exists a positive real* l *(or* $l = +\infty$*), such that* $\lim_{n\to+\infty} \frac{u_n}{v_n} = l$, *we then have:*

1. *If $l = 0$ and the series $\sum\limits_{n\geq 0} v_n$ converge, the series $\sum\limits_{n\geq 0} u_n$ converges.*
2. *If $l = +\infty$ and the series $\sum\limits_{n\geq 0} v_n$ diverge, the series $\sum\limits_{n\geq 0} u_n$ diverges.*
3. *If $l \neq 0$ and $\neq +\infty$, both $\sum\limits_{n\geq 0} v_n$ and $\sum\limits_{n\geq 0} u_n$ are of the same nature.*

Proof. 1. By definition

$$\lim_{n\to+\infty} \frac{u_n}{v_n} = 0 \Longleftrightarrow \forall \epsilon > 0,\ \exists n_0 \in \mathbb{N},\ \forall n \in \mathbb{N},\ n \geq n_0,\ \text{we have } \frac{u_n}{v_n} < \epsilon. \tag{1.16}$$

Let us choose $\epsilon = 1$, so we have $u_n < v_n$. Since the series $\sum\limits_{n\geq 0} v_n$ converges, the series $\sum\limits_{n\geq 0} u_n$ also converges.

2. Similarly

$$\lim_{n\to+\infty} \frac{u_n}{v_n} = +\infty \Leftrightarrow \lim_{n\to+\infty} \frac{v_n}{u_n} = 0 \Longleftrightarrow \forall \epsilon > 0,\ \exists n_0 \in \mathbb{N},\ \forall n \in \mathbb{N},\ n \geq n_0,\ \text{we have } \frac{v_n}{u_n} < \epsilon. \tag{1.17}$$

Let us choose $\epsilon = 1$, so we have $u_n > v_n$. Since the series $\sum\limits_{n\geq 0} v_n$ diverges, the series $\sum\limits_{n\geq 0} u_n$ also diverges.

3. If $l \neq 0$ and $\neq +\infty$,

$$\lim_{n\to+\infty} \frac{u_n}{v_n} = l \Longleftrightarrow \forall \epsilon > 0,\ \exists n_0 \in \mathbb{N},\ \forall n \in \mathbb{N},\ n \geq n_0,\ \text{we have } \left(\frac{u_n}{v_n} - l\right) < \epsilon. \tag{1.18}$$

Let us choose $\epsilon < l$, we therefore have $(l-\epsilon)v_n < u_n < (l+\epsilon)v_n$. Using Corollary 1.2.1 confirms us the reresult, by taking $\alpha = l - \epsilon > 0$ and $\beta = l + \epsilon$. □

1.2.2 Usual rules of convergence

Riemann's rule

Riemann's rule amounts to comparing a series with given positive terms to a Riemann series.

Definition 1.2.2. *A Riemann series is any numerical series whose general term*

$$u_n = \frac{1}{n^\alpha},\ \alpha \in \mathbb{R}$$

.

Proposition 1.2.2. *The Riemann series converges, for all $\alpha > 1$.*

Proof. If $\alpha \geq 0$, $\lim_{n\to+\infty} u_n \neq 0$, the series $\sum_{n\geq 1} u_n$ is therefore divergent.
Let us now assume that $\alpha > 0$, and consider the function f, which is defined on $]0, +\infty[$ by $f(x) = \dfrac{1}{x^\alpha}$.
This function is positive defined, continuous and decreasing on $]0, +\infty[$ .By Theorem 5.6.2 (see Chapter 5), the series $\sum_{n\geq 1} u_n$ and the generalized integral $\int_1^{+\infty} f(x)dx$ are the same nature. Let

$$F(x) = \int_1^y f(x)dx = \begin{cases} \ln(y), \text{ if } \alpha = 1, \\ \dfrac{1}{(1-\alpha)y^{\alpha-1}} - \dfrac{1}{1-\alpha}, \text{ if } \alpha \neq 1. \end{cases} \tag{1.19}$$

The function F has a finite limit, if and only if $\alpha > 1$, which shows that the series $\sum_{n\geq 1} u_n$ converges if and only if $\alpha > 1$. □

Proposition 1.2.3. *(Riemann's rule)*
Let $\sum_{n\geq 1} u_n$ be a series with positive real terms and let $\alpha \in \mathbb{R}$. Suppose there exists a positive real number l (or $l = +\infty$) , such that $\lim n^\alpha u_n = l$. We then have:

1. *If $l = 0$ and $\alpha > 1$, the series $\sum_{n\geq 1} u_n$ converges.*
2. *If $l = +\infty$ and $\alpha \leq 1$, the series $\sum_{n\geq 1} u_n$ diverges.*
3. *If $l \neq 0$ and if $l \neq +\infty$, the two series $\sum_{n\geq 1} u_n$ and $\sum_{n\geq 1} \dfrac{1}{n^\alpha}$ are the same nature.*

Proof. 1. By definition

$$\lim_{n\to+\infty} n^\alpha u_n = 0 \iff \forall \epsilon > 0,\ \exists n_0 \in \mathbb{N},\ \forall n \in \mathbb{N},\ n \geq n_0,\ \text{we have } n^\alpha u_n < \epsilon. \tag{1.20}$$

Let us choose $\epsilon = 1$, we therefore have $u_n < \dfrac{1}{n^\alpha}$. Since $\sum_{n\geq 1} \dfrac{1}{n^\alpha}$ converges for all $\alpha > 1$, the series $\sum_{n\geq 1} u_n$ also converges for all $\alpha > 1$.
2. If $l = +\infty$ and $\alpha \leq 1$, still according to the definition of the limit

$$\exists n_1 \in \mathbb{N},\ \forall n \in \mathbb{N},\ n \geq n_1,\ \text{we have } n^\alpha u_n > \epsilon. \tag{1.21}$$

Let us choose $\epsilon = 1$, we then have $u_n > \dfrac{1}{n^\alpha}$. Since $\sum_{n\geq 1} \dfrac{1}{n^\alpha}$ diverges for all $\alpha \leq 1$, the series $\sum_{n\geq 1} u_n$ also diverges for all $\alpha \leq 1$.

3. The third property is the intersection of properties 1 and 2. □

D'Alembert's rule

D'Alembert's rule amounts to comparing a series with positive terms to a geometric series.

Proposition 1.2.4. *Let $\sum_{n\geq 1} u_n$ be a series with strictly positive real terms. Suppose that there exists a positive real number l (or $l = +\infty$), such that $\lim_{n\to+\infty} \frac{u_{n+1}}{u_n} = l$. We then have:*

1. *If $l < 1$, the series $\sum_{n\geq 1} u_n$ converges .*
2. *If $l > 1$, the series $\sum_{n\geq 1} u_n$ diverges.*

Proof. By definition

$$\lim_{n\to+\infty} \frac{u_{n+1}}{u_n} = l. \Longleftrightarrow \forall \epsilon > 0,\ \exists n_0 \in \mathbb{N},\ \forall n \in \mathbb{N},\ n \geq n_0, \text{ we have } \left|\frac{u_{n+1}}{u_n} - l\right| < \epsilon. \tag{1.22}$$

1. If $l < 1$, let us choose ϵ, such that $l + \epsilon = k < 1$, we therefore have $\frac{u_{n+1}}{u_n} < k = \frac{v_{n+1}}{v_n}$(by setting $v_n = k^n$). We therefore have:

$$\frac{u_{n+1}}{v_{n+1}} < \frac{u_n}{v_n} < .... < \frac{u_{n_0}}{v_{n_0}} = a\ (a > 0). \tag{1.23}$$

That is, $u_n < av_n$. Since $\sum_{n\geq 1} v_n$ converges, the series $\sum_{n\geq 1} u_n$ converges .
2. If $l > 1$, let us choose ϵ, such that $l - \epsilon = k \geq 1$, we therefore have $\frac{u_{n+1}}{u_n} \geq 1$. Since (u_n) is increasing non-identically zero, we therefore have $\lim_{n\to+\infty} u_n \neq 0$, and then the series $\sum_{n\geq 1} u_n$ diverges. □

Cauchy's rule

Cauchy's rule also amounts to comparing a series with positive terms to a geometric series.

Proposition 1.2.5. *Let $\sum\limits_{n\geq 1} u_n$ be a series with strictly positive real terms. Suppose that there exists a positive real number l (or $l=+\infty$), such that $\lim_{n\to+\infty}\sqrt[n]{u_n}=l$. We then have:*

1. *If $l<1$, the series $\sum\limits_{n\geq 1} u_n$ converges .*
2. *If $l>1$, the series $\sum\limits_{n\geq 1} u_n$ diverges.*

Proof. By definition

$$\lim_{n\to+\infty}\sqrt[n]{u_n}=l \Longleftrightarrow \forall\epsilon>0,\ \exists n_0\in\mathbb{N},\ \forall n\in\mathbb{N},\ n\geq n_0,\ \text{we have } \left|\sqrt[n]{u_n}-l\right|<\epsilon. \tag{1.24}$$

1. If $l<1$, let's choose ϵ, such that $l+\epsilon=k<1$, we then have $u_n<k^n$. Since $\sum\limits_{n\geq 1} k^n$ converges for all $k<1$ (geometric series), the series $\sum\limits_{n\geq 1} u_n$ also converges .

2. If $l>1$, let us choose ϵ, such that $l-\epsilon=k\geq 1$, we therefore have $u_n\geq 1$. Since $\lim_{n\to+\infty}u_n\neq 0$, the series $\sum\limits_{n\geq 1} u_n$ diverges . □

Raabe and Duhamel rule

The Raabe-Duhamel rule amounts to comparing a given series with positive terms to a Riemann series.

Proposition 1.2.6. *Let $\sum\limits_{n\geq 1} u_n$ be a series with positive terms. Assume that the following limit exists:*

$$l=\lim_{n\to+\infty} n\left(\frac{u_n}{u_{n+1}}-1\right), \tag{1.25}$$

we then have:

1. *If $l>1$, the series $\sum\limits_{n\geq 1} u_n$ converges.*
2. *If $l<1$, the series $\sum\limits_{n\geq 1} u_n$ diverges.*

Proof. By definition:

$$\lim_{n\to+\infty} n\left(\frac{u_n}{u_{n+1}}-1\right)=l \Leftrightarrow \forall\epsilon>0,\ \exists n_0\in\mathbb{N},\ \forall n\in\mathbb{N},\ n\geq n_0,\ \text{we have: } l-\epsilon<n\left(\frac{u_n}{u_{n+1}}-1\right)<l+\epsilon. \tag{1.26}$$

1. Suppose that $l>1$ and choose ϵ, such that $n\left(\frac{u_n}{u_{n+1}}-1\right)>l-\epsilon=q>1$.

Let $m\in\mathbb{R}_+^*$ such that $m\in\,]1,q[$, the series of general term $w_n=\dfrac{1}{n^m}$ is therefore convergent.

We can write:

$$\frac{w_n}{w_{n+1}} = \left(\frac{n+1}{n}\right)^m = \left(1+\frac{1}{n}\right)^m. \tag{1.27}$$

By performing the limited development of the function $x \mapsto \dfrac{1}{1+x}$ in the neighborhood of 0, we obtain:

$$\frac{w_n}{w_{n+1}} = 1 + \frac{m}{n} + \frac{1}{n^2}\delta(n), \tag{1.28}$$

where $\dfrac{\delta(n)}{n} \to 0$, when $n \to +\infty$. That is to say:

$$\text{For all } \eta = q - m > 0,\ \exists n_1 \in \mathbb{N},\ \forall n \in \mathbb{N},\ n \geq n_1,\ \text{we have: } \frac{\delta(n)}{n} < q - m. \tag{1.29}$$

We then have the following inequalities:

$$m + \frac{\delta(n)}{n} < q < n\left(\frac{u_n}{u_{n+1}} - 1\right),\ \text{for all } n \geq \max(n_0, n_1) = n_2, \tag{1.30}$$

or in an equivalent manner:

$$\frac{w_n}{w_{n+1}} = 1 + \frac{m}{n} + \frac{1}{n^2}\delta(n) \leq \frac{u_n}{u_{n+1}}. \tag{1.31}$$

That is to say:

$$\frac{u_{n+1}}{w_{n+1}} \leq \frac{u_n}{w_n} \leq \ldots \leq \frac{u_{n_2}}{w_{n_2}} = a\ (a > 0 \tag{1.32}$$

Since $\sum_{n\geq 1} w_n$ is convergent, it follows from the comparison theorem that $\sum_{n\geq 1} u_n$ is also convergent.

2. Now, suppose that $l < 1$ and choose ϵ, such that:

$$n\left(\frac{u_n}{u_{n+1}} - 1\right) < l + \epsilon = q \leq 1,\ \text{for all } n \geq n_0. \tag{1.33}$$

Let us also consider the series with general term $w_n = \frac{1}{n}$, which is divergent. From the inequality (1.33), we can easily see:

$$\frac{u_n}{u_{n+1}} \leq 1 + \frac{1}{n} = \frac{n+1}{n} = \frac{w_n}{w_{n+1}}. \tag{1.34}$$

Since $\sum_{n\geq 1} w_n$ is divergent, it follows from the previous method that $\sum_{n\geq 1} u_n$ is also divergent. □

Bertrand Series

Definition 1.2.3. *A Bertrand series is any numerical series whose general term*

$$u_n = \frac{\ln^{\beta}(n)}{n^{\alpha}},\ \alpha \in \mathbb{R}_*^+ \text{ and } \beta \in \mathbb{R}.$$

Proposition 1.2.7. *The Bertrand series is:*

1. *convergent, if and only if*

   1.1. $\alpha > 1$ *and* $\beta \in \mathbb{R}$,

   or else

   1.2. $\alpha = 1$ *and* $\beta < -1$.

2. *divergent if and only if*

   2.1. $\alpha < 1$ *and* $\beta \in \mathbb{R}$,

   or else

   2.2. $\alpha = 1$ *and* $\beta \geq -1$.

Proof. 1.1. Let's assume that $\alpha > 1$ and $\beta \in \mathbb{R}$. So it exists $\gamma \in]1, \alpha[$ verifying

$$\lim_{n\to+\infty} n^{\gamma} u_n = \lim_{n\to+\infty} \frac{\ln^{\beta}(n)}{n^{\alpha-\gamma}} = 0,\ (\text{since } \alpha - \gamma > 0). \tag{1.35}$$

Since $\gamma > 1$, the use of Proposition 1.2.3 confirms the convergence of the considered series.

1.2 and 2.2. Now suppose that $\alpha = 1$. We can write $u_n = \frac{\ln^{\beta}(n)}{n}$.

* If $\beta = 0$, $\sum_{n\geq1} u_n = \sum_{n\geq1} \frac{1}{n}$ which is divergent.

 * If $\beta > 0$, we have:

$$\lim_{n\to+\infty} n u_n = \lim_{n\to+\infty} \ln^{\beta}(n) = +\infty, \tag{1.36}$$

and the series $\sum_{n\geq1} u_n$ is divergent by applying comparaison theorem.

 * If $\beta < 0$, consider the function f defined by:

$$f(x) = \frac{\ln^{\beta}(x)}{x^{\alpha}},\ x \in]\delta, +\infty[\ (\delta > 1). \tag{1.37}$$

This function is positive defined, continuous and decreasing on $]\delta, +\infty[$. According to Cauchy's theorem, the series $\sum_{n\geq2} \frac{\ln^{\beta}(n)}{n^{\alpha}}$ and the generalized

integral $\int_{\delta}^{+\infty} \frac{\ln^{\beta}(x)}{x^{\alpha}} dx$ are of the same nature.
We know that

$$\int_{\delta}^{t} \frac{\ln^{\beta}(x)}{x^{\alpha}} dx = \begin{cases} \frac{1}{\beta+1}\left[\ln^{\beta+1}(t) - \ln^{\beta+1}(\delta)\right], \text{ if } \beta \neq -1, \\ \ln\left(\frac{\ln t}{\ln \delta}\right), \text{ if } \beta = -1. \end{cases} \tag{1.38}$$

Which gives:

$$\lim_{t\to+\infty} \int_{\delta}^{t} \frac{\ln^{\beta}(x)}{x^{\alpha}} dx = \begin{cases} -\frac{\ln^{\beta+1}(\delta)}{\beta+1}, \text{ if } \beta < -1, \\ +\infty, \text{ if } \beta > -1, \\ +\infty, \text{ if } \beta = -1 \end{cases} \tag{1.39}$$

So, if$\alpha = 1$ and $\beta < -1$, the integral $\int_{\delta}^{+\infty} \frac{\ln^{\beta}(x)}{x^{\alpha}} dx$ converges, whereas if $\alpha = 1$ and $\beta \geq -1$, the integral $\int_{\delta}^{+\infty} \frac{\ln^{\beta}(x)}{x^{\alpha}} dx$ diverges.
2.2 Now, let us suppose that $\alpha < 1$ and $\beta \in \mathbb{R}$. So it exists $\gamma \in]\alpha, 1[$ verifying

$$\lim_{n\to+\infty} n^{\gamma} u_n = \lim_{n\to+\infty} n^{\gamma-\alpha} \ln^{\beta}(n) = +\infty, \text{ since } \gamma - \alpha > 0. \tag{1.40}$$

Since $\gamma < 1$, the use of Proposition 1.2.3 confirms the divergence of the series $\sum_{n\geq 1} u_n$. □

1.3 Series of arbitrary sign

1.3.1 Convergence rules for series of arbitrary sign

Abel's Rule

Proposition 1.3.1. *Let (b_n) be a positive sequence decreasing towards 0, and let (a_n) be a sequence verifying:*

$$\exists M > 0, \forall n \in \mathbb{N}, \left| A_n = \sum_{k=1}^{n} a_k \right| \leq M. \tag{1.41}$$

Then, the series of general term $u_n = a_n b_n$ is convergent.

Proof. We can write:

$$\begin{aligned}\sum_{n\geq 1}^{+\infty} u_n &= a_1(b_1-b_2)+(a_1+a_2)(b_2-b_3)+...+(a_1+a_2+...+a_n)(b_n-b_{n-1})+...\\ &= \sum_{n\geq 1}^{+\infty}(a_1+a_2+...+a_n)(b_n-b_{n+1})\\ &= \sum_{n\geq 1}^{+\infty} A_n(b_n-b_{n+1}). \qquad (1.42)\end{aligned}$$

We can then see the series of general terms appear:

$$v_n = A_n(b_n - b_{n+1}). \qquad (1.43)$$

We just need to show that this series is convergent. Indeed:

$$\begin{aligned}|v_n| &= |A_n(b_n-b_{n+1})|\\ &\leq M(b_n-b_{n+1}), \text{ car } b_n-b_{n+1}\geq 0 \ ((b_n) \text{ is decreasing}). \qquad (1.44)\end{aligned}$$

The series $\sum_{n\geq 1}^{+\infty}(b_n - b_{n+1})$ is convergent, since the sequence of its partial sums is verified:

$$\sum_{k=1}^{n}(b_k - b_{k+1}) = b_1 - b_{n+1} \to b_1, \text{ as } n \to +\infty. \qquad (1.45)$$

The comparison theorem asserts the absolute convergence of the series $\sum_{n\geq 1}^{+\infty} v_n$. It follows that the starting series $\sum_{n\geq 1}^{+\infty} u_n$ is convergent. □

1.3.2 Alternating series

Definition 1.3.1. *An alternating series is any series whose general term*

$$u_n = (-1)^n a_n, a_n \geq 0, \textit{ for all } n \in \mathbb{N}. \qquad (1.46)$$

Convergence rule for alternating series

Proposition 1.3.2. *(Leibniz criterion)*

Let $\sum_{n\geq 0} u_n$ *be an alternating series. If* $(|u_n|$ *is a sequence decreasing towards* 0*, then* *the series* $\sum_{n\geq 0} u_n$ *is convergent, moreover we have:*

$$\left|\sum_{k=n+1}^{+\infty}(-1)^k u_k\right| \leq u_{n+1}. \tag{1.47}$$

Proof. The series $\sum_{n\geq 0} u_n$ is alternating, so we can write:

$$u_n = u_n = a_n \times b_n, \text{ such thate } b_n = (-1)^n \text{ and } a_n \geq 0, \text{ for all } n \in \mathbb{N}. \tag{1.48}$$

Since $\left|\sum_{k=0}^{n} b_k\right| \leq 1$ and ($|u_n| = a_n$) a sequence decreasing towards 0, the use of Abel's criterion shows the convergence of the series $\sum_{n\geq 0} u_n$.
Let now (S_n) be the partial sum of the series $\sum_{n\geq 0} u_n$, we then have

$$S_{2n+2} - S_{2n} = u_{2n+2} - u_{2n+1} \leq 0, \tag{1.49}$$

$$S_{2n+1} - S_{2n-1} = -u_{2n+1} + u_{2n} \geq 0, \text{ for all } n \in \mathbb{N}. \tag{1.50}$$

So the sequence (S_{2n}) is decreasing and the sequence (S_{2n+1}) is increasing. Moreover we have:

$$S_{2n+1} - S_{2n} = -u_{2n+1}, \text{ for all } n \in \mathbb{N}. \tag{1.51}$$

Which shows that the sequence $(S_{2n+1} - S_{2n})$ tends to 0. The sequences (S_{2n}) and (S_{2n+1}) are therefore adjacent and thus both converge to the same finite limit S. Consequently the sequence (S_n) converges to S, which shows once again that the series $\sum_{n\geq 0} u_n$ converges. Furthermore, we have:

$$S_{2n+1} \leq S \leq S_{2n+2} = S_{2n+1} + u_{2n+2}, \text{ for all } n \in \mathbb{N}. \tag{1.52}$$

That is to say

$$S - S_{2n+2} =\leq u_{2n+2}, \text{ for all } n \in \tag{1.53}$$

We also have

$$S_{2n} - u_{2n+1} = S_{2n+1} \leq S \leq S_{2n}, \text{ for all } n \in \mathbb{N}, \tag{1.54}$$

or in an equivalent manner:

$$-u_{2n+1} \leq S - S_{2n} \leq 0, \text{ for all } n \in \mathbb{N}. \tag{1.55}$$

It follows that, for all $n \in \mathbb{N}$:

$$\left|\sum_{k=n+1}^{+\infty}(-1)^k u_k\right| = |S - S_n| \leq u_{n+1}, \tag{1.56}$$

where S is the sum of the series. □

1.3.3 Absolutely convergent series

Definition 1.3.2. *A series with general term u_n is said to be absolutely convergent if the series with general term $|u_n|$ converges.*

Proposition 1.3.3. *If the numerical series with general term u_n converges absolutely, then this series is convergent.*

Proof. Suppose that the series of general term u_n converges absolutely. Applying the Cauchy criterion to the series of general term $|u_n|$, we find:

$$\forall \epsilon > 0,\ \exists n_0 \in \mathbb{N},\ \forall p, q \in \mathbb{N},\ p > q \geq n_0,\ \text{we have} \sum_{k=q+1}^{p} |u_k| < \epsilon. \tag{1.57}$$

On the other hand, we have:

$$\left|\sum_{k=q+1}^{p} u_k\right| \leq \sum_{k=q+1}^{p} |u_k| < \epsilon. \tag{1.58}$$

The series of the general term u_n then verifies the Cauchy criterion. This series is therefore indeed convergent. □

Remak 1.4. *The converse of this proposition is false. For example, the series with general term $\dfrac{(-1)^n}{n}$ converges, while it does not converge absolutely.*

Proposition 1.3.4. *Let $\sum_{n\geq 0} u_n$ be a numerical series with arbitrary terms. Suppose that there exists a positive numerical sequence verifying:*

$$\exists n_0 \in \mathbb{N}, \forall n \geq n_0,\ \text{we have}\ |u_n| \leq v_n, \tag{1.59}$$

then, if $\sum_{n\geq 0} v_n$ converges, $\sum_{n\geq 0} u_n$ converges absolutely.

Proof. The proof proceeds immediately, using the comparison theorem of series with positive terms. □

1.3.4 Semi-convergent series

Definition 1.3.3. *A series with general term u_n is said to be semi-convergent, when it converges without being absolutely convergent.*

We also have the following immediate properties:

Proposition 1.3.5. *Suppose that the two series of respective general terms u_n and v_n are absolutely convergent (respectively semi-convergent), then the sum series of general term $(u_n + v_n)$ is absolutely convergent (respectively semi-convergent), and the series produced by a scalar of general term αu_n is absolutely convergent (respectively semi-convergent), for all $\alpha \in \mathbb{k}(\mathbb{k} = \mathbb{R}$ or $\mathbb{C})$.*

Proof. The proof of this proposition proceeds immediately using the Cauchy criterion. □

Example 1.3.1. *It can be easily shown that the alternating Riemann series $\sum_{n\geq 1} \frac{(-1)^n}{n^\alpha}$, $\alpha \in \mathbb{R}$ is:*

* *divergent for all $\alpha \leq 0$,*

* *absolutely convergent, for all $\alpha > 1$,*

* *semi-convergent, for all $0 < \alpha \leq 1$.*

1.3.5 Additional properties of series convergent

Property 1: Use of the D'Alembert and Cauchy criteria

We know that these two criteria apply a priori to series with positive terms. Let $\sum_{n\geq 0} u_n$ be a series with of arbitray sign. We can therefore perfectly use these criteria with any series of terms, but we must be very careful not to forget the absolute values.
Let us now suppose that $l_1 = \lim_{n\to+\infty} \left|\frac{u_{n+1}}{u_n}\right|$ exists (respectively $l_2 = \lim_{n\to+\infty} \sqrt[n]{|u_n|}$ exists).

* If $l_1 < 1$ (respect. $l_2 < 1$), the D'Alembert criterion (resp. the Cauchy criterion) asserts that the series $\sum_{n\geq 0} |u_n|$ is convergent. The $\sum_{n\geq 0} u_n$ is then absolutely convergent.

* If $l_1 > 1$ (respect. $l_2 > 1$), the D'Alembert criterion (resp. the Cauchy criterion) asserts that the general term $|u_n|$ tends to $+\infty$. The series $\sum_{n\geq 0} u_n$ is then divergent.

Example 1.3.2. *Consider the series of general term* $u_n = \left(\dfrac{3n+1}{n+1}\right)^{3n} \alpha^n$, $\alpha \in \mathbb{R}$ *and* $n \geq 0$.

We have $\lim_{n\to+\infty} \sqrt[n]{|u_n|} = 27\,|\alpha|$. *the Cauchy criterion states that:*

1. *If* $|\alpha| < \dfrac{1}{27}$, *the considered series is absolutely convergent.*
2. *1. If* $|\alpha| > \dfrac{1}{27}$, *the considered series is divergent.*
3. *If 1. If* $|\alpha| = \dfrac{1}{27}$, *the value absolute of general term becomes*

$$|u_n| = \left(\frac{3n+1}{3n+3}\right)^{3n} = \left(1 - \frac{2}{3n+3}\right)^{3n} \to \exp(-2)(\neq 0), \textit{ when } n \to +\infty,$$

and the considered series is divergent.

Property 3: Use of Limited Developments

By performing a limited development of the general term u_n of the series $\sum_{n\geq 0} u_n$ in the neighborhood of infinity at a sufficiently high order (to have an absolutely convergent remainder), we can quickly conclude on the nature of this series.

As an example, the series of general term $u_n = \dfrac{(-1)^n}{n+(-1)^n}$. We can éwrite:

$$u_n = \frac{(-1)^n}{n} \times \frac{1}{1+\dfrac{(-1)^n}{n}}. \tag{1.60}$$

By performing the limited development of the function $x \mapsto \dfrac{1}{1+x}$ in the neighborhood of 0, we obtain:

$$\begin{aligned} u_n &= \frac{(-1)^n}{n}\left(1 - \frac{(-1)^n}{n} + \frac{1}{n}\epsilon(n)\right) \\ &= \frac{(-1)^n}{n} - \frac{1}{n^2} + \frac{(-1)^n}{n^2}\epsilon(n), \text{ où } \epsilon(n) \to 0, \text{ when } n \to +\infty. \end{aligned} \tag{1.61}$$

We therefore have the sum of two convergent series and an absolutely convergent remainder. The considered series is therefore convergent.

1.4 Cauchy product of series

Definition 1.4.1. *We consider two numerical series with general terms u_n and v_n, respectively. We call the Cauchy product of these two series the series with general term:*

$$w_n = \sum_{k=0}^{n} u_k v_{n-k}. \tag{1.62}$$

Proposition 1.4.1. *Let $\sum_{n\geq 0} u_n$ and $\sum_{n\geq 0} u_n$ be two absolutely convergent series. Then the Cauchy product series with general term defined by (1.62) is absolutely convergent, and moreover, we have:*

$$\sum_{n\geq 0} w_n = \left(\sum_{n\geq 0} u_n\right) \times \left(\sum_{n\geq 0} u_n\right). \tag{1.63}$$

Proof. For all $n \in \mathbb{N}$, Let us consider the following partial sums:

$$S_n = \sum_{k=0}^{n} u_k, T_n = \sum_{k=0}^{n} v_n \text{ and } R_n = \sum_{k=0}^{n} w_n. \tag{1.64}$$

Let us also consider the following notations:

$$S = \sum_{n\geq 0} |u_n| \text{ and } T = \sum_{n\geq 0} |v_n|. \tag{1.65}$$

The two sequences (S_n) and (T_n) are convergent, so they satisfy the following Cauchy criterion:

$$\forall \epsilon > 0,\ \exists n_0 \in \mathbb{N},\ \forall p, n \in \mathbb{N},\ p > n \geq n_0, \text{ we have } \left|\sum_{k=n+1}^{p} u_k\right| < \epsilon \text{ and } \left|\sum_{k=n+1}^{p} v_k\right| < \epsilon, \tag{1.66}$$

On the one hand, $R_{2n} - S_n T_n$ can be rewritten in the form:

$$\begin{aligned} R_{2n} - S_n T_n &= u_0(v_{n+1} + ... + v_{2n}) + u_1(v_{n+1} + ... + v_{2n-1}) + ... + u_{n-1}v_{n+1} \\ &\quad + v_0(u_{n+1} + ... + u_{2n}) + v_1(u_{n+1} + ... + u_{2n-1}) + ... + v_{n-1}u_{n+1}. \end{aligned}$$

For all $p, n \in \mathbb{N}$, $p > n \geq n_0$, we then have

$$\begin{aligned} |R_{2n} - S_n T_n| &\leq \epsilon\left(\sum_{k=0}^{n} |u_k| \ \sum_{k=0}^{n} |v_k|\right) \\ &\leq \epsilon(S + T). \end{aligned} \tag{1.67}$$

So (R_{2n}) is convergent, and furthermore, we have:

$$\lim_{n\to+\infty} R_{2n} = \lim_{n\to+\infty} S_n T_n = \left(\sum_{n\geq 0} u_n\right) \times \left(\sum_{n\geq 0} u_n\right). \tag{1.68}$$

On the other hand, for all $n \geq n_0$, we have:

$$\begin{aligned} |R_{2n+1} - R_{2n}| &= |(u_0 v_{2n+1} + ... + u_{n+1} v_n) + (v_0 u_{2n+1} + ... + v_{n+1} u_n)| \\ &\leq \epsilon(S+T), \end{aligned} \tag{1.69}$$

which ensures the convergence of (R_{2n+1}).
Since (R_{2n}) and (R_{2n+1}) are convergent, (R_n) is also convergent, and moreover:

$$\lim_{n\to+\infty} R_n = \lim_{n\to+\infty} R_{2n+1} = \lim_{n\to+\infty} R_{2n} = \left(\sum_{n\geq 0} u_n\right) \times \left(\sum_{n\geq 0} u_n\right). \tag{1.70}$$

That is to say:

$$\left(\sum_{n\geq 0} w_n\right) = \left(\sum_{n\geq 0} u_n\right) \times \left(\sum_{n\geq 0} u_n\right).$$

□

1.5 Exercises about chapter 1

Exercise 1.5.1. *Show that the following numerical series are convergent and calculate their sums:*

$$1) \sum_{n=2}^{+\infty} \frac{1}{n(n-1)} \qquad 2) \sum_{n=0}^{+\infty} \frac{n^2}{n!} \qquad 3)\sum_{n=0}^{+\infty} (-1)^{n+1} \frac{\cos(nx)}{2^n},\ x \in \mathbb{R}.$$

Exercise 1.5.2. *Study the nature of the following numerical series:*

$$1) \sum_{n=1}^{+\infty} n \sin\left(\frac{1}{n}\right) \qquad 2) \sum_{n=1}^{+\infty} \arctan\left(\frac{1}{n^2}\right) \qquad 3) \sum_{n=1}^{+\infty} \frac{\alpha^n}{\alpha^{2n} + \alpha^n + 1}\ (\alpha \geq 0)$$

$$4) \sum_{n=1}^{+\infty} \left(\frac{n+a}{n+b}\right)^{n^2},\ a \text{ et } b \in \mathbb{R} \qquad 5) \sum_{n=1}^{+\infty} \frac{\ln(n)}{n^2+2} \qquad 6) \sum_{n=1}^{+\infty} \frac{1 \times 3 \times ... \times (2n-1)}{2 \times 4 \times ... \times (2n)}$$

$$7) \sum_{n\geq 1} \frac{2^n}{n^2} \sin^{2n}(\theta),\ \theta \in \left[0, \frac{\pi}{2}\right] \qquad 8) \frac{\exp(inx)}{n},\ x \in \mathbb{R}.$$

Exercise 1.5.3. *Let θ be a real number. Study, according to the value of θ, the absolute convergence, the semi-convergence and the divergence of the following numerical series:*

$$1)\sum_{n=1}^{+\infty}\frac{(-1)^n}{n^\theta},\qquad 2)\sum_{n=1}^{+\infty}\left(\frac{2n-1}{n+1}\right)^{2n}\theta^n\,,\qquad 3)\sum_{n=1}^{+\infty}\frac{\cos(n)}{n^\theta+\cos(n)}.$$

Chapter 2

Sequences and series of functions

2.1 Sequences of functions

Let $\mathbb{k}$ be one of the fields $\mathbb{R}$ or $\mathbb{C}$ and let E and F be two non-empty subsets of $\mathbb{k}$.

Definition 2.1.1. *We call a sequence of functions any application f_n: $\mathbb{N} \to £$, where $£ = £(E, \mathbb{k})$ is the set of applications of E in F.*

2.1.1 Simple convergence of a sequence of functions

In general, to study the simple convergence of a sequence of functions $f_n(x)$ on a subset E of $\mathbb{R}$, we will try to fix the real x and we will study the corresponding numerical sequence.

Definition 2.1.2. *We say that the sequence of functions $(f_n)_{n\in\mathbb{N}}$ simply converges on E to a function $f(x_0)$, when the numerical series $(f_n(x_0))_{n\in\mathbb{N}}$ is convergent, for* *all $x_0 \in E$. We thus define a function f on the domain E by:*

$$f(x) = \lim_{n \to +\infty} f_n(x). \tag{2.1}$$

We will then say that f is the simple limit of the sequence of functions $(f_n)_{n \in \mathbb{N}}$.

Example 2.1.1. *Let the sequence of functions f_n be defined on $\mathbb{R}$ by*

$$f_n(x) = \frac{x}{x^2 + n}, \ n \in \mathbb{N}. \tag{2.2}$$

If $x = 0$, $f_n(0) = 0$, and the sequence converges to 0, and if $x \neq 0$, $\lim_{n \to +\infty} f_n(x) = \lim_{n \to +\infty} \dfrac{x}{n} = 0$.
Finally, the sequence f_n simply converges on $\mathbb{R}$, and its limit is $f(x) = 0$.

2.1.2 Uniform convergence of a sequence of functions

Definition 2.1.3. *We say that the sequence of functions $(f_n)_{n \in \mathbb{N}}$ converges uniformly on E to a function f if and only if:*

$$\forall \epsilon > 0, \ \exists n_0 \in \mathbb{N}, \ \forall n \geq n_0 \text{ and } \forall x \in E, \text{ we have } \left| f_n(x) - f(x) \right| < \epsilon. \tag{2.3}$$

Remak 2.1. *This integer n_0 obviously depends only on ϵ and not on x. On the other hand, if n_0 depends a priori on both x and ϵ, we will say that the convergence is simple on E.*

Remak 2.2. *Uniform convergence on E implies simple convergence on E.*

2.1.3 A sufficient condition for uniform convergence (convergence normal

Proposition 2.1.1. *Let $(f_n)_{n \in \mathbb{N}}$ be a sequence of functions which simply converges on E to a function f. If there exists a positive sequence (b_n) that converges to 0, such that*

$$\forall x \in E, \ \left| f_n(x) - f(x) \right| < b_n, \tag{2.4}$$

then the sequence of functions $(f_n)_{n\in\mathbb{N}}$ is uniformly convergent on E.
In this case, we say that the sequence of functions $(f_n)_{n\in\mathbb{N}}$ is normally convergent on E.

Proof. Suppose that b_n tends to 0, when $n \to +\infty$, that is:

$$\forall \epsilon > 0,\ \exists n_0 \in \mathbb{N},\ \text{such that } \forall n \geq n_0 \text{ we have } b_n < \frac{\epsilon}{2}. \tag{2.5}$$

Since:

$$\left|f_n(x) - f(x)\right| < b_n < \frac{\epsilon}{2},\ \text{for all } x \in E. \tag{2.6}$$

We find

$$\sup_{x\in E}\left|f_n(x) - f(x)\right| \leq \frac{\epsilon}{2} < \epsilon,\ \text{for all } x \in E. \tag{2.7}$$

□

2.1.4 A necessary and sufficient condition for uniform convergence

Proposition 2.1.2. *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions that simply converges on E to a function f. For $(f_n)_{n\in\mathbb{N}}$ to be uniformly convergent to f on E, it is necessary and sufficient that the numerical sequence (a_n) which is defined by:*

$$a_n = \sup_{x\in E}\left|f_n(x) - f(x)\right|, \tag{2.8}$$

is convergent to 0.

Proof. $\underline{\Rightarrow}$ Suppose that $(f_n)_{n\in\mathbb{N}}$ be uniformly convergent to f, then:

$$\forall \epsilon > 0,\ \exists n_0 \in \mathbb{N},\ \forall n \geq n_0 \text{ and } \forall x \in E,\ \left|f_n(x) - f(x)\right| < \frac{\epsilon}{2}. \tag{2.9}$$

As a result:

$$a_n = \sup_{x\in E}\left|f_n(x) - f(x)\right| \leq \frac{\epsilon}{2} < \epsilon. \tag{2.10}$$

$\underline{\Leftarrow}$ Snow let's assume that a_n tends to 0, when $n \to +\infty$, we then have:

$$\forall \epsilon > 0,\ \exists n_0 \in \mathbb{N},\ \forall n \geq n_0,\ a_n < \epsilon. \tag{2.11}$$

As a result:

$$\left|f_n(x) - f(x)\right| \leq \sup_{x\in E}\left|f_n(x) - f(x)\right| = a_n < \epsilon, \text{ for all } x \in E. \tag{2.12}$$

□

Remak 2.3. *A sufficient condition for the sequence of functions $(f_n)_{n\in\mathbb{N}}$ not to converge uniformly to f on E is the existence of a sequence of points $(x_n) \subset E$, verifying:*

$$\left|f_n(x_n) - f(x_n)\right| \nrightarrow 0, \text{ when } n \to +\infty. \tag{2.13}$$

Example 2.1.2. *Let the sequence of functions f_n be defined on $[0, 1]$ by:*

$$f_n(x) = x^n(1 - x), \ n \in \mathbb{N}. \tag{2.14}$$

** If $x = 0$ or $x = 1$, $f_n(0) = f_n(1) = 0$, and the sequence converges to 0.*
** If $x \in]0, 1[$, $\lim_{n\to+\infty} f_n(x) = 0$.*
Finally, the sequence f_n simply converges on $[0, 1]$ and its limit is the zero function $f(x) = 0$.
By performing a simple calculation, we find

$$\sup_{x\in[0,1]}\left|f_n(x) - f(x)\right| = a_n, \text{ such that } a_n = \frac{1}{n+1}\left(\frac{n}{n+1}\right)^n. \tag{2.15}$$

This last quantity is equivalent to the neighborhood of infinity by $\frac{1}{ne}$. Since this quantity tends to 0, when n tends to $+\infty$, the sequence of functions considered converges uniformly to 0 on the segment $[0, 1]$.

2.1.5 Cauchy criterion for uniform convergence

Proposition 2.1.3. *For the sequence of functions $(f_n)_{n\in\mathbb{N}}$ to be uniformly convergent to f on E, it is necessary and sufficient that:*

$$\forall\epsilon > 0, \ \exists n_0 \in \mathbb{N}, \ \forall p, q \in \mathbb{N}, \ p > q \geq n_0 \text{ and } \forall x \in E, \ \left|f_p(x) - f_q(x)\right| < \epsilon. \tag{2.16}$$

Proof. $\Rightarrow$ Suppose that $(f_n)_{n\in\mathbb{N}}$ be uniformly convergent to f, then:

$$\forall \epsilon > 0,\ \exists n_0 \in \mathbb{N},\ \forall n \geq n_0 \text{ and } \forall x \in E,\ \left|f_n(x) - f(x)\right| < \frac{\epsilon}{2}. \tag{2.17}$$

Let $\epsilon > 0$, then for all $p > q \geq n_0$, we have:

$$\begin{aligned} \left|f_p(x) - f_q(x)\right| &\leq \left|f_p(x) - f(x)\right| + \left|f_q(x) - f(x)\right| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \text{ for all } x \in E. \end{aligned}$$

$\Leftarrow$ Now let's assume that:

$$\forall \epsilon > 0,\ \exists n_0 \in \mathbb{N},\ \forall p, q \in \mathbb{N},\ p > q \geq n_0 \text{ and } \forall x \in E,\ \left|f_p(x) - f_q(x)\right| < \epsilon. \tag{2.18}$$

Let p tend towards $+\infty$, we fin the result. □

2.1.6 Properties of uniformly convergent sequences of functions

Continuity of the uniform limit of a sequence of functions

Proposition 2.1.4. *Let (f_n) be a sequence of continuous functions on a segment $[a, b]$, converging uniformly on the same segment to a function f. Then f is a continuous function on $[a, b]$.*

Proof. Let x_0 be any point of $[a, b]$.
f_n is continuous at the point x_0, then:

$$\forall \epsilon > 0,\ \exists \delta > 0,\ \forall x \in [a, b],\ |x - x_0| < \delta \Rightarrow \left|f_n(x) - f_n(x_0)\right| < \frac{\epsilon}{3}. \tag{2.19}$$

$(f_n)_{n\in\mathbb{N}}$ converges uniformly to f, then:

$$\forall \epsilon > 0,\ \exists n_0 \in \mathbb{N},\ \forall n \geq n_0 \text{ and } \forall x \in E,\ \left|f_n(x) - f(x)\right| < \frac{\epsilon}{3}. \tag{2.20}$$

We can write:

$$\begin{aligned} \left|f(x) - f(x_0)\right| &= \left|f(x) - f_n(x) + f_n(x) - f_n(x_0) + f_n(x_0) - f(x_0)\right| \\ & \quad \left|f(x) - f_n(x)\right| + \left|f_n(x) - f_n(x_0)\right| + \left|f_n(x_0) - f(x_0)\right| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon. \end{aligned}$$

□

Remak 2.4. *Under the conditions of the previous proposition, we can write:*

$$\lim_{x\to x_0}\lim_{n\to+\infty} f_n(x) = \lim_{n\to+\infty}\lim_{x\to x_0} f_n(x) = f(x_0). \tag{2.21}$$

Integration of the uniform limit of a sequence of functions

Proposition 2.1.5. *Let (f_n) be a sequence of continuous functions on a segment $[a,b]$, converging uniformly on the same segment to a function f. Then f is an integrable function on $[a,b]$, and moreover:*

$$\int_a^b f(x)dx = \lim_{n\to+\infty}\int_a^b f_n(x)dx. \tag{2.22}$$

Proof. Under the assumptions of proposition 2.1.5, the uniform limit f is also continuous, which ensures the integrability of $f_n(x)$ and f.
$(f_n)_{n\in\mathbb{N}}$ converges uniformly to f, then:

$$\forall\epsilon>0,\ \exists n_0\in\mathbb{N},\ \forall n\geq n_0 \text{ and } \forall x\in E,\ \left|f_n(x)-f(x)\right| < \frac{\epsilon}{b-a}. \tag{2.23}$$

We can write:

$$\begin{aligned}\left|\int_a^b f_n(x)dx - \int_a^b f(x)dx\right| &\leq \int_a^b \left|(f_n(x)-f(x)\right|dx \\ &< \frac{\epsilon}{b-a}\left|\int_a^b dx\right| = \epsilon.\end{aligned}$$

□

Corollary 2.1.1. *Under the assumptions of proposition 2.1.5, we deduce that the sequence of integrals $\left(\int_a^x f_n(y)dy\right)_n$ is uniformly convergent to $\left(\int_a^x f(y)dy\right)_n$, for all $x\in[a,b]$.*

Proof. Since the rank n_0 in the relation (2.23) does not depend on b, it suffices to replace b by x. □

Differentiability of the uniform limit of a sequence of functions

Proposition 2.1.6. *Let (f_n) be a sequence of functions defined on a segment $[a, b]$ and verify the following three conditions:*
1. f_n, $n = 0, 1, \ldots$ are of class C^1 on a segment $[a, b]$.
2. (f_n) simply converges on the same segment to a function f.
3. The sequence of derivatives $(\dot{f_n})$ converges uniformly to a function g.
Then, the sequence of functions (f_n) converges uniformly to a derivable function f and moreover $\dot{f} = g$.

Proof. Since $(\dot{f_n})$ is a sequence of continuous functions on a segment $[a, b]$, converging uniformly on the same segment to a function g, then the use of the proposition of integration affirms that $\dot{f_n}$ is an integrable function on $[a, b]$, and moreover:

$$f_n(x) = f_n(a) + \int_a^x \dot{f_n}(x)dx. \tag{2.24}$$

According to Corollary 2.1.1, the sequence $\left(\int_a^x \dot{f_n}(x)dx\right)$ f_n converges uniformly, and the numerical sequence $(f_n(a))$ is also convergent, $(f_n(x))$ is therefore the sum of two uniformly convergent sequences, so it is uniformly convergent.
We have:

$$\lim_{n\to+\infty} \int_a^x \dot{f_n}(x)dx = \int_a^x g(x)dx, \tag{2.25}$$

On the other hand:

$$\begin{aligned} \lim_{n\to+\infty} \int_a^x \dot{f_n}(x)dx &= \lim_{n\to+\infty} (f_n(x) = f_n(a)) \\ &= f(x) - f(a). \end{aligned} \tag{2.26}$$

Using (2.25) and (2.26), we get:

$$\int_a^x g(x)dx = f(x) - f(a). \tag{2.27}$$

We derive this last equality, we find:

$$g(x) = \dot{f}(x). \tag{2.28}$$

□

Remak 2.5. *Under the conditions of the previous proposition, we can write:*

$$\lim_{n\to+\infty}\left(\frac{\partial}{\partial x}f_n(x)\right)=\frac{\partial}{\partial x}\left(\lim_{n\to+\infty}f_n(x)\right)=\acute{f}(x_0). \tag{2.29}$$

2.2 Series of functions

Definition 2.2.1. *Let (f_n) be a sequence of functions from E to $\mathbb{k}$. A series of functions with general term f_n is any expression of the form $=\sum_{k=0}^{+\infty} f_n(x)$.*
Let $S_n(x)=\sum_{k=0}^{n} f_n(x)$, $n\in\mathbb{N}$ and $x\in E$.
S_n is called the partial sum of order n of the series $\sum_{n\geq 0} f_n(x)$.

2.2.1 Simple convergence

Definition 2.2.2. *A series of functions with general term f_n is aid to ne simply convergent on a subset E of $\mathbb{R}$, if for all $x\in E$, the numerical series with general term $f_n(x)$ converges.*
If the series simply converges, the term:

$$R_n(x)=S(x)-S_n(x)=\sum_{k=n+1}^{+\infty} f_n(x),\ \ n\in\mathbb{N} \text{ and } x\in E. \tag{2.30}$$

is called the rest of order n of the series with general term f_n.
The convergence of the series of general term f_n is then expressed by the convergence of the sequence of partial sums $(S_n(x))$. That is to say:

$$\forall\epsilon>0,\ \exists n_0\in\mathbb{N},\ \forall n\geq n_0 \text{ and } \forall x\in E,\ \ |S_n(x)-S(x)|=|R_n(x)|<\epsilon. \tag{2.31}$$

Example 2.2.1. *Let us study the series of functions with a general term*

$$f_n(x)=\frac{x^n}{\sqrt{n}+1},\ n\geq 0 \text{ and } x\in\mathbb{R}. \tag{2.32}$$

For $x\neq 0$, the d'Alembert criterion gives us $\lim_{n\to+\infty}\left|\frac{f_{n+1}(x)}{f_n(x)}\right|=|x|$. The series converges when $|x|<1$ and diverges when $|x|>1$.

If $x = -1$, the series becomes alternating and verifies the convergence criterion. If $x = 1$, it diverges.
Finally, the series of functions simply converges on $[-1, 1[$.

2.2.2 Uniform convergence

Definition 2.2.3. *A series of functions with general term f_n, converges uniformly on a subset E of $\mathbb{R}$ and has the sum S, when the sequence of its partial sums is uniformly convergent on E, that is:*

$$\forall \epsilon > 0,\ \exists n_0 \in \mathbb{N},\ \forall n \geq n_0,\ \sup_{x \in E} |S_n(x) - S(x)| = \sup_{x \in E} |R_n(x)| < \epsilon. \tag{2.33}$$

To say that the sequence of partial sums converges uniformly on E therefore means that $(R_n)_{n \in \mathbb{N}}$ converges uniformly to 0 on E.

Remak 2.6. *We can define a norm of uniform convergence of S_n on E by:*

$$||S_n|| = \sup_{x \in E} |S_n(x)|\,. \tag{2.34}$$

The series of functions with a general term f_n converges uniformly and with a sum S if and only if the numerical sequence $(||S_n - S||)_{n \in \mathbb{N}}$ converges to 0.

2.2.3 Cauchy criterion for uniform convergence

Theorem 2.2.1. *For the series of functions with general term f_n to be uniformly convergent on E, it is necessary and sufficient that:*

$$\forall \epsilon > 0,\ \exists n_0 \in \mathbb{N},\ \forall p, q \in \mathbb{N},\ p > q \geq n_0 \text{ and } \forall x \in E,\ \sup_{x \in E} \left| \sum_{k=q+1}^{p} f_k(x) \right| < \epsilon. \tag{2.35}$$

Proof. The proof of this theorem is the same as for sequences by reasoning on the sequence of partial sums. □

Corollary 2.2.1. *The use of the uniform Cauchy criterion is often by its contraposition, to show that a series of functions does not converge uniformly.*

2.2.4 A necessary condition for uniform convergence

Proposition 2.2.1. *For a series of functions to be uniformly convergent, it is necessary that its general term tends to* 0 *uniformly.*

Proof. It suffices to apply the uniform Cauchy criterion on:

$$\left\| f_n \right\| = \sup_{x \in E} \left\| f_n(x) \right\| = \sup_{x \in E} \left\| S_n(x) - S_{n-1} \right\|. \quad (2.36)$$

□

2.2.5 A sufficient condition for uniform convergence (Weierstass criterion)

Proposition 2.2.2. *(Proposition and definition) Let $\sum_{n \geq 0} f_n(x)$ be a series of functions defined on E. If there exists a positive series $\sum_{n \geq 0} b_n$ that converges, such that:*

$$\forall x \in E, \ \left| f_n(x) \right| < b_n, \quad (2.37)$$

*then the series of functions $\sum_{n \geq 0} f_n(x)$ is absolutely and uniformly convergent on E. In this case, we say that the series of functions $(f_n)_{n \in \mathbb{N}}$ is **normally** convergent on E.*

Proof. From the inequality (2.37) and the comparison theorem, we deduce absolute convergence.
On the other hand, the numerical series $\sum_{n \geq 0} b_n$ converges, that is:

$$\forall \epsilon > 0, \ \exists n_0 \in \mathbb{N}, \text{ such thay if} \forall n \geq n_0 \sum_{k \geq n+1} b_k < \epsilon. \quad (2.38)$$

So

$$\forall \epsilon > 0, \ \exists n_0 \in \mathbb{N}, \ \forall n \geq n_0 \ \left| \sum_{k \geq n+1} f_k(x) \right| \leq \sum_{k \geq n+1} \left| f_k(x) \right| \leq \sum_{k \geq n+1} b_k < \epsilon. \quad (2.39)$$

This latter quantity independent of x, the rest of the series $\sum_{n \geq 0} f_n(x)$ converges uniformly to 0, the series $\sum_{n \geq 0} f_n(x)$ is therefore uniformly convergent. □

Example 2.2.2. *The series of functions* $\sum_{n\geq 0} \frac{\sin(nx)}{\alpha^n}$, $\alpha > 1$ *is normally convergent on* $\mathbb{R}$, *since* $\left|\frac{\sin(nx)}{\alpha^n}\right| \leq \left(\frac{1}{\alpha}\right)^n$, *general term of a convergent geometric series.*

2.2.6 Necessary and sufficient condition for normal convergence

Proposition 2.2.3. *For the series of functions* $\sum_{n\geq 0} f_n$ *to be normally convergent on* E, *it is necessary and sufficient that the numerical series* (a_n) *with general term:*

$$a_n = \sup_{x\in E} |f_n(x)|, \tag{2.40}$$

be convergent.

Proof. $\Rightarrow$ When the series of functions $\sum_{n\geq 0} f_n$ is normally convergent on E, there exists a positive convergent series of term b_n verifying:

$$\forall x \in E, \ |f_n(x)| \leq b_n, \tag{2.41}$$

As result

$$a_n = \sup_{x\in E} |f_n(x)| \leq b_n, \tag{2.42}$$

and the series $\sum_{n\geq 0} a_n$ is convergent.
$\Leftarrow$ Now let's assume that $\sum_{n\geq 0} a_n$ is convergent, then we have:

$$|f_n(x)| \leq \sup_{x\in E} |f_n(x)| = a_n < \epsilon, \ x \in E, \tag{2.43}$$

this is the definition of a normally convergent series. □

Example 2.2.3. *The series of functions* $\sum_{n\geq 0} f_n(x)$ *defined on* $[0, 1]$, *such that:*

$$f_n(x) = \begin{cases} x^n \ln^2 x \text{ if } x \in]0, 1], \\ 0, \text{ if } x = 0 \end{cases} \tag{2.44}$$

We have:

$$\dot{f_n}(x) = \ln(x)\,(2 + n\ln(x))\,x^{n-1} = 0, \text{ if } x = \exp(\frac{-2}{\pi}) = x_n. \tag{2.45}$$

As result

$$a_n = \sup_{x\in[0,1]} \left|f_n(x)\right| = f_n(x_n) = \frac{4}{n^2 e^2}, \tag{2.46}$$

general term of a convergent series. The series of functions is normally convergent on $[0,1]$.

Proposition 2.2.4. *The normal convergence of a series of functions on a subset E of $\mathbb{R}$ implies the uniform convergence of this series on E, and the converse is false.*

Proof. When the series of functions $\sum_{n\geq 0} f_n$ is normally convergent on E, the proof proceeds from the inequality:

$$\sup_{x\in E}\left|\sum_{k=q+1}^{p} f_k(x)\right| \leq \sum_{k=q+1}^{p} \sup_{x\in E}\left|f_k(x)\right| \tag{2.47}$$

and the Cauchy criterion.
The converse of this proposition is false. As an example, we take the series of functions with a general term

$$f_n(x) = \frac{(-1)^n}{n+x},\ x \in [0,1] \text{ and } n \geq 1. \tag{2.48}$$

This series is uniformly convergent without being normally convergent on $[0,1]$.
On the other hand:

$$|R_n(x)| \leq \left|f_{n+1}(x)\right| = \frac{1}{n+1+x} < \frac{1}{n+1} < \epsilon, \text{ for all } x \in [0,1], \tag{2.49}$$

which shows uniform convergence on $[0,1]$.
By against:

$$\sup_{x\in[0,1]}\left|f_n(x)\right| = \frac{1}{n}, \tag{2.50}$$

general term of a divergent series, the series is therefore not normally convergent. □

2.3 Properties of uniformly convergent series of functions

2.3.1 Continuity of the sum of a series of functions

Proposition 2.3.1. *Let be a series of functions of general term f_n, defined on the interval $[a, b]$, which converges uniformly and of sum S on $[a, b]$. If f_n is continuous on $[a, b]$, for all $n \in \mathbb{N}$, then S is also continuous on $[a, b]$, and moreover, we have the following éequality:*

$$\lim_{x \to x_0} \sum_{n \geq 0} f_n(x) = \sum_{n \geq 0} \lim_{x \to x_0} f_n(x) = S(x_0), \text{ for all } x_0 \in [a, b], \tag{2.51}$$

which is a case ***of inversion of limit and infinite sum.***

Proof. It suffices to apply Proposition 2.1.4 to the sequence (S_n) of partial sums of the series $\sum_{n \geq 0} f_n$, which are continuous as finite sums of continuous functions. □

Remak 2.7. *The condition of uniform convergence of the series of functions is sufficient but not necessary to ensure the continuity of the sums.*

Remak 2.8. *When the series of continuous functions of general term f_n simply converges on $[a, b]$ and has as sum a discontinuous function S, then $\sum_{n \geq 0} f_n$ does not converge uniformly on this interval.*

Example 2.3.1. *The series of general term continuous functions:*

$$f_n(x) = \sin^2(x) \cos^n(x), \; x \in \left[0, \frac{\pi}{2}\right], n \in \mathbb{N}. \tag{2.52}$$

converges simply on $\left[0, \frac{\pi}{2}\right]$ and has the sum:

$$S(x) = \begin{cases} \dfrac{\sin^2(x)}{1 - \cos(x)}, \; \text{if } x \in \left]0, \dfrac{\pi}{2}\right] \\ 0, \; si \; x = 0. \end{cases} \tag{2.53}$$

Since S is discontinuous at 0, $\sum_{n \geq 0} f_n$ does not converge uniformly on $\left[0, \frac{\pi}{2}\right]$.

2.3.2 Integration of the sum of a series of functions

Proposition 2.3.2. *Let a series of functions with general term f_n, defined on $[a, b]$, converges uniformly and with sum S on $[a, b]$. If f_n is continuous on $[a, b]$, for all $n \in \mathbb{N}$, then, the numerical series with general term $\int_a^b f_n(x)dx$ converges and has the sum $\int_a^b S(x)dx$, and moreover, we have the following equality:*

$$\int_a^b S(x)dx = \sum_{n\geq 0} \int_a^b f_n(x)dx = \int_a^b \left(\sum_{n\geq 0} f_n(x)dx \right),$$

which is a case of ***interversion sum and integral****.*

Proof. It suffices to apply Proposition 2.1.5 to the sequence of partial sums $(S_n)_{n\in\mathbb{N}}$ of the series $\sum_{n\geq 0} f_n(x)$. □

Example 2.3.2. *Let the series of functions with general term:*

$$f_n(x) = \frac{x^{2n}}{(2n)!},\ x \in [0, 1]\,.$$

This series converges uniformly on $[0, 1]$, since $\left|f_n(x)\right| \leq \frac{1}{(2n)!}$, for all $x \in [0, 1]$. According to the previous proposition, we then have:

$$\begin{aligned} \int_0^x \left(\sum_{n\geq 0} \frac{x^{2n}}{(2n)!} dx \right) &= \sum_{n\geq 0} \int_0^x \frac{x^{2n}}{(2n)!} dx \\ &= \sum_{n\geq 0} \frac{x^{2n+1}}{(2n+1)!} \\ &= \sinh(x), \textit{ for all } x \in [0, 1]\,. \end{aligned}$$

2.3.3 Derivability of the sum of a series of functions

Proposition 2.3.3. *We consider a series of general term functions f_n, derivable on the segment $[a, b]$ and verifying:*

1. The series of functions $\sum_{n\geq 0} f_n(x)$ simply converges on $[a, b]$.

2. The series of derivatives of general term $\dot{f}_n$ converges uniformly on $[a, b]$ and has

as sum a function g .
Then, the series of general term f_n is derivable term by term, and we have:

$$\dot{S}(x) = \frac{\partial}{\partial x}\left(\sum_{n\geq 0} f_n(x)\right) = \sum_{n\geq 0} \frac{\partial}{\partial x} f_n(x) = g(x),$$

Proof. It suffices to apply Proposition 2.1.5 to the sequence of partial sums $(S_n)_{n\in\mathbb{N}}$ of the series of general term f_n, which are derivable as finite sums of derivable functions. □

2.4 Exercises about chapter 2

Exercise 2.4.1. *Let (f_n) be a sequence of functions defined on the set $E_i \in \mathbb{R}$. Study the simple and uniform convergence of this sequence of functions on E_i in the following cases:*
1. $f_n(x) = \dfrac{1-nx^2}{1+nx^2}$, *$E_1 = [-a, a]$, then on $E_2 = [a, +\infty[\ (a > 0)$.*
2. $f_n(x) = \dfrac{x}{1+nx}$, *$E_3 = [0, 1]$.*
3. $f_n(x) = \cos(\dfrac{5+nx}{n})$, *$E_4 = \mathbb{R}$.*
4. $f_n(x) = \dfrac{\sin(nx)}{nx}$ *and $f_n(0) = 0$, $E_5 = \mathbb{R}$, then on $E_6 = [a, +\infty[\ (a > 0)$.*

Exercise 2.4.2. *Let the series of functions with general term:*

$$f_n(x) = \sin^2(x)\cos^n(x), \text{ for } n \geq 1 \text{ and } x \in \left[0, \frac{\pi}{2}\right].$$

1. Prove that the series of functions $\sum f_n(x)$ converges simply on $\left[0, \frac{\pi}{2}\right]$ and calculate its sum.
2. Is the series uniformly convergent on $\mathbb{R}$?.

Exercise 2.4.3. *Let*

$$f_n(x) = \frac{x}{(1+x^2)^n}, \ n \geq 1 \text{ and } x \in \mathbb{R}.$$

1. Prove that the series of functions $\sum f_n(x)$ converges simply on $\mathbb{R}$ and calculate its sum.
2. Is the series uniformly convergent on $\mathbb{R}$?.

3. *Study the normal convergence on* $[a, b]$*, then on* $[a, +\infty[$ *,* $(0 < a < b)$*.*
4. *Calculate* $\sum_{n\geq 1} \int_1^e f_n(x)dx$*.*

Chapter 3

Power series

In this chapter, we will study a power series which are special forms of the series of functions of real or complex variables. For this, x denotes a real variable and z a complex variable.

3.1 Real (or complex) power series

Definition 3.1.1. *A real (resp. complex) power series is any series of functions whose general term:*

$$f_n(x) = a_n x^n, \tag{3.1}$$

where $a_0, a_1, ..., a_n, ...$ are real numbers and $x \in \mathbb{R}$ (resp.

$$f_n(x) = a_n z^n, \tag{3.2}$$

where $a_0, a_1, ..., a_n, ...$ are complex numbers and $z \in \mathbb{C}$.)

To unify the presentation of the following results, we consider the case where $x \in \mathbb{R}$.

Lemma 3.1.1. *(Abel's Lemma) If the power series $\sum a_n x^n$ converges at the point $x_0 \neq 0$, then it converges absolutely for all $x \in \mathbb{R}$, such that $|x| < x_0$.*

Proof. Since the power series $\sum a_n x_0^n$ converges, its general term is bounded, there then exists $M > 0$, such that:

$$\text{for all } n \in \mathbb{N},\ \left|a_n x_0^n\right| \leq M. \tag{3.3}$$

For all $x \in \mathbb{R}$, such that $|x| < x_0$, we thus have:

$$\begin{aligned} |a_n x^n| &= \left|a_n x_0^n\right| \times \left|\frac{x}{x_0}\right|^n \\ &\leq M\left|\frac{x}{x_0}\right|^n, \end{aligned} \tag{3.4}$$

and $\left|\frac{x}{x_0}\right|^n$ is the general term of a convergent geometric series ($\left|\frac{x}{x_0}\right| < 1$); we deduce that the series $\sum a_n x^n$ converges absolutely. □

3.1.1 Radius of convergence of a power series

Theorem 3.1.1. *(theorem and definition) If a power series $\sum a_n x^n$ converges to the point $x_0 \neq 0$, then there exists a unique element $R \in \mathbb{R}_+ \cup \{+\infty\}$ verifying the following two conditions:*
1. For all $x \in \mathbb{R}$, such that $|x| < R$, the power series $\sum a_n x^n$ absolutely converges .
2. For all $x \in \mathbb{R}$, such that $|x| > R$, the power series $\sum a_n x^n$ diverge.
The number R is called the radius of convergence of the series, and the set $]-R, R[$ is called the interval of convergence.

Proof. Suppose there exists at least one real $x_0 \neq 0$, such that the series $\sum a_n x_0^n$ converges and one real x_1 such that the series $\sum a_n x_1^n$ diverges.
Since absolute convergence on $[0, R[$ implies convergence on $]-R, 0]$, and divergence on $]R, +\infty[$ implies divergence on $]-\infty, -R[$, we will study the nature of $\sum a_n x^n$ on $\mathbb{R}_+$.
Let us then consider the set D of positive reals defined by:

$$D = \left\{x \in \mathbb{R}_+, \sum a_n x^n \text{ converge.}\right\} \tag{3.5}$$

Since the series $\sum a_n x_0^n$ converges, D is therefore non-empty.
According to the relation (3.5), the set D is majorized, it therefore admits a non-zero upper bound $R = \sup_{x \in \mathbb{R}_+} D$.

1. Prove that for all $x \in \mathbb{R}_+$, such that $x < R$, the power seires $\sum a_n x^n$ converges absolutely.
The second property of the upper bound states that there exists $r = x_0$ between x and R, such that $\sum a_n x^n$ converges at the point x_0, so according to Abel, it is absolutely convergent for all $x \in \mathbb{R}_+$, such that $x < x_0$.
2. Let us now show that for all $x \in \mathbb{R}_+$, such that $x > R$, the power series $\sum a_n x^n$ diverges.
Suppose by contradiction that $\sum a_n x^n$ converges, and consider $y = \dfrac{R+x}{2}$. Since $0 < y < x$, Abel's lemma states that the series $\sum a_n y^n$ converges, y is therefore a point of convergence, that is $y \in D$. Consequently $y \leq R$, and this is false, because by construction $y = \dfrac{R+x}{2} > R$, and the series $\sum a_n x^n$ diverges. □

3.1.2 Cauchy-Hadamard rule

Theorem 3.1.2. *The radius of convergence of a power series $\sum a_n x^n$ is given by:*

$$R = \lim_{n \to +\infty} \left(\sqrt{|a_n|} \right)^{-1} \text{ (when this limit exists).} \tag{3.6}$$

Proof. It suffices to apply the Cauchy criterion on the series of functions $\sum |a_n x^n|$ □

Example 3.1.1. *The power series $\sum_{n \geq 1} (\dfrac{n+1}{n})^{n^2} x^n$*

3.1.3 D'Alembert's rule

Theorem 3.1.3. *The radius of convergence of a power series $\sum a_n x^n$ is given by:*

$$R = \lim_{n \to +\infty} \left(\left| \frac{a_{n+1}}{a_n} \right| \right)^{-1} \text{ (when this limit exists).} \tag{3.7}$$

Proof. It suffices to apply D'Alembert's criterion on the series of functions $\sum |a_n x^n|$ □

Example 3.1.2. *The integer series $\sum_{n \geq 1} \dfrac{n!}{n^n} x^n$ has for the radius of convergence $R = e$.*

3.1.4 Normal convergence (Weierstrass rule)

Theorem 3.1.4. *Any power series $\sum a_n x^n$ converges normally in any compact contained in the domain of convergence $]-R, R[$ $(R > 0)$.*

Proof. Let $[-\alpha, \alpha] \subset]-R, R[$ $(\alpha > 0)$. In the segment $[-\alpha, \alpha]$, the series $\sum a_n x^n$ is bounded above in absolute value by the positive series $\sum |a_n| \alpha^n$, which is convergent, the series $\sum a_n x^n$ is therefore normally convergent. □

3.2 Properties of power series

3.2.1 Continuity of the sum of a power series

Theorem 3.2.1. *Let $\sum a_n x^n$ be a power series with a non-zero radius of convergence R; then the sum of the series $S(x) = \sum a_n x^n$ is a continuous function on any compact set contained in the domain of convergence $]-R, R[$.*

Proof. For all $n \in \mathbb{N}$, each function $f_n(x) = a^n x^n$ is continuous on $[-\alpha, \alpha]$ of $]-R, R[$ and the series $\sum a_n x^n$ converges uniformly on $[-\alpha, \alpha]$. By the property of the continuity of series of functions, the sum of the integer series $\sum a_n x^n$ is a continuous function. □

Theorem 3.2.2. *(Abel's Theorem) Let $\sum a_n x^n$ be a power series with radius of convergence R. If this series converges for $x = R$ (resp. for $x = -R$), then this series is uniformly convergent on $[0, R]$ (resp. on $[-R, 0]$) and the sum S of this series is continuous t̀o the left of $x = R$ (resp. t̀o the right of $x = -R$), that is:*

$$\lim_{x \to R^-} \sum a_n x^n = \sum a_n R^n = S(R), \tag{3.8}$$

(resp.

$$\lim_{x \to -R^+} \sum a_n x^n = \sum a_n (-1)^n R^n = S(-R). \tag{3.9}$$

Proof. We demonstrate this in the case where the series converges for $x = R$. Consider the new power series $\sum_{n \geq 0} a_n R^n y^n$ of the variable $y \in [0, 1]$.

For $y = 1$, the series becomes $\sum_{n\geq 0} a_n R^n$, which is convergent, so it is uniformly convergent.
Let us now assume that $y \in [0, 1[$. We use the Abel transformation, we can write:

$$\sum_{n\geq 0} a_n R^n y^n = \sum_{n\geq 0} (a_0 + a_1 R + ... + a_n R^n)\left(y^n - y^{n+1}\right) \tag{3.10}$$

We can then see the series with terms general:

$$g_n(y) = (a_0 + a_1 R + ... + a_n R^n)\left(y^n - y^{n+1}\right).$$

We just need to show that this series is uniformly convergent.
Indeed:

$$\begin{aligned} \left|g_n(y)\right| &= \left|(a_0 + a_1 R + ... + a_n R^n)\left(y^n - y^{n+1}\right)\right| \\ &\leq M(y^n - y^{n+1}), \end{aligned} \tag{3.11}$$

because $y^n - y^{n+1} \geq 0$ ((y^n) is decreasing) and the sequence of general term $\sum_{k=0}^{n} a_k R^k$ is bounded.
For all $y \in [0, 1[$, the sequence of general term y^n converges uniformly to 0, hence according to the telescopic property, the series $\sum\limits_{n\geq 1}^{+\infty} (y^n - y^{n+1})$ is uniformly convergent. The comparison theorem therefore asserts the uniform convergence of the series $\sum\limits_{n\geq 0}^{+\infty} g_n$. It follows that the initial series $\sum\limits_{n\geq 0}^{+\infty} a_n R^n y^n$ is uniformly convergent on $[0, 1[$.
We then deduce the uniform convergence of $\sum\limits_{n\geq 0}^{+\infty} a_n R^n y^n$ on $[0, 1]$.
Since each function $a_n R^n y^n$ is continuous on $[0, 1]$, it results in the continuity of the sum of this series on $[0, 1]$, and moreover:

$$\sum_{n\geq 0}^{+\infty} a_n R^n y^n = \begin{cases} S(yR), \text{ if } y \in [0, 1[\\ \sum_{n\geq 0} a_n R^n, \text{ if } y = 1. \end{cases} \tag{3.12}$$

Continuity on the left at $y = 1$, then gives us:

$$\lim_{y\to 1^-} S(yR) = S(R) = \sum_{n\geq 0} a_n R^n \tag{3.13}$$

□

3.2.2 Derivability of power series

Theorem 3.2.3. *Let $\sum_{n\geq 0}^{+\infty} a_n x^n$ be a power series of non-zero radius of convergence R, then its sum S is a function derivable on any compact $[a,b]$ contained in the domain of convergence $]-R,R[$, and for any $x \in [a,b]$, we have:*

$$\dot{S}(x) = \frac{\partial}{\partial x}\left(\sum_{n\geq 0}^{+\infty} a_n x^n\right) = \sum_{n\geq 0}^{+\infty} \frac{\partial}{\partial x}(a_n x^n) = \sum_{n\geq 1}^{+\infty} n a_n x^{n-1}. \tag{3.14}$$

Proof. It suffices to show that the power series $\sum_{n\geq 0}^{+\infty} a_n x^n$ and its derivative series $\sum_{n\geq 1}^{+\infty} n a_n x^n$ have the same radius of convergence R; then the theorem of derivation of series of functions applies since a power series converges uniformly on any compact contained in the domain of convergence. Indeed, let R be the radius of convergence of the series $\sum (n+1)a_{n+1}x^n$.

1. If $|x| < R$, the series $\sum (n+1)a_{n+1}x^n$ is convergent.

Since:

$$\left|a_{n+1}x^{n+1}\right| \leq \left|(n+1)a_{n+}x^{n+1}\right| = (n+1)\left|a_{n+}x^n\right||x|, \tag{3.15}$$

the series $\sum \left|a_{n+}x^{n+1}\right|$ is convergent, the series $\sum a_{n+1}x^{n+1}$(or simply $\sum a_n x^n$) is therefore convergent.

2. If $|x| > R$, the series $\sum (n+1)a_{n+1}x^n$ is divergent. Let $y = \dfrac{R+|x|}{2} \in]R,|x|[$, the series $\sum (n+1)a_{n+1}y^n$ diverges and its general term is not bounded.

We can write:

$$\left|a_{n+1}x^{n+1}\right| = \left|(n+1)a_{n+1}y^n\right| \frac{1}{n+1}\left(\frac{|x|}{y}\right)^n. \tag{3.16}$$

Since $\dfrac{|x|}{y} > 1$, $\lim_{n\to+\infty} \frac{1}{n+1}\left(\dfrac{|x|}{y}\right)^n = +\infty$, so $\lim_{n\to+\infty}\left|a_{n+1}x^{n+1}\right| \neq 0$.

The series $\sum a_{n+1}x^{n+1}$(or simply $\sum a_n x^n$) is therefore divergent. □

Corollary 3.2.1. *Let $\sum_{n\geq 0}^{+\infty} a_n x^n$ be a power series of non-zero radius of convergence R, then its sum S is an infinitely derivable function on any compact contained in the domain of convergence $]-R,R[$, and for any $x \in]-R,R[$ and $k \geq 1$ we have:*

$$S^{(k)}(x) = \sum_{n\geq 0}^{+\infty}(n+k)(n+k-1)...(n+1)a_{n+k}x^n. \tag{3.17}$$

Proof. It suffices to show by recurrence that the series $\sum_{n\geq 0}^{+\infty}(n+k)(n+k-1)...(n+1)a_{n+k}x^n$, $k = 1, 2, ...$ have the same radius of convergence R. □

3.2.3 Integration of a power series

Theorem 3.2.4. *Any power series* $\sum_{n\geq 0}^{+\infty} a_n x^n$ *is term by term integrable over any compact contained in the domain of convergence* $]-R, R[$ *. In particular, its sum* S *verifies:*

$$\int_0^x S(t)dx = \sum_{n\geq 0}^{+\infty} a_n \frac{x^{n+1}}{n+1}, \text{for all } x \in]-R, R[. \tag{3.18}$$

Proof. Let $x \in]-R, R[$. Since the power series $\sum_{n\geq 0}^{+\infty} a_n x^n$ converges uniformly on $[0, x]$, the sum S is then a continuous function, , and therefore integrable on $[0, x]$. The Equ. (3.18) is therefore well defined. Moreover, if we derivate the series (3.18), we find:

$$S(x) = \sum_{n\geq 0}^{+\infty} a_n x^n, \text{ for all } x \in]-R, R[. \tag{3.19}$$

The two integer series $\sum_{n\geq 0}^{+\infty} a_n x^n$ and $\sum_{n\geq 1}^{+\infty} n a_n x^n$ then have the same convergence radius R . □

Example 3.2.1. *Consider the power series of general term:*

$$a_n x^n = \frac{x^n}{n},\ n \geq 1. \tag{3.20}$$

The d'Alembert criterion shows that this series is absolutely convergent on $]-1, 1[$ *and has the sum* S*.*
For all $x \in]-1, 1[$ *, the series* $\sum_{n\geq 1}^{+\infty} a_n x^n$ *is derivable term by term. We then have:*

$$\dot{S}(x) = \sum_{n\geq 0}^{+\infty} x^n = \frac{1}{1-x}, \text{ for all } x \in]-1, 1[. \tag{3.21}$$

S *is continuous on* $[0, x]$*, so it is integrable on this interval. We then have:*

$$S(x) = -\ln(1 - x). \tag{3.22}$$

On the other hand, if $x = -1$, the numerical series $\sum_{n\geq 1}^{+\infty} \frac{(-1)^n}{n}$ converges, we can then apply Abel's theorem 3.2.2, we deduce that:

$$\sum_{n\geq 1}^{+\infty} \frac{(-1)^n}{n} = -\ln 2. \tag{3.23}$$

3.3 Sums and products of power series

3.3.1 Sum of two power series

Let $\sum a_n x^n$ and $\sum b_n x^n$ be two power series with radius of convergence R_a and R_b respectively, we then have:

Proposition 3.3.1. *The radius of convergence R of the power series $\sum(a_n + b_n)x^n$ verifies:*

$$\begin{aligned} R &\geq \inf(R_a, R_b), \text{ if } R_a = R_b. \qquad (3.24)\\ R &= \inf(R_a, R_b), \text{ if } R_a \neq R_b \end{aligned}$$

Moreover, for all $|x| < \inf(R_a, R_b)$, we have:

$$\sum (a_n + b_n)x^n = \sum a_n x^n + \sum b_n x^n. \tag{3.25}$$

Proof. 1. When $|x| < \inf(R_a, R_b)$, the two series $\sum a_n x^n$ and $\sum b_n x^n$ are convergent, the series $\sum(a_n + b_n)x^n$ is therefore convergent, we deduce that:

$$R \geq \inf(R_a, R_b). \tag{3.26}$$

Let $R_a \neq R_b$. Suppose for example that $R_a < R_b$, and let $x \in \mathbb{R}$, such that $R_a < |x| < R_b$, the series $\sum a_n x^n$ is therefore divergent while the series $\sum b_n x^n$ is convergent. The series $\sum(a_n + b_n)x^n$ is then divergent, and moreover:

$$R \leq R_a = \inf(R_a, R_b). \tag{3.27}$$

From (3.26) and (3.27), we deduce that $R = \inf(R_a, R_b)$.

2. If $R_a = R_b$, we cannot conclude anything about the radius of convergence

of the series $\sum(a_n + b_n)x^n$.
As an example, the two power series $\sum \left(\frac{n}{n+1}\right)^n x^n$ and $-\sum \left(\frac{n}{n+1}\right)^n x^n$ have the same radius of convergence $R_a = R_b = e$, while the sum series is the series with a zero general term, and therefore $R = +\infty$. □

3.3.2 Product of two power series

Proposition 3.3.2. *The radius of convergence R of the series with general term:*

$$c_n x^n = \left(\sum_{k=0}^{n} a_k b_{n-k}\right) x^n \tag{3.28}$$

verifies $R \geq \inf(R_a, R_b)$. In addition, for all $|x| < \inf(R_a, R_b)$, we have:

$$\sum_{n\geq 0} c_n x^n = \left(\sum_{n\geq 0} a_n x^n\right) \times \left(\sum_{n\geq 0} b_n x^n\right). \tag{3.29}$$

Proof. let $|x| < \inf(R_a, R_b)$. Since the two series $(\sum_{n\geq 0} a_n x^n)$ and $(\sum_{n\geq 0} b_n x^n)$ are absolutely convergent, the Cauchy product series of general term:

$$\sum_{k=0}^{n} \left(a_k x^k\right)\left(b_{n-k} x^{n-k}\right) = \left(\sum_{k=0}^{n} a_k b_{n-k}\right) x^n,$$

is also absolutely convergent, and for all $|x| < \inf(R_a, R_b)$, we have:

$$\sum_{n\geq 0} c_n x^n = \sum_{n\geq 0}\left[\sum_{k=0}^{n} \left(a_k x^k\right)\left(b_{n-k} x^{n-k}\right)\right] = \left(\sum_{n\geq 0} a_n x^n\right) \times \left(\sum_{n\geq 0} b_n x^n\right). \tag{3.30}$$

□

3.4 Functions developable in a power series (Taylor series)

In this section, we will study the problem in reverse.

3.4.1 Functions developable in a power series

Definition 3.4.1. *Let x_0 be a given real number and let $f : \mathbb{R} \to \mathbb{R}$ be a function defined in the neighborhood of x_0. We say that f is developable in a power series at the point x_0, if there exists a power series $\sum_{n\geq 0} a_n x^n$ with radius of convergence $R > 0$, such that:*

$$\text{for all } x \in \mathbb{R}, |x - x_0| < R,\ f(x) = \sum_{n\geq 0} a_n (x - x_0)^n \tag{3.31}$$

By performing the change of variable $X = x - x_0$, we then speak of a function that is developable in a power series at the origin.

Definition 3.4.2. *A function f of a complex variable z is said to be developable in a power series at the point z_0, if there exists a power series $\sum_{n\geq 0} a_n (z - z_0)^n$, with radius of convergence $R > 0$, such that:*

$$\text{for all } z \in \mathbb{C}, |z - z_0| < R,\ f(z) = \sum_{n\geq 0} a_n (z - z_0)^n. \tag{3.32}$$

Definition 3.4.3. *If f is indefinitely differentiable, the power series with general term $\dfrac{f^{(k)}(0)}{k!} x^n$ is called Taylor series of f.*

3.4.2 Necessary condition for development in power series

Theorem 3.4.1. *When a function f is developable in power series, then f is of class $C^{+\infty}$ on any compact contained in the domain of convergence $]-R, R[$ and f coincides with its Taylor series. Moreover, if the power series development exists it is unique.*

Proof. Suppose that f is developable in a power series at the origin, then there exists a power series $\sum_{n\geq 0} a_n x^n$ with a non-zero radius of convergence R, such that:

$$\text{for all } x \in \mathbb{R}, |x| < R, \text{ we have } f(x) = \sum_{n\geq 0} a_n x^n = S(x), \tag{3.33}$$

where S is the sum of this series.
According to the theorem of the derivation of power series, we deduce that f is of class $C^{+\infty}$ on $]-R, R[$ and moreover:

$$f^{(k)}(x) = \sum_{n\geq 0}^{+\infty}(n+k)(n+k-1)...(n+1)a_{n+k}x^n. \tag{3.34}$$

It follows that:

$$f^{(k)}(0) = a_k k!, \text{ i.e. } a_k = \frac{f^{(k)}(0)}{k!}, \tag{3.35}$$

which ensures the uniqueness of the development. □

Remak 3.1. *The converse of the previous theorem is false. Indeed, the condition that f is of class $C^{+\infty}$ on any compact contained in the domain of convergence $]-R, R[$, is not sufficient to ensure that this function is developable in a power series, even if its Taylor series converges. As an example, we consider the function f defined on $\mathbb{R}$ by:*

$$f(x) = \begin{cases} \exp(-\frac{1}{x^2}), \ si\ x > 0, \\ 0, \ si\ x \leq 0. \end{cases} \tag{3.36}$$

By recurrence, we can easily verify that this function is of class $C^{+\infty}$ on $\mathbb{R}$. Moreover for all $k \in \mathbb{N}$, the derivative of order k of f at point 0 is zero. So, if we assume that f is developable in a power series, its development is the zero series, which is impossible since $f(x) \neq 0$, for all $x \in]-R, R[$.

3.4.3 Sufficient condition for development in power series

Theorem 3.4.2. *Let f be an indefinitely derivable function on an interval $]-r, +r[$. A sufficient condition for f to be developable in a power series is the following:*

$$\exists M > 0, \ \forall n \in \mathbb{N}, \forall x \in]-r, +r[, \ \left|f^{(n)}(x)\right| \leq M. \tag{3.37}$$

In addition, for all $x \in]-r, +r[$, we have:

$$f(x) = \sum_{n\geq 0}^{+\infty} \frac{f^{(n)}(0)}{n!}x^n. \tag{3.38}$$

Proof. Since f is indefinitely derivable on $]-r,+r[$, the formula of Mac-Laurin gives:

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{n!}x^k + \frac{x^{n+1}}{(+1)!}f^{(n+1)}(\theta x),\ \theta \in]0,1[\,. \tag{3.39}$$

It is enough to show that $\lim_{n\to+\infty} \frac{x^{n+1}}{(+1)!}f^{(n+1)}(\theta x) = 0$. Indeed, by hypothesis, we can write:

$$0 \leq \left|\frac{x^{n+1}}{(+1)!}f^{(n+1)}(\theta x)\right| \leq M\left|\frac{x^{n+1}}{(n+1)!}\right|. \tag{3.40}$$

Since $\frac{x^{n+1}}{(n+1)!}$ is the general term of a convergent series, we therefore have:

$$\lim_{n\to+\infty} \frac{x^{n+1}}{(n+1)!} = 0. \tag{3.41}$$

Consequently:

$$\lim_{n\to+\infty} \frac{x^{n+1}}{(+1)!}f^{(n+1)}(\theta x) = 0. \tag{3.42}$$

The function f is indeed the sum of its Taylor series on $]-r,+r[$. □

3.5 Development in power series of usual functions

3.5.1 The sine and cosine functions

These two functions are of class $C^{+\infty}$ on $\mathbb{R}$. By recurrence, we can easily verify that their n th derivatives are:

$$\sin^{(n)}(x) = \sin(x + n\frac{\pi}{2}) \tag{3.43}$$

$$\cos^{(n)}(x) = \cos(x + n\frac{\pi}{2}), \tag{3.44}$$

are indeed majored by $M = 1$, for all $x \in \mathbb{R}$. They are therefore developable in power series on $\mathbb{R}$, which means that $R = +\infty$. We therefore have:

$$\sin(x) = \sum_{n\geq 0} \frac{(-1)^n}{(2n+1)!}x^{2n+1},\ \text{for all } x \in \mathbb{R}. \tag{3.45}$$

$$\cos(x) = \sum_{n\geq 0} \frac{(-1)^n}{(2n)!}x^{2n},\ \text{for all } x \in \mathbb{R}. \tag{3.46}$$

3.5.2 The exponential function $x \mapsto \exp(x)$

This function is of class $C^{+\infty}$ on any interval $]-r, r[$. By recurrence, we can easily verify that its n th derivative is also equal to $\exp(x)$, and is well bounded above $\exp(r)$. It is therefore developable in a power series on any interval $]-r, r[$. Since r is arbitrary, we deduce that $R = +\infty$. We therefore have:

$$\exp(x) = \sum_{n\geq 0} \frac{x^n}{n!}, \text{ for all } x \in \mathbb{R}. \tag{3.47}$$

3.5.3 The logarithm function $x \mapsto \ln(1-x)$

The function $x \mapsto \dfrac{1}{1-x}$ is developable in a power series on $]-1, 1[$. Indeed, let $(x^n)_{n\in\mathbb{N}}$ be a geometric sequence, we can write:

$$\frac{1}{1-x} = \sum_{k=0}^{n} x^k + \frac{x^{n+1}}{1-x}, x \in \mathbb{R}/\{1\}. \tag{3.48}$$

We deduce that:

$$\frac{1}{1-x} = \sum_{n\geq 0} x^n, x \in]-1, 1[. \tag{3.49}$$

Integrating term to term, we obtain:

$$\ln(1-x) = -\sum_{n\geq 0} \frac{x^{n+1}}{n+1}, x \in [-1, 1[\text{ (because } \sum_{n\geq 0} \frac{(-1)^{n+1}}{n+1} \text{ converge).} \tag{3.50}$$

Remak 3.2. *The techniques of the previous parts can be applied to obtain other developments from these cases. These techniques are adapted to the following functions:*

$$\cosh(x) = \frac{\exp(x) + \exp(-x)}{2} = \sum_{n\geq 0} \frac{x^{2n}}{(2n)!}, \ x \in \mathbb{R}. \tag{3.51}$$

$$\sinh(x) = \frac{\exp(x) - \exp(-x)}{2} = \sum_{n\geq 0} \frac{x^{2n+1}}{(2n+1)!}, \ x \in \mathbb{R}. \tag{3.52}$$

$$\frac{1}{ax+b} = \frac{1}{b}\frac{1}{1-(-\frac{a}{b}x)} = \frac{1}{b}\sum_{n\geq 0}(-1)^n(\frac{a}{b})^n x^n, \ x \in \left]-\left|\frac{b}{a}\right|, \left|\frac{b}{a}\right|\right[\text{ and } a, b \neq 0. \tag{3.53}$$

$$\frac{1}{1+x^2} = \frac{1}{1+(-x^2)} = \sum_{n\geq 0}(-1)^n x^{2n}, \ x \in]-1, 1[. \tag{3.54}$$

$$\arctan(x) = \int_0^x \frac{dt}{1+t^2} = \sum_{n\geq 0} \frac{(-1)^n x^{2n+1}}{2n+1}, x \in [-1,1]. \tag{3.55}$$

$$\arg th(x) = \frac{1}{2}\ln(\frac{1+x}{1-x}) = \sum_{n\geq 0} \frac{x^{2n+1}}{2n+1}, x \in [-1,1[. \tag{3.56}$$

3.5.4 Rational functions

The decomposition of a rational function into simple elements and the use of the power series development of the function $x \mapsto f(x) = \frac{1}{1-x}$, allow us to develop a rational function in a power series.

Example 3.5.1. *We consider the rational function* $f(x) = \frac{1}{2-x}$.
We then have

$$f(x) = -\frac{1}{2} \times \frac{1}{1-\frac{x}{2}} = -\frac{1}{2}\sum_{n\geq 0}\left(\frac{x}{2}\right)^n, \ x \in]-2,2[. \tag{3.57}$$

3.6 Application to the resolution of certain differential equations

We will present here an example of a differential equation, a method that allows us to find a solution in the form of a function that can be developed in a power series over a certain interval $]-r,r[$.
Let us then consider the differential equation:

$$2x\dot{y} + y - \frac{1}{1-x} = 0. \tag{3.58}$$

Suppose there exists a power series $y(x) = \sum_{n\geq 0} a_n x^n$ with radius of convergence $r > 0$.
For all $x \in]-1,1[$,

$$\frac{1}{1-x} = 1 + \sum_{n\geq 1} x^n. \tag{3.59}$$

So, we have:

$$2x\dot{y} + y - \frac{1}{1-x} = a_0 + \sum_{n\geq 0}(2n+1)a_n x^n - \left(1 + \sum_{n\geq 1} x^n\right) = 0. \tag{3.60}$$

We deduce that:

$$a_0 = 1 \text{ and } a_n = \frac{1}{2n+1}. \tag{3.61}$$

Therefore:

$$y(x) = \sum_{n\geq 0} \frac{x^n}{2n+1}, \ \ ]-1,1[\tag{3.62}$$

3.7 Exercises about chapter 3

Exercise 3.7.1. *Give the radius of convergence and calculate the sum of each of the following power series:*

$$1)\ f_1(x) = \sum_{n=0}^{+\infty} a^n x^n \qquad 2)\ f_2(x) = \sum_{n=0}^{+\infty} (n+1)x^n$$
$$3)\ f_3(x) = \sum_{n=0}^{+\infty} n^2 x^n \qquad 4)\ f_4(x) = \sum_{n=0}^{+\infty} (-1)^n \frac{2^n}{n+1} x^n\ .$$

Exercise 3.7.2. *Let the power series* $\sum_{n=2}^{+\infty} \frac{x^n}{n(n-1)}$.
1. Find the radius of convergence of this series and calculate its sum S.
2. By passing to the suitably justified limit, calculate $S(1)$ *and* $S(-1)$.

Exercise 3.7.3. *Consider a power series with general term* $a_n t^n$, *radius of convergence* $R > 0$, *and sum* S. *We assume that* S *is a solution of the differential equation:*

$$(1+t^2)\ddot{f}(t) = 2f(t).$$

1) Establish a relation linking for each $n \in \mathbb{N}$ *the coefficients* a_n *and* a_{n+2}.
2) Determine the value of a_4 *then of* a_{2p}, *for all* $p > 2$.
3) We now assume that $S(0) = 0$ *and* $\dot{S}(0) = 1$. *Calculate* a_0, a_2 *and the value of* a_{2p+1}, *for all* $p \in \mathbb{N}$.
4) Prove that the power series with general term $a_n t^n$ *converges normally on the interval* $[-1,+1]$. *What is its radius of convergence?*
5) Let $g(0) = 0$ *and* $g(t) = \frac{\dot{S}(t)-1}{t}$ *for* $t \neq 0$. *Calculate the derivative* $\dot{g}$ *of* g *(we will find a simple rational fraction).*
6) Deduce from 5) an explicit expression of the function S.

Exercise 3.7.4. *Let* $f(t,x) = \dfrac{x\sin(t)}{x^2+1-2x\cos(t)}$.
1) Develop f *into a power series according to the powers of* x.
2) Calculate $\int_0^{\pi} f(t,x)dt$.

Chapter 4

Fourier series

4.1 Periodic functions

Definition 4.1.1. *Let f be a function defined on $\mathbb{R}$. We say that f is periodic with period T (or $T-$periodic), if and only if:*

$$f(x+T) = f(x), \text{ for all } x \in \mathbb{R}. \tag{4.1}$$

Corollary 4.1.1. *Let f be a $T-$periodic function. We then have the following properties:*
1. The number $-T$ is also a period of f.
2. The number nT, $n \in \mathbb{Z}$ is also a period of f.

Proof. 1. Since f is a $T-$periodic function, we can then write:

$$\begin{aligned} f(x-T) &= f(x-T+T) \\ &= f(x), \text{ for } x \in \mathbb{R}. \end{aligned} \tag{4.2}$$

2. If $n \geq 0$, by recurrence, we can write:

$$\begin{aligned} f(x+nT) &= f(x+(n-1)T+T) = f(x+(n-1)T) \\ &= \ldots \\ & \quad \ldots \\ & \quad \ldots \\ &= f(x+T) = f(x), \text{ for all } x \in \mathbb{R}. \end{aligned} \tag{4.3}$$

Now, suppose that $n < 0$. Since $-T$ is also a period of f, using the previous proof gives:

$$f(x+nT) = f(x+(-n)(-T)) = f(x), \text{ for all } x \in \mathbb{R}. \tag{4.4}$$

□

Proposition 4.1.1. *Let f be a T-periodic function, then the function g defined on $\mathbb{R}$ by $g(x) = f(\alpha x + \beta)$ $(\alpha \neq 0)$ is $\dfrac{T}{\alpha}$- periodic.*

Proof. Since f is $T-$periodic, we can write

$$\begin{aligned} g\left(x + \frac{T}{\alpha}\right) &= f\left[\alpha\left(x + \frac{T}{\alpha}\right) + \beta\right] \\ &= f(\alpha x + \beta + T) = f(\alpha x + \beta) = g(x). \end{aligned} \tag{4.5}$$

□

Proposition 4.1.2. *Let f be a $T-$periodic function and integrable on an interval $[\lambda, \lambda + T]$ (interval of length T), then we have:*

$$\int_{\lambda}^{\lambda+T} f(x)dx = \int_0^T f(x)dx, \textit{ for all } \lambda \in \mathbb{R}. \tag{4.6}$$

Proof. Indeed:

$$\begin{aligned} \int_{\lambda}^{\lambda+T} f(x)dx &= \int_{\lambda}^{T} f(x)dx + \int_{T}^{\lambda+T} f(x)dx \\ &= \int_{\lambda}^{T} f(x)dx + \int_{T}^{\lambda+T} f(x-T)dx \ \ (\text{because } -T \text{ is also a period of} f\ ) \\ &= \int_{\lambda}^{T} f(x)dx + \int_{0}^{\lambda} f(x)dx \ (\text{by posing } x - T = X) \\ &= \int_{0}^{T} f(x)dx, \text{ for all } \lambda \in \mathbb{R}. \end{aligned} \tag{4.7}$$

□

4.2 Trigonometric series

Definition 4.2.1. *A trigonometric series is a series of functions that can be written in the form:*

$$\frac{a_0}{2} + \sum_{n\geq 1} a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right), \tag{4.8}$$

where $(a_n)_n$ and $(b_n)_n$ are two sequences of scalars, real or complex, $T = 2l$ is the period of the series.

Definition 4.2.2. *(Complex form of a trigonometric series) Since $\sin 0 = 0$, we can assume that $b_0 = 0$.*

$$c_n = \frac{a_n - ib_n}{2} \text{ and } c_{-n} = \frac{a_n + ib_n}{2}, \tag{4.9}$$

the expression (4.8) can be rewritten in the complex form:

$$\sum_{n\in\mathbb{Z}} c_n \exp\left(i\frac{n\pi x}{l}\right). \tag{4.10}$$

Thus, a trigonometric series can be considered as a series of functions in $\mathbb{C}$ of the form $\sum_{n\in\mathbb{Z}} c_n \exp\left(i\frac{n\pi x}{l}\right)$.

4.2.1 Rules of convergence

Proposition 4.2.1. *1) If the series $\sum_{n\geq1} a_n$ and $\sum_{n\geq1} b_n$ are absolutely convergent, then the trigonometric series (4.8) is normally (therefore uniformly) convergent on $\mathbb{R}$, and its sum is a continuous function on $\mathbb{R}$.*
2) If the sequences (a_n) and (a_n) are positive, decreasing and converge to 0, the trigonometric series (4.8) simply converges on $\mathbb{R} - 2l\mathbb{Z}$ and it is uniformly convergent on any interval of the form $[2k\pi + \lambda, 2(k+1)\pi - \lambda]$, where $k \in \mathbb{Z}$ and $0 < \lambda < \pi$.

Proof. 1) We can write:

$$\left|a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right)\right| \leq |a_n| + b_n, \text{ for all } x \in \mathbb{R}. \tag{4.11}$$

The trigonometric series (4.8) is therefore normally convergent on $\mathbb{R}$ and therefore uniformly convergent on $\mathbb{R}$ by the Weierstrass criterion.

The continuity of the sum of this series occurs because the functions $x \mapsto a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right)$ are continuous and the series (4.8) is uniformly convergent.

2) This is an immediate consequence of Abel's criterion. □

4.2.2 Calculation of coefficients of a trigonometric series

Proposition 4.2.2. *Suppose that the numerical series $\sum_{n\geq 1} a_n$ and $\sum_{n\geq 1} b_n$ are absolutely convergent, and let S be the sum of the trigonometric series (4.8). Then the coefficients of this series are given by:*

$$a_n = \frac{1}{l}\int_{-l}^{l} S(x)\cos\left(\frac{n\pi x}{l}\right)dx,\ n \geq 0, \tag{4.12}$$

$$b_n = \frac{1}{l}\int_{-l}^{l} S(x)\sin\left(\frac{n\pi x}{l}\right)dx,\ n \geq 1, \tag{4.13}$$

Proof. We multiply the general term of the trigonometric series (4.8) by $\cos\left(\frac{m\pi x}{l}\right)$ or by $\sin\left(\frac{m\pi x}{l}\right)$ and under the following inequalities:

$$\left|\left[a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right)\right]\cos\left(\frac{m\pi x}{l}\right)\right| \leq |a_n| + |b_n|,\ x \in \mathbb{R}, \tag{4.14}$$

$$\left|\left[a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right)\right]\sin\left(\frac{m\pi x}{l}\right)\right| \leq |a_n| + |b_n|,\ x \in \mathbb{R}, \tag{4.15}$$

we can ensure the uniform convergence of the following series of functions:

$$\sum_{n\geq 1}\left[a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right)\right]\ \cos\left(\frac{m\pi x}{l}\right) \tag{4.16}$$

and

$$\sum_{n\geq 1}\left[a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right)\right]\sin\left(\frac{m\pi x}{l}\right) \tag{4.17}$$

Therefore, we can integrate term by term.

The properties (4.12) and (4.13) are therefore consequences of the following calculation:

$$\int_{-l}^{l}\cos\left(\frac{n\pi x}{l}\right)\cos\left(\frac{m\pi x}{l}\right)dx = \int_{-l}^{l}\sin\left(\frac{n\pi x}{l}\right)\sin\left(\frac{m\pi x}{l}\right)dx = 0,\ n \neq m, \tag{4.18}$$

$$\int_{-l}^{l} \cos^2\left(\frac{n\pi x}{l}\right) dx = \int_{-l}^{l} \sin^2\left(\frac{n\pi x}{l}\right) dx = l, \tag{4.19}$$

$$\int_{-l}^{l} \cos\left(\frac{n\pi x}{l}\right) \sin\left(\frac{m\pi x}{l}\right) dx = \int_{-l}^{l} \cos\left(\frac{m\pi x}{l}\right) \sin\left(\frac{n\pi x}{l}\right) dx = 0, \; n, m \geq 1. \tag{4.20}$$

□

We can now define the Fourier series of a periodic function.

4.3 Fourier series

Definition 4.3.1. *Let f be a function defined on $\mathbb{R}$, $2l-$periodic and integrable on the interval $[-l, l]$. We call the Fourier series of f and we denote $SF(f)$ the trigonomic series:*

$$\frac{a_0}{2} + \sum_{n \geq 1} a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right). \tag{4.21}$$

The numbers a_n and b_n defined by:

$$a_n = \frac{1}{l} \int_{-l}^{l} f(x) \cos\left(\frac{n\pi x}{l}\right) dx, \; n \geq 0, \tag{4.22}$$

$$b_n = \frac{1}{l} \int_{-l}^{l} f(x) \sin\left(\frac{n\pi x}{l}\right) dx, \; n \geq 1, \tag{4.23}$$

are called Fourier coefficients of f .

Corollary 4.3.1. *To calculate the Fourier coefficients of a function, we can calculate the integrals over any interval of the type $[\lambda, \lambda + 2l]$ instead of $[-l, l]$, and the Fourier coefficients become:*

$$a_n = \frac{1}{l} \int_{\lambda}^{\lambda+2l} f(x) \cos\left(\frac{n\pi x}{l}\right) dx, \; n \geq 0, \tag{4.24}$$

$$b_n = \frac{1}{l} \int_{\lambda}^{\lambda+2l} f(x) \sin\left(\frac{n\pi x}{l}\right) dx, \; n \geq 1, \tag{4.25}$$

Proof. This is an immediate consequence of the Proposition 4.1.2 . □

4.3.1 Fourier series of even or odd functions

Proposition 4.3.1. *1) If f is even on $\mathbb{R}$, the Fourier coefficients are given by:*

$$\begin{cases} b_n = 0, \text{ for } n \geq 1 \\ a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx, \ n \geq 0. \end{cases} \tag{4.26}$$

2) If f is odd on $\mathbb{R}$, the Fourier coefficients are given by:

$$\begin{cases} a_n = 0, \ n \geq 0, \\ b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx, \text{ for } n \geq 1. \end{cases} \tag{4.27}$$

Proof. 1) Since f is even, we can then write:

$$\begin{aligned} a_n &= \frac{1}{l}\int_{-l}^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx \\ &= \frac{1}{l}\left[\int_{-l}^{0} f(x)\cos\left(\frac{n\pi x}{l}\right)dx + \int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx\right] \\ &= \frac{1}{l}\left[\int_{l}^{0} f(-x)\cos\left(-\frac{n\pi x}{l}\right)d(-x) + \int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx\right] \\ &= \frac{1}{l}\left[\int_{0}^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx + \int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx\right] \\ &= \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx. \end{aligned}$$

On the other hand, if f is even on $\mathbb{R}$, the function $f\sin\left(\frac{n\pi}{l}x\right)$ is odd for all $n \geq 1$, and therefore:

$$\begin{aligned} b_n &= \frac{1}{l}\int_{-l}^{l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx = -\frac{1}{l}\int_{l}^{-l} f(x)\sin\left(\frac{n\pi x}{l}\right)d(-x) \\ &= -\frac{1}{l}\int_{-l}^{l} f(x)\sin\left(\frac{n\pi x}{l}\right)d(x) = -\frac{1}{l}b_n, \end{aligned}$$

Therefore $b_n = 0$, for all $n \geq 1$.

2) The proof is analogous for the case where f is odd. □

Example 4.3.1. *Consider the f 2π -periodic function defined by:*

$$f(x) = x, \ x \in \left]-\pi, \pi\right[. \tag{4.28}$$

f is odd, so for all $n \geq 0$, $a_n = 0$, and

$$b_n = \frac{2}{\pi}\int_0^{\pi} x\sin(nx)\,dx = 2\frac{(-1)^{n-1}}{n},\ n \geq 1. \tag{4.29}$$

The Fourier series of f is therefore:

$$SF(f(x)) = 2\sum_{n\geq 1}\frac{(-1)^{n-1}}{n}\sin(nx). \tag{4.30}$$

Example 4.3.2. *Consider the f 2π -periodic function defined by:*

$$f(x) = x^2,\ x \in [-\pi, \pi]. \tag{4.31}$$

f is even, so for all $n \geq 0$, $b_n = 0$, and

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x^2 dx = 4\frac{\pi^2}{3},\ a_n = \frac{2}{\pi}\int_0^{\pi} x^2\cos(nx)\,dx = 4\frac{(-1)^n}{n},\ n \geq 1. \tag{4.32}$$

The Fourier series of f is therefore:

$$SF(f(x)) = \frac{2}{3}\pi^2 + 4\sum_{n\geq 1}\frac{(-1)^n}{n^2}\cos(nx). \tag{4.33}$$

4.3.2 Riemann-Lebesgue Lemma (Necessary Convergence Condition)

Lemma 4.3.1. *Let f be a function integrable on an interval $[a, b]$, we then have:*

$$\lim_{n\to+\infty}\int_a^b f(x)\cos\left(\frac{n\pi x}{l}\right)dx = \lim_{n\to+\infty}\int_a^b f(x)\sin\left(\frac{n\pi x}{l}\right)dx = 0.$$

Proof. It is enough to show that:

$$\lim_{n\to+\infty}\int_a^b f(x)\exp\left(i\frac{n\pi x}{l}\right)dx = 0. \tag{4.34}$$

Suppose that f is integrable on $[a, b]$, that is, that there exists a subdivision of the interval $[a, b]$ by a finite number of points:$a = x_0 < x_1 < < x_n = b$ and a staircase function φ defined by $\varphi(x) = m_j$, $x \in \left[x_{j-1}, x_j\right[$, $j = 1, 2, ..., n$, such that:

$$\left|f(x) - \varphi(x)\right| \leq \frac{\epsilon}{2(b-a)},\ \text{for all } \epsilon > 0. \tag{4.35}$$

We can write :

$$\int_a^b \left|[f(x)-\varphi(x)]\exp\left(i\frac{n\pi x}{l}\right)\right|dx \leq \int_a^b \left|f(x)-\varphi(x)\right| < \epsilon, \text{ for all } \epsilon > 0. \quad (4.36)$$

Therefore:

$$\begin{aligned}
\lim_{n\to+\infty}\int_a^b f(x)\exp\left(i\frac{n\pi x}{l}\right)dx &= \lim_{n\to+\infty}\int_a^b \varphi(x)\exp\left(i\frac{n\pi x}{l}\right)dx \\
&= \lim_{n\to+\infty}\sum_{j=1}^{n}\int_{x_{j-1}}^{x_j} m_j\exp\left(i\frac{n\pi x}{l}\right)dx \\
&= \sum_{j=1}^{n}\lim_{n\to+\infty}\int_a^b m_j\exp\left(i\frac{n\pi x}{l}\right)dx \\
&= \sum_{j=1}^{n}\lim_{n\to+\infty}\left[\frac{m_j l}{n\pi}\exp\left(i\frac{n\pi x}{l}\right)\right]_a^b \\
&= 0 \qquad (4.37)
\end{aligned}$$

□

Corollary 4.3.2. *Let f be a function defined on $\mathbb{R}$, $2l-$periodic and integrable on $[-l, l]$, then the sequences of Fourier coefficients (a_n) and (a_n) converge to 0 when $n \to +\infty$.*

4.3.3 Dirichlet Theorem (Sufficient Convergence Condition)

In this section, we will study a case of convergence of Fourier series.

Definition 4.3.2. *Let f be a function defined on an interval $[a, b]$. We say that f is piecewise continuous on $[a, b]$, if there exists a subdivision $\left\{\left[x_{j-1}, x_j\right[,\ j = 1, 2, ..., n\right\}$ of $[a, b]$ such that:*

1. f is continuous on each interval $\left[x_{j-1}, x_j\right[$, $j = 1, 2, ..., n$.

2. f admits discontinuities of the first kind at the points x_j, $j = 1, 2, ..., n$.

We recall that f admits a discontinuity of the first kind at a point x_0, when it admits at this point a right limit $f(x_0^+)$ and a left limit $f(x_0^-)$, such that $f(x_0^+) \neq f(x_0^-)$.

Definition 4.3.3. *We say that f is of class C^1 piecewise on $[a,b]$, if there exists a subdivision $\left\{\left[x_{j-1},x_j\right[,\ j=1,2,...,n\right\}$ of $[a,b]$ such that:*
1. f is of class C^1 on each interval $\left[x_{j-1},x_j\right[,\ j=1,2,...,n$.
2. f admits right-hand derivatives and left-hand derivatives at the points x_j, $j=1,2,...,n$ which are distinct..

Definition 4.3.4. *(Dirichlet kernel) We call the Dirichlet kernel the function D_n dédefined by:*

$$D_n(x)=\begin{cases}\dfrac{\sin\left(\left(n+\frac{1}{2}\right)\frac{\pi x}{l}\right)}{2\sin\left(\frac{\pi x}{2l}\right)}, & \text{if } x\neq 2ml,\ m\in\mathbb{Z}\\ n+\dfrac{1}{2}, & \text{if } x=2ml,\ m\in\mathbb{Z}.\end{cases} \tag{4.38}$$

Proposition 4.3.2. *The Dirichlet kernel D_n has the following properties:*
1. D_n is an even and periodic function of period $2l$.
2. D_n is a continuous function on $\mathbb{R}$.
3. D_n can be represented by the formula:

$$D_n(x)=\frac{1}{2}+\sum_{k=1}^{n}\cos(\frac{k\pi x}{l}),\ x\in\mathbb{R}, \tag{4.39}$$

and furthermore, we have:

$$\frac{1}{l}\int_0^l D_n(x)dx=\frac{1}{2}. \tag{4.40}$$

Proof. 1. D_n is an even function (obvious).
On the other hand, for any $x\neq 2ml$, $m\in\mathbb{Z}$, we have:

$$\begin{aligned} D_n(x+2l) &= \frac{\sin\left[\left(n+\frac{1}{2}\right)\frac{\pi x}{l}+(2n+1)\pi\right]}{2\sin\left(\frac{\pi x}{2l}+\pi\right)} \\ &= \frac{-\sin\left(\left(n+\frac{1}{2}\right)\frac{\pi x}{l}\right)}{-2\sin\left(\frac{\pi x}{2l}\right)}=\frac{\sin\left(\left(n+\frac{1}{2}\right)\frac{\pi x}{l}\right)}{2\sin\left(\frac{\pi x}{2l}\right)} \\ &= D_n(x). \end{aligned} \tag{4.41}$$

The function D_n is therefore $2l$ -periodic.

2. It suffices to show that D_n is continuous at the point $x_0 = 0$. Indeed,

$$\begin{aligned}
\lim_{x\to 0} D_n(x) &= \lim_{x\to 0} \frac{\sin\left(\left(n+\frac{1}{2}\right)\frac{\pi x}{l}\right)}{2\sin\left(\frac{\pi x}{2l}\right)} \\
&= \lim_{x\to 0} \frac{\sin\left(\left(n+\frac{1}{2}\right)\frac{\pi x}{l}\right)}{\left(n+\frac{1}{2}\right)\frac{\pi x}{l}} \times \frac{\frac{\pi x}{2l}}{\sin\left(\frac{\pi x}{2l}\right)} \times \frac{\left(n+\frac{1}{2}\right)\frac{\pi x}{l}}{2\times\frac{\pi x}{2l}} \\
&= n+\frac{1}{2} = D_n(0).
\end{aligned}$$

By periodicity of the function D_n, we then deduce that D_n is continuous at all points $x = 2ml$, $m \in \mathbb{Z}$. Therefore D_n is continuous on $\mathbb{R}$.

3. For all $x \neq 2ml$, $m \in \mathbb{Z}$, we have:

$$\begin{aligned}
D_n(x) &= \frac{1}{2} + \sum_{k=1}^{n} \cos(k\frac{\pi x}{l}) = \frac{1}{2} + \Re\left(\sum_{k=1}^{n} \exp(ik\frac{\pi x}{l})\right) \\
&= \frac{1}{2} + \Re\left[\exp(i\frac{\pi x}{l})\left(\frac{1-\exp(in\frac{\pi x}{l})}{1-\exp(i\frac{\pi x}{l})}\right)\right] \\
&= \frac{1}{2} + \Re\left[\exp(i\frac{\pi x}{l})\left(\frac{1-\cos\left(\frac{n\pi x}{l}\right)-i\sin\left(\frac{n\pi x}{l}\right)}{1-\cos(\frac{\pi x}{l})-i\sin(\frac{\pi x}{l})}\right)\right] \\
&= \frac{1}{2} + \Re\left[\exp(i\frac{\pi x}{l})\left(\frac{2\sin^2(\frac{n\pi x}{2l})-2i\cos\left(\frac{n\pi x}{2l}\right)\sin\left(\frac{n\pi x}{2l}\right)}{2\sin^2\left(\frac{\pi x}{2l}\right)-2i\cos\left(\frac{\pi x}{2l}\right)\sin\left(\frac{\pi x}{2l}\right)}\right)\right] \\
&= \frac{1}{2} + \Re\left[\exp(i\frac{\pi x}{l})\left(\frac{\cos\left(\frac{n\pi x}{2l}\right)+i\sin\left(\frac{n\pi x}{2l}\right)}{\cos\left(\frac{n\pi x}{2l}\right)+i\sin\left(\frac{n\pi x}{2l}\right)}\right)\left(\frac{\sin\left(\frac{n\pi x}{2l}\right)}{\sin\left(\frac{n\pi x}{2l}\right)}\right)\right] \\
&= \frac{1}{2} + \Re\left[\exp\left(i\left(\frac{n+1}{2}\right)\frac{\pi x}{l}\right)\left(\frac{\sin(\frac{n\pi x}{2l})}{\sin(\frac{n\pi x}{2l})}\right)\right] \\
&= \frac{1}{2} + \cos\left(\left(\frac{n+1}{2}\right)\frac{\pi x}{l}\right)\left(\frac{\sin\left(\frac{n\pi x}{2l}\right)}{\sin\left(\frac{n\pi x}{2l}\right)}\right) \\
&= \frac{1}{2} + \frac{1}{2}\left[\frac{\sin\left(\left(\frac{2n+1}{2}\right)\frac{\pi x}{l}\right)-\sin\left(\frac{\pi x}{2l}\right)}{\sin\left(\frac{\pi x}{2l}\right)}\right] \\
&= \frac{\sin\left(\left(n+\frac{1}{2}\right)\frac{\pi x}{l}\right)}{\sin\left(\frac{\pi x}{2l}\right)} = D_n(x). \qquad (4.42)
\end{aligned}$$

If $x = 2ml$, $m \in \mathbb{Z}$, we have:

$$D_n(x) = \frac{1}{2} + \sum_{k=1}^{n} \cos(2km\pi) = \frac{1}{2} + n. \qquad (4.43)$$

In addition, we have:

$$\frac{1}{l}\int_0^l D_n(x)dx = \frac{1}{2l}\int_0^l dx + \frac{1}{l}\sum_{k=1}^{n}\int_0^l \cos(\frac{k\pi x}{l})dx = \frac{1}{2}. \qquad (4.44)$$

□

Here is now the Dirichlet theorem:

Theorem 4.3.1. *(Dirichlet theorem) Let f be a $2l-$ periodic function of class C^1 piecewise on $\mathbb{R}$, then the Fourier series of f converges simply at every point x of $\mathbb{R} - 2l\mathbb{Z}$ and has the sum:*

$$S(x) = \frac{f(x^+) + f(x^-)}{2}. \tag{4.45}$$

Moreover, if f is continuous, the Fourier series of f converges uniformly on $\mathbb{R}$ and $S(x) = f(x)$, for all $x \in \mathbb{R}$.

Proof. Let's consider for all $n \in \mathbb{N}$, the sequence of partial sums of the Fourier series:

$$S_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} a_n \cos(\frac{n\pi x}{l}) + b_n \sin(\frac{n\pi x}{l}). \tag{4.46}$$

Let x be any point of $\mathbb{R}$. It is enough to demonstrate:

$$\lim_{n \to +\infty} S_n(x) = \frac{f(x^+) + f(x^-)}{2}. \tag{4.47}$$

Indeed:

$$\begin{aligned}
S_n(x) &= \frac{a_0}{2} + \sum_{k=1}^{n} a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right) \\
&= \frac{1}{2l}\int_{-l}^{l} f(t)dt + \sum_{k=1}^{n}\int_{-l}^{l} f(t)\left(\cos\left(\frac{n\pi t}{l}\right)\cos\left(\frac{n\pi x}{l}\right) + \sin\left(\frac{n\pi x}{l}\right)\sin\left(\frac{n\pi x}{l}\right)\right)dt \\
&= \frac{1}{l}\int_{-l}^{l} f(t)\left(\frac{1}{2} + \sum_{k=1}^{n}\cos\left(\frac{n\pi (t-x)}{l}\right)\right)dt \\
&= \frac{1}{l}\int_{-l}^{l} f(t)D_n(t-x)\,dt \\
&= \frac{1}{l}\int_{-l-x}^{l-x} f(x+y)D_n(y)\,dy \ \ \text{(using change } y = t-x\text{)} \\
&= \frac{1}{l}\int_{-l}^{l} f(x+y)D_n(y)\,dy \ \ \text{(By Proposition 4.1.2)} \\
&= \frac{1}{l}\left(\int_{-l}^{0} f(x+y)D_n(y)\,dy + \int_{0}^{l} f(x+y)D_n(y)\,dy\right) \\
&= \frac{1}{l}\int_{0}^{l} (f(x-y)+f(x+y))\,D_n(y)\,dy \ \text{(by making the change } u \mapsto -u\text{).}
\end{aligned} \tag{4.48}$$

The use of formulas $\frac{1}{l}\int_0^l D_n(y)dy = \frac{1}{l}\int_0^l \left(\frac{\sin\left(\left(n+\frac{1}{2}\right)\frac{\pi y}{l}\right)}{2\sin\left(\frac{\pi y}{2l}\right)}\right)dy = \frac{1}{2}$, we get:

$$\begin{aligned}
S_n(x) - \frac{f(x^+)+f(x^-)}{2} &= \frac{1}{l}\int_0^l (f(x-y)+f(x+y))\left(\frac{\sin\left(\left(n+\frac{1}{2}\right)\frac{\pi y}{l}\right)}{2\sin\left(\frac{\pi y}{2l}\right)}\right)dy + \\
&\quad -\left(\frac{f(x^+)+f(x^-)}{2l}\right)\int_0^l \left(\frac{\sin\left(\left(n+\frac{1}{2}\right)\frac{\pi y}{l}\right)}{\sin\left(\frac{\pi y}{2l}\right)}\right)dy. \\
&= \frac{1}{2\pi}\int_0^l \frac{f(x-y)-f(x^-)}{y} \times \frac{\frac{\pi y}{l}}{\sin\left(\frac{\pi y}{2l}\right)} \times \sin\left(\left(n+\frac{1}{2}\right)\frac{\pi y}{l}\right)dy + \\
&\quad + \frac{1}{2\pi}\int_0^l \frac{f(x+y)-f(x^+)}{y} \times \frac{\frac{\pi y}{l}}{\sin\left(\frac{\pi y}{2l}\right)} \times \sin\left(\left(n+\frac{1}{2}\right)\frac{\pi y}{l}\right)dy.
\end{aligned}$$

Since the function f is of class C^1 piecewise and $\lim \dfrac{\frac{\pi y}{2l}}{\sin\left(\frac{\pi y}{2l}\right)} = 1$, the functions:

$$y \mapsto \frac{f(x-y)-f(x^-)}{y} \times \frac{\frac{\pi y}{l}}{\sin\left(\frac{\pi y}{2l}\right)} \text{ and } y \mapsto \frac{f(x+y)-f(x^+)}{y} \times \frac{\frac{\pi y}{l}}{\sin\left(\frac{\pi y}{2l}\right)}, \tag{4.49}$$

are bounded.

The result is therefore a consequence of the Riemann-Lebesgue lemma. □

4.3.4 Parseval's formula

Theorem 4.3.2. *Let f be a function defined on $\mathbb{R}$, periodic of period $2l$, integrable on $[-l, l]$ and let*

$$f(x) = \frac{a_0}{2} + \sum_{n\geq 1}\left(a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right)\right), \tag{4.50}$$

Then Parseval's formula is given by:

$$\frac{1}{l}\int_{-l}^{l} f^2(x)dx = \frac{a_0^2}{2} + \sum_{n\geq 1}\left(a_n^2 + b_n^2\right). \tag{4.51}$$

Proof. We demonstrate this theorem when

$$\sum_{n\geq 1} a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right), \tag{4.52}$$

is uniformly convergent. Let us then assume that the numerical series $\sum_{n\geq 1} a_n$ and $\sum_{n\geq 1} b_n$ are absolutely convergent, and let (S_n) be the sequence of partial sums of the Fourier series such that

$$S_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n}\left(a_k \cos\left(\frac{k\pi x}{l}\right) + b_k \sin\left(\frac{k\pi x}{l}\right)\right). \tag{4.53}$$

For all $x \in \mathbb{R}$, we have:

$$\left|f(x) - S_n(x)\right| \leq \sum_{k\geq n+1} |a_k + b_k|, \tag{4.54}$$

quantity tending towards 0, when $n \to +\infty$. We deduce:

$$\begin{aligned}
\left|f^2(x) - S_n^2(x)\right| &= \left|f(x) - S_n(x)\right|\left|f(x) + S_n(x)\right| \\
&\leq \left|f(x) - S_n(x)\right| \times 2\left|f(x)\right| \\
&\leq 2\sum_{k\geq n+1}|a_k + b_k|\left(\left|\frac{a_0}{2}\right| + \sum_{k\geq 1}|a_k + b_k|\right) \to 0, \text{ when } n \to +\infty.
\end{aligned} \tag{4.55}$$

The sequence of functions S_n^2 is therefore uniformly convergent towards f^2. This uniform convergence allows us to write:

$$\frac{1}{l}\int_{-l}^{l} f^2(x)dx = \lim_{n\to+\infty}\left(\frac{1}{l}\int_{-l}^{l} S_n^2(x)dx\right). \tag{4.56}$$

Using the properties (4.18), (4.19), (4.20) and the fact that:

$$\left(\sum_{i=1}^{n}\alpha_i\right)^2 = \sum_{i=1}^{n}\alpha_i^2 + 2\sum_{i,j=1_{i\neq j}}^{n}\alpha_i\alpha_j, \tag{4.57}$$

$$\frac{1}{l}\int_{-l}^{l}\frac{a_0^2}{4}dx = \frac{a_0^2}{2}, \tag{4.58}$$

we can clearly deduce that

$$\frac{1}{l}\int_{-l}^{l} S_n^2(x)dx = \frac{a_0^2}{2} + \sum_{k=1}^{n}\left(a_k^2 + b_k^2\right), \tag{4.59}$$

By making n tend to $+\infty$, the expression (4.56), becomes

$$\frac{1}{l}\int_{-l}^{l} f^2(x)dx = \frac{a_0^2}{2} + \sum_{n\geq 1}\left(a_n^2 + b_n^2\right). \tag{4.60}$$

□

Remak 4.1. *The preceding Parseval theorem remains true even if f is not the sum of its Fourier series.*

4.4 Some applications of Fourier series

The use of the continuity of the sum of the Fourier series on certain intervals and Parseval's theorem gives us remarkable new identities:

Indeed, in the example 4.3.1, the Fourier series of x is given by:

$$SF(x) = 2\sum_{n\geq 1} \frac{(-1)^{n-1}}{n} \sin(nx). \tag{4.61}$$

$x_0 = \frac{\pi}{2}$ is a point of continuity, we then have $\frac{\pi}{2} = 2\sum_{n\geq 1} \frac{(-1)^{n-1}}{n} \sin\left(n\frac{\pi}{2}\right)$, so

$$\sum_{n\geq 0} \frac{(-1)^n}{2n+1} = \frac{\pi}{4} \tag{4.62}$$

On the other hand, using Parseval's theorem gives us

$$\frac{1}{\pi}\int_{-\pi}^{\pi} x^2 dx = \frac{2}{3}\pi^2 = 4\sum_{n\geq 1} \frac{1}{n^2}. \tag{4.63}$$

So:

$$\sum_{n\geq 1} \frac{1}{n^2} = \frac{\pi^2}{6}. \tag{4.64}$$

Similarly, in example 4.3.1, the Fourier series of x^2 is given by:

$$SF(x^2) = \frac{2}{3}\pi^2 + 4\sum_{n\geq 1} \frac{(-1)^n}{n^2} \cos(nx). \tag{4.65}$$

$x_0 = 0$ is a point of continuity, we then have $0 = \frac{2}{3}\pi^2 + 4\sum_{n\geq 1} \frac{(-1)^n}{n^2}$, hence

$$\sum_{n\geq 1} \frac{(-1)^n}{n^2} = -\frac{\pi^2}{6} \tag{4.66}$$

On the other hand, using Parseval's theorem gives us:

$$\frac{1}{\pi}\int_{-\pi}^{\pi} x^4 dx = \frac{2}{5}\pi^4 = \left(\frac{2}{9}\pi^4 + 16\sum_{n\geq 1} \frac{1}{n^4}\right), \tag{4.67}$$

So:

$$\sum_{n\geq 1} \frac{1}{n^4} = \frac{\pi^4}{90}. \tag{4.68}$$

4.5 Exercises about chapter 4

Exercise 4.5.1. *Consider the function 2π-periodic f defined by:*

$$f(x) = |x|, \ x \in [-\pi, \pi]. \tag{4.69}$$

1. *Represent* f.
2. *Prove that* f *is developable in Fourier series and explain its sum on* $[-\pi, \pi]$.
3. *Deduce:*

$$a)\sum_{n\geq 0}\frac{1}{(2n+1)^2},\quad b)\sum_{n\geq 1}\frac{1}{n^2},\quad c)\sum_{n\geq 0}\frac{1}{(2n+1)^4},\quad d)\sum_{n\geq 1}\frac{1}{n^4}.$$

$n\geq 1$

Exercise 4.5.2. *Consider the* 2π*-periodic even function* f *defined by:*

$$f(x) = x - \pi,\ x \in [0, \pi]. \tag{4.70}$$

1. *Represent* f.
2. *Prove that* f *is developable in a Fourier series and explain its sum on* $[-\pi, \pi]$.
3. *Deduce the sum of the series* $\sum_{n\geq 0}\frac{1}{(2n+1)^2}$ *by choosing a particular value of* x.

4) *Calculate by applying Parseval's formula the sum* $\sum_{n\geq 0}\frac{1}{(2n+1)^4}$.

Exercise 4.5.3. *Let* α *be a non-integer real number and let* f *be the* $2\pi-$ *periodic function defined on* $\mathbb{R}$ *by:*

$$f(x) = \cos(\alpha x),\ \ x \in [-\pi, \pi]. \tag{4.71}$$

1. *Prove that* f *is developable in a Fourier series and determine this series.*
2. *Is the Fourier series of* f *uniformly convergent on* $\mathbb{R}$*?*
3. *Deduce the following identities:*

$$\begin{aligned} a)\ \frac{\pi}{\sin(\alpha\pi)} &= \frac{1}{\alpha} + 2\alpha\sum_{n\geq 0}\frac{(-1)^n}{\alpha^2 - n^2}.\\ b)\ \pi\cot(\alpha) &= \frac{1}{\alpha} + 2\alpha\sum_{n\geq 0}\frac{1}{\alpha^2 - n^2}.\\ c)\ \frac{\pi^2}{\sin^2(\alpha\pi)} &= \sum_{n\in\mathbb{Z}}\frac{1}{(\alpha - n)^2}. \end{aligned}$$

Chapter 5

Generalized (improper) integrals

This chapter mainly consists in generalizing the notion of Riemann integrals to unbounded functions defined on intervals that are not necessarily bounded.

5.1 Convergence of generalized integrals

Definition 5.1.1. *Let $f : [a, b[$(or $b = +\infty$)$\to \mathbb{R}$ be a locally integrable function on $[a, b[$ (i.e. its restriction to each compact of $[a, b[$ is Riemann-integrable), and let F be the function defined on $[a, b[$ by:*

$$F(x) = \int_a^x f(t)dt, \text{for all } x \in [a, b[\,. \tag{5.1}$$

We say that the generalized integral $\int_a^b f(t)dt$ converges, if and only if $\lim_{x\to b^-} F(x)$ exists. Otherwise, we say that the integral $\int_a^b f(t)dt$ diverges.

Remak 5.1. *Let $f : [a, b[$(or $b = +\infty$)$\to \mathbb{R}$ be a locally integrable function on $[a, b[$. For all $c \in [a, b[$, we can write:*

$$\int_a^b f(t)dt = \int_a^c f(t)dt + \int_c^b f(t)dt. \tag{5.2}$$

Since $\int_a^c f(t)dt$ *always converges (Riemann integral), the generalized integrals* $\int_a^b f(t)dt$ *and* $\int_c^b f(t)dt$ *are therefore of the same nature.*

Definition 5.1.2. *Now, let* $f :]a, b]$*(or* $a = -\infty$*)*$\longrightarrow \mathbb{R}$ *(or* $\mathbb{C}$ *) be a locally integrable function on* $]a, b]$ *and let F be the function defined on* $]a, b]$ *by:*

$$F(x) = \int_x^b f(t)dt, \text{for all } x \in]a, b]\,. \tag{5.3}$$

The generalized integral $\int_a^b f(t)dt$ *is said to be convergent if and only if* $\lim_{x\to a^+} F(x)$ *exists. Otherwise, the integral* $\int_a^b f(t)dt$ *is said to diverge.*

Definition 5.1.3. *Let* f *be a locally integrable function on* $]a, b[$*où* $a \in \mathbb{R} \cup \{-\infty\}$ *and* $b \in \mathbb{R} \cup \{+\infty\}$ *and let* $c \in]a, b[$.
The generalized integral $\int_a^b f(t)dt$ *is said to be convergent, if and only the two generalizedl integrals* $\int_a^c f(t)dt$ *and* $\int_c^b f(t)dt$ *are convergent. Otherwise, this integral is said to be divergent.*

Example 5.1.1. *Consider the generalized integral* $\int_0^{+\infty} \frac{dt}{t^2}$. *Since:*

$$\int_x^1 \frac{dt}{t^2} = -1 + \frac{1}{x} \longrightarrow +\infty, \text{ when } x \longrightarrow 0^+, \tag{5.4}$$

the integral $\int_0^1 \frac{dt}{t^2}$ *diverges, however*

$$\int_1^x \frac{dt}{t^2} = 1 - \frac{1}{x} \longrightarrow 1, \text{ when } x \longrightarrow +\infty. \tag{5.5}$$

So the integral $\int_1^{+\infty} \frac{dt}{t^2}$ *converges. We deduce that the integral* $\int_0^{+\infty} \frac{dt}{t^2}$ *diverges.*

Proposition 5.1.1. *Let* f *and* g *be two locally integrable functions on* $[a, b[$.
1. If the integrals $\int_a^b f(t)dt$ *and* $\int_a^b g(t)dt$ *converge, then the integral* $\int_a^b (f(t) + g(t))\,dt$ *is also converged, and moreover:*

$$\int_a^b (f(t) + g(t))\,dt = \int_a^b f(t)dt + \int_a^b g(t)dt. \tag{5.6}$$

On the other hand, if one of the two integrals $\int_a^b f(t)dt$ *or* $\int_a^b g(t)dt$ *diverges, then* $\int_a^b (f(t) + g(t))\,dt$ *diverges.*

2. *Let $\alpha \in \mathbb{R}$, then the integral $\int_a^b f(t)dt$ converges if and only if the integral $\int_a^b \alpha f(t)dt$ also converges and moreover:*

$$\int_a^b \alpha f(t)dt = \alpha \int_a^b f(t)dt. \tag{5.7}$$

Proposition 5.1.2. *The proof of this proposition is based on the linearity of Riemann integrals and on the theorems on the sum and product of limits.*

5.2 Integration formulas for generalized integrals

The following two theorems are very useful in the study of generalized integrals:

5.2.1 Integration by parts

Theorem 5.2.1. *Let f and g be two functions of class C^1 on $]a, b[$. If the integral $\int_a^b \dot{f}(t)g(t)dt$ converges and if the function fg has a limit on the right of a and a limit on the left of b, then the integral $\int_a^b f(t)\dot{g}(t)dt$ converges and we have:*

$$\int_a^b \dot{f}(t)g(t)dt = \left(\lim_{x\to b^-} f(x)g(x) - \lim_{x\to a^+} f(x)g(x)\right) - \int_a^b \dot{f}(t)g(t)dt \tag{5.8}$$

Proof. It is based on the fact that the function fg is the primitive of the function $\dot{f}g + f\dot{g}$ on the compact interval $[a, x]$ and on the fact that the function fg has a limit on the right of a and a limit on the left of b, □

5.2.2 Change of variables

Theorem 5.2.2. *Let f be a continuous function on $]a, b[$ and let φ be a bijective function of class C^1 on $]\alpha, \beta[$, verifying $a = \lim_{x\to\alpha^+} \varphi(x)$ and $b = \lim_{x\to\beta^-} \varphi(x)$. Then the integrals $\int_a^b f(t)dt$ and $\int_\alpha^\beta \dot{\varphi}(x)f(\varphi(x))dx$ are of the same nature, moreover,* *if they converge, we have:*

$$\int_a^b f(t)dt = \int_\alpha^\beta \dot{\varphi}(x) f(\varphi(x))dx. \tag{5.9}$$

Proof. It is based on the change of variable $t = \varphi(x)$ and the fact that $a = \lim_{x\to\alpha^+}\varphi(x)$ et $b = \lim_{x\to\beta^-}\varphi(x)$.

□

5.3 Generalized integral of functions of constant sign

In the following, we are interested in the case where the functions f and g are locally integrable and of constant sign on the interval $[a, b[$ or $]a, b]$.
In particular, we will state all the results in the case where the functions f and g are positive. If the functions f and g are negative, we will study the integral of the functions $-f$ and $-g$.

Theorem 5.3.1. *Let f be a positive and locally integrable function on $[a, b[$. Then the integral $\int_a^b f(t)dt$ converges if and only if the function*

$$x \mapsto F(x) = \int_a^x f(t)dt, x \in [a, b[\,, \tag{5.10}$$

is majorized $[a, b[$, and moreover, we have

$$\textit{for all } x \in [a, b[\,; F(x) \leq \int_a^b f(t)dt, \tag{5.11}$$

Proof. Since f is positive, the function F is therefore increasing. But for $\lim_{x\to b^-} F(x)$ to exist, it is necessary and sufficient that F be majorised and we have:

$$F(x) = \int_a^x f(t)dt \leq \int_a^b f(t)dt. \tag{5.12}$$

□

5.3.1 Comparison of generalized integrals of two positive functions

Theorem 5.3.2. *(comparison theorem) Let f and g be two positive and locally integrable functions on $[a,b[$ or $]a,b]$ verifying:*

$$0 \leq f(t) \leq g(t). \tag{5.13}$$

1. If the integral $\int_a^b g(t)dt$ is convergent, then the integral $\int_a^b f(t)dt$ is also convergent, and we have:

$$\int_a^b f(t)dt \leq \int_a^b g(t)dt. \tag{5.14}$$

2. If the integral $\int_a^b f(t)dt$ is divergent, then the integral $\int_a^b g(t)dt$ is also divergent.

Proof. To prove this theorem, suppose for example that f and g are defined on $[a,b[$.

For all $x \in [a,b[$, we have

$$F(x) = \int_a^x f(t)dt \leq \int_a^x g(t)dt = G(x). \tag{5.15}$$

1. If the integral $\int_a^b g(t)dt$ is convergent, G is majorized, F is also majorized, hence the result.

2. If the integral $\int_a^b f(t)dt$ diverges, this means that $\lim_{x\to b^-} F(x) = +\infty$, so $\lim_{x\to b^-} G(x) = +\infty$ and the integral $\int_a^b g(t)dt$ diverges. □

Corollary 5.3.1. *Let f and g be two positive and locally integrable functions on $[a,b[$ verifying:*

$$f(t) = O(g(t)), \text{ when } t \to b^-. \tag{5.16}$$

1. If the integral $\int_a^b g(t)dt$ is convergent, then the integral $\int_a^b f(t)dt$ is also convergent

2. If the integral $\int_a^b f(t)dt$ is divergent, then the integral $\int_a^b g(t)dt$ is also divergent.

Proof. By definition $f(t) = O(g(t))$, when $t \to b^-$ if and only if:

$$\exists t_0 \in [a,b[\text{, and } \exists c > 0, \text{ such that } \forall t \in [t_0,b[\text{, we have } f(t) \leq cg(t). \tag{5.17}$$

The rest of the proof follows immediately from Theorem 5.3.2. □

Corollary 5.3.2. *Let f and g be two positive and locally integrable functions on $[a, b[$ verifying:*

$$\lim_{t \to b^-} \frac{f(t)}{g(t)} = 0. \tag{5.18}$$

1. If the integral $\int_a^b g(t)dt$ is convergent, then the integral $\int_a^b f(t)dt$ is also convergent
2. If the integral $\int_a^b f(t)dt$ is divergent, then the integral $\int_a^b g(t)dt$ is also divergent.

Proof. By definition $\lim_{t \to b^-} \dfrac{f(t)}{g(t)} = 0$ if and only if:

$$\forall \epsilon > 0, \exists t_0 \in [a, b[\,, \text{ such that } \forall t \in [t_0, b[\,, \text{ on a } \frac{f(t)}{g(t)} < \epsilon. \tag{5.19}$$

The rest of the proof follows immediately from the theorem 5.3.2, taking $\epsilon = 1$. □

5.3.2 Generalized integral of two positive equivalent functions

Definition 5.3.1. *Recall that two functions f and g are equivalent in the neighborhood of a point t_0, if and only if:*

$$\lim_{t \to t_0} \frac{f(t)}{g(t)} = 1. \tag{5.20}$$

Theorem 5.3.3. *Let f and g be two positive functions equivalent to the end of the integration interval $[a, b[$ or $]a, b]$. Then the two integrals $\int_a^b f(t)dt$ and dt $\int_a^b g(t)dt$ are of the same nature.*

Proof. Suppose for example that f and g are defined on $[a, b[$.
By definition, $\lim_{t \to b} \dfrac{f(t)}{g(t)} = 1 \Leftrightarrow$ if and only if:

$$\forall \epsilon > 0, \exists \alpha \in [a, b[\,, \text{ such that } \forall t \in [\alpha, b[\,, (1-\epsilon)g(t) \leq f(t) \leq (1+\epsilon)g(t). \tag{5.21}$$

For $\epsilon = \dfrac{1}{2}$ fixed, we then have:

$$\forall t \in [\alpha, b[\,, \frac{1}{2}g(t) \leq f(t) \leq \frac{3}{2}g(t). \tag{5.22}$$

We can therefore apply the theorem 5.3.2: If $\int_a^b g(t)dt$ converges, $\int_a^b f(t)dt$ also converges by the inequality on the right and if $\int_a^b f(t)dt$ converges, $\int_a^b g(t)dt$ also converges by the inequality on the left.
We make an analogous demonstration to show that if one of the integrals diverges, then the same is true for the other. □

Proposition 5.3.1. *(Riemann functions)*
1. Let f be a locally integrable function on $[1, +\infty[$. Then the Riemann integral $\int_1^{+\infty} \frac{dt}{t^\alpha}$ converges if and only if $\alpha > 1$.
2. Now, let f be a locally integrable function on $]0, 1]$. Then the Riemann integral $\int_0^1 \frac{dt}{t^\alpha}$ converges if and only if $\alpha < 1$.

Proof. 1. A simple calculation, we find:

$$\int_1^x \frac{dt}{t^\alpha}dt = \begin{cases} \frac{1}{1-\alpha}\left(\frac{1}{x^{\alpha-1}} - 1\right), \text{ if } \alpha \neq 1, \\ \ln x, \text{ if } \alpha = 1. \end{cases}$$

These functions admit finite limits in the neighborhood of infinity only in the case where $\alpha > 1$.
2. Similarly, for $x \in]0, 1[$

$$\int_x^1 \frac{dt}{t^\alpha}dt = \begin{cases} \frac{-1}{1-\alpha}\left(\frac{1}{x^{\alpha-1}} - 1\right), \text{ if } \alpha \neq 1, \\ -\ln x, \text{ if } \alpha = 1. \end{cases} \tag{5.23}$$

These functions admit finite limits in the neighborhood of zero only in the case where $\alpha < 1$. □

Remak 5.2. *Using a change of variable $t \mapsto t - t_0$, allows us to also apply the preceding arguments to the functions $t \mapsto \frac{1}{(t-t_0)^\alpha}$ on the half-open intervals $]t_0, b]$ and $[b, +\infty[$, such that $b > t_0$.*

5.4 General convergence criteria

5.4.1 Cauchy criterion

Theorem 5.4.1. *Let f a locally integrable function on $[a,b[$. For the integral $\int_a^b f(t)dt$ to be convergent, it is necessary and sufficient that:*

$$\forall \epsilon > 0, \exists \delta > 0,\ \forall x, y \in [a,b[, b-\delta < x < y < b, \left|\int_x^y f(t)dt\right| < \epsilon. \tag{5.24}$$

Proof. It suffices to apply the Cauchy criterion to the function $x \mapsto F(x) = \int_a^x f(t)dt$ and the fact that $F(x) - F(y) = \int_x^y f(t)dt$. □

5.4.2 Abel-Dirichlet criterion

Lemma 5.4.1. *Let f be a function of class C^1, positive, decreasing on $[a,b[$ and tends to 0 when x tends to b, and let g be a continuous function on $[a,b[$ verifying the property:*

$$\exists M > 0, \forall x, y \in [a,b[\,, \left|\int_x^y g(t)dt\right| \leq M. \tag{5.25}$$

Then the integral $\int_a^b f(t)g(t)dt$ converges, and for all $x \in [a,b[$, we have:

$$\left|\int_x^b f(t)g(t)dt\right| \leq Mf(x). \tag{5.26}$$

Proof. By definition, we have:

$$\lim_{x\to b^-} f(x) = 0 \Leftrightarrow \forall \epsilon > 0, \exists \delta > 0,\ \forall x \in [b-\delta, b[\,,\ f(x) < \frac{\epsilon}{M}. \tag{5.27}$$

Now let x be fixed in $[a,b[$ and let $G(y) = \int_x^y g(t)dt$.
By doing an integration by parts, we obtain:

$$\int_x^y f(t)g(t)dt = f(y)G(y) + \int_x^y (-\dot{f}(t))G(t)dt. \tag{5.28}$$

So for all $y \geq x \geq a$, we have:

$$\begin{aligned} \left|\int_x^y f(t)g(t)dt\right| &\leq \left|f(y)G(y)\right| + \int_x^y \left|(-\dot{f}(t))G(t)\right| dt \\ &\leq Mf(y) + \int_x^y (-\dot{f}(t))Mdt \\ &= Mf(y) + Mf(x) - Mf(y) \\ &= Mf(x). \end{aligned} \tag{5.29}$$

Using the inequality (5.27), we find:

$$\left|\int_x^y f(t)g(t)dt\right| < \epsilon. \tag{5.30}$$

The Cauchy criterion then allows us to assert that the integral $\int_a^b f(t)g(t)dt$ converges. Moreover, by making y tend towards b in the inequality (5.30), we obtain the inequality (5.26). □

5.5 Absolute convergence or semi-convergence

Let f be a locally integrable function on $[a, b[$.

Definition 5.5.1. *The integral $\int_a^b f(t)dt$ is said to converge absolutely if and only if the integral $\int_a^b \left|f(t)\right| dt$ converges.*

Definition 5.5.2. *The integral $\int_a^b f(t)dt$ is said to be semi-convergent when it is convergent without being absolutely convergent.*

Theorem 5.5.1. *If the integral $\int_a^b f(t)dt$ converges absolutely, then $\int_a^b f(t)dt$ converges and moreover:*

$$\left|\int_a^b f(t)dt\right| \leq \int_a^b \left|f(t)\right| dt. \tag{5.31}$$

Proof. The proof proceeds using the Cauchy criterion and the fact that:

$$\left|\int_x^y f(t)dt\right| \leq \int_x^y \left|f(t)\right| dt,\ x, y \in [a, b[\tag{5.32}$$

□

Example 5.5.1. *Consider the function f defined on $\left[\frac{\pi}{2}, +\infty\right[$ by $f(x) = \frac{\sin t}{t^2}$. For all $t \geq \frac{\pi}{2}$, we have:*

$$\left|\int_x^y f(t)dt\right| \leq \frac{1}{t^2}. \tag{5.33}$$

Since the integral $\int_{\frac{\pi}{2}}^{+\infty} \frac{dt}{t^2}$ converges, we deduce that $\int_{\frac{\pi}{2}}^{+\infty} f(t)dt$ converges absolutely, so it converges.

The following theorem is widely used in practice.

Theorem 5.5.2. *Let f be a locally integrable function on the interval $[a, +\infty[$, where $a > 0$, such that there exists $\alpha > 1$ verifying $\lim_{t\to+\infty} t^\alpha \left|f(t)\right| = 0$. Then the integral $\int_a^{+\infty} f(t)dt$ converges absolutely.*

Proof. By definition:

$$\lim_{t\to+\infty} t^\alpha \left|f(t)\right| = 0 \Leftrightarrow \forall \epsilon > 0, \exists A > a \text{ such that } \forall t \in [A, +\infty[\,, \left|f(t)\right| < \frac{\epsilon}{t^\alpha}.$$

Since the Riemann integral $\int_a^{+\infty} \frac{dt}{t^\alpha}$ converges, using the comparison theorem 5.3.2 shows the absolute convergence of the integral $\int_a^{+\infty} f(t)dt$. □

5.6 Generalized integrals and numerical series

In this section, we will give some results specifying the link between generalized integrals and numerical series.

Theorem 5.6.1. *Let f be a locally integrable function on the interval $[a, +\infty[$. Then the following properties are equivalent:*

1. The integral $\int_a^{+\infty} f(t)dt$ converges.

2. The numerical sequence with general term $F(x_n) = \int\limits_a^{x_n} f(x)dx$ converges, , where $(x_n)_n$ is a sequence of elements of $[a, +\infty[$ with limit $+\infty$, when $n \to +\infty$.

3. The numerical series with term general $u_n = \int\limits_{x_n}^{x_{n+1}} f(x)dt$ converges, where $(x_n)_n$ is a sequence of elements of $[a, +\infty[$ with limit $+\infty$, when $n \to +\infty$.

Proof. Let's show 1⇒ 2. For all $x \in [a, +\infty[$, the integral $\int_a^{+\infty} f(t)dt$ converges if and only:

$$\forall \epsilon > 0, \exists \delta > 0, \forall x \geq \delta, \text{ we have } \left| \int_a^{+\infty} f(t)dt - \int_a^{x} f(t)dt \right| = \left| \int_x^{+\infty} f(t)dt \right| < \epsilon. \tag{5.34}$$

Let now $(x_n)_n$ be a sequence of élements of $[a, +\infty[$ of limit $+\infty$, when $n \longrightarrow +\infty$, i.e.

$$\forall A > 0, \exists n_0 \in \mathbb{N}, \forall n \geq n_0, \; x_n \geq A \tag{5.35}$$

Let's take $A = \delta$, so for all $n \geq n_0$, and for all $x_n \geq \delta$, we have:

$$\left| \int_a^{+\infty} f(t)dt - F(x_n) \right| = \left| \int_a^{+\infty} f(t)dt - \int_a^{x_n} f(t)dt \right| = \left| \int_x^{+\infty} f(t)dt \right| < \epsilon. \tag{5.36}$$

Hence the sequence $F(x_n) = \int\limits_a^{x_n} f(n)dt$ converges.

Let us now show 2 ⇒ 1. Suppose that $F(x_n) = \int\limits_a^{x_n} f(x)dt$ converges and that the integral $\int_a^{+\infty} f(t)dt$ does not converge, we then have:

$$\exists \epsilon > 0, \forall \delta > 0, \text{ we can find } x > \delta, \text{ such that } \left| \int_a^{+\infty} f(t)dt - \int_a^{x} f(t)dt \right| \geq \epsilon. \tag{5.37}$$

As a result, we can find a sequence of elements (x_n) of $[a, +\infty[$verifying::

$$\left| \int_a^{+\infty} f(t)dt - \int_a^{x_n} f(t)dt \right| = \left| \int_a^{+\infty} f(t)dt - F(x_n)dt \right| \geq \epsilon,$$

which contradicts the hypothesis that $(F(x_n))_n$ converges.
Finally, let us show that 2 ⇔ 3. For all $n \in \mathbb{N}$:

$$F(x_n) = \int\limits_a^{x_n} f(n)dt = \int\limits_a^{x_0} f(n)dt + \sum_{k=0}^{n-1} \int\limits_{x_k}^{x_{k+1}} f(n)dt. \tag{5.38}$$

Therefore the numerical sequence $(F(x_n))_n$ converges if and only if the series of general term $u_n = \int\limits_{x_n}^{x_{n+1}} f(n)dt$ converges. □

Corollary 5.6.1. *Similarly, when f is a locally integrable function on the interval $[a, +b[$, we can easily demonstrate that the integral $\int_a^b f(t)dt$ converges if and only* *if the numerical sequence with general term* $F^*(x_n) = \int_a^{b-x_n} f(x)dt$ *converges, where* $(x_n)_n$ *is a sequence of elements of* $[a, +b[$ *with limit* 0*, when* $n \to +\infty$*.*

Theorem 5.6.2. *(Chauchy's theorem) Let* $f :]0, +\infty[\to \mathbb{R}$ *be a positive, continuous and decreasing definite function, then the series with positive terms* $\sum_{n\geq 1} f(n)$ *and the generalized integral* $\int_1^{+\infty} f(x)dx$ *are of the same nature.*

Proof. Let (S_n) be the sequence of partial sums of the series $\sum_{n\geq 1} f(n)$.
Since f is decreasing on $]0, +\infty[$, we can write:

$$\text{for all } k = 1, 2, ..., \ x \in]0, +\infty[\,, \text{ such that } k \leq x \leq k+1, \text{ we have } f(k+1) \leq f(x) \leq f(k). \tag{5.39}$$

Integrating over $[k, k+1]$, we find:

$$f(k+1) \leq \int_k^{k+1} f(x)dx \leq f(k), \text{ for all } k = 1, 2, ... \tag{5.40}$$

By adding these last equalities, we obtain:

$$S_{n+1} - f(1) \leq \int_1^{n+1} f(x)dx \leq S_n. \tag{5.41}$$

* Suppose that $\int_1^{+\infty} f(x)dx$ converges, we can see then :

$$S_{n+1} - f(1) \leq \int_1^{n+1} f(x)dx \leq \int_1^{+\infty} f(x)dx. \tag{5.42}$$

Which shows that the sequence (S_{n+1}) is majorized, and consequently the series $\sum_{n\geq 1} f(n)$ converges.
* Suppose that $\sum_{n\geq 1} f(n)$ converges. We know well that

$$n \leq x < n+1, \text{ for } x \geq 1, \tag{5.43}$$

where n represents the integer part of x. We then have:

$$\int_1^x f(x)dx \leq \int_1^{n+1} f(x)dx \leq S_n. \tag{5.44}$$

Since (S_n) is majorized, the integral $\int_1^x f(x)dx$ is also majorized, which ensures the existence of the integral $\int_1^{+\infty} f(x)dx$.

It also follows by contraposition that the divergence of the series $\sum_{n\geq 1} f(n)$ entails the divergence of the integral $\int_1^{+\infty} f(x)dx$.

□

5.7 Generalized integrals and numerical sequences

Theorem 5.7.1. *(Dominated convergence theorem) Let $(f_n)_n$ be a sequence of locally integrable functions on $[a,b[$, which converges uniformly locally on $[a,b[$ to a function f, and let g be a positive and locally integrable function on $[a,b[$ verifying the following two properties:*

1. The integral $\int_a^b g(t)dt$ converge.

2. For all $n \in \mathbb{N}$ and for all $x \in [a,b[$, we have:

$$\left|f_n(t)\right| \leq g(t). \tag{5.45}$$

So the two integrals $\int_a^b f_n(t)dt$ ($n \in \mathbb{N}$) and $\int_a^b f(t)dt$ converge absolutely, and we have:

$$\lim_{n\to+\infty} \int_a^b f_n(t)dt = \int_a^b f(t)dt. \tag{5.46}$$

Proof. The absolute convergence of the integral $\int_a^b f_n(t)dt$ proceeds from the inequality (5.45) and the use of the comparison theorem. Moreover, by making n tend to $+\infty$ in (5.45), we obtain:

$$\text{for all } x \in [a,b[\,,\ \left|f(t)\right| \leq g(t). \tag{5.47}$$

We then deduce the absolute convergence of the integral $\int_a^b f_n(t)dt$.
Let us now show that $\lim_{n\to+\infty}\int_a^b f_n(t)dt = \int_a^b f(t)dt$.
By definition, the integral $\int_a^b g(t)dt$ converges if and only if:

$$\forall \epsilon > 0, \exists \delta > 0, \text{ such that } \forall x \in [b-\delta, b[\,, \text{ we have } \left|\int_x^b g(t)dt\right| < \frac{\epsilon}{3}. \tag{5.48}$$

On the other hand, the sequence $(f_n)_n$ converges uniformly locally to f on $[a, b[$, so it converges uniformly to f on any compact $[a, x]$, for any x fixed in $[b-\delta, b[$ i.e:

$$\forall \epsilon > 0, \exists n_0 \in \mathbb{N}, \forall n \in \mathbb{N}, \forall t \in [a, x]\ n \geq n_0,\ \left|f_n(t) - f(t)\right| < \frac{\epsilon}{3(x-a)}. \tag{5.49}$$

We then have:

$$\begin{aligned}
\left|\int_a^b f_n(t)dt - \int_a^b f(t)dt\right| &= \left|\int_a^b (f_n(t)dt - f(t))\,dt\right| \\
&= \left|\int_a^x (f_n(t)dt - f(t))\,dt + \int_x^b (f_n(t)dt - f(t))\,dt\right| \\
&\leq \int_a^x \left|f_n(t) - f(t)\right| dt + \int_x^b \left|f_n(t)\right| dt + \int_x^b \left|f(t)\right| dt \\
&\leq \int_a^x \left|f_n(t) - f(t)\right| dt + 2\int_x^b g(t)dt \\
&\leq \frac{\epsilon(x-a)}{3(x-a)} + \frac{2\epsilon}{3} = \epsilon.
\end{aligned}$$

□

5.8 Mean value theorems for integrals

5.8.1 First formula of the mean value

Theorem 5.8.1. *Let f and g be two functions verifying f integrable and with a constant sign on $[a, b]$ and g is continuous on the same segment, then there exists $c \in [a, b]$ verifying:*

$$\int_a^b f(t)g(t)dt = g(c)\int_a^b f(t)dt. \tag{5.50}$$

Proof. Let us place ourselves in the case where $f(x) > 0$, for all $x \in [a,b]$.
Let $m = \inf_{x\in[a,b]} g(x)$ and $M = \sup_{x\in[a,b]} g(x)$, we can then write:

$$m\int_a^b f(t)dt \leq \int_a^b f(t)g(t)dt \leq M\int_a^b f(t)dt, \tag{5.51}$$

Hence:

$$m \leq \frac{\int_a^b f(t)g(t)dt}{\int_a^b f(t)dt} \leq M. \tag{5.52}$$

That is to say that $\frac{\int_a^b f(t)g(t)dt}{\int_a^b f(t)dt} \in \left[\inf_{x\in[a,b]} g(x), \sup_{x\in[a,b]} g(x)\right]$.
Since g is continuous, we can then find $c \in [a,b]$ verifying:

$$\frac{\int_a^b f(t)g(t)dt}{\int_a^b f(t)dt} = g(c), \tag{5.53}$$

which completes the proof of the theorem. □

5.8.2 Second formula for the mean value

Theorem 5.8.2. *Let f be a function of class C^1, positive and decreasing on $[a,b]$, and let g be a continuous function on $[a,b]$, then there exists $c \in [a,b]$ verifying:*

$$\int_a^b f(t)g(t)dt = f(a)\int_a^c f(t)dt. \tag{5.54}$$

Proof. Let $G(x) = \int_a^x g(t)dt$.
By doing an integration by parts, we obtain:

$$\begin{aligned}\int_a^b f(t)g(t)dt &= f(x)G(x) + \int_a^b (-\dot{f}(t))G(t)dt \\ &= f(b)G(b) + \int_a^b (-\dot{f}(t))G(t)dt \text{ (because } G(a)=0).\end{aligned} \tag{5.55}$$

Since f is increasing (i.e $\dot{f}(x) \leq 0$ on $[a,b]$) and $f(b) > 0$, we can then write:

$$f(b)\times m+\int_a^b (-\dot{f}(t))\times mdt \leq f(b)G(b)+\int_a^b (-\dot{f}(t))G(t)dt \leq f(b)\times M+\int_a^b (-\dot{f}(t))\times Mdt, \tag{5.56}$$

where $m = \inf_{x\in[a,b]} G(x)$ and $M = \sup_{x\in[a,b]} G(x)$, or in an equivalent manner:

$$mf(a) \leq \int_a^b f(t)g(t)dt \leq Mf(b). \tag{5.57}$$

That is to say that: $\dfrac{\int_a^b f(t)g(t)dt}{f(a)} \in \left[\inf_{x\in[a,b]} G(x), \sup_{x\in[a,b]} G(x)\right]$.
Since G is continuous, we can then find $c \in [a,b]$ verifying:

$$\frac{\int_a^b f(t)g(t)dt}{f(a)} = G(c) = \int_a^c g(t)dt, \tag{5.58}$$

which completes the proof of the theorem. □

5.9 Cauchy principal value

Definition 5.9.1. *Let f be a locally integrable function on $]-\infty, +\infty[$. The Cauchy principal value (or principal value of the divergent integral $\int_{-\infty}^{+\infty} f(t)dt$), denoted $V.P\left(\int_{-\infty}^{+\infty} f(t)dt\right)$ is the element of $\mathbb{R}$ defined by :*

$$V.P\left(\int_{-\infty}^{+\infty} f(t)dt\right) = \lim_{a\longrightarrow+\infty} \int_{-a}^{a} f(t)dt) \text{ (when this limit exists).} \tag{5.59}$$

Definition 5.9.2. *Now, let f be a function, wich has a singular point $c \in]a,b[$. The Cauchy principal value (or principal value of the divergent integral $\int_a^b f(t)dt$), denoted $V.P\left(\int_a^b f(t)dt\right)$ is the element of $\mathbb{R}$ defined by:*

$$V.P\left(\int_a^b f(t)dt\right) = \lim_{\epsilon\longrightarrow0}\left(\int_a^{c-\epsilon} f(t)dt) + \int_{c+\epsilon}^b f(t)dt\right) \text{ (when this limit exists).} \tag{5.60}$$

Example 5.9.1. *As an example, we take the integral $\int_0^2 \dfrac{dt}{t-1}$. It is clear that this integral is divergent. We then have:*

$$\begin{aligned} V.P\left(\int_0^2 \frac{dt}{t-1}\right) &= \lim_{\epsilon\longrightarrow0}\left(\int_0^{1-\epsilon} \frac{dt}{t-1}) + \int_{1+\epsilon}^2 \frac{dt}{t-1}dt\right) \\ &= \lim_{\epsilon\longrightarrow0}\left(\ln\frac{\epsilon}{\epsilon} + \ln\frac{1}{1}\right) = 0. \end{aligned} \tag{5.61}$$

5.10 Exercises about chapter 5

Exercise 5.10.1. *Returning to the definition, study whether the following generalized integrals have meaning or not and give their possible value:*

$$I_1 = \int_0^{+\infty} x^n \exp(-x)dx,\ n \in \mathbb{N}, \quad I_2 = \int_4^{+\infty} x\sin(2x)dx.$$

$$I_3 = \int_0^{+\infty} (\ln(x))^n dx,\ n \in \mathbb{N}, \quad I_4 = \int_0^{2} \frac{dx}{\sqrt{(2x - x^2)}}.$$

Exercise 5.10.2. *Study the convergence of the following generalized integrals:*

$$I_1 = \int_0^{\frac{1}{2}} \frac{\sin x}{x^{\alpha} |\ln x|^{\beta}} dx, \quad I_2 = \int_0^{\frac{1}{2}} \frac{\ln(1 + x\sin x)}{x |\ln x|^{\delta}} dx.$$

$$I_3 = \int_0^{\frac{1}{2}} \frac{\sin(x^2) + \cos(x^3)}{x(\ln x)^{\frac{3}{2}}} dx, \quad I_4 = \int_1^{+\infty} \frac{\exp(-\frac{x}{5})\sin(\ln(x))}{(x-1)^{\frac{3}{2}}} dx.$$

Exercise 5.10.3. *Study the absolute convergence and semi-convergence of the following generalized integrals:*

$$I_1 = \int_1^{+\infty} \frac{\sin x}{x^{\theta}} dx,\ \theta \in \mathbb{R}_+^*, \quad I_2 = \int_0^{+\infty} x^2 \cos(\exp(x))dx.$$

Chapter 6

Functions defined by an integral

This chapter consists of studying functions of the form:

$$F(y) = \int_a^b f(x,y)dx, \text{ or } a \in \mathbb{R} \cup \{-\infty\} \text{ and } b \in \mathbb{R} \cup \{+\infty\},$$

in particular, we ask ourselves the question of knowing on what conditions on f we have the continuity, the differentiability and the integrability of the function F. We will distinguish the cases of proper integrals where f is defined on a compact interval of $\mathbb{R}$ and improper integrals.

6.1 Functions defined by a proper integral

In this section we consider a function $f = f(x,y) : [a,b] \times I \to \mathbb{R}$ (I an open interval of $\mathbb{R}$) Riemann-integrable with respect to the first variable x on $[a,b]$, for all $y \in I$.

6.1.1 Properties of a function defined by a proper integral

Continuity

In this paragraph, we are interested in the continuity of the function defined by:

$$F(y) = \int_a^b f(x,y)dx, y \in I. \tag{6.1}$$

Theorem 6.1.1. *Let $f : [a,b] \times I \to \mathbb{R}$ be a continuous function, then the function F defined by the relation (6.1) is also continuous on I. In particular, for all $y_0 \in I$, we have:*

$$\begin{aligned} \lim_{y\to y_0} F(y) &= \lim_{y\to y_0} \int_a^b f(x,y)dx = \int_a^b \lim_{y\to y_0} f(x,y)dx \\ &= \int_a^b f(x,y_0)dx = F(y_0), \end{aligned}$$

which is a case of ***interversion of limit and integral****.*

Proof. Since f is continuous, the function $x \mapsto f(x,y)$ is integrable on $[a,b]$. Let now y_1, y_2 be any two points of I and let $V \subset I$ be a compact interval containing y_1 and y_2.
Since f is continuous on $[a,b] \times I$, it is therefore uniformly continuous on the compact $[a,b] \times V$, i.e;
$\forall \epsilon > 0, \exists \delta > 0$, such that $\forall (x_1,y_1), (x_2,y_2) \in [a,b] \times V$, we have:

$$\left\|(x_1,y_1) - (x_2,y_2)\right\| < \delta \Rightarrow \left|f(x_1,y_1) - f(x_2,y_2)\right| < \frac{\epsilon}{b-a}. \tag{6.2}$$

In particular, if we set $x = x_1 = x_2$, we find, for all $y_1, y_2 \in V$:

$$\left|y_1 - y_2\right| < \delta \Rightarrow \left|f(x,y_1) - f(x,y_2)\right| < \frac{\epsilon}{b-a}. \tag{6.3}$$

Hence, for all $y_1, y_2 \in V$, such that $\left|y_1 - y_2\right| < \delta$, we have:

$$\begin{aligned} \left|F(y_1) - F(y_2)\right| &= \left|\int_a^b \left(f(x,y_1) - f(x,y_2)\right) dx\right| \\ &\le \int_a^b \left|f(x,y_1) - f(x,y_2)\right| dx < \epsilon. \end{aligned} \tag{6.4}$$

Consequently F is uniformly continuous on V, so it is continuous on I. □

Derivability

Theorem 6.1.2. *Let $f : [a,b] \times I \to \mathbb{R}$ be a continuous function on $[a,b] \times I$. We assume that the partial derivative $\frac{\partial f}{\partial y}$ exists and is continuous on $[a,b] \times I$, then the function F defined by the relation (6.1) is derivable on I, and we have:*

$$\dot{F}(y) = \frac{\partial}{\partial y}\left(\int_a^b f(x,y)dx\right) = \int_a^b \frac{\partial}{\partial y} f(x,y)dx, \tag{6.5}$$

which is a case of ***inversion of derivative and integral****.*

Proof. Let us first note that the functions $x \mapsto f(x,y)$ and $x \mapsto \frac{\partial}{\partial y} f(x,y)$ are integrable on $[a,b]$ because f and $\frac{\partial}{\partial y} f$ are continuous.
As before, let y be any point of I and let $V \subset I$ be a compact interval containing y. It suffices to show that:

$$\lim_{h\to 0}\left(\frac{F(y+h) - F(y)}{h}\right) - \int_a^b \frac{\partial}{\partial y} f(x,y)dx = 0. \tag{6.6}$$

Indeed, if the theorem of finite increments applies, we have then:

$$\begin{aligned} \frac{F(y+h)-F(y)}{h} - \int_a^b \frac{\partial}{\partial y} f(x,y)dx &= \int_a^b \left(\frac{f(x,y+h)-f(x,y)}{h} - \frac{\partial}{\partial y} f(x,y)\right)dx \quad (6.7)\\ &= \int_a^b \left(\frac{\partial}{\partial y} f(x,y+\theta h) - \frac{\partial}{\partial y} f(x,y)\right)dx,\ \theta \in \left]0,1\right[. \end{aligned}$$

Since $\frac{\partial}{\partial y} f$ is uniformly continuous on the compact $[a,b] \times V$, we then have, $\forall \epsilon > 0, \exists \delta > 0$, such that $\forall x \in [a,b], \forall y, h \in V$:

$$\left|(y+h) - y\right| = |h| < \delta \Rightarrow \left|\frac{\partial}{\partial y} f(x,y+\theta h) - \frac{\partial}{\partial y} f(x,y)\right| < \frac{\epsilon}{b-a}. \tag{6.8}$$

Consequently:

$$\left|\frac{F(y+h)-F(y)}{h} - \int_a^b \frac{\partial}{\partial y} f(x,y)dx\right| \le \int_a^b \left|\frac{\partial}{\partial y} f(x,y+\theta h) - \frac{\partial}{\partial y} f(x,y)\right| < \epsilon, \tag{6.9}$$

which proves the differentiability of F at y, and since y is considered arbitrary in I, this clearly shows the differentiability of F on I. □

Integration

Theorem 6.1.3. *Let $f : [a, b] \times I \to \mathbb{R}$ be a continuous function on $[a, b] \times I$, then the function F defined by the relation (6.1) is integrable on I. In particular for all $y \in I$, we have:*

$$\begin{aligned} \int_I F(y)dy &= \int_I dy\left(\int_a^b f(x, y)dx\right) \\ &= \int_a^b dx\left(\int_I f(x, y)dy\right), \end{aligned} \tag{6.10}$$

which is a case of ***interversion of integrals****.*

Proof. We are going to demonstrate a more general formula. That is, we want to demonstrate that:

$$G(z) = \int_I dy\left(\int_a^z f(x, y)dx\right) = \int_a^z dx\left(\int_I f(x, y)dy\right) = H(z), \text{ for allt } z \in [a, b], \tag{6.11}$$

where G, H are continuous functions on $[a, b]$.
On the one hand, if we set $\int_a^z f(x, y)dx = F(z, y)$. It is clear that F is continuous by the theorem 6.1.1, and moreover:

$$\frac{\partial}{\partial z}F(z, y) = f(z, y). \tag{6.12}$$

We deduce from the theorem 6.1.2 that:

$$\dot{G}(z) = \int_I \frac{\partial}{\partial z}F(z, y)dy = \int_I f(z, y)dy. \tag{6.13}$$

On the other hand, if we set $\int_c^d f(x, y)dy = K(x)$, we can then write:

$$H(z) = \int_a^z K(x)dx. \tag{6.14}$$

Since k is continuous, we deduce:

$$\dot{H}(z) = K(z) = \int_c^d f(z, y)dy \tag{6.15}$$

From (6.13) and (6.15), we find:

$$\dot{G}(z) = \dot{H}(z) \tag{6.16}$$

By integrating this last equality from a to t, and using the fact that $G(a) = H(a) = 0$, we deduce that $G(z) = H(z)$, for all $z \in [a, b]$. □

6.2 Functions defined by a generalized integral

Let $f : [a, b[\times I \to \mathbb{R}$ (I an open interval of $\mathbb{R}$) be a function of two variables. $b \in \mathbb{R} \cup \{+\infty\}$ and that for some $y \in I$, the function f is not defined at b. We also assume that the generalized integral $\int_a^b f(x, y)dx$ converges and we are interested in the properties of the functions defined on I by the following generalized integrals:

$$F(y) = \int_a^{+\infty} f(x, y)dx = \lim_{z \to +\infty} \int_a^z f(x, y)dx \tag{6.17}$$

and:

$$F^*(y) = \int_a^b f(x, y)dx = \lim_{\gamma \to 0} \int_a^{b-\gamma} f(x, y)dx. \tag{6.18}$$

6.2.1 Uniform convergence of generalized integrals

Definition 6.2.1. *We say that the generalized integral (6.17) is uniformly convergent on I, if and only if:*

$$\forall \epsilon > 0, \exists \delta > 0, \ \forall y \in I, \ \forall z \in [\delta, +\infty[, \ \left| F(y) - \int_a^z f(x, y)dx \right| < \epsilon. \tag{6.19}$$

Definition 6.2.2. *We say that the generalized integral (6.18) is uniformly convergent on I, if and only if:*

$$\forall \epsilon > 0, \exists \delta > 0, \ \forall y \in I, \ \forall \gamma \in]0, \delta[, \ \left| F^*(y) - \int_a^{b-\gamma} f(x, y)dx \right| < \epsilon. \tag{6.20}$$

Theorem 6.2.1. *Let $f : [a, +\infty[\times I \to \mathbb{R}$ (I be an open interval of $\mathbb{R}$) an integrable function (in the generalized sense) on the interval $[a, +\infty[$. Then the following properties are equivalent:*

1. The integral $\int_a^{+\infty} f(x, y)dx$ converges uniformly on I.

2. The sequence of functions with general term $F_n(y) = \int\limits_a^{y_n} f(x, y)dx$ converges uniformly on I, where $(y_n)_n$ is a sequence of elements of $[a, +\infty[$ with limit $+\infty$, when $n \to +\infty$.

Proof. The proof of this theorem follows immediately from the proof of Theorem 5.6.1 (See Chapter 5). □

Corollary 6.2.1. *Similarly, if f is an integrable function (in the generalized sense on the interval $[a, b[$. Then the following properties are equivalent:*
1. The integral $\int_a^b f(x,y)dx$ converges uniformly on I.
2. The sequence of functions with general term $F_n^*(y) = \int\limits_a^{\gamma_n} f(x,y)dx$ converges uniformly on I, where $(\gamma_n)_n$ is a sequence of elements of $[a, +b[$ of limit 0, when $n \longrightarrow +\infty$.

6.2.2 Uniform convergence criteria of generalized integrals

Cauchy criterion

Theorem 6.2.2. *A necessary and sufficient condition for the generalized integral (6.17) to be uniformly convergent on I is:*

$$\forall \epsilon > 0, \exists \delta > 0,\ \forall y \in I,\ \forall z_2 > z_1 \geq \delta,\ \left| \int_{z_1}^{z_2} f(x,y)dx \right| < \epsilon. \tag{6.21}$$

Proof. * Suppose that the generalized integral (6.17) is uniformly convergent on $[a, +\infty[$. We then have:

$$\forall \epsilon > 0, \exists \delta > 0,\ \forall y \in I,\ \forall z_2 > z_1 \geq \delta,\ \left| \int_{z_1}^{+\infty} f(x,y)dx \right| < \frac{\epsilon}{2} \text{ and } \left| \int_{z_2}^{+\infty} f(x,y)dx \right| < \frac{\epsilon}{2}.$$

For all $y \in I$, and for all $z_2 > z_1 \geq \delta$, we can write:

$$\begin{aligned} \left| \int_{z_1}^{z_2} f(x,y)dx \right| &= \left| \int_{z_1}^{+\infty} f(x,y)dx - \int_{z_2}^{+\infty} f(x,y)dx \right| \\ &\leq \left| \int_{z_1}^{+\infty} f(x,y)dx \right| + \left| \int_{z_2}^{+\infty} f(x,y)dx \right| < \epsilon. \end{aligned} \tag{6.22}$$

* Now, suppose that the generalized integral is realized (6.17) verifies the Cauchy criterion, i.e:

$$\forall \epsilon > 0, \exists \delta > 0,\ \forall y \in I,\ \forall z_2 > z_1 \geq \delta, \left| \int_{z_1}^{z_2} f(x,y)dx \right| < \epsilon.$$

By passing to the limit, when $z_2 \to +\infty$, we obtain the requested result. □

Corollary 6.2.2. *In the same way, we can demonstrate that the generalized integral (6.18) is uniformly convergent on I, if and only if:*

$$\forall \epsilon > 0, \exists \delta > 0,\ \forall y \in I,\ \forall \gamma_1, \gamma_2 \in]0, \delta[\,, (\gamma_1 > \gamma_2),\ \left| \int_{b-\gamma_1}^{b-\gamma_2} f(x,y)dx \right| < \epsilon. \tag{6.23}$$

Weierstrass criterion

Theorem 6.2.3. *Let $f : [a, b[\times I \to \mathbb{R}$ be an integrable function on $[a, b[$ (I an open interval of $\mathbb{R}$ and $b \in \mathbb{R} \cup \{+\infty\}$). Suppose that there exists a real function g locally integrable on $[a, b[$ (called the upper bound function) verifying:*

$$\begin{cases} 1.\ \left|f(x,y)\right| \le g(x), \text{for all } x \in [a,b[, \\ 2.\ \int_a^b g(x)dx \text{ converge.} \end{cases}$$

Then, for all $y \in I$, the generalized integral $\int_a^b f(x,y)dx$ converges absolutely and uniformly on I.

Proof. The absolute convergence of the integral $\int_a^b f(x,y)dx$ follows immediately from the first condition.
The uniform convergence of the integral $\int_a^b f(x,y)dx$ follows immediately from the inequality:

$$\left| \int_{z_1}^{z_2} f(x,y)dx \right| \le \int_{z_1}^{z_2} \left|f(x,y)\right| dx \le \int_{z_1}^{z_2} \left|g(x)\right| dx < \epsilon,$$

and the Cauchy criterion. □

Abel-Dirichlet criterion

Lemma 6.2.1. *Let $f : [a, b[\times I \to \mathbb{R}$ be an integrable, positive and decreasing function with respect to x on $[a, b[$ and tending to 0 uniformly when x tends to b, and let $g : [a, b[\times I \to \mathbb{R}$ be an integrable function on $[a, b[$ and verifies the property:*

$$\exists M > 0, \forall z \in [a, b[\,, \forall y \in I, \text{ we have } \left| \int_a^z g(x,y)dx \right| \le M. \tag{6.24}$$

Then the integral $\int_a^b f(x,y)g(x,y)dx$ converges uniformlyon I.

Proof. For all $z_1, z_2 \in [a, b[$, using the second formula of the mean value, gives us:

$$\int_{z_1}^{z_2} f(x,y)g(x,y)dx = f(z_1,y)\int_{z_1}^{k} g(x,y)dx + f(z_2,y)\int_{k}^{z_2} g(x,y)dx,\ k \in [z_1, z_2]. \tag{6.25}$$

By hypothesis,

$$\lim_{x\to b} f(x,y) = 0 \Leftrightarrow \forall\epsilon > 0, \exists\delta > 0,\ \forall y \in I,\ \forall x \in [\delta, b[\,,\ \left|f(x,y)\right| < \frac{\epsilon}{4M}. \tag{6.26}$$

On the other hand, using the hypothesis (6.24), allows us to write:

$$\left|\int_{z_1}^{k} g(x,y)dx\right| = \left|\int_{a}^{k} g(x,y)dx - \int_{a}^{z_1} g(x,y)dx\right| \leq 2M, \text{ for all } k \in [z_1, z_2]. \tag{6.27}$$

and

$$\left|\int_{k}^{z_2} g(x,y)dx\right| = \left|\int_{a}^{z_2} g(x,y)dx - \int_{a}^{k} g(x,y)dx\right| \leq 2M, \text{ for all } k \in [z_1, z_2]. \tag{6.28}$$

So, for all $x, z_1, z_2 \in [\delta, b[$ and for all $k \in [z_1, z_2]$, we can write:

$$\begin{aligned} \left|\int_{z_1}^{z_2} f(x,y)g(x,y)dx\right| &\leq \left|f(z_1,y)\right|\left|\int_{z_1}^{k} g(x,y)dx\right| + \left|f(z_2,y)\right|\left|\int_{k}^{z_2} g(x,y)dx\right| \\ &< \frac{\epsilon}{4M} \times 2M + \frac{\epsilon}{4M} = \epsilon. \end{aligned}$$

□

6.3 Properties of a function defined by a generalized integral

In this section, we are interested in properties of the function defined for $y \in I$ by:

$$F(y) = \int_{a}^{+\infty} f(x,y)dx, \tag{6.29}$$

The case where b is fixed in $\mathbb{R}$ this deals with the same way.

6.3.1 Continuity

Theorem 6.3.1. *Let $f : [a, +\infty[\times I \to \mathbb{R}$ be a continuous function and let the integral (6.29) converge uniformly on I, then the function F defined by the relation (6.29) is also continuous on I. In particular, for all $y_0 \in I$, we have:*

$$\begin{aligned} \lim_{y\to y_0} F(y) &= \lim_{y\to y_0} \int_a^{+\infty} f(x,y)dx = \int_a^{+\infty} \lim_{y\to y_0} f(x,y)dx \\ &= \int_a^{+\infty} f(x,y_0)dx = F(y_0), \end{aligned} \tag{6.30}$$

which is a case of ***interversion of limit and integral.***

Proof. According to the theorem 6.2.1, the uniform convergence of the integral (6.29) implies the uniform convergence of the sequence of functions (F_n) with general term

$$F_n(y) = \int_a^{y_n} f(x,y)dx,$$

where (y_n) is a sequence of elements of $[a, +\infty[$ with limit $+\infty$, when $n \to +\infty$. Using the theorem 6.1.1 shows that F_n is a continuous function on I, ; that is, $F(y) = \lim_{n\to+\infty} \int_a^{z_n} f(x,y)dx$ is a continuous function on I, and moreover:

$$\begin{aligned} \lim_{y\to y_0} F(y) &= \lim_{y\to y_0} \lim_{n\to+\infty} \int_a^{y_n} f(x,y)dx = \lim_{n\to+\infty} \lim_{y\to y_0} \int_a^{y_n} f(x,y)dx \\ &= \lim_{n\to+\infty} \int_a^{y_n} \lim_{y\to y_0} f(x,y)dx = \int_a^{+\infty} f(x,y_0)dx = F(y_0) \end{aligned} \tag{6.31}$$

□

6.3.2 Derivability

Theorem 6.3.2. *Let $f, \dfrac{\partial f}{\partial y} : [a, +\infty[\times I \to \mathbb{R}$ be two continuous functions. We assume that the integral (6.29) converges and that the integral $\int_a^{+\infty} \dfrac{\partial f}{\partial y} dx$ converges uniformly on I, then the function F defined by the relation (6.29) is derivable on I , and we have:*

$$\dot{F}(y) = \frac{\partial}{\partial y}\left(\int_a^{+\infty} f(x,y)dx\right) = \int_a^{+\infty} \frac{\partial}{\partial y} f(x,y)dx, \tag{6.32}$$

which is a case ***of inversion of derivative and integral.***

Proof. According to the theorem 6.2.1, the convergence of the integral (6.29) leads to the convergence of the sequence of functions (F_n) of general term

$$F_n(y) = \int_a^{y_n} f(x, y)dx, \tag{6.33}$$

where (y_n) is a sequence of elements of $[a, +\infty[$ with limit $+\infty$, when $n \to +\infty$. Since $f, \dfrac{\partial f}{\partial y} : [a, +\infty[\times I \to \mathbb{R}$ are continuous functions, using the theorem 6.1.2 shows that:

$$\dot{F}_n(y) = \int_a^{z_n} \frac{\partial f}{\partial y}(x, y)dx, \tag{6.34}$$

and moreover $\dot{F}_n$ is continuous on I (according to the theorem 6.1.1).
On the other hand, $\dot{F}_n$ is uniformly convergent on I because $\int_a^{+\infty} \dfrac{\partial}{\partial y} f(x, y)dx$ is also.
Now, using the theorem of the derivability of sequences of functions ensures the derivability of the function F on I, and moreover, we have:

$$\dot{F}(y) = \lim_{n\to+\infty} \dot{F}_n(y) = \lim_{n\to+\infty} \int_a^{z_n} \frac{\partial f}{\partial y}(x, y)dx = \int_a^{+\infty} \frac{\partial}{\partial y} f(x, y)dx.$$

□

6.3.3 Integration

Theorem 6.3.3. *Lete $f : [a, +\infty[\times I \to \mathbb{R}$ be a continuous function and let the integral (6.29) converge uniformly on I, then the function F defined by the relation (6.29) is also integrable on I, and moreover, we have:*

$$\int_I F(y)dy = \int_I dy \left(\int_a^{+\infty} f(x, y)dx \right) = \int_a^{+\infty} dx \left(\int_I f(x, y)dy \right). \tag{6.35}$$

which is a case of ***interversion of the two integrals.***

Proof. According to the theorem 6.2.1, the uniform convergence of the integral (6.29) implies the uniform convergence of the sequence of functions (F_n) with general term

$$F_n(y) = \int_a^{y_n} f(x, y)dx, \tag{6.36}$$

where (y_n) is a sequence of elements of $[a, +\infty[$ with limit $+\infty$, when $n \to +\infty$. Using theorem 6.1.3 shows that F_n is an integrable function on I; that is, $F(y) = \lim_{n\to+\infty} \int_a^{z_n} f(x,y)dx$ is an integrable function on I (limit of a continuous and uniformly convergent sequence of functions, so it is integrable), and moreover:

$$\begin{aligned}\int_I F(y)dy &= \int_I dy\left(\lim_{n\to+\infty}\int_a^{z_n} f(x,y)dx\right) = \lim_{n\to+\infty}\int_I dy\left(\int_a^{z_n} f(x,y)dx\right)\\ &\lim_{n\to+\infty}\int_a^{z_n} dx\left(\int_I f(x,y)dy\right).\end{aligned} \tag{6.37}$$

We then deduce:

$$\int_I F(y)dy = \int_I dy\left(\int_a^{+\infty} f(x,y)dx\right) = \int_a^{+\infty} dx\left(\int_I f(x,y)dy\right). \tag{6.38}$$

□

6.4 Special Functions

6.4.1 Euler's Gamma Function

Definition 6.4.1. *We call Euler's gamma function the special function Γ defined by:*

$$\Gamma(\alpha) = \int_0^{+\infty} x^{\alpha-1}\exp(-x)dx,\ \alpha > 0. \tag{6.39}$$

Remak 6.1. *The Euler gamma function defined by the relation (6.39) is well defined on $]0, +\infty[$.*

Indeed: In the neighborhood of zero, $x^{\alpha-1}\exp(-x) \overset{V(0)}{\sim} x^{\alpha-1}$.

Since $\int_0^1 x^{\alpha-1}dx$ converges if and only if $\alpha > 0$, we deduce that $\int_0^1 x^{\alpha-1}\exp(-x)dx$ converges if and only if $\alpha > 0$.

In the neighborhood of infinity, if we set for example $g(x) = x^{-2}$, we then find:

$$\lim_{x\to+\infty}\frac{x^{\alpha-1}\exp(-x)}{g(x)} = \lim_{x\to+\infty} x^{\alpha+1}\exp(-x) = 0, \textit{ for all } \alpha \in \mathbb{R}. \tag{6.40}$$

Since $\int_1^{+\infty} x^{-2}dx$ *converges, we deduce that* $\int_1^{+\infty} x^{\alpha-1}\exp(-x)dx$ *converges, for all* $\alpha \in \mathbb{R}$.
Finally $\int_0^{+\infty} x^{\alpha-1}\exp(-x)dx$ *converges if and only if* $\alpha > 0$.

Theorem 6.4.1. *The function* Γ *having the following property:*

$$\Gamma(\alpha+1) = \alpha\Gamma(\alpha), \text{ for all } \alpha > 0. \tag{6.41}$$

In particular $\Gamma(n+1) = n!$, *for all* $n \in \mathbb{N}$.

Proof. An integration by parts gives us:

$$\begin{aligned} \Gamma(\alpha+1) &= \int_0^{+\infty} x^{\alpha}\exp(-x)dx \\ &= -\lim_{t\to+\infty}[x^{\alpha}\exp(-x)]_0^t + \alpha\int_0^{+\infty} x^{\alpha-1}\exp(-x)dx = \alpha\Gamma(\alpha). \end{aligned} \tag{6.42}$$

On the other hand $\Gamma(0) = \int_0^{+\infty}\exp(-x)dx = 1$, we deduce that $\Gamma(n+1) = n!$, for all $n \in \mathbb{N}$. □

Remak 6.2. *The function* Γ *can be extended to a function defined on the set of real numbers, except for* $\alpha = 0, -1, -2, -3, \ldots$.
Indeed, from the relation (6.41), we can write:

$$\Gamma(\alpha-1) = \frac{\Gamma(\alpha)}{\alpha-1} \qquad -1 < \alpha-1 < 0$$

$$\Gamma(\alpha-2) = \frac{\Gamma(\alpha-1)}{\alpha-2} \qquad -2 < \alpha-2 < -1.$$

In this way, we can find:

$$\Gamma(\alpha) = \frac{\Gamma(\alpha+1)}{\alpha} \qquad -n < \alpha < -(n-1). \tag{6.43}$$

Figure 6.1 shows the graph of Euler's gamma function.

Theorem 6.4.2. *The function* Γ *is infinitely derivable on* $\mathbb{R}_+^*$ *and furthermore, we have:*

$$\Gamma^{(n)}(\alpha) = \int_0^{+\infty} \ln^n(x)x^{\alpha-1}\exp(-x)dx. \tag{6.44}$$

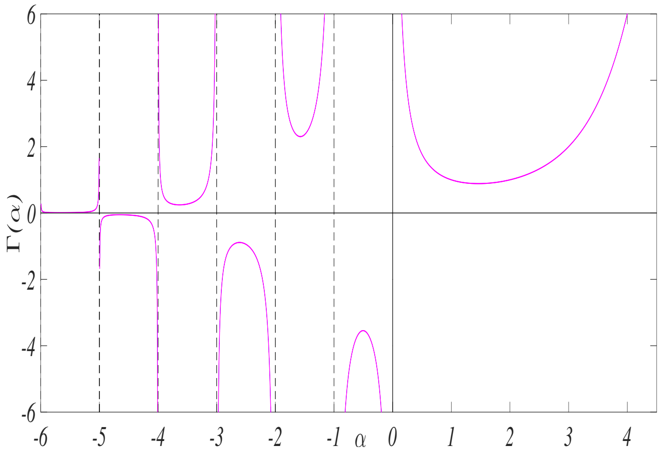

Figure 6.1: Graph of the gamma function.

Proof. Let us set $f(\alpha, x) = x^{\alpha-1}\exp(-x)$, $(\alpha, x) \in (\mathbb{R}_+^*)^2$.
For all $n \in \mathbb{N}$, f is of class C^n on $(\mathbb{R}_+^*)^2$ and furthermore, we have:

$$\frac{\partial^n f}{(\partial\alpha)^n}(\alpha, x) = \ln^n(x)x^{\alpha-1}\exp(-x). \tag{6.45}$$

Consider any compact interval $[a, b]$ of $]0, +\infty[$. It suffices to show that the integral $\int_0^{+\infty} \frac{\partial^n f}{(\partial\alpha)^n}(\alpha, x)dx$ is uniformly convergent on any segment $[a, b]$. Indeed, for all $x \in]0, 1]$, the function $\alpha \mapsto x^{\alpha-1}$ is decreasing, so for all $\alpha \in [a, b]$, we have $0 < f(\alpha, x) \leq x^{a-1}$, and for all $x > 1$, the function $\alpha \mapsto x^{\alpha-1}$ is increasing, so for all $\alpha \in [a, b]$, we have $0 < f(\alpha, x) \leq x^{b-1}\exp(-x)$.
In the neighborhood of 0, we then have

$$\left|\frac{\partial^n f}{(\partial\alpha)^n}(\alpha, x)\right| = \left|\ln^n(x)\right| f(\alpha, x) = o\left(\frac{1}{x^{1-\frac{a}{2}}}\right), \tag{6.46}$$

and in the neighborhood of infinity,

$$\left|\frac{\partial^n f}{(\partial\alpha)^n}(\alpha, x)\right| = \left|\ln^n(x)\right| f(\alpha, x) = o\left(\frac{1}{x^2}\right). \tag{6.47}$$

Since the two integrals $\int_0^1 \frac{dx}{x^{1-\frac{a}{2}}}$ and $\int_1^{+\infty} \frac{dx}{x^2}$ are convergent, the Weierstrass criterion shows that the integral $\int_0^{+\infty} \frac{\partial^n f}{(\partial\alpha)^n}(\alpha, x)dx$ is also uniformly convergent on any segment $[a, b]$, which completes the establishment that Γ is of class C^n on $\mathbb{R}_+^*$ for all $n \in \mathbb{N}$, so that Γ is of class $C^{+\infty}$ on $\mathbb{R}_+^*$. ◻

6.4.2 Euler's Beta function

Definition 6.4.2. *We call Euler's beta function the special function B defined by:*

$$B(\alpha, \beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\,dx, \text{ for } \alpha,\ \beta > 0. \tag{6.48}$$

Remak 6.3. *The Euler function beta defined by the relation (6.48) is well defined on* $(\mathbb{R}_+^*)^2$.
In the neighborhood of zero, $x^{\alpha-1}(1-x)^{\beta-1} \overset{V(0)}{\sim} x^{\alpha-1}$.
Since $\int_0^{\frac{1}{2}} x^{\alpha-1}dx$ *converges if and only if* $\alpha > 0$*, we deduce that* $\int_0^{\frac{1}{2}} x^{\alpha-1}(1-x)^{\beta-1}\,dx$ *converges if and only if* $\alpha > 0$ *and* $\beta \in \mathbb{R}$.
In the neighborhood of 1, $x^{\alpha-1}(1-x)^{\beta-1} \overset{V(0)}{\sim} x^{\beta-1}$.
Since $\int_{\frac{1}{2}}^1 x^{\alpha-1}dx$ *converges if and only if* $\beta > 0$*, we deduce that* $\int_{\frac{1}{2}}^1 x^{\alpha-1}(1-x)^{\beta-1}\,dx$ *converges if and only if* $\beta > 0$ *and* $\alpha \in \mathbb{R}$.
Finally $\int_0^1 x^{\alpha-1}\exp(-x)dx$ *converges if and only if* $\alpha > 0$ *and* $\beta > 0$.

Theorem 6.4.3. *(Symmetry) The function B is symmetric, that is, for all* $\alpha, \beta > 0$*, on a:*

$$B(\alpha, \beta) = B(\beta, \alpha). \tag{6.49}$$

Proof. Indeed, by making the change of variable $x = 1 - t$, we immediately find the result. ◻

Theorem 6.4.4. *(Another formula for the beta function) The beta function can be represented by the following formula:*

$$B(\alpha, \beta) = \int_0^{+\infty} \frac{t^{\alpha-1}}{(1+t)^{\alpha+\beta}}dt. \tag{6.50}$$

Proof. Indeed, by performing the change of variable $x = \frac{t}{1+t}$, we immediately find the result. □

6.4.3 Relationship between gamma and beta functions

Theorem 6.4.5. *1) For all $\alpha, \beta > 0$, we have:*

$$B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}. \tag{6.51}$$

2) The functions B and Γ verify the following Euler reflection formula:

$$B(\alpha, 1-\alpha) = \Gamma(\alpha)\Gamma(1-\alpha) = \frac{\pi}{\sin(\alpha\pi)}, \alpha \in \left]0, 1\right[. \tag{6.52}$$

Proof. By making the change of variable $x = (1+z)y$ $(z > 0)$ in the relation (6.39), we find:

$$\frac{\Gamma(\alpha+\beta)}{(1+z)^{\alpha+\beta}} = \int_0^{+\infty} y^{\alpha+\beta-1} \exp\left(-(1+z)\,y\right) dy, \alpha, \beta, z > 0. \tag{6.53}$$

Multiplying both sides of the equality (6.53) by $z^{\alpha-1}$, then integrating with respect to z from 0 to $+\infty$, we find:

$$\Gamma(\alpha+\beta) \int_0^{+\infty} \frac{z^{\alpha-1}}{(1+z)^{\alpha+\beta}} dz = \int_0^{+\infty} z^{\alpha-1} dz \left(\int_0^{+\infty} y^{\alpha+\beta-1} \exp\left(-(1+z)\,y\right) dy \right), \text{ for all } \alpha, \beta, z > 0, \tag{6.54}$$

or in an equivalent manner:

$$\begin{aligned}
\Gamma(\alpha+\beta) \times B(\alpha, \beta) &= \int_0^{+\infty} z^{\alpha-1} \left(\int_0^{+\infty} y^{\alpha+\beta-1} \exp\left(-(1+z)\,y\right) dy \right) dz \\
&= \int_0^{+\infty} y^{\alpha+\beta-1} \exp(-y) \left(\int_0^{+\infty} z^{\alpha-1} \exp(-zy)\, dz \right) dy \\
&= \int_0^{+\infty} y^{\alpha+\beta-1} \exp(-y) \frac{\Gamma(\alpha)}{y^\alpha} dy = \Gamma(\alpha) \int_0^{+\infty} y^{\beta-1} \exp(-y)\, dy \\
&= \Gamma(\alpha)\Gamma(\beta)
\end{aligned}$$

2) The equality $B(\alpha, 1-\alpha) = \Gamma(\alpha)\Gamma(1-\alpha)$ follows immediately from the relation (6.51) by setting $\beta = 1-\alpha$.
Using the second formula of the beta function, we find:

$$B(\alpha, 1-\alpha) = \int_0^{+\infty} \frac{t^{\alpha-1}}{1+t} dt \tag{6.55}$$

Let us now calculate the integral (6.55) by the theorem of residues. We then define the following path, for $f(z) = \dfrac{z^{\alpha-1}}{1+z}$ and $0 < \epsilon < 1 < R$:
a) C_ϵ the half circle of radius ϵ on the half plane $\Re(z) < 0$.
b) The two segments $S^{\pm}_{\epsilon,R} = \left\{\pm i\epsilon, \ \pm i\epsilon + \sqrt{R^2-\epsilon^2}\right\}$.
c) The arc of a circle $\Gamma_{\epsilon,R} = \left\{R\exp(i\theta),\ \theta \in \left[\arctan\dfrac{\epsilon}{\sqrt{R^2-\epsilon^2}}, 2\pi - \arctan\dfrac{\epsilon}{\sqrt{R^2-\epsilon^2}}\right]\right\}$.
Let us choose ϵ and R such that $z_0 = -1$ is in the loop. Using the residue theorem gives us:

$$\int_{C_\epsilon} f(z)dz + \int_{S^-_{\epsilon,R}} f(z)dz + \int_{\Gamma_{\epsilon,R}} f(z)dz + \int_{S^+_{\epsilon,R}} f(z)dz = 2\pi i \times \text{Res}(f, -1). \tag{6.56}$$

Passing to the limit, when $\epsilon \to 0$ and $R \to +\infty$, it comes by Jordan's lemma that:

$$\lim_{\substack{\epsilon\to 0\\ R\to+\infty}} \int_{C_\epsilon} f(z)dz + \int_{\Gamma_{\epsilon,R}} f(z)dz = 0 + 0 = 0. \tag{6.57}$$

On the other hand, for all $t > 0$, we have:

$$\lim_{\epsilon\to 0}(t+i\epsilon)^{1-\alpha} = t^{1-\alpha} \text{ et } \lim_{\epsilon\to 0}(t-i\epsilon)^{1-\alpha} = t^{1-\alpha}\exp(-2\pi i\alpha). \tag{6.58}$$

So:

$$\lim_{\epsilon\to 0}(t+i\epsilon)^{\alpha-1} = t^{\alpha-1} \text{ et } \lim_{\epsilon\to 0}(t-i\epsilon)^{\alpha-1} = t^{\alpha-1}\exp(2\pi i\alpha). \tag{6.59}$$

From (6.56), (6.57) and (6.59), we can then write:

$$\exp(2\pi i\alpha)\int_{+\infty}^{0} \frac{z^{\alpha-1}}{1+z}dz + \int_0^{+\infty} \frac{z^{\alpha-1}}{1+z}dz = 2\pi i \times Res(f, -1). \tag{6.60}$$

We deduce:

$$\begin{aligned}(1-\exp(2\pi i\alpha))\int_0^{+\infty} \frac{z^{\alpha-1}}{1+z}dz &= 2\pi i \times Res(f,-1) = 2\pi i \times \lim_{z\to -1} z^{\alpha-1} \\ &= 2\pi i \times \lim_{z\to -1} \frac{1}{z^{1-\alpha}} = -\exp(i\pi\alpha),\end{aligned}$$

so, after simplification, we find:

$$\int_0^{+\infty} \frac{z^{\alpha-1}}{1+z}dz = \frac{\pi}{\sin(\alpha\pi)}, \ \alpha > 0. \tag{6.61}$$

□

Example 6.4.1. *Consider the integral* $I = \int_0^{+\infty} \frac{t^{\frac{2}{3}}}{(1+t)^3}dt$

We have:

$$\begin{aligned} I &= \int_0^{+\infty} \frac{t^{\frac{5}{3}-1}}{(1+t)^{\frac{5}{3}+\frac{4}{3}}}dt = B\left(\frac{5}{3}\frac{4}{3}\right) = \frac{\Gamma(\frac{5}{3})\Gamma(\frac{4}{3})}{\Gamma(3)} \\ &= \frac{\Gamma(\frac{2}{3}+1)\Gamma(\frac{1}{3}+1)}{2} = \frac{\frac{2}{3}\times\Gamma(\frac{2}{3})\times\frac{1}{3}\Gamma(\frac{1}{3})}{2} \\ &= \frac{1}{9}\frac{\Gamma(1-\frac{1}{3})\times\Gamma(\frac{1}{3})}{\Gamma(1-\frac{1}{3}+\frac{1}{3})} = B\left(\frac{1}{3}, 1-\frac{1}{3}\right) \\ &= \frac{\pi}{\sin(\frac{\pi}{3})} = \frac{2\pi}{\sqrt{3}}. \end{aligned} \tag{6.62}$$

6.5 Exercises about chapter 6

Exercise 6.5.1. *Returning to the definition, study whether the following limits make sense or not and give their possible value:*

$$1. \lim_{y\to 0}\int_{-1}^{1} \frac{dx}{x^2+y^2+1}, \quad 2.\lim_{y\to 0}\int_0^{\frac{\pi}{2}} \frac{\cos(x(y+1))}{2+\sin(x(y+1))}dx$$

Exercise 6.5.2. *1. Study whether the following integral makes sense or not and give their possible value:*

$$I(\alpha,\beta) = \int_0^{+\infty} \exp(-\alpha x)\ \frac{\sin(\beta x)}{x}dx, \ \alpha \geq 0.$$

2. Passing to the suitably justified limit, find the value of the Dirichlet integral:

$$D(\beta) = \int_0^{+\infty} \frac{\sin(\beta x)}{x}dx.$$

3. *Deduce the values of the following integrals:*

$$I_1 = \int_0^{+\infty} \frac{\sin(\alpha x)\cos(\beta x)}{x}dx, \quad I_2 = \int_0^{+\infty} \frac{\sin(x^2)}{x}dx, \quad I_3 = \int_0^{+\infty} \frac{\sin^3(\alpha x)}{x}dx.$$

Exercise 6.5.3. *Using the special functions, calculate the following integrals:*

$$J_1 = \int_0^{\frac{1}{2}} x^2\sqrt{1-4x^2}dx \ \ \textit{(by posing } x = \frac{y}{2}\textit{).}$$

$$J_2 = \int_1^{+\infty} \frac{\ln(\ln(x))\,dx}{x\sqrt{\ln x}\,(1+\ln x)} \ \ \textit{(by posing } y = \ln x\textit{).}$$

$$J_3 = \int_0^{+\infty} \exp(-3x)\,(\exp(x)-1)^{\frac{3}{2}}\,dx \ \ \textit{(by posing } \exp x = 1+y\textit{).}$$

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