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Course of Master 1 (first year) Fundamental and Applied Mathematics

DISCRETE DYNAMICAL SYSTEM

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CONTENTS

INTRODUCTION

This comprehensive study reveals the mathematical structure of discrete-time dynamical systems, where simple recursive rules of the form $\boxed{x_{n+1} = f(x_n)}$ give rise to surprisingly rich and complex behaviors.

Our exploration begins with the fundamental bridge between continuous and discrete systems (Section 1.1), where we develop the conceptual framework and precise terminology (Section 1.2) that underpin our entire investigation. Through careful analysis of system orbits (Section 1.3), we discover the ordered beauty of fixed points and periodic solutions (Section 1.4), while the powerful concept of topological equivalence (also in Section 1.4) reveals hidden connections between seemingly different systems.

We illustrate this using vivid graphical methods (Section 1.6), where one-dimensional cobweb plots (Section 1.6.1) and two-dimensional phase portraits (Section 1.6.2) transform abstract concepts into visual intuition.

Chapter 2: One-Dimensional Systems

Chapter 2 focuses intensively on one-dimensional systems, where we develop a complete analytical toolkit:

- Fixed point characterization (Section 2.1)
- Graphical iteration techniques (Sections 2.2, 2.3)
- Stability criteria (Section 2.4), distinguishing:
  - Hyperbolic cases (Section 2.4.1)
  - Non-hyperbolic cases (Section 2.4.2)
- Periodic points and their structure (Section 2.5)

Chapter 3: Two-Dimensional Systems

In Chapter 3, we transition to **two-dimensional discrete dynamical systems**, where the mathematical framework becomes significantly more intricate and powerful. These systems exhibit rich behaviors such as spirals, saddles, and centers, analyzed using matrix dynamics and eigenvalue techniques.

Section 3.1 explores linear systems via matrix iterations $\mathbf{x}_{n+1} = A\mathbf{x}_n$, including:

- Linear maps vs. linear systems (Section 3.1.1)
- Computing A^n via diagonalization and Jordan form (Section 3.1.2)
- Characterization via eigenvalues and eigenvectors (Section 3.1.3)
- Trace-determinant classification (Section 3.1.4)

Section 3.2 introduces nonlinear systems. Then, in **Section 3.3**, we study **stability via linearization** using Jacobian matrices. Finally, **Section 3.4** introduces **Liapunov functions** as a global stability tool, independent of linear approximation.

Chapter 4: Bifurcations and Chaos Onset

Chapter 4 explores **bifurcations**, where small parameter changes produce qualitative system changes. In Section 4.1, we study:

- Saddle-node bifurcations (Section 4.1.1)
- Pitchfork bifurcations (Section 4.1.2)
- Emergence of periodic points (Section 4.1.3)
- Period-doubling route to chaos (Section 4.1.4)

Section 4.2 introduces **Feigenbaum's constants** and their universality. **Section 4.3** explores odd-period orbits and **Sharkovskii's theorem**, while **Section 4.4** generalizes to **higher-dimensional Neimark bifurcations** where invariant circles appear.

Chapter 5: Chaos Theory

Our journey culminates in Chapter 5 with **chaos theory**, where deterministic systems exhibit unpredictable outcomes due to sensitive dependence on initial conditions.

Throughout this intellectual adventure, we maintain a careful balance between rigorous mathematical foundations and practical computational tools, creating a versatile framework applicable to physics, biology, economics, and engineering.

CHAPTER 1

DISCRETE-TIME DYNAMICAL SYSTEMS

Discrete-time modelling can be imposed either by the nature of the process or by the need to 'discretise' a continuous-time model in order to process it numerically. The evolution of the system is observed by choosing certain moments in time, which are assumed to be equidistant. In all cases, the choice of time unit is an important part of modelling the system. In the model, time is therefore denoted by a variable that takes the integer values n = - - - - 2, -1, 0, 1, 2, ... Here is a basic example of a discrete-time dynamic process.

Let us consider the one-dimensional difference equation

$$x(n+1) = f(x(n)) = f^{[n]}(x_0), \quad n = 0, 1, \ldots \tag{1.1}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a given nonlinear function in $x(n)$.

When studying the motion of difference equations, we try to determine equilibrium points and periodic points, analyse their stability and asymptotic stability, and determine aperiodic points and chaotic behaviour. We call equation (1.1) a scalar (or one-dimensional) dynamical system. The function f is called the *map* associated with equation (2.2.1).

A *solution* of equation (1.1) is a sequence $\{x_n\}_{n=0}^{\infty}$ which satisfies the equation for all $n = 0, 1, \ldots$. If an initial condition $x(0) = x_0$ is given, the problem of solving equation (1.1) so that the solution satisfies the initial condition is called the *initial value problem*.

The *general solution* to equation (1.1) is a sequence $\{x_n\}_{n=0}^{\infty}$ which satisfies equation (1.1) for all $n =$

$0, 1, \ldots$ and contains a constant C which can be determined once an initial value is given. A *particular solution* is a sequence $\{x_n\}_{n=0}^{\infty}$ which satisfies equation (1.1) for all $n = 0, 1, \ldots$.

Example 1.0.1 ***Discrete-Time Rabbit Population Model***

Suppose we have a rabbit population that at the start of our experiment has $x(0)$ rabbits. We know that in one year the population increases by 10%. Let $x(n)$ be the number of rabbits in the n-th year.

After one year, we obtain $x(1)$ rabbits:

$$x(1) = x(0) + 0.1x(0) = 1.1x(0)$$

In the second year:

$$x(2) = x(1) + 0.1x(1) = 1.1x(1)$$

Continuing this pattern, we find that for any given year n:

$$x(n + 1) = x(n) + 0.1x(n) = 1.1x(n)$$

Thus we can see that for each time period:

$$x(n + 1) = p(x(n))$$

where

$$p(x) = 1.1x$$

In other words, the population dynamics can be described, as in the previous example, by the iteration of a function $p(x)$. Knowing this function, we can reconstruct the state of the system at any moment in time. The closed-form solution to this recurrence relation is:

$$x(n) = x(0) \cdot (1.1)^n$$

which demonstrates exponential growth of the rabbit population.

1.1 Going from Continuous Time to Discrete Time

There are several techniques for discretizing (sampling) systems. Here is a simple example, often used: **Euler's method**. Consider a first-order differential equation:

$$\dot{x} = f(x)$$

We want to study the trajectory of this equation only at specific, equally spaced time instants $t_n =$

$t_0+n\cdot\Delta$. If the sampling period Δt is chosen to be small enough, the derivative of $x(t)$ can be approximated by the difference:

$$\dot{x} \approx \frac{x(t_n+1) - x(t_n)}{\Delta t}$$

Then, the continuous-time dynamic system can be approximated by the following discrete-time dynamic system:

$$x(n+1) = x(n) + \Delta t \cdot f(x(n))$$

1.2 definitions

In the general case, a discrete dynamical system is described by a system of finite difference equations, in other words, by a recurrence. As in the continuous case, there are several types of systems.

Definition 1.2.1 ***First-Order Discrete Dynamical Systems (DDS) in Dimension m***
Let $D \in \mathbb{R}^m$ be a set and $f : D \to D$ a continuous and differentiable function. The following recurrence is called a first-order discrete dynamical system in dimension m:

$$x(0) = x_0 \in D, \quad x(n+1) = f(x(n)), \quad n \geq 0$$

We will often use the notation (f, D) to denote the dynamical system defined by the function f on the set D.

When the system has multiple state variables, we can represent it in vector form.

Let

$$\vec{x}(n) = \begin{pmatrix} x_1(n) \\ x_2(n) \\ \vdots \\ x_m(n) \end{pmatrix}$$

be the vector of the system's states. The space formed by these states is called the *phase space* of the system.

Let $\vec{f} : \mathbb{R}^m \to \mathbb{R}^m$ be a continuous and differentiable mapping:

$$\vec{f}(\vec{x}) = \begin{pmatrix} f_1(\vec{x}) \\ f_2(\vec{x}) \\ \vdots \\ f_m(\vec{x}) \end{pmatrix}$$

Then the system (f, D) can be written in the form:

$$\vec{x}(0) = \vec{x}_0 \in D, \quad \vec{x}(n+1) = \vec{f}(\vec{x}(n)), \quad n \geq 0$$

Non-Autonomous Discrete Dynamical Systems

If the function $\vec{f}$ depends on the state $\vec{x}$ and the time variable n, then the system is called **non-autonomous**:

$$\vec{x}(0) = \vec{x}_0, \quad \vec{x}(n+1) = \vec{f}(n, \vec{x}(n)), \quad n \geq 0$$

Higher-Order Discrete Dynamical Systems

These systems are described by finite difference equations of order $r \geq 2$, either autonomous or non-autonomous:

$$\vec{x}(n+r) = \vec{f}(\vec{x}(n), \vec{x}(n+1), \ldots, \vec{x}(n+r-1)), \quad n \geq 0 \tag{1.2}$$

There exists a simple procedure that allows transforming any higher-order system into a first-order system. For this, it is enough to define a new *phase space* formed by vectors of the form:

$$\vec{y}(n) = \begin{pmatrix} \vec{x}(n) \\ \vec{x}(n+1) \\ \vdots \\ \vec{x}(n+r-1) \end{pmatrix}$$

The dimension of this space is $m \cdot r$. In this space, we define a mapping $\vec{g} : \mathbb{R}^{m \cdot r} \to \mathbb{R}^{m \cdot r}$ by the formula:

$$\vec{g}(\vec{y}) = \begin{pmatrix} \vec{g}_1(\vec{y}) \\ \vec{g}_2(\vec{y}) \\ \vdots \\ \vec{g}_{r-1}(\vec{y}) \end{pmatrix}$$

where for $k = 1, \ldots, r-1$, each $\vec{g}_k(\vec{y})$ is defined as:

$$\vec{g}_k(\vec{y}) = \begin{pmatrix} y_{k \cdot m+1} \\ y_{k \cdot m+2} \\ \vdots \\ y_{k \cdot m+m} \end{pmatrix}$$

Then, equation (1.2)becomes equivalent to the following first-order equation for $\vec{y}(n)$:

$$\vec{y}(n+1) = \vec{g}(\vec{y}(n))$$

In certain cases (especially linear ones), this transformation allows us to apply to higher-order systems the same analysis methods used for first-order systems.

1.3 Notion of the orbit of a system

From now on we will only study first order systems. Our aim is to describe how the states of the system evolve from the initial conditions.

We therefore need to introduce the concept of the *trajectory* or *orbit* of the system.

Let a first-order discrete dynamical system be defined by the iteration of a function $f(x)$:

$$x(0) = x_0, \quad x(n+1) = f(x(n)), \quad n \geq 0 \tag{1.3}$$

Definition 1.3.1 *orbit*

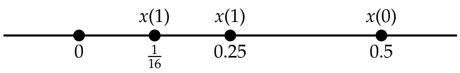
*Given the starting point x_0, the **orbit** (or **trajectory**) of the system (1.3) is the sequence:*

$$\mathcal{O}(x_0) = \{x(0) = x_0,\ x(1) = f(x_0),\ x(2) = f(x(1)),\ \ldots,\ x(n+1) = f(x(n)),\ \ldots\}$$

Example 1.3.1 *Let a first-order discrete dynamical system of dimension 1 be defined by the function $f(x) = x^2$ on the interval $[0, +\infty)$. Let the initial condition be $x_0 = \frac{1}{2}$. The corresponding orbit is*

$$x(0) = x_0 = \frac{1}{2}, \quad x(1) = f(x(0)) = \left(\frac{1}{2}\right)^2 = \frac{1}{4}, \quad x(2) = f(x(1)) = \left(\frac{1}{4}\right)^2 = \frac{1}{16}$$

We observe that



$$x(n) = f(x(n-1)) = f^{(n)}(x_0) = \left(\frac{1}{2}\right)^{2^n} \longrightarrow 0 \quad \textit{as } n \to \infty$$

Let us take a different starting point, $x_0 = 2$. Then

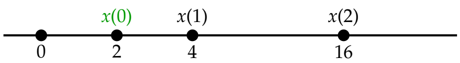
$$x(0) = 2, \quad x(1) = f(x(0)) = 4, \quad x(2) = f(x(1)) = 16$$

In this case, as $n \to \infty$, we have:

$$x(n) = f(x(n-1)) = f^{(n)}(x_0) = 2^{2^n} \longrightarrow \infty$$

Finally, if we choose the starting point $x_0 = 1$, we observe that:

$$\mathcal{O}(x_0) = \{1, 1, x(n) = 1^{2^n} = 1, \ldots\}$$



Remark 1.3.1 *We thus observe three different behaviours of the same system, depending on the chosen starting point. This allows us to speak of the* ***properties of a system*** *by describing all its possible orbits.*

1.4 stationary point,Periodic orbits

Definition 1.4.1 ***stationary point***

A point x^* is called a stationary point of equation (1.1) if

$$x^* = f(x^*). \tag{1.4}$$

Each x^* can be regarded either as a state of the dynamical system

$$x(n+1) = f(x(n))$$

satisfying equation (1.4) , or as a solution to the system of equations

$$x = f(x).$$

We also call x a fixed (or stationary or equilibrium) point of f.

Example 1.4.1 *Every steady state of the system*

$$x(n+1) = ax(n)(1 - x(n)) \tag{1.5}$$

must satisfy the equation

$$x = ax(1 - x). \tag{1.6}$$

We see that $x^* = 0$ is a stationary state regardless of the value of a. Another stationary point is given by solving

$$x = ax(1-x) \Rightarrow ax(1-x) - x = 0 \Rightarrow x(a(1-x)-1) = 0.$$

Set the second factor to zero:

$$a(1-x) - 1 = 0 \Rightarrow a(1-x) = 1 \Rightarrow 1 - x = \frac{1}{a} \Rightarrow x = 1 - \frac{1}{a}.$$

So the second fixed point is

$$x^* = 1 - \frac{1}{a}. \tag{1.7}$$

Definition 1.4.2 ***Periodic orbits***

A point $p \in \mathbb{R}$ is called a periodic point of period k if

$$f^k(p) = p.$$

The point p is called a periodic point of minimal period k (or a prime period k point) if

$$f^k(p) = p,$$

and k is the smallest positive integer for which this holds.

If p is a periodic point, then the set $O(p)$ is called the periodic orbit of p. Orbits that are not periodic are said to be aperiodic.

The choice of "minimal" is motivated by the fact that the orbit cannot be decomposed into smaller loops. A fixed point can be regarded as a periodic point of period 1. Since

$$f^p(x_0) = x_0 \quad \text{implies} \quad f^m(x_0) \neq x_0 \quad \text{for all } 0 < m < p,$$

whenever p is the period of $O(x_0)$, we also have $f^m(x_0) \neq x_0$ for every $m < p$.

Every point

$$x(t), \quad t = 0, 1, \ldots, p-1,$$

of the periodic orbit $O(x_0)$ of period p is periodic of the same period. Thus, $O(x_0)$ contains exactly p distinct periodic points of period p. Sometimes, we use O_p to denote a periodic orbit of period p.

The periodic orbits of period 2 of f are given by the intersections of the graph of $f(f(x))$ with the line $y = x$.

In particular, any **fixed point**, being a periodic point of period $p = 1$, is a periodic point of any period

Example 1.4.2 *Let us consider a one-dimensional system defined by the function (1.5) Here, a is a parameter which we will assume lies in the interval $a \in (0, 4)$.*

Let's investigate whether this system has periodic points of fundamental period 2. These points must be solutions of the equation:

$$f(f(x)) = x \tag{1.8}$$

We must exclude from the start the fixed points (i.e., points of period 1) which are solutions to the equation:

$$ax(1-x) = f(x) = x$$

From this, we deduce that:

$$x \neq 0, \quad x \neq 1 - \frac{1}{a} \tag{1.9}$$

Let's now focus on equation (1.8):

$$a^2x(1-x)(1-ax(1-x)) = x$$

The periodic points we're looking for are therefore the roots of a degree 4 polynomial:

$$a^3x^4 - 2a^3x^3 + a^2(1+a)x^2 - (a^2-1)x = 0$$

We already know two of its roots: these are the two fixed points. To eliminate them and find the remaining two roots more easily, we factor this polynomial by dividing it by the polynomial of the fixed points' equation:

$$a^3x^4 - 2a^3x^3 + a^2(1+a)x^2 - (a^2-1)x = (ax^2 - (a-1)x)(a^2x^2 - (a^2+a)x + a + 1)$$

So the periodic points we are looking for are the real solutions of the equation:

$$a^2x^2 - (a^2+a)x + a + 1 = 0$$

The roots of this polynomial are of the form:

$$x_{1,2} = \frac{a+1}{2a} \pm \frac{1}{2a}\sqrt{(a-3)(a+1)}$$

Therefore, if $a > 3$*, there are two distinct periodic points. They then belong to the same periodic orbit of period 2.*

If $a < 3$*, there are no periodic points.*

Finally, if $a = 3$*, there is only one periodic point, which coincides with one of the fixed points.*
In this example, we can observe a very important phenomenon in the theory of dynamical systems: the change in characteristics of a system depending on the choice of its parameters. We will study this phenomenon later, in the courses to follow.

1.5 Topological Equivalence of Dynamical Systems

In this section, we will define a notion of equivalence between two systems, which is crucial for the study of dynamical systems, particularly for understanding complex behaviors.

Let D and E be two metric spaces, and let $f : D \to D$ and $g : E \to E$ be two maps defining dynamical systems on D and E, respectively.

Definition 1.5.1 *Two dynamical systems (D, f) and (E, g) are said to be* ***topologically conjugate*** *if there exists a homeomorphism (a continuous and bijective map) $h : D \to E$ such that:*

$$h \circ f = g \circ h \tag{1.10}$$

Remark 1.5.1 *The condition (1.10) can be written explicitly as: for all $x \in D$,*

$$h(f(x)) = g(h(x))$$

This equivalence can be represented by the following commutative diagram:

The following theorem demonstrates the importance of this definition.

$$\begin{tikzcd} D \arrow[r, "f"] \arrow[d, "h"'] & D \arrow[d, "h"] \\ E \arrow[r, "g"] & E \end{tikzcd} \quad \textbf{ou} \quad \begin{tikzcd} x \arrow[r, "f"] \arrow[d, "h"'] & f(x) \arrow[d, "h"] \\ h(x) \arrow[r, "\sigma"] & g(h(x)) \end{tikzcd}$$

Theorem 1.5.1 *Let (D, f) and (E, g) be two dynamical systems. Suppose they are topologically conjugate via a homeomorphism $h : D \to E$. Then:*

(a) The inverse map $h^{-1} : E \to D$ also satisfies the definition and thus ensures topological equivalence between (D, f) and (E, g).

(b) $h \circ f^{(n)} = g^{(n)} \circ h$ for all $n \in \mathbb{N}$.

(c) If $p \in D$ is a periodic point of f with fundamental period k, then $h(p) \in E$ is a periodic point of g with fundamental period k.

Remark 1.5.2 *The map $h : D \to E$ simply corresponds to a change of variables that transforms f into g. Indeed, suppose $\{x(n), n = 0, 1, \ldots\}$ is an arbitrary orbit of the system (D, f). If we set for all $n = 0, 1, \ldots,$*

$$y(n) = h(x(n)),$$

then we easily observe that

$$y(n+1) = h(x(n+1)) = h(f(x(n))) = g(h(x(n))) = g(y(n)).$$

This means that the sequence $\{y(n), n = 0, 1, \ldots\}$, the image of $\{x(n), n = 0, 1, \ldots\}$ under h, is an orbit of the system (E, g).

1.6 Graphical study of dynamic systems

In this section, we'll look at some very simple ways of visualizing the behavior of certain systems. These representations will help us better understand the phenomena we're about to study.

1.6.1 1-Dimensional Discrete Dynamical Systems

Consider a 1-dimensional discrete dynamical system (DDS) defined by a function:

$$f : \mathbb{R} \to \mathbb{R}, \quad x(0) = x_0, \quad x(n+1) = f(x(n))$$

The evolution of an orbit $O(x_0)$ can be visualized in the (x, y)-plane using the graph of f and the line $y = x$.

As an example, take the function:

$$f(x) = 4.5x - 3.5x^2$$

We will represent the orbit starting at $x_0 = 0.2$. First, we plot the graph of f and the line $y = x$ (see Figure (1.2)).

In the (x, y)-plane, the orbit begins at the point $(x_0, 0)$. We then draw:

1. A **vertical line** from $(x(0), 0)$ to the graph of f, intersecting at $(x(0), x(1))$ where $x(1) = f(x(0))$.
2. A **horizontal line** from $(x(0), x(1))$ to $(x(1), x(1))$ on the line $y = x$.

From this point, we draw another **vertical line** to the graph of $f(x)$ to find the next point $x(2) = f(x(1))$ (see Figure 1.2). By repeating this process, we can track the orbit's evolution for as many steps as desired.

This graphical representation is particularly useful because:

- It clearly shows **fixed points** (intersections of $f(x)$ and $y = x$).
- It reveals orbit behavior near fixed points (e.g., convergence, divergence, or cycles), as illustrated in Figures (1.1) .

In upcoming lectures, we will use this representation to explore key concepts in 1D dynamical systems theory.

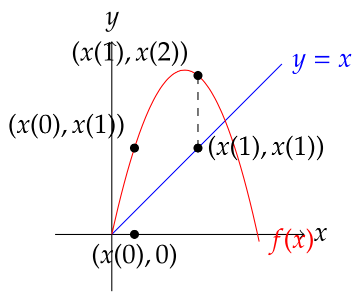


Figure 1.1: Orbit of the system $x(n+1) = 4.5x(n) - 3.5x^2(n)$: Second step.

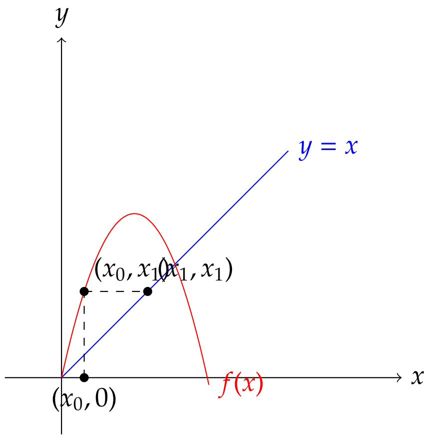


Figure 1.2: Visualization of the orbit $O(x_0)$ for $f(x) = 4.5x - 3.5x^2$.

1.6.2 Two-Dimensional Discrete Dynamical:Phase portraits

A discrete dynamic system of dimension 2 is described by two equations:

$$x_1(n+1) = f_1(x_1(n), x_2(n))$$

$$x_2(n+1) = f_2(x_1(n), x_2(n))$$

To study these systems, phase portraits are often used. To plot the phase portrait of a dynamic system defined by the map $f : \mathbb{R}^2 \to \mathbb{R}^2$, where:

$$f(x_1, x_2) = (f_1(x_1, x_2), f_2(x_1, x_2))$$

a dense grid of points (x_1, x_2) is chosen on the plane, and at each point, the direction of the orbit starting from that point is plotted. This direction for an initial point $\mathbf{x}(0) = (x_1, x_2)$ is defined by the vector:

$$\mathbf{x}(1) - \mathbf{x}(0) = f(\mathbf{x}(0)) - \mathbf{x}(0)$$

This provides an overview of all possible orbits of the system. If we are interested in a specific orbit, we can trace it on the phase portrait by following the direction vectors starting from the initial point of that orbit.

Fixed points of the system can be observed on a phase portrait. These are the points where $f(\mathbf{x}) = \mathbf{x}$. Hence, the direction vector in the phase portrait must be zero at a fixed point. The behavior of orbits around a fixed point is important, and the phase portrait provides a first qualitative analysis of this behavior. In the figure, some orbits starting from points close to the fixed points are shown.

One can also observe periodic orbits on the phase portrait, if the system has any. In this case, closed curves formed by a group of direction vectors can be distinguished.

This provides an overview (see Figure 1.3) of all possible system orbits. To study a particular orbit, we can trace it on the phase portrait by following the vector field directions starting from the orbit's initial point.

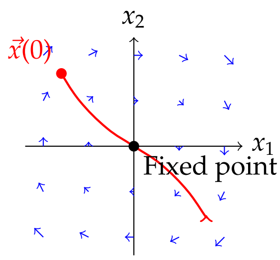


Figure 1.3: Phase portrait showing vector field (blue arrows) and a sample orbit (red curve) starting from $\vec{x}(0)$

Remark 1.6.1 *More details and examples will be discussed in Chapter (3).*

CHAPTER 2

DISCRETE ONE-DIMENSIONAL DYNAMICAL SYSTEMS

The aim of dynamic systems theory is to model processes that evolve over time and to study their behavior. This study must enable us to predict the system's behavior and regulate it to obtain the desired results. To develop a model, we first need to define the values that evolve over time, the system states. Next, we need to find the mathematical equations that describe their evolution. Generally speaking, these are differential equations (if time is considered continuous) or finite-difference equations (if model time is discrete). The parameters of the model are the coefficients of these equations and the initial conditions. In the courses that follow, we will focus on discrete-time dynamical systems. It's important to stress, however, that all the notions we'll be discussing here are also defined in the case of continuous-time systems, and form the basis for the appropriate studies.
A discrete one-dimensional dynamical system is a system subjected to a single equation of this type

$$x(n+1) = f(x(n)) \tag{2.1}$$

where $x \in I \subseteq \mathbb{R}$ and f is a function of x. The variable n is in general considered as the time, but in discrete systems the time takes only discrete values, so that it is possible to take $t \in \mathbb{Z}$.
A trajectory is a set $\{x(n)\}_{n=0}^{\infty}$ of points satisfying the above equation. It is evident that the initial point $x_0 = x(0)$ determines the entire trajectory. The behaviour of the dynamical system is therefore given by all the trajectories $\{x(n) : x(0) = x_0\}$ for all initial values $x_0 \in I$.
A dynamical system depending on a parameter is described by a family $\{f_a\}$ of functions parametrized

by a, where $a \in A \subseteq \mathbb{R}$.

$$x(n+1) = f_a(x(n)) \tag{2.2}$$

2.1 Fixed points

Definition 2.1.1 *Let $\bar{x} \in I$ be a point of the dynamical system 2.1 satisfying $f(\bar{x}) = \bar{x}$. Consider a trajectory starting at $x_0 = \bar{x}$. It is evident that the entire trajectory is formed by the unique point $\bar{x}$, i.e. $x(n) = \bar{x} \forall t \geq 0$. A point $\bar{x}$ satisfying $f(\bar{x}) = \bar{x}$ is called a fixed point or a equilibrium point of the system (2.1).*

Graphically speaking, a fixed point of a map f is a point where the curve $y = f(x)$ intersects the diagonal line $y = x$. For example, the fixed points of the cubic map $f(x) = x^3$ can be obtained by solving the equation $x^3 = x$ or $x^3 - x = 0$. Hence, there are three fixed points -1, 0, 1 for this map (see Fig. (2.1)).

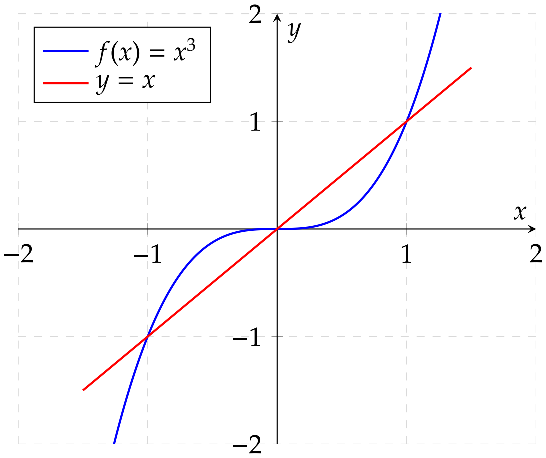


Figure 2.1: Fixed Points of $f(x) = x^3$ and their Geometric Interpretation

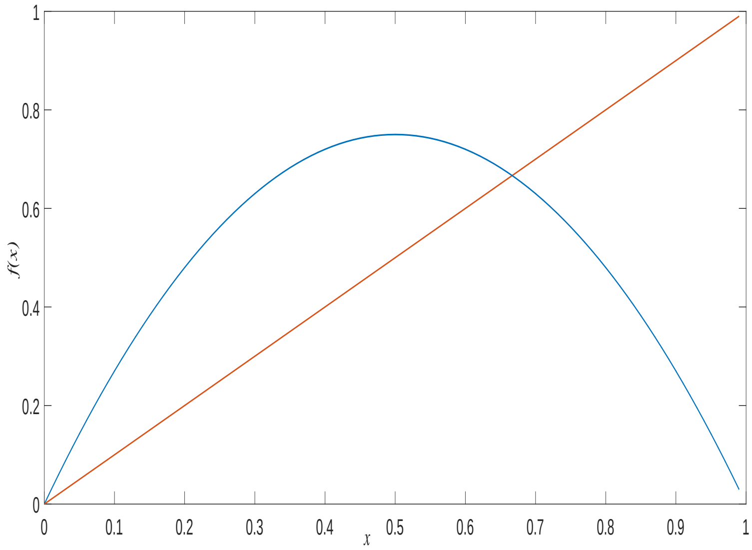


Figure 2.2: The fixed points of the mapping $f(x) = 3x(1 - x)$.

2.2 Graphical study of one-dimensional dynamical systems

We now describe a graphical method for analyzing the trajectories of a dynamical system, known as the Koenigs Lemeray or Cobweb method. Recall that the fixed points of an application f are the abscissas of the points of intersection between the graph of f and the straight line $y = x$. Assume that the parameter a in (2.2) is fixed. The trajectory $\mathcal{O}(x_0)$ can be visualized in the plane by drawing a vertical segment from the point $(x_0, 0)$ to $(x_0, f(x_0)) = (x_0, x_1)$ on the graph of f, then a segment to the line $y = x$ at the point (x_1, x_1), and again to (x_1, x_2) on the graph of f. Continue this process until you have sufficient information about the orbit behavior $\mathcal{O}(x_0)$. The image that provides this information is called **the cobweb diagram** figures (2.3,2.4,2.6, 2.5). For example, when the orbit $\mathcal{O}(x_0)$ converges towards a fixed point x^*, the cobweb diagram starting from the point $(x_0, 0)$ will be a sequence of vertical and horizontal segments spiralling towards (x^*, x^*). This situation is illustrated in figure (2.3), by a cobweb diagram of the logistics application

$$f(x) = 2.9x(1 - x).$$

The fixed point of f is $x^* = \dfrac{19}{29}$ and the initial condition is $x_0 = 0.1$. Translated with DeepL.com (free version) Similarly, if the orbit $\mathcal{O}(x_0)$ converges to a periodic orbit of period p $(z_0, z_1, ..., z_{p-1})$, the spider's

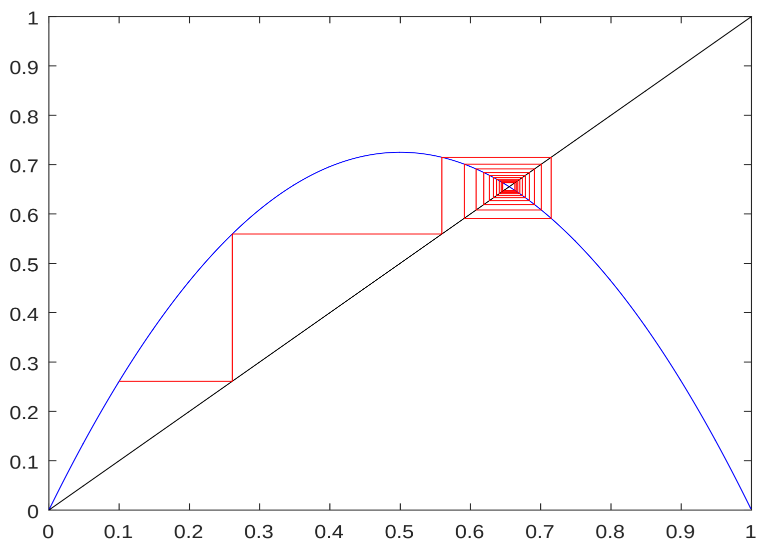


Figure 2.3: The cobweb diagram (or web plot) of the dynamical system defined by $f(x) = 2.9x(1 - x)$.

web diagram starts from $(x_0, 0)$ and approaches the closed cycle $\{(z_0, z_0), (z_0, z_1), ..., (z_{p-1}, z_{p-1}), (z_{p-1}, z_0)$. Figure (2.4) illustrates this situation using the logistic application $f(x) = 3.4x(1 - x)$. In this case $\mathcal{O}(0.1)$ converges to a 2-periodic orbit. Figure (2.5) shows that the trajectory of the dynamical system

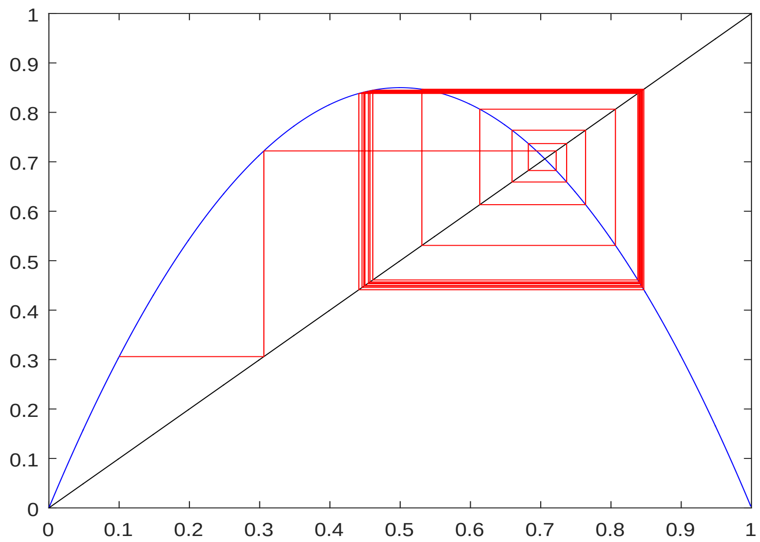


Figure 2.4: The cobweb diagram (or web plot) of the dynamical system defined by $f(x) = 3.4x(1 - x)$.

$$f(x) = 3.4495x(1 - x),$$

converges to a periodic orbit of period 4. Finally, figure (2.6) describes the chaotic behaviour of the trajectory $O(0.1)$ of the dynamical system $f(x) = 4x(1 - x)$.

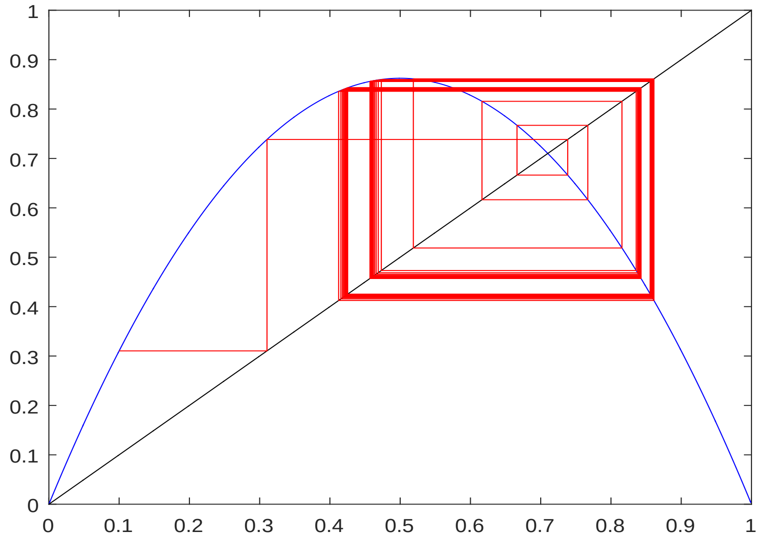


Figure 2.5: The cobweb diagram (or web plot) of the dynamical system defined by $f(x) = 3.4495x(1 - x)$.

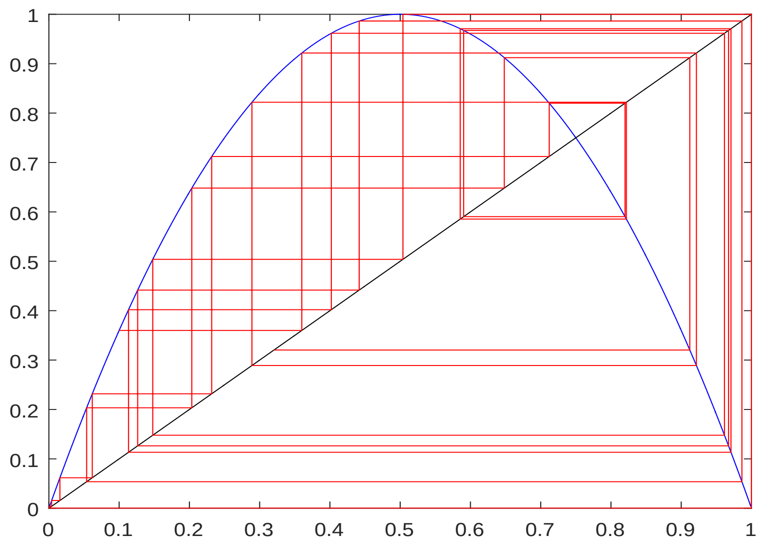


Figure 2.6: The cobweb diagram (or web plot) of the dynamical system defined by $f(x) = 4x(1 - x)$.

2.3 Graphical Iteration and Stability

One of the main objectives in the theory of dynamical systems is the study of the behaviour of orbits near fixed points, i.e. the behaviour of solutions of a difference equation near equilibrium points. Such a programme of investigation is called stability theory, which will be our main focus. We begin our exposition by introducing the basic notions of stability. Let $\mathbb{Z}^+$ denote the set of non-negative integers.

Definition 2.3.1 *The trajectory of the system (2.1) starting at x_0 is the set $\{x_0, f(x_0), f(f(x_0)), \ldots\}$, i.e., the succession of points $\{x(n)\}_{n=0}^{\infty}$ determined by the recurrence 2.1 with the initial condition $x(0) = x_0$.*

We are now interested in the trajectories starting at points which are near $\bar{x}$. In order to better understand the trajectories of the one-dimensional dynamical system we introduce the graphical solution. Consider the graph of the function $f(x)$. The abscissa represents $x(n)$ and the ordinate $x(t+1)$.

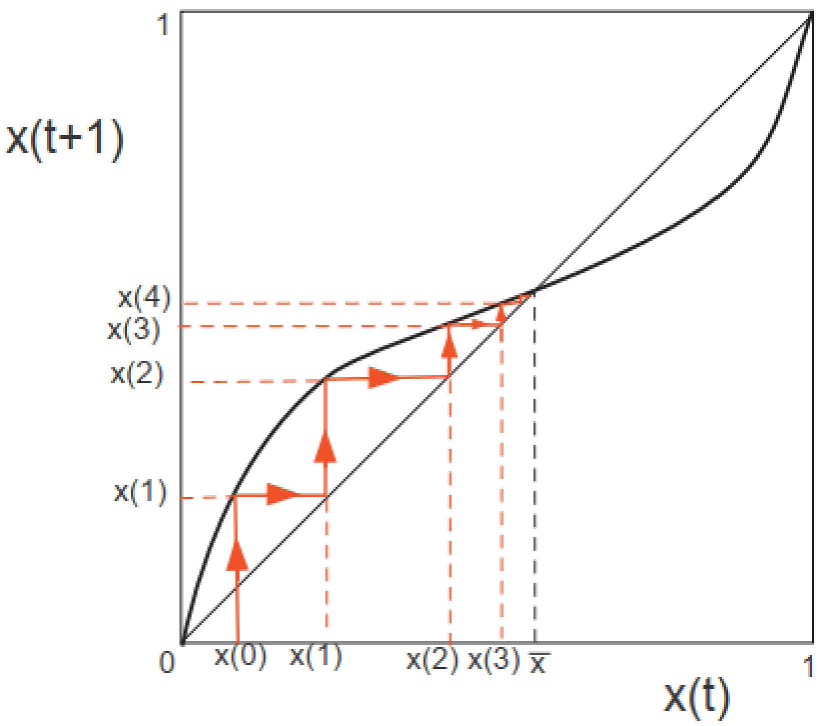


Figure 2.7: Graphical method to obtain a trajectory

Consider now a fixed point $\bar{x}$ and suppose that $f(x)$ be smooth at $\bar{x}$. Then there is a neighbourhood U of the point $\bar{x}$ where all trajectories starting at a point of U remain in U and approach $\bar{x}$ or all trajectories starting at a point of U move away from $\bar{x}$ and exit from U.

In the first case the fixed point is said to be an attracting point or a stable equilibrium point and in the second case a repelling point or an unstable equilibrium point.

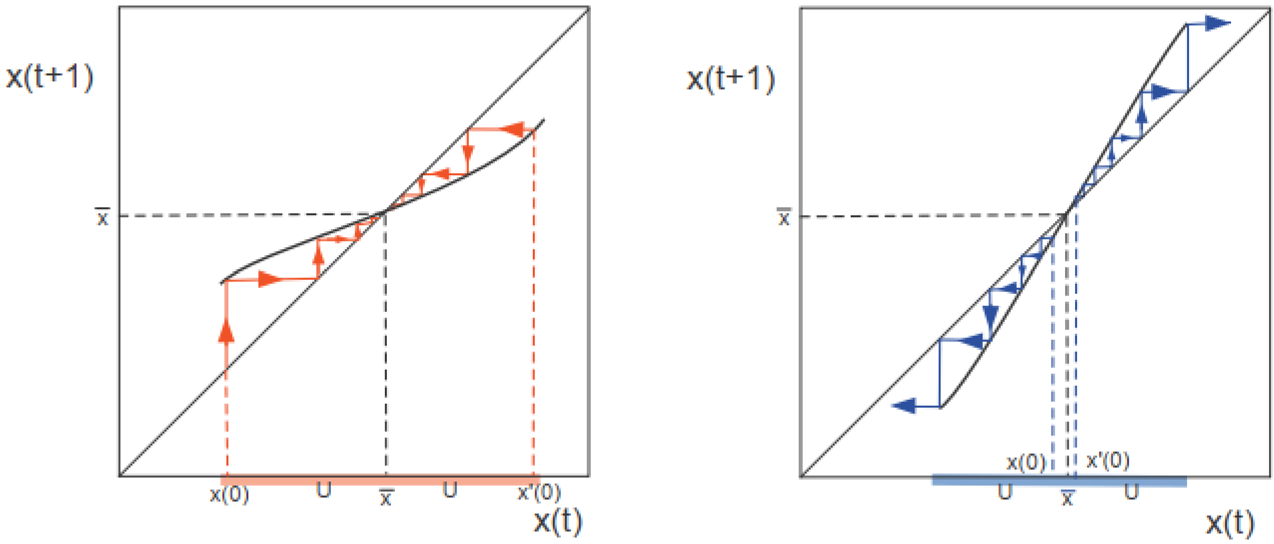


Figure 2.8: Attracting (left) and repelling (right) flxed point

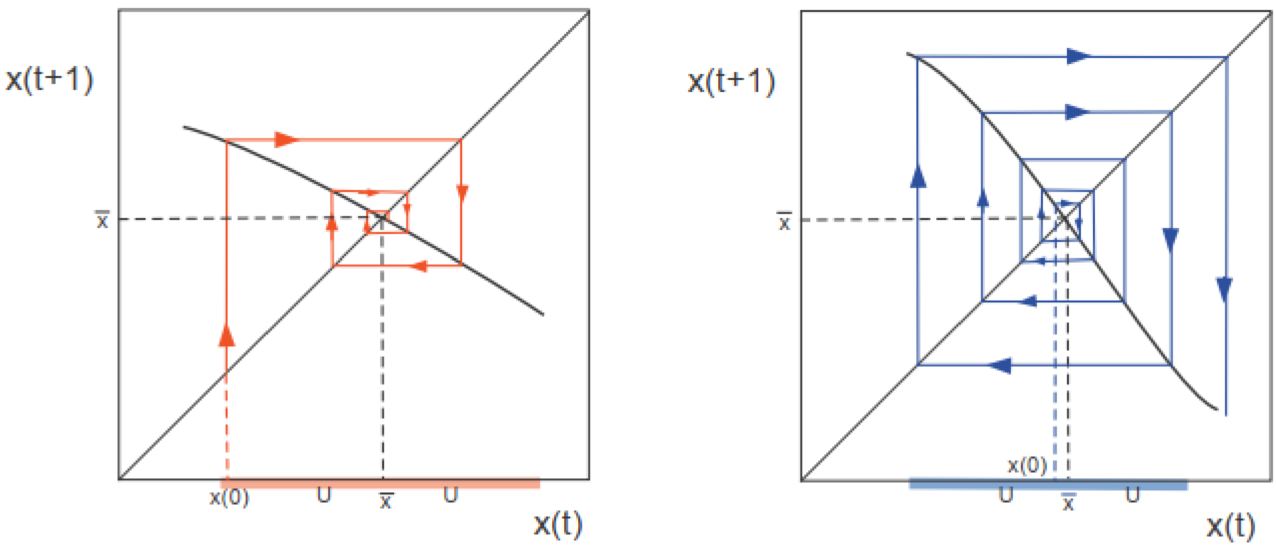


Figure 2.9: Attracting (left) and repelling (right) flxed point

Problem 1. Give a graphical example where a fixed point $\bar{x}$ is neither attracting nor repelling, and f is smooth with $\left|f'(\bar{x})\right| = 1$.

Problem 2. Give a graphical example where a fixed point $\bar{x}$ is neither attracting nor repelling, and f is not smooth at $\bar{x}$.

2.4 Criteria for Stability

Question. Observe Figures (2.8) and (2.9). Which property of f at the fixed point $\bar{x}$ determines whether $\bar{x}$ is attracting or repelling?

In this section,answers the question above we will establish some simple but powerful criteria for local stability of fixed points. Fixed (equilibrium) points may be divided into two types: **hyperbolic** and **nonhyperbolic**. A fixed point x^* of a map f is said to be hyperbolic if $\left|f'(x^*)\right| \neq 1$. Otherwise, it is nonhyperbolic. We will treat the stability of each type separately.

2.4.1 Hyperbolic Fixed Points

The following result is the main tool in detecting local stability.

Theorem 2.4.1 *Let x^* be a hyperbolic fixed point of a map f, where f is continuously differentiable at x^*. The following statements then hold true: 1. If $\left|f'(x^*)\right| < 1$, then x^* is asymptotically stable. 2. If $\left|f'(x^*)\right| > 1$, then x^* is unstable.*

Proof. Suppose that $\left|f'(x^*)\right| < M < 1$ for some $M > 0$. Then, there is an open interval $I = (x^* - \varepsilon, x^* + \varepsilon)$ such that $\left|f'(x)\right| \leq M < 1$ for
all $x \in I$ (Why? Problem 10). By the mean value theorem, for any $x_0 \in I$, there exists c between x_0 and x^* such that

$$\left|f(x_0) - x^*\right| = \left|f(x_0) - f(x^*)\right| = \left|f'(c)\right| |x_0 - x^*| \leq M |x_0 - x^*| . \tag{2.3}$$

Since $M < 1$, inequality (2.3) shows that $f(x_0)$ is closer to x^* than x_0. Consequently, $f(x_0) \in I$. Repeating the above argument on $f(x_0)$ instead of x_0, we can show that

$$\left|f^2(x_0) - x^*\right| \leq M \left|f(x_0) - x^*\right| \leq M^2 |x_0 - x^*| . \tag{2.4}$$

By mathematical induction, we can show that for all $n \in \mathbb{Z}^+$,

$$\left|f^n(x_0) - x^*\right| \leq M^n |x_0 - x^*| . \tag{2.5}$$

To prove the stability of x^*, for any $\varepsilon > 0$, we let $\delta = \min(\varepsilon, \tilde{\varepsilon})$. Then, $|x_0 - x^*| < \delta$ implies that $\left|f^n(x_0) - x^*\right| \leq M^n |x_0 - x^*| < \varepsilon$, which establishes stability. Furthermore, from Inequality (2.5) $\lim_{n\to\infty} \left|f^n(x_0) - x^*\right| = 0$ and thus $\lim_{n\to\infty} f^n(x_0) = x^*$, which yields asymptotic stability. . ■

The following examples illustrate the applicability of the above theorem.

Example 2.4.1 *Consider the map $G_\lambda(x) = 1 - \lambda x^2$ defined on the interval $[-1, 1]$, where $\lambda \in (0, 2]$. Find the fixed points of $G_\lambda(x)$ and determine their stability.*

SOLUTION To find the fixed points of $G_\lambda(x)$ we solve the equation $1-\lambda x^2 = x$ or $\lambda x^2 + x - 1 = 0$. There are two fixed points:

$$x_1^* = \frac{-1-\sqrt{1+4\lambda}}{2\lambda} \text{ and } x_2^* = \frac{-1+\sqrt{1+4\lambda}}{2\lambda}.$$

Observe that $G'_\lambda(x) = -2\lambda x$. Thus, $\left|G'_\lambda\left(x_1^*\right)\right| = 1+\sqrt{1+4\lambda} > 1$, and hence, x_1^* is unstable for all $\lambda \in (0,2]$..
Furthermore, $\left|G'_\lambda\left(x_2^*\right)\right| = \sqrt{1+4\lambda}-1 < 1$ if and only if $\sqrt{1+4\lambda} < 2$.
Solving the latter inequality for λ, we obtain $\lambda < \frac{3}{4}$. This implies by Theorem (2.4.1) that the fixed point x_2^* is asymptotically stable if $0 < \lambda < \frac{3}{4}$ and unstable if $\lambda > \frac{3}{4}$ When $\lambda = \frac{3}{4}, G'_\lambda\left(x_2^*\right) = -1$.
(see Fig. (2.10)).

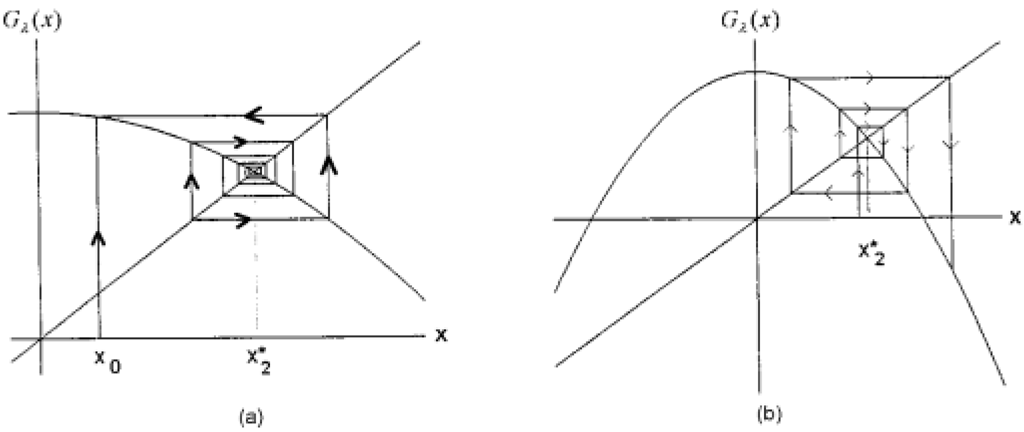


Figure 2.10: (a) $\lambda = \frac{1}{2}$, x_2^* is asymptotically stable while (b) $\lambda = \frac{3}{2}$, x_2^* is unstable.

2.4.2 Nonhyperbolic Fixed Points

The stability criteria for nonhyperbolic fixed points are more involved. They will be summarized in the next two results, the first of which treats the case when $f'(x^*) = 1$ and the second for $f'(x^*) = -1$.

Theorem 2.4.2 *Let x^* be a fixed point of a map f such that $f'(x^*) = 1$. If $f'(x)$, $f''(x)$, and $f'''(x)$ are continuous at x^*, then the following statements hold:*

1. *If $f''(x^*) \neq 0$, then x^* is unstable (semistable).*
2. *If $f''(x^*) = 0$ and $f'''(x^*) > 0$, then x^* is unstable.*
3. *If $f''(x^*) = 0$ and $f'''(x^*) < 0$, then x^* is asymptotically stable.*

Proof. [10] ∎

Example 2.4.2 *Let $f(x) = -x^3 + x$. Then $x^* = 0$ is the only fixed point of f. Note that $f'(0) = 1, f''(0) = 0, f'''(0) < 0$. Hence by Theorem (2.4.2) is asymptotically stable.*
The preceding theorem may be used to establish stability criteria for the case when $f'(x^*) = -1$. But before doing so, we need to introduce the notion of the ***Schwarzian derivative****.*

Definition 2.4.1 *The Schwarzian derivative, Sf, of a function f is defined by*

$$Sf(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left[\frac{f''(x)}{f'(x)}\right]^2 \tag{2.6}$$

And if $f'(x^*) = -1$, then

$$Sf(x^*) = -f'''(x^*) - \frac{3}{2}\left[f''(x^*)\right]^2 \tag{2.7}$$

Theorem 2.4.3 *Let x^* be a fixed point of a map f such that $f'(x^*) = -1$. If $f'(x)$, $f''(x)$, and $f'''(x)$ are continuous at x^*, then the following statements hold:*

1. *If $Sf(x^*) < 0$, then x^* is asymptotically stable.*
2. *If $Sf(x^*) > 0$, then x^* is unstable.*

Proof. The main idea of the proof is to create an associated function g with the property that $g'(x^*) = 1$, so that we can use Theorem (2.4.2). This function is indeed $g = f \circ f = f^2$. Two important facts need to be observed here. First, if x^* is a fixed point of f, then it is also a fixed point of g. Second, if x^* is asymptotically stable (unstable) with respect to g, then it is also asymptotically stable (unstable) with respect to f. By the chain rule:

$$g'(x) = \frac{d}{dx} f(f(x)) = f'(f(x))f'(x) \tag{2.8}$$

Hence,

$$g'(x) = (f''(x^*))^2$$

and Theorem now applies. For this reason we compute $g''(x^*)$. From Equation (2.8) , we have

$$g''(x) = f'(f(x))f''(x) + f''(f(x))(f'(x))^2 \tag{2.9}$$

$$g''(x^*) = f'(x^*)\, f''(x^*) + f''(x^*)\,(f'(x^*))^2$$

$$= 0 \quad (\text{ since } f'(x^*) = -1)\,. \tag{2.10}$$

Computing $g'''(x)$ from Equation (2.10), we get

$$g'''(x^*) = -2f'''(x^*) - 3(f''(x^*))^2 \tag{2.11}$$

It follows from Equation (2.8)

$$g'''(x^*) = 2\,\mathrm{S}f(x^*) \tag{2.12}$$

Statements 1 and 2 now follow immediately from Theorem (2.4.2) ■

Remark 2.4.1 *Note that if $f'(x^*) = -1$ and $g = f \circ f$, then from (2.10) we have*

$$\mathrm{S}f(x^*) = \frac{1}{2}g'''(x^*)\,* \tag{2.13}$$

Furthermore,

$$g''(x^*) = 0 \tag{2.14}$$

We are now ready to give an example of a nonhyperbolic fixed point.

Example 2.4.3 *Consider the map $f(x) = x^2 + 3x$ on the interval $[-3, 3]$. Find the equilibrium points and then determine their stability.*

SOLUTION The fixed points of f are obtained by solving the equation $x^2 + 3x = x$. Thus, there are two fixed points: $x_1^* = 0$ and $x_2^* = -2$. So for x_1^*, $f'(0) = 3$, which implies by Theorem (2.4.3) that x_1^* is unstable.

For x_2^*, we have $f'(-2) = -1$, which requires the employment of Theorem (2.4.3) We observe that

$$\mathrm{S}f(-2) = -f'''(-2) - \frac{3}{2}\left[f''(-2)\right]^2 = -6 < 0$$

Hence, x_2^* is asymptotically stable .

2.5 Periodic Points and their Stability

The notion of periodicity is one of the most important notion in the field of dynamical systems. Its importance stems from the fact that many physical phenomena have certain patterns that repeat themselves. These patterns produce cycles (or periodic cycles), where a cycle is understood to be the orbit of a periodic point. In this section, we address the questions of existence and stability of periodic points.

Definition 2.5.1 *Let $\bar{x}$ be in the domain of a map f. Then,*

1. *$\bar{x}$ is said to be a **periodic point** of f with period k if $f^k(\bar{x}) = \bar{x}$ for some positive integer k. In this case $\bar{x}$ may be called k-periodic. If in addition $f^r(\bar{x}) \neq \bar{x}$ for $0 < r < k$, then k is called the **minimal period** of $\bar{x}$. Note that $\bar{x}$ is k-periodic if it is a fixed point of the map f^k.*

The orbit of a k-periodic point is the set

$$O(\bar{x}) = \left\{\bar{x}, f(\bar{x}), f^2(\bar{x}), \ldots, f^{k-1}(\bar{x})\right\}$$

and is often called a k-periodic cycle. Graphically, a k-periodic point is the x coordinate of a point at which the graph of the map f^k meets the diagonal line $y = x$.
Next we turn our attention to the question of stability of periodic points.

Definition 2.5.2 *Let $\bar{x}$ be a periodic point of f with minimal period k. Then,*

1. *$\bar{x}$ is **stable** if it is a stable fixed point of f^k.*

2. *$\bar{x}$ is **asymptotically stable** if it is an asymptotically stable fixed point of f^k.*

3. *$\bar{x}$ is **unstable** if it is an unstable fixed point of f^k.*

Thus, the study of the stability of k-periodic solutions of the difference equation

$$x(n+1) = f(x(n)) \tag{2.15}$$

reduces to studying the stability of the equilibrium points of the associated difference equation

$$y(n+1) = g(y(n)) \tag{2.16}$$

where $g = f^k$.
The next theorem gives a practical criteria for the stability of periodic points based on Theorem (**??**) in the preceding section.

Theorem 2.5.1 *Let $O(\bar{x}) = \left\{\bar{x}, f(\bar{x}), \ldots, f^{k-1}(\bar{x})\right\}$ be the orbit of the k-periodic point $\bar{x}$, where f is a continuously differentiable function at $\bar{x}$. Then the following statements hold true:*

1. *$\bar{x}$ is asymptotically stable if*

$$\left|f'\left(\bar{x}_1\right) f'\left(f\left(\bar{x}_2\right)\right) \ldots f'\left(f^{k-1}\left(\bar{x}_k\right)\right)\right| < 1 \tag{2.17}$$

2. *$\bar{x}$ is unstable if*

$$\left|f'(\bar{x}) f'(f(\bar{x})) \ldots f'\left(f^{k-1}(\bar{x})\right)\right| > 1 \tag{2.18}$$

Proof. By using the chain rule, we can show that

$$\frac{d}{dx} f^k(\bar{x}) = f'(\bar{x}) f'(f(\bar{x})) \ldots f'\left(f^{k-1}(\bar{x})\right)$$

Conditions (2.17) and (2.18) now follow immediately by application of Theorem (**??**) to the composite map $g = f^k$. ∎

Example 2.5.1 *Consider the difference equation $x(n+1) = f(x(n))$ where $f(x) = 1 - x^2$ is defined on the interval* $[-1, 1]$. *Find all the 2-periodic cycles and determine their stability.*

SOLUTION First, let us calculate the fixed points of f out of the way. Solving the equation $x^2 + x - 1 = 0$, we find that the fixed points of f are $x_1^* = -\frac{1}{2} - \frac{\sqrt{5}}{2}$ and $x_2^* = -\frac{1}{2} + \frac{\sqrt{5}}{2}$. Only x_2^* is in the domain of f. The fixed point x_2^* is unstable. To find the two cycles, we find f^2 and put $f^2(x) = x$. Now, $f^2(x) = 1 - \left(1 - x^2\right)^2 = 2x^2 - x^4$ and $f^2(x) = x$ yields the equation

$$x\left(x^3 - 2x + 1\right) = x(x-1)\left(x^2 + x - 1\right) = 0$$

Hence, we have the 2-periodic cycle $\{0, 1\}$; the other two roots are the fixed points of f. To check the stability of this cycle, we compute $\left|f'(0) f'(1)\right| = 0 < 1$. Hence, by Theorem (2.5.1), the cycle is asymptotically stable .

Example 2.5.2 *Consider the quadratic function:*

$$f(x) = x^2 - 2.$$

To find the fixed points, we solve $f(x) = x$:

$$x^2 - 2 = x \quad \Rightarrow \quad x^2 - x - 2 = 0 \quad \Rightarrow \quad (x-2)(x+1) = 0.$$

Thus, the fixed points are $x = 2$ and $x = -1$.

To find the period-2 points, we solve $f^2(x) = x$, or equivalently $f^2(x) - x = 0$. Since the fixed points are also solutions to this equation, we know $(x-2)(x+1)$ is a factor.
By computation:

$$f^2(x) = f(f(x)) = (x^2-2)^2 - 2 = x^4 - 4x^2 + 2.$$

Then:

$$f^2(x) - x = x^4 - 4x^2 + 2 - x = x^4 - 4x^2 - x + 2.$$

We factor:

$$f^2(x) - x = (x-2)(x+1)(x^2+x-1).$$

Solving the quadratic factor:

$$x = \frac{-1 \pm \sqrt{5}}{2}.$$

So the 2-cycle is:

$$\left\{\frac{-1+\sqrt{5}}{2}, \frac{-1-\sqrt{5}}{2}\right\}.$$

To check the stability, we compute:

$$\left| f'\left(\frac{-1-\sqrt{5}}{2}\right) \cdot f'\left(\frac{-1+\sqrt{5}}{2}\right) \right| = \left| \left(-1-\sqrt{5}\right) \cdot \left(-1+\sqrt{5}\right) \right| = |1-5| = 4.$$

Correction: That evaluates to:

$$|(-1-\sqrt{5})(-1+\sqrt{5})| = |1-5| = 4 > 1.$$

*Therefore, the 2-cycle is **unstable**.*

Example 2.5.3 *Find the period of the point $\frac{1}{8}(5+\sqrt{5})$ for the map*

$$f(x) = 4x(1-x), \quad x \in [0,1].$$

Also determine its stability.

Solution. *Given map is $f(x) = 4x(1-x)$, $x \in [0,1]$. This is a quadratic map. Now,*

$$\begin{aligned} f\left(\frac{1}{8}(5+\sqrt{5})\right) &= 4 \cdot \frac{1}{8}(5+\sqrt{5})\left(1 - \frac{1}{8}(5+\sqrt{5})\right) \\ &= \frac{1}{16}(5+\sqrt{5})(3-\sqrt{5}) = \frac{1}{8}(5-\sqrt{5}), \\ f\left(\frac{1}{8}(5-\sqrt{5})\right) &= 4 \cdot \frac{1}{8}(5-\sqrt{5})\left(1 - \frac{1}{8}(5-\sqrt{5})\right) \\ &= \frac{1}{16}(5-\sqrt{5})(3+\sqrt{5}) = \frac{1}{8}(5+\sqrt{5}). \end{aligned}$$

Again,

$$f^2\left(\frac{1}{8}(5+\sqrt{5})\right)=f\left(f\left(\frac{1}{8}(5+\sqrt{5})\right)\right)=f\left(\frac{1}{8}(5-\sqrt{5})\right)=\frac{1}{8}(5+\sqrt{5}).$$

This shows that the point $\frac{1}{8}(5+\sqrt{5})$ *is a fixed point of the map* f^2 *and hence it is a periodic point of period-2 of the given map. We shall now examine the stability of this periodic-2 point. We have*

$$\begin{aligned} f^2(x) &= f(f(x)) = f(4x(1-x)) = 4\cdot 4x(1-x)\,(1-4x(1-x)) \\ &= 16x-80x^2+128x^3-64x^4. \end{aligned}$$

We shall use the derivative test for finding the stability character of the periodic point of the map. We see that

$$(f^2)'(x)=16-160x+384x^2-256x^3.$$

Since

$$\left|(f^2)'\left(\frac{1}{8}(5+\sqrt{5})\right)\right|=16(244+105\sqrt{5})>1,$$

the periodic-2 point $\frac{1}{8}(5+\sqrt{5})$ *of* f *is* ***unstable****.*

CHAPTER 3

TWO DIMENSIONAL DISCRETE DYNAMICAL SYSTEMS

This chapter presents a comprehensive study of two-dimensional discrete dynamical systems described by difference equations of the form:

$$x_i(t+1) = f_i(x_1(t), x_2(t)), \quad i = 1, 2. \tag{3.1}$$

which can be compactly represented in vector form as:

$$\mathbf{x}(t+1) = \mathbf{f}(\mathbf{x}(t)) \tag{3.2}$$

3.1 linear Discrete Dynamical Systems

3.1.1 Linear Maps vs. Linear Systems

The characterisation of trajectories in two-dimensional first-order autonomous linear systems provides the conceptual basis for generalising the analysis to higher-order, non-autonomous, non-linear dynamical systems.

Consider the following discrete linear dynamical system

$$\begin{cases} x_{1,n+1} = a_{11}x_{1,n} + a_{12}x_{2,n} \\ x_{2,n+1} = a_{21}x_{2,n} + a_{22}x_{2,n} \end{cases}, \qquad -\infty < x_{1.}, x_{2.} < +\infty. \tag{3.3}$$

with initial condition (x_{10}, x_{20}). Where $a_{11}, a_{12}, a_{21}, a_{22}$ are real constants. The system (3.3) can be written in matrix form as

$$\begin{pmatrix} x_{1,n+1} \\ x_{2,n+1} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_{1,n} \\ x_{2,n} \end{pmatrix} = A \begin{pmatrix} x_{1,n} \\ x_{2,n} \end{pmatrix}. \tag{3.4}$$

where $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$. The origin O is the only fixed point of the application f (if $\det(I - A) \neq 0$) defined by

$$f\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} f_1\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ f_2\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} a_{11}x_{1,n} + a_{12}x_{2,n} \\ a_{21}x_{2,n} + a_{22}x_{2,n} \end{pmatrix}.$$

A solution of the linear system (3.4), is a trajectory $\{X_n\}_{n\geq 0}$ où $X_n = \begin{pmatrix} x_{1,n} \\ x_{2,n} \end{pmatrix}$ for any positive integer n. So the value of X_n in periods $1, 2, 3, ..., n$ is

$$\begin{aligned} X_1 &= AX_0, \\ X_2 &= AX_1 = A^2X_0, \\ &. \\ &. \\ &. \\ X_n &= AX_{n-1} = A^nX_0. \end{aligned}$$

Example 3.1.1 *Consider the following two-dimensional discrete dynamic system* $X_{n+1} = AX_n$ *où* $X_n = \begin{pmatrix} x_n \\ y_n \end{pmatrix}$ *and* $A = \begin{pmatrix} 2 & 0 \\ 0 & 0.5 \end{pmatrix}$, *with the initial condition* $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$. *As the matrix A is a diagonal matrix, the evolution of each* *of the state variables x_n, y_n are independent of each other. In addition*

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} 2^n & 0 \\ 0 & (0.5)^n \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix},$$

or

$$x_n = 2^n x_0,$$
$$y_n = (0.5)^n \, y_0.$$

The fixed point of the system is $P = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. *We have $x_n \to +\infty$ at $n \to +\infty$ and $y_n \to 0$ when $n \to +\infty$ The origin is called point* ***saddle (col)****.*

So if the matrix A is a diagonal matrix, then there is no interdependence between the different state variables. The matrix A^n is also a diagonal matrix, and the evolution of each state variable can be analysed separately using the method developed for one-dimensional dynamical systems.

If the matrix A is not a diagonal matrix and there are dependencies in the evolution of the state variables, there are linear algebraic procedures (Jardon's normal form) which allow to transform the system with interdependent state variables into a system with independent or partially independent state variables.

In the next section, we will develop the necessary machinery to compute A^n for any matrix of order two.

3.1.2 Computing A^n

Consider a matrix $A = \left(a_{ij}\right)$ of order 2×2. Then, $p(\lambda) = \det(A - \lambda I)$ is called the characteristic polynomial of A and its zeros are called the eigenvalues of A. Associated with each eigenvalue λ of A a nonzero eigenvector $V \in \mathbb{R}^2$ with $AV = \lambda V$.

Example 3.1.2 *Find the eigenvalues and the eigenvectors of the matrix*

$$A = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$$

SOLUTION First we find the eigenvalues of A by solving the characteristic equation $\det(A - \lambda I) = 0$ or

$$\begin{vmatrix} 2-\lambda & 3 \\ 1 & 4-\lambda \end{vmatrix} = 0$$

which is

$$\lambda^2 - 6\lambda + 5 = 0.$$

Hence, $\lambda_1 = 1$ and $\lambda_2 = 5$. To find the corresponding eigenvector V_1, we solve the vector equation $AV_1 = \lambda V_1$ or $(A - \lambda_1 I)\, V_1 = 0$. For $\lambda_1 = 1$, we have

$$\begin{pmatrix} 1 & 3 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} v_{11} \\ v_{21} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Hence, $v_{11} + 3v_{21} = 0$. Thus, $v_{11} = -3v_{21}$. So, if we let $v_{21} = 1$, then $v_{11} = -3$. It follows that the eigenvector V_1 corresponding to λ_1 is given by $V_1 = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$.

For $\lambda_2 = 5$, the corresponding eigenvector may be found by solving the equation $(A - \lambda_2 I)\, V_2 = 0$. This yields

$$\begin{pmatrix} -3 & 3 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} v_{12} \\ v_{22} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Thus, $-3v_{12} + 3v_{22} = 0$ or $v_{12} = v_{22}$. It is then appropriate to let $v_{12} = v_{22} = 1$ and hence $V_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

To find the general form for A^n for a general matrix A is a formidable task even for a 2×2 matrix such as in Example (3.1.2). Fortunately, however, we may be able to transform a matrix A to another simpler matrix B whose nth power B^n can easily be computed. The essence of this process is captured in the following definition.

Definition 3.1.1 *The matrices A and B are said to be similar if there exists a nonsingular matrix P such that*

$$P^{-1}AP = B.$$

We note here that the relation "similarity" between matrices is an equivalence relation, i.e.,

1. A is similar to A.

2. If A is similar to B then B is similar to A.

3. If A is similar to B and B is similar to C, then A is similar to C.

The most important feature of similar matrices, however, is that they possess the same eigenvalues. that $\det P \neq 0$, *where det denotes determinant.*

Theorem 3.1.1 *Let A and B be two similar matrices. Then A and B have the same eigenvalues.*

Proof. [10]. ∎

Theorem 3.1.2 *Let A be a 2×2 real matrix. Then A is similar to one of the following matrices:*

1. $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$
2. $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$
3. $\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$

Proof. [10] ■ Theorem (3.1.2) gives us a simple method of computing the general form of A^n for any 2×2 real matrix. In the first case, when $P^{-1}AP = D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$, we have

$$\begin{aligned} A^n &= \left(PDP^{-1}\right)^n \\ &= PD^nP^{-1} \\ &= P\begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix}P^{-1}. \end{aligned} \tag{3.5}$$

In the second case, when $P^{-1}AP = J = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$, then

$$\begin{aligned} A^n &= PJ^nP^{-1} \\ &= P\begin{pmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{pmatrix}P^{-1}. \end{aligned} \tag{3.6}$$

Equation (3.6) may be easily proved by mathematical induction. In the third case, we have $P^{-1}AP = J = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$. Let $\omega = \arctan(\beta/\alpha)$. Then $\cos\omega = \alpha/|\lambda_1|, \sin\omega = \beta/|\lambda_1|$. Now, we write the matrix J in the form

$$J = |\lambda_1|\begin{pmatrix} \alpha/|\lambda_1| & \beta/|\lambda_1| \\ -\beta/|\lambda_1| & \alpha/|\lambda_1| \end{pmatrix} = |\lambda_1|\begin{pmatrix} \cos\omega & \sin\omega \\ -\sin\omega & \cos\omega \end{pmatrix}.$$

By mathematical induction one may show that

$$J^n = |\lambda_1|^n\begin{pmatrix} \cos n\omega & \sin n\omega \\ -\sin n\omega & \cos n\omega \end{pmatrix}. \tag{3.7}$$

and thus

$$A^n = |\lambda_1|^n \, P \begin{pmatrix} \cos n\omega & \sin n\omega \\ -\sin n\omega & \cos n\omega \end{pmatrix} P^{-1} \tag{3.8}$$

Example 3.1.3 *Solve the system of difference equations*

$$X(n+1) = AX(n) \tag{3.9}$$

where

$$A = \begin{pmatrix} -4 & 9 \\ -4 & 8 \end{pmatrix}, X(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

SOLUTION The eigenvalues of A are repeated: $\lambda_1 = \lambda_2 = 2$. The only eigenvector that we are able to find is $V_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$. To construct P we need to find a generalized eigenvector V_2. This is accomplished by solving the equation $(A - 2I)V_2 = V_1$. Then, V_2 may be taken as any vector $\begin{pmatrix} x \\ y \end{pmatrix}$, with $3y - 2x = 1$. We take $V_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Now if we put $P = \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}$, then $P^{-1}AP = J = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$. Thus, the solution of Equation (3.9) is given by

$$\begin{aligned} X(n) &= PJ^nP^{-1}x(0) \\ &= \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 2^n & n2^{n-1} \\ 0 & 2^n \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= 2^n \begin{pmatrix} 1-3n \\ -2n \end{pmatrix}. \end{aligned}$$

Remark 3.1.1 *If a map $f : \mathbb{R}^2 \to \mathbb{R}^2$ is given by $f(X_0) = AX_0$, then $f^n(X_0) = A^nX_0 = PJ^nP^{-1}X_0$. In particular, if $X_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, then $f^n(X_0) = 2^n \begin{pmatrix} 1-3n \\ -2 \end{pmatrix}$ for all $n \in \mathbb{Z}^+$.*

3.1.3 Characterization of Solutions to a Linear Dynamical System

To characterize the solutions of the two-dimensional dynamical system

$$X_{n+1} = AX_n, \tag{3.10}$$

with $X \in \mathbb{R}^2$ and A be a 2×2 square matrix. We are initially interested in the system:

$$Y_{n+1} = DY_n, \tag{3.11}$$

With $X = QY$ and $D = Q^{-1}AQ$ (where D and Q are the matrices defined in theorem (3.1.2)), we deduce the characterization of the solutions of system (3.10) using the relation $X = QY$.

To simplify the notation in what follows, we set $Y = \begin{pmatrix} x \\ y \end{pmatrix}$.

♦ **Distinct real eigenvalues**

Consider the dynamical system

$$Y_{n+1} = DY_n, \tag{3.12}$$

whit $D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ and $\lambda_1 \neq \lambda_2 \in \mathbb{R}$. So $Y_n = D^n Y_0$ with $D^n = \begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix}$ *i.e.*

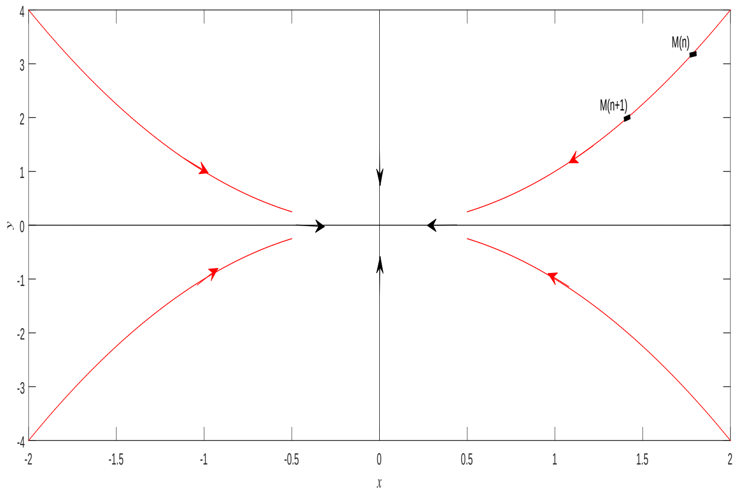
$$\begin{cases} x_n = (\lambda_1)^n x_0, \\ y_n = (\lambda_2)^n y_0, \end{cases} \tag{3.13}$$

with $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ is an initial condition of the vector $\begin{pmatrix} x \\ y \end{pmatrix}$. If $\lambda_1 > 0$ and $\lambda_2 > 0$, By eliminating n between the two relations, we obtain the equation of a family of curves:

$$y = cx^{ln(\lambda_1)/ln(\lambda_2)}, \; c = \text{constant determined by } x_0, y_0, \tag{3.14}$$

which are invariant under the application of the transformation T defined by (3.12), i.e., the change of x to $\lambda_1 x$ and y to $\lambda_2 y$ does not modify (3.14).

⋆ If $0 < \lambda_2 < \lambda_1 < 1$, these invariant curves have a parabolic shape with a common tangent along the ox-axis, and a common asymptotic direction along oy, as shown in figure (3.1).



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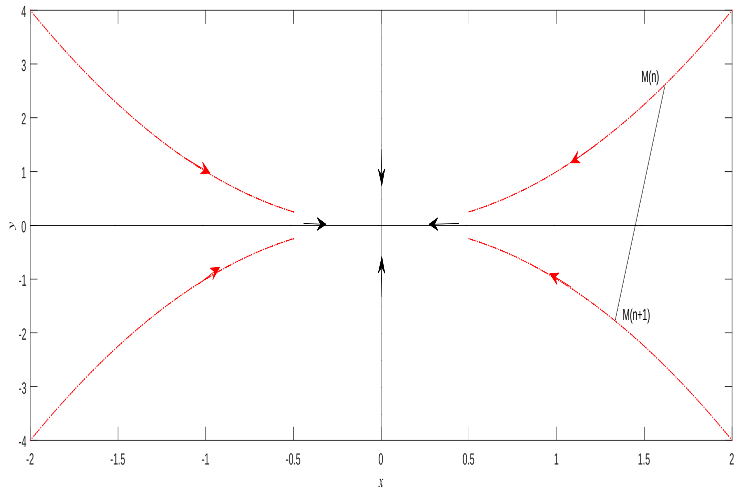

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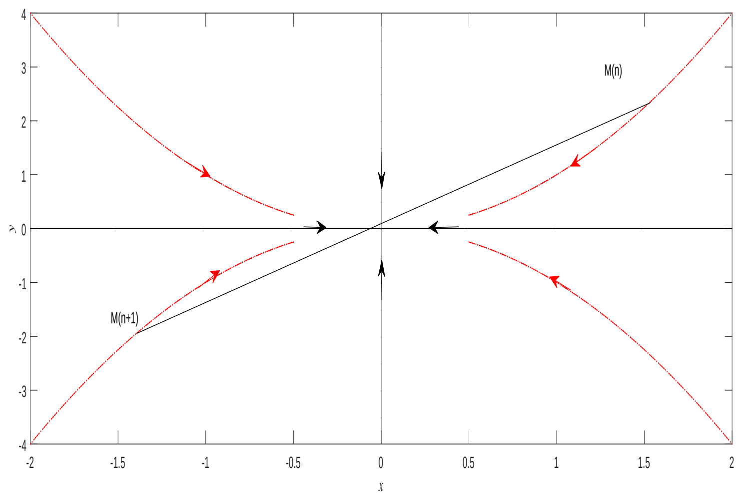


[3]

Figure 3.1: Phase portrait of the (3.12) if $|\lambda_1| < |\lambda_2| < 1$ (a) Type 1 stable node $0 < \lambda_2 < \lambda_1$. (b) Type 2 stable node $0 < \lambda_1$ and $\lambda_2 < 0$. (c) Type 3 stable node$\lambda_1 < 0$ et $\lambda_2 < 0$.

⋆ If $0 < \lambda_2 < 1 < \lambda_1$ These invariant curves have a hyperbolic shape, with asymptotes along ox and oy, as shown in figure (3.3).

In both cases, the x-axis and the y-axis are particular invariant curves. ⋆ If one of the multipliers λ_1, λ_2 is negative, or if both multipliers are negative, the curves (3.14) are then invariant under the application of T^2, which has both of its multipliers positive.

1. **If** $|\lambda_2| < |\lambda_1| < 1$

   The curves invariant under the application of T or T^2 therefore have the shape shown in figure (3.1). Starting from an initial point $M_0(x_0, y_0)$, the sequence of points generated by (3.12), i.e., the discrete trajectory originating from M_0, lies on the invariant curve passing through M_0 if $\lambda_1 > 0, \lambda_2 > 0$; and on two invariant curves — the one passing through M_0 and the one passing through M_1 — if one of the multipliers is negative or both multipliers are negative.

   This sequence is such that:

   $$\lim_{n \longrightarrow +\infty} x_n = \lim_{n \longrightarrow +\infty} y_n = 0$$

   The point 0 is an attractive fixed point, or asymptotically stable (***a stable node***).

   (a) If $\lambda_1 > 0, \lambda_2 > 0$, the sequence of points generated from $M_0(x_0, y_0)$ tends toward 0 without oscillating around either axis OX or OY. The point 0 is called a ***type 1 stable node*** — see figure (3.1) (a).

   (b) If $\lambda_1 < 0, \lambda_2 > 0$, or $\lambda_1 > 0, \lambda_2 < 0$, the sequence of points generated from $M_0(x_0, y_0)$ tends toward 0 while oscillating around OY in the first case, and around OX in the second. The point 0 is called a ***type 2 stable node*** — see figure (3.1) (b).

   (c) If $\lambda_1 < 0, \lambda_2 < 0$, the sequence of points generated from $M_0(x_0, y_0)$ tends toward 0 while oscillating around both axes OX and OY. The point 0 is called a ***type 3 stable node*** — see figure (3.1) (c).

2. **If** $|\lambda_2| > |\lambda_1| > 1$, The discrete trajectories are still located on the same parabolic-shaped curves, but the sequence of points obtained from an initial point $M_0(x_0, y_0)$ moves away from 0.

   $$\lim_{n \longrightarrow +\infty} x_n = \pm\infty, \lim_{n \longrightarrow +\infty} y_n = \pm\infty.$$

   0 is called a ***unstable node***, which, depending on the signs of λ_1, λ_2, can be of type 1, 2, or 3. It is also referred to as a ***repelling node*** — see figure (3.2).

[1]

[2]

[3]

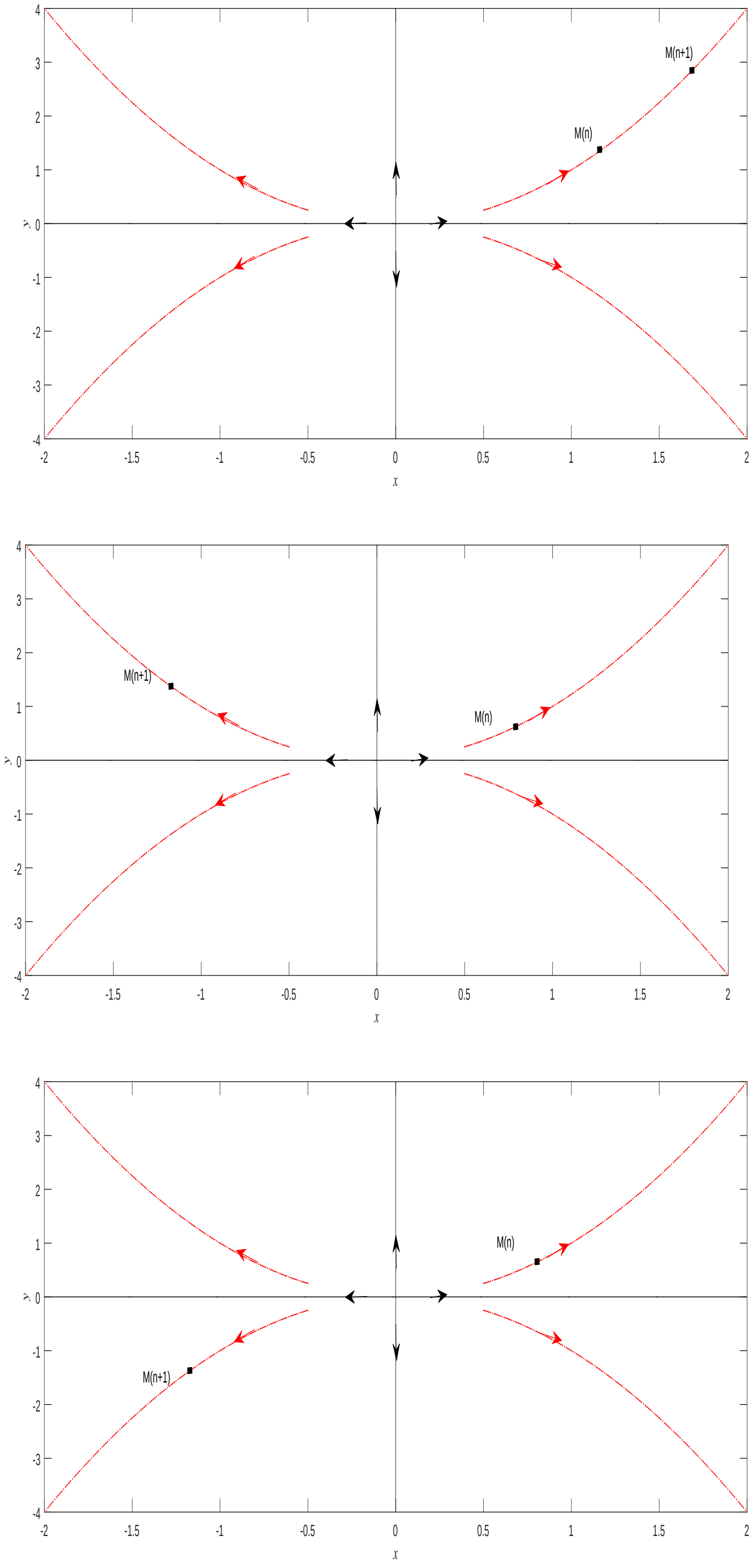


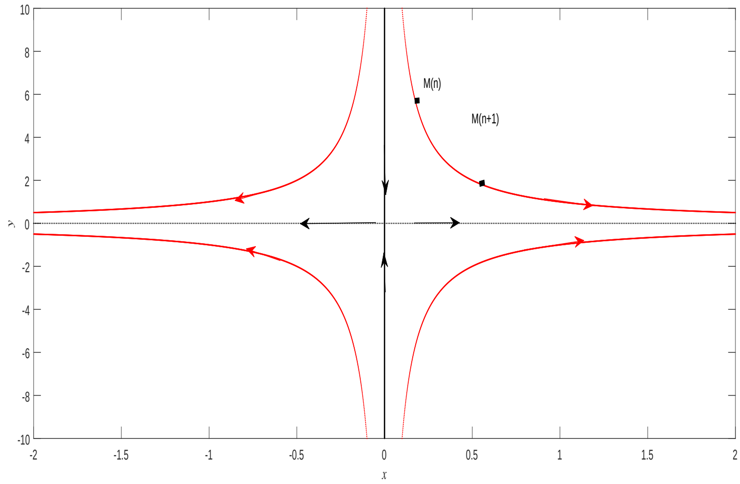
Figure 3.2: Phase portrait of the (3.12) if $|\lambda_1| > |\lambda_2| > 1$ (a) Type 1 unstable node $\lambda_1 > 0, \lambda_2 > 0$. (b) Type 2 stable node $\lambda_1 < 0, \lambda_2 > 0$. (c) Type 3 stable node $\lambda_1 < 0, \lambda_2 < 0$.

3. **If** $|\lambda_2| < 1$**,** $|\lambda_1| > 1$, the discrete trajectories are located on the invariant curves with a hyperbolic shape, as shown in figure (3.3), and the origin 0 is an unstable fixed point, which will be called a ***col***.

   For initial conditions $M_0(x_0 = 0, y_0)$ on the OY-axis, the sequence of points M_n remains on OY and tends towards 0; for $M_0(x_0, y_0 = 0)$, the sequence of points M_n remains on OX and moves away from 0.

   The axes OX, OY correspond to two invariant curves passing through 0; these are the only invariant curves passing through the fixed point.

   The signs of λ_1, λ_2 further distinguish the ***cols of type 1, 2, or 3*** — see figure (3.3).



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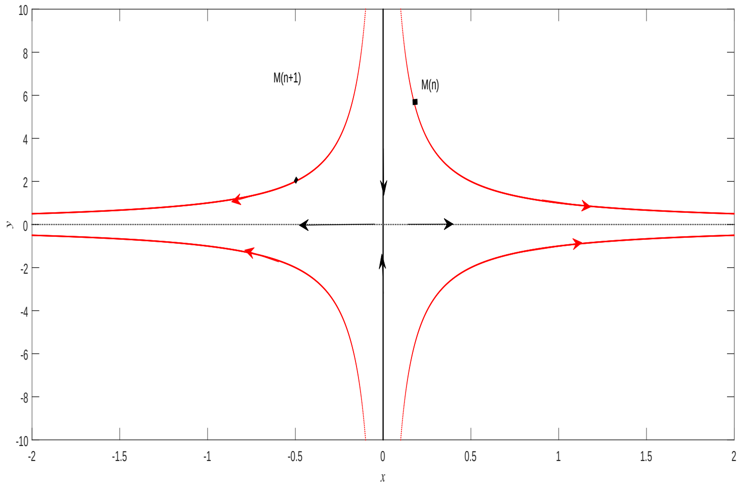

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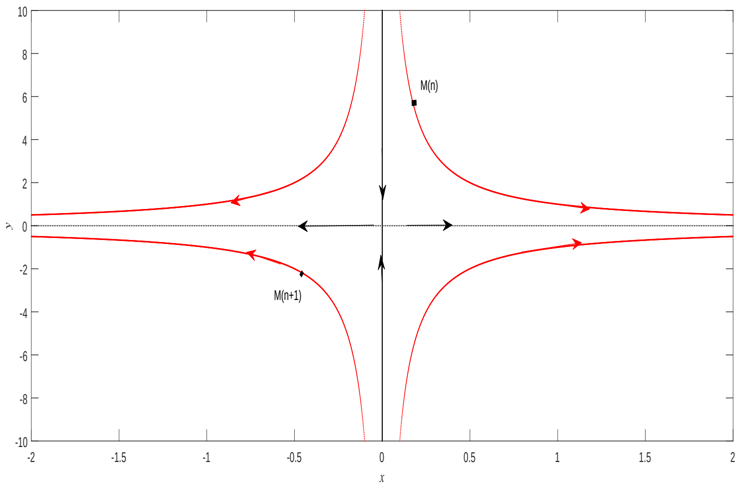


Figure 3.3: Phase portrait of the (3.12) if $|\lambda_2| < 1$, $|\lambda_1| > 1$ (a) Col of type1$\lambda_1 > 0$, $\lambda_2 > 0$. (b) Col of type 2 $\lambda_1 > 0$, $\lambda_2 < 0$. (c) Col of type 3 $\lambda_1 < 0$, $\lambda_2 < 0$.

4. **If** $|\lambda_1| = |\lambda_2|$, The invariant curves of the transformation T for $\lambda_1 = \lambda_2 > 0$, of the transformation T^2 for $\lambda_1 = -\lambda_2$, $\lambda_1 = \lambda_2 < 0$, starting from different points M_0, are straight lines passing through 0. The discrete trajectories lie on these invariant curves, and the fixed point 0 is called a ***star node of type 1*** ($\lambda_1 = \lambda_2 > 0$), ***of type 2*** ($\lambda_1 = -\lambda_2$), or ***of type 3*** ($\lambda_1 = \lambda_2 < 0$) attractive (figure (3.4)), or repelling (figure (3.5)), depending on whether $|\lambda_1| < 1$ or $|\lambda_1| > 1$.

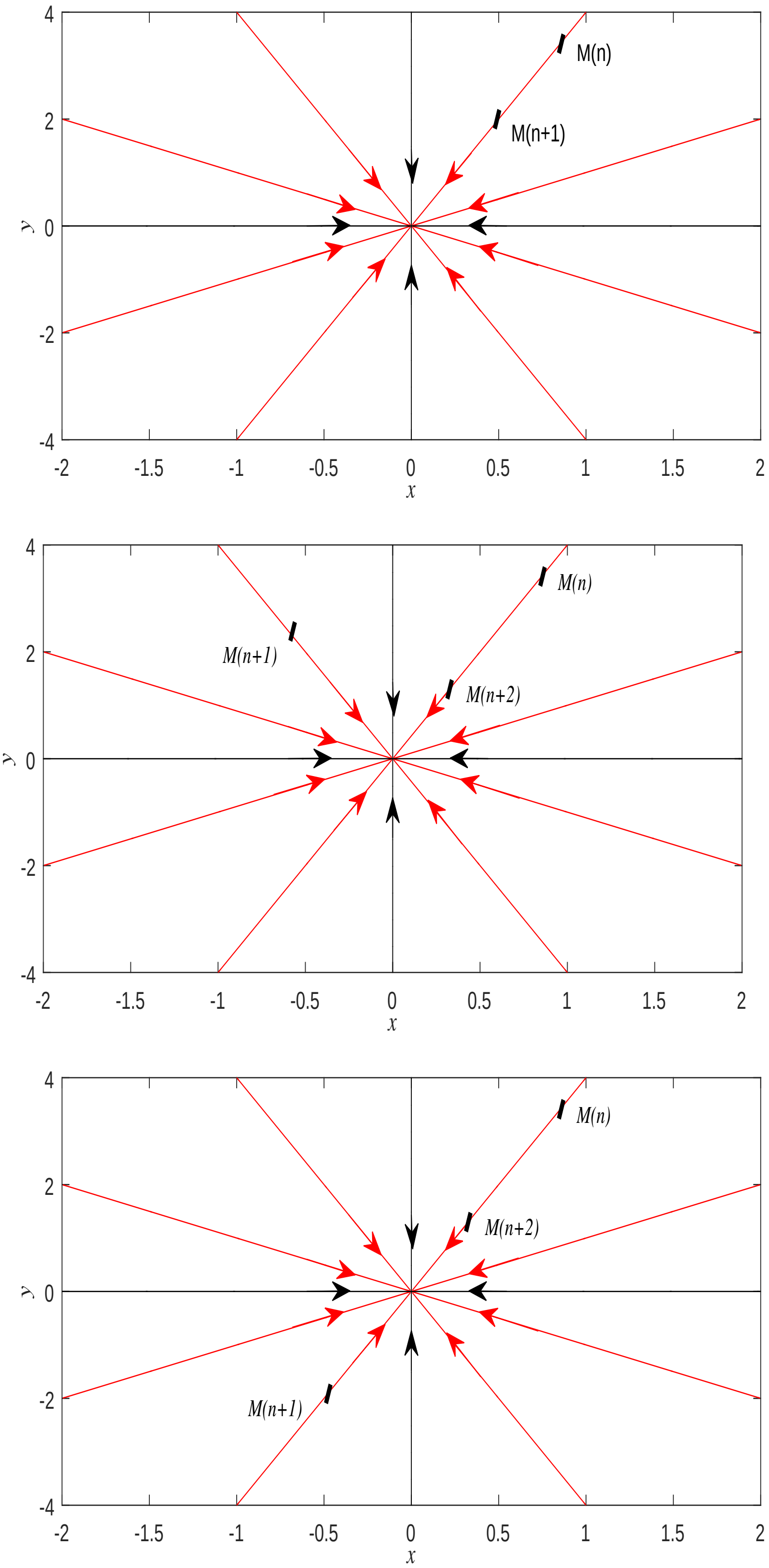
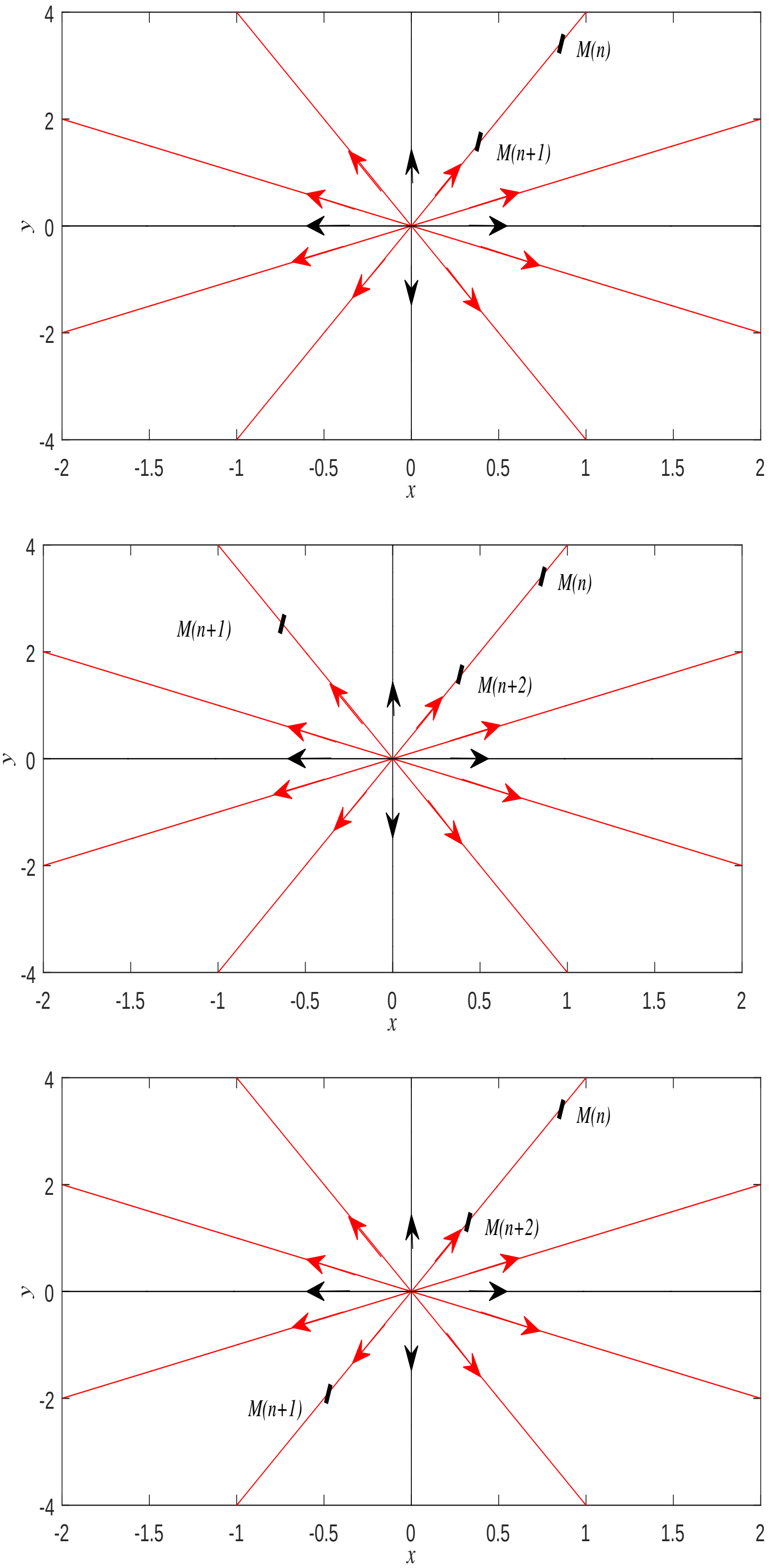


Figure 3.4: Phase portrait of the (3.12) if $|\lambda_1| = |\lambda_2| < 1$ (a) Stable star node of type 1 $\lambda_1 = \lambda_2 > 0$. (b) type 2 $\lambda_1 = -\lambda_2$. (c) type 3 $\lambda_1 = \lambda_2 < 0$.



[1] [2]

[3]

Figure 3.5: Phase portrait of the (3.12) if $|\lambda_1| = |\lambda_2| > 1$ (a) Unstable star node de type 1 $\lambda_1 = \lambda_2 > 0$. (b) type 2 $\lambda_1 = -\lambda_2$. (c) type 3 $\lambda_1 = \lambda_2 < 0$.

♦ **Equal real eigenvalues**

Consider the dynamical system (3.12) whit $D = \begin{pmatrix} \lambda & 0 \\ 1 & \lambda \end{pmatrix}$ et $\lambda \in \mathbb{R}$.

So $Y_n = D^n Y_0$ avec $D^n = \begin{pmatrix} \lambda^n & 0 \\ n\lambda^{n-1} & \lambda^n \end{pmatrix}$ *i.e.*

$$\begin{cases} x_n = \lambda^n x_0 \\ y_n = n\lambda^{n-1}x_0 + \lambda^n y_0. \end{cases} \tag{3.15}$$

Whit $\lambda > 0$ et $\dfrac{x}{x_0} > 0$ The trajectories are located on the invariant curves with the equation

$$y = \frac{y_0}{x_0}x + \frac{ln(\frac{x}{x_0})}{\lambda ln(\lambda)}x. \tag{3.16}$$

1. If λ, the fixed point 0 is then called a ***an improper attractive node*** — see figure (3.6) (a).

2. If λ, the fixed point 0 is then called a ***an improper repelling node*** — see figure (3.6) (b).

♦ **Complex eigenvalues**

Consider the dynamical system (3.12) with $D = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$, $a, b \in \mathbb{R}$.Consider the geometric representation of the eigenvalues, by setting $r = \sqrt{a^2 + b^2}$, $a = r\cos\theta$ and $b = r\sin\theta$ with $0 < \theta < \pi$ so $\lambda_{1,2} = a \pm ib = re^{\pm i\theta}$ (figure 3.7).

[1]

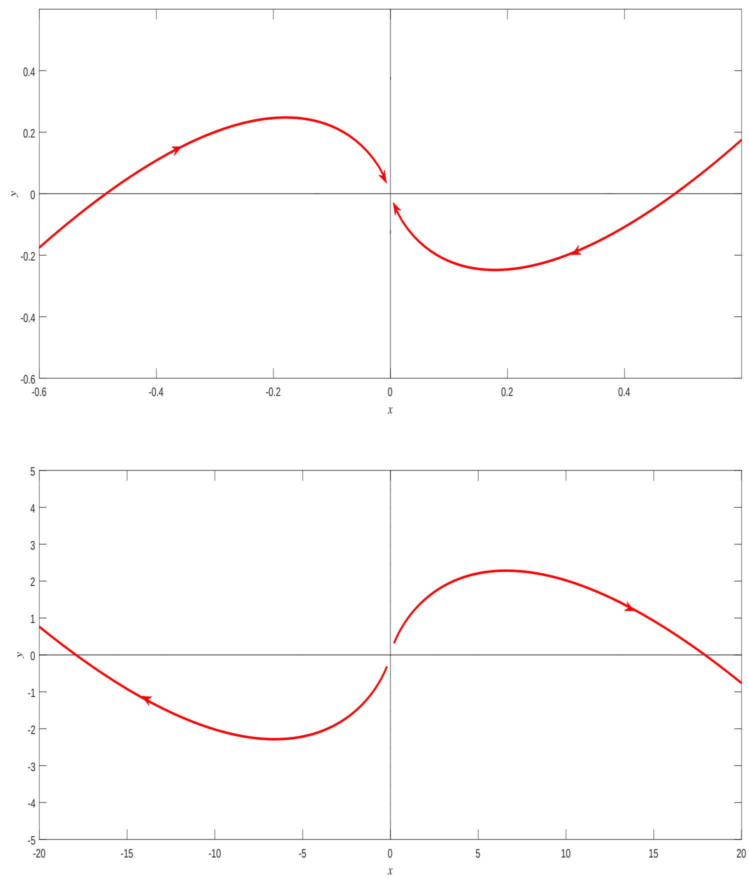


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Figure 3.6: Phase portrait of the (3.12) if $\lambda_1 = \lambda_2 = \lambda$ (a) Improper stable node $0 < \lambda < 1$. (b) Improper unstable node $1 < \lambda < \infty$.

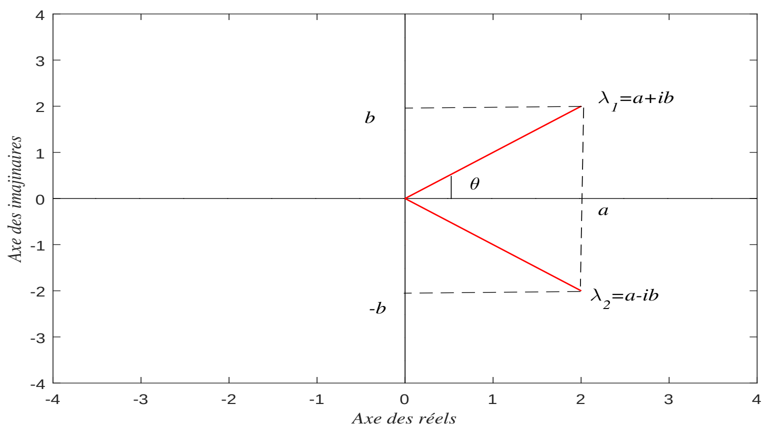


Figure 3.7: Geometric representation of the eigenvalues $\lambda_{1,2} = a \pm ib = re^{\pm i\theta}$.

Hence

$$D = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} = r \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \tag{3.17}$$

and

$$D^n = r^n \begin{pmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{pmatrix}, \tag{3.18}$$

therefore the vector Y_n is given by

$$\begin{aligned} \begin{pmatrix} x_n \\ y_n \end{pmatrix} &= r^n \begin{pmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}. \\ &= r^n \begin{pmatrix} x_0 \cos n\theta - y_0 \sin n\theta \\ x_0 \sin n\theta + y_0 \cos n\theta \end{pmatrix} \\ &= r^n \sqrt{x_0^2 + y_0^2} \begin{pmatrix} \frac{x_0}{\sqrt{x_0^2+y_0^2}} \cos n\theta - \frac{y_0}{\sqrt{x_0^2+y_0^2}} \sin n\theta \\ \frac{x_0}{\sqrt{x_0^2+y_0^2}} \sin n\theta + \frac{y_0}{\sqrt{x_0^2+y_0^2}} \cos n\theta \end{pmatrix} \end{aligned}$$

Hence

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = r^n \rho_0 \begin{pmatrix} \cos(\varphi_0 + n\theta) \\ \sin(\varphi_0 + n\theta) \end{pmatrix}, \tag{3.19}$$

where $\rho_0 = \sqrt{x_0^2 + y_0^2}$, $x_0 = \rho_0 \cos(\varphi_0)$ et $y_0 = \rho_0 \sin(\varphi_0)$.

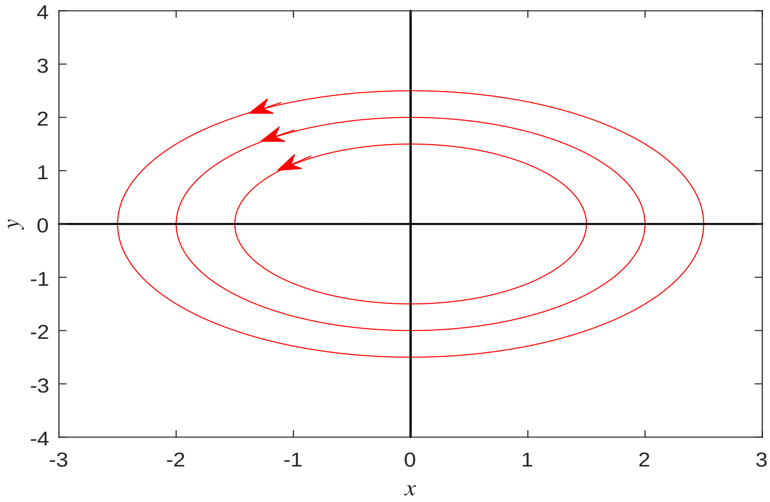
1. if $r = 1$, we have

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \rho_0 \begin{pmatrix} \cos(\varphi_0 + n\theta) \\ \sin(\varphi_0 + n\theta) \end{pmatrix},$$

or after eliminating n

$$x_n^2 + y_n^2 = \rho_0^2, \tag{3.20}$$

The circles centered at 0 are invariant curves for the system (3.12). The fixed point 0 is called a ***center*** (a fixed point that is a ***center*** is ***stable*** but not ***asymptotically stable***). If $\theta \neq \frac{2k\pi}{q}$ (where k and q are integers with no common divisor), a trajectory starting from a point $M_0 = (x_0, y_0)$ consists of a sequence of points dense on the circle of radius ρ_0. If $\theta = \frac{2k\pi}{q}$ (where k and q are integers with no common divisor), a trajectory starting from a point $M_0 = (x_0, y_0)$ consists of the q points of a cycle of order q, located on the circle of radius ρ_0. In both cases, the direction of the motion is ***counterclockwise*** if $b > 0$, and ***clockwise*** if $b < 0$, as shown in figure (3.8).



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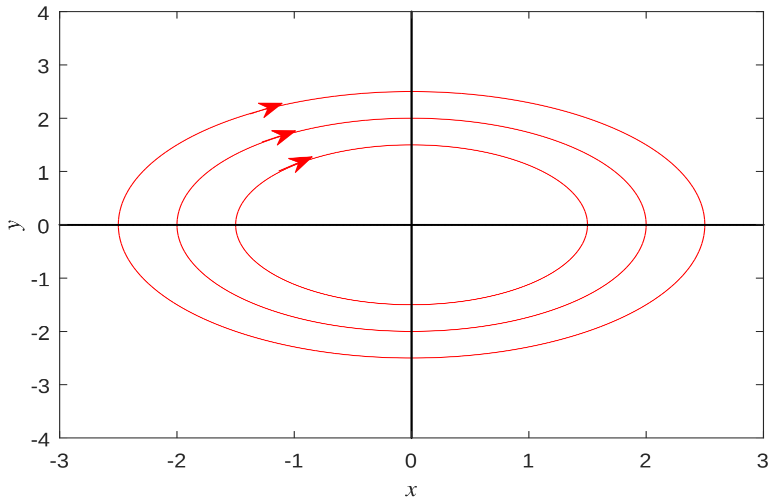


[2]

Figure 3.8: Center fixed point $r = 1$. (a) $b > 0$. (b) $b < 0$.

2. If $r \neq 1$, by introducing polar coordinates $x = \rho\cos\varphi$, $y = \rho\sin\varphi$, so the system becomes

$$\begin{pmatrix} \rho_{n+1}\cos\varphi_{n+1} \\ \rho_{n+1}\sin\varphi_{n+1} \end{pmatrix} = r\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \rho_n\cos\varphi_n \\ \rho_n\sin\varphi_n \end{pmatrix}$$
$$= \rho_n r\begin{pmatrix} \cos(\theta+\varphi_n) \\ \sin(\theta+\varphi_n) \end{pmatrix},$$

so

$$\begin{cases} \rho_{n+1} = r\rho_n \\ \varphi_{n+1} = \varphi_n + \theta \end{cases}$$

where the solution is

$$\begin{cases} \rho_n = r^n\rho_0 \\ \varphi_n = n\theta + \varphi_0 \end{cases} \tag{3.21}$$

then

$$\rho = r^{(\varphi-\varphi_0)/\theta}\rho_0$$

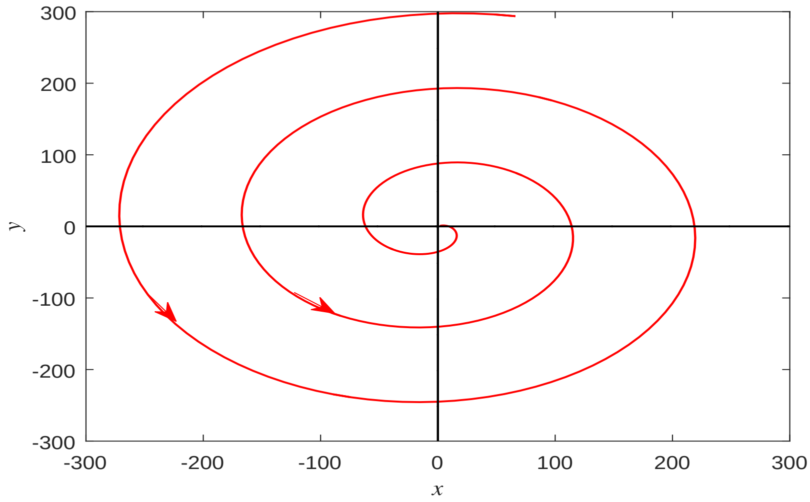
or in the Cartesian coordinates

$$x^2 + y^2 = \rho_0 r^{2(\varphi-\varphi_0)/\theta}. \tag{3.22}$$

Logarithmic spirals (3.22) are invariant curves for the system (3.12). Moreover

(a) If $|r| < 1$, then $\lim_{n\to+\infty}\rho_n = 0$, and φ_n increases as n increases if $b > 0$, and decreases as n increases if $b < 0$.

Thus, the trajectories spiral toward the origin. In this case, the origin is called a *stable focus*, and the dynamics of the trajectories from a point $M_0(x_0, y_0)$ are in the ***counterclockwise*** direction if $b > 0$, and in the ***clockwise*** direction if $b < 0$ — see figure (3.9).



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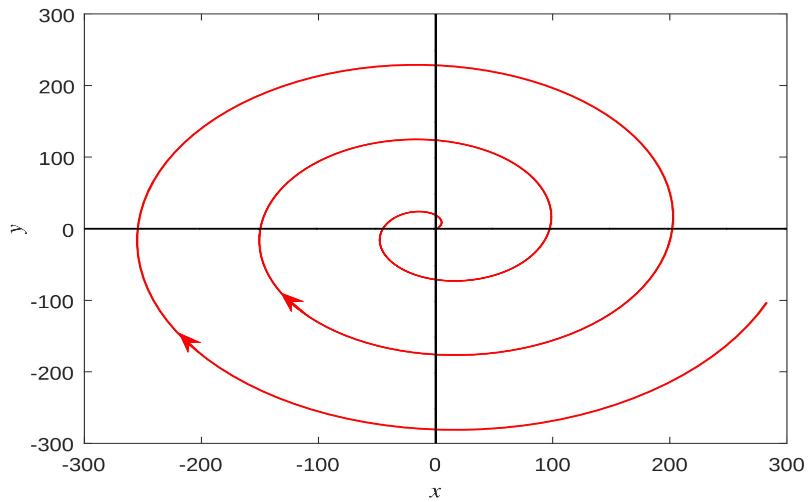

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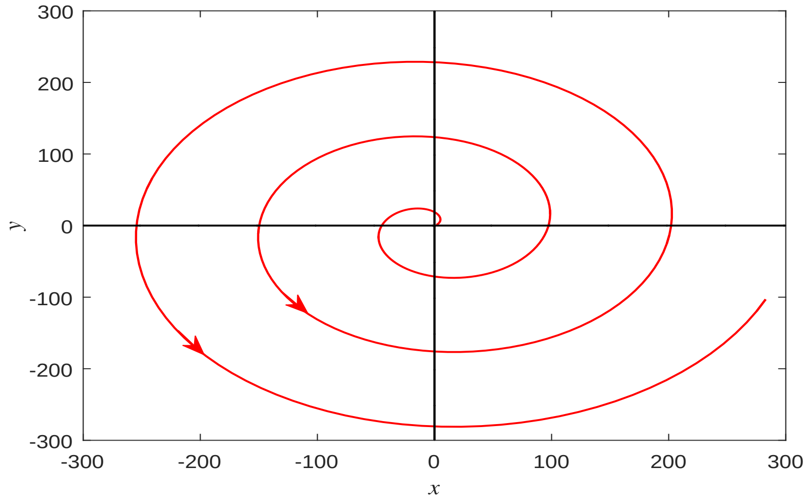
Figure 3.9: Stable focus. (a) $b > 0$. (b) $b < 0$.



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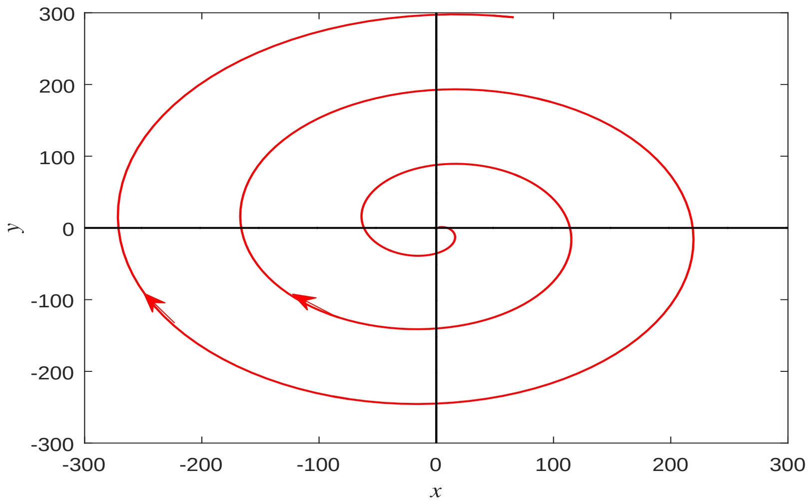


[2]

Figure 3.10: Stable focus. (a) $b > 0$. (b) $b < 0$.

(b) If $|r| > 1$, then $\lim_{n\to+\infty} \rho_n = +\infty$, and φ_n increases as n increases if $b > 0$, and decreases as n increases if $b < 0$.

Thus, the trajectories spiral away from the origin. In this case, the origin is called an *unstable focus*, and the dynamics of the trajectories from a point $M_0(x_0, y_0)$ are in the ***counterclockwise*** direction if $b > 0$, and in the ***clockwise*** direction if $b < 0$ — see figure (3.10).

Remark 3.1.2

1. *For all the fixed points considered, such that* $|\lambda_1| < 1$ *and* $|\lambda_2| < 1$*, the domain of stability consists of the entire plane* (y_1, y_2) *except the points at infinity.*

2. *When one of the eigenvalues has modulus equal to 1, for example* $|\lambda_1| = 1$*, then equation 3.11 shows that the X-axis is a curve made up of an infinite number of fixed points if* $\lambda_1 = 1$*, or an infinite number of order-2 cycles (except the fixed point 0) if* $\lambda_1 = -1$*, attractive when* $|\lambda_2| < 1$*, repelling when* $|\lambda_2| > 1$*.*

3.1.4 Characterization of the Solutions Based on the Trace and Determinant of A

Let us consider the two-dimensional linear dynamical system $X_{n+1} = AX_n$. The qualitative properties of this system can be classified based on the values of $\mathrm{tr}(A)$ and $\det(A)$. The eigenvalues of the matrix A are obtained as the solutions of the characteristic equation:

$$P(\lambda) = \lambda^2 - \mathrm{tr}(A)\lambda + \det(A) = 0.$$

1. If $(\mathrm{tr}(A))^2 > 4\det(A)$, the eigenvalues are real.

2. If $(\mathrm{tr}(A))^2 < 4\det(A)$, the eigenvalues are complex.

Theorem 3.1.3 *Figure (3.11).*
Consider the two-dimensional linear dynamical system $X_{n+1} = AX_n$*, and let* λ_1, λ_2 *be the eigenvalues of* A*.*

1. *If* $(\mathrm{tr}(A))^2 > 4\det(A)$*, then* λ_1, λ_2 *are real* ($\lambda_1 > \lambda_2$)*, and moreover:*

   (a) *The origin* 0 *is a saddle point (i.e.,* $\lambda_1 > 1$ *and* $|\lambda_2| < 1$*, or* $|\lambda_1| < 1$ *and* $\lambda_2 < -1$*) if and only if:*

   $$\begin{cases} P(1) < 0 \text{ and } P(-1) > 0 \\ \quad\quad \text{or} \\ P(1) > 0 \text{ and } P(-1) < 0 \end{cases}$$

   i.e., if and only if:

   $$\begin{cases} -\mathrm{tr}(A) - 1 < \det(A) < \mathrm{tr}(A) - 1 \\ \quad\quad \text{or} \\ \mathrm{tr}(A) - 1 < \det(A) < -\mathrm{tr}(A) - 1 \end{cases}$$

   (b) *The origin* 0 *is a stable node* ($|\lambda_{1,2}| < 1$) *if and only if:*

   $$P(1) > 0 \text{ and } P(-1) > 0,$$

   i.e., if and only if:

   $$\det(A) > \mathrm{tr}(A) - 1 \text{ and } \det(A) > -\mathrm{tr}(A) - 1.$$

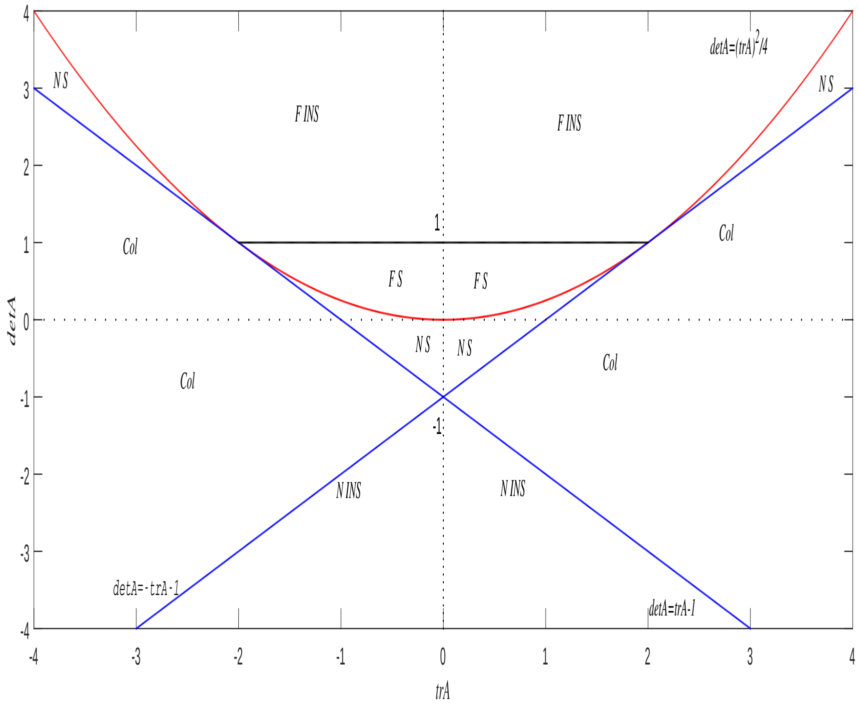


Figure 3.11: Caractérisation des solutions en fonction de *tr(A) et det(A).*

(c) *The origin* 0 *is an unstable node* ($|\lambda_{1,2}| > 1$) *if and only if:*

$$P(1) < 0 \text{ and } P(-1) < 0,$$

i.e., if and only if:

$$\det(A) < \operatorname{tr}(A) - 1 \text{ and } \det(A) < -\operatorname{tr}(A) - 1.$$

2. *If* $(\operatorname{tr}(A))^2 < 4\det(A)$*, then* λ_1, λ_2 *are complex, and:*

   (a) *The origin* 0 *is a stable focus if and only if* $\det(A) < 1$.

   (b) *The origin* 0 *is an unstable focus if and only if* $\det(A) > 1$.

Example 3.1.4 *Consider the one-parameter family of linear systems* $X(n+1) = AX(n)$*, where*

$$A = \begin{pmatrix} -1 & a \\ -2 & 1 \end{pmatrix}$$

which depends on the parameter a*. As* a *varies, the determinant of the matrix,* $\det A$*, is always* $2a - 1$*, while the trace of the matrix,* $\operatorname{tr} A$*, is always* 0 *. As we vary the parameter* a *from negative to positive values, the corresponding point* (T, D) *moves vertically along the line* $T = 0$*. Now if* $D < -1$*, which occurs if* $2a - 1 < -1$

or $a < 0$, we have a degenerate case, $\lambda_1 = 1$ and $\lambda_2 = -1$ with corresponding eigenvectors $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Thus every point on the y-axis is a fixed point and every other point in the plane is periodic of period 2. For $0 < a \leq \frac{1}{2}$, we have a sink, and for $\frac{1}{2} < a < 1$ we have a spiral sink. At exactly $a = 1$ we have a center, and if $a > 1$ we have a spiral source .

The values of a where critical dynamical changes occur are called bifurcation values. In this example, the bifurcation values of a are $0, \frac{1}{2}, 1$

In the next chapter, we will discuss the definition and types of bifurcation..

3.2 Nonlinear Discrete Dynamical Systems

A two-dimensional discrete nonlinear system is given by:

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = f\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} f_1\begin{pmatrix} x_n \\ y_n \end{pmatrix} \\ f_2\begin{pmatrix} x_n \\ y_n \end{pmatrix} \end{pmatrix},$$

or

$$\begin{cases} x_{n+1} = f_1(x_n, y_n), \\ y_{n+1} = f_2(x_n, y_n), \end{cases} \tag{3.23}$$

$f_1(x, y)$, $f_2(x, y)$ where f and g are continuous, single-valued, nonlinear functions of the real variables x and y. The properties of solutions to system (3.23) are considerably more complex than those of recurrence (3.3). In particular, except in special cases, the solutions cannot be expressed in closed form using known transcendental functions, and must instead be characterized through the system's singularities.(3.23).

3.3 Stability via Linearization

Fixed points :

The nonlinear case thus leads to the possibility of multiple fixed points. Let X^* be a fixed point for system (3.23). If $f_1(x, y)$ and $f_2(x, y)$ are at least once differentiable (C^1 class) at the point $X^* = (x^*, y^*)$, a Taylor expansion in the neighborhood of X^* gives:

$$\begin{pmatrix} x_{n+1} - x^* \\ y_{n+1} - y^* \end{pmatrix} \approx J(X^*) \begin{pmatrix} x_n - x^* \\ y_n - y^* \end{pmatrix} + \mathcal{O}(\|X_n - X^*\|^2)$$

where $J(X^*)$ is the Jacobian matrix evaluated at X^*:

$$J(X^*) = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix}_{(x^*, y^*)}$$

The **Jacobian matrix** of f at X^*. As in the linear case, in general, the multipliers (eigenvalues of J) λ_1, λ_2 determine the behavior of the system's trajectories, but here only for initial conditions taken in a (sufficiently small) neighborhood of the fixed point X^*.

The cases where one multiplier (or both) has (have) a modulus equal to one, which in the linear case appear as boundary cases separating two different structures of discrete trajectories, no longer allow us to determine the nature of the invariant curves based on the linear approximation of (3.23). Indeed, if $|\lambda_1| = 1$, or $|\lambda_2| = 1$, or $|\lambda_1| = |\lambda_2| = 1$, this boundary characteristic - between two qualitatively different behaviors - implies that the trajectories of the linear approximation are generally not preserved, no matter how small the neighborhood $\mathcal{D}$ of the fixed point X^* is.

It is the nonlinear terms $\bar{X}$, $\bar{Y}$ that determine the shape of these trajectories within $\mathcal{D}$. In this case, we say there is a *critical case in the sense of Lyapunov*.

Theorem 3.3.1 *Let $f : G \subset \mathbb{R}^2 \to \mathbb{R}^2$ be a C^1 map, where G is an open subset of $\mathbb{R}^2$, X^* is a fixed point of f, and $A = Df(X^*)$. Then the following statements hold true:*

1. If $\rho(A) < 1$, then X^* is asymptotically stable.

2. If $\rho(A) > 1$, then X^* is unstable.

3. If $\rho(A) = 1$, then X^* may or may not be stable.

1. If all eigenvalues of the Jacobian matrix J have moduli strictly less than one, then the fixed point X^* of system (3.23) is locally asymptotically stable.

2. If the Jacobian matrix J has at least one eigenvalue with modulus strictly greater than one, then the fixed point X^* is unstable.

3. If some eigenvalues of matrix J lie on the unit circle while the others are inside, we cannot determine the local stability of the fixed point X^*.

Example 3.3.1 *Pielou Logistic Delay Equation*

$$x(n+1) = \frac{\alpha x(n)}{1 + \beta x(n-1)}. \tag{3.24}$$

We now write Equation (3.24) in system form. Let $x_1(n) = x(n-1)$, and $x_2(n) = x(n)$. Then,

$$\begin{pmatrix} x_1(n+1) \\ x_2(n+1) \end{pmatrix} = \begin{pmatrix} x_2(n) \\ \frac{\alpha x_2(n)}{1-\beta x_1(n)} \end{pmatrix} = \begin{pmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{pmatrix} \tag{3.25}$$

There are two fixed points $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ *and* $\begin{pmatrix} (\alpha-1)/\beta \\ (\alpha-1)/\beta \end{pmatrix}$.

1. The fixed point $Z_1^* = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. *Here,*

$$A = Df(0) = \begin{pmatrix} 0 & 1 \\ 0 & \alpha \end{pmatrix}$$

with eigenvalues 0 and α. Since $\alpha > 1$, the origin is unstable by Theorem (3.3.1).

2. The fixed point $Z_2^* = \begin{pmatrix} (\alpha-1)/\beta \\ (\alpha-1)/\beta \end{pmatrix}$. *In this case,*

$$A = Df\left(z_2^*\right) = \begin{pmatrix} 0 & 1 \\ \frac{1-\alpha}{\alpha} & 1 \end{pmatrix}.$$

By Theorem (3.1.3) , $\rho(A) < 1$ if and only if

$$|\operatorname{tr} A| < 1 + \det A < 2$$

if and only if

$$1 < 1 + \frac{\alpha-1}{\alpha} < 2$$

if and only if

$$0 < \frac{\alpha-1}{\alpha} < 1$$

Clearly this is satisfied if $\alpha > 1$. Hence, by Theorem 4.11, z_2^* is asymptotically stable .

3.4 Liapunov Functions for Nonlinear Maps

Since linearization theory is inherently local, it cannot address global stability properties. In this section, we present an alternative approach known as **Lyapunov's second method** or the **direct method**. This approach is called "direct" because it does not require explicit knowledge of solutions to the system of difference equations.

The method determines the stability characteristics of critical points by constructing an appropriate auxiliary function, called a *Lyapunov function*. This framework provides a global approach for analyzing the asymptotic behavior of solutions.

The method generalizes two fundamental physical principles for conservative systems:

1. A rest position is stable if the potential energy represents a local minimum, and unstable otherwise;
2. The total energy remains constant throughout any motion.

The existence of a Lyapunov function demonstrates that solutions originating from an extensive region will converge to an equilibrium point. Modern formulations extend this concept through the following key results:

Consider the autonomous difference equation (3.23) where

$$f : G \to \mathbb{R}^n, \quad G \subset \mathbb{R}^n, \tag{3.26}$$

is continuous. We assume that x^* is a fixed point of equation (3.23), that is

$$f(x^*) = x^*. \tag{3.27}$$

Let $V : \mathbb{R}^n \to \mathbb{R}$ be a real-valued function. The variation of V relative to equation (3.23) is defined as

$$\Delta V(x) = V(f(x)) - V(x), \tag{3.28}$$

and

$$\Delta V(x(t)) = V(f(x(t))) - V(x(t)) = V(x(t+1)) - V(x(t)). \tag{3.29}$$

If $\Delta V(x) \leq 0$, then V is nonincreasing along solutions of equation (3.23). The function V is said to be a *Lyapunov function* on a subset H of $\mathbb{R}^n$ if:

(i) V is continuous on H, and

(ii) $\Delta V(x) \leq 0$, whenever x and $f(x)$ belong to H.

Theorem 3.4.1 *Suppose that V is a positive definite Liapunov function defined on an open ball $G = B(X^*, \gamma)$ around a fixed point X^* of a continuous map $f : \mathbb{R}^2 \to \mathbb{R}^2$. Then:*

(i) X^* is stable.

(ii) If, in addition, $\Delta V(X) < 0$ whenever $X, f(X) \in G$ with $X \neq X^*$, then X^* is asymptotically stable on G.

(iii) Moreover, if $G = \mathbb{R}^2$ and $V(X) \to \infty$ as $\|X\| \to \infty$, then X^* is globally asymptotically stable.

Example 3.4.1 *Consider the discrete-time system:*

$$x_1(t+1) = 2x_2(t) - 2x_2(t)x_2(t) \tag{3.30}$$

$$x_2(t+1) = -x_1(t) + x_1(t)x_2^2(t) \tag{3.31}$$

Equilibrium points: $(0,0)$ *and others. Using:*

$$V(x(t)) = x_1^2(t) + 4x_2^2(t) \tag{3.32}$$

we find:

$$\Delta V(x_1(t), x_2(t)) = 4x_1^2(t)x_2^2(t)\left[x_1^2(t) + x_2^2(t) - 1\right] \tag{3.33}$$

When $x_1^2 + x_2^2 < 1$, $\Delta V < 0$, *so* $(0,0)$ *is stable*

Theorem 3.4.2 (Instability Criterion) *If* $\Delta V(x)$ *is positive definite in a neighborhood of the origin and there exists a sequence* $\{a_i\}$ *with* $a_i \to 0$ *and* $V(a_i) > 0$ *for each* i, *then the zero solution of equation* (3.23) *is unstable.*

Example 3.4.2 (Unstable Nonlinear System) *Consider the two-dimensional system:*

$$x_1(t+1) = 4x_2(t) - 2x_2(t)x_1^2(t) \tag{3.34}$$

$$x_2(t+1) = -\frac{1}{2}x_1(t) + x_1(t)x_2^2(t) \tag{3.35}$$

Define the Lyapunov function candidate:

$$V(x_1, x_2) = x_1^2 + 4x_2^2 \tag{3.36}$$

The variation of V *along solutions is:*

$$\Delta V(x_1, x_2) = V(f(x_1, x_2)) - V(x_1, x_2) \tag{3.37}$$

$$= 3x_1^2 + 16x_2^2x_1^2 + 4x_1^2x_2^2 > 0 \quad \text{when } x_1 \neq 0 \tag{3.38}$$

Since:

1. ΔV *is positive definite (it vanishes only at the origin)*
2. V *itself is positive definite*
3. *There exist points* $(a_i, 0)$ *with* $a_i \to 0$ *and* $V(a_i, 0) = a_i^2 > 0$

By Theorem 3.4.2, *the equilibrium* $(0,0)$ *is unstable.*

Example 3.4.3 *Consider the system:*

$$\begin{cases} x_1(t+1) = x_1(t), \\ x_2(t+1) = \dfrac{\alpha x_2(t)}{1+\beta x_2^2(t)}, \end{cases} \qquad \textit{where } \beta > 0.$$

We choose the Lyapunov candidate:

$$V(x(t)) = x_1^2(t) + x_2^2(t).$$

Then,

$$\Delta V(x(t)) = V(x(t+1)) - V(x(t)) = \left[\frac{\alpha^2}{(1+\beta x_2^2(t))^2} - 1\right] x_2^2(t).$$

There are three equilibrium points,

$$(0,0), \quad (\pm\beta^*, \pm\beta^*),$$

if $\alpha > 1$, *where* $\beta^* \equiv \sqrt{\frac{\alpha-1}{\beta}}$. *Consider the stability of the equilibrium point* $(0,0)$. *Let*

$$V(x(t)) = x_1^2(t) + x_2^2(t).$$

This is continuous and positive definite on $\mathbb{R}^2$.

$$\Delta V(x(t)) = \left(\frac{\alpha^2}{\left(1+\beta x_2^2(t)\right)^2} - 1\right) x_2^2(t) \leq (\alpha^2 - 1) x_2^2(t).$$

If $\alpha^2 \leq 1$, *according to Theorem* (3.4.1), *the unique equilibrium point* $(0,0)$ *is stable.*

Since

$$V(x) \to \infty \quad \textit{as} \quad \|x\| \to \infty,$$

- ***Case*** $\alpha^2 = 1$***:***

  $$\Delta V(x(t)) = 0 \textit{ along the } x_1\textit{-axis}.$$

  Theorem (3.4.1) *is inconclusive. In fact, it can be shown that the zero solution is* ***not asymptotically stable*** *in this case.*

- ***Case*** $\alpha^2 < 1$***:***

  $$\Delta V(x(t)) < 0 \quad \textit{for all } x(t) \neq 0.$$

  Hence, the origin is ***asymptotically stable****.*

- ***Case*** $\alpha^2 > 1$***:***

  $$\Delta V(x(t)) > 0 \quad \textit{in some region around the origin}.$$

  Stability of the origin is ***indeterminate****;*

CHAPTER 4

BIFURCATION THEORY

The goal of bifurcation theory is to study the changes that dynamical systems undergo when parameters change. In this chapter we provide some details about this concept

4.1 Bifurcations in one-dimensional discrete systems

In this chapter we will study the simplest dynamical systems. We will see, however, that even in this case the scenario of different possible dynamics is very rich. In particular, we will consider dynamical systems depending on a parameter. When the value of the parameter changes continuously, the behaviour of the system may change in a discontinuous way. One says that a bifurcation occurs for an isolate value of the parameter at which the type of dynamic changes.

Loss of stability : bifurcation

We have seen that the fixed points of the dynamical system (3.23) are the points x satisfying $f(x) = x$. Let us suppose that our system is given by a function f , and that such function belongs to a family f_a of functions depending continuously on a parameter a. Let $f = f_0$. Hence, for $a = 0$ there is only one attracting fixed point and two repelling fixed points at the extremes of the interval where f is defined. Let us suppose that all the functions of the family satisfy $f(0) = 0, f(1) = 1$ and $f'(1) = f'(1) > 1$. By a continuous change of the function f_0 into f_a, the attracting fixed point $\bar{x}_a$ (satisfying $f_a(\bar{x}_a) = \bar{x}_a$ moves continuously. However, for some isolate value of the parameter a, something may happen which changes the dynamic.

4.1.1 Saddle-node bifurcation

As shown in Figure (4.1), it may happen that the graph of f_a, for some isolated value a^* of a becomes tangent to the diagonal (the graph of the function $h(x) = x$). At the point of tangency, say x^*, the derivative of the function f_{a^*} is equal to 1 , and therefore the equilibrium point x^* is non hyperbolic. For $a > a^*$ two new fixed points exist. Observe that necessarily one is stable and the other one is unstable.

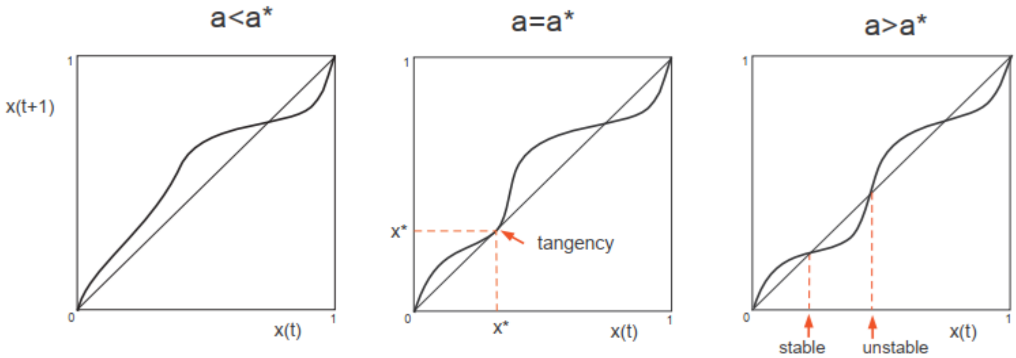


Figure 4.1: Saddle-node bifurcation in the family f_a.

The bifurcation diagram is the graph of a multivalued function, showing for every value of the parameter a in a neighbourhood of the bifurcation value a^* the fixed points of f_a in a neighbourhood of x^*. Stable fixed points are marked by a continuous line, unstable points by a dotted line.

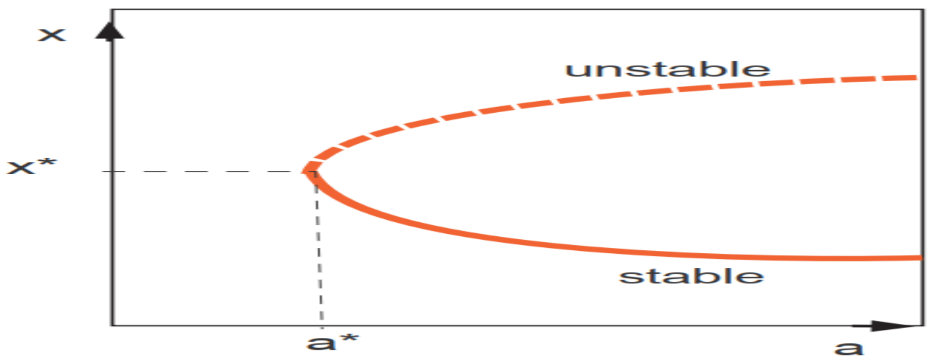


Figure 4.2: Bifurcation diagram of a saddle-node bifurcation.

4.1.2 Pitch-fork bifurcation

In this case the derivative of the fixed point $\bar{x}_a$ of f_a changes passing through the value 1 (or -1) (see Figure (4.3)). At that point, say x^*, the graph of f_{a^*} is tangent to the diagonal, with an order-2 tangency. When a increases, the point of tangency disappears, the fixed point that was stable (derivative higher than zero and less than one) becomes unstable (derivative higher than one) and two other fixed points exist at right and at left of the unstable fixed point. These two points are stable. Figure (4.4) shows the

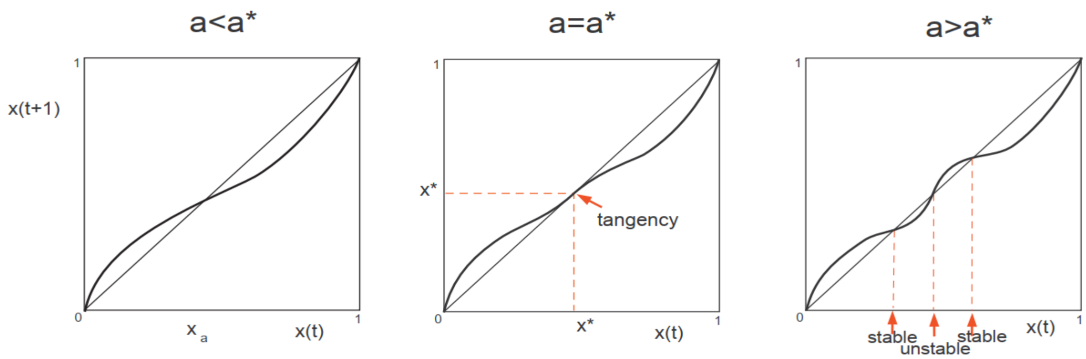


Figure 4.3: Bifurcation diagram of a saddle-node bifurcation.

bifurcation diagram. At x^* the stable fixed point becomes unstable and about it new stable fixed points appear.

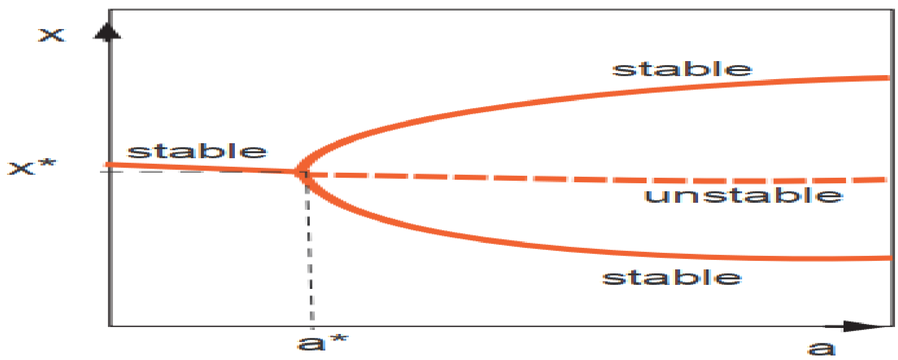


Figure 4.4: Bifurcation diagram of a pitch-fork bifurcation..

4.1.3 Periodic points

In order to introduce another typical phenomenon of the discrete one-dimensional systems we study the dynamics determined by the family of smooth functions:

$$f_a = ax(1-x)$$

defined on the unit interval $I = [0, 1]$ for $a \in (0, 4]$.

Evidently, $x = 0$ is a fixed point, and since $f'(0) = a$, it is stable for all values of a less than 1 .

For $a = 1$ the origin is therefore a non hyperbolic fixed point and for $a > 1$ it is unstable. We will denote by a_0 the value $a = 1$.

The equation $f_a(x) = x$ has as solution, besides $x = 0$, the point $\bar{x}_a = 1 - 1/a$, which is in the interval $[0, 1]$ for $a > 1$. The derivative at such point is $a(1 - 2\bar{x}_a) = 2 - a$, therefore $\bar{x}_a$ is stable for $1 < a < 3$. The point $\bar{x}_a$ becomes unstable at $a = 3$. A trajectory starting near the equilibrium point $\bar{x}_a$ is like that in Figure 3,

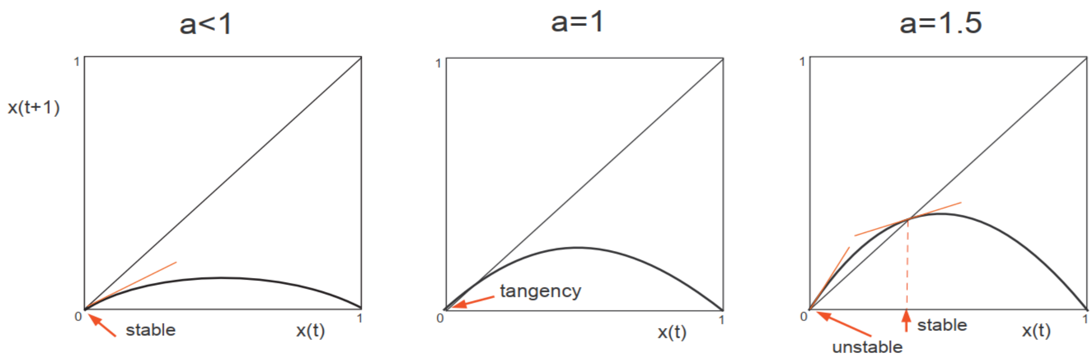


Figure 4.5: The logistic map for $a < 3$

left, for $a < 3$ and like that in figure 3, right, for $a > 3$. We will denote the value $a = 3$ by a_1. But the

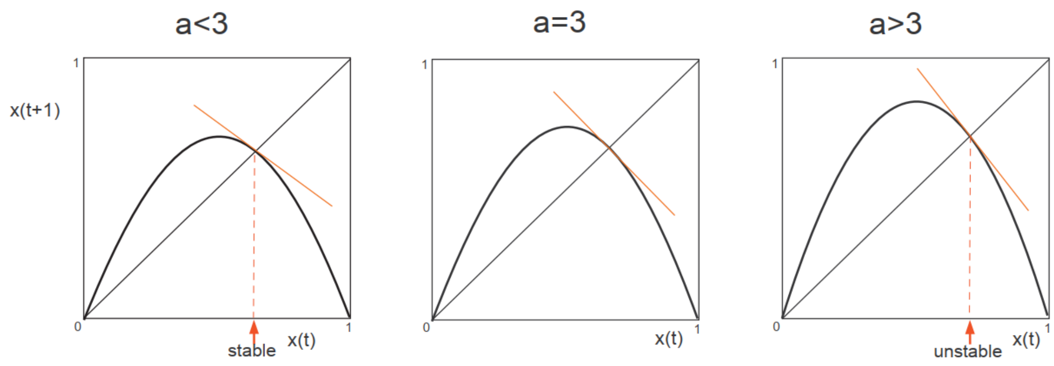


Figure 4.6: The logistic map for $a = 3$

question now is: where "the trajectory is going", i.e., does the succession $x(t)$, starting near $\bar{x}$, approach some set of points? In other words, does it exist an attracting set, which is not a fixed point?

The answer is yes. There is a value $a_2 > 3$ such that for $a_1 < a < a_2$, all trajectories starting at points different from 0 and non containing $\bar{x}_a$ are attracted towards a cycle of two points (see Figure (4.7)). In fact, for every value of a between a_0 and a_1 there are two points x_1 and x_2 such that $f(x_1) = x_2$ and $f(x_2) = x_1$. The trajectory staring at x_1 or x_2 is therefore formed by $\{x_1, x_2, x_1, x_2, x_1, x_2, \ldots\}$. Moreover, 'almost' all other trajectories tend to such a cycle. How to prove this?

We will consider, instead of the map f_a, the map $f_a^{(2)} := x \to f_a(f_a(x))$, the second iterate. It is evident that x_1 and x_2 satisfy

$$x_i = f_a^{(2)}(x_i) \qquad i = 1, 2$$

i.e., they are fixed points for this map : if they are attracting (repelling) for $f_a^{(2)}$, the cycle (x_1, x_2) will be attracting (repelling).

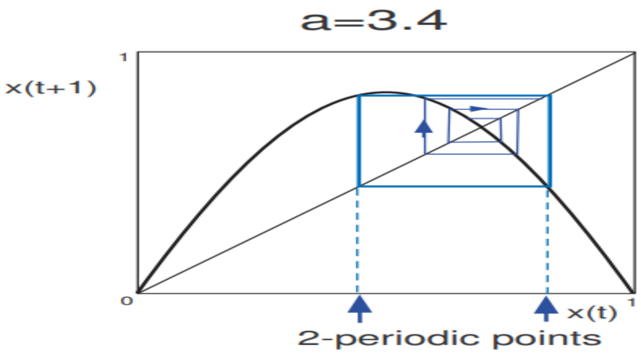


Figure 4.7: The attracting 2-cycle for $a = 3,4$

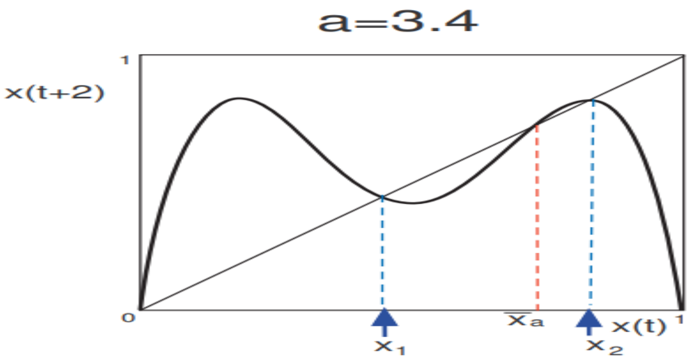


Figure 4.8: The stability of the attracting 2-cycle for $a = 3,4$

In Figure 11 we see that the absolute value of the slope of $f_a^{(2)}$ at x_1 and x_2 is less than 1 .

4.1.4 Period doubling Bifurcation

As a increases, the absolute value of the slope of $f_a^{(2)}$ at x_1 and x_2 increases (see Figure (4.9)), till the value $a_2 = 1 + \sqrt{6} \approx 3.4495$ when it becomes equal to 1 . For such a value of a the 2-cycle (x_1, x_2) loses stability. Observe that $f_a^{(2)}$, for $a > a_2$ has always four fixed points $(0, \bar{x}_a, x_1, x_2)$ but they are all unstable. Again, we ask: where the trajectories are going?

We observe that, locally, i.e. in a neighbourhood of x_1 or of x_2, the function $f_a^2(x)$ looks like the function $f_a(x)$ about $\bar{x}_a$ (its graph intersects the diagonal, the slope varying about -1). Therefore, if we now consider the iterate of $f_a^{(2)}(x)$, i.e. the fourth iterate $f_a^{(4)}(x) = f_a\left(f_a\left(f_a\left(f_a(x)\right)\right)\right)$, we expect a similar phenomenon near the points x_1 and x_2. I.e., for $a = a_2$ the function $f_a^{(4)}$ has contemporarily 2 pitchforks bifurcations in correspondence of the points x_1 and x_2, see Figure (4.10) .

A cycle similar to that of Figure 10 continues to exist, but it is unstable and a double cycle of 4 points is the attracting set (see Figure (4.11)).

This phenomenon repeats when a increases: for a value $a_3 > a_2$ the 4 -cycle loses stability: the map $f_a^{(4)}\left(f_a^{(4)}(x) = f^{(8)}(x)\right.$ has 4 pitch-fork bifurcations and 8 new fixed points appear (i.e. 8 -periodic points

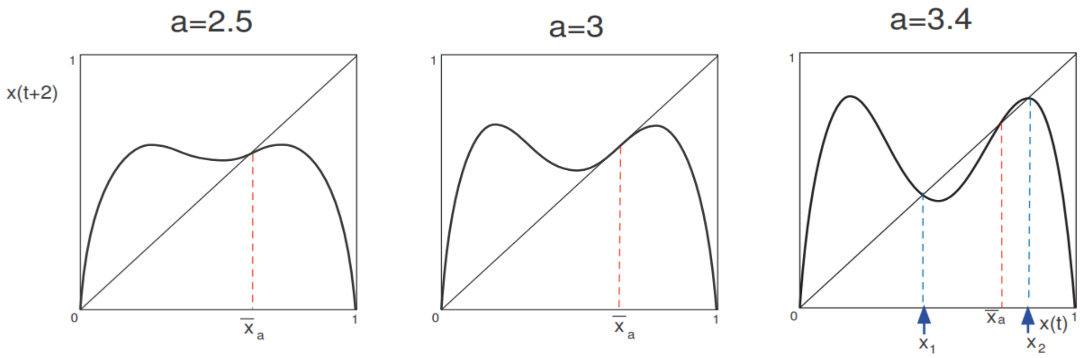


Figure 4.9: The pitchfork bifurcation of f_a^2 at $a = a1$

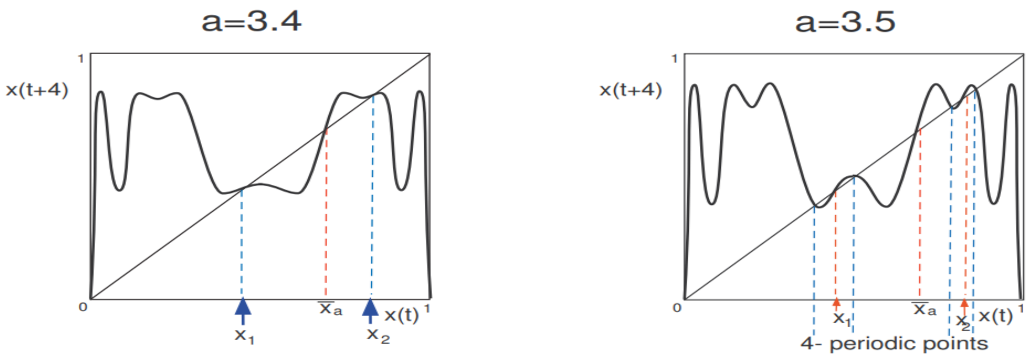


Figure 4.10: The loss of stability of x_1 and x_2 and birth of 4 stable 4-periodic points

for f_a).

What we observe in the behaviour of the map f_a when a varies is not the pitch-fork bifurcation (which is visible in the 2^n-iterate of f_a), but a phenomenon which is called period doubling bifurcation, see figure (4.12) .

This phenomenon occurs for a succession of values a_i, (where the 2^{i-1}-cycle loses stability and the stable 2^i-cycle appears), which is converging to a value $a_\infty = 3.569946\ldots$, and whose first values are

$$a_1 = 3, \quad a_2 \approx 3.49949, \quad a_3 \approx 3.54409, \quad a_4 \approx 3.5644, \quad a_5 \approx 3.5687$$

4.2 Universality and Feigenbaum constants

This phenomenon of a succession of period adding bifurcations is not peculiar of the logistic map. Indeed, Feigenbaum proved in 1975 that every family $F_a = aF(x)$ of functions defined on the unit

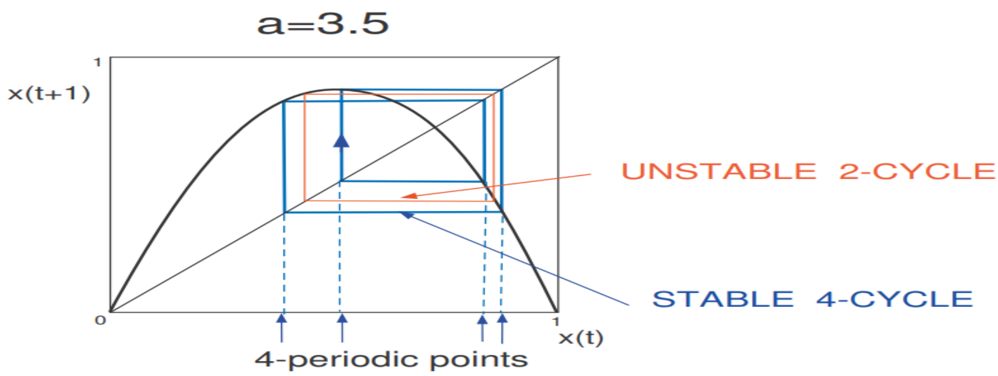


Figure 4.11: The stable 4-cycle and the unstable 2-cycle.

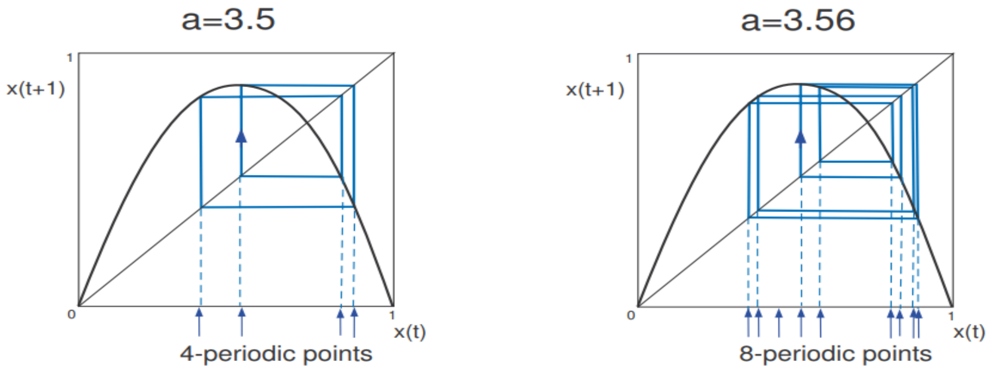


Figure 4.12: By a period doubling bifurcation a 4-cycle loses stability and appears a stable 8-cycle.

interval, such that F is at least 3 times differentiable and has a unique maximum in $[0, 1]$, exhibits the same behaviour. Such functions are said unimodal Moreover, he found two 'universal constants', that are characteristics only of the cascade of doubling periods bifurcation, and not depend on the particular map we are using. These constants are denoted by δ and α :

$$\delta = \lim_{n\to\infty} \frac{a_n - a_{n-1}}{a_{n+1} - a_n} = 4.66920160910299067185320382\ldots$$

The windows of the parameter values between successive bifurcation values decreases very rapidly. The constant α is given by

$$\alpha = \lim_{n\to\infty} \frac{d_n}{d_{n+1}} = 2.502907875095892822283902873218\ldots$$

where d_n is the distance between two branching points (coming from the preceding bifurcation) at the value $a = a_n$.

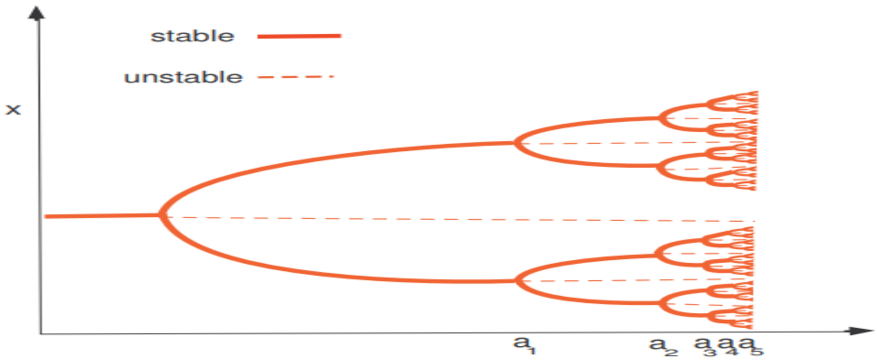
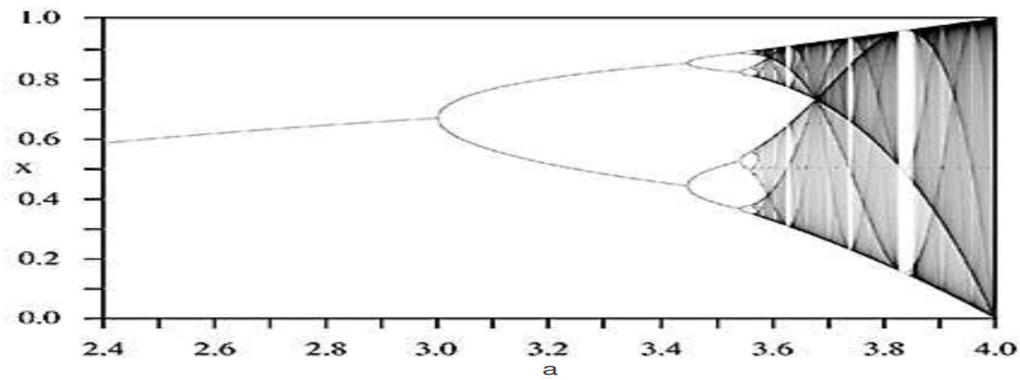


Figure 4.13: Scheme of the cascade of period doubling bifurcations

4.3 Chaos and other periods

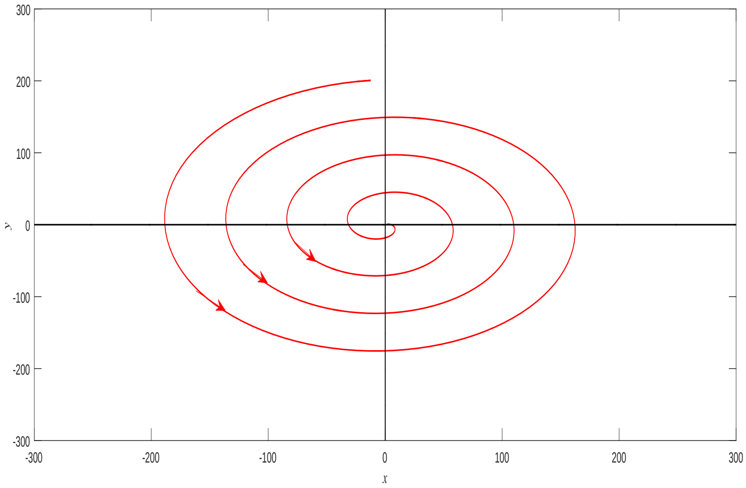
At the value a_∞ the 'periodic cycle' is an infinite set of points which is called Feigenbaum attractor and has a fractal dimension equal to 0.538. This dimension is the same for unimodal maps. For values of $a > a_\infty$ the map f_a has chaotic behaviour, but there are intervals where there are attracting stable cycles, as shown in this bifurcation diagram, where the stable attracting set is plotted versus a. The period 3 loses stability by a doubling period cascade, so that there are all $3 \cdot 2^n$ periodic points, characterised by the Feigenbaum constants.

Remark 4.3.1 *The ratio between the diameters of successive circles on the real axis of the Mandelbrot set converges to the Feigenbaum constant δ*



4.4 Neimark bifurcation

The **Neimark-Sacker bifurcation** occurs in systems of dimension greater than 1. This bifurcation is characterized by the emergence (when the bifurcation parameter μ crosses the critical value μ_0) of a closed invariant curve from a fixed point, as the fixed point changes stability via a pair of complex eigenvalues crossing the unit circle. The bifurcation can be **supercritical** (Figure 4.14) or **subcritical** (Figure **??**), giving rise to a stable or unstable closed invariant curve, respectively.



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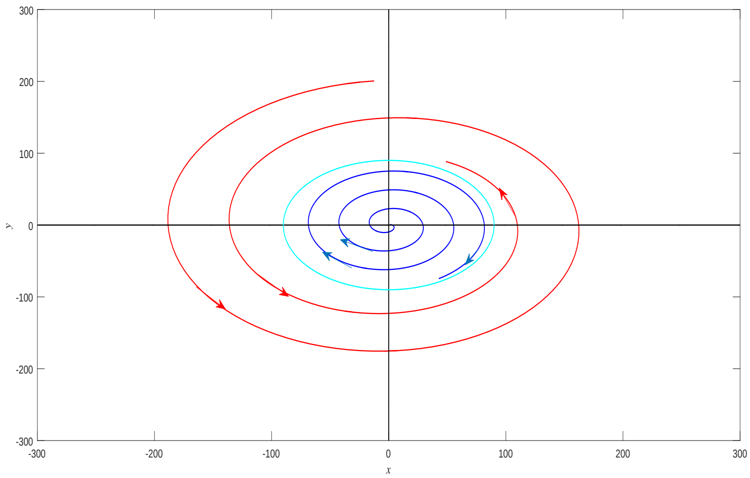

Figure 4.14: Supercritical Neimark–Sacker bifurcation. (a) Stable focus for $\mu < \mu_0$. (b) Stable closed invariant curve for $\mu > \mu_0$.

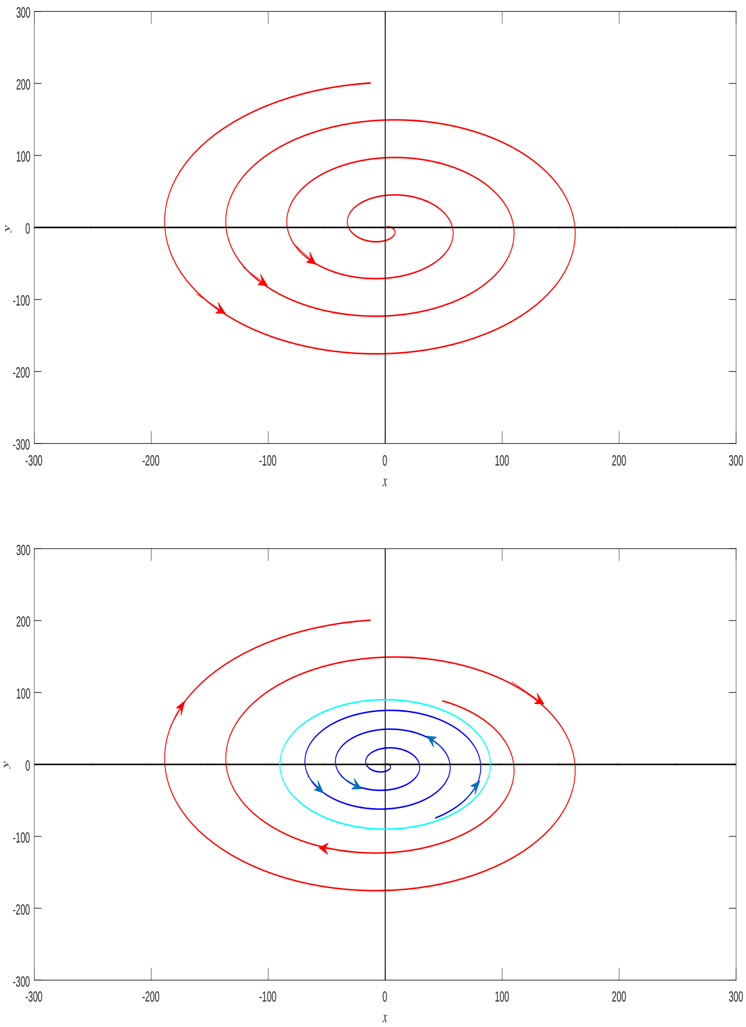

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Figure 4.15: Subcritical Neimark–Sacker bifurcation. (a) Stable focus for $\mu < \mu_0$. (b) Unstable closed invariant curve for $\mu > \mu_0$.

CHAPTER 5

INTRODUCTION TO CHAOS THEORY.

The world around us often appears unpredictable, disordered, random, and chaotic. A chaotic system is a simple or complex system that is sensitive to initial conditions and exhibits repetitive behavior with strong recurrence. A small disturbance can lead to immense instability or imbalance that is not predictable in the long term. Thus, simple systems can give rise to complex phenomena. A chaotic system is the opposite of a perfectly regular system. In the following section, we attempt to give a definition of chaos, as formulated by R.L. Devaney [6].

5.0.1 Chaotic Dynamical Systems

In the literature, several mathematical definitions of chaos can be found, but up to now, there is no universally accepted mathematical definition of chaos. Before presenting a definition of chaos proposed by R.L. Devaney [6], a few basic definitions are necessary.

Definition 5.0.1
A function $f : J \to J$ is said to be ***topologically transitive*** *if, for every pair of open sets $U, V \subset J$, there exists an integer $k > 0$ such that*

$$f^k(U) \cap V \neq \emptyset.$$

Definition 5.0.2
A function $f : J \to J$ has ***sensitive dependence on initial conditions*** *if there exists a constant $\delta > 0$ such that, for every $x \in J$ and for every neighborhood N_x of x, there exists a point $y \in N_x$ and an integer $n > 0$ such that*

$$|f^n(x) - f^n(y)| > \delta.$$

Intuitively, a function has sensitive dependence on initial conditions if there are points arbitrarily close to x that eventually separate from x by at least δ under iteration by f.

We now turn to one of the central themes in dynamical systems: the notion of **chaos**. We present here a particular definition given by R.L. Devaney [6], as it applies to a wide range of examples and is generally easy to verify.

Definition 5.0.3

*Let V be a set. A function $f : V \to V$ is said to be **chaotic on** V if:*

1. *f has sensitive dependence on initial conditions,*
2. *f is topologically transitive,*
3. *The set of periodic points is dense in V.*

To summarize, a chaotic function possesses three essential properties: **unpredictability**, **indecomposability**, and an element of **regularity**.

A chaotic system is unpredictable due to its sensitivity to initial conditions. It is indecomposable because it cannot be split into two disjoint, non-interacting invariant open subsets—thanks to topological transitivity. And finally, amidst this apparent randomness, there is still a regular structure: the periodic points are dense in the space.

5.0.2 Attractors, Basins of Attraction, and Boundaries

The analysis of the long-term behavior of a dynamical system requires a definition of attractors that applies to sets more general than just fixed points and periodic orbits.

Definition 5.0.4

*Let $U \supset \mathbb{R}^q$ be an open set, and let $f : U \longrightarrow U$ be a function. A closed and bounded set $A \subset U$ is called an **attractor** if $f(A) = A$ and there exists $r > 0$ such that $d(x_0, A) < r$ implies*

$$\lim_{n \to +\infty} d(x_n, A) = 0.$$

Attractors play a fundamental role in the study of the long-term behavior of dynamical systems. Some of these attractors have extremely complex geometry, to the point that some have been called **strange attractors** [25]. Attractors can be classified into three categories [1]:

1. A fixed point (a single point),
2. A periodic orbit (a finite set of points),
3. A chaotic attractor (strange attractor), which includes all other types of attractors.

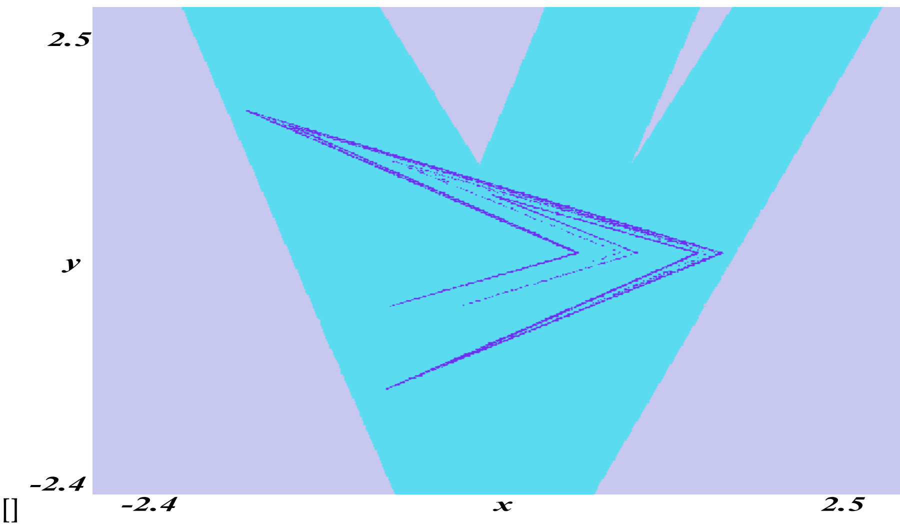

Figure 5.1: Lozi attractor obtained for $a = 1.7$ and $b = 0.5$.

Definition 5.0.5

Given an attractor A, we call the ***basin of attraction*** *of A the set of all initial conditions* x_0 *such that*

$$\lim_{n \to +\infty} d(x_n, A) = 0.$$

Different basins of attraction are separated by basin boundaries. The geometry of these boundaries is often as complex as the geometry of the attractors themselves.

Figure (5.1) presents the attractor of the Lozi map, defined by (5.1):

$$L \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 - a|x| + by \\ x \end{pmatrix}. \tag{5.1}$$

where a and b are bifurcation parameters.

5.0.3 Lyapunov Exponents

In the analysis of a dynamical system, the Lyapunov exponent [30] is used to quantify the stability or instability of its trajectories. A Lyapunov exponent is either a real (finite) number, or it equals $+\infty$ or $-\infty$. An unstable motion has a positive Lyapunov exponent, while a stable motion has a negative Lyapunov exponent. Bounded motions of a linear system have a Lyapunov exponent that is negative or zero.

The Lyapunov exponent can be used to study the stability (or instability) of equilibrium points in nonlinear systems.

In this subsection, we will define the notion of the Lyapunov exponent and show how it can be used to study chaotic systems and even detect the presence of chaos within a system.

1. **Case of a one-dimensional dynamical system:**

   Let f be a function from $\mathbb{R}$ to $\mathbb{R}$. We choose two very close initial conditions, x_0 and $x_0 + \epsilon$, and observe how the trajectories diverge over time. Suppose they diverge on average at an exponential rate. Then, there exists a real number μ such that after n iterations:

   $$|f^n(x_0 + \epsilon) - f^n(x_0)| \simeq \epsilon \exp(n\mu),$$

   from which we get:

   $$\log\left|\frac{f^n(x_0 + \epsilon) - f^n(x_0)}{\epsilon}\right| \simeq n\mu,$$

   Letting ϵ tend toward zero, we find:

   $$\mu \simeq \log\left|\frac{df^n(x_0)}{dx_0}\right|.$$

   Finally, letting n tend to infinity and using the chain rule:

   $$\frac{d}{dx} f^n(x_0) = \prod_{i=0}^{n-1} f'(x_i),$$

   we obtain:

   $$\mu = \lim_{n \to +\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log\left|f'(x_i)\right|. \tag{5.2}$$

   μ is called the Lyapunov exponent. By its definition, the Lyapunov exponent characterizes the stability of an orbit $O(x_0)$. If $\mu > 0$, the orbit $O(x_0)$ is unstable. If $\mu < 0$, it is stable.

   Figure (5.2) shows the Lyapunov exponent of the logistic map (5.3) for $3 < a \leq 4$:

   $$x_{n+1} = ax_n(1 - x_n). \tag{5.3}$$

2. **Case of a two-dimensional dynamical system:**

   Let us now generalize the notion of the Lyapunov exponent to the multidimensional case. Let $f : \mathbb{R}^m \longrightarrow \mathbb{R}^m$ and consider

   $$x_{n+1} = f(x_n), \tag{5.4}$$

   the dynamical system defined by f.

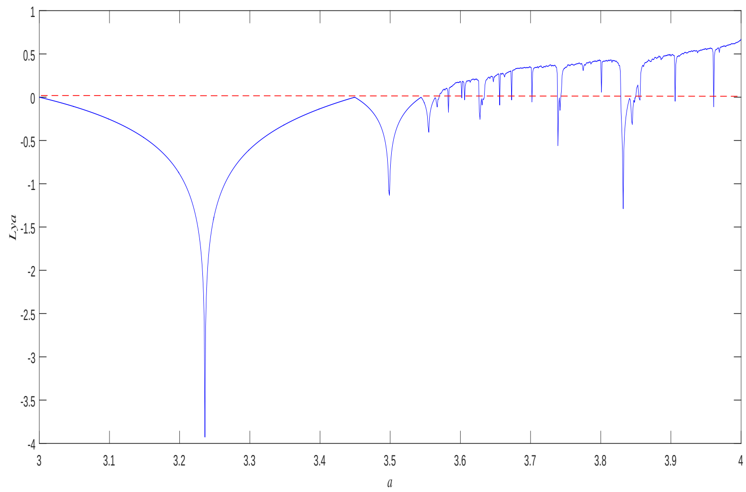

Figure 5.2: Lyapunov exponent of the logistic map (5.3) as a function of the parameter a.

First, note that an m-dimensional system will have m Lyapunov exponents μ_i, for $i = 1, 2, \ldots, m$. Each exponent measures the divergence rate along one of the system's axes. For each $n \geq 1$, consider the Jacobian matrix of the function f^n (denoted by $J_n(x_0)$) evaluated at a point x_0. As in the one-dimensional case, we have:

$$J_n(x_0) = \prod_{i=1}^{n} J(x_i), \tag{5.5}$$

where $x_i = f^i(x_0)$ are the points along the orbit $O(x_0)$, and $J(x)$ is the Jacobian matrix of f evaluated at the point x.

To compute the Lyapunov exponents μ_i, we consider:

$$f^n(x_0 + \epsilon) - f^n(x_0)$$

Let $x_0' = x_0 + \epsilon$, and perform a first-order Taylor expansion of $f^n(x_0')$ around x_0:

$$x_n - x_n' = J_n(x_0)(x_0 - x_0').$$

If $J_n(x_0)$ is diagonalizable, then there exists an invertible matrix P_t such that $D_n(x_0) = P_t^{-1} J_n(x_0) P_t$, where $D_n(x_0)$ is a diagonal matrix containing the eigenvalues of $J_n(x_0)$. Let these eigenvalues be denoted by θ_i, $i = 1, \ldots, m$. The m Lyapunov exponents are then defined by:

$$\mu_i = \lim_{n \to +\infty} \frac{1}{n} \ln(\theta_i^n), \quad i = 1, \ldots, m.$$

Figure (5.3) shows the two Lyapunov exponents of the Hénon map defined by:

$$\begin{cases} x_{n+1} = 1 - ax_n^2 + by_n \\ y_{n+1} = x_n \end{cases} \tag{5.6}$$

for $b = 0.3$ and $0.2 < a \leq 1.5$.

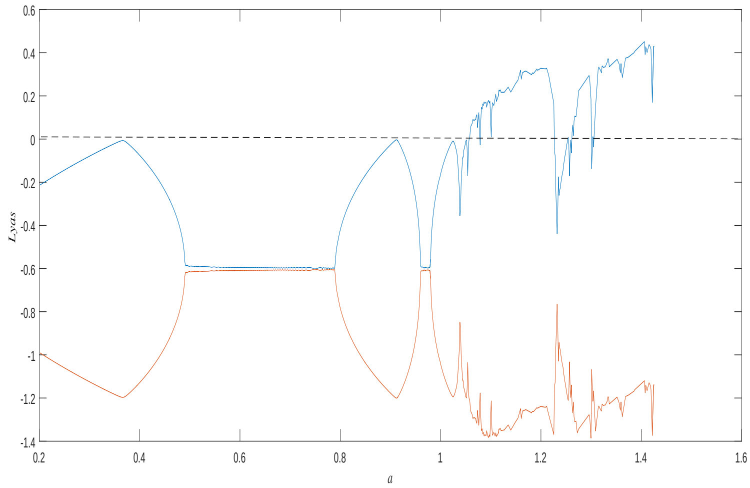

Figure 5.3: Lyapunov exponents of the Hénon map for $b = 0.3$ and $0.2 < a \leq 1.5$.

The Lyapunov exponents of an attractive periodic orbit are also negative. If one of the exponents is zero, it indicates a bifurcation point. If one of the exponents is positive, this implies sensitivity to initial conditions—i.e., chaos.

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