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Department of mathematics

Course Of

Introduction To Dynamics Systems.

Master 1 (first year) fundamental and applied mathematics

The first semester

By: ROUIBAH KHAOULA

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CONTENTS

INTRODUCTION

This course covers the fundamental topics necessary for a thorough understanding of the qualitative theory of ordinary differential equations and the concept of dynamical systems. It is specifically tailored for first-year master's students.

The course begins with an introduction to linear systems of ordinary differential equations, a topic that students who have completed a basic course in differential equations are likely familiar with. Chapter 1 presents an effective method for solving any linear system of ordinary differential equations.

The core focus of the course is on nonlinear systems of ordinary differential equations and dynamical systems. Given that most nonlinear differential equations cannot be solved analytically, the course prioritizes the qualitative or geometric analysis of such systems. This perspective, pioneered by Henri Poincaré in the late 19th century, integrates modern dynamical system concepts, emphasizing the structural and functional properties of solution sets in nonlinear differential equations.

The main objective of this course is to examine the qualitative characteristics of differential equation solutions, including invariant sets and the asymptotic behavior of the flows or dynamical systems defined by these equations

CHAPTER 1

ORDINARY DIFFERENTIAL EQUATIONS

We begin our study of nonlinear systems of differential equations

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \tag{1.1}$$

where $\mathbf{f} : E \to \mathbf{R}^n$ and E is an open subset of $\mathbf{R}^n$. We show that under certain conditions on the function f, the nonlinear system (1.1) has a unique solution through each point $\mathrm{x}_0 \in E$ defined on a maximal interval of existence $(\alpha, \beta) \subset \mathbf{R}$.

We shall only consider autonomous systems of ordinary differential equations (1.1) as opposed to nonautonomous systems

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t) \tag{1.2}$$

where the function f can depend on the independent variable t; however, any nonautonomous system (1.2)with $\mathbf{x} \in \mathbf{R}^n$ can be written as an autonomous system (1.1) with $\mathbf{x} \in \mathbf{R}^{n+1}$ simply by letting $x_{n+1} = t$ and $\dot{x}_{n+1} = 1$. The fundamental theory for (1.1) and (1.2)does not differ significantly although

Before stating and proving the fundamental existence-uniqueness theorem for the nonlinear system (1.1), it is first necessary to define some terminology and notation concerning the derivative $D\mathbf{f}$ of a function $\mathbf{f} : \mathbf{R}^n \to \mathbf{R}^n$.

Definition 1.0.1 *The function* $\mathbf{f}:\mathbf{R}^n \to \mathbf{R}^n$ *is differentiable at* $\mathbf{x}_0 \in \mathbf{R}^n$ *if there is a linear transformation* $D\mathbf{f}(\mathrm{x}_0) \in L(\mathbf{R^n})$ *that satisfies*

$$\lim_{|\mathbf{h}|\to 0} \frac{|\mathbf{f}(\mathbf{x}_0+\mathbf{h}) - \mathbf{f}(\mathbf{x}_0) - D\mathbf{f}(\mathbf{x}_0)\,\mathbf{h}|}{|\mathbf{h}|} = 0$$

The linear transformation $D\mathbf{f}(\mathbf{x}_0)$ *is called the derivative of* $\mathbf{f}$ *at* $\mathbf{x}_0$.

Theorem 1.0.1 *If* $\mathbf{f}:\mathbf{R^n} \to \mathbf{R^n}$ *is differentiable at* $\mathbf{x}_0$*, then the partial derivatives* $\dfrac{\partial f_i}{\partial x_j}, i, j = 1, \ldots, n$*, all exist at* x_0 *and for all* $\mathbf{x} \in \mathbf{R}^n$,

$$D\mathbf{f}(\mathbf{x}_0)\,\mathbf{x} = \sum_{j=1}^{n} \frac{\partial \mathbf{f}}{\partial x_j}(\mathbf{x}_0)\, x_j.$$

Thus, if $\mathbf{f}$ *is a differentiable function, the derivative* $D\mathbf{f}$ *is given by the* $n \times n$ *Jacobian matrix*

$$D\mathbf{f} = \left[\frac{\partial f_i}{\partial x_j}\right]$$

Definition 1.0.2 *Suppose that* V_1 *and* V_2 *are two normed linear spaces with respective norms* $\|\cdot\|_1$ *and* $\|\cdot\|_2$*; i.e.,* V_1 *and* V_2 *are linear spaces with norms* $\|\cdot\|_1$ *and* $\|\cdot\|_2$ *satisfying a-c in Section 1.3 of Chapter 1 . Then*

$$F\colon V_1 \to V_2$$

is continuous at $\mathrm{x}_0 \in V_1$ *if for all* $\varepsilon > 0$ *there exists a* $\delta > 0$ *such that* $\mathbf{x} \in V_1$ *and* $\|\mathbf{x} - \mathbf{x}_0\|_1 < \delta$ *implies that*

$$\|\mathbf{F}(\mathbf{x}) - \mathbf{F}(\mathbf{x}_0)\|_2 < \varepsilon.$$

And $\mathbf{F}$ *is said to be continuous on the set* $E \subset V_1$ *if it is continuous at each point* $\mathbf{x} \in E$*. If* $\mathbf{F}$ *is continuous on* $E \subset V_1$*, we write* $\mathbf{F} \in C(E)$.

Definition 1.0.3 *Suppose that* $\mathbf{f}: E \to \mathbf{R}^n$ *is differentiable on E. Then* $\mathbf{f} \in C^1(E)$ *if the derivative* $D\mathbf{f}: E \to L(\mathbf{R}^n)$ *is continuous on E.*

The next theorem, gives a simple test for deciding whether or not a function $\mathrm{f}: E \to \mathbf{R}^n$ belongs to $C^1(E)$.

Theorem 1.0.2 *Suppose that E is an open subset of* $\mathbf{R^n}$ *and that* $\mathbf{f}: E \to \mathbf{R}^n$*. Then* $\mathrm{f} \in C^1(E)$ *iff the partial derivatives* $\dfrac{\partial f_i}{\partial x_j}, i, j = 1, \ldots, n$*, exist and are continuous on E.*

Remark 1.0.1 *For E an open subset of* $\mathbf{R}^n$*, the higher order derivatives* $D^k\mathbf{f}(\mathbf{x}_0)$ *of a function* $\mathbf{f}: E \to \mathbf{R}^n$ *are* *defined in a similar way and it can be shown that* $\mathrm{f} \in C^k(E)$ *if and only if the partial derivatives*

$$\frac{\partial^k f_i}{\partial x_{j_1} \cdots \partial x_{j_k}}$$

with $i, j_1, \ldots, j_k = 1, \ldots, n$*, exist and are continuous on* E*. Furthermore,* $D^2\mathbf{f}(\mathbf{x}_0) : E \times E \to \mathbf{R}^n$ *and for* $(\mathbf{x}, \mathbf{y}) \in E \times E$ *we have*

$$D^2\mathbf{f}(\mathbf{x}_0)(\mathbf{x}, \mathbf{y}) = \sum_{j_1, j_2 = 1}^{n} \frac{\partial^2 \mathbf{f}(\mathbf{x}_0)}{\partial x_{j_1} \partial x_{j_2}} x_{j_1} y_{j_2}.$$

Similar formulas hold for $D^k\mathbf{f}(\mathbf{x}_0) : (E \times \cdots \times E) \to \mathbf{R}^n$*.*

A function $\mathbf{f} : E \to \mathbf{R}^n$ is said to be analytic in the open set $E \subset \mathbf{R}^n$ if each component $f_j(\mathbf{x})$, $j = 1, \ldots, n$, is analytic in E, i.e., if for $j = 1, \ldots, n$ and $\mathbf{x}_0 \in E$, $f_j(\mathbf{x})$ has a Taylor series which converges to $f_j(\mathbf{x})$ in some neighborhood of $\mathbf{x}_0$ in E.

1.1 The Fundamental Existence-Uniqueness Theorem

In this section, we establish the fundamental existence-uniqueness theorem for a nonlinear autonomous system of ordinary differential equations (1.1) under the hypothesis that $\mathbf{f} \in C^1(E)$ where E is an open subset of $\mathbf{R^n}$. Picard's classical method of successive approximations is used to prove this theorem. The more modern approach based on the contraction mapping principle is relegated to the problems at the end of this section. The method of successive approximations not only allows us to establish the existence and uniqueness of the solution of the initial value problem associated with (1.1), but it also allows us to establish the continuity and differentiability of the solution with respect to initial conditions and parameters. In order to apply the method of successive approximations to establish the existence of a solution of (1.1), we need to define the concept of a Lipschitz condition and show that C^1 functions are locally Lipschitz.

Definition 1.1.1 *Let* E *be an open subset of* $\mathbf{R}^n$*. A function* $\mathrm{f} : E \to \mathbf{R}^n$ *is said to satisfy a Lipschitz condition on* E *if there is a positive constant* K *such that for all* $x, y \in E$

$$|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})| \leq K|\mathbf{x} - \mathbf{y}|.$$

The function $\mathbf{f}$ is said to be locally Lipschitz on E if for each point $\mathbf{x}_0 \in E$ there is an ε-neighborhood of $\mathrm{x}_0, N_\varepsilon(\mathrm{x}_0) \subset E$ and a constant $K_0 > 0$ such that for all $\mathbf{x}, \mathbf{y} \in N_\varepsilon(x_0)$

$$|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})| \leq K_0|\mathbf{x} - \mathbf{y}|.$$

By an ε-neighborhood of a point $\mathbf{x}_0 \in \mathbf{R}^n$, we mean an open ball of positive radius ε; i.e.,

$$N_\varepsilon(x_0) = \{\mathbf{x} \in \mathbf{R}^n \| \mathbf{x} - \mathrm{x}_0 \,| < \varepsilon\}.$$

Lemma 1.1.1 *Let E be an open subset of $\mathbf{R}^\mathbf{n}$ and let $\mathbf{f} : E \to \mathbf{R}^n$. Then, if $\mathbf{f} \in C^1(E)$, $\mathbf{f}$ is locally Lipschitz on E.*

Proof. Since E is an open subset of $\mathbf{R}^n$, given $\mathrm{x}_0 \in E$, there is an $\varepsilon > 0$ such that $N_\varepsilon(\mathrm{x}_0) \subset E$. Let

$$K = \max_{|\mathrm{x}-\mathrm{x}_0| \le \varepsilon/2} \|D\mathbf{f}(\mathrm{x})\|,$$

the maximum of the continuous function $D\mathbf{f}(\mathbf{x})$ on the compact set $|\, x - \mathrm{x}_0 \,| \le \varepsilon/2$. Let N_0 denote the $\varepsilon/2$-neighborhood of $\mathrm{x}_0, N_{\varepsilon/2}(\mathrm{x}_0)$. Then for $\mathbf{x}, \mathbf{y} \in N_0$, set $\mathbf{u} = \mathbf{y} - \mathbf{x}$. It follows that $\mathbf{x} + s\mathbf{u} \in N_0$ for $0 \le s \le 1$ since N_0 is a convex set. Define the function $F : [0, 1] \to \mathbf{R}^n$ by

$$\mathbf{F}(s) = \mathbf{f}(\mathbf{x} + s\mathbf{u}).$$

Then by the chain rule,

$$\mathbf{F}'(s) = D\mathbf{f}(\mathbf{x} + s\mathbf{u})\mathbf{u}$$

and therefore

$$\begin{aligned} \mathbf{f}(\mathbf{y}) - \mathbf{f}(\mathbf{x}) &= \mathbf{F}(1) - \mathbf{F}(0) \\ &= \int_0^1 \mathbf{F}'(s) ds = \int_0^1 D\mathbf{f}(\mathbf{x} + s\mathbf{u})\mathbf{u} ds. \end{aligned}$$

It then follows from the lemma that

$$\begin{aligned} |\mathbf{f}(\mathbf{y}) - \mathbf{f}(\mathbf{x})| &\le \int_0^1 |D\mathbf{f}(\mathbf{x} + s\mathrm{u})\mathrm{u}| ds \\ &\le \int_0^1 \|D\mathbf{f}(\mathbf{x} + s\mathrm{u})\| |\mathbf{u}| ds \\ &\le K|\mathbf{u}| = K|\mathbf{y} - \mathbf{x}|. \end{aligned}$$

And this proves the lemma. ■

Definition 1.1.2 *Let V be a normed linear space. Then a sequence $\{u_k\} \subset V$ is called a Cauchy sequence if for all $\varepsilon > 0$ there is an N such that $k, m \ge N$ implies that*

$$\|u_k - \mathbf{u}_m\| < \varepsilon.$$

The space V is called complete if every Cauchy sequence in V converges to an element in V.

The following theorem, establishes the completeness of the normed linear space $C(I)$ with $I = [-a, a]$.

Theorem 1.1.1 (The Fundamental Existence-Uniqueness Theorem) *Let E be an open subset of $\mathbf{R}^n$ containing $\mathbf{x}_0$ and assume that $\mathbf{f} \in C^1(E)$. Then there exists an $a > 0$ such that the initial value problem*

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}) \\ \mathbf{x}(0) &= \mathbf{x}_0 \end{aligned} \tag{1.3}$$

has a unique solution $\mathbf{x}(t)$ on the interval $[-a, a]$.

Proof. Since $\mathrm{f} \in C^1(E)$, it follows from the lemma that there is an ε neighborhood $N_\varepsilon(\mathrm{x}_0) \subset E$ and a constant $K > 0$ such that for all $\mathbf{x}, \mathbf{y} \in N_\varepsilon(\mathbf{x}_0)$,

$$|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})| \leq K|\mathbf{x} - \mathbf{y}|.$$

Let $b = \varepsilon/2$. Then the continuous function $\mathbf{f}(\mathbf{x})$ is bounded on the compact set

$$N_0 = \{\mathbf{x} \in \mathbf{R}^n \| \mathbf{x} - \mathbf{x}_0 | \leq b\}.$$

Let

$$M = \max_{\mathbf{x} \in N_0} |\mathbf{f}(\mathbf{x})|.$$

Let the successive approximations $u_k(t)$ be defined by

$$\begin{aligned} \mathbf{u}_0(t) &= \mathbf{x}_0 = x(0) \\ \mathbf{u}_{k+1}(t) &= \mathbf{x}_0 + \int_0^t \mathbf{f}(\mathbf{u}_k(s))\, ds \quad k = 0, 1, \ldots \end{aligned} \tag{1.4}$$

. Then assuming that there exists an $a > 0$ such that $u_k(t)$ is defined and continuous on $[-a, a]$ and satisfies

$$\max_{[-a,a]} |u_k(t) - x_0| \leq b, \tag{1.5}$$

it follows that $\mathrm{f}(\mathrm{u}_k(t))$ is defined and continuous on $[-a, a]$ and therefore that

$$\mathbf{u}_{k+1}(t) = \mathrm{x}_0 + \int_0^t \mathrm{f}(\mathbf{u}_k(s))\, ds$$

is defined and continuous on $[-a, a]$ and satisfies

$$|\mathrm{u}_{k+1}(t) - \mathrm{x}_0| \leq \int_0^t |\mathbf{f}(\mathrm{u}_k(s))|\, ds \leq Ma$$

for all $t \in [-a, a]$. Thus, choosing $0 < a \leq b/M$, it follows by induction that $u_k(t)$ is defined and continuous and satisfies (1.5) for all $t \in [-a, a]$ and $k = 1, 2, 3, \ldots$.

Next, since for all $t \in [-a, a]$ and $k = 0, 1, 2, 3, \ldots, \mathbf{u}_k(t) \in N_0$, it follows from the Lipschitz condition satisfied by f that for all $t \in [-a, a]$

$$\begin{aligned}
|\mathbf{u}_2(t) - \mathbf{u}_1(t)| &\leq \int_0^t |\mathbf{f}(\mathbf{u}_1(s)) - \mathbf{f}(\mathbf{u}_0(s))| \, ds \\
&\leq K \int_0^t |\mathbf{u}_1(s) - \mathbf{u}_0(s)| \, ds \\
&\leq Ka \max_{[-a,a]} |\mathbf{u}_1(t) - \mathbf{x}_0| \\
&\leq Kab.
\end{aligned}$$

And then assuming that

$$\max_{[-a,a]} \left|\mathbf{u}_j(t) - \mathbf{u}_{j-1}(t)\right| \leq (Ka)^{j-1} b \tag{1.6}$$

for some integer $j \geq 2$, it follows that for all $t \in [-a, a]$

$$\begin{aligned}
\left|\mathbf{u}_{j+1}(t) - \mathbf{u}_j(t)\right| &\leq \int_0^t \left|\mathbf{f}\left(\mathbf{u}_j(s)\right) - \mathbf{f}\left(\mathbf{u}_{j-1}(s)\right)\right| ds \\
&\leq K \int_0^t \left|\mathbf{u}_j(s) - \mathbf{u}_{j-1}(s)\right| ds \\
&\leq Ka \max_{[-a,a]} \left|\mathbf{u}_j(t) - \mathbf{u}_{j-1}(t)\right| \\
&\leq (Ka)^j b.
\end{aligned}$$

Thus, it follows by induction that (1.6) holds for $j = 2, 3, \ldots$. Setting $\alpha =$ and choosing $0 < a < 1/K$, we see that for $m > k \geq N$ and $t \in [-a, a$

$$\begin{aligned}
|\mathbf{u}_m(t) - \mathbf{u}_k(t)| &\leq \sum_{j=k}^{m-1} \left|\mathbf{u}_{j+1}(t) - \mathbf{u}_j(t)\right| \\
&\leq \sum_{j=N}^{\infty} \left|\mathbf{u}_{j+1}(t) - \mathbf{u}_j(t)\right| \\
&\leq \sum_{j=N}^{\infty} \alpha^j b = \frac{\alpha^N}{1 - \alpha} b.
\end{aligned}$$

This last quantity approaches zero as $N \to \infty$. Therefore, for all ε there exists an N such that $m, k \geq N$ implies that

$$\|\mathbf{u}_m - \mathbf{u}_k\| = \max_{|-a,a|} |\mathbf{u}_m(t) - \mathbf{u}_k(t)| < \varepsilon;$$

i.e., $\{u_k\}$ is a Cauchy sequence of continuous functions in $C([-a,a])$. I lows from the above theorem that $u_k(t)$ converges to a continuous funs $\mathbf{u}(t)$ uniformly for all $t \in [-a,a]$ as $k \to \infty$. And then taking the lim both sides of equation (1.4)defining the successive approximations, w that the continuous function

$$\mathbf{u}(t) = \lim_{k\to\infty} \mathbf{u}_k(t) \tag{1.7}$$

satisfies the integral equation

$$\mathbf{u}(t) = \mathbf{x}_0 + \int_0^t \mathbf{f}(\mathbf{u}(s))ds \tag{1.8}$$

for all $t \in [-a,a]$. We have used the fact that the integral and the limit can be interchanged since the limit in (1.7) is uniform for all $t \in [-a,a]$. Then since $\mathbf{u}(t)$ is continuous, $\mathbf{f}(\mathbf{u}(t))$ is continuous and by the fundamental theorem of calculus, the right-hand side of the integral equation (1.8) is differentiable and

$$\mathbf{u}'(t) = \mathbf{f}(\mathbf{u}(t))$$

for all $t \in [-a,a]$. Furthermore, $\mathbf{u}(0) = \mathbf{x}_0$ and from (1.5) it follows that $\mathbf{u}(t) \in N_c(\mathbf{x}_0) \subset E$ for all $t \in [-a,a]$. Thus $\mathbf{u}(t)$ is a solution of the initial value problem (1.3) on $[-a,a]$. It remains to show that it is the only solution.

Let $\mathbf{u}(t)$ and $\mathbf{v}(t)$ be two solutions of the initial value problem (1.3) on $[-a,a]$. Then the continuous function $|\mathbf{u}(t) - \mathbf{v}(t)|$ achieves its maximum at some point $t_1 \in [-a,a]$. It follows that

$$\begin{aligned}
||\mathbf{u} - \mathbf{v}|| &= \max_{[-a,a]} |\mathbf{u}(t) - \mathbf{v}(t)| \\
&= \left| \int_0^{t_1} \mathbf{f}(\mathbf{u}(s)) - \mathbf{f}(\mathbf{v}(s))ds \right| \\
&\leq \int_0^{|t_1|} |\mathbf{f}(\mathbf{u}(s)) - \mathbf{f}(\mathbf{v}(s))|ds \\
&\leq K \int_0^{|t_1|} |\mathbf{u}(s) - \mathbf{v}(s)|ds \\
&\leq Ka \max |\mathbf{u}(t) - \mathbf{v}(t)| \\
&\leq Ka||\mathbf{u} - \mathbf{v}||
\end{aligned}$$

But $Ka < 1$ and this last inequality can only be satisfied if $||\mathbf{u} - \mathbf{v}|| = 0$. Thus, $\mathbf{u}(t) = \mathbf{v}(t)$ on $[-a,a]$. We have shown that the successive approximations (1.4) converge uniformly to a unique solution of the initial value problem (1.3) on the interval $[-a,a]$ where a is any number satisfying $0 < a < \min\left(\frac{b}{M}, \frac{1}{K}\right)$

■

1.2 Dependence on Initial Conditions and Parameters

In this section we investigate the dependence of the solution of the initial value

$$\begin{aligned} \dot{\mathbf{x}} &= f(\mathbf{x}, \mu) \\ \mathbf{x}(0) &= \mathbf{y} \end{aligned} \tag{1.9}$$

depends on the initial conditions $\mathbf{y}$ and a parameters $\mu \in R^m$

Lemma 1.2.1 (Gronwall) *Suppose that $g(t)$ is a continuous real valued function that satisfies $g(t) \geq 0$ and*

$$g(t) \leq C + K \int_0^t g(s)ds$$

for all $t \in [0, a]$ where C and K are positive constants. It then follows that for all $t \in [0, a]$,

$$g(t) \leq Ce^{Kt}$$

Proof. Let $G(t) = C + K \int_0^t g(s)ds$ for $t \in [0, a]$. Then $G(t) \geq g(t)$ and $G(t) > 0$ for all $t \in [0, a]$. It follows from the fundamental theorem of calculus that

$$G'(t) = Kg(t)$$

and therefore that

$$\frac{G'(t)}{G(t)} = \frac{Kg(t)}{G(t)} \leq \frac{KG(t)}{G(t)} = K$$

for all $t \in [0, a]$. And this is equivalent to saying that

$$\frac{d}{dt}(\log G(t)) \leq K$$

or

$$\log G(t) \leq Kt + \log G(0)$$

or

$$G(t) \leq G(0)e^{Kt} = Ce^{Kt}$$

for all $t \in [0, a]$, which implies that $g(t) \leq Ce^{Kt}$ for all $t \in [0, a]$. ■

Theorem 1.2.1 (Dependence on Initial Conditions) *Let E be an open subset of $\mathbf{R}^n$ containing $\mathbf{x}_0$ and assume that $\mathbf{f} \in C^1(E)$. Then there exists an $a > 0$ and a$\delta > 0$ such that for all $\mathbf{y} \in N_\delta(\mathbf{x}_0)$ the initial value problem*

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}) \\ \mathbf{x}(0) &= \mathbf{y} \end{aligned} \tag{1.10}$$

has a unique solution $\mathbf{u}(t, \mathbf{y})$ with $\mathbf{u} \in C^1(G)$ where $G = [-a, a] \times N_6(\mathbf{x}_0) \subset \mathbf{R}^{n+1}$; furthermore, for each $\mathbf{y} \in N_\delta(\mathbf{x}_0)$, $\mathbf{u}(t, \mathbf{y})$ is a twice continuously differentiable function of t for $t \in [-a, a]$.

Proof. The uniqueness of the solution $\mathbf{u}(t, y)$ follows from the fundamental theorem in Section 1.1.
For all $(t, y) \in G$; i.e., $\mathbf{u}(t, y)$ is a twice continuously differentiable function of t which satisfies the initial value problem (1.10) for all $(t, y) \in G$. It follows that

$$\dot{\mathbf{u}}(t, \mathbf{y}) = \mathbf{f}(\mathbf{u}(t, \mathbf{y}))$$

and that

$$\ddot{\mathbf{u}}(t, \mathbf{y}) = D\mathbf{f}(\mathbf{u}(t, \mathbf{y}))\dot{\mathbf{u}}(t, \mathbf{y})$$

We now show that $\mathbf{u}(t, y)$ is a continuously differentiable function of $\mathbf{y}$ for all $(t, y) \in [-a, a] \times N_{\delta/2}(x_0)$. In order to do this, fix $\mathbf{y}_0 \in N_{\delta/2}(\mathbf{x}_0)$ and choose $\mathbf{h} \in \mathbf{R^n}$ such that $|\mathbf{h}| < \delta/2$. Then $\mathbf{y}_0 + \mathbf{h} \in N_\delta(\mathbf{x}_0)$.
Let $\mathbf{u}(t, \mathbf{y}_0)$ and $\mathbf{u}(t, \mathbf{y}_0 + \mathbf{h})$ be the solutions of the initial value problem (1.10) with $\mathbf{y} = \mathbf{y}_0$ and with $y = y_0 + \mathbf{h}$ respectively. It then follows that

$$\begin{aligned} \left|\mathbf{u}(t, \mathbf{y}_0 + \mathbf{h}) - \mathbf{u}(t, \mathbf{y}_0)\right| &\le |\mathbf{h}| + \int_0^t \left|\mathbf{f}(\mathbf{u}(s, \mathbf{y}_0 + \mathbf{h})) - \mathbf{f}(\mathbf{u}(s, \mathbf{y}_0))\right| ds \\ &\le |\mathbf{h}| + K \int_0^t \left|\mathbf{u}(s, \mathbf{y}_0 + \mathbf{h}) - \mathbf{u}(s, \mathbf{y}_0)\right| ds \end{aligned}$$

for all $t \in [-a, a]$. Thus, it follows from Gronwall's Lemma that

$$\left|\mathbf{u}(t, \mathbf{y}_0 + \mathbf{h}) - \mathbf{u}(t, \mathbf{y}_0)\right| \le |\mathbf{h}| e^{K|t|} \tag{1.11}$$

for all $t \in [-a, a]$. We next define $\Phi(t, y_0)$ to be the fundamental matrix solution of the initial value problem

$$\begin{aligned} \dot{\Phi} &= A(t, \mathbf{y}_0)\Phi \\ \Phi(0, \mathbf{y}_0) &= I \end{aligned} \tag{1.12}$$

with $A(t, y_0) = D\mathbf{f}(\mathbf{u}(t, \mathbf{y}_0))$ and I the $n \times n$ identity matrix. The existence and continuity of $\Phi(t, y_0)$ on some interval $[-a, a]$ follow from the method of successive approximations as in problem (1.11).

It then follows from the initial value problems for $\mathbf{u}(t, y_0)$, $\mathbf{u}(t, \mathbf{y}_0 + \mathbf{h})$ and $\Phi(t, \mathbf{y}_0)$ and Taylor's Theorem,

$$\mathbf{f}(\mathbf{u}) - \mathbf{f}(\mathbf{u}_0) = D\mathbf{f}(\mathbf{u}_0)(\mathbf{u} - \mathbf{u}_0) + \mathbf{R}(\mathbf{u}, \mathbf{u}_0)$$

where $|\mathbf{R}(\mathbf{u}, \mathbf{u}_0)| / |\mathbf{u} - \mathbf{u}_0| \to 0$ as $|\mathbf{u} - \mathbf{u}_0| \to 0$, that

$$\begin{aligned}\left|\mathbf{u}(t, \mathbf{y}_0) - \mathbf{u}(t, \mathbf{y}_0 + \mathbf{h}) + \Phi(t, \mathbf{y}_0)\mathbf{h}\right| &\leq \int_0^t \mathbf{f}(\mathbf{u}(s, \mathbf{y}_0)) - \mathbf{f}(\mathbf{u}(s, \mathbf{y}_0 + \mathbf{h})) + D\mathbf{f}(\mathbf{u}(s, \mathbf{y}_0))\Phi(s, \mathbf{y}_0)\mathbf{h} \mid ds \\ &\leq \int_0^t \left\|D\mathbf{f}(\mathbf{u}(s, \mathbf{y}_0))\right\| \left|\mathbf{u}(s, \mathbf{y}_0) - \mathbf{u}(s, \mathbf{y}_0 + \mathbf{h}) + \Phi(s, \mathbf{y}_0)\mathbf{h}\right| ds \\ &+ \int_0^t \left|\mathbf{R}(\mathbf{u}(s, \mathbf{y}_0 + \mathbf{h}), \mathbf{u}(s, \mathbf{y}_0))\right| ds\end{aligned} \tag{1.13}$$

Since $|\mathbf{R}(u, u_0)| / |u - u_0| \to 0$ as $|\mathbf{u} - \mathbf{u}_0| \to 0$ and since $\mathbf{u}(s, \mathbf{y})$ is continuous on G, it follows that given any $\varepsilon_0 > 0$, there exists a $\delta_0 > 0$ such that if $|\mathbf{h}| < \delta_0$ then $\left|\mathbf{R}(\mathbf{u}(s, \mathbf{y}_0), \mathbf{u}(s, \mathbf{y}_0 + \mathbf{h}))\right| < \varepsilon_0 \left|\mathbf{u}(s, \mathbf{y}_0) - \mathbf{u}(s, \mathbf{y}_0 + \mathbf{h})\right|$ for all $s \in [-a, a]$. Thus, if we let

$$g(t) = \left|\mathbf{u}(t, \mathbf{y}_0) - \mathbf{u}(t, \mathbf{y}_0 + \mathbf{h}) + \Phi(t, \mathbf{y}_0)\mathbf{h}\right|$$

it then follows from (1.11) and (1.13) that for all $t \in [-a, a]$, $y_0 \in N_{\delta/2}(x_0)$ and $|\mathbf{h}| < \min(\delta_0, \delta/2)$ we have

$$g(t) \leq M_1 \int_0^t g(s)ds + \varepsilon_0|\mathbf{h}|ae^{Ka}.$$

Hence, it follows from Gronwall's Lemma that for any given $\varepsilon_0 > 0$

$$g(t) \leq \varepsilon_0|\mathbf{h}|a\mathrm{e}^{Ka}e^{M_1\mathrm{a}}$$

for all $t \in [-a, a]$ provided $|\mathbf{h}| < \min(\delta_0, \delta/2)$. Thus,

$$\lim_{|\mathbf{h}| \to 0} \frac{\left|\mathbf{u}(t, \mathbf{y}_0) - \mathbf{u}(t, \mathbf{y}_0 + \mathbf{h}) + \Phi(t, \mathbf{y}_0)\mathbf{h}\right|}{|\mathbf{h}|} = 0$$

uniformly for all $t \in [-a, a]$. Therefore, according to Definition 1 in Section 2.1,

$$\frac{\partial \mathbf{u}}{\partial \mathbf{y}}(t, \mathbf{y}_0) = \Phi(t, \mathbf{y}_0)$$

for all $t \in [-a, a]$ where $\Phi(t, y_0)$ is the fundamental matrix solution of the initial value problem (5) which is continuous in t and in y_0 for all $t \in [-a, a]$ and $y_0 \in N_{\delta/2}(x_0)$. This completes the proof of the theorem.

■

Corollary 1.2.1 . *Under the hypothesis of the above theorem,*

$$\Phi(t, \mathbf{y}) = \frac{\partial \mathbf{u}}{\partial \mathbf{y}}(t, \mathbf{y})$$

for $t \in [-a, a]$ *and* $\mathbf{y} \in N_\delta(\mathrm{x}_0)$ *if and only if* $\Phi(t, y)$ *is the fundamental matrix solution of*

$$\Phi = D\mathbf{f}[\mathbf{u}(t, \mathbf{y})]\Phi$$
$$\Phi(0, \mathbf{y}) = I$$

for $t \in [-a, a]$ *and* $\mathrm{y} \in N_\delta(\mathrm{x}_0)$.

Remark 1.2.1 *A similar proof shows that if* $f \in C^r(E)$ *then the solution* $\mathbf{u}(t, \mathbf{y})$ *of the initial value problem* (1.9) *is in* $C^r(G)$ *where* G *is defined as in the above theorem. And if* $\mathbf{f}(\mathbf{x})$ *is a (real) analytic function for* $\mathbf{x} \in E$ *then* $\mathbf{u}(t, \mathbf{y})$ *is analytic in the interior of* G.

Remark 1.2.2 *If* x_0 *is an equilibrium point of* (1.9), *i.e., if* $f(x_0) = 0$ *so that* $\mathbf{u}(t, \mathbf{x}_0) = \mathbf{x}_0$ *for all* $t \in \mathbf{R}$, *then*

$$\Phi(t, \mathbf{x}_0) = \frac{\partial \mathrm{u}}{\partial \mathbf{x}_0}(t, \mathbf{x}_0)$$

satisfies

$$\dot{\Phi} = D\mathbf{f}(\mathbf{x}_0)\Phi$$
$$\Phi(0, \mathbf{x}_0) = I.$$

And according to the Fundamental Theorem for Linear Systems

$$\Phi(t, \mathrm{x}_0) = e^{D\mathbf{f}(\mathrm{x}_0)t}.$$

Theorem 1.2.2 (Dependence on Parameters) *Let* E *be an open subset of* $\mathbf{R}^{n+m}$ *containing the point* $(\mathbf{x}_0, \mu_0)$ *where* $\mathbf{x}_0 \in \mathbf{R}^n$ *and* $\mu_0 \in \mathbf{R}^m$ *and assume that* $\mathbf{f} \in C^1(E)$. *It then follows that there exists an* $a > 0$ *and a* $\delta > 0$ *such that for all* $\mathbf{y} \in N_\delta(\mathbf{x}_0)$ *and* $\mu \in N_\delta(\mu_0)$, *the initial value problem*

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}, \mu) \\ \mathbf{x}(0) &= \mathbf{y} \end{aligned} \tag{1.14}$$

has a unique solution $\mathbf{u}(t, \mathbf{y}, \boldsymbol{\mu})$ *with* $\mathbf{u} \in C^1(G)$ *where* $G = [-a, a] \times N_\delta(\mathbf{x}_0) \times N_\delta(\mu_0)$.

This theorem follows immediately from the previous theorem by replacing the vectors $\mathbf{x}_0, \mathbf{x}, \dot{\mathbf{x}}$ and $\mathbf{y}$ by the vectors $(\mathbf{x}_0, \mu_0), (\mathbf{x}, \mu), (\dot{\mathbf{x}}, \mathbf{0})$ and $(\mathbf{y}, \boldsymbol{\mu})$ or it can be proved directly using Gronwall's Lemma and the method of successive approximations.

1.3 The Fundamental Theorem for Linear Systems

Let A be an $n \times n$ matrix. In this section we establish the fundamental fact that for $x_0 \in \mathbb{R}^n$ the initial value problem

$$\begin{aligned}\dot{\mathbf{x}} &= A\mathbf{x} \\ \mathbf{x}(0) &= \mathbf{x}_0.\end{aligned} \tag{1.15}$$

has a unique solution for all $t \in R^n$ which is given by

$$x(t) = \exp(At)x_0. \tag{1.16}$$

Lemma 1.3.1 *Let A be a square matrix, then*

$$\frac{d}{dt}e^{At} = Ae^{At}.$$

Proof.

$$\begin{aligned}\frac{d}{dt}e^{At} &= \lim_{h\to 0}\frac{e^{A(t+h)} - e^{At}}{h} \\ &= \lim_{h\to 0} e^{At}\frac{\left(e^{Ah} - I\right)}{h} \\ &= e^{At}\lim_{h\to 0}\lim_{k\to\infty}\left(A + \frac{A^2h}{2!} + \cdots + \frac{A^kh^{k-1}}{k!}\right) \\ &= Ae^{At}.\end{aligned}$$

■

Theorem 1.3.1 (The Fundamental Theorem for Linear Systems) *Let A be an $n \times n$ matrix. Then for a given $\mathbf{x}_0 \in \mathbf{R}^n$, the initial value problem*

$$\begin{aligned}\dot{\mathbf{x}} &= A\mathbf{x} \\ \mathbf{x}(0) &= \mathbf{x}_0\end{aligned} \tag{1.17}$$

has a unique solution given by

$$\mathbf{x}(t) = e^{At}\mathbf{x}_0. \tag{2}$$

Proof. By the preceding lemma, if $\mathbf{x}(t) = e^{At}\mathbf{x}_0$, then

$$\mathbf{x}'(t) = \frac{d}{dt}e^{At}\mathbf{x}_0 = Ae^{At}\mathbf{x}_0 = A\mathbf{x}(t)$$

for all $t \in \mathbf{R}$. Also, $\mathbf{x}(0) = I\mathbf{x}_0 = \mathbf{x}_0$. Thus $\mathbf{x}(t) = e^{At}\mathbf{x}_0$ is a solution.

To see that this is the only solution, let $\mathbf{x}(t)$ be any solution of the initial value problem (1.17) and set

$$\mathbf{y}(t) = e^{-At}\mathbf{x}(t).$$

Then from the above lemma and the fact that $\mathbf{x}(t)$ is a solution of (1.17)

$$\begin{aligned}\mathbf{y}'(t) &= -Ae^{-At}\mathbf{x}(t) + e^{-At}\mathbf{x}'(t)\\ &= -Ae^{-At}\mathbf{x}(t) + e^{-At}A\mathbf{x}(t)\\ &= 0\end{aligned}$$

for all $t \in \mathbf{R}$ since e^{-At} and A commute. Thus, $\mathbf{y}(t)$ is a constant.

Setting $t = 0$ shows that $y(t) = x_0$ and therefore any solution of the initial value problem (1.17) is given by $\mathbf{x}(t) = e^{At}\mathbf{y}(t) = e^{At}\mathbf{x}_0$. This completes the proof of the theorem. ■

Example 1.3.1 *Solve the initial value problem*

$$\begin{aligned}\dot{\mathbf{x}} &= A\mathbf{x}\\ \mathbf{x}(0) &= \begin{bmatrix} 1 \\ 0 \end{bmatrix}\end{aligned}$$

for

$$A = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$$

and sketch the solution curve in the phase plane $\mathbf{R}^2$.

By the above theorem and Corollary 3 of the last section, the solution is given by

$$\mathbf{x}(t) = e^{At}\mathbf{x}_0 = e^{-2t}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = e^{-2t}\begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

1.3.1 Nonhomogeneous Linear Systems

In this section we solve the nonhomogeneous linear system

$$\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{b}(t) \tag{1.18}$$

where A is an $n \times n$ matrix and $\mathbf{b}(t)$ is a continuous vector valued function.

Definition 1.3.1 *A fundamental matrix solution of*

$$\dot{\mathbf{x}} = A\mathbf{x} \tag{1.19}$$

is any nonsingular $n \times n$ matrix function $\Phi(t)$ that seatisfies

$$\Phi'(t) = A\Phi(t) \text{ for all } t \in \mathbf{R}.$$

$\Phi(t) = e^{At}$ is a fundamental matrix solution which satisfies $\Phi(0) = I$, the $n \times n$ identity matrix.

Once we have found a fundamental matrix solution to equation (1.19), solving the nonhomogeneous system (1.18) becomes straightforward. The result is provided in the following theorem.

Theorem 1.3.2 *If $\Phi(t)$ is a fundamental matrix solution of equation (1.19), then the solution to the nonhomogeneous linear system (1.18) with the initial condition $\mathbf{x}(0) = \mathbf{x}_0$ is unique and is given by:*

$$\mathbf{x}(t) = \Phi(t)\Phi^{-1}(0)\mathbf{x}_0 + \int_0^t \Phi(t)\Phi^{-1}(\tau)\mathbf{b}(\tau)d\tau. \tag{1.20}$$

Proof. For the function $\mathbf{x}(t)$ described above, defined above,

$$\begin{aligned}\mathbf{x}'(t) =& \Phi'(t)\Phi^{-1}(0)\mathbf{x}_0 + \Phi(t)\Phi^{-1}(t)\mathbf{b}(t)\\ &+ \int_0^t \Phi'(t)\Phi^{-1}(\tau)\mathbf{b}(\tau)d\tau\end{aligned}$$

And since $\Phi(t)$ is a fundamental matrix solution of (1.19), it follows that

$$\begin{aligned}\mathbf{x}'(t) &= A\left[\Phi(t)\Phi^{-1}(0)\mathbf{x}_0 + \int_0^t \Phi(t)\Phi^{-1}(\tau)\mathbf{b}(\tau)d\tau\right] + \mathbf{b}(t)\\ &= A\mathbf{x}(t) + \mathbf{b}(t)\end{aligned}$$

for all $t \in \mathbf{R}$. And this completes the proof of the theorem. ■

Remark 1.3.1 *With $\Phi(t) = e^{At}$, the solution of the nonhomogeneous linear system (1.18), as given in the above theorem, has the form*

$$\mathbf{x}(t) = e^{At}\mathbf{x}_0 + e^{At}\int_0^t e^{-A\tau}\mathbf{b}(\tau)d\tau.$$

Example 1.3.2 *Find the solution of the nonhomogeneous system $\dot{x} = x + y + t, \dot{y} = -y + 1$ with the initial conditions $x(0) = 1, y(0) = 0$.*

Solution

In matrix notation, the system takes the form $\underset{\sim}{\dot{x}}(t) = \underset{\sim}{x}(t) + \underset{\sim}{b}(t)$, where $A = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$ and $\underset{\sim}{b}(t) = \binom{t}{1}$.

The initial conditions become $\underset{\sim}{x}(0) = \underset{\sim}{x}0$, *where* $\underset{\sim}{x}0 = \binom{1}{0}$. *Matrix A has eigenvalues* $\lambda_1 = 1, \lambda_2 = -1$ *with corresponding eigenvectors* $\binom{1}{0}$ *and* $\binom{1}{-2}$. *Therefore*

$$\Phi(t) = \begin{pmatrix} e^t & e^{-t} \\ 0 & -2e^{-t} \end{pmatrix}$$

This gives

$$\Phi^{-1}(t) = \frac{1}{2}\begin{pmatrix} 2e^{-t} & e^{-t} \\ 0 & -e^t \end{pmatrix}, \Phi(0) = \begin{pmatrix} 1 & 1 \\ 0 & -2 \end{pmatrix} \text{ and } \Phi^{-1}(0) = \frac{1}{2}\begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix}.$$

Therefore the required solution is

$$\begin{aligned}
\underset{\sim}{x}(t) &= \Phi(t)\Phi^{-1}(0)\underset{\sim}{x}0 + \Phi(t)\int_0^t \Phi^{-1}(\alpha)\underset{\sim}{b}(\alpha)\mathrm{d}\alpha \\
&= \frac{1}{2}\Phi(t)\left\{\begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \int_0^t \begin{pmatrix} 2e^{-\alpha} & e^{-\alpha} \\ 0 & -e^{x} \end{pmatrix}\begin{pmatrix} \alpha \\ 1 \end{pmatrix}\mathrm{d}\alpha\right\} \\
&= \frac{1}{2}\Phi(t)\left\{\begin{pmatrix} 2 \\ 0 \end{pmatrix} + \begin{pmatrix} 3-(2t+3)e^{-t} \\ 1-e^t \end{pmatrix}\right\} \\
&= \frac{1}{2}\begin{pmatrix} e^t & e^{-t} \\ 0 & -2e^{-t} \end{pmatrix}\begin{pmatrix} 5-(2t+3)e^{-t} \\ 1-e^t \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 5e^t - 2t - 4 + e^{-t} \\ 2 - 2e^{-2t} \end{pmatrix}
\end{aligned}$$

CHAPTER 2

CONCEPTS OF DYNAMICAL SYSTEMS

The second section aims to introduce the basic mathematical tools for analysing ordinary differential equations (ODEs) from a dynamical systems perspective. Unlike traditional approaches that seek exact analytical solutions, which are often unattainable for complex systems, this framework emphasises qualitative methods to extract meaningful insights. Emphasis is placed on long-term behaviour (e.g. attractors, stability) and local dynamics near critical points such as equilibria.

A dynamical system is defined as a set of n first-order ODEs governing time evolution in $\mathbb{R}^n$. Systems are classified according to properties such as determinism (future/past uniquely determined by present state), dimensionality (finite/infinite), and time dependence (continuous/discrete). Continuous time systems are described by $\dot{x} = f(x, t)$, while discrete time systems follow $x_{n+1} = g(x_n)$.

A critical distinction is autonomy: systems without explicit time dependence ($\dot{x} = f(x)$) exhibit time-invariant trajectories, while non-autonomous systems ($\dot{x} = f(x, t)$) can be converted to autonomous form by introducing time as an additional variable. Examples include linear oscillators (e.g. damped harmonic $\ddot{x} + \alpha\dot{x} + \beta x = 0$), nonlinear models (e.g. pendulum $\ddot{x} + \omega^2 \sin x = 0$) and ecological systems (Lotka-Volterra model). Non-autonomous cases include forced oscillators such as the Duffing equation ($\ddot{x} + \alpha\dot{x} + \omega_0^2 x + \beta x^3 = f \sin \omega t$), where external periodic forcing enriches the dynamics.

This approach prioritises the understanding of structural behaviour such as stability and bifurcation over exact solutions, enabling the analysis of inherently non-linear phenomena in physics, biology and engineering.

2.1 Flows and Evolution

The time-evolutionary process can be understood as the flow of a vector field. In general, the term **flow** is used to describe the overall dynamics of a system rather than its evolution at a specific point. For a system represented by

$$\dot{x} = f(x),$$

the solution $x(t)$, which satisfies $x(t_0) = x_0$, provides both the past ($t < t_0$) and future ($t > t_0$) behavior of the system.

Mathematically, the flow is defined as $\varphi_t(x) : U \to \mathbb{R}^n$, where $\varphi_t(x) = \varphi(t, x)$ is a smooth vector function of $x \in U \subseteq \mathbb{R}^n$ and $t \in I \subseteq \mathbb{R}$. The flow satisfies the differential equation:

$$\frac{d}{dt}\varphi_t(x) = f(\varphi_t(x)),$$

for all t where the solution exists, with the initial condition $\varphi(0, x) = x$. The flow $\varphi_t(x)$ satisfies the following properties:

- $\varphi_0 = \text{Id}$ (identity mapping),
- $\varphi_{t+s} = \varphi_t \circ \varphi_s$ (composition property).

In some cases, the flow also satisfies:

$$\varphi(t + s, x) = \varphi(t, \varphi(s, x)) = \varphi(s, \varphi(t, x)) = \varphi(s + t, x).$$

Flows in $\mathbb{R}$

Consider a one-dimensional autonomous system described by

$$\dot{x} = f(x),$$

where $x \in \mathbb{R}$. Imagine a hypothetical fluid flowing along the real line, with its local velocity determined by $f(x)$. This hypothetical fluid is referred to as the **phase fluid**, and the real line is called the **phase line**.

To solve the system $\dot{x} = f(x)$ starting from an initial position x_0, we can imagine placing a hypothetical particle, called a **phase point**, at x_0 and observing how it moves along the phase line over time t. As time progresses, the phase point (x, t) in the one-dimensional system $\dot{x} = f(x)$ with $x(0) = x_0$ moves along the x-axis according to a function $\varphi(t, x_0)$. This function is called the **trajectory** for the given initial state x_0.

The set

$$\{\varphi(t, x_0) \mid t \in I \subseteq \mathbb{R}\}$$

is known as the **orbit** of $x_0 \in \mathbb{R}$. The complete set of qualitative trajectories of the system is referred to as the **phase portrait**.

Flows in $\mathbb{R}^2$

Consider a two-dimensional system of differential equations:

$$\dot{x} = f(x, y), \quad \dot{y} = g(x, y), \quad (x, y) \in \mathbb{R}^2.$$

In this system, an imaginary particle moves within the plane $\mathbb{R}^2$, known as the *phase plane*. As the system evolves, the variables x and y trace a parametric curve over time, $x = x(t)$ and $y = y(t)$. This curve, which passes through an initial point $P(x(t_0), y(t_0))$, is called a *phase path*.

The set of points:

$$\{\varphi(t, x_0) \mid t \in I \subset \mathbb{R}\}$$

represents the *orbit* of the initial state x_0 in $\mathbb{R}^2$. While the phase plane can contain infinitely many trajectories, the overall behavior of the system can often be understood by plotting a few trajectories with different initial conditions.

The *phase portrait* visually illustrates the systems behavior, showing how x and y change with time. If a trajectory satisfies $x(t + p) = x(t)$ for all t, it is called *periodic*, and the smallest positive value p satisfying this condition is the *prime period* of the orbit. It is worth noting that in $\mathbb{R}^1$, flows cannot form oscillatory or closed paths.

Flows in $\mathbb{R}^n$

A system of n autonomous ordinary differential equations can be expressed as:

$$\begin{aligned}
\dot{x}_1 &= f_1(x_1, x_2, \ldots, x_n), \\
\dot{x}_2 &= f_2(x_1, x_2, \ldots, x_n), \\
&\vdots \\
\dot{x}_n &= f_n(x_1, x_2, \ldots, x_n),
\end{aligned}$$

or more compactly as:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}),$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ represents the state vector, and $\mathbf{f} = (f_1, f_2, \ldots, f_n)$ is the vector field.

The solution of this system, starting from the initial condition $\mathbf{x}(0) = \mathbf{x}_0$, describes a continuous trajectory in the *phase space* $\mathbb{R}^n$, parameterized by time $t \in I \subset \mathbb{R}$. The set of all possible states of the system is represented as an n-dimensional vector field in $\mathbb{R}^n$.

The trajectories originating from different initial conditions form a family of curves in the phase space, collectively known as the *phase portrait* of the system. At every point in the phase space, the vector field $\mathbf{f}(\mathbf{x})$ is tangent to these trajectories, with its orientation indicating the direction of the system evolution over time.

Evolution

Consider the system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^n,$$

with initial condition

$$\mathbf{x}(t_0) = \mathbf{x}_0.$$

Let $E \subseteq \mathbb{R}^n$ be an open set and $\mathbf{f} \in C^1(E)$. For $\mathbf{x}_0 \in E$, the solution $\varphi(t, \mathbf{x}_0)$ of the system over its maximal interval of existence $I(\mathbf{x}_0) \subseteq \mathbb{R}$ is referred to as the *evolution operator* of the system.

The operator $\varphi_t : \mathbb{R}^n \to \mathbb{R}^n$ is defined by

$$\varphi_t(\mathbf{x}_0) = \varphi(t, \mathbf{x}_0).$$

For a linear system $\dot{\mathbf{x}} = A\mathbf{x}$ with initial condition $\mathbf{x}(t_0) = \mathbf{x}_0$, the flow is expressed as:

$$\varphi_t = e^{At},$$

where e^{At} represents the matrix exponential.

The evolution operators φ_t for both linear and nonlinear systems satisfy the following properties:

1. $\varphi_0(\mathbf{x}) = \mathbf{x}$ *(Identity property).*

2. $\varphi_s(\varphi_t(\mathbf{x})) = \varphi_{s+t}(\mathbf{x})$ for all $s, t \in \mathbb{R}$.

3. $\varphi_t(\varphi_{-t}(\mathbf{x})) = \varphi_{-t}(\varphi_t(\mathbf{x})) = \mathbf{x}$ for all $t \in \mathbb{R}$.

In general, a dynamical system can be viewed as a family of linear or nonlinear operators evolving as:

$$\{\varphi_t(\mathbf{x}) \mid t \in \mathbb{R}, \mathbf{x} \in \mathbb{R}^n\}.$$

The following group properties of dynamical systems hold:

1. **Closure**: $\varphi_t \circ \varphi_s \in \{\varphi_t(\mathbf{x}) \mid t \in \mathbb{R}, \mathbf{x} \in \mathbb{R}^n\}$.

2. **Associativity**: $(\varphi_t \circ \varphi_s) \circ \varphi_r = \varphi_t \circ (\varphi_s \circ \varphi_r)$.

3. **Identity**: $\varphi_0(\mathbf{x}) = \mathbf{x}$, where φ_0 is the identity operator.

4. **Inverse**: $\varphi_t \circ \varphi_{-t} = \varphi_{-t} \circ \varphi_t = \varphi_0$.

In some cases, the flow also satisfies the commutative property:

$$\varphi_t \circ \varphi_s = \varphi_s \circ \varphi_t.$$

2.1.1 Fixed Points of a System

The concept of fixed points is crucial for understanding the local behavior of a system. A *fixed point* is a constant solution, also referred to as an equilibrium or invariant solution. A point $\mathbf{x}$ is a fixed point of the flow generated by an autonomous system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^n,$$

if and only if:

$$\varphi(t, \mathbf{x}) = \mathbf{x}, \quad \forall t \in \mathbb{R}.$$

For continuous systems, this implies:

$$\dot{\mathbf{x}} = 0 \implies \mathbf{f}(\mathbf{x}) = 0.$$

In non-autonomous systems, fixed points can be defined over a specific time interval.

Fixed points are also known as *critical points*, *equilibrium points*, or *stationary points*. With respect to the flow φ_t in $\mathbb{R}^n$, a fixed point is sometimes called a *stagnation point*.

For flows on the real line, the number of fixed points can vary:

- **No fixed points**: $\dot{x} = 5$.
- **One fixed point**: $\dot{x} = x$.
- **Finite fixed points**: $\dot{x} = x^2 - 1$ (two fixed points).
- **Infinite fixed points**: $\dot{x} = \sin(x)$ (an infinite number of fixed points).

2.2 Phase Portraits and Dynamics

In applications, the differential equation $\dot{x} = X(x)$ models the time dependence of a property, x, of some physical system. We say that the state of the system is specified by x. For example, the equation:

$$\dot{p} = ap, \quad p, a > 0 \tag{2.1}$$

models the growth of the population, p, of an isolated species. Within this model, the state of the species at time t is given by the number of individuals, $p(t)$, living at that time.

Another example is Newton's law of cooling, where the temperature, T, of a body cooling in a draught with temperature t is given by:

$$\dot{T} = -a(T - t), \quad a > 0. \tag{2.2}$$

Here, the state of the body is determined by its temperature.

We can represent the state $x(t_0)$ of a model at any time t_0 by a point on the phase line of $\dot{x} = X(x)$. As time increases, the state changes, and the phase point representing it moves along the line with velocity $\dot{x} = X(x)$. Thus, the dynamics of the physical system are represented by the motion of a phase point on the phase line.

The phase portrait records only the direction of the velocity of the phase point and therefore represents the dynamics in a qualitative way. Such qualitative information can be helpful when constructing models. For example, consider the model (2.1) of an isolated population. Observe that for $p > 0$, the phase portrait in Fig(2.1)(a) shows that the population increases indefinitely. This feature is clearly unrealistic; the environment in which the species live must have limits and could not support an ever-increasing population.

Figure 2.1: The phase portraits for the differential equations $\dot{p} = ap$ and $\dot{p} = p(a - bp)$, $P_c = \frac{a}{b}$, are shown in Figures (a) and (b), respectively. In both cases, we are interested only in the behavior for non-negative populations ($p \geq 0$).

Let us suppose that the environment can support a population P_c. Then how could (2.1) be modified to take this into account? Obviously, the indefinite increase of p should be interrupted. One possibility is to introduce an attractor at P_c, as shown in Fig.(2.1)(b). This means that populations greater than P_c decline, while populations less than P_c increase. Finally, equilibrium is reached at $P = P_c$. The fixed points at $P = P_c$ as well as $P = 0$ require a nonlinear $X(P)$ in (2.1). The form:

$$\dot{p} = p(a - bp) \tag{2.3}$$

has the advantage of reducing to (2.1) when $b = 0$; otherwise, $P_c = \frac{a}{b}$. The population P_c is known as the carrying capacity of the environment.

Of course, models of physical systems frequently involve more than a single state variable. If we are to be able to use qualitative ideas in modeling these systems, then we must examine autonomous equations involving more than one variable.

2.2.1 Autonomous Systems in the Plane

Consider the differential equation:

$$\dot{x} = \frac{dx}{dt} = X(x), \tag{1.19}$$

where $x = (x_1, x_2)$ is a vector in $\mathbb{R}^2$. This equation is equivalent to the system of two coupled equations:

$$\begin{aligned} \dot{x}_1 &= X_1(x_1, x_2), \\ \dot{x}_2 &= X_2(x_1, x_2), \end{aligned}$$

where $X(x) = (X_1(x_1, x_2), X_2(x_1, x_2))$, because $x = (x_1, x_2)$.

A solution to (1.19) consists of a pair of functions $(x_1(t), x_2(t))$, $t \in \mathbb{R}$, which satisfy (1.20). In general, both $x_1(t)$ and $x_2(t)$ involve an arbitrary constant, so there is a two-parameter family of solutions.

The qualitative behavior of dynamical systems in the plane is characterized by the evolution of state variables (x_1, x_2) as the time parameter t increases. This two-dimensional phase plane representation provides significantly more information than the one-dimensional phase line analysis, capturing the complete system dynamics through a family of oriented solution curves. These curves, known as **trajectories** or **orbits**, represent possible evolutions of the system from different initial conditions, with arrows indicating the direction of motion as time progresses.

The phase portrait serves as a powerful visual tool that reveals key dynamical features: equilibrium points (where $\dot{x}_1 = \dot{x}_2 = 0$), stability properties through the convergence or divergence of nearby trajectories, and the overall flow structure of the vector field $(f_1(x_1, x_2), f_2(x_1, x_2))$. For any autonomous planar system of the form:

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2) \\ \dot{x}_2 &= f_2(x_1, x_2) \end{aligned}$$

the phase portrait consists of the integral curves of this vector field, providing immediate insight into the system's long-term behavior without requiring explicit solutions.

To examine qualitative behavior in the plane, we begin by looking at fixed points of the system $\dot{x} = X(x)$. These are constant solutions of the form $x(t) = C = (C_1, C_2)$ that occur when:

$$\mathrm{X}_1(C_1, C_2) = 0 \quad \text{and} \quad \mathrm{X}_2(C_1, C_2) = 0. \tag{1.21}$$

The corresponding trajectory in the phase plane is simply the point (C_1, C_2). As in one-dimensional systems, the nature of these fixed points determines the phase portrait.

2.2.2 Phase portraits of various planar systems

Let us consider examples of isolated fixed points in the plane. Figures 2.2 – 2.7 illustrate some possible configurations. Consider 2.2 , showing the system:

$$\dot{x}_1 = -x_1, \quad \dot{x}_2 = -x_2 \tag{1.22}$$

which has a fixed point at $(0, 0)$. The solutions are:

$$x_1(t) = C_1 e^{-t}, \quad x_2(t) = C_2 e^{-t} \tag{1.23}$$

where $C_1, C_2 \in \mathbb{R}$. All solutions satisfy:

$$x_2(t) = K x_1(t), \quad K = C_2/C_1 \tag{1.24}$$

for all t, meaning each trajectory lies on a radial line in the x_1x_2-plane. As t increases, both $|x_1(t)|$ and $|x_2(t)|$ decrease monotonically to zero, indicated by arrows pointing toward the origin.

Figure 1.24 shows a variation where $x_2 = Kx_1^2$, changing the trajectory shapes while maintaining their inward direction. In contrast, Figure 2.4 presents a different case with solutions:

$$x_1(t) = C_1 e^{-t}, \quad x_2(t) = C_2 e^{t} \tag{1.25}$$

Here $|x_1(t)|$ decreases while $|x_2(t)|$ increases as t grows.

$$C_1,\ C_2 \text{ real; so that} \quad x_2 = Kx_1^{-1}, \tag{1.26}$$

with $K = C_1C_2$. In this case, only two special trajectories approach the fixed point at (0,0), the remainder all turn away sooner or later and $|x_2| \to \infty$ as $|x_1| \to 0$. This qualitative behaviour is obviously quite different from that in Figs 2.2 and 2.3.

In Fig. 2.5 the trajectories close on themselves so that the same set of points in the phase plane recur time and time again as t increases. , we show that the system

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1 \tag{1.27}$$

has solutions

$$x_1(t) = C_1 \cos(-t + C_2), \quad x_2(t) = C_1 \sin(-t + C_2). \tag{1.28}$$

It follows that

$$x_1^2 + x_2^2 = C_1^2 \tag{1.29}$$

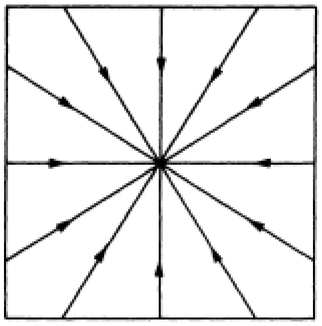

Figure 2.2: $\dot{x}_1 = -x_1,\ \dot{x}_2 = -x_2$.

and the trajectories are a family of concentric circles centred on the fixed point at $(0, 0)$. This obviously corresponds to yet another kind of qualitative behaviour. The fact that $x_1(t)$ and $x_2(t)$ are periodic with the same period is reflected in the closed trajectories.

These examples show that qualitatively different solutions, $(x_1(t),\ x_2(t))$, lead to trajectories with different geometrical properties. The problem of recognizing different types of fixed points becomes one of recognizing 'distinct' geometrical configurations of trajectories, we must decide

What we mean by 'distinct' and is there an element of choice in the criteria that we set.

For example, in Figs 2.2 and 2.3 all the trajectories are directed towards the origin. It would be reasonable to argue that this is the dominant qualitative feature and that the differences in shape of the trajectories are unimportant. We would then say that the nature of the fixed point at $(0, 0)$ was the same in both cases. Of course, its nature would be completely changed if we replaced x_1 by $-x_1$ and x_2 by $-x_2$. Under these circumstances all trajectories would be directed away from the origin corresponding to quite different qualitative behaviour of the solutions.

Let us compare Figs 2.4 and 2.6. Are the fixed points of the same nature? In both cases $|x_1(t)|$ tends to zero while $|x_2(t)|$ becomes infinite and only two special trajectories approach the fixed point itself. Yes, we would argue, they are the same. If the orientation of the trajectories is reversed in these examples is the nature of the fixed point changed as in our previous example? Orientation reversal would mean that the roles x_1 and x_2 were interchanged. However, the features which distinguish 2.4 and 2.6 from the remaining ten diagrams still persist and we conclude that the nature of the fixed point does not change. Similarly, we would say that Figs 2.5 , 2.7 and their counterparts with orientation reversed all had the same kind of fixed point at the origin.

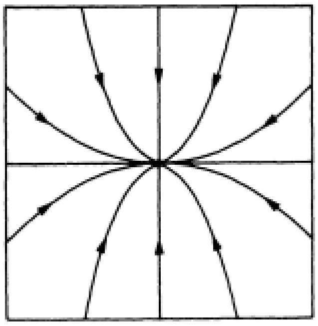

Figure 2.3: $\dot{x}_1 = -x_1,\ \dot{x}_2 = -2x_2$.

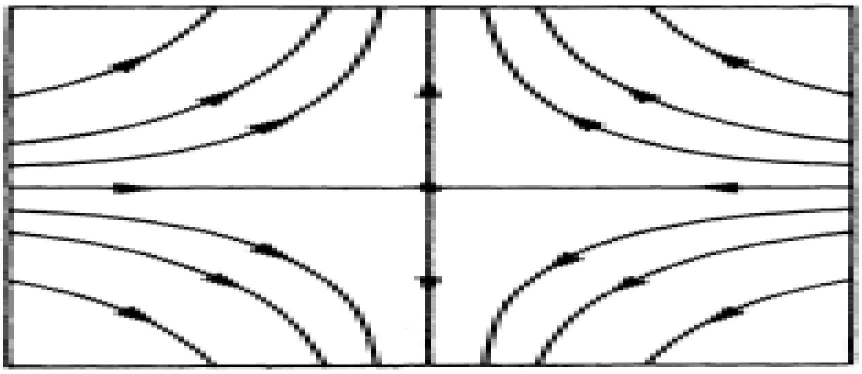

Figure 2.4: $\dot{x}_1 = -x_1,\ \dot{x}_2 = x_2$.

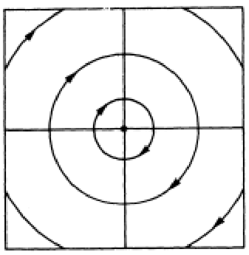

Figure 2.5: $\dot{x}_1 = x_2,\ \dot{x}_2 = -x_1$.

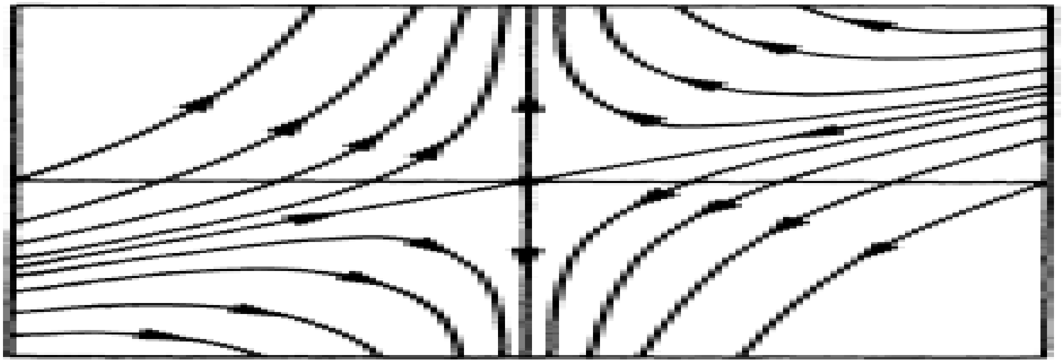

Figure 2.6: $\dot{x}_1 = -x_1,\ \dot{x}_2 = -x_1 + x_2$.

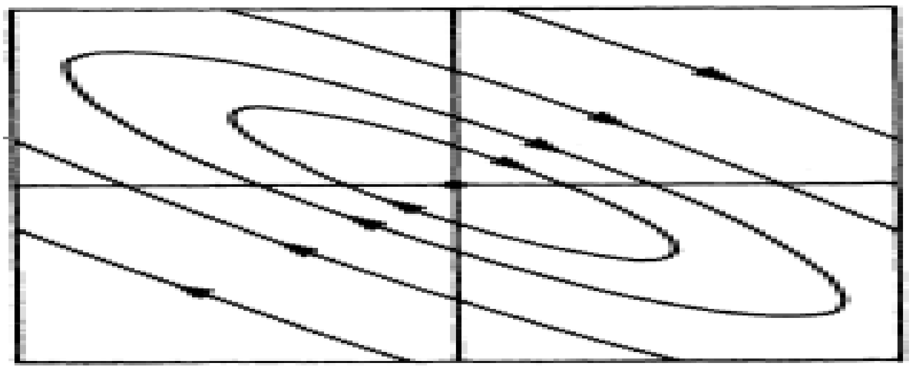

Figure 2.7: $\dot{x}_1 = 3x_1 + 4x_2,\ \dot{x}_2 = -3x_1 - 3x_2$.

Examples

1. **The Simple Pendulum:**
$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin(\theta) = 0,$$

Phase diagrams are graphical representations used to study the behavior of dynamical systems. They plot the state variables of the system against each other, showing how the system evolves over time. In this document, we will explore the phase diagram of a simple pendulum.

The motion of a simple pendulum is governed by the following second-order differential equation:

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin(\theta) = 0,$$

where:

- θ is the angle of the pendulum from the vertical,
- g is the acceleration due to gravity,
- L is the length of the pendulum.

To analyze this system, we convert it into a system of first-order differential equations by introducing the angular velocity $\omega = \frac{d\theta}{dt}$:

$$\frac{d\theta}{dt} = \omega,$$

$$\frac{d\omega}{dt} = -\frac{g}{L}\sin(\theta).$$

Phase Diagram

The phase space for the simple pendulum is the (θ, ω) plane. The phase diagram plots θ on the horizontal axis and ω on the vertical axis.

Fixed Points

The fixed points occur where $\frac{d\theta}{dt} = 0$ and $\frac{d\omega}{dt} = 0$. This gives:

$$\omega = 0 \quad \text{and} \quad \sin(\theta) = 0.$$

The solutions are $\theta = n\pi$, where n is an integer. Thus, the fixed points are at $(\theta, \omega) = (n\pi, 0)$.

Stability Analysis

- For even n (e.g., $\theta = 0, 2\pi, \ldots$), the fixed points are stable (centers). Small perturbations result in oscillations around these points.
- For odd n (e.g., $\theta = \pi, 3\pi, \ldots$), the fixed points are unstable (saddles). Small perturbations cause the pendulum to move away from these points.

Phase diagrams provide a powerful tool for analyzing the behavior of dynamical systems. The simple pendulum example illustrates how fixed points, stability, and trajectories can be visualized to understand the system's long-term behavior. By studying phase diagrams, one can gain deeper insights into the fundamental behaviors of dynamical systems and predict their responses under various conditions.

2. Consider the following two-dimensional dynamical system:

$$\dot{x} = x - y, \quad \dot{y} = x + y$$

where $\dot{x}$ and $\dot{y}$ represent the time derivatives of x and y, respectively.

Step 1: Find Fixed Points

Fixed points occur where $\dot{x} = 0$ and $\dot{y} = 0$. Solve the system:

$$\begin{cases} x - y = 0 \\ x + y = 0 \end{cases}$$

From the first equation, $x = y$. Substituting into the second equation:

$$x + x = 0 \implies 2x = 0 \implies x = 0$$

Thus, $y = 0$. The only fixed point is $(x^*, y^*) = (0, 0)$.

Step 2: Linear Stability Analysis

The Jacobian matrix of the system is:

$$J = \begin{pmatrix} \frac{\partial \dot{x}}{\partial x} & \frac{\partial \dot{x}}{\partial y} \\ \frac{\partial \dot{y}}{\partial x} & \frac{\partial \dot{y}}{\partial y} \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

The eigenvalues λ of J satisfy the characteristic equation:

$$\det(J - \lambda I) = 0$$

$$\det \begin{pmatrix} 1 - \lambda & -1 \\ 1 & 1 - \lambda \end{pmatrix} = (1 - \lambda)^2 + 1 = 0$$

$$(1 - \lambda)^2 + 1 = 0 \implies \lambda^2 - 2\lambda + 2 = 0$$

Solving the quadratic equation:

$$\lambda = \frac{2 \pm \sqrt{(-4)}}{2} = 1 \pm i$$

The eigenvalues are $\lambda = 1 + i$ and $\lambda = 1 - i$.

Step 3: Interpret Eigenvalues

The eigenvalues are complex conjugates with a positive real part ($\mathrm{Re}(\lambda) = 1 > 0$). This indicates that the fixed point $(0, 0)$ is an **unstable spiral** (trajectories spiral away from the fixed point).

Step 4: Phase Diagram

The phase diagram for this system can be sketched as follows:

- The fixed point $(0, 0)$ is an unstable spiral.
- Trajectories spiral outward from the origin.
- The direction of the spiral is counterclockwise (determined by the imaginary part of the eigenvalues).

The phase diagram reveals that the fixed point $(0, 0)$ is an unstable spiral. Trajectories starting near the origin spiral outward, indicating that the system diverges away from the fixed point over time.

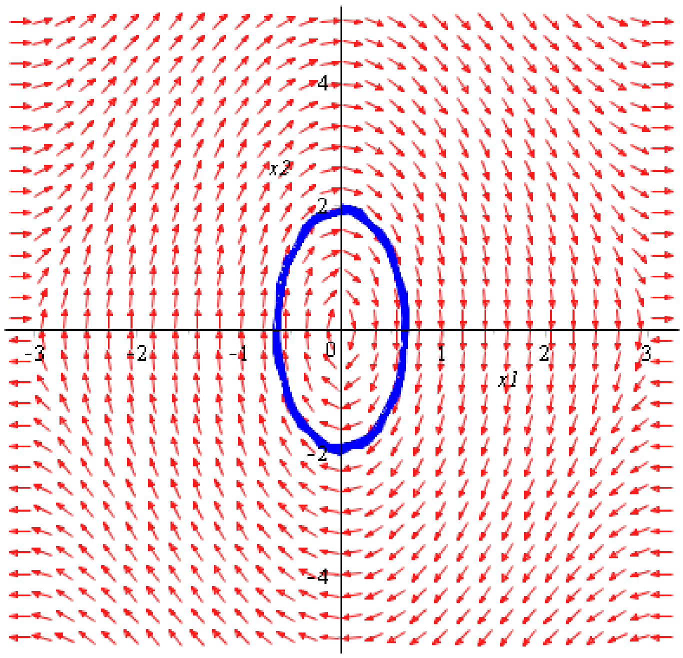

Figure 2.8: Phase Diagram for Simple Pendulum.

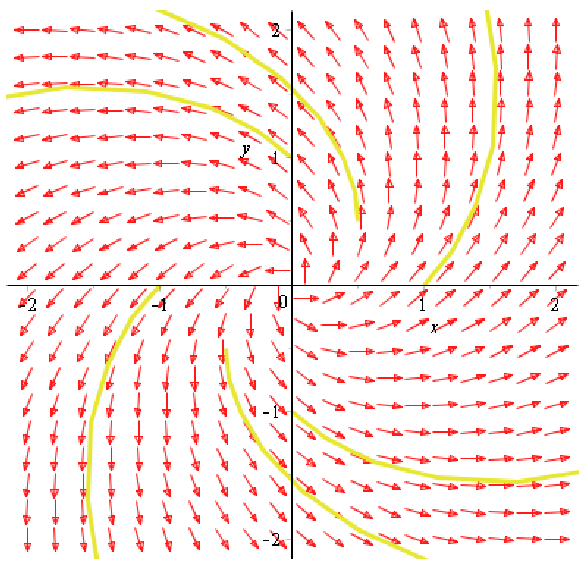

Figure 2.9: Phase diagram for the system $\dot{x} = x - y$, $\dot{y} = x + y$. Trajectories spiral outward from the origin.

CHAPTER 3

STABILITY THEORY

Stability of solutions is a fundamental qualitative property in both linear and nonlinear systems. This chapter aims to introduce various methods for analysing the stability of a system, emphasising its importance in the dynamics of the system. Stability plays a crucial role in determining the behaviour of solutions, especially in the context of differential equations. However, rigorous mathematical definitions often prove too restrictive when analysing the stability of solutions. Over time, various methods for assessing stability have been developed in the theory of differential equations. We begin by looking at the stability analysis of linear systems.

The concept of stability theory originated in classical mechanics, where it was first used to understand the behaviour of physical systems over time. In the context of differential equations, the stability of solutions remains a cornerstone of both theoretical and applied mathematics, particularly in areas such as control theory, dynamical systems and engineering.

Methods for analysing stability can differ significantly depending on whether the system is linear or non-linear. These methods usually combine both qualitative and quantitative approaches. Below we provide a structured overview of the main concepts and techniques used for stability analysis of differential equations.

3.1 Stability of Linear Systems

3.1.1 Stability of Linear Systems in $\mathbf{R}^2$

In this section, we analyze the phase portraits of linear systems in R^2 of the form:

$$\dot{\mathbf{x}} = A\mathbf{x} \tag{3.1}$$

where $\mathbf{x} \in \mathbb{R}^2$ and A is a 2×2 matrix. The phase portrait describes the behavior of the system's trajectories in the plane, and it depends on the eigenvalues and eigenvectors of A. To simplify the analysis, we first consider the system:

$$\dot{\mathbf{x}} = B\mathbf{x} \tag{3.2}$$

where $B = P^{-1}AP$ is in one of the canonical forms . The phase portrait for the original system (3.1) can then be obtained by applying the linear transformation $\mathbf{x} = P\mathbf{y}$ to the phase portrait of (3.2).

Canonical Forms of B

The matrix B can take one of the following forms, depending on the eigenvalues and structure of A:

1. **Diagonal Form**:
$$B = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}$$
This occurs when A has two distinct real eigenvalues λ and μ.

2. **Jordan Form**:
$$B = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$$
This occurs when A has a repeated real eigenvalue λ but only one linearly independent eigenvector.

3. **Complex Form**:
$$B = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$
This occurs when A has complex conjugate eigenvalues $\lambda = a \pm ib$.

The solution to the initial value problem $\dot{\mathbf{x}} = B\mathbf{x}$ with $\mathbf{x}(0) = \mathbf{x}_0$ is given by:

1. **Diagonal Form**:
$$\mathbf{x}(t) = \begin{bmatrix} e^{\lambda t} & 0 \\ 0 & e^{\mu t} \end{bmatrix} \mathbf{x}_0$$

The solution consists of exponential growth or decay along the eigenvectors, depending on the signs of λ and μ.

2. **Jordan Form**:

$$\mathbf{x}(t) = e^{\lambda t} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \mathbf{x}_0$$

The solution includes a linear term t due to the defective eigenvalue.

3. **Complex Form**:

$$\mathbf{x}(t) = e^{at} \begin{bmatrix} \cos bt & -\sin bt \\ \sin bt & \cos bt \end{bmatrix} \mathbf{x}_0$$

The solution involves oscillatory terms due to the complex eigenvalues.

3.1.2 Phase Portraits for Linear Systems

Diagonal Form Systems

Consider the linear system $\dot{\mathbf{x}} = A\mathbf{x}$ where A is a diagonal matrix:

$$A = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}$$

Case 1: Both Eigenvalues Positive ($\lambda > 0$, $\mu > 0$)

The equilibrium at $\mathbf{x} = \mathbf{0}$ is an **unstable node**. All trajectories move away from the origin exponentially in the directions of the eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$. The phase portrait shows curves diverging from the origin, with the rate of divergence determined by the eigenvalue magnitudes. The general solution takes the form $\mathbf{x}(t) = c_1 e^{\lambda t}\mathbf{v}_1 + c_2 e^{\mu t}\mathbf{v}_2$.

Case 2: Both Eigenvalues Negative ($\lambda < 0$, $\mu < 0$)

Here the origin becomes a **stable node**, with all trajectories converging exponentially toward $\mathbf{0}$ along the eigenvector directions. The phase portrait consists of curves approaching the origin, where the convergence rate depends on how negative the eigenvalues are.

Case 3: Eigenvalues with Opposite Signs ($\lambda\mu < 0$)

This configuration produces a **saddle point** at the origin. Trajectories approach the origin along the stable eigenvector (negative eigenvalue) while diverging along the unstable eigenvector (positive eigenvalue). The phase portrait shows hyperbolic curves, characteristic of saddle points.

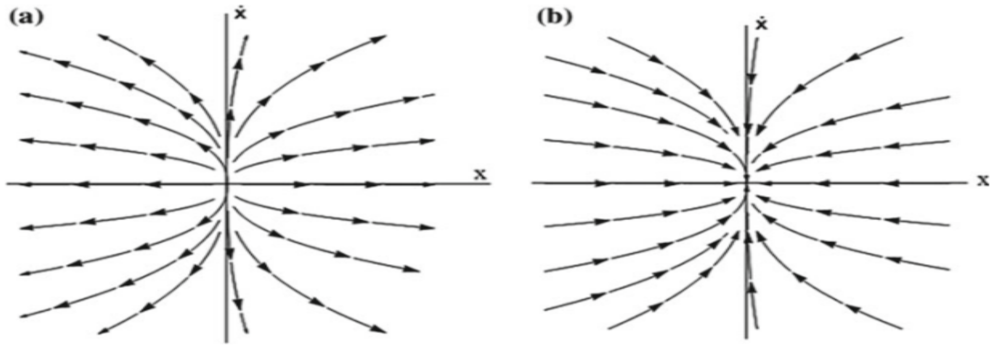

Figure 3.1: Phase diagrams for nonzero real, distinct eigenvalues of same sign, a unstable node, b stable node.

Case 4: One Eigenvalue Zero ($\lambda = 0$, $\mu \neq 0$)

When one eigenvalue is zero ($\lambda = 0$) while the other is non-zero ($\mu \neq 0$), the system exhibits degenerate behavior. The equilibrium points form an entire line along the eigenvector $\mathbf{v}_1$ corresponding to $\lambda = 0$. Trajectories behave uniformly along the direction of the non-zero eigenvalue's eigenvector $\mathbf{v}_2$: they diverge from the origin if $\mu > 0$ or converge toward it if $\mu < 0$. The resulting phase portrait shows parallel lines of trajectories moving along the $\mathbf{v}_2$ direction, with each point on the $\mathbf{v}_1$ axis being a fixed point. Mathematically, the solution takes the form $\mathbf{x}(t) = c_1\mathbf{v}_1 + c_2e^{\mu t}\mathbf{v}_2$, clearly showing the static behavior in the $\mathbf{v}_1$ direction and exponential behavior in the $\mathbf{v}_2$ direction.

Case 5: Both Eigenvalues Zero ($\lambda = \mu = 0$)

In the completely degenerate case where both eigenvalues vanish, every point in the phase plane becomes an equilibrium point. The system is completely static with $\dot{\mathbf{x}} = \mathbf{0}$ for all initial conditions, resulting in a phase portrait where no trajectories move - every point is fixed. This corresponds to the trivial case where the matrix A is the zero matrix. The general solution is simply $\mathbf{x}(t) = \mathbf{x}_0$, reflecting that all solutions remain at their initial conditions indefinitely. While mathematically simple, this case serves as an important limiting scenario in the classification of linear systems.

Summary of Phase Portraits for Diagonal Form

The phase portraits for the diagonal form $\dot{\mathbf{x}} = A\mathbf{x}$ with $A = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}$ are summarized as follows:

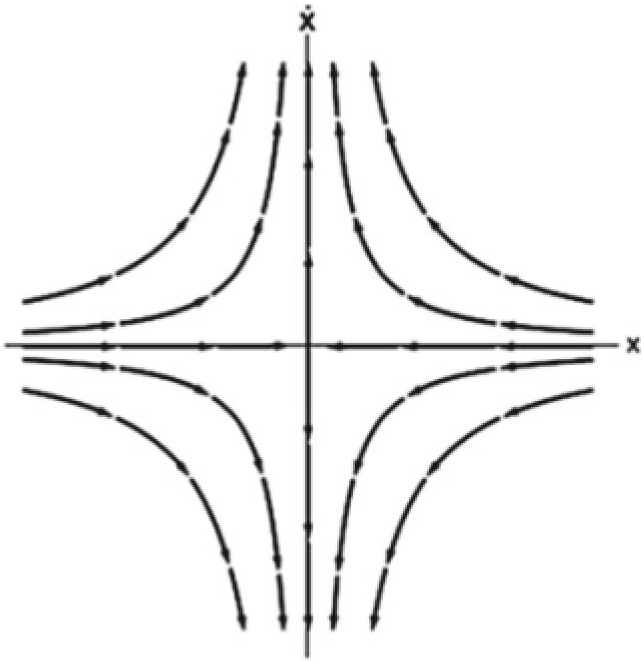

Figure 3.2: Phase diagram for nonzero real, distinct eigenvalues of opposite signs.

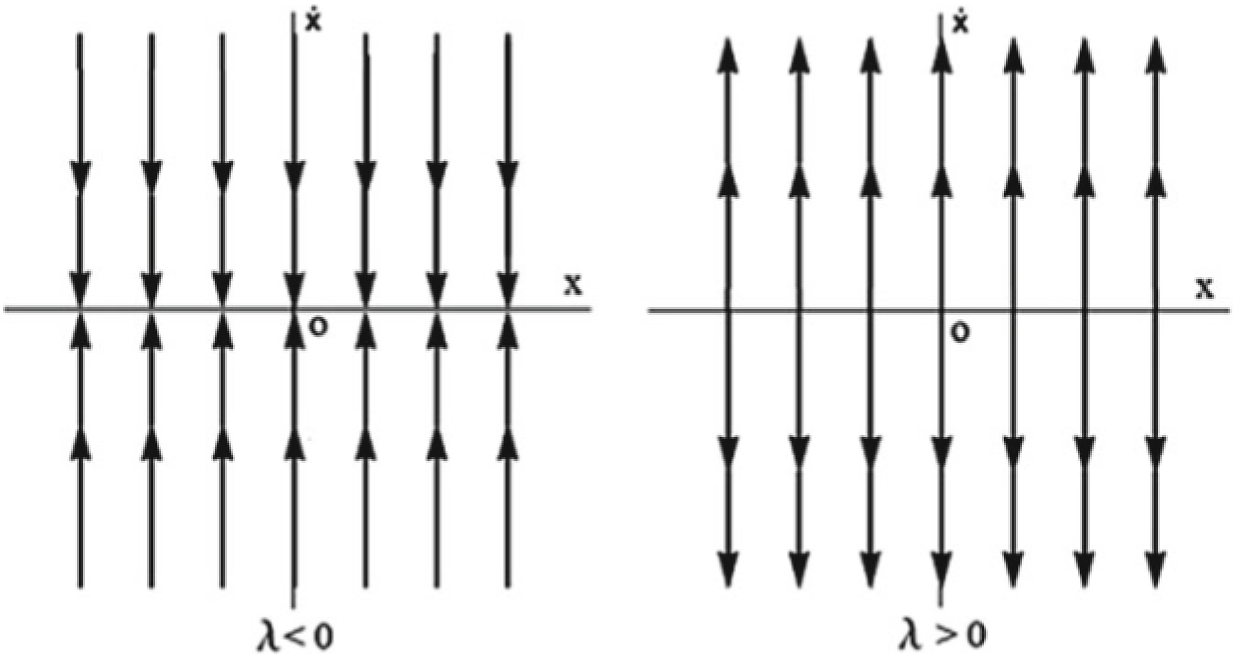

Figure 3.3: Phase portraits when only one eigenvalue is zero.

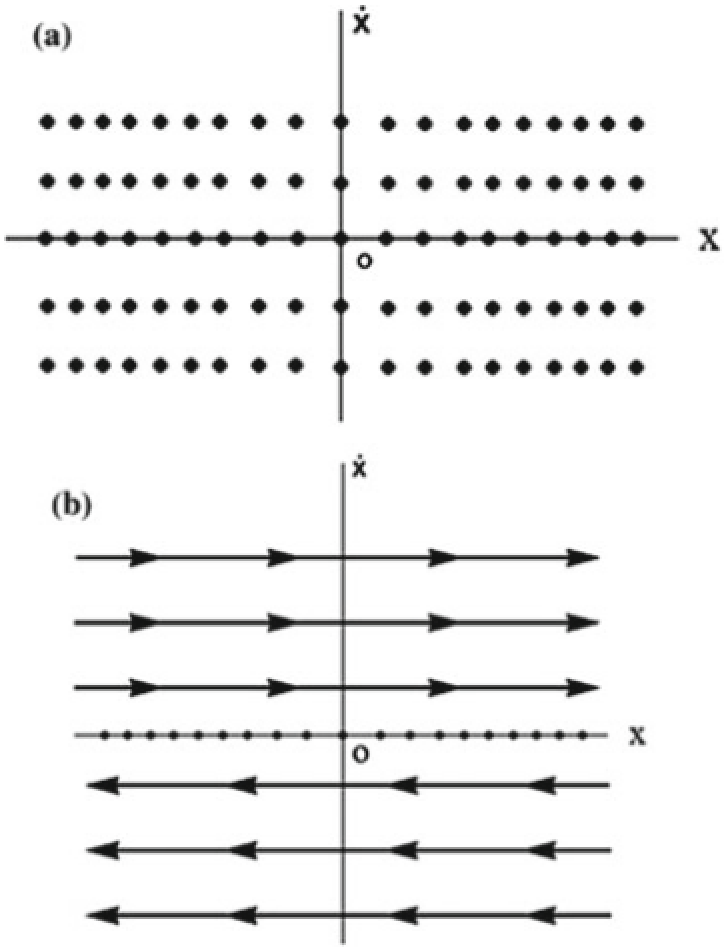

Figure 3.4: a Phase portrait when both eigenvalues are zero. b A typical phase portrait when all eigenvalues are zero.

Eigenvalues	**Equilibrium Type**	**Phase Portrait**
$\lambda > 0, \mu > 0$	Unstable node	Diverging trajectories
$\lambda < 0, \mu < 0$	Stable node	Converging trajectories
$\lambda > 0, \mu < 0$	Saddle point	Hyperbolic trajectories
$\lambda = 0, \mu \neq 0$	Line of equilibria	Parallel trajectories
$\lambda = 0, \mu = 0$	Plane of equilibria	Static system

Example: Unstable Node Dynamics

The linear system $\dot{\mathbf{x}} = A\mathbf{x}$ with diagonal matrix $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ demonstrates a classic **unstable node** at the origin. With distinct positive eigenvalues $\lambda_1 = 2$ and $\lambda_2 = 3$, the system exhibits exponential growth along both principal axes. The general solution takes the form:

$$\mathbf{x}(t) = c_1 e^{2t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

where the eigenvectors $\mathbf{v}_1 = [1, 0]^\top$ and $\mathbf{v}_2 = [0, 1]^\top$ define the directions of fastest ($\lambda_2 = 3$) and slower ($\lambda_1 = 2$) expansion. In the phase portrait, all trajectories radiate outward from the origin, with the y-direction dominating as $t \to \infty$ due to the larger eigenvalue. The absence of off-diagonal elements makes this a particularly simple case of an unstable node, with no rotational components in the flow.

Example: Saddle Point Dynamics

The system $\dot{\mathbf{x}} = A\mathbf{x}$ with $A = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$ presents a fundamental **saddle point** configuration. Its eigenvalues $\lambda_1 = 1$ (unstable) and $\lambda_2 = -2$ (stable) produce the solution:

$$\mathbf{x}(t) = c_1 e^{t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

This solution reveals exponential growth along the x-axis (unstable manifold) and exponential decay along the y-axis (stable manifold). The phase portrait shows characteristic hyperbolic trajectories: solutions approach the origin along the vertical axis while simultaneously diverging along the horizontal axis. The saddle point's distinctive feature is this simultaneous attraction and repulsion along different eigendirections, with the stable manifold (y-axis) acting as a separatrix between different classes of trajectories.

Jordan Form Systems

For systems with defective eigenvalues (geometric multiplicity less than algebraic multiplicity), the Jordan form reveals important behavior. When the matrix A cannot be diagonalized and takes the Jordan block form $A = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$, the equilibrium at $\mathbf{x} = \mathbf{0}$ becomes an **improper node**. The solution to $\dot{\mathbf{x}} = A\mathbf{x}$ involves the matrix exponential $e^{At} = e^{\lambda t} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$, yielding the general solution $\mathbf{x}(t) = e^{\lambda t} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \mathbf{x}_0$. This solution structure produces characteristic twisted trajectories due to the linear t term in the matrix exponential.

Stable Improper Node ($\lambda < 0$)

For $\lambda < 0$ in the Jordan block $A = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$, the system exhibits a **stable improper node**. All trajectories spiral into the origin as $t \to \infty$, but unlike a stable spiral, they approach along a preferred direction determined by the generalized eigenvectors. The phase portrait shows curves that initially align with the eigenvector direction but exhibit a characteristic "twist" as they converge, resulting from the nilpotent part of the Jordan decomposition.

Unstable Improper Node ($\lambda > 0$)

When $\lambda > 0$ in the Jordan form, the equilibrium becomes an **unstable improper node**. The solution $\mathbf{x}(t) = e^{\lambda t} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \mathbf{x}_0$ shows that trajectories grow exponentially while simultaneously twisting away from the eigenvector direction. The phase portrait features curves that diverge from the origin with increasing separation, maintaining a dominant direction of expansion but with a characteristic curvature introduced by the off-diagonal term.

Degenerate Case ($\lambda = 0$)

The special case $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ produces a **line of equilibria** along the x-axis. The solution simplifies to $\mathbf{x}(t) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \mathbf{x}_0$, showing linear rather than exponential growth in the x-direction. The phase portrait consists of horizontal trajectories (parallel to the line of equilibria) with constant vertical spacing, representing a shear flow in the phase plane. Each point on the x-axis is an equilibrium, while all other points move horizontally with velocity proportional to their y-coordinate.

Summary of Phase Portraits for Jordan Form

The phase portraits for the Jordan form $\dot{\mathbf{x}} = A\mathbf{x}$ with $A = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$ are summarized as follows:

Eigenvalue	Equilibrium Type	Phase Portrait
$\lambda < 0$	Stable improper node	Spiraling into the origin
$\lambda > 0$	Unstable improper node	Spiraling away from the origin
$\lambda = 0$	Line of equilibria	Horizontal lines

Example: Stable Improper Node

Consider the system $\dot{\mathbf{x}} = A\mathbf{x}$ with $A = \begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix}$. This system has a repeated eigenvalue $\lambda = -2$, making the origin a **stable improper node**. The solution takes the form $\mathbf{x}(t) = e^{-2t} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \mathbf{x}_0$, revealing both exponential decay and a characteristic linear twist due to the defective eigenvalue. In the phase portrait, trajectories spiral into the origin while maintaining a dominant direction determined by the single eigenvector. The t-term in the solution matrix causes the characteristic "twisting" behavior as trajectories approach the origin, distinguishing it from a standard stable node.

Example: Unstable Improper Node

The system $\dot{\mathbf{x}} = A\mathbf{x}$ with $A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ demonstrates an **unstable improper node**, with repeated eigenvalue $\lambda = 2$. Its solution $\mathbf{x}(t) = e^{2t} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \mathbf{x}_0$ shows exponential growth combined with linear divergence from the eigenvector direction. The phase portrait features trajectories that spiral outward from the origin, with the off-diagonal term causing trajectories to curve away from the principal direction as they diverge. This creates a characteristic "twisted" divergence pattern unique to improper nodes.

Example: Zero Eigenvalue

For the system $\dot{\mathbf{x}} = A\mathbf{x}$ where $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, the repeated zero eigenvalue produces a **line of equilibria** along the entire x-axis. The solution simplifies to $\mathbf{x}(t) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \mathbf{x}_0$, showing linear rather than exponential time dependence. In the phase portrait, all points on the x-axis are equilibrium points, while other trajectories form horizontal lines moving with constant vertical velocity. This creates a shear flow where points above the x-axis move rightward and points below move leftward, with velocity proportional to their y-coordinate.

Example: Diagonal Form (Saddle Point)

The diagonal system $\dot{\mathbf{x}} = A\mathbf{x}$ with $A = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$ has eigenvalues $\lambda_1 = 2$ and $\lambda_2 = -1$, with corresponding eigenvectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. The general solution $\mathbf{x}(t) = c_1 e^{2t}\mathbf{v}_1 + c_2 e^{-t}\mathbf{v}_2$ reveals exponential growth along the x-axis and exponential decay along the y-axis, characteristic of a **saddle point** equilibrium. The phase portrait shows hyperbolic trajectories that approach the origin along the stable manifold (y-axis) while diverging along the unstable manifold (x-axis), with the eigenvalues' magnitudes determining the relative rates of convergence and divergence.

Complex Eigenvalue Systems

Linear systems in $\mathbb{R}^2$ with complex eigenvalues exhibit particularly interesting dynamics. When the matrix A takes the form $A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ where $a, b \in \mathbb{R}$, the eigenvalues become $\lambda = a \pm ib$. The system's behavior is fundamentally determined by the real part a of these complex eigenvalues, leading to three distinct cases:

Center Dynamics ($a = 0$)

For purely imaginary eigenvalues ($a = 0$), the system matrix reduces to $A = \begin{bmatrix} 0 & -b \\ b & 0 \end{bmatrix}$. This configuration produces a **center** at the origin, where trajectories form concentric closed orbits (perfect circles when properly normalized) around the equilibrium point. The solutions are purely oscillatory, taking the form $\mathbf{x}(t) = c_1 \begin{bmatrix} \cos(bt) \\ \sin(bt) \end{bmatrix} + c_2 \begin{bmatrix} -\sin(bt) \\ \cos(bt) \end{bmatrix}$, representing continuous rotation with constant angular velocity b. The phase portrait shows these nested periodic orbits, with no tendency to approach or diverge from the origin.

Stable Spiral ($a < 0$)

When the real part is negative ($a < 0$), the system exhibits a **stable spiral**. The matrix maintains its general form $A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ but now includes exponential decay. Solutions combine rotation with decay: $\mathbf{x}(t) = e^{at}\left(c_1 \begin{bmatrix} \cos(bt) \\ \sin(bt) \end{bmatrix} + c_2 \begin{bmatrix} -\sin(bt) \\ \cos(bt) \end{bmatrix} \right)$. In the phase portrait, trajectories spiral inward toward the origin

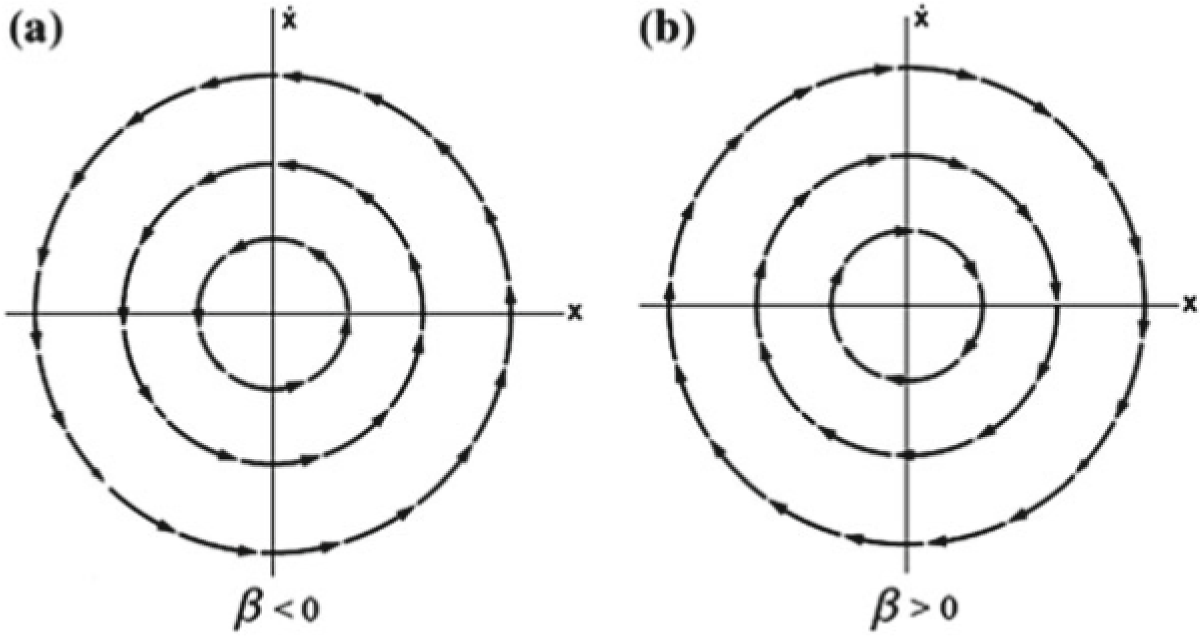

Figure 3.5: Phase portraits for purely imaginary eigenvalues.

while maintaining their angular frequency b, with the decay rate determined by $|a|$. The negative real part ensures all trajectories asymptotically approach the equilibrium.

Unstable Spiral ($a > 0$)

The case of positive real part ($a > 0$) creates an **unstable spiral**, where trajectories spiral outward from the origin. While maintaining the same rotational component as the stable case, the solution now includes exponential growth: $\mathbf{x}(t) = e^{at}\left(c_1\begin{bmatrix}\cos(bt)\\ \sin(bt)\end{bmatrix} + c_2\begin{bmatrix}-\sin(bt)\\ \cos(bt)\end{bmatrix}\right)$. The phase portrait shows trajectories unwinding from the origin with increasing amplitude, where the growth rate is governed by a and the rotation rate by b. This represents systems where small perturbations lead to oscillatory divergence from equilibrium.

Summary of Phase Portraits for Complex Form

The phase portraits for the complex form $\dot{\mathbf{x}} = A\mathbf{x}$ with $A = \begin{bmatrix}a & -b\\ b & a\end{bmatrix}$ are summarized as follows:

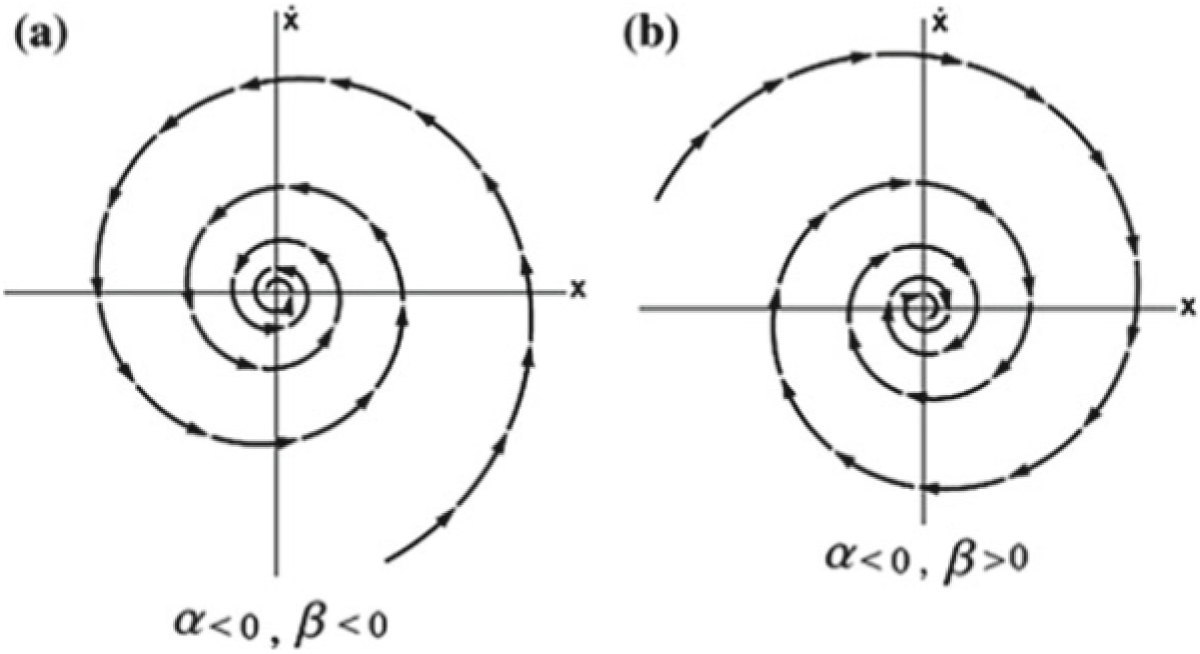

Figure 3.6: Phase portraits for complex eigenvalues with negative real part.

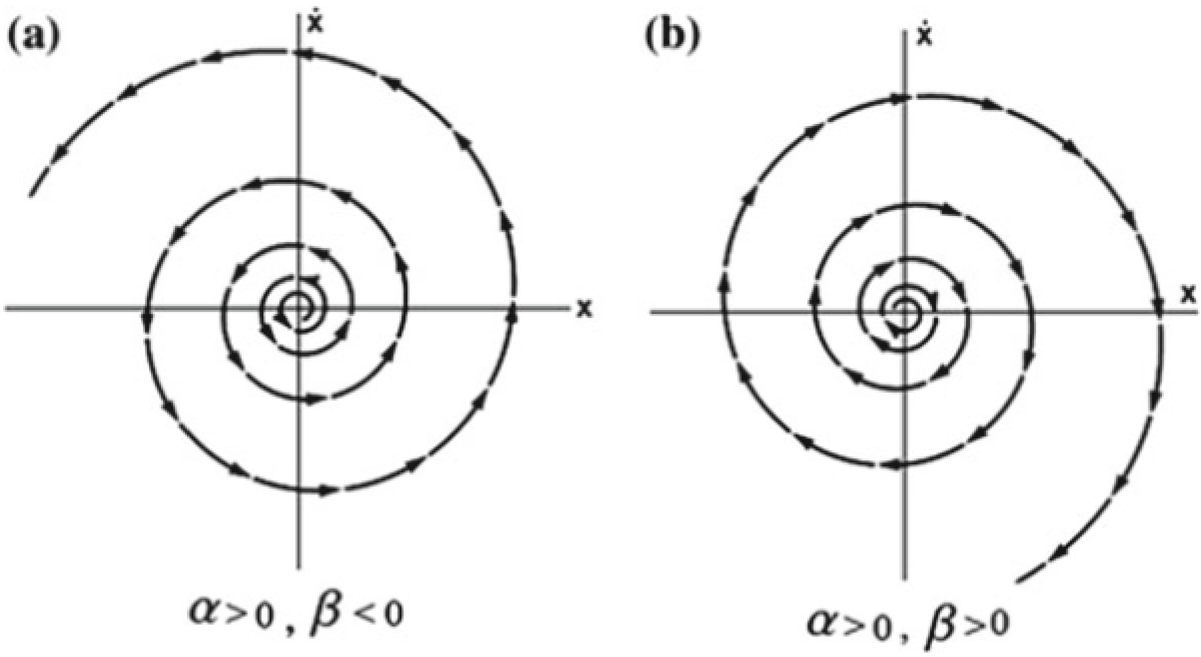

Figure 3.7: Phase portraits for complex eigenvalues with positive real part.

Real Part (a)	Equilibrium Type	Phase Portrait
$a = 0$	Center	Closed orbits (circles or ellipses)
$a < 0$	Stable spiral	Spirals converging to the origin
$a > 0$	Unstable spiral	Spirals diverging from the origin

Example: Center Dynamics

The canonical center system $\dot{\mathbf{x}} = A\mathbf{x}$ with $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ exhibits purely imaginary eigenvalues $\lambda = \pm i$, resulting in a **center** equilibrium at the origin. The solution takes the rotational form $\mathbf{x}(t) = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} \mathbf{x}_0$, representing perfect periodic motion with period 2π. In the phase plane, trajectories trace concentric circles about the origin, each corresponding to a different initial condition $\mathbf{x}_0$. The angular velocity is constant (1 radian per unit time), and the system conserves energy, with all solutions remaining at fixed distances from the origin for all time.

Example: Stable Spiral Dynamics

For the dissipative system $A = \begin{bmatrix} -1 & -2 \\ 2 & -1 \end{bmatrix}$, the complex eigenvalues $\lambda = -1 \pm 2i$ (negative real part) create a **stable spiral**. The solution combines exponential decay with oscillation: $\mathbf{x}(t) = e^{-t} \begin{bmatrix} \cos(2t) & -\sin(2t) \\ \sin(2t) & \cos(2t) \end{bmatrix} \mathbf{x}_0$. Here, the real part -1 governs the decay rate while the imaginary part 2 determines the oscillation frequency. Phase portrait trajectories spiral inward toward the origin, making increasingly tight rotations as they approach equilibrium. The negative real part ensures all solutions asymptotically stabilize to the origin.

Example: Unstable Spiral Dynamics

The system $A = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$ with eigenvalues $\lambda = 1 \pm 2i$ (positive real part) demonstrates an **unstable spiral**. Its solution $\mathbf{x}(t) = e^{t} \begin{bmatrix} \cos(2t) & -\sin(2t) \\ \sin(2t) & \cos(2t) \end{bmatrix} \mathbf{x}_0$ shows exponential growth modulated by rotational motion. The positive real part causes amplitude growth, while the imaginary part produces oscillations at frequency 2. In the phase plane, trajectories spiral outward from the origin with ever-increasing radius. This represents systems where small perturbations lead to growing oscillations, with the divergence rate controlled by the real part of the eigenvalues.

3.1.3 Multiple Eigenvalues

The solution to the linear system $\dot{\mathbf{x}} = A\mathbf{x}$ with initial condition $\mathbf{x}(0) = \mathbf{x}_0$ is $\mathbf{x}(t) = e^{At}\mathbf{x}_0$. When A has distinct eigenvalues, e^{At} can be computed directly. For matrices with multiple eigenvalues, generalized eigenvectors and nilpotent matrices are used.

Definition 1. Let λ be an eigenvalue of A with multiplicity $m \leq n$. A nonzero vector $\mathbf{v}$ satisfying $(A - \lambda I)^k \mathbf{v} = 0$ for some $k \leq m$ is called a *generalized eigenvector* of A.

Definition 2. An $n \times n$ matrix N is *nilpotent of order* k if $N^{k-1} \neq 0$ and $N^k = 0$.

Theorem 1. Let A be a real $n \times n$ matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$ (repeated by multiplicity). Then:

1. There exists a basis of generalized eigenvectors for $\mathbb{R}^n$.
2. If $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is such a basis, the matrix $P = [\mathbf{v}_1 \cdots \mathbf{v}_n]$ is invertible, and $A = S + N$, where:
   - $P^{-1}SP = \text{diag}[\lambda_j]$,
   - $N = A - S$ is nilpotent of order $k \leq n$,
   - S and N commute ($SN = NS$).

Corollary 1. The solution to $\dot{\mathbf{x}} = A\mathbf{x}$ with $\mathbf{x}(0) = \mathbf{x}_0$ is:

$$\mathbf{x}(t) = P \cdot \text{diag}\left[e^{\lambda_j t}\right] \cdot P^{-1} \cdot \left[I + Nt + \cdots + \frac{N^{k-1}t^{k-1}}{(k-1)!}\right]\mathbf{x}_0.$$

If λ is an eigenvalue of multiplicity n, then $S = \text{diag}[\lambda]$ and $N = A - S$, simplifying the solution to:

$$\mathbf{x}(t) = e^{\lambda t}\left[I + Nt + \cdots + \frac{N^k t^k}{k!}\right]\mathbf{x}_0.$$

Theorem 2. Let A be a real $2n \times 2n$ matrix with complex eigenvalues $\lambda_j = a_j + ib_j$ and $\overline{\lambda}_j = a_j - ib_j$ for $j = 1, \ldots, n$. Then:

1. There exist generalized complex eigenvectors $\mathbf{w}_j = \mathbf{u}_j + i\mathbf{v}_j$ and $\overline{\mathbf{w}}_j = \mathbf{u}_j - i\mathbf{v}_j$, such that $\{\mathbf{u}_1, \mathbf{v}_1, \ldots, \mathbf{u}_n, \mathbf{v}_n\}$ is a basis for $\mathbb{R}^{2n}$.
2. For such a basis, $P = [\mathbf{v}_1\ \mathbf{u}_1\ \cdots\ \mathbf{v}_n\ \mathbf{u}_n]$ is invertible, and $A = S + N$, where:
   - $P^{-1}SP = \text{diag}\begin{bmatrix} a_j & -b_j \\ b_j & a_j \end{bmatrix}$,
   - $N = A - S$ is nilpotent of order $k \leq 2n$,
   - S and N commute.

Corollary 2. The solution to $\dot{\mathbf{x}} = A\mathbf{x}$ with $\mathbf{x}(0) = \mathbf{x}_0$ is:

$$\mathbf{x}(t) = P \cdot \text{diag}\left[e^{a_j t}\begin{pmatrix} \cos b_j t & -\sin b_j t \\ \sin b_j t & \cos b_j t \end{pmatrix}\right] \cdot P^{-1} \cdot \left[I + \cdots + \frac{N^k t^k}{k!}\right]\mathbf{x}_0.$$

3.2 Stability Theory

In this section, we define the **stable subspace** E^s, the **unstable subspace** E^u, and the **center subspace** E^c for a linear system of the form (3.1).

Let $\mathbf{w}_j = \mathbf{u}_j + i\mathbf{v}_j$ be a generalized eigenvector of the real matrix A, corresponding to an eigenvalue $\lambda_j = a_j + ib_j$. Note that if $b_j = 0$ (i.e., the eigenvalue is real), then $\mathbf{v}_j = \mathbf{0}$. Consider the basis

$$B = \{\mathbf{u}_1, \ldots, \mathbf{u}_k, \mathbf{u}_{k+1}, \mathbf{v}_{k+1}, \ldots, \mathbf{u}_m, \mathbf{v}_m\}$$

for $\mathbb{R}^n$ (where $n = 2m - k$).

Definition 1. Let $\lambda_j = a_j + ib_j$, $\mathbf{w}_j = \mathbf{u}_j + i\mathbf{v}_j$, and B be as defined above. Then:

$$\begin{aligned}
E^s &= \operatorname{Span}\left\{\mathbf{u}_j, \mathbf{v}_j \mid a_j < 0\right\}, \\
E^c &= \operatorname{Span}\left\{\mathbf{u}_j, \mathbf{v}_j \mid a_j = 0\right\}, \\
E^u &= \operatorname{Span}\left\{\mathbf{u}_j, \mathbf{v}_j \mid a_j > 0\right\}.
\end{aligned}$$

Definition 2. If all eigenvalues of the $n \times n$ matrix A have nonzero real parts, then the flow $e^{At} : \mathbb{R}^n \to \mathbb{R}^n$ is called a **hyperbolic flow**, and the linear system (1.1) is referred to as a **hyperbolic linear system**.

Definition 3. A subspace $E \subset \mathbb{R}^n$ is said to be **invariant** under the flow $e^{At} : \mathbb{R}^n \to \mathbb{R}^n$ if, for all $t \in \mathbb{R}$, the flow maps E into itself; that is, $e^{At}E \subset E$.

Example: Find the Linear Subspaces

Consider the system of differential equations:

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x}, \quad \mathbf{x}(0) = \mathbf{x}_0,$$

where the matrix A is given by:

$$A = \begin{pmatrix} -3 & 0 & 0 \\ 0 & 3 & -2 \\ 0 & 1 & 1 \end{pmatrix}.$$

Solution:

The eigenvalues λ are found by solving $\det(A - \lambda I) = 0$:

$$\det\begin{pmatrix} -3-\lambda & 0 & 0 \\ 0 & 3-\lambda & -2 \\ 0 & 1 & 1-\lambda \end{pmatrix} = 0.$$

This simplifies to:

$$(-3-\lambda)(\lambda^2-4\lambda+5)=0.$$

The eigenvalues are:

$$\lambda_1=-3, \quad \lambda_2=2+i, \quad \lambda_3=2-i.$$

Eigenvector for $\lambda_1=-3$ Solve $(A-\lambda_1 I)\mathbf{v}_1=0$:

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 6 & -2 \\ 0 & 1 & 4 \end{pmatrix}\begin{pmatrix} v_{11} \\ v_{12} \\ v_{13} \end{pmatrix}=0.$$

This gives:

$$\mathbf{v}_1=\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Eigenvectors for $\lambda_2=2+i$ **and** $\lambda_3=2-i$ Solve $(A-\lambda_2 I)\mathbf{v}_2=0$:

$$\begin{pmatrix} -5-i & 0 & 0 \\ 0 & 1-i & -2 \\ 0 & 1 & -1-i \end{pmatrix}\begin{pmatrix} v_{21} \\ v_{22} \\ v_{23} \end{pmatrix}=0.$$

From the first row, $v_{21}=0$. Solving the remaining system:

$$\mathbf{v}_2=\begin{pmatrix} 0 \\ 1+i \\ 1 \end{pmatrix}, \quad \mathbf{v}_3=\begin{pmatrix} 0 \\ 1-i \\ 1 \end{pmatrix}.$$

So

- **Stable Subspace (E^s):**

$$E^s=\text{Span}\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right\}.$$

- **Unstable Subspace (E^u):**

$$E^u = \text{Span}\left\{\begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}\right\}.$$

- **Center Subspace (E^c):**

$$E^c = \{0\}.$$

Lemma 3.2.1 *Let E be the generalized eigenspace of A corresponding to an eigenvalue λ. Then $AE \subset E$.*

Proof. To show that $AE \subset E$, we need to prove that for any vector $v \in E$, the vector Av is also in E.

The generalized eigenspace E corresponding to an eigenvalue λ is defined as:

$$E = \{v \in V \mid (A - \lambda I)^k v = 0 \text{ for some integer } k \geq 1\}.$$

Here, V is the vector space on which A acts, and I is the identity operator.

Let $v \in E$. By definition, there exists an integer $k \geq 1$ such that:

$$(A - \lambda I)^k v = 0.$$

Consider Av. We want to show that $Av \in E$, i.e., there exists an integer $m \geq 1$ such that:

$$(A - \lambda I)^m (Av) = 0.$$

Notice that A and $(A - \lambda I)$ commute:

$$A(A - \lambda I) = (A - \lambda I)A.$$

This implies that:

$$(A - \lambda I)^k A = A(A - \lambda I)^k.$$

Using the commutativity, we have:

$$(A - \lambda I)^k (Av) = A(A - \lambda I)^k v.$$

Since $v \in E$, $(A - \lambda I)^k v = 0$, so:

$$(A - \lambda I)^k (Av) = A \cdot 0 = 0.$$

This shows that Av satisfies $(A - \lambda I)^k(Av) = 0$, which means $Av \in E$. Therefore, $AE \subset E$. ■

Theorem 3.2.1 *Consider a system $\dot{x} = Ax$, where A is a $n \times n$ matrix with real entries. Then phase space $\mathbb{R}^n$ can be decomposed as*

$$\mathbb{R}^n = E^u \oplus E^s \oplus E^c,$$

where E^u, E^s, and E^c are the unstable, stable, and center subspaces of the system, respectively. Furthermore, these subspaces are invariant with respect to the flow.

Definition 3.2.1 *Consider the linear system:*

$$\dot{x} = Ax \tag{3.3}$$

*If all of the eigenvalues of A have negative (positive) real parts, the origin is called a **sink** (**source**) for the linear system* (3.3).

Example 3.2.1 *Consider the linear system:*

$$\dot{x} = Ax,$$

where A is a 2×2 matrix given by:

$$A = \begin{pmatrix} -2 & 0 \\ 0 & -3 \end{pmatrix}.$$

The eigenvalues of A are the solutions to the characteristic equation:

$$\det(A - \lambda I) = 0.$$

For the given matrix A, this becomes:

$$\det \begin{pmatrix} -2 - \lambda & 0 \\ 0 & -3 - \lambda \end{pmatrix} = (-2 - \lambda)(-3 - \lambda) = 0.$$

The eigenvalues are:

$$\lambda_1 = -2, \quad \lambda_2 = -3.$$

*Since both eigenvalues ($\lambda_1 = -2$ and $\lambda_2 = -3$) have **negative real parts**, the origin is a **sink** for this system. Trajectories will converge to the origin as $t \to \infty$.*

Example 3.2.2 *If we change the matrix A to:*

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix},$$

*the eigenvalues become $\lambda_1 = 1$ and $\lambda_2 = 2$. Since both eigenvalues have **positive real parts**, the origin is a **source** for this system. Trajectories will diverge from the origin as $t \to \infty$.*

Theorem 3.2.2 *Let A be a real $n \times n$ matrix, and consider the linear system of differential equations:*

$$\dot{x} = Ax.$$

The following statements are equivalent:

(a) For all initial conditions $x_0 \in \mathbb{R}^n$, the solution $x(t) = e^{At}x_0$ satisfies:

$$\lim_{t\to\infty} x(t) = 0,$$

and for $x_0 \neq 0$, the solution grows unbounded as $t \to -\infty$:

$$\lim_{t\to-\infty} |x(t)| = \infty.$$

(b) All eigenvalues of A have negative real parts. That is, if λ is an eigenvalue of A, then $\mathrm{Re}(\lambda) < 0$.

(c) There exist positive constants a, c, m, and M such that for all $x_0 \in \mathbb{R}^n$, the solution $x(t) = e^{At}x_0$ satisfies:

$$|x(t)| \leq Me^{-ct}|x_0| \quad \textit{for } t \geq 0,$$

and

$$|x(t)| \geq me^{-at}|x_0| \quad \textit{for } t \leq 0.$$

Proof. We prove the equivalence of the three statements by showing $(a) \implies (b)$, $(b) \implies (c)$, and $(c) \implies (a)$.

$(a) \implies (b)$ Assume statement (a) holds. That is, for all $x_0 \in \mathbb{R}^n$, $\lim_{t\to\infty} e^{At}x_0 = 0$, and for $x_0 \neq 0$, $\lim_{t\to-\infty} |e^{At}x_0| = \infty$.

- If A had an eigenvalue λ with $\mathrm{Re}(\lambda) \geq 0$, then there would exist a solution $x(t) = e^{At}x_0$ that either does not decay to zero as $t \to \infty$ (if $\mathrm{Re}(\lambda) > 0$) or remains bounded but does not grow as $t \to -\infty$ (if $\mathrm{Re}(\lambda) = 0$). This contradicts statement (a). - Therefore, all eigenvalues of A must have negative real parts. This proves (b).

$(b) \implies (c)$ Assume all eigenvalues of A have negative real parts. We show that there exist positive constants a, c, m, and M such that the exponential decay and growth estimates hold.

- By the Jordan canonical form, we can write $A = PJP^{-1}$, where J is a block-diagonal matrix consisting of Jordan blocks corresponding to the eigenvalues of A. - For each Jordan block, the exponential e^{Jt} can be computed explicitly. Since all eigenvalues have negative real parts, each block contributes terms of the form $t^k e^{\lambda t}$, where $\mathrm{Re}(\lambda) < 0$ and k is a nonnegative integer. - These terms decay exponentially as $t \to \infty$

and grow exponentially as $t \to -\infty$. Thus, there exist constants a, c, m, and M such that:

$$|e^{At}x_0| \leq Me^{-ct}|x_0| \quad \text{for } t \geq 0,$$

and

$$|e^{At}x_0| \geq me^{-at}|x_0| \quad \text{for } t \leq 0.$$

This proves (c).

$(c) \implies (a)$ Assume statement (c) holds. That is, there exist positive constants a, c, m, and M such that:

$$|x(t)| \leq Me^{-ct}|x_0| \quad \text{for } t \geq 0,$$

and

$$|x(t)| \geq me^{-at}|x_0| \quad \text{for } t \leq 0.$$

- For $t \geq 0$, the inequality $|x(t)| \leq Me^{-ct}|x_0|$ implies that $\lim_{t\to\infty} x(t) = 0$. - For $t \leq 0$, the inequality $|x(t)| \geq me^{-at}|x_0|$ implies that $\lim_{t\to-\infty} |x(t)| = \infty$ for $x_0 \neq 0$. - Thus, statement (a) holds.

We have shown that $(a) \implies (b)$, $(b) \implies (c)$, and $(c) \implies (a)$. Therefore, the three statements are equivalent. ■

Theorem 3.2.3 *Let A be a real $n \times n$ matrix, and consider the linear system of differential equations:*

$$\dot{x} = Ax.$$

The following statements are equivalent:

(a) For all initial conditions $x_0 \in \mathbb{R}^n$, the solution $x(t) = e^{At}x_0$ satisfies:

$$\lim_{t\to-\infty} x(t) = 0,$$

and for $x_0 \neq 0$, the solution grows unbounded as $t \to \infty$:

$$\lim_{t\to\infty} |x(t)| = \infty.$$

(b) All eigenvalues of A have positive real parts. That is, if λ is an eigenvalue of A, then $\mathrm{Re}(\lambda) > 0$.

(c) There exist positive constants a, c, m, and M such that for all $x_0 \in \mathbb{R}^n$, the solution $x(t) = e^{At}x_0$ satisfies:

$$|x(t)| \leq Me^{ct}|x_0| \quad \textit{for } t \leq 0,$$

and

$$|x(t)| \geq me^{at}|x_0| \quad \text{for } t \geq 0.$$

Proof. We prove the equivalence of the three statements by showing $(a) \implies (b)$, $(b) \implies (c)$, and $(c) \implies (a)$.

$(a) \implies (b)$ Assume statement (a) holds. That is, for all $x_0 \in \mathbb{R}^n$, $\lim_{t\to-\infty} e^{At}x_0 = 0$, and for $x_0 \neq 0$, $\lim_{t\to\infty} |e^{At}x_0| = \infty$.

- If A had an eigenvalue λ with $\mathrm{Re}(\lambda) \leq 0$, then there would exist a solution $x(t) = e^{At}x_0$ that either does not decay to zero as $t \to -\infty$ (if $\mathrm{Re}(\lambda) < 0$) or remains bounded but does not grow as $t \to \infty$ (if $\mathrm{Re}(\lambda) = 0$). This contradicts statement (a). - Therefore, all eigenvalues of A must have positive real parts. This proves (b).

$(b) \implies (c)$ Assume all eigenvalues of A have positive real parts. We show that there exist positive constants a, c, m, and M such that the exponential decay and growth estimates hold.

- By the Jordan canonical form, we can write $A = PJP^{-1}$, where J is a block-diagonal matrix consisting of Jordan blocks corresponding to the eigenvalues of A. - For each Jordan block, the exponential e^{Jt} can be computed explicitly. Since all eigenvalues have positive real parts, each block contributes terms of the form $t^k e^{\lambda t}$, where $\mathrm{Re}(\lambda) > 0$ and k is a nonnegative integer. - These terms decay exponentially as $t \to -\infty$ and grow exponentially as $t \to \infty$. Thus, there exist constants a, c, m, and M such that:

$$|e^{At}x_0| \leq Me^{ct}|x_0| \quad \text{for } t \leq 0,$$

and

$$|e^{At}x_0| \geq me^{at}|x_0| \quad \text{for } t \geq 0.$$

This proves (c).

$(c) \implies (a)$ Assume statement (c) holds. That is, there exist positive constants a, c, m, and M such that:

$$|x(t)| \leq Me^{ct}|x_0| \quad \text{for } t \leq 0,$$

and

$$|x(t)| \geq me^{at}|x_0| \quad \text{for } t \geq 0.$$

- For $t \leq 0$, the inequality $|x(t)| \leq Me^{ct}|x_0|$ implies that $\lim_{t\to-\infty} x(t) = 0$. - For $t \geq 0$, the inequality $|x(t)| \geq me^{at}|x_0|$ implies that $\lim_{t\to\infty} |x(t)| = \infty$ for $x_0 \neq 0$. - Thus, statement (a) holds.

We have shown that $(a) \implies (b)$, $(b) \implies (c)$, and $(c) \implies (a)$. Therefore, the three statements are equivalent.

■

CHAPTER 4

NONLINEAR SYSTEMS: LOCAL THEORY

When a system is disturbed from its equilibrium or steady-state position, does it naturally return to that state over time, or does the disturbance grow, leading to a significant deviation from the original state? In other words: **Stable System** if the system returns to its equilibrium or steady state after a small perturbation, it is considered stable. The effects of the disturbance diminish over time. **Unstable System** if the system does not return to equilibrium and the disturbance grows over time, leading to larger deviations, the system is considered unstable. Even a small perturbation can have significant and potentially catastrophic consequences, this concept is fundamental in understanding the behavior of physical, engineering, and dynamical systems, as it determines whether a system can maintain its desired state or if it is prone to divergence and failure.

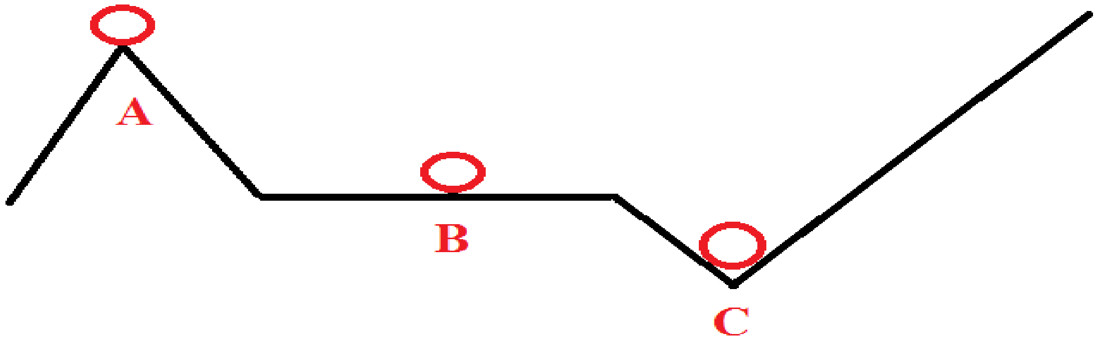

Figure 4.1: A) Unstable B) Locally stable C) Asymptotically stable.

Consider the following autonomous system:

$$\dot{x} = f(x) \tag{4.1}$$

where $f \in C^1(E)$ and E is an open subset of $\mathbb{R}^n$. A point a in E such that $f(a) = 0$ is called an equilibrium point or a critical point of the system (4.1). Critical points correspond to constant solutions of the differential system. We use the notation $\phi(t, x_0)$ to denote the unique solution $x(t)$ of (4.1) that satisfies $x(0) = x_0$. The parametrized map $\phi_t = \phi(t, \cdot) : \mathbb{R}^n \to \mathbb{R}^n$ is called the flow of the system (4.1).

An important property of orbits is given in the following theorem.

Theorem 4.0.4 *(Property of semi-groups) Let $x_0 \in \mathbb{R}^n$ and (α, ω) be the maximal interval of existence of $\phi(t, x_0)$. Then*

$$\phi(t + \tau, x_0) = \phi(t, \phi(\tau, x_0)),$$

for $t, \tau, t + \tau \in (\alpha, \omega)$.

Proof. Let $x_0 \in \mathbb{R}^n$ and let (α, ω) be the maximal interval of existence of $\phi(t, x_0)$. Suppose $t, \tau, t + \tau \in (\alpha, \omega)$. The function $\psi(t) = \phi(t + \tau, x_0)$ is a solution of equation (4.1) on the interval $(\alpha - \tau, \omega - \tau)$ (see Theorem 3.4 in [2]W. G. Kelley and A. C. Peterson, The theory of differential equatioe001 Classical and qualitaive, Springer, New York. 2010).

On the other hand, $\phi(t, \phi(\tau, x_0))$ is also a solution of equation (4.1) and satisfies the same initial condition at $t = 0$. Since solutions to (4.1) are unique, it follows that

$$\psi(t) = \phi(t + \tau, x_0) = \phi(t, \phi(\tau, x_0)).$$

■

Definition 4.0.2 *(Local Stability)*
- An equilibrium point a of equation (4.1) is stable in the see001e of Lyapunov if, for every $\epsilon > 0$, there exists $\eta > 0$ such that for all $x \in E$ satisfying $\|x - a\| \leq \eta$, it follows that $\|\phi(t, x) - a\| \leq \epsilon$ for all $t \geq 0$.
- An equilibrium point a of equation (4.1) is asymptotically stable in the sense of Lyapunov if it is stable in the Lyapunov sense and, moreover, for all x sufficiently close to a, we have

$$\lim_{t \to +\infty} \phi(t, x) = a.$$

We now present two methods to study the stability of a nonlinear system:
- **Indirect method**, based on linearization.
- **Direct method**, which involves using a function known as a Lyapunov function.

4.1 Indirect Method (Linearization)

To study the stability of an equilibrium point of equation (4.1), we first translate the equilibrium point to the origin by defining a new variable $x = x - a$, where a is the equilibrium point. The Taylor expansion of the function $f(x)$ around $x = 0$ is given by:

$$f(x) = Df(0)x + \frac{1}{2!}D^2 f(0)(x, x) + \ldots$$

For x sufficiently close to the origin, the higher-order nonlinear terms become negligible compared to the linear term. Lyapunov's indirect method for studying stability around the equilibrium point 0 involves analyzing the corresponding linear system:

$$\dot{x} = Ax \tag{4.2}$$

where $A = Df(0)$ is the Jacobian matrix of f evaluated at 0, given by:

$$A = Df(0) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(0) & \frac{\partial f_1}{\partial x_2}(0) & \cdots & \frac{\partial f_1}{\partial x_n}(0) \\ \frac{\partial f_2}{\partial x_1}(0) & \frac{\partial f_2}{\partial x_2}(0) & \cdots & \frac{\partial f_2}{\partial x_n}(0) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1}(0) & \frac{\partial f_n}{\partial x_2}(0) & \cdots & \frac{\partial f_n}{\partial x_n}(0) \end{pmatrix}$$

The system (4.2) is referred to as the linearized system of the original nonlinear system (4.1) at the equilibrium point 0.

Definition 4.1.1 *An equilibrium point a of equation (4.1) is called a* hyperbolic point *if none of the eigenvalues of the Jacobian matrix $A = Df(a)$ have a real part equal to zero.*

Definition 4.1.2 *The following definition establishes a foundation for the subsequent discussions:*
An equilibrium point a of equation (4.1) is classified as one of the following types, based on the eigenvalues of the Jacobian matrix $A = Df(a)$:
Sink *if all the eigenvalues of the matrix $A = Df(a)$ have negative real parts.*
Source *if all the eigenvalues of the matrix $A = Df(a)$ have positive real parts.*
Saddle Point*: if at least one eigenvalue of the matrix $A = Df(a)$ has a positive real part, and at least one has a negative real part.*

Definition 4.1.3 *Two autonomous systems are said to be* topologically equivalent *in a neighborhood of the origin (or to have the same structure) if there exists a homeomorphism H that maps an open set U containing the origin to an open set V containing the origin. This homeomorphism must transform the trajectories of the first system in U into the trajectories of the second system in V, while preserving the orientation of the trajectories with respect to time.*

Example 4.1.1 *Consider the following two linear systems:*

$$\dot{x} = Ax \tag{4.3}$$

and

$$\dot{y} = By \tag{4.4}$$

where $A = \begin{pmatrix} -1 & -3 \\ -3 & -1 \end{pmatrix}$ *and* $B = \begin{pmatrix} 2 & 0 \\ 0 & -4 \end{pmatrix}$.
Let $H(x) = Rx$, where

$$R = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad R^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

It follows that

$$B = RAR^{-1}.$$

Now, let $y = H(x) = Rx$, or equivalently $x = R^{-1}y$. Then, the time derivative of y is related to the time derivative *of x as follows:*

$$\begin{aligned}\dot{y} &= R\dot{x}\\ &= RAx\\ &= RAR^{-1}y\\ &= By.\end{aligned}$$

Thus, if $x(t) = e^{At}x_0$ *is the solution of the first system with initial condition* x_0*, then the solution* $y(t) = H(x(t)) = Rx(t) = Re^{At}x_0 = e^{Bt}Rx_0$ *is the solution of the second system with initial condition* $y_0 = Rx_0$*.*
Therefore, the transformation $H(x) = Rx$ *is a simple* $45°$ *rotation, which is clearly a homeomorphism.*

Theorem 4.1.1 *(Hartman-Grobman)*
Let U, V *be two open sets of* $\mathbb{R}^n$ *containing the origin, let* $f \in C^1(U)$*, and let* ϕ_t *be the flow of the nonlinear system (4.1). Suppose that the origin is a hyperbolic equilibrium point. Then, there exists a homeomorphism H from the open set U to the open set V such that for each* $x_0 \in U$*, there is an open interval* $I_0 \subset \mathbb{R}$ *containing 0, and for all* $t \in I_0$*,*

$$H \circ \phi_t(x_0) = e^{At}H(x_0),$$

i.e., H maps the trajectories of the nonlinear system (4.1) to the trajectories of its linearized system (4.2) and preserves the direction of time.

The theorem asserts that, under certain conditions, in the neighborhood of a point a where $f(a) = 0$, the nonlinear system

$$\dot{x} = f(x)$$

is topologically equivalent to its linearized system

$$\dot{x} = Ax,$$

where $A = Df(a)$ is the Jacobian matrix at the equilibrium point a. Although the original statement of the theorem is typically given for $a = 0$, we can always shift the system to this case by defining a new function

$$g(x) = f(x + a) - f(a),$$

which moves the equilibrium point to the origin. In this new system, the origin becomes the equilibrium point, and the analysis can proceed as described in the theorem.

Thus, near an equilibrium point, the nonlinear system behaves in a manner similar to the linearized system, and the topological structure of the trajectories is preserved. This result shows that locally (in the neighborhood of a hyperbolic equilibrium point), the nonlinear dynamics are qualitatively the same as the linear dynamics. **Proof.** See [6]. ■

It is evident that a corollary exists.

Corollary 4.1.1 *Consider the system (4.1) and its linearized version (4.2). If all the eigenvalues of A have negative real parts, then a is locally asymptotically stable.*
If there is at least one eigenvalue of A with a positive real part, then a is unstable.

Example 4.1.2 *Consider the system of a pendulum with friction*

$$\begin{cases} \dot{x} = y \\ \dot{y} = -ry - \frac{g}{L}\sin(x) \end{cases} \tag{4.5}$$

with equilibrium points at $(n\pi, 0)$ for any integer n. The Jacobian matrix at the point $(n\pi, 0)$ is

$$\begin{pmatrix} 0 & 1 \\ \frac{g}{L}(-1)^{n+1} & -r \end{pmatrix}$$

with the eigenvalues

$$\lambda_{1,2} = \frac{-r \pm \sqrt{r^2 + (-1)^{n+1} 4g/L}}{2}.$$

If n is even, then both eigenvalues have negative real parts, and the equilibrium point is locally asymptotically stable.
If n is odd, then the two eigenvalues are real with opposite signs

$$\lambda_1 = \frac{-r - \sqrt{r^2 + 4g/L}}{2} < 0 < \lambda_2 = \frac{-r + \sqrt{r^2 + 4g/L}}{2}$$

which means the equilibrium point is a saddle point (unstable). Figure (4.2) illustrates the corresponding phase portrait.

4.2 Direct Method (Lyapunov Function)

The local stability of a hyperbolic equilibrium point a of (4.1) is clearly determined by the signs of the real parts of the eigenvalues of the Jacobian matrix $Df(a)$.

The stability of a non-hyperbolic equilibrium point is typically more difficult to determine.

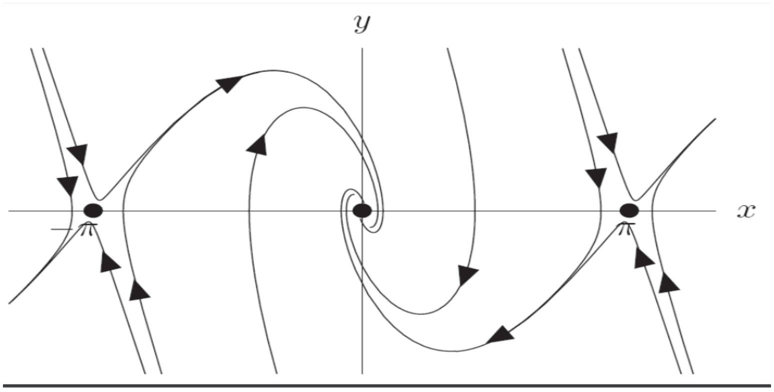

Figure 4.2: Phase portrait of the pendulum with friction

In this section, we present the second Lyapunov method, which is very useful for determining the stability of an equilibrium point.

Definition 4.2.1 *If $V : \mathbb{R}^n \to \mathbb{R}$ has partial derivatives with respect to each component of x, then we define the gradient of V as the vector function*

$$grad(V(x)) = \left[\frac{\partial V}{\partial x_1}(x)\ \frac{\partial V}{\partial x_2}(x)\ \dots\ \frac{\partial V}{\partial x_n}(x)\right].$$

Definition 4.2.2 *Let a be an equilibrium point of (4.1). A continuously differentiable function V defined on an open set $U \subset \mathbb{R}^n$ containing a is called a Lyapunov function for the system (4.1) on U if*

- $V(a) = 0,$
- $V(x) > 0$ *for* $x \in U \setminus \{a\}$, *and*
- $$grad(V(x)).f(x) \leq 0 \tag{4.6}$$

If the inequality (4.6) is strict for $x \in U \setminus \{a\}$, then V is called a strict Lyapunov function for the system (4.1) on U.

Note that (4.6) implies that if $x \in U$, then

$$\frac{d}{dt}V(\phi(t,x)) = \text{grad}(V(\phi(t,x))) \cdot f(\phi(t,x)) \leq 0$$

along the trajectory $\phi(t,x)$ in U, This signifies that V decreases along the orbits residing in U.

Theorem 4.2.1 *If V is a Lyapunov function for the system (4.1) in the open set U containing the equilibrium point a, then a is stable.*
If V is a strict Lyapunov function, then a is asymptotically stable.

Proof. [4]

. Suppose V is a Lyapunov function for the system (4.1) in the open set U containing the equilibrium point a.

Fix $r > 0$ sufficiently small such that $B(a, r) \subset U$, and define

$$m = \min\{V(x) : |x - a| = r\} > 0$$

Then,

$$W = \left\{x : V(x) < \frac{m}{2}\right\} \cap B(a, r)$$

is an open set containing a.Choose $s > 0$ such that $B(a, s) \subset W$.

For $x \in B(a, s)$, we have

$$V(\phi(t, x)) < \frac{m}{2},$$

so that $\phi(t, x)$ stays within W because $V(\phi(t, x))$ is decreasing.

Thus, $\phi(t, x)$ cannot cross the boundary of $B(a, r)$ for $t \geq 0$, meaning $\phi(t, x)$ remains within $B(a, r)$ for $t \geq 0$, and a is stable.

Now suppose V is a strict Lyapunov function for the system (4.1) in the open set U containing the equilibrium point a, but a is not asymptotically stable.

Then there exists $x \in B(a, s)$ such that $\phi(t, x)$ does not approach a as $t \to \infty$.

Since the orbit is bounded, there exists $x_1 \neq a$ and a sequence $t_k \to \infty$ such that $\phi(t_k, x) \to x_1$ as $k \to \infty$.

By the property of semigroups (Theorem 4.0.4), we have

$$\phi(t_k + 1, x) = \phi(1, \phi(t_k, x))$$

and since $\phi(t, x)$ is continuous with respect to x (Theorem **??**), we have

$$V(\phi(t_k + 1, x)) = V(\phi(1, \phi(t_k, x))) \to V(\phi(1, x_1)) < V(x_1),$$

so there exists an integer N such that

$$V(\phi(t_N + 1, x)) < V(x_1).$$

Choose k such that $t_k > t_N + 1$. Thus,

$$V(x_1) \leq V(\phi(t_k, x)) < V(\phi(t_N + 1, x)),$$

which is a contradiction. Therefore, a is asymptotically stable. ■

Example 4.2.1 *Consider the following system:*

$$\begin{cases} \dot{x} = -x - xy^2 \\ \dot{y} = -y + 3x^2 y \end{cases} \tag{4.7}$$

The origin is an equilibrium point for this system.

We propose the simple function $V(x, y) = ax^2 + by^2$, where a and b are two positive real numbers to be determined.

We have $V(0, 0) = 0$ and $V(x, y) > 0$ if $(x, y) \neq (0, 0)$.

Also, $\nabla V(x) \cdot f(x) = -2ax^2 - 2ax^2y^2 - 2by^2 + 6bx^2y^2$.

Lets choose $a = 3$ and $b = 1$ to eliminate two terms. In this case,

$$\nabla V(x) \cdot f(x) = -6x^2 - 2y^2 < 0, \quad \text{if } (x, y) \neq (0, 0).$$

Thus, $V(x, y) = 3x^2 + y^2$ is a strict Lyapunov function for the system (4.7) in $\mathbb{R}^2$, and the equilibrium point $(0, 0)$ is asymptotically stable.

4.3 Hyperbolicity and Structural Stability

Flows near hyperbolic fixed points exhibit distinctive dynamical properties collectively referred to as hyperbolicity. This behavior is encapsulated by two key theorems. First, the Hartman-Grobman theorem (4.1.1) establishes a topological conjugacy between nonlinear flows and their linearizations near hyperbolic points, ensuring the preservation of time orientation through a continuous invertible mapping. Second, the stable manifold theorem guarantees that nonlinear flows retain the same local manifold structure as their linear counterparts, with well-defined stable (W^s_{loc}) and unstable (W^u_{loc}) manifolds intersecting transversally at the fixed point. These manifolds are formally defined as the sets of points whose trajectories asymptotically approach the fixed point under forward (stable) or backward (unstable) time evolution within some neighborhood. Together, these theorems demonstrate how hyperbolic structure persists under nonlinear perturbations, providing a robust foundation for understanding the local dynamics near such points.

4.3.1 Stable Manifold

Definition 4.3.1 *(Local Stable Manifold):*

Let U be a neighborhood of a hyperbolic fixed point x^*. The local stable manifold, denoted by $W^s_{loc}(x^*)$, is defined as:

$$W^s_{loc}(x^*) = \left\{ x \in U \,\middle|\, \phi_t(x) \to x^* \text{ as } t \to \infty,\ \phi_t(x) \in U \ \forall t \geq 0 \right\}.$$

Definition 4.3.2 *(Local Unstable Manifold):*
The local unstable manifold, denoted by $W^u_{loc}(x^*)$, is defined as:

$$W^u_{loc}(x^*) = \left\{x \in U \,\middle|\, \phi_t(x) \to x^* \text{ as } t \to -\infty,\ \phi_t(x) \in U\, \forall t \leq 0\right\}.$$

Definition 4.3.3 *(Stable Manifold Theorem):*
The stable manifold theorem states that the local stable and unstable manifolds exist and have the same dimension as the stable and unstable manifolds of the corresponding linearized system:

$$\dot{x} = Ax,$$

*where x^* is a hyperbolic equilibrium point. Furthermore, these manifolds are tangent to the stable and unstable manifolds**

Definition 4.3.4 ***Hyperbolic Flow***
*If all the eigenvalues of an $n \times n$ matrix A are nonzero, the flow $e^{At} : \mathbb{R}^n \to \mathbb{R}^n$ is called a **hyperbolic flow**, and the linear system $\dot{x} = Ax$ is referred to as a **hyperbolic linear system**.*

Definition 4.3.5 ***Invariant Manifold***
*A subset $D \subseteq \mathbb{R}^n$ is called a C^r **(for** $r \geq 1$**) invariant manifold** if D has the structure of a C^r differentiable manifold. Positively and negatively invariant manifolds are defined similarly. More formally, a subspace $D \subseteq \mathbb{R}^n$ is **invariant** if any flow starting in this subspace remains in it for all future time.*
The linear subspaces E^s, E^u, and E^c are invariant subspaces of the linear system $\dot{x} = Ax$ under the flow e^{At}.

Definition 4.3.6 *Let E be an open subset of $\mathbb{R}^n$, and let $f \in C^1(E)$. Let $\Phi_t : E \to E$ represent the flow of the system $\dot{x} = f(x)$. A set $S \subseteq E$ is called invariant with respect to Φ_t if $\Phi_t(S) \subseteq S$ for all $t \in \mathbb{R}$. Furthermore, S is positively invariant if $\Phi_t(S) \subseteq S$ for $t \geq 0$ and negatively invariant if $\Phi_t(S) \subseteq S$ for $t \leq 0$. If S is invariant with respect to Φ_t, it is both positively and negatively invariant.*

Theorem 4.3.1 (Stable Manifold Theorem) *Let $x^* = 0$ be a **hyperbolic equilibrium point** of the system $\dot{x} = f(x)$, where $f \in C^1$. Let E^s and E^u be the stable and unstable manifolds of the corresponding linear system $\dot{x} = Ax$. Then there exist **local stable** and **unstable manifolds**, denoted by $W^s_{loc}(0)$ and $W^u_{loc}(0)$, of the nonlinear system. These manifolds have the same dimension as E^s and E^u, respectively. Moreover, these manifolds are **tangential to** E^s and E^u at the origin and are as smooth as the function f.*

Let x_0 be a **hyperbolic fixed point** of a nonlinear system. The classification of x_0 is based on the eigenvalues of the corresponding linearized system:

- x_0 is called a **sink** if all eigenvalues have strictly negative real parts.
- x_0 is called a **source** if all eigenvalues have strictly positive real parts.

- If the eigenvalues have real parts of mixed signs (some positive, some negative), x_0 is classified as a **saddle point**.

Example 4.3.1 ***Local Stable and Unstable Manifolds***
Find the local stable and unstable manifolds of the system:

$$\dot{x} = x - y^2, \quad \dot{y} = -y.$$

Solution *The fixed point* (x_0, y_0) *satisfies:* $\dot{x} = 0 \quad \text{and} \quad \dot{y} = 0$. *From* $\dot{y} = -y$, *we get* $y = 0$. *Substituting* $y = 0$ *into* $\dot{x} = x - y^2$, *we get* $x = 0$. *Thus, the fixed point is* $(x_0, y_0) = (0, 0)$.
The system has a unique equilibrium point at the origin, $(0, 0)$. *The origin is a* ***saddle equilibrium point*** *of the corresponding linearized system:*

$$\dot{x} = x, \quad \dot{y} = -y,$$

with the invariant linear stable and unstable subspaces:

$$E^s(0, 0) = \{(x, y) : x = 0\}, \quad E^u(0, 0) = \{(x, y) : y = 0\}.$$

By the ***Stable Manifold Theorem****, the system has local stable and unstable manifolds:*

$$W^s_{loc}(0, 0) = \{(x, y) : x = S(y), \ \frac{\partial S}{\partial y}(0) = 0\},$$

$$W^u_{loc}(0, 0) = \{(x, y) : y = U(x), \ \frac{\partial U}{\partial x}(0) = 0\}.$$

We now find these manifolds explicitly.
Stable Manifold *For the local stable manifold, we expand* $S(y)$ *as a power series in the neighborhood of the origin:*

$$S(y) = \sum_{i \geq 0} s_i y^i = s_0 + s_1 y + s_2 y^2 + s_3 y^3 + \cdots .$$

Since $S(0) = 0$ *and* $\frac{\partial S}{\partial y}(0) = 0$, *we have* $s_0 = s_1 = 0$. *Thus,*

$$x = S(y) = \sum_{i \geq 2} s_i y^i = s_2 y^2 + s_3 y^3 + s_4 y^4 + \cdots .$$

Now, substituting $x = S(y)$ *into* $\dot{x} = x - y^2$, *we get:*

$$\dot{x} = s_2 y^2 + s_3 y^3 + s_4 y^4 + \cdots - y^2 = (s_2 - 1) y^2 + s_3 y^3 + s_4 y^4 + \cdots .$$

On the other hand, from $x = S(y)$, we have:

$$\dot{x} = \frac{\partial S}{\partial y}\dot{y} = \left(2s_2y + 3s_3y^2 + 4s_4y^3 + \cdots\right)(-y).$$

Expanding this gives:

$$\dot{x} = -\left(2s_2y^2 + 3s_3y^3 + 4s_4y^4 + \cdots\right).$$

Equating terms of like powers of y, we get:

$$(s_2 - 1)y^2 + s_3y^3 + s_4y^4 + \cdots = -\left(2s_2y^2 + 3s_3y^3 + 4s_4y^4 + \cdots\right).$$

From the coefficients, we find:

$$s_2 = \frac{1}{3}, \quad s_3 = s_4 = \cdots = 0.$$

Thus:

$$x = S(y) = \frac{1}{3}y^2.$$

Therefore, the local stable manifold of the nonlinear system in the neighborhood of the equilibrium point is:

$$W^s_{loc}(0,0) = \{(x, y) : x = \frac{1}{3}y^2\}.$$

Unstable Manifold *For the local unstable manifold, we expand $U(x)$ as a power series in the neighborhood of the origin:*

$$U(x) = \sum_{i\geq 0} u_i x^i = u_0 + u_1x + u_2x^2 + u_3x^3 + \cdots .$$

Since $U(0) = 0$ and $\frac{\partial U}{\partial x}(0) = 0$, we have $u_0 = u_1 = 0$. Thus,

$$y = U(x) = \sum_{i\geq 2} u_i x^i = u_2x^2 + u_3x^3 + u_4x^4 + \cdots .$$

Now, substituting $y = U(x)$ into $\dot{y} = -y$, we get:

$$\dot{y} = -\left(u_2x^2 + u_3x^3 + u_4x^4 + \cdots\right).$$

On the other hand, from $y = U(x)$, we have:

$$\dot{y} = \frac{\partial U}{\partial x}\dot{x} = \left(2u_2x + 3u_3x^2 + 4u_4x^3 + \cdots\right)(x - y^2).$$

Expanding this and equating coefficients of like powers of x, we find:

$$u_2 = u_3 = u_4 = \cdots = 0.$$

Thus:

$$y = U(x) = 0.$$

Therefore, the local unstable manifold of the nonlinear system in the neighborhood of the origin is:

$$W^u_{loc}(0,0) = \{(x,y) : y = 0\}.$$

4.3.2 Center Manifold

In the previous section, we have seen that near a hyperbolic fixed point, the nonlinear system and its corresponding linear system have the same qualitative features locally. Another important feature is that hyperbolic equilibrium points retain their character under sufficiently small perturbations.

Let the origin be a hyperbolic fixed point of the linear system:

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n.$$

Consider the perturbed system:

$$\dot{x} = f(x) + \varepsilon g(x), \tag{4.11}$$

where $g(x)$ is a smooth vector field defined in $\mathbb{R}^n$, and ε is a sufficiently small perturbation parameter.

The fixed points of (4.11) are given by:

$$f(x) + \varepsilon g(x) = 0.$$

Expanding $f(x)$ and $g(x)$ in Taylor series about $x = 0$, and using $f(0) = 0$, we have:

$$Df(0)x + \varepsilon g(0) + \varepsilon Dg(0)x + O(|x|^2) = 0.$$

Simplifying:

$$[Df(0) + \varepsilon Dg(0)]\, x + \varepsilon g(0) + O(|x|^2) = 0.$$

Since the origin is hyperbolic, the eigenvalues of $Df(0)$ are nonzero. Thus, for sufficiently small ε, the eigenvalues of $Df(0) + \varepsilon Dg(0)$ are also nonzero, ensuring that:

$$\det [Df(0) + \varepsilon Dg(0)] \neq 0.$$

Therefore, $[Df(0) + \varepsilon Dg(0)]^{-1}$ exists, and the fixed points of (4.11) are given by:

$$x = -\varepsilon \left[Df(0) + \varepsilon Dg(0)\right]^{-1} g(0) + O(|x|^2).$$

Since ε is small, the eigenvalues of $Df(x) + \varepsilon Dg(x)$ have nonzero real parts for sufficiently small x. Thus, for sufficiently small ε, the eigenvalues of the perturbed system do not change, and the equilibrium points retain their hyperbolic nature. This proves that the character of a hyperbolic fixed point remains unchanged under small perturbations.

Theorem 4.3.2 (Center Manifold Theorem) *Consider a nonlinear system:*

$$\dot{x} = f(x),$$

where $f \in C^r(E)$, $r \geq 1$, and E is an open subset of $\mathbb{R}^n$ containing a non-hyperbolic fixed point $x = 0$. Suppose that the Jacobian matrix $J = Df(0)$ at the origin has:

- *j eigenvalues with positive real parts,*
- *k eigenvalues with negative real parts,*
- *$m = n - j - k$ eigenvalues with zero real parts.*

Then, there exist:

- *a j-dimensional C^r-class unstable manifold $W^u(0)$,*
- *a k-dimensional C^r-class stable manifold $W^s(0)$,*
- *an m-dimensional C^r-class center manifold $W^c(0)$,*

tangent to the subspaces E^u, E^s, and E^c of the corresponding linear system $\dot{x} = Ax$ at the origin, respectively. These manifolds are invariant under the flow ϕ_t of the nonlinear system. The stable and unstable manifolds are unique, but the local center manifold $W^c(0)$ is not unique.

Example 4.3.2 *Find the manifolds of the system $\dot{x} = x$, $\dot{y} = y^2$*
Solution
The system has a non-hyperbolic fixed point at the origin. The unstable subspace $E^u(0,0)$ of the linearized system at the origin is the x-axis, and the center subspace $E^c(0,0)$ is the y-axis. No stable subspace exists for this system. Using the power series expansion technique discussed in Example 4.13, we find that the unstable manifold at the origin is the x-axis, and the center manifold is the y-axis, i.e., the line $x = 0$. However, other center manifolds also exist.
From the equations, we have:

$$\frac{dy}{dx} = \frac{y^2}{x}.$$

The solution to this equation is:

$$x = ke^{-1/y}, \quad \text{for } y \neq 0.$$

Thus, the center manifold at the origin is:

$$W^c_{loc}(0,0) = \{(x,y) \in \mathbb{R}^2 : x = ke^{-1/y},\ y > 0 \text{ or } x = 0,\ y \leq 0\}.$$

This represents a one-parameter family (k) of center manifolds at the origin. Note that if we use the power series expansion technique for the center manifold, we only obtain $x = 0$ as the center manifold. This example shows that the center manifold is not unique.

Example 4.3.3 *Consider the system*

$$\dot{x} = x, \quad \dot{y} = -x^2 y.$$

We aim to prove that the set

$$S := \{(x,y) \in \mathbb{R}^2 \mid y = -\frac{x^2}{3}\}$$

is invariant with respect to the flow of this system, assuming initial conditions $x(0) = c_1$ and $y(0) = c_2$.

Solution

To solve the system, we first find $x(t)$ by solving $\dot{x} = x$, which leads to $\ln x = t + C$, and thus $x(t) = c_1 e^t$. For $y(t)$, substituting $x(t) = c_1 e^t$ into $\dot{y} = -x^2 y$, we get $\dot{y} = -(c_1 e^t)^2 y$. Dividing and integrating yields

$$\ln|y| = -\frac{c_1^2}{2} e^{2t} + K,$$

so

$$y(t) = K(t) e^{-c_1^2 e^{2t}}.$$

Using the initial condition $y(0) = c_2$, we find $K = c_2 e^{c_1^2/3}$, which gives

$$y(t) = c_2 e^{-c_1^2 e^{2t} + c_1^2/3}.$$

Now, we verify the invariance of S. The set S is defined as $y = -\frac{x^2}{3}$. Substituting $x(t) = c_1 e^t$ into S, we find

$$y = -\frac{(c_1 e^t)^2}{3} = -\frac{c_1^2}{3} e^{2t},$$

which matches the condition for $y(t)$. Thus, the set S remains invariant under the flow of the system.

In conclusion, the set

$$S := \{(x,y) \in \mathbb{R}^2 \mid y = -\frac{x^2}{3}\}$$

is invariant under the flow of the system, and the flow is represented as

$$\Phi_t(c_1, c_2) = \begin{pmatrix} c_1 e^t \\ -\frac{c_1^2}{3} e^{2t} \end{pmatrix}.$$

Example 4.3.4 *Show that the differential equation*

$$\dot{x} = x^2, \quad \ddot{y} = -y \tag{4.8}$$

has infinitely many smooth centre manifolds.
The linearisation of (4.8) *at the origin is*

$$\dot{x} = DX(0)x = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \tag{4.9}$$

The eigenvalues of $DX(0)$ *are* 0 *and* -1*, with eigenvectors along the x- and y-axes, respectively. The Centre Manifold Theorem predicts the existence of a curve invariant under the flow and tangent to the x-axis at* $x = 0$*. The x-axis itself is clearly such a centre manifold because* $y \equiv 0$ *implies* $\dot{y} = 0$*.*

Some examples of the eigenspaces E^u, E^s and E^c for flows on $\mathbb{R}^3$:

(a) $\dot{x} = -x, \dot{y} = 0, \dot{z} = z$: at $x = 0, E^u = z$-axis, $E^s = x$-axis, $E^c = y$-axis;

(b) $\dot{x} = -y, \dot{y} = x, \dot{z} = -z$: $E^u = \{0\}, E^s = z$-axis, $E^c = xy$-plane;

(c) $\dot{x} = y, \dot{y} = -x, \dot{z} = -z$: at $x = 0, E^u = \{0\}, E^s = z$-axis, $E^c = xy$-plane.

4.4 alysis and classification of fixed points in nonlinear dynamical systems: The Linear vs. Nonlinear Relationship

Fixed points, where a system's dynamics remain constant, are crucial to understanding the stability and long-term behaviour of both linear and nonlinear systems. While linearisation provides valuable insights, nonlinear systems can exhibit richer dynamics, including centres, foci and limit cycles, which require deeper analysis. This discussion explores how to classify fixed points, when linear approximations suffice, and where nonlinear methods become essential to bridge theory with practical stability analysis.

Definition 4.4.1 *The origin is called a center for the nonlinear system* (4.1) *if there exists a* $\delta > 0$ *such that every solution curve of (2) in the deleted neighborhood* $N_\delta(0) \setminus \{0\}$ *is a closed curve with 0 in its interior.*

Definition 4.4.2 *The origin is called a center-focus for (2) if there exists a sequence of closed solution curves Γ_n, with Γ_{n+1} in the interior of Γ_n, such that $\Gamma_n \to 0$ as $n \to \infty$ and such that every trajectory between Γ_n and Γ_{n+1} spirals toward Γ_n or Γ_{n+1} as $t \to \pm\infty$.*

Definition 4.4.3 *The origin is called a stable focus for (2) if there exists a $\delta > 0$ such that for $0 < r_0 < \delta$ and $\theta_0 \in \mathbb{R}$, $r(t, r_0, \theta_0) \to 0$ and $|\theta(t, r_0, \theta_0)| \to \infty$ as $t \to \infty$. It is called an unstable focus if $r(t, r_0, \theta_0) \to 0$ and $|\theta(t, r_0, \theta_0)| \to \infty$ as $t \to -\infty$. Any trajectory of (2) which satisfies $r(t) \to 0$ and $|\theta(t)| \to \infty$ as $t \to \pm\infty$ is said to spiral toward the origin as $t \to \pm\infty$.*

Definition 4.4.4 *The origin is called a stable node for (2) if there exists a $\delta > 0$ such that for $0 < r_0 < \delta$ and $\theta_0 \in \mathbb{R}$, $r(t, r_0, \theta_0) \to 0$ as $t \to \infty$ and*

$\lim_{t\to\infty} \theta(t, r_0, \theta_0)$ *exists; i.e., each trajectory in a deleted neighborhood of the origin approaches the origin along a well-defined tange*

The origin is called an unstable node if $r(t, r_0, \theta_0) \to 0$ as $t \to -\infty$ and

$$\lim_{t\to-\infty} \theta(t, r_0, \theta_0) \text{ exists for all } r_0 \in (0, \delta) \text{ and } \theta_0 \in \mathbb{R}.$$

The origin is called a proper node for (2) if it is a node and if every ray through the origin is tangent to some trajectory of (2).

Definition 4.4.5 *The origin is a (topological) saddle for (2) if there exist two trajectories Γ_1 and Γ_2 which approach 0 as $t \to \infty$ and two trajectories Γ_3 and Γ_4 which approach 0 as $t \to -\infty$ and if there exists a $\delta > 0$ such that all other trajectories which start in the deleted neighborhood of the origin $N_\delta(0) \setminus \{0\}$ leave $N_\delta(0)$ as $t \to \pm\infty$. The special trajectories $\Gamma_1, \ldots, \Gamma_4$ are called separatrices.*

Definition 4.4.6 *The origin is a saddle point for the nonlinear system $\dot{x} = f(x)$ if there exist two trajectories Γ_1 and Γ_2 that approach the origin as $t \to +\infty$, and two trajectories Γ_3 and Γ_4 that approach the origin as $t \to -\infty$. These trajectories $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4$ are called **separatrices**.*

Remark 4.4.1 *For a saddle point:*

- *The **stable manifold** is $S = \Gamma_1 \cup \Gamma_2 \cup \{0\}$.*
- *The **unstable manifold** is $U = \Gamma_3 \cup \Gamma_4 \cup \{0\}$.*

Theorem 4.4.1 *Suppose that E is an open subset of $\mathbb{R}^2$ containing the origin and that $f \in C^1(E)$. If the origin is a hyperbolic equilibrium point of the nonlinear system*

$$\dot{x} = f(x),$$

then the origin is a (topological) saddle for the nonlinear system if and only if the origin is a saddle for the linear system

$$\dot{x} = Ax,$$

where $A = Df(0)$.

Theorem 4.4.2 *Let E be an open subset of $\mathbb{R}^2$ containing the origin, and let $f \in C^2(E)$. Suppose that the origin is a hyperbolic critical point of the nonlinear system*

$$\dot{x} = f(x).$$

Then:

- *The origin is a stable (or unstable) node for the nonlinear system if and only if it is a stable (or unstable) node for the linear system*

  $$\dot{x} = Ax,$$

  where $A = Df(0)$.

- *The origin is a stable (or unstable) focus for the nonlinear system if and only if it is a stable (or unstable) focus for the linear system*

  $$\dot{x} = Ax,$$

  where $A = Df(0)$.

Theorem 4.4.3 *Let E be an open subset of $\mathbb{R}^2$ containing the origin, and let $f \in C^1(E)$ with $f(0) = 0$. Suppose that the origin is a center for the linear system*

$$\dot{x} = Ax,$$

where $A = Df(0)$. Then the origin is either:

- *a center,*
- *a center-focus, or*
- *a focus*

for the nonlinear system

$$\dot{x} = f(x).$$

Proof. [6] ■

Corollary 4.4.1 *Let E be an open subset of $\mathbb{R}^2$ containing the origin and let f be analytic in E with $f(0) = 0$. Suppose that the origin is a center for the linear system*

$$\dot{x} = Ax,$$

where $A = Df(0)$. Then the origin is either a center or a focus for the nonlinear system

$$\dot{x} = f(x).$$

Definitions 4.4.1 *The system*

$$\dot{x} = f(x)$$

is said to be symmetric with respect to the x-axis if it is invariant under the transformation $(t, y) \mapsto (-t, -y)$. It is said to be symmetric with respect to the y-axis if it is invariant under the transformation $(t, x) \mapsto (-t, -x)$.

Example 4.4.1

$$\dot{x} = y + xy, \quad \dot{y} = P(x, y)$$

$$\dot{y} = x + y^2, \quad \dot{y} = Q(x, y)$$

Substitute:

$$x'(t) = y(t) + x(t)y(t), \quad y'(t) = x(t) + y^2(t)$$

For the transformation $(t, y) \to (-t, -y)$:

1. *Substitute into the system:*

$$-\dot{-x}(t) = -y(-t) + x(-t)y(-t)$$

$$\dot{-y}(t) = x(-t) + y(-t)^2$$

With the substitution, we get:

$$x'(-t) = y(-t) - x(-t)y(-t)$$

$$y'(-t) = x(-t) + y(-t)^2$$

Finally, the system becomes symmetric with respect to the x-axis.

Theorem 4.4.4 *Let E be an open subset of $\mathbb{R}^2$ containing the origin and let $f \in C^1(E)$ with $f(0) = 0$. If the nonlinear system*

$$\dot{x} = f(x)$$

is symmetric with respect to the x-axis or the y-axis, and if the origin is a center for the linear system

$$\dot{x} = Ax,$$

where $A = Df(0)$, then the origin is a center for the nonlinear system.

Example 4.4.2 ***Showing that*** $(0, 0)$ ***is a Center for the Nonlinear System***

We analyze the system:

$$\begin{cases} x' = -y - x^2 y, \\ y' = x + xy^3. \end{cases}$$

Step 1: Linearization at $(0, 0)$

The Jacobian matrix at $(0, 0)$ *is:*

$$J(0,0) = \begin{pmatrix} \frac{\partial x'}{\partial x} & \frac{\partial x'}{\partial y} \\ \frac{\partial y'}{\partial x} & \frac{\partial y'}{\partial y} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

since:

The eigenvalues of $J(0, 0)$ *are* $\lambda = \pm i$*, so the linearized system has a* ***center*** *at* $(0, 0)$*.*

To confirm that $(0, 0)$ *remains a center for the nonlinear system, we check for* ***reversibility****:*

1. ***Symmetry with respect to the*** x***-axis****:*

$$(t, y) \mapsto (-t, -y)$$

$$\begin{cases} x' &= y + x^2 y, \\ y' &= x - xy^3. \end{cases}$$

   This is ***not identical*** *to the original system.*

2. ***Symmetry with respect to the*** y***-axis****:*

$$(t, x) \mapsto (-t, -x)$$

$$\begin{cases} x' &= -y - x^2 y, \\ y' &= x + xy^3. \end{cases}$$

This is Since:

1. *The linearized system has a center,*

2. *the nonlinear system has a* ***center*** *at* $(0, 0)$*.*

4.5 Blowing-up Techniques on $\mathbb{R}^2$

Blowing-up techniques are a mathematical approach that involves introducing changes of coordinates to expand, or 'blow up,' a non-hyperbolic fixed point (commonly assumed to be at $x = 0$) into a curve along which a number of singularities are distributed. These techniques are particularly useful in analyzing and understanding the behavior of dynamical systems near such fixed points. The topological structure of each singularity on this curve is then investigated, typically using tools like the Hartman-Grobman Theorem.

The coordinate transformations employed in blowing-up are inherently singular at the fixed point because they collapse the curve into a single point. Away from the fixed point, however, these transformations are smooth and can be regarded as diffeomorphisms. A well-known and straightforward example of such a transformation is the use of plane polar coordinates, which offers an intuitive way to analyze the behavior of systems in a radial-angular framework.

Polar Blowing-up (Dumortier, 1978; Guckenheimer & Holmes, 1983)

Consider a differential equation of the form $\dot{x} = X(x)$, $x \in \mathbb{R}^2$. This equation can be elegantly expressed in terms of polar coordinates (r, θ), where $x = (r\cos\theta, r\sin\theta)$. Transforming to polar coordinates serves to desingularize the system by isolating radial and angular components, enabling a clearer study of the dynamics near the fixed point.

In polar coordinates, the system takes the form:

$$\dot{r} = f(r, \theta), \quad \dot{\theta} = g(r, \theta),$$

where f and g are smooth functions that describe the radial and angular dynamics, respectively. This formulation not only simplifies the local analysis near the fixed point but also provides a powerful framework for classifying and understanding the nature of singularities that arise in the system.

Let $x = (x, y)^T$, $f_1(x) = P(x, y)$, and $f_2(x) = Q(x, y)$. The nonlinear system can be expressed as:

$$\begin{aligned} \dot{x} &= P(x, y), \\ \dot{y} &= Q(x, y). \end{aligned}$$

By setting $r^2 = x^2 + y^2$ and $\theta = \tan^{-1}(y/x)$, the system can be reformulated in polar coordinates as:

$$r\dot{r} = x\dot{x} + y\dot{y},$$

and

$$r^2\dot{\theta} = x\dot{y} - y\dot{x}.$$

For $r > 0$, the nonlinear system becomes:

$$\begin{aligned} \dot{r} &= P(r\cos\theta, r\sin\theta)\cos\theta + Q(r\cos\theta, r\sin\theta)\sin\theta, \\ r\dot{\theta} &= Q(r\cos\theta, r\sin\theta)\cos\theta - P(r\cos\theta, r\sin\theta)\sin\theta, \end{aligned}$$

or equivalently:

$$\frac{dr}{d\theta} = \frac{r[P(r\cos\theta, r\sin\theta)\cos\theta + Q(r\cos\theta, r\sin\theta)\sin\theta]}{Q(r\cos\theta, r\sin\theta)\cos\theta - P(r\cos\theta, r\sin\theta)\sin\theta}.$$

The polar blowing-up technique is particularly effective when applied to systems where linearization fails to provide sufficient insight. By converting the system into a polar framework, the method reveals geometric structures and invariant manifolds that are otherwise obscured in Cartesian coordinates. Rewriting the system in polar coordinates often clarifies the nature of equilibrium points, as demonstrated in the following example.

Example 4.5.1 *Use polar blowing-up to find the topological type of the singularity at the origin of the system*

$$\dot{x} = x^2 - 2xy, \quad \dot{y} = y^2 - 2xy.$$

Solution. *In polar coordinates,* (4.5.1) *becomes*

$$\dot{r} = r^2(\cos^3\theta - 2\cos^2\theta\sin\theta - 2\cos\sin^2\theta + \sin^3\theta) = r^2R(r,\theta), \tag{2.8.9}$$

$$\dot{\theta} = 3r\cos\theta\sin\theta(\sin\theta - \cos\theta) = r\Theta(r,\theta).$$

In order to examine the $r = 0$ *circle, observe that (2.8.9) is topologically equivalent to*

$$\dot{r} = rR(r,\theta), \quad \dot{\theta} = \Theta(r,\theta).$$

Setting $r = 0$ *in (2.8.10) gives the flow on the* $r = 0$ *circle shown in Figure*(4.3). *Singularities occur at* $\theta = 0$, π, $\pi/2$, $3\pi/2$, $\pi/4$, *and* $5\pi/4$, *and Hartman s Theorem, applied at each of these points in turn, gives the topological types shown. For example, for* $\theta = 0$ *we have*

$$\begin{pmatrix} \dot{r} \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -3 \end{pmatrix} \begin{pmatrix} r \\ \theta \end{pmatrix}.$$

Thus $(r,\theta) = (0,0)$ *is a saddle point with unstable manifold tangent to the outward radial direction, and so on. In this case, these linearizations are easily done, but they can be rather tedious in some examples.*
Finally, we can contract the $r = 0$ *circle in Figure*(4.3) *onto the origin to obtain the local phase portrait shown in Figure* (4.4).

Example 4.5.2 *Write the system:*

$$\dot{x} = -y - xy,$$
$$\dot{y} = x + x^2,$$

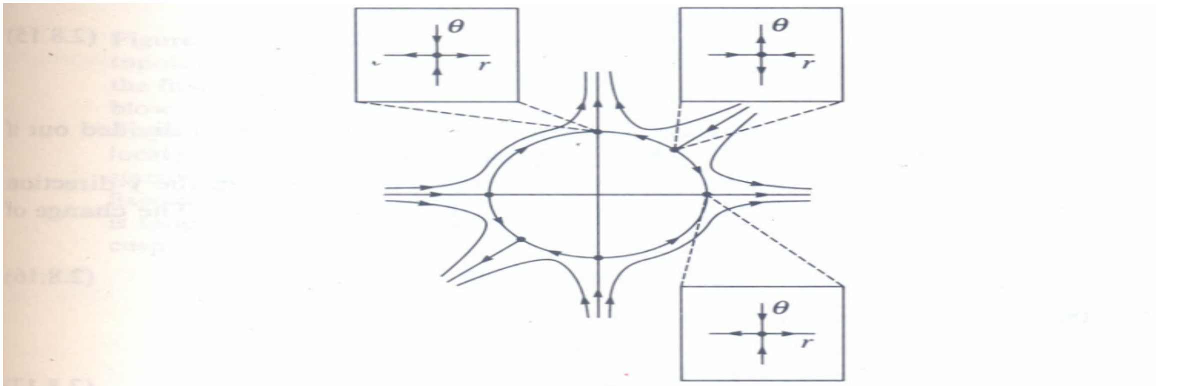

Figure 4.3: The flow on, and near, the $r = 0$-circle for (4.5.1). The topological types of the fixed points are obtained by using HartmanâĂŹs Theorem. Since $\dot{r}$ and $\dot{\theta}$ change sign when $\theta \to \theta - \pi$, it is sufficient to consider only the singularities indicated.

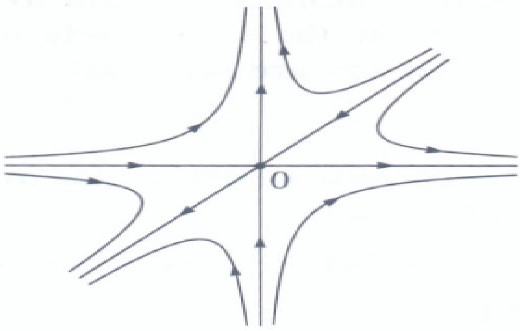

Figure 4.4: Local phase portrait for (4.5.1) at the origin obtained by allowing the radius of the $r = 0$ circle in (4.3) to shrink to zero.

in polar coordinates. For $r > 0$, we have:

$$\dot{r} = \frac{x\dot{x} + y\dot{y}}{r} = \frac{-xy - x^2y + xy + x^2y}{r} = 0,$$

and

$$\dot{\theta} = \frac{x\dot{y} - y\dot{x}}{r^2} = \frac{x^2 + x^3 + y^2 + xy^2}{r^2} = 1 + x > 0,$$

for $x > -1$.

Thus, along any trajectory of this system in the half-plane $x > -1$, $r(t)$ is constant and $\theta(t)$ increases without bound as $t \to \infty$. That is, the origin is called a center for this nonlinear system.

Example 4.5.3 ***Solution***

We are given the system:

$$\dot{x} = -y - x(x^2 + y^2),$$

$$\dot{y} = x - y(x^2 + y^2).$$

In polar coordinates, we define:

$$x = r\cos\theta, \quad y = r\sin\theta, \quad r^2 = x^2 + y^2, \quad \theta = \tan^{-1}\left(\frac{y}{x}\right).$$

The derivatives transform as:

$$r\dot{r} = x\dot{x} + y\dot{y}, \quad r^2\dot{\theta} = x\dot{y} - y\dot{x}.$$

Substitute $x = r\cos\theta$ and $y = r\sin\theta$ into the system:

$$r\dot{r} = x\dot{x} + y\dot{y} = (-y - xr^2)x + (x - yr^2)y.$$

Simplify:

$$r\dot{r} = -xy - x^2r^2 + xy - y^2r^2 = -r^2(x^2 + y^2).$$

Since $r^2 = x^2 + y^2$, we get:

$$\dot{r} = -r^3.$$

$$r^2\dot{\theta} = x\dot{y} - y\dot{x} = x(x - yr^2) - y(-y - xr^2).$$

Simplify:

$$r^2\dot{\theta} = x^2 - xyr^2 + y^2 + xyr^2 = r^2.$$

Thus:

$$\dot{\theta} = 1.$$

The system in polar coordinates becomes:

$$\dot{r} = -r^3,$$
$$\dot{\theta} = 1.$$

This indicates that r decreases over time due to the negative cubic term, while θ increases linearly, meaning trajectories spiral inward toward the origin.
From $\dot{\theta} = 1$:

$$\theta = t + \theta_0.$$

For r:

$$\dot{r} = -r^3 \implies \frac{-dr}{r^3} = dt \implies \frac{1}{2r^2} = t + C.$$

Rearranging for r^2:

$$r^2 = \frac{1}{2t + K}.$$

Applying the initial condition $r(0) = r_0$:

$$r_0^2 = \frac{1}{K} \implies K = \frac{1}{r_0^2}.$$

Substituting K back:

$$r^2 = \frac{1}{2t + \frac{1}{r_0^2}} = \frac{r_0^2}{2tr_0^2 + 1}.$$

Solving for r:

$$r = r_0 \sqrt{\frac{1}{2tr_0^2 + 1}}.$$

Condition for r to exist:

$$2tr_0^2 + 1 > 0 \implies t > -\frac{1}{2r_0^2}.$$

Hence:

$$r \text{ exists if } t \in \left] -\frac{1}{2r_0^2}, +\infty \right[.$$

As $t \to +\infty$:

$$|\theta(t, r_0, \theta_0)| \to +\infty, \quad r(t, r_0, \theta_0) \to 0.$$

$(0, 0)$ *is a **stable focus** for the system.*

Example 4.5.4 *We are given the system:*

$$\dot{x} = -y + x(x^2 + y^2),$$
$$\dot{y} = x + y(x^2 + y^2).$$

In polar coordinates, we define:

$$x = r\cos\theta, \quad y = r\sin\theta, \quad r^2 = x^2 + y^2, \quad \theta = \tan^{-1}\left(\frac{y}{x}\right).$$

The derivatives transform as:

$$r\dot{r} = x\dot{x} + y\dot{y}, \quad r^2\dot{\theta} = x\dot{y} - y\dot{x}.$$

Substitute $x = r\cos\theta$ and $y = r\sin\theta$ into the system:

$$r\dot{r} = x\dot{x} + y\dot{y} = (-y + xr^2)x + (x + yr^2)y.$$

Simplify:

$$r\dot{r} = -xy + x^2r^2 + xy + y^2r^2 = r^2(x^2 + y^2).$$

Since $r^2 = x^2 + y^2$, we get:

$$\dot{r} = r^3.$$

$$r^2\dot{\theta} = x\dot{y} - y\dot{x} = x(x + yr^2) - y(-y + xr^2).$$

Simplify:

$$r^2\dot{\theta} = x^2 + xyr^2 + y^2 + xyr^2 = r^2 + 2xyr^2.$$

Thus:

$$\dot{\theta} = 1 + 2xy.$$

The system in polar coordinates becomes:

$$\dot{r} = r^3,$$

$$\dot{\theta} = 1.$$

From $\dot{r} = r^3$*:*

$$\frac{dr}{r^3} = dt \implies -\frac{1}{2r^2} = t + C.$$

Rearranging for r^2*:*

$$r^2 = \frac{1}{-2t + K}.$$

Applying the initial condition $r(0) = r_0$*:*

$$r_0^2 = \frac{1}{K} \implies K = \frac{1}{r_0^2}.$$

Substituting K back:

$$r^2 = \frac{1}{-2t + \frac{1}{r_0^2}} = \frac{r_0^2}{-2tr_0^2 + 1}.$$

Solving for r:

$$r = r_0\sqrt{\frac{1}{-2tr_0^2 + 1}}.$$

Condition for r to exist:

$$-2tr_0^2 + 1 > 0 \implies t < \frac{1}{2r_0^2}.$$

Hence:

$$r \textit{ exists if } t \in \left]-\infty, \frac{1}{2r_0^2}\right[.$$

$$\theta(t) = t + \theta_0,$$

$$\frac{dr}{r^3} = -dt \implies -\frac{1}{2r^2} = t + C \implies r^2 = \frac{1}{2t + K}.$$

From the initial condition $r(0) = r_0$*:*

$$r_0^2 = \frac{1}{K} \implies K = \frac{1}{r_0^2} \implies r^2 = \frac{1}{2t + \frac{1}{r_0^2}} = \frac{r_0^2}{1 + 2tr_0^2}.$$

Thus:

$$r = \frac{r_0}{\sqrt{1 - 2tr_0^2}}.$$

r exists if:

$$t \in \left]-\infty, \frac{1}{2r_0^2}\right[.$$

If $t \to \frac{1}{2r_0^2}$*:*

$$|\theta(t, r_0, \theta_0)| \to +\infty, \quad r(t, r_0, \theta_0) \to 0.$$

Therefore, $(0, 0)$ *is an unstable focus for the system.*

4.6 Examples of applications

Consider the following ecological model, which represents the interaction between two species: a predator and a prey. Let $x(t)$ be the population of prey at time t, and $y(t)$ be the population of predators at time t.

The system of equations governing the interaction between these two species is given by:

$$\begin{cases} \dot{x} = \alpha x - \beta xy \\ \dot{y} = \delta xy - \gamma y \end{cases} \tag{4.10}$$

where: - α is the growth rate of the prey, - β is the rate at which predators capture prey, - δ is the rate at which predators reproduce based on the availability of prey, - γ is the natural death rate of the predators.

The system describes how the population of prey and predators change over time due to their interactions. The prey population increases exponentially when predators are absent, while the predator population grows as it consumes prey. However, both populations are limited by their interaction rates.

To study the stability of the system, we can examine the equilibrium points by setting $\dot{x} = 0$ and $\dot{y} = 0$. This gives us the following system:

$$\begin{cases} \alpha x - \beta xy = 0 \\ \delta xy - \gamma y = 0 \end{cases}$$

The nontrivial equilibrium points are found by solving this system, and analyzing the stability of these points can provide insight into the long-term behavior of the ecosystem.

Ecological Model: Predator-Prey System

We are given the following system of differential equations representing a predator-prey model:

$$\begin{cases} \dot{x} = \alpha x - \beta xy \\ \dot{y} = \delta xy - \gamma y \end{cases}$$

Where:

- $x(t)$ is the prey population at time t,
- $y(t)$ is the predator population at time t,
- α is the growth rate of the prey,
- β is the rate at which predators capture prey,
- δ is the rate at which predators reproduce based on prey availability,
- γ is the natural death rate of the predators.

This system represents a typical predator-prey model (Lotka-Volterra equations).
To find the equilibrium points, we set both $\dot{x} = 0$ and $\dot{y} = 0$. This corresponds to situations where both the prey and predator populations stop changing, i.e., their populations are constant over time.

$$\begin{cases} \alpha x - \beta xy = 0 \\ \delta xy - \gamma y = 0 \end{cases}$$

Equation 1: $\alpha x - \beta xy = 0$
This equation can be factored as:

$$x(\alpha - \beta y) = 0$$

So, either: 1. $x = 0$ (no prey), or 2. $\alpha - \beta y = 0$, which simplifies to $y = \frac{\alpha}{\beta}$.

Equation 2: $\delta xy - \gamma y = 0$

This equation can be factored as:

$$y(\delta x - \gamma) = 0$$

So, either: 1. $y = 0$ (no predators), or 2. $\delta x = \gamma$, which simplifies to $x = \frac{\gamma}{\delta}$.

Now we combine the results from both equations:

1. Case 1: $x = 0$ and $y = 0$ - The first equilibrium point is $(x, y) = (0, 0)$, which corresponds to the scenario where both the prey and predator populations are extinct.

2. Case 2: $x = \frac{\gamma}{\delta}$ and $y = \frac{\alpha}{\beta}$ - The second equilibrium point is $\left(\frac{\gamma}{\delta}, \frac{\alpha}{\beta}\right)$, which corresponds to a stable population of both prey and predators, where the prey population is $\frac{\gamma}{\delta}$ and the predator population is $\frac{\alpha}{\beta}$.

Thus, the equilibrium points are: - $(0, 0)$ âĂŤ both species extinct, - $\left(\frac{\gamma}{\delta}, \frac{\alpha}{\beta}\right)$ âĂŤ a non-zero equilibrium where both species coexist.

To determine the stability of the equilibrium points, we can linearize the system around each equilibrium point and examine the eigenvalues of the Jacobian matrix.

The Jacobian matrix for the system is given by:

$$J(x, y) = \begin{pmatrix} \frac{\partial}{\partial x}(\alpha x - \beta xy) & \frac{\partial}{\partial y}(\alpha x - \beta xy) \\ \frac{\partial}{\partial x}(\delta xy - \gamma y) & \frac{\partial}{\partial y}(\delta xy - \gamma y) \end{pmatrix}$$

Let's compute the partial derivatives: - $\frac{\partial}{\partial x}(\alpha x - \beta xy) = \alpha - \beta y$ - $\frac{\partial}{\partial y}(\alpha x - \beta xy) = -\beta x$ - $\frac{\partial}{\partial x}(\delta xy - \gamma y) = \delta y$ - $\frac{\partial}{\partial y}(\delta xy - \gamma y) = \delta x - \gamma$

So, the Jacobian matrix is:

$$J(x, y) = \begin{pmatrix} \alpha - \beta y & -\beta x \\ \delta y & \delta x - \gamma \end{pmatrix}$$

Case 1: At $(0, 0)$

Substitute $x = 0$ and $y = 0$ into the Jacobian matrix:

$$J(0, 0) = \begin{pmatrix} \alpha & 0 \\ 0 & -\gamma \end{pmatrix}$$

The eigenvalues of this matrix are α and $-\gamma$. Since $\alpha > 0$ (prey grow) and $\gamma > 0$ (predators die), we have one positive eigenvalue and one negative eigenvalue. This indicates that the equilibrium point $(0, 0)$ is a **saddle point** and is **unstable**.

Case 2: At $\left(\frac{\gamma}{\delta}, \frac{\alpha}{\beta}\right)$

Substitute $x = \frac{\gamma}{\delta}$ and $y = \frac{\alpha}{\beta}$ into the Jacobian matrix:

$$J\left(\frac{\gamma}{\delta}, \frac{\alpha}{\beta}\right) = \begin{pmatrix} \alpha - \beta \cdot \frac{\alpha}{\beta} & -\beta \cdot \frac{\gamma}{\delta} \\ \delta \cdot \frac{\alpha}{\beta} & \delta \cdot \frac{\gamma}{\delta} - \gamma \end{pmatrix}$$

Simplifying:

$$J\left(\frac{\gamma}{\delta}, \frac{\alpha}{\beta}\right) = \begin{pmatrix} 0 & -\frac{\beta\gamma}{\delta} \\ \frac{\alpha\delta}{\beta} & 0 \end{pmatrix}$$

The eigenvalues of this matrix are the solutions to the characteristic equation:

$$\lambda^2 + \left(\frac{\alpha\gamma}{\beta\delta}\right) = 0$$

This gives purely imaginary eigenvalues $\lambda = \pm i\sqrt{\frac{\alpha\gamma}{\beta\delta}}$. Therefore, the equilibrium point $\left(\frac{\gamma}{\delta}, \frac{\alpha}{\beta}\right)$ is a **center** and the population oscillates around this point, indicating **neutral stability**.

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