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Study of existence, uniqueness, stability and numerical resolutions of differential equations and inclusions of fractional order

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May Allah bestow blessings upon his noble messenger, his family, and companions, and may He bless our lives abundantly.

Benzahi Ahlem

Dedication

I dedicate this thesis to my beloved parents, whose unwavering love, support, and sacrifices have been the foundation of my journey.

To my dear husband, family and friends, your encouragement and understanding have fueled my determination to reach this milestone.

To all those who have touched my life, directly or indirectly, and have contributed to my development, I express my heartfelt gratitude.

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Abstract

This thesis comprehensively addresses theoretical and numerical aspects of a specific class of fractional differential equations. Initially, we discuss results related to the uniqueness, existence and stability in the Hyers-Ulam sense for a category of initial value problems concerning nonlinear implicit fractional differential equations with non-instantaneous impulses including the Caputo-Fabrizio fractional derivative. Furthermore, our focus extends to exploring existence and uniqueness results for a class of fractional integro-differential equations (FIDEs) with non-instantaneous impulses under the Caputo fractional derivative. To achieve the existence and uniqueness results, we employed the fixed point theorems of Krasnoselskii, Darbo combined with the Kuratowski's measure of noncompactness as well as the Banach contraction principle. Adding a numerical dimension, we delve into the resolution of linear Fredholm fractional integro-differential equations, where the fractional derivative is considered in the Caputo sense. To establish this, we utilize the least squares method (LSM) alongside spectral approximation, employing a compact combination of shifted Chebyshev polynomials (SCP) of the first kind. Throughout the thesis, various examples are provided to validate and elucidate both theoretical and numerical results discussed in each chapter.

Key words: Fractional differential equation, fractional integro-differential equations, Caputo fractional derivative, Caputo-Fabrizio fractional integral, Caputo-Fabrizio fractional derivative, non-instantaneous impulse, fixed point theorems, measure of noncompactness, Hyers-Ulam stability, least squares approximation, Chebyshev polynomials, Chebyshev spectral method.

AMS Subject Classification: 26A33, 34A12, 34A37, 34K37, 35A01, 41A50, 93E24.

ملخص

تتناول هذه الأطروحة بشكل شامل الجوانب النظرية والعددية لفئة معينة من المعادلات التفاضلية الكسرية. في البداية، نناقش النتائج المتعلقة بالوجود والوحدانية واستقرارالحلول لفئة من المعادلات التفاضلية الكسرية الضمنية غير الخطية ذات النبضات غير اللحظية وتتضمن مشتق كابوتو-فابريزيو الجزئي. علاوة على ذلك، يمتد تركيزنا إلى استكشاف نتائج الوجود والوحدانية لفئة من المعادلات التفاضلية التكاملية الكسرية ذات النبضات غير اللحظية تحت مشتقة كابوتو الكسرية. ولتحقيق نتائج الوجود والوحدانية، تم استخدام نظريتي النقطة الصامدة لكراسنوسيلسكي، وداربو المرتبطة بقياس عدم التراص، وكذا نظرية انكماش بناخ . بإضافة البعد العددي، فإننا نتعمق في حل معادلات فريدهولم التكاملية التفاضلية الخطية، حيث يتم اعتبار المشتقة الكسرية من نوع كابوتو. ولتحقيق ذلك، قمنا بتقديم طريقة المربعات الصغرى باستخدام مجموعة مدمجة من كثيرات حدود تشيبيشيف المزاحة من النوع الأول. في كل فصل من الأطروحة، يتم تقديم أمثلة مختلفة للتحقق من صحة ودقة النتائج النظرية والعددية المتحصل عليها.

كلمات مفتاحية: المعادلات التفاضلية الكسرية، المعادلات التفاضلية التكاملية الكسرية، المشتقة الكسرية لكابوتو، تكامل كابوتو-فابريزيو من الرتبة الكسرية، المشتقة الكسرية لكابوتو-فابريزيو، نبضات غيرلحظية، قياس عدم التراص، نظريات النقطة الصامدة، استقرار أولام-هايرز، كثيرات حدود تشيبيشيف، طريقة تشيبيشيف الطيفية، طريقة المربعات الصغرى.

Résumé

Cette thèse aborde de manière exhaustive les aspects théoriques et numériques d'une classe spécifique d'équations différentielles fractionnaires. Tout d'abord, on examine les résultats liés à l'existence, l'unicité et la stabilité de type Hyers-Ulam de solutions pour une catégorie de problèmes de valeurs initiales des équations différentielles fractionnaires non linéaires avec des impulsions non instantanées, incorporant la dérivée fractionnaire de Caputo-Fabrizio. De plus, notre exploration se porte sur les résultats d'existence et d'unicité de solutions pour une classe d'équations intégro-différentielles fractionnaires avec des impulsions non instantanées sous la dérivée fractionnaire de Caputo. Pour parvenir aux résultats concernant l'existence et d'unicité des solutions, on applique le théorème de point fixe de Krasnoselskii et le théorème de point fixe de Darbo combiné avec la mesure de non compacité de Kuratowski, ainsi que le principe de contraction de Banach. En ajoutant une dimension numérique, la résolution des équations intégro-différentielles fractionnaires linéaires de Fredholm est présentée où la dérivée fractionnaire est considérée au sens de Caputo. Pour réaliser cela, on introduit la méthode des moindres carrés en employant une combinaison compacte des polynômes de Chebyshev décalés de première espèce. Tout au long de la thèse, divers exemples sont présentés pour valider et élucider les résultats théoriques et numériques discutés dans chaque chapitre.

Mots clés : Équation différentielle fractionnaire, équations intégro-différentielles fractionnaires, dérivée fractionnaire de Caputo, intégrale de Caputo-Fabrizio d'ordre fractionnaire, dérivée fractionnaire de Caputo-Fabrizio, impulsion non instantanée, mesure de non compacité, théorèmes de point fixe, stabilité de Hyers-Ulam, polynômes de Tchebychev, méthode spectrale de Tchebychev, approximation au sens des moindres carrés.

Publications

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Acronym List

AE	Absolute Error
CFfd	Caputo-Fabrizio fractional derivative
FC	Fractional Calculus
FD	Fractional Derivative
FDEs	Fractional Differential Equations
FIDEs	Fractional Integro-Differential Equations
FPT	Fixed Point Theorem
HU	Hyers-Ulam
IVP	Initial Value Problem
KMNC	Kuratowski's Measure of Noncompactness
LSM	Least Squares Method
NIIs	Non-Instantaneous Impulses
OM	Our Method
SCP	Shifted Chebyshev polynomials

General introduction

The fractional calculus, known as FC, is a branch of mathematics that deals with generalizing the concepts of differentiation and integration to non-integer orders. It finds its origins in a correspondence that unfolded over several months in 1695 between Leibniz and L'Hospital. During that year, Leibniz penned a letter to L'Hospital, which marked the inception of FC, posing the following inquiry [80] :

"Can the meaning of derivatives with integer order be generalized to derivatives with non-integer orders?" L'Hopital was somewhat curious about the above question and replied by another simple one to Leibniz: "What if the order will be 1/2?". Leibniz in a letter dated September 30, 1695, replied: "It will lead to a paradox, from which one day useful consequences will be drawn."

The question Leibniz posed regarding a fractional derivative (specifically, a semi-derivative) continued to captivate attention in the subsequent decades [80, 35]. Subsequent to L'Hopital's and Leibniz's initial exploration, fractional calculus emerged as an exclusive domain of inquiry for Europe's most adept mathematical thinkers. Euler, in 1730, expressed in his writings [35]:

"When n is a positive integer and p is a function of $\upsilon, p = p(\upsilon)$, the ratio of $d^n p$ to $d\upsilon^n$ can always be expressed algebraically. But what kind of ratio can then be made if n be a fraction?"

Across the expanse of time, numerous esteemed mathematicians have lent their expertise to shape the theory of fractional calculus. Remarkably, the precise birth of FC is attributed to September 30, 1695. Its foundational roots delve into the seminal works of Bernoulli (1697), Euler (1730), and Lagrange (1772). Subsequent to these pioneers, a succession of luminaries including Laplace in 1812, Lacroix in 1819, Fourier in 1822, Abel in 1823, and Liouville in 1832, Riemann in 1847, Green in 1859, Grunwald in 1867, Letnikov in 1868, Nekrasov (1888), Laurent (1884), Hadamard (1892), Weyl (1917), Riesz (1922), Kober (1940), Kuttner (1953), M. Caputo (1967), K.S. Miller, B.Ross (1993), and numerous others have advanced the foundational principle of fractional calculus [70, 64, 53].

The derivative, a cornerstone of applied mathematics, quantifies the rate of change in functions, essential for constructing models addressing real-world issues. Over the last three decades, fractional derivatives ascended in significance because of their superior suitability in tackling diverse practical problems in numerous fields including acoustics, control theory, chaos, signal processing, economics, bioengineering, and more [81, 88, 73].
The convergence of preceding contributions has given rise to various forms of fractional order derivatives featuring different kernels, whether singular or nonsingular [17]. The famous and most commonly used in theory of fractional calculus are Riemann-Liouville fractional derivative (FD) [59] and Caputo FD [26]. There are also other types, among them, Caputo-Fabrizio FD [25], generalized FD [45], Hadamard and Hilfer FD [46], Atangana-Baleanu FD [14], conformable FD [66], and more.

Especially noteworthy is Michele Caputo's groundbreaking concept the Caputo derivative which emerged in the 1967s and played a transformative role in reshaping FC. Unlike traditional derivatives, this innovation broadened differentiation to non-integer orders, revolutionizing our comprehension of intricate systems. It excels in capturing dynamics marked by memory, non-locality, and unconventional behaviors. By encompassing a function's entire history rather than just its immediate state, the Caputo derivative becomes indispensable across physics, engineering, biology, and finance. Its applications range from modeling viscoelastic materials and neuronal dynamics to refining option pricing models in finance. This unique attribute of accommodating fractional orders empowers scientists, engineers, and mathematicians, unlocking previously inaccessible phenomena. [43, 71, 9, 10, 11, 74].

Building on this foundation, the Caputo-Fabrizio fractional derivative (CFfd) emerged as a groundbreaking solution to overcome limitations in classical constitutive equations, particularly in describing the intricate behavior of modern materials used in advanced technologies. Caputo and Fabrizio presented a novel fractional derivative in 2015, incorporating an exponential function as its kernel. This innovation was a response to address the limitations imposed by singular kernels found in traditional Caputo derivatives. By replacing the problematic kernel with $\exp(-s(\upsilon-y)/(1-s))$ and $\frac{1}{\sqrt{2\pi(1-s^2)}}$, they effectively eliminated singularities at $\upsilon = y$, enhancing the derivative's applicability. This alteration was implemented to address the challenge posed by singular kernels inherent in the traditional Caputo FD. In this modification, the original kernel $(\upsilon-y)^{-s}$ was substituted with the function $\exp\left(-\frac{s(\upsilon-y)}{1-s}\right)$, and $\frac{1}{\Gamma(1-s)}$ was replaced by $\frac{1}{\sqrt{2\pi(1-s^2)}}$. The crucial distinction between the former and updated definitions lies in the ability of the new kernel to eliminate the singularity at $\upsilon = y$ [25]. The CFfd finds broad applications in biology, infectious disease studies, and has been particularly effective in modeling the evolution of the COVID-19 pandemic, as evidenced by various research sources [27, 14, 15, 8, 48].

FC has surged in prominence across diverse domains, unlocking new possibilities and refining approaches in various applications. In electromagnetic theory has seen significant growth. Engheta's introduction of fractional curl operators in 1998 [34], further expanded by Naqvi and Abbas [68], created the foundation for "Fractional Paradigms in Electromagnetic Theory". This pioneering work has led to widespread acceptance and extensive utilization of FC in contemporary electromagnetic research. Notably, Faryad and Naqvi's investigation of a rectangular Waveguide showcased innovative applications of FC [36].

Moreover, in control engineering, the increasing demand for precise and efficient task execution by robots emphasizes the need for resilient control systems to reduce production time. Flexible robots operating within expansive workspaces are affected by nonlinear and fractional-order dynamic effects [79, 103, 98, 89].

Outside the realm of engineering, the challenges associated with the diffusion of biological populations manifest nonlinearly, leading to an increasing prevalence of fractional-order differential equations in various research domains [33, 90, 91, 75].

FC, especially in fractional reaction-diffusion equations, outperforms traditional mathematical models by offering more accurate descriptions of real-world phenomena. These equations excel in capturing intricate behaviors present in complex systems. Specialized analytical and numerical methods, including finite element, finite difference, Adomian decomposition, and spectral methods [101, 28, 78, 57] have emerged to handle fractional equations effectively.

In specific circumstances, a system undergoes external impulses that can impact it briefly or over prolonged periods during its motion. Modeling such systems often involves impulsive differential equations, incorporating an additional condition to define applied impulses, these equations have been extensively explored in [56, 21]. Impulsive differential equations generally fall into two categories: "instantaneous", where impulses occur momentarily and are modeled through abrupt changes at a singular time point, and "non-instantaneous", where impulses extend over periods, modeled by specific conditions on a quantity during defined time intervals. Non-instantaneous equations offer a more realistic portrayal of reality, as nothing in practical situations occurs instantaneously. However, they emerged relatively recently, making their appearance in 2013 [42]. The initial exploration of non-instantaneous impulses (NIIs) fractional differential equations (FDEs) seems to have occurred around 2013 [72]. Research in this area has persisted in recent years, evident from the research publications [100, 16, 31], and the wealth of related literature.

A commonly studied mathematical model that incorporates non-instantaneous impulses is the Fractional integro-differential equations (FIDEs). They involve fractional derivatives and integrals which capture the memory effects and long-range interactions present in the system. They have found extensive utility in modeling diverse processes across applied sciences, encompassing domains like biology, engineering, finance, physics, and more. These equations enable the modeling of systems characterized by memory effects, long-range interactions, and abrupt changes that endure for a finite duration [18, 63, 55, 12]. These equations enable the representation of systems characterized by memory effects, long-range interactions, and abrupt changes that endure for a finite duration. Several studies, including Kataria et al. [50], Khan et al. [51], Abbass [2], and Benkhettou et al. [24], investigate non-instantaneous impulsive functional integro-differential equations within Banach spaces. These investigations employ tools such as operator semigroup theory, Banach contraction principles, Leray-Schauder nonlinear alternative and Krasnoselskii's fixed point theorem.

Fixed point theory, originating from the iterative method of Picard for solving differential equations, has evolved into a significant mathematical concept. Banach's fixed point theorem emerged from this framework and has since developed into a standalone subject. Lately, there has been a notable resurgence in the application of fixed point theorems, specifically, the Banach and Krasnoselskii fixed point theorems, as vital instruments in resolving FDEs. These theorems stand as pivotal theories, extensively utilized by numerous scientists, as evidenced by their applications in various studies. Notable book references such as [3, 4, 104] and articles [22, 32, 41, 5, 55] highlight this prevalent approach, showcasing the application of fixed point theorems in solving these FDEs, indicating their relevance and effectiveness in addressing complex phenomena in diverse fields.

The renaissance of fixed point theorems in solving FDEs has spurred innovative unions with noncompactness measures. These theorems fuse the principles of fixed point theory with quantitative measures of noncompactness, notably Kuratowski's measure $\eta(\mathcal{F})$, which evaluates the spread within a bounded set $\mathcal{F}$. These measures, exemplified in recent propositions like Darbo's theorem, have attracted attention in Banach spaces and associated properties. Noteworthy literature, including references such as [85, 23, 1, 19, 7, 13], not only demonstrates the application of fixed point theorems in proving existence and uniqueness solutions for FDEs but also showcases the integration of noncompactness measures within these theorems, underscoring their collaborative power in addressing intricate problems across diverse scientific disciplines.

Concurrently, stability analysis, which finds its roots in Ulam's seminal investigation in

1940 [94, 93], has undergone significant evolution and advancement. Ulam's pioneering work laid the groundwork for subsequent developments in the field, serving as a catalyst for further exploration into the stability of mathematical systems. One notable outcome of this exploration is the concept of Hyers-Ulam stability, introduced by Donald H. Hyers in 1941 [44]. This stability notion offers a straightforward and practical method for obtaining approximate solutions to differential equations, proving instrumental in both theoretical inquiries and real-world applications. Moreover, the scope of Hyers-Ulam stability has expanded beyond traditional differential equations to encompass the study of stability for fractional differential equations. In the recent past, a significant cohort of mathematicians has devoted their efforts to scrutinizing the Hyers-Ulam stability of FDEs, yielding a considerable number of findings [49, 95, 96, 85, 86, 87] and the books [76, 29] and references therein.

The intricate challenges posed by solving FDEs, particularly FIDES, analytically have driven the widespread adoption of numerical methods. Notably, spectral methods such as the Legendre collocation method [83], Chebyshev pseudo spectral method [52], and Legendre spectral element method [30], and others in [77, 60, 97]. Complementing these techniques, the versatile and precise least squares method has emerged as a valuable tool for tackling FIDEs, well-documented in the studies by Mohammed et al. [65], Nanware et al. [67], Sabeg et al. [84], and Jia et al. [47]. In this method, the selection of appropriate basis functions is crucial. Spectral analysis aids in identifying these functions, often comprising compact combinations of orthogonal polynomials. Using this selection of basis function, the dimension of the approximation space will be diminished. Integrating these methods allows the least squares approach to efficiently tackle these complex equations by leveraging the streamlined basis provided by spectral analysis, showcasing their collaborative synergy in numerical problem-solving.

Below, we present an overview of the organization of the thesis, comprising four chapters that delineate the contributions made in this work.

Chapter 1: The chapter begins with an exploration of basic notions pertinent to functional spaces and basic definitions and theorems employed within this thesis. Following this, it proceeds to delve into Kuratowski's measure of noncompactness, shedding light on properties related to this measure. The subsequent sections were dedicated to more advanced topics, such as the Gamma and Beta functions, as well as the theory of fractional calculus, including definitions and fundamental lemmas, theorems, and properties. Additionally, fixed point theorems were extensively discussed. Moving forward, The chapter concludes with the final section, wherein we elaborate on properties of Chebyshev polynomials that will be employed in this thesis.

Chapter 2:

Within this chapter, we introduce two primary findings. Firstly, we explore the examination of the existence and uniqueness outcomes. Subsequently, we derive Hyers-Ulam stability results for the following class of Caputo-Fabrizio FDEs with non-instantaneous impulses

$$\begin{cases} \text{For} \quad \upsilon \in \theta_{\mathtt{m}},\ \mathtt{m} = 0, 1, \ldots, \mathrm{n},\ \mu \in]0,1[,\ \lambda > 0, \\ {}^{CF}\mathfrak{D}^{\mu}_{\mathfrak{s}_{\mathtt{m}},\upsilon}\varphi(\upsilon) = -\lambda\varphi(\upsilon) + \mathrm{g}(\upsilon, \varphi(\upsilon)), \\ \\ \text{For} \quad \upsilon \in J_{\mathtt{m}}, \mathtt{m} = 1, 2, \ldots, \mathrm{n},\ \sigma \in]0,1[,\ \sigma \neq \mu, \\ \varphi(\upsilon) = p + {}^{CF}\mathfrak{I}^{\sigma}_{\upsilon_{\mathtt{m}},\upsilon}\mathrm{h}_{\mathtt{m}}(\upsilon, \varphi(\upsilon)) - {}^{CF}\mathfrak{I}^{\mu}_{0,\mathfrak{s}_{\mathtt{m}}}\mathrm{g}\left(\mathfrak{s}_{\mathtt{m}}, \varphi\left(\mathfrak{s}_{\mathtt{m}}\right)\right), \\ \\ \varphi(0) = \varphi_0. \end{cases} \tag{1}$$

Where $\theta_0 = (0, \upsilon_1]$, $J_{\mathtt{m}} = (\upsilon_{\mathtt{m}}, \mathfrak{s}_{\mathtt{m}}]$, for all $\mathtt{m} = 1, 2, \ldots, \mathrm{n}$, $\theta_{\mathtt{m}} = (\mathfrak{s}_{\mathtt{m}}, \upsilon_{\mathtt{m}+1}]$, for all $\mathtt{m} = 0, 1, \ldots, \mathrm{n}$. ${}^{CF}\mathfrak{D}^{\mu}_{\mathfrak{s}_{\mathtt{m}},\upsilon}$ is the CFfd of order $\mu \in]0,1[$, with the lower limit $\mathfrak{s}_{\mathtt{m}}$, $0 = \mathfrak{s}_0 < \upsilon_1 \leq \mathfrak{s}_1 \leq \upsilon_2 < \cdots < \upsilon_{\mathrm{n}} \leq \mathfrak{s}_{\mathrm{n}} \leq \upsilon_{\mathrm{n}+1} = \mathcal{T}$ are prefixed numbers, $\mathrm{g} : [0, \mathcal{T}] \times \mathbb{R} \to \mathbb{R}$ and $\mathrm{h}_{\mathtt{m}} : [\upsilon_{\mathtt{m}}, \mathfrak{s}_{\mathtt{m}}] \times \mathbb{R} \to \mathbb{R}$, $\mathtt{m} = 1, 2, \ldots, \mathrm{n}$ are continuous, $\lambda > 0$ and p is a real number.
$\mathfrak{I}^{\sigma}_{\upsilon_{\mathtt{m}},\upsilon}\mathrm{h}_{\mathtt{m}}$ and $\mathfrak{I}^{\mu}_{0,\mathfrak{s}_{\mathtt{m}}}\mathrm{g}$ are presented by the following expressions:

$$\begin{aligned} {}^{CF}\mathfrak{I}^{\gamma}_{\upsilon_{\mathtt{m}},\upsilon}\mathrm{h}_{\mathtt{m}}(\upsilon, \varphi(\upsilon)) &= \frac{2(1-\gamma)}{M(\gamma)(2-\gamma)}\mathrm{h}_{\mathtt{m}}(\upsilon, \varphi(\upsilon)) + \frac{2\gamma}{M(\gamma)(2-\gamma)}\int_{\upsilon_{\mathtt{m}}}^{\upsilon}\mathrm{h}_{\mathtt{m}}(\theta, u(\theta))d\theta, \\ {}^{CF}\mathfrak{I}^{\mu}_{0,\mathfrak{s}_{\mathtt{m}}}\mathrm{g}\left(\mathfrak{s}_{\mathtt{m}}, \varphi\left(\mathfrak{s}_{\mathtt{m}}\right)\right) &= \frac{2(1-\mu)}{M(\mu)(2-\mu)}\mathrm{g}(\mathfrak{s}_{\mathtt{m}}, \varphi(\mathfrak{s}_{\mathtt{m}})) + \frac{2\mu}{M(\mu)(2-\mu)}\int_0^{\mathfrak{s}_{\mathtt{m}}}\mathrm{g}(\theta, \varphi(\theta))d\theta. \end{aligned}$$

The existence and uniqueness results are established through the application of both the Banach contraction principle and Darbo's fixed point theorem (FPT) combined with the Kuratowski's measure of noncompactness (KMNC). Following this, we demonstrate that the above problem is Hyers-Ulam stable. To confirm the validity of our findings, we present two illustrative examples.

Chapter 3: In this chapter, we investigate the existence of at least one solution for the following non-instantaneous impulsive fractional integro-differential equations

$$\begin{cases} {}^{C}\mathfrak{D}^{\mu}\varphi(\upsilon) = \xi(\upsilon, \varphi(\upsilon)) + \displaystyle\int_0^{\upsilon}\Psi(\upsilon, r, \varphi(r))dr, \quad \upsilon \in (\delta_{\mathtt{m}}, \upsilon_{\mathtt{m}+1}], \mathtt{m} = 0, \ldots n, \\ \varphi(\upsilon) = \dfrac{1}{\Gamma(\mu)}\displaystyle\int_{\upsilon_{\mathtt{m}}}^{\upsilon}(\upsilon - r)^{\mu-1}\mathbb{G}_{\mathtt{m}}(r, \varphi(r_{\mathtt{m}}^{-}))dr, \quad \upsilon \in (\upsilon_{\mathtt{m}}, \delta_{\mathtt{m}}], \mathtt{m} = 1, \ldots n, \\ \alpha_1\varphi(0) + \alpha_2\varphi(\upsilon) = \eta(0). \end{cases} \tag{2}$$

Where ${}^{C}\mathfrak{D}^{\mu}$ is Caputo's differential operator of order $\mu \in (0,1]$, $\theta = [0,\mathcal{T}], \mathcal{T} > 0$, $\alpha_1, \alpha_2 \in \mathbb{R}$. $\theta_0 = [0, \upsilon_1]$, $\theta_{\mathtt{m}} = (\delta_{\mathtt{m}}, \upsilon_{\mathtt{m}+1}]; \mathtt{m} = 0, \ldots, n$, $J_{\mathtt{m}} = (\upsilon_{\mathtt{m}}, \delta_{\mathtt{m}}]; \mathtt{m} = 1, \ldots n$, $\xi : \theta \times \mathbb{R} \to \mathbb{R}, \Psi : \theta \times \theta \times \mathbb{R} \to \mathbb{R}$, $\mathbb{G}_{\mathtt{m}} : J_{\mathtt{m}} \times \mathbb{R} \to \mathbb{R}$ are continuous functions with $\xi(\upsilon, \varphi(\upsilon))_{\upsilon=0} = 0$. We consider the split of the interval θ with respect to $\upsilon_{\mathtt{m}}, \delta_{\mathtt{m}}$ such that $0 < \upsilon_{\mathtt{m}} < \delta_{\mathtt{m}} < \mathcal{T}$ for $\mathtt{m} = 1, 2, 3, \ldots, n$ and assume $\upsilon_{n+1} = \mathcal{T}$.

The results are founded upon the application of both Krasnoselskii FPT and Darbo's FPT combined with the KMNC. Finally, two concrete examples are provided to substantiate the significant findings of the study.

Chapter 4: This chapter is devoted to presenting an approximation to the solution for the following linear FIDE, considering two distinct cases for the initial conditions.

$$ {}^{C}\mathfrak{D}^{\mu}\varphi(\upsilon) = f(\upsilon) + \int_0^1 \mathcal{K}(\upsilon, \tau)\varphi(\tau)d\tau, \quad 0 \le \upsilon \le 1, \tag{3} $$

with two cases of the initial conditions. For the first initial condition where $0 < \mu \le 1$, we impose $\varphi(0) = 0$. Meanwhile, in the second initial conditions where $1 < \mu \le 2$, we impose $\varphi(0) = \varphi'(0) = 0$. Functions $f(\upsilon)$ and $\mathcal{K}(\upsilon, \tau)$ are predefined, while $\varphi(\upsilon)$ denotes the unknown function to be determined.

To solve this class of FIDEs numerically, we will apply the LSM using a compact combination of SCP of the first Kind.

In conclusion, we wrap up our thesis by summarizing key findings and suggesting potential directions for future research.

Chapter 1

Preliminaries

This chapter serves as a foundation, introducing the requisite mathematical tools, notations, and concepts that form the cornerstone of the subsequent chapters. Within this context, we delve into the vital attributes of fractional differential operators, scrutinizing their essential properties. Additionally, we undertake a comprehensive review of the fundamental characteristics inherent to measures of noncompactness and fixed point theorems. Alongside these discussions, we also present notable properties of Chebyshev polynomials. These aspects hold pivotal importance in the context of our forthcoming discussions concerning fractional differential equations.

1.1 Basic notions

Let $\mathcal{C}(\theta, \mathbb{R})$ denote the Banach space consisting of all continuous functions from $\theta = [0, \mathcal{T}]$, $\mathcal{T} > 0$, to $\mathbb{R}$ equipped with the norm

$$\|g\|_\infty = \sup_{\upsilon \in \theta} |g(\upsilon)|,$$

and denote $L^1(\theta)$ as the Banach space of measurable functions $g : \theta \to \mathbb{R}$ that are Lebesgue integrable, equipped with the norm

$$\|g\|_{L^1} = \int_0^{\mathcal{T}} |g(\upsilon)| d\upsilon.$$

Definition 1.1 ([54]). *We say that g is Caratheodory if*

- *$g(., t)$ is measurable for each $t \in \mathbb{R}$.*
- *$g(\upsilon, .)$ is continuous for each $\upsilon \in \theta$.*

Definition 1.2. *Let f be a function defined on $\mathbb{R}_+^*$, $\mathtt{m} \in \mathbb{R}$ and $n \in \mathbb{N}$. We define:*

- $f \in \mathcal{C}_{\mathtt{m}}$ *if there exists a real* $b > \mathtt{m}$, *such that* $f(\upsilon) = \upsilon^b f_1(\upsilon)$, *where* $f_1(\upsilon) \in \mathcal{C}[0, \infty)$.
- $f \in \mathcal{C}_{\mathtt{m}}^n$ *if and only if* $f^{(n)} \in \mathcal{C}_{\mathtt{m}}$.

Definition 1.3. *[61] Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces. A linear operator T from $\mathcal{X}$ into $\mathcal{Y}$ is bounded if T(B) is a bounded subset of $\mathcal{Y}$ whenever B is a bounded subset of $\mathcal{X}$.*

Definition 1.4. *[61] Let $\mathcal{E}$ be a subset of a Banach space $\mathcal{Y}$. $\mathcal{E}$ is considered relatively compact if its closure $\overline{\mathcal{E}}$ is compact, where the closure $\overline{\mathcal{E}}$ of $\mathcal{E}$ is the union of $\mathcal{E}$ and its limit points.*

Definition 1.5. *[61] Assume $\mathcal{X}$ and $\mathcal{Y}$ are Banach spaces. A linear operator T from $\mathcal{X}$ to $\mathcal{Y}$ is deemed compact if for any bounded subset B of $\mathcal{X}$, the image T(B) is a relatively compact subset of $\mathcal{Y}$.*

Theorem 1.1 (Lebesgue dominated convergence theorem [6])**.** *Suppose g is Lebesgue integrable on E. The sequence $\{f_n\}$ of measurable functions satisfies:*

1. $|f_n| \leq g$ *almost everywhere on E for $n \in \mathbb{N}$;*
2. $\lim\limits_{n\to\infty} f_n = f$ *almost everywhere on E.*

Then, $f \in L(E)$ and

$$\lim_{n\to\infty} \int_E f_n = \int_E f. \tag{1.1}$$

Theorem 1.2 (Arzelà-Ascoli theorem [6])**.** *Consider the subset $\mathcal{G}$ of $\mathcal{C}(\theta, X)$. Then, $\mathcal{G}$ is considered relatively compact in $\mathcal{C}(\theta, X)$ if and only if the subsequent conditions hold:*

i) The set $\mathcal{G}$ is uniformly bounded, i.e., there exists a constant $K > 0$ in which

$$\|g(\upsilon)\| \leq K, \quad \text{for all } \upsilon \in \theta, \quad \text{and all } g \in \mathcal{G}.$$

ii) The set $\mathcal{G}$ is equicontinuous, which means for all $\varepsilon > 0$, there exists $a > 0$ so that

$$|\upsilon_1 - \upsilon_2| < a \Rightarrow |g(\upsilon_1) - g(\upsilon_2)| < \epsilon, \quad \forall \upsilon_1, \upsilon_2 \in \theta, \forall g \in \mathcal{G}.$$

1.2 Kuratowski's measure of noncompactness

In this section, we introduce Kuratowski's measure of noncompactness (referred to as **KMNC**) and outline its fundamental properties.
Let $\mathcal{Y}$ be a metric space and $F_{\mathcal{Y}}$ represent the family of all bounded subsets of this space.

Definition 1.6 ([19])**.** *A map $\eta : F_{\mathcal{Y}} \to [0,\infty)$ is called a measure of noncompactness on $\mathcal{Y}$ if it satisfies the following properties for all $B,\ B_1, B_2 \in F_{\mathcal{Y}}$.*

- $\eta(B) = 0$ *if and only if B is precompact (regularity).*
- $\eta(B) = \eta(\bar{B})$ *(invariance under closure).*
- $\eta\left(B_1 \cup B_2\right) = \max\left\{\eta\left(B_1\right), \eta\left(B_2\right)\right\}$ *(semi-additivity).*

Definition 1.7 ([19])**.** *Consider $\mathcal{Y}$ as a Banach space, and let $W_{\mathcal{Y}}$ be the family of bounded subsets of $\mathcal{Y}$. The KMNC is the map $\eta : W_{\mathcal{Y}} \to [0,\infty)$ defined by*

$$\eta(\mathcal{S}) = \inf\left\{\epsilon > 0 : \mathcal{S} \subset U_{j=1}^{m} \mathcal{S}_j,\ \operatorname{diam}\left(\mathcal{S}_j\right) \leq \epsilon\right\},$$

where $\mathcal{S} \in W_{\mathcal{Y}}$.

The measure η satisfies the following properties

- $\eta(\mathcal{S}) = 0 \Leftrightarrow \overline{\mathcal{S}}$ is compact ($\mathcal{S}$ is relatively compact).
- $\eta(\mathcal{S}) = \eta(\overline{\mathcal{S}})$.
- $\mathcal{S}_1 \subset \mathcal{S}_2 \Rightarrow \eta\left(\mathcal{S}_1\right) \leq \eta\left(\mathcal{S}_2\right)$.
- $\eta\left(\mathcal{S}_1 + \mathcal{S}_2\right) \leq \eta\left(\mathcal{S}_1\right) + \eta\left(\mathcal{S}_2\right)$.
- $\eta(b\mathcal{S}) = |b|\eta(\mathcal{S}),\ b \in \mathbb{R}$.
- $\eta(\operatorname{conv} \mathcal{S}) = \eta(\mathcal{S})$.

Lemma 1.1 ([40])**.** *Suppose $W \subseteq \mathcal{C}(\theta, \mathbb{R})$ is a equicontinuous and bounded set, hence*

- *Function $\upsilon \to \eta(W(\upsilon))$ is continuous on θ, with*
$$\eta(W) = \sup_{0 \leq \upsilon \leq \mathcal{T}} \eta(W(\upsilon)). \tag{1.2}$$
- $\eta\left(\int_0^{\mathcal{T}} g(r)dr :\ g \in W\right) \leq \int_0^{\mathcal{T}} \eta(W(r))dr,$ *where*
$$W(r) = \{g(r) :\ g \in W\},\ r \in \theta. \tag{1.3}$$

1.3 Gamma and Beta functions

1.3.1 Gamma function

The Gamma function, symbolized by $\Gamma(\mu)$, plays a crucial role in fractional calculus, extending the notion of the factorial $s!$ to encompass real and complex values of s.

Definition 1.8. *([73]) We define the Gamma function as follows:*

$$\Gamma(\mu)=\int_0^{+\infty} \upsilon^{\mu-1}e^{-\upsilon}\mathrm{d}\upsilon,$$

converge in only one the right half of the complex plane when $\mathrm{Re}(\mu)>0$.
It has the following property:

$$\Gamma(\mu+1)=\mu\Gamma(\mu),$$

hence, for positive integer values of s, we find that $\Gamma(s)=(s-1)!$.

Example 1.1. *A notable specific value of the function is* $\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$, *indeed*

$$\Gamma\left(\frac{1}{2}\right)=\int_0^{\infty}\frac{e^{-\upsilon}}{\sqrt{\upsilon}}d\upsilon$$

Let $\mathfrak{e}=\sqrt{\upsilon}$, *then* $\upsilon=\mathfrak{e}^2$ *and* $d\upsilon=2\mathfrak{e}\ d\mathfrak{e}$,
thus:

$$\Gamma\left(\frac{1}{2}\right)=\int_0^{\infty}e^{-\mathfrak{e}^2}d\mathfrak{e}$$

knowing that

$$\int_0^{\infty}e^{-\mathfrak{e}^2}d\mathfrak{e}=\frac{\sqrt{\pi}}{2},\left(\textit{Gauss integral}:\int_{-\infty}^{+\infty}\mathbb{E}^{-\mu\upsilon^2}d\upsilon=\sqrt{\frac{\pi}{\mu}}\quad \mu\in\mathbb{R}_+^*\right),$$

it comes that

$$\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}.$$

Lemma 1.2. *For any $\mu\in\mathbb{C}$ such that* $\mathrm{Re}(\mu)>0$, *we have:*

- $\Gamma(\mu+1)=\mu\Gamma(\mu)$.
- $\Gamma(s)=(s-1)!,\quad \forall s\in\mathbb{N}^*$.
- $\Gamma\left(s+\frac{1}{2}\right)=\dfrac{(2s!)\sqrt{\pi}}{2^{2s}s!},\quad \forall s\in\mathbb{N}$.

1.3.2 Beta function

In some cases, using the Beta function is convenient.

Definition 1.9. *([73]) The Beta function is expressed as follows:*

$$\mathrm{B}(r,s)=\int_0^1 \upsilon^{r-1}(1-\upsilon)^{s-1}\mathrm{d}\upsilon, \quad r,s>0.$$

The subsequent formula establishes the connection between both Beta and Gamma functions:

$$\mathrm{B}(r,s)=\frac{\Gamma(r)\Gamma(s)}{\Gamma(r+s)}, \quad r,s>0.$$

1.4 Fundamental fractional order calculus

In this part, we introduce definitions related to fractional order calculus, accompanied by a collection of properties, propositions and lemmas that hold significance in the context of this thesis.

1.4.1 Integrals of fractional order

Definition 1.10 ([53])**.** *Consider a function $g \in \mathcal{C}_\nu$ with $\nu \geq -1$ and let $\mu > 0$. The Riemann-Liouville integral operator of order μ is given by:*

$$\begin{aligned}\mathfrak{I}^{\mu} g(\upsilon) &= \frac{1}{\Gamma(\mu)}\int_0^{\upsilon}(\upsilon-s)^{\mu-1}g(s)ds, \quad \mu>0,\ \upsilon>0,\\ \mathfrak{I}^{0} g(\upsilon) &= g(\upsilon).\end{aligned}$$

Considering $a,b \geq 0$, $c \geq -1$, $\nu \geq -1$ and $g \in \mathcal{C}_\nu$, the Riemann-Liouville fractional integration verifies the following properties:

- $\mathfrak{I}^{a}\mathfrak{I}^{b}g(\upsilon)=\mathfrak{I}^{a+b}g(\upsilon)$,
- $\mathfrak{I}^{a}\mathfrak{I}^{b}g(\upsilon)=\mathfrak{I}^{b}\mathfrak{I}^{a}g(\upsilon)$,
- $\mathfrak{I}^{a}\upsilon^{c}=\frac{\Gamma(c+1)}{\Gamma(a+c+1)}\upsilon^{c+a}$.

Definition 1.11 ([58])**.** *Let $0<\mu<1$ and $g \in L^1(\theta)$, The Caputo-Fabrizio fractional integral of order μ is defined by:*

$${}^{CF}\mathfrak{I}^{\mu}g(\upsilon)=\frac{2(1-\mu)}{M(\mu)(2-\mu)}g(\upsilon)+\frac{2\mu}{M(\mu)(2-\mu)}\int_0^{\upsilon}g(s)ds,\ \upsilon\geq 0,$$

here, $M(\mu)>0$ denotes the normalization constant which is dependent on μ, and fulfills to the conditions: $M(0)=M(1)=1$.

Below are the fundamental characteristics of the operator ${}^{CF}\mathfrak{J}^{\mu}$.

- ${}^{CF}\mathfrak{J}^{\mu}g(\upsilon) = g(\upsilon),$ where $\mu = 0$,
- ${}^{CF}\mathfrak{J}^{\mu}(g(\upsilon) + h(\upsilon)) = {}^{CF}\mathfrak{J}^{\mu}g(\upsilon) + {}^{CF}\mathfrak{J}^{\mu}h(\upsilon)$,
- ${}^{CF}\mathfrak{J}^{\mu}[{}^{CF}\mathfrak{D}^{\mu}g(\upsilon)] = g(\upsilon) - g(0)$.

1.4.2 Derivatives of fractional order

Definition 1.12. *For $\mu > 0$, let $l \in \mathbb{N}$ be the smallest integer greater than μ, and let $g \in \mathcal{C}^{l}_{-1}$ be a function. We define the Caputo FD of order μ as*

$$
{}^{C}\mathfrak{D}^{\mu}g(\upsilon) = \mathfrak{J}^{l-\mu}\mathfrak{D}^{l}g(\upsilon) = \begin{cases} \dfrac{1}{\Gamma(l-\mu)}\displaystyle\int_0^{\upsilon}(\upsilon - s)^{l-\mu-1}g^{(r)}(s)ds, & \upsilon > 0,\ l-1 < \mu < l, \\ \dfrac{d^{l}g(\upsilon)}{d\upsilon^{l}}, & \mu = l. \end{cases}
$$

Here are some Caputo FD properties:

- ${}^{C}\mathfrak{D}^{\mu}c = 0$, c is a constant.
- Let $[\mu]$ denote the integer part of μ, then we have:

$$
{}^{C}\mathfrak{D}^{\mu}\upsilon^{l} = \begin{cases} 0, & l \in \mathbb{N}_0 = \{0, 1, 2, \dots\}, l < [\mu], \\ \dfrac{\Gamma(l+1)}{\Gamma(l+1-\mu)}\upsilon^{l-\mu}, & l \in \mathbb{N}_0, l \geq [\mu]. \end{cases}
$$

- The linearity of Caputo fractional differentiation is expressed as follows:

$$
{}^{C}\mathfrak{D}^{\mu}(ag + bh)(\upsilon) = a{}^{C}\mathfrak{D}^{\mu}g(\upsilon) + b{}^{C}\mathfrak{D}^{\mu}h(\upsilon), \quad \forall a, b \in \mathbb{R}.
$$

Definition 1.13 ([25, 58])**.** *For a function $g \in \mathcal{C}^{1}(\theta)$ and $0 < \mu < 1$, the Caputo-Fabrizio FD of order μ is defined as:*

$$
{}^{CF}\mathfrak{D}^{\mu}g(\upsilon) = \frac{(2-\mu)M(\mu)}{2(1-\mu)}\int_0^{\upsilon}\exp\left(-\frac{\mu}{1-\mu}(\upsilon - s)\right)g'(s)ds,\ \upsilon \in \theta.
$$

Here are the fundamental characteristics of the operator ${}^{CF}\mathfrak{D}^{\mu}$.

- ${}^{CF}\mathfrak{D}^{\mu}g(\upsilon) = g(\upsilon),$ where $\mu = 0$,
- ${}^{CF}\mathfrak{D}^{\mu}(ag(\upsilon) + bh(\upsilon)) = a{}^{CF}\mathfrak{D}^{\mu}g(\upsilon) + b{}^{CF}\mathfrak{D}^{\mu}h(\upsilon)$, where $a, b \in \mathbb{R}$,
- ${}^{CF}\mathfrak{D}^{\mu}c = 0$, where c is constant.

1.4.3 Necessary lemmas, theorems and propositions

Lemma 1.3. *[102] Let $\mu > 0$ and $g \in \mathcal{C}(\theta) \cap L(\theta)$. Then the FDE ${}^{C}\mathfrak{D}^{\mu} g(\upsilon) = 0$ admits solutions*

$$g(\upsilon) = a_0 + a_1\upsilon + a_2\upsilon^2 + \cdots + a_{m-1}\upsilon^{m-1}, a_i \in \mathbb{R}, i = 0, 1, 2, \ldots, m-1, \ m = [\mu] + 1.$$

Lemma 1.4. *[102] Let $\mu > 0$, then*

$$\mathfrak{I}^{\mu}\, {}^{C}\mathfrak{D}^{\mu} g(\upsilon) = g(\upsilon) + a_0 + a_1\upsilon + a_2\upsilon^2 + \cdots + a_{m-1}\upsilon^{m-1},$$

for $m = [\mu] + 1$, and $a_i \in \mathbb{R}, i = \overline{0, m-1}$.

Lemma 1.5. *[58] Let $0 < \mu < 1$. The solution of the subsequent FDE*

$${}^{CF}\mathfrak{D}^{\mu} g(\upsilon) = f(\upsilon), \ \upsilon \geq 0, \tag{1.4}$$

is given by:

$$g(\upsilon) = a_{\mu}[f(\upsilon) - f(0)] + b_{\mu} \int_0^{\upsilon} f(s)ds + g(0), \ \upsilon \geq 0, \tag{1.5}$$

where

$$a_{\mu} = \frac{2(1-\mu)}{(2-\mu)M(\mu)}, \quad b_{\mu} = \frac{2\mu}{(2-\mu)M(\mu)}.$$

Proof. Using Laplace formula transformation, we get

$$\mathscr{L}\left[{}^{\mathrm{CF}}\mathfrak{D}^{\mu} g(\upsilon)\right](s) = \mathscr{L}[f(\upsilon)](s), \quad s > 0.$$

In other words, utilizing

$$\mathscr{L}\left[{}^{\mathrm{CF}}\mathfrak{D}^{\mu} g(\upsilon)\right](s) = \frac{(2-\mu)M(\mu)}{2(s+\mu(1-s))}(s\mathscr{L}[g(\upsilon)](s) - g(0)), \quad s > 0, \tag{1.6}$$

where $\mathscr{L}[g(\upsilon)]$ denotes the Laplace transformation of the function g, we have that

$$\frac{(2-\mu)M(\mu)}{2(s+\mu(1-s))}(s\mathscr{L}[g(\upsilon)](s) - g(0)) = \mathscr{L}[f(\upsilon)](s), \quad s > 0,$$

or equivalently,

$$\mathscr{L}[g(\upsilon)](s) = \frac{1}{s}g(0) + \frac{2\mu}{s(2-\mu)M(\mu)}\mathscr{L}[f(\upsilon)](s) + \frac{2(1-\mu)}{(2-\mu)M(\mu)}\mathscr{L}[f(\upsilon)](s), \quad s > 0.$$

Therefore, leveraging the recognized properties of the inverse Laplace transformation, one can conclude:

$$g(\upsilon) = \frac{2(1-\mu)}{(2-\mu)M(\mu)}f(\upsilon) + \frac{2\mu}{(2-\mu)M(\mu)}\int_0^{\upsilon} f(s)ds + g(0), \quad \upsilon \geq 0. \tag{1.7}$$

To put it another way, the function defined as

$$g(\upsilon) = \frac{2(1-\mu)}{(2-\mu)M(\mu)} f(\upsilon) + \frac{2\mu}{(2-\mu)M(\mu)} \int_0^{\upsilon} f(s)ds + c, \quad \upsilon \geq 0,$$

is also a solution of (1.4) where $c \in \mathbb{R}$ is a constant.
We can alternatively express FDE (1.4) as

$$\frac{(2-\mu)M(\mu)}{2(1-\mu)} \int_0^{\upsilon} \exp\left(-\frac{\mu}{1-\mu}(\upsilon - s)\right) g'(s)ds = f(\upsilon), \quad \upsilon \geq 0,$$

or equivalently,

$$\int_0^{\upsilon} \exp\left(\frac{\mu}{1-\mu} s\right) g'(s)ds = \frac{2(1-\mu)}{(2-\mu)M(\mu)} \exp\left(\frac{\mu}{1-\mu}\upsilon\right) f(\upsilon), \quad \upsilon \geq 0.$$

Upon differentiating both sides of the latter equation, we derive that

$$g'(\upsilon) = \frac{2(1-\mu)}{(2-\mu)M(\mu)} \left(f'(\upsilon) + \frac{\mu}{1-\mu} f(\upsilon)\right), \quad \upsilon \geq 0.$$

Therefore, integrating now from 0 to υ, we conclude as shown in (1.7), that

$$g(\upsilon) = \frac{2(1-\mu)}{(2-\mu)M(\mu)}[f(\upsilon) - f(0)] + \frac{2\mu}{(2-\mu)M(\mu)} \int_0^{\upsilon} f(s)ds + g(0), \quad \upsilon \geq 0.$$

□

Lemma 1.6. *[58] Consider $0 < \mu < 1$ and g denote a solution of the following FDE:*

$$^{CF}\mathfrak{D}^{\mu} g(\upsilon) = 0, \; \upsilon \geq 0, \tag{1.8}$$

subsequently, g is a constant function and then the converse is also true.

Proof. Using the property $^{CF}\mathfrak{I}^{\mu}[^{CF}\mathfrak{D}^{\mu} g(\upsilon)] = g(\upsilon) - g(0)$, we deduce that any solution to equation (1.8) must fulfill the condition $g(\upsilon) = g(0)$ for all $\upsilon \geq 0$. Consequently, it becomes evident that g is bound to be constant function. □

Proposition 1.1. *[58] The unique solution of the following IVP with $0 < \mu < 1$,*

$$^{CF}\mathfrak{D}^{\mu} g(\upsilon) = f(\upsilon), \; \upsilon \geq 0, \tag{1.9}$$

$$g(0) = g_0 \in \mathbb{R}, \tag{1.10}$$

is

$$g(\upsilon) = g_0 + a_{\mu}(f(\upsilon) - f(0)) + b_{\mu}\mathfrak{I}^1 f(\upsilon), \; \upsilon \geq 0, \tag{1.11}$$

where $\mathfrak{I}^1$ denotes a primitive of g.

Proof. Suppose that the IVP (1.9)-(1.10) has two solutions, g_1 and g_2. In that case, we have that

$$^{CF}\mathfrak{D}^{\mu}g_1(\upsilon) - {}^{CF}\mathfrak{D}^{\mu}g_2(\upsilon) = [^{CF}\mathfrak{D}^{\mu}g_1 - g_2](\upsilon) = 0 \text{ and } (g_1 - g_2)(0) = 0.$$

So, by Lemma 1.6, we have that $g_1 - g_2 = 0$. That is $g_1(\upsilon) = g_2(\upsilon)$ for all $\upsilon \geq 0$.
From equation (1.7), it's evident that the function given by the equation (1.11) is a solution of the fractional differential equation (1.9). Additionally, substituting υ with 0 in equation (1.11), we obtain g_0.
Thus, the function given by (1.11) is the unique solution of the IVP (1.9)-(1.10). □

Remark 1.1. *When μ equals 1, the solution of equation (1.9) corresponds to usual primitive of f.*

Proposition 1.2. *[58] For any real number λ and $\mu \in]0,1[$. Therefore, the IVP represented by*

$$\begin{cases} ^{CF}\mathfrak{D}^{\mu}g(\upsilon) = \lambda g(\upsilon) + W(\upsilon), \quad \upsilon \geq 0, \\ g(0) = g_0 \in \mathbb{R}, \end{cases} \tag{1.12}$$

admits a unique solution.

Lemma 1.7. *Let $W \in L^1(\theta)$. Then the IVP*

$$\begin{cases} ^{CF}\mathfrak{D}^{\mu}_{0,\upsilon}g(\upsilon) = -\lambda g(\upsilon) + W(\upsilon), \quad \upsilon \in \theta, \\ g(0) = g_0 \in \mathbb{R}, \end{cases} \tag{1.13}$$

has the following unique solution

$$g(\upsilon) = g_0 - A^{\lambda}_{\mu}W(0) + A^{\lambda}_{\mu}W(\upsilon) + B^{\lambda}_{\mu}\int_0^{\upsilon}[-\lambda g + W](\theta)d\Theta, \tag{1.14}$$

in which

$$A^{\lambda}_{\mu} = \frac{a_\mu}{1+\lambda a_\mu}, \quad B^{\lambda}_{\mu} = \frac{b_\mu}{1+\lambda a_\mu}.$$

Proof. Suppose that g satisfies (1.13). Using Proposition 1.1 and Proposition 1.2, we get

$$^{CF}\mathfrak{D}^{\mu}_{0,\upsilon}g(\upsilon) = -\lambda g(\upsilon) + W(\upsilon),$$

which implies

$$g(\upsilon) - g(0) = a_\mu\left(\lambda(g_0 - g(\upsilon)) + W(\upsilon) - W(0)\right) + b_\mu\int_0^{\upsilon}[-\lambda g + W](\Theta)d\Theta,$$

or

$$g(\upsilon) = g(0) + \frac{a_\mu}{1+\lambda a_\mu}W(\upsilon) - \frac{a_\mu}{1+\lambda a_\mu}W(0) + \frac{b_\mu}{1+\lambda a_\mu}\int_0^{\upsilon}[-\lambda g + W](\Theta)d\Theta.$$

From the initial condition $g(0) = g_0$, we obtain

$$g(\upsilon) = g_0 - A^{\lambda}_{\mu}W(0) + A^{\lambda}_{\mu}W(\upsilon) + B^{\lambda}_{\mu}\int_0^{\upsilon}[-\lambda g + W](\Theta)d\Theta,$$

which satisfies (1.14). □

1.5 Fixed point theorems

This part introduces the fixed point theorems utilized in our study.

Theorem 1.3 (**Banach's** FPT [39]). *Given a closed and non-empty subset $\mathcal{D}$ of a Banach space $\mathcal{Y}$, any contraction mapping T from $\mathcal{D}$ to itself admits a unique fixed point.*

Theorem 1.4 (**Darbo's** FPT [38]). *Consider a closed, nonempty, bounded, and convex subset $\mathcal{D}$ of a Banach space $\mathcal{Y}$. Let* N *be a continuous mapping of $\mathcal{D}$ into itself, , satisfying the condition of being a γ-contraction, i.e., for any nonempty subset* h *of $\mathcal{D}$, we have*

$$\eta(\mathrm{N}(\mathrm{h})) \leq \gamma\eta(\mathrm{h}),\ 0 \leq \gamma < 1,$$

where η denotes the KMNC on $\mathcal{Y}$. Therefore, N *admits a fixed point in $\mathcal{D}$.*

Theorem 1.5. *(**Krasnoselskii's** FPT [39]) Consider a nonempty, convex and closed subset $\mathcal{D}$ of a Banach space $\mathcal{Y}$ and $\mathcal{B}$ and $\mathcal{G}$ denote the operators as*

1. $\mathcal{B}\upsilon + \mathcal{G}x \in \mathcal{D},\ \forall \upsilon, x \in \mathcal{D}$,

2. *$\mathcal{B}$ is continuous and compact,*

3. *$\mathcal{G}$ is a contraction mapping.*

Hence there exists $\upsilon^* \in \mathcal{D}$ where $\upsilon^* = \mathcal{B}\upsilon^* + \mathcal{G}\upsilon^*$.

1.6 Fundamental properties of Chebyshev polynomials

Chebyshev polynomials are a family of orthogonal polynomials named after the Russian mathematician Pafnuty Chebyshev in the 1850s as part of his research into number theory and the theory of differential equations. These polynomials are defined on a specific interval, usually [-1, 1], although variations on this interval are also possible. Chebyshev polynomials possess numerous advantageous properties, such as being easily computable, numerically stable, and featuring rapid convergence rates. Due to these characteristics, they find significant utility across various domains like numerical analysis, approximation theory, signal processing, and other fields within mathematics and physics. The definition of Chebyshev polynomial of the first kind and some of its properties are presented in this section. For more details see [92, 69, 62].

Definition 1.14. *The Chebyshev polynomial $\mathbb{T}_{m}(\upsilon)$ of the first kind, defined by the relation*

$$\mathbb{T}_{m}(\upsilon) = \cos(m \ \arccos \upsilon), \quad \textit{with } m \in \mathbb{N}, \tag{1.15}$$

is a polynomial in υ of degree m.

If the variable υ ranges over the interval [-1, 1], then the corresponding variable β can be considered to range over $[0, \pi]$. These ranges are traversed in opposite directions, given that $\upsilon = -1$ corresponds to $\beta = \pi$ and $\upsilon = 1$ corresponds to $\beta = 0$.
As a consequence of De Moivre's theorem, it is well known that $\cos(\mathtt{m}\beta)$ is a polynomial of degree $\mathtt{m}$ in $\cos(\beta)$. Indeed, we are familiar with the elementary formulas

$$\cos(0\beta) = 1, \ \cos(1\beta) = \cos(\beta), \ \cos(2\beta) = 2\cos^2(\beta - 1),$$

$$\cos(3\beta) = 4\cos^3(\beta - 3\cos\beta), \ \cos(4\beta) = 8\cos^4(\beta - 8\cos^2\beta + 1), \ldots$$

Chebyshev polynomial of the first kind satisfies the following properties:

Property 1.1. *The polynomial $\mathbb{T}_{m}$ of degree m in υ with leading coefficient $a_{m} = 2^{m-1}$, follows the subsequent recurrence relation*

$$\mathbb{T}_{m+1} = 2\upsilon\mathbb{T}_{m} - \mathbb{T}_{m-1}, \quad m = 1, 2, 3 \ldots, \mathbb{T}_0 = 1, \mathbb{T}_1 = \upsilon, \tag{1.16}$$

Proof. Equation (1.16) is derived through direct computation:

$$\mathbb{T}_0(\upsilon) = \cos(0) = 1, \mathbb{T}_1(\upsilon) = \cos\beta = \upsilon,$$

and

$$\mathbb{T}_{\mathtt{m}+1}(\upsilon) + \mathbb{T}_{\mathtt{m}-1}(\upsilon) = \cos(\mathtt{m}+1)\beta + \cos(\mathtt{m}-1)\beta = 2\cos(\beta)\cos(\mathtt{m}\beta) = 2\upsilon\mathbb{T}_{\mathtt{m}}(\upsilon),$$

utilizing the formula $\cos(c + d) = \cos c \ \cos d - \sin c \ \sin d$.
Subsequently, we proceed to demonstrate by induction that $\mathbb{T}_{\mathtt{m}}$ represents a polynomial of degree $\mathtt{m}$ in the variable υ, with $2\mathtt{m} - 1$ serving as the leading coefficient, i.e.,

$$\mathbb{T}_{\mathtt{m}}(\upsilon) = 2^{\mathtt{m}-1}\upsilon^{\mathtt{m}} + \text{lower degree terms}, \mathtt{m} \geq 1. \tag{1.17}$$

For $\mathtt{m} = 1$, equation (1.17) holds true since $\mathbb{T}_1(\upsilon) = \upsilon = 2^{1-1}\upsilon$, and its degree is 1.
Given that equation (1.17) holds for a fixed integer $\mathtt{m} \geq 1$, we are able to express $\mathbb{T}_{\mathtt{m}}$ as

$$\mathbb{T}_{\mathtt{m}}(\upsilon) = 2^{\mathtt{m}-1}\upsilon_{\mathtt{m}} + A_{\mathtt{m}-1}(\upsilon),$$

where $A_{\mathtt{m}-1}$ is a polynomial of degree at most $\mathtt{m}-1$ in the variable υ.
To conclude the demonstration, we utilize relation (1.16) to derive that

$$\mathbb{T}_{\mathtt{m}+1}(\upsilon) = 2\upsilon\mathbb{T}_{\mathtt{m}}(\upsilon)\mathbb{T}_{\mathtt{m}-1}(\upsilon) = 2\upsilon(2^{\mathtt{m}-1}\upsilon^{\mathtt{m}} + A_{\mathtt{m}-1}(\upsilon)) - \mathbb{T}_{\mathtt{m}-1}(\upsilon) = 2^{\mathtt{m}}\upsilon^{\mathtt{m}+1} + \widetilde{A}_{\mathtt{m}}(\upsilon),$$

where $\widetilde{A}_{\mathtt{m}}$ is a polynomial of degree at most $\mathtt{m}$ in υ. Thus, $\mathbb{T}_{\mathtt{m}}(\upsilon)$ is a polynomial of degree $\mathtt{m}$ in the variable υ with $2^{\mathtt{m}-1}$ as leading coefficient.
Using the three-term recurrence relation (1.16), any $\mathbb{T}_{\mathtt{m}}(\upsilon)$ can be generated. Specifically, the first 5 Chebyshev polynomials are as follows:

$$\begin{aligned}
&\mathbb{T}_0(\upsilon) = 1,\\
&\mathbb{T}_1(\upsilon) = \upsilon,\\
&\mathbb{T}_2(\upsilon) = 2\upsilon^2 - 1,\\
&\mathbb{T}_3(\upsilon) = 4\upsilon^3 - 3\upsilon,\\
&\mathbb{T}_4(\upsilon) = 8\upsilon^4 - 8\upsilon^2 + 1,\\
&\mathbb{T}_5(\upsilon) = 16\upsilon^5 - 20\upsilon^3 + 5\upsilon.
\end{aligned}$$

□

Property 1.2. *$\mathbb{T}_m$ satisfies the following orthogonality relation*

$$\int_0^{\pi} \cos(m\beta)\cos(n\beta)d\beta = k_m\delta_{m,n} = \int_{-1}^{1} \mathbb{T}_m(\upsilon)\mathbb{T}_n(\upsilon)\frac{d\upsilon}{\sqrt{1-\upsilon^2}}, \tag{1.18}$$

with $k_0 = \pi, k_m = \frac{\pi}{2}, m \geq 1$ and $\delta_{m,n}$ denotes the Kronecker delta.

Proof. Relation (1.18) is established through direct computation, employing one more time the addition formula

$$2\cos(\mathtt{m}\beta)\cos(m\beta) = \cos(\mathtt{m}+m)\beta + \cos(\mathtt{m}-m)\beta,$$

and as $\upsilon = \cos\beta, 0 < \beta < \pi \Rightarrow d\upsilon = -\sin\beta d\beta = -\sqrt{1-\cos^2\beta}d\beta$. □

Property 1.3. *The polynomial $\mathbb{T}_m$ satisfies the second-order holonomic differential equation*

$$(1-\upsilon^2)\mathbb{T}''_m(\upsilon) - \upsilon\mathbb{T}'_m(\upsilon) + m^2\mathbb{T}_m(\upsilon) = 0,\ m \geq 0. \tag{1.19}$$

Proof. Relation (1.19) is also demonstrated by direct computation. Indeed

$$\mathbb{T}'_{\mathtt{m}}(\upsilon) = \frac{d}{d\upsilon}\mathbb{T}_{\mathtt{m}}(\upsilon),$$

$$
\begin{aligned}
&= \frac{d\beta}{d\upsilon}\frac{d}{d\beta}\mathbb{T}_{\mathtt{m}}(\upsilon),\\
&= \frac{-1}{\sin\beta}\frac{d}{d\beta}\cos(\mathtt{m}\beta),\\
&= \frac{\mathtt{m}\sin(\mathtt{m}\beta)}{\sin\beta},\ \mathtt{m}\geq 1.
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{T}''_{\mathtt{m}}(\upsilon) &= \frac{d}{d\upsilon}\frac{d}{d\upsilon}\mathbb{T}_{\mathtt{m}}(\upsilon),\\
&= \frac{d\beta}{d\upsilon}\frac{d}{d\beta}\left(\frac{d\beta}{d\upsilon}\frac{d}{d\beta}\mathbb{T}_{\mathtt{m}}(\upsilon)\right),\\
&= \frac{-1}{\sin\beta}\frac{d}{d\beta}\left(\frac{-1}{\sin\beta}\frac{d}{d\beta}\cos(\mathtt{m}\beta)\right),\\
&= \frac{\mathtt{m}\cos\beta\sin(\mathtt{m}\beta)}{\sin\beta\sin^2\beta} + \frac{-\mathtt{m}^2\cos(\mathtt{m}\beta)}{\sin^2\beta},\\
&= \frac{\upsilon\mathbb{T}'_{\mathtt{m}}(\upsilon)}{1-\upsilon^2} + \frac{-\mathtt{m}^2\mathbb{T}_{\mathtt{m}}(\upsilon)}{1-\upsilon^2}, \mathtt{m}\geq 1.
\end{aligned}
$$

□

Property 1.4. *For any $m \geq 1$, $\mathbb{T}_m$ has exactly m zeros, all within the interval of orthogonality $(-1, 1)$. These zeros, arranged in ascending order, are determined by*

$$\upsilon_{m,\ell} = \cos\left(\frac{2(m-\ell)+1}{2m}\pi\right),\ 1 \leq \ell \leq m,\ m \geq 1. \tag{1.20}$$

Property 1.5. *The zeros $\upsilon_{m,\ell}$ of $\mathbb{T}_m$ satisfy*

$$\upsilon_{m,\ell} \neq \upsilon_{m+1,\ell}, \forall m \geq 1,\ 1 \leq \ell \leq m,\ 1 \leq j \leq m+1, \tag{1.21}$$

$$\upsilon_{m+1,\ell} < \upsilon_{m,\ell},\ 1 \leq \ell \leq m. \tag{1.22}$$

Chapter 2

Study of Fractional Differential Equations with Non Instantaneous Impulses under Caputo-Fabrizio Derivative

2.1 Introduction

This chapter focuses on investigating the existence, uniqueness and stability in the Ulam-Hyers sense of solutions to a specific type of mathematical problem called nonlinear implicit fractional differential equations (FDEs) with non-instantaneous impulses (NIIs). These equations involve a fractional derivative (FD) known as the Caputo-Fabrizio FD, which captures non-local and memory-dependent effects in the behavior of the system under study. The intricacy of the problem stems from its implicit nature, where the relation among variables are not explicitly defined. This complexity is significantly heightened by the existence of NIIs, sudden and intermittent changes that occur at specific points within the system, causing abrupt transitions. At its essence, the initial value problem involves deciphering a system's behavior from its initial conditions, requiring a solution that complies to the given equation while takes into account the presence of NIIs. In deriving the existence outcomes, we apply well-known mathematical techniques, notably the Banach's standard fixed point theorem (FPT) and the Darbo's FPT combined with the Kuratowski's measure of noncompactness (KMNC). Additionally, we discuss the Hyers-Ulam (HU) stability of the given problem, which further enhances the practical usefulness and dependability of these solutions. To showcase the versatility of the uncovered findings across different values of μ, we provide several examples towards the conclusion. These results carry practical significance across diverse domains,

facilitating enhanced modeling and comprehension of intricate systems.
Inspired by the works of [99, 20, 58], we delve into the examination of the subsequent Caputo-Fabrizio FDEs with NIIs

$$\begin{cases} \text{For} \quad \upsilon \in \theta_{\mathtt{m}}, \ \mathtt{m} = 0, 1, \ldots, \mathrm{n}, \ \mu \in]0,1[, \ \lambda > 0, \\ {}^{CF}\mathfrak{D}^{\mu}_{\mathfrak{s}_{\mathtt{m}},\upsilon}\varphi(\upsilon) = -\lambda\varphi(\upsilon) + \mathrm{g}(\upsilon, \varphi(\upsilon)), \\ \\ \text{For} \quad \upsilon \in J_{\mathtt{m}}, \mathtt{m} = 1, 2, \ldots, \mathrm{n}, \ \sigma \in]0,1[, \ \sigma \neq \mu, \\ \varphi(\upsilon) = p + {}^{CF}\mathfrak{I}^{\sigma}_{\upsilon_{\mathtt{m}},\upsilon}\mathrm{h}_{\mathtt{m}}(\upsilon, \varphi(\upsilon)) - {}^{CF}\mathfrak{I}^{\mu}_{0,\mathfrak{s}_{\mathtt{m}}}\mathrm{g}\left(\mathfrak{s}_{\mathtt{m}}, \varphi\left(\mathfrak{s}_{\mathtt{m}}\right)\right), \\ \\ \varphi(0) = \varphi_0. \end{cases} \tag{2.1}$$

Where $\theta_0 = (0, \upsilon_1]$, $J_{\mathtt{m}} = (\upsilon_{\mathtt{m}}, \mathfrak{s}_{\mathtt{m}}]$, for all $\mathtt{m} = 1, 2, \ldots, \mathrm{n}$, $\theta_{\mathtt{m}} = (\mathfrak{s}_{\mathtt{m}}, \upsilon_{\mathtt{m}+1}]$, for all $\mathtt{m} = 0, 1, \ldots, \mathrm{n}$. ${}^{CF}\mathfrak{D}^{\mu}_{\mathfrak{s}_{\mathtt{m}},\upsilon}$ is the CFfd of order $\mu \in]0,1[$, with the lower limit $\mathfrak{s}_{\mathtt{m}}$, $0 = \mathfrak{s}_0 < \upsilon_1 \leq \mathfrak{s}_1 \leq \upsilon_2 < \cdots < \upsilon_{\mathrm{n}} \leq \mathfrak{s}_{\mathrm{n}} \leq \upsilon_{\mathrm{n}+1} = \mathcal{T}$ are prefixed numbers, $\mathrm{g} : [0, \mathcal{T}] \times \mathbb{R} \to \mathbb{R}$ and $\mathrm{h}_{\mathtt{m}} : [\upsilon_{\mathtt{m}}, \mathfrak{s}_{\mathtt{m}}] \times \mathbb{R} \to \mathbb{R}$, $\mathtt{m} = 1, 2, \ldots, \mathrm{n}$ are continuous, $\lambda >0$ and p is a real number.
$\mathfrak{I}^{\sigma}_{\upsilon_{\mathtt{m}},\upsilon}\mathrm{h}_{\mathtt{m}}$ and $\mathfrak{I}^{\mu}_{0,\mathfrak{s}_{\mathtt{m}}}\mathrm{g}$ are presented by the following expressions:

$$\begin{aligned} {}^{CF}\mathfrak{I}^{\sigma}_{\upsilon_{\mathtt{m}},\upsilon}\mathrm{h}_{\mathtt{m}}(\upsilon, \varphi(\upsilon)) &= \frac{2(1-\sigma)}{M(\sigma)(2-\sigma)}\mathrm{h}_{\mathtt{m}}(\upsilon, \varphi(\upsilon)) + \frac{2\sigma}{M(\sigma)(2-\sigma)}\int_{\upsilon_{\mathtt{m}}}^{\upsilon}\mathrm{h}_{\mathtt{m}}(\Theta, u(\Theta))d\Theta, \\ {}^{CF}\mathfrak{I}^{\mu}_{0,\mathfrak{s}_{\mathtt{m}}}\mathrm{g}\left(\mathfrak{s}_{\mathtt{m}}, \varphi\left(\mathfrak{s}_{\mathtt{m}}\right)\right) &= \frac{2(1-\mu)}{M(\mu)(2-\mu)}\mathrm{g}(\mathfrak{s}_{\mathtt{m}}, \varphi(\mathfrak{s}_{\mathtt{m}})) + \frac{2\mu}{M(\mu)(2-\mu)}\int_0^{\mathfrak{s}_{\mathtt{m}}}\mathrm{g}(\Theta, \varphi(\Theta))d\Theta. \end{aligned}$$

2.2 Existence and uniqueness outcomes

In this section, we address both the existence and uniqueness of solutions for the problem (2.1). Let

$$\mathcal{PC}1 = \Big\{\varphi : \theta \to \mathbb{R} : \varphi|_{J_{\mathtt{m}}} ; \ \mathtt{m} = 1, 2, \ldots, \mathrm{n}, \ \varphi|_{\theta_{\mathtt{m}}} ; \mathtt{m} = 0, 1, \ldots, \mathrm{n} \ \text{ are continuous and there exist} \\ \varphi\left(\mathfrak{s}_{\mathtt{m}}^-\right), \varphi\left(\mathfrak{s}_{\mathtt{m}}^+\right), \varphi\left(\upsilon_{\mathtt{m}}^-\right) \text{ and } \varphi\left(\upsilon_{\mathtt{m}}^+\right)\Big\},$$

a Banach space equipped with the norm

$$\|\varphi\|_{\mathcal{PC}1} = \sup_{\upsilon \in \theta} |\varphi(\upsilon)|.$$

Lemma 2.1. *Let $W : \theta \to \mathbb{R}$ be a continuous function, $\varphi \in \mathcal{PC}1(\theta, \mathbb{R})$ is a solution of the FDEs*

$$\begin{cases} \varphi(0) = \varphi_0, \\ \varphi(\upsilon) = \varphi_m - A_\mu^\lambda \mathrm{g}(\mathfrak{s}_m) + A_\mu^\lambda W(\upsilon) + B_\mu^\lambda \int_{\mathfrak{s}_m}^{\upsilon} [-\lambda\varphi + W](\Theta)d\Theta, \ \upsilon \in \theta_m, \ m = \overline{0, \mathrm{n}}, \\ \varphi(\upsilon) = p + {}^{CF}\mathfrak{I}_{\upsilon_m,\upsilon}^{\sigma} \mathrm{h}_m(\upsilon) - {}^{CF}\mathfrak{I}_{0,\mathfrak{s}_m}^{\mu} W(s_m), \ \upsilon \in J_m, \ m = \overline{1, \mathrm{n}}, \end{cases} \tag{2.2}$$

if and only if φ is a solution of the problem

$$\begin{cases} {}^{CF}\mathfrak{D}_{\mathfrak{s}_m,\upsilon}^{\mu} \varphi(\upsilon) = -\lambda\varphi(\upsilon) + W(\upsilon), & \upsilon \in \theta_m, \ m = \overline{0, \mathrm{n}}, \mu \in]0, 1[, \\ \varphi(\upsilon) = p + {}^{CF}\mathfrak{I}_{\upsilon_m,\upsilon}^{\sigma} \mathrm{h}_m(\upsilon) - {}^{CF}\mathfrak{I}_{0,\mathfrak{s}_m}^{\mu} W(\mathfrak{s}_m), & \upsilon \in J_m, \ m = \overline{1, \mathrm{n}}, \sigma \in]0, 1[, \\ \varphi(0) = \varphi_0 \in \mathbb{R}. \end{cases} \tag{2.3}$$

Proof. Assuming that φ fulfills equation (2.3), for $\upsilon \in \theta_0 = (0, \upsilon_1]$,

$${}^{CF}\mathfrak{D}_{0,\upsilon}^{\mu} \varphi(\upsilon) = -\lambda\varphi(\upsilon) + W(\upsilon).$$

Through integration of the final equation over the interval $[0, \upsilon]$, and utilizing Definition 1.11 and Lemma 1.7, we derive

$$\varphi(\upsilon) = \varphi_0 - A_\mu^\lambda W(0) + A_\mu^\lambda W(\upsilon) + B_\mu^\lambda \int_0^{\upsilon} [-\lambda\varphi + W](\Theta)d\Theta.$$

For $\upsilon \in (\upsilon_1, \mathfrak{s}_1], \varphi(\upsilon) = p + {}^{CF}\mathfrak{I}_{\upsilon_1,\upsilon}^{\sigma} \mathrm{h}_1(\upsilon) - {}^{CF}\mathfrak{I}_{0,\mathfrak{s}_1}^{\mu} W(\mathfrak{s}_1)$.
For $\upsilon \in \theta_1 = (\mathfrak{s}_1, \upsilon_2], \ {}^{CF}\mathfrak{D}_{\mathfrak{s}_1,\upsilon}^{\mu} \varphi(\upsilon) = -\lambda\varphi(\upsilon) + W(\upsilon)$.
When integrating the final equation from $\mathfrak{s}_1$ to υ, and subsequently applying Definition 1.11 and Lemma 1.7, we get

$$\begin{aligned} \varphi(\upsilon) &= \varphi(\mathfrak{s}_1) - A_\mu^\lambda W(\mathfrak{s}_1) + A_\mu^\lambda W(\upsilon) + B_\mu^\lambda \int_{\mathfrak{s}_1}^{\upsilon} [-\lambda\varphi + W](\Theta)d\Theta, \\ &= \varphi_1 - A_\mu^\lambda W(\mathfrak{s}_1) + A_\mu^\lambda W(\upsilon) + B_\mu^\lambda \int_{\mathfrak{s}_1}^{\upsilon} [-\lambda\varphi + W](\Theta)d\Theta. \end{aligned}$$

Ultimately, for any $\upsilon \in \theta_m$ and $\upsilon \in J_m$, utilizing Definition 1.11 and Lemma 1.7, we derive (2.2). The converse portion of the demonstration can be established through standard steps. □

Lemma 2.2. *We say that φ is a solution to the problem described by equation (2.1) if and only if φ satisfies the following integral equation*

$$\begin{cases} \varphi(0) = \varphi_0, \\ \textit{For} \quad \upsilon \in \theta_m, \ m = 0, 1, \dots, \mathrm{n}, \\ \varphi(\upsilon) = C_m^{\mu,\lambda} + A_\mu^\lambda \mathrm{g}(\upsilon, \varphi(\upsilon)) - \lambda B_\mu^\lambda \int_{\mathfrak{s}_m}^{\upsilon} \varphi(\Theta)d\Theta + B_\mu^\lambda \int_{\mathfrak{s}_m}^{\upsilon} \mathrm{g}(\Theta, \varphi(\Theta))d\Theta, \\ \\ \textit{For} \quad \upsilon \in J_m, \ m = 1, 2, \dots, \mathrm{n}, \\ \varphi(\upsilon) = \mathcal{H}_m^\mu + a_\sigma \mathrm{h}_m(\upsilon, \varphi(\upsilon)) + b_\sigma \int_{\upsilon_m}^{\upsilon} \mathrm{h}_m(\Theta, \varphi(\Theta))d\Theta - b_\mu \int_0^{\mathfrak{s}_m} \mathrm{g}(\Theta, \varphi(\Theta))d\Theta, \end{cases} \tag{2.4}$$

where

$$C_{m}^{\mu,\lambda} = \varphi_{m} - A_{\mu}^{\lambda}\mathrm{g}(\mathfrak{s}_{m}, \varphi(\mathfrak{s}_{m})), \qquad \mathcal{H}_{m}^{\mu} = p - a_{\mu}\mathrm{g}(\mathfrak{s}_{m}, \varphi(\mathfrak{s}_{m})),$$

$$a_{\sigma} = \frac{2(1-\sigma)}{(2-\sigma)M(\sigma)}, \ b_{\sigma} = \frac{2\sigma}{(2-\sigma)M(\sigma)}.$$

The following assumptions will be utilized subsequently.

(A_1) $\mathrm{g} : \theta \times \mathbb{R} \to \mathbb{R}$ fulfills the Caratheodory conditions.

(A_2) There exists a positive constant L_{g} in such a way that

$$|\mathrm{g}(\upsilon, \varphi_1) - \mathrm{g}(\upsilon, \varphi_2)| \leq L_{\mathrm{g}}|\varphi_1 - \varphi_2|, \text{ for each } \upsilon \in \theta \text{ and all } \varphi_1, \varphi_2 \in \mathbb{R}.$$

(A_3) Let $\mathrm{h}_{\mathtt{m}} \in C\left(J_{\mathtt{m}} \times \mathbb{R} \to \mathbb{R}\right)$, $\mathtt{m} = 1, 2, \ldots, \mathrm{n}$, and there are positive constants $L_{\mathrm{h}_{\mathtt{m}}}, \mathtt{m} = 1, 2, \ldots, \mathrm{n}$, such that

$$|\mathrm{h}_{\mathtt{m}}(\upsilon, \varphi_1) - \mathrm{h}_{\mathtt{m}}(\upsilon, \varphi_2)| \leq L_{\mathrm{h}_{\mathtt{m}}}|\varphi_1 - \varphi_2|, \text{ for each } \upsilon \in J_{\mathtt{m}} \text{ and all } \varphi_1, \varphi_2 \in \mathbb{R}.$$

(H_1) There exist two functions $\varrho \in L^q\left(\theta_{\mathtt{m}}, \mathbb{R}_+\right)$ $\left(q > \frac{1}{\mu}, \ \mathtt{m} = 0, 1, \ldots, \mathrm{n}\right)$ and $\psi : (0, \infty] \to (0, \infty]$, continuous and nondecreasing continuous, respectively, where

$$|\mathrm{g}(\upsilon, \varphi)| \leq \varrho(\upsilon)\psi(\|\varphi\|), \ \ \upsilon \in \theta, \ \varphi \in \mathbb{R}.$$

(H_2) Let $\mathrm{h}_{\mathtt{m}} \in \mathcal{C}\left(J_{\mathtt{m}} \times \mathbb{R} \to \mathbb{R}\right)$, $\mathtt{m} = 0, 1, \ldots, \mathrm{n}$, and there exists a continuous function $\mathcal{H}_{\mathtt{m}} \in \mathcal{C}(J_{\mathtt{m}}, \mathbb{R}_+)$, with

$$|\mathrm{h}_{\mathtt{m}}(\upsilon, \varphi)| \leq \mathcal{H}_{\mathtt{m}}(\upsilon), \ \upsilon \in J_{\mathtt{m}}, \ \varphi \in \mathbb{R}.$$

(H_3) There exists a continuous function $\Phi \in L^q\left(\theta_{\mathtt{m}}, \mathbb{R}_+\right), q > \frac{1}{\mu}$, $\mathtt{m} = 0, 1, \ldots, \mathrm{n}$, such that for each bounded set $\mathcal{G} \subset \mathbb{R}$, we put

$$\eta(\mathrm{g}(\upsilon, \mathcal{G})) \leq \Phi(\upsilon)\eta(\mathcal{G}), \ \upsilon \in \theta_{\mathtt{m}}.$$

(H_4) For each bounded set $\mathcal{G} \subset \mathbb{R}$, we have

$$\eta(\mathrm{h}_{\mathtt{m}}(\upsilon, \mathcal{G})) \leq \mathcal{H}_{\mathtt{m}}(\upsilon)\eta(\mathcal{G}), \ \upsilon \in J_{\mathtt{m}}.$$

Ultimately, we pose

$$\varrho^* = \sup_{\upsilon \in \theta} \varrho(\upsilon),$$

$$\mathcal{H}^* = \max_{\mathtt{m}=0,\ldots,\mathrm{n}} (\sup_{\upsilon \in J_{\mathtt{m}}} \mathcal{H}_{\mathtt{m}}(\upsilon)),$$

$$\Phi^* = \sup_{\upsilon \in \theta} \Phi(\upsilon).$$

Theorem 2.1. *Assume that* $(\mathrm{A}_1)-(\mathrm{A}_3)$ *are satisfied. If the inequality*

$$\chi=\max\left\{\left((A_\mu^\lambda+B_\mu^\lambda\mathcal{T})L_{\mathrm{g}}+\lambda B_\mu^\lambda\mathcal{T}\right),((a_\sigma+b_\sigma\mathcal{T})L_{\mathrm{h}_m}+(a_\mu+b_\mu\mathcal{T})L_{\mathrm{g}}))\right\}<1 \tag{2.5}$$

is satisfied, thus the problem (2.1) *admits a unique solution.*

Proof. Consider the operator $\mathrm{N}:\mathcal{PC}1\to\mathcal{PC}1$ defined by

$$(\mathrm{N}\varphi)(\upsilon)=\begin{cases}\text{For}\quad \upsilon\in\theta_{\mathtt{m}},\ \text{and}\quad \mathtt{m}=0,1,\ldots,\mathrm{n},\\ C_{\mathtt{m}}^{\mu,\lambda}+A_\mu^\lambda\mathrm{g}(\upsilon,\varphi(\upsilon))-\lambda B_\mu^\lambda\int_{\mathfrak{s}_{\mathtt{m}}}^{\upsilon}\varphi(\Theta)d\Theta+B_\mu^\lambda\int_{\mathfrak{s}_{\mathtt{m}}}^{\upsilon}\mathrm{g}(\Theta,\varphi(\Theta))d\Theta,\\ \text{For}\quad \upsilon\in J_{\mathtt{m}},\ \text{and}\quad \mathtt{m}=0,1,\ldots,\mathrm{n},\\ \mathcal{H}_{\mathtt{m}}^{\mu}+a_\sigma\mathrm{h}_{\mathtt{m}}(\upsilon,\varphi(\upsilon))+b_\sigma\int_{\upsilon_{\mathtt{m}}}^{\upsilon}\mathrm{h}_{\mathtt{m}}(\Theta,\varphi(\Theta))d\Theta-b_\mu\int_0^{\mathfrak{s}_{\mathtt{m}}}\mathrm{g}(\Theta,\varphi(\Theta))d\Theta.\end{cases} \tag{2.6}$$

Since the functions g and $\mathrm{h}_{\mathtt{m}}$ are continuous and by the properties of fractional integrals, the operator $\mathrm{N}:\mathcal{PC}1\to\mathcal{PC}1$ defined in (2.6) is clearly well-defined. Consequently, we prove that N is a contraction mapping.

Case 1 Let $\varphi_1,\varphi_2\in\mathcal{PC}1(\theta,\mathbb{R})$ and $\upsilon\in\theta_{\mathtt{m}}$, $\mathtt{m}=0,1,\ldots,\mathrm{n}$, we observe

$$\begin{aligned}|(\mathrm{N}\varphi_1)(\upsilon)-(\mathrm{N}\varphi_2)(\upsilon)|&=|C_{\mathtt{m}}^{\mu,\lambda}+A_\mu^\lambda\mathrm{g}(\upsilon,\varphi_1(\upsilon))-\lambda B_\mu^\lambda\int_{\mathfrak{s}_{\mathtt{m}}}^{\upsilon}\varphi_1(\Theta)d\Theta+B_\mu^\lambda\int_{\mathfrak{s}_{\mathtt{m}}}^{\upsilon}\mathrm{g}(\Theta,\varphi_1(\Theta))d\Theta\\ &\quad-C_{\mathtt{m}}^{\mu,\lambda}-A_\mu^\lambda\mathrm{g}(\upsilon,\varphi_2(\upsilon))-\lambda B_\mu^\lambda\int_{\mathfrak{s}_{\mathtt{m}}}^{\upsilon}\varphi_2(\Theta)d\Theta-B_\mu^\lambda\int_{\mathfrak{s}_{\mathtt{m}}}^{\upsilon}\mathrm{g}(\Theta,\varphi_2(\Theta))d\Theta,\\ &\leq A_\mu^\lambda|\mathrm{g}(\upsilon,\varphi_1(\upsilon))-\mathrm{g}(\upsilon,\varphi_2(\upsilon))|+\lambda B_\mu^\lambda\int_{\mathfrak{s}_{\mathtt{m}}}^{\upsilon}|\varphi_1(\Theta)-\varphi_2(\Theta)d\Theta|\\ &\qquad+B_\mu^\lambda\int_{\mathfrak{s}_{\mathtt{m}}}^{\upsilon}|\mathrm{g}(\Theta,\varphi_1(\Theta))-\mathrm{g}(\Theta,\varphi_2(\Theta))d\Theta|,\\ &\leq A_\mu^\lambda L_{\mathrm{g}}\|\varphi_1-\varphi_2\|_{\mathcal{PC}1}+\lambda B_\mu^\lambda\mathcal{T}\|\varphi_1-\varphi_2\|_{\mathcal{PC}1}+B_\mu^\lambda\mathcal{T}L_{\mathrm{g}}\|\varphi_1-\varphi_2\|_{\mathcal{PC}1},\\ &\leq\left((A_\mu^\lambda+B_\mu^\lambda\mathcal{T})L_{\mathrm{g}}+\lambda B_\mu^\lambda\mathcal{T}\right)\|\varphi_1-\varphi_2\|_{\mathcal{PC}1},\end{aligned}$$

which implies that

$$\|\mathrm{N}\varphi_1-\mathrm{N}\varphi_2\|_{\mathcal{PC}1}\leq\chi_1\|\varphi_1-\varphi_2\|_{\mathcal{PC}1},\quad\text{with}\quad\chi_1=(A_\mu^\lambda+B_\mu^\lambda\mathcal{T})L_{\mathrm{g}}+\lambda B_\mu^\lambda\mathcal{T}.$$

Case 2 Let $\varphi_1,\varphi_2\in\mathcal{PC}1(\theta,\mathbb{R})$, for $\upsilon\in J_{\mathtt{m}}$, $\mathtt{m}=0,1,\ldots,\mathrm{n}$, we get

$$\begin{aligned}|(\mathrm{N}\varphi_1)(\upsilon)-(\mathrm{N}\varphi_2)(\upsilon)|&=\Bigg|\mathcal{H}_{\mathtt{m}}^{\mu}+a_\sigma\mathrm{h}_{\mathtt{m}}(\upsilon,\varphi_1(\upsilon))+b_\sigma\int_{\upsilon_{\mathtt{m}}}^{\upsilon}\mathrm{h}_{\mathtt{m}}(\Theta,\varphi_1(\Theta))d\Theta-b_\mu\int_0^{\mathfrak{s}_{\mathtt{m}}}\mathrm{g}(\Theta,\varphi_1(\Theta))d\Theta\\ &\quad-\mathcal{H}_{\mathtt{m}}^{\mu}-a_\sigma\mathrm{h}_{\mathtt{m}}(\upsilon,\varphi_2(\upsilon))-b_\sigma\int_{\upsilon_{\mathtt{m}}}^{\upsilon}\mathrm{h}_{\mathtt{m}}(\Theta,\varphi_2(\Theta))d\Theta+b_\mu\int_0^{\mathfrak{s}_{\mathtt{m}}}\mathrm{g}(\Theta,\varphi_2(\Theta))d\Theta\Bigg|,\\ &\leq a_\sigma|\mathrm{h}_{\mathtt{m}}(\upsilon,\varphi_1(\upsilon))-\mathrm{h}_{\mathtt{m}}(\upsilon,\varphi_2(\upsilon))|\end{aligned}$$

$$
\begin{aligned}
&+ b_\sigma \int_{\upsilon_{\mathtt{m}}}^{\upsilon} |\mathrm{h}_{\mathtt{m}}(\Theta, \varphi_1(\Theta)) - \mathrm{h}_{\mathtt{m}}(\Theta, \varphi_2(\Theta))| d\Theta \\
&\quad + b_\mu \int_0^{\mathfrak{s}_{\mathtt{m}}} |\mathrm{g}(\Theta, \varphi_1(\Theta)) - \mathrm{g}(\Theta, \varphi_2(\Theta))| d\Theta, \\
&\leq a_\sigma L_{\mathrm{h}_{\mathtt{m}}} \|\varphi_1 - \varphi_2\|_{\mathcal{PC}1} + b_\sigma \mathcal{T} L_{\mathrm{h}_{\mathtt{m}}} \|\varphi_1 - \varphi_2\|_{\mathcal{PC}1} + b_\mu \mathcal{T} L_{\mathrm{g}} \|\varphi_1 - \varphi_2\|_{\mathcal{PC}1}, \\
&\leq \left((a_\sigma + b_\sigma \mathcal{T}) L_{\mathrm{h}_{\mathtt{m}}} + b_\mu \mathcal{T} L_{\mathrm{g}}\right) \|\varphi_1 - \varphi_2\|_{\mathcal{PC}1}, \\
&\leq \left((a_\sigma + b_\sigma \mathcal{T}) L_{\mathrm{h}_{\mathtt{m}}} + (a_\mu + b_\mu \mathcal{T}) L_{\mathrm{g}})\right) \|\varphi_1 - \varphi_2\|_{\mathcal{PC}1},
\end{aligned}
$$

which implies that

$$
\|\mathrm{N}\varphi_1 - \mathrm{N}\varphi_2\|_{\mathcal{PC}1} \leq \chi_2 \|\varphi_1 - \varphi_2\|_{\mathcal{PC}1}, \quad \text{where} \quad \chi_2 = (a_\sigma + b_\sigma \mathcal{T}) L_{\mathrm{h}_{\mathtt{m}}} + (a_\mu + b_\mu \mathcal{T}) L_{\mathrm{g}}.
$$

From the above cases, we obtain

$$
\|\mathrm{N}\varphi_1 - \mathrm{N}\varphi_2\|_{\mathcal{PC}1} \leq \chi \|\varphi_1 - \varphi_2\|_{\mathcal{PC}1}, \quad \text{where} \quad \chi = \max\{\chi_1, \chi_2\}.
$$

In conclusion, given the condition (2.5), we conclude that N is a contraction mapping. Consequently, N has a fixed point, implying that problem (2.1) has a unique solution. □

Now, we will demonstrate the existence of at least one solution for problem (2.1) by applying the Darbo's FPT.

Theorem 2.2. *Suppose that hypotheses $(H_1) - (H_4)$ and (A_1) are fulfilled. If the inequality*

$$
\gamma = \max\left\{ \left((A_\mu^\lambda + B_\mu^\lambda \mathcal{T}) \Phi^* + \lambda B_\mu^\lambda \mathcal{T} \right), \left((a_\sigma + b_\sigma \mathcal{T}) \mathcal{H}^* + b_\mu \mathcal{T} \Phi^* \right) \right\} < 1 \tag{2.7}
$$

holds, then the problem (2.1) has at least one solution defined on θ.

Proof. Let's consider the operator $\mathrm{N} : \mathcal{PC}1 \to \mathcal{PC}1$ as defined by equation (2.6). Now, let $\mathrm{R} > 0$ with the condition

$$
\mathrm{R} \geq \max\left\{ \frac{|C_{\mathtt{m}}^{\mu,\lambda}| + (A_\mu^\lambda + B_\mu^\lambda \mathcal{T}) \varrho^* \psi(R)}{1 - \lambda B_\mu^\lambda \mathcal{T}} \ , \ |\mathcal{H}_{\mathtt{m}}^\mu| + (a_\sigma + b_\sigma \mathcal{T}) \mathcal{H}^* + b_\mu \mathcal{T} \varrho^* \psi(\mathrm{R}) \right\},
$$

we define

$$
\mathcal{B}_{\mathrm{R}} := \mathcal{B}(0, \mathrm{R}) = \{\upsilon \in \mathcal{PC}1 : \|\upsilon\|_{\mathcal{PC}1} \leq \mathrm{R}\}.
$$

Obviously, $\mathcal{B}_{\mathrm{R}}$ is nonempty, convex, bounded and closed, as evident from the context. Proceeding, to demonstrate that N fulfills the conditions of Theorem 1.4, the proof will be presented in 4 steps.

Step 1. For $\mathrm{N}(\mathcal{B}_{\mathrm{R}}) \subseteq \mathcal{B}_{\mathrm{R}}$, for each $\upsilon \in \theta_{\mathtt{m}}$, $\mathtt{m} = 0, 1, \ldots, \mathrm{n}$, and $\varphi \in \mathcal{PC}1$, we get

$$\begin{aligned}
|(\mathrm{N}\varphi)(\upsilon)| &= |C_{\mathtt{m}}^{\mu,\lambda} + A_{\mu}^{\lambda}\mathrm{g}(\upsilon, \varphi(\upsilon)) - \lambda B_{\mu}^{\lambda}\int_{\mathfrak{s}_{\mathtt{m}}}^{\upsilon}\varphi(\Theta)d\Theta + B_{\mu}^{\lambda}\int_{\mathfrak{s}_{\mathtt{m}}}^{\upsilon}\mathrm{g}(\Theta, \varphi(\Theta))d\Theta|,\\
&\leq |C_{\mathtt{m}}^{\mu,\lambda}| + A_{\mu}^{\lambda}|\mathrm{g}(\upsilon, \varphi(\upsilon))| + \lambda B_{\mu}^{\lambda}\int_{\mathfrak{s}_{\mathtt{m}}}^{\upsilon}|\varphi(\Theta)|d\Theta + B_{\mu}^{\lambda}\int_{\mathfrak{s}_{\mathtt{m}}}^{\upsilon}|\mathrm{g}(\Theta, \varphi(\Theta))|d\Theta,\\
&\leq |C_{\mathtt{m}}^{\mu,\lambda}| + A_{\mu}^{\lambda}\varrho(\upsilon)\psi(\|\varphi\|) + \lambda B_{\mu}^{\lambda}\mathcal{T}\|\varphi\|_{\mathcal{PC}1} + B_{\mu}^{\lambda}\mathcal{T}\varrho(\upsilon)\psi(\|\varphi\|),\\
&\leq |C_{i}^{\mu,\lambda}| + A_{\mu}^{\lambda}\varrho^{*}\psi(\|\varphi\|) + \lambda B_{\mu}^{\lambda}\mathcal{T}\|\varphi\|_{\mathcal{PC}1} + B_{\mu}^{\lambda}\ \mathcal{T}\varrho^{*}\psi(\|\varphi\|),\\
&\leq |C_{i}^{\mu,\lambda}| + A_{\mu}^{\lambda}\varrho^{*}\psi(\mathrm{R}) + \lambda B_{\mu}^{\lambda}\mathcal{T}(\mathrm{R}) + B_{\mu}^{\lambda}\ \mathcal{T}\varrho^{*}\psi(\mathrm{R}),\\
&\leq |C_{\mathtt{m}}^{\mu,\lambda}| + (A_{\mu}^{\lambda} + B_{\mu}^{\lambda}\mathcal{T})\varrho^{*}\psi(\mathrm{R}) + \lambda B_{\mu}^{\lambda}\mathcal{T}\mathrm{R},\\
&\leq \mathrm{R}.
\end{aligned}$$

Moreover, for each $\upsilon \in J_{\mathtt{m}}$, $\mathtt{m} = 0, 1, \ldots, \mathrm{n}$, and $\varphi \in \mathcal{PC}1$, we obtain

$$\begin{aligned}
|(\mathrm{N}\varphi)(\upsilon)| &= |\mathcal{H}_{\mathtt{m}}^{\mu} + a_{\sigma}\mathrm{h}_{\mathtt{m}}(\upsilon, \varphi(\upsilon)) + b_{\sigma}\int_{\upsilon_{\mathtt{m}}}^{\upsilon}\mathrm{h}_{\mathtt{m}}(\Theta, \varphi(\Theta))d\Theta - b_{\mu}\int_{0}^{\mathfrak{s}_{\mathtt{m}}}\mathrm{g}(\Theta, \varphi(\Theta))d\Theta|,\\
&\leq |\mathcal{H}_{\mathtt{m}}^{\mu}| + a_{\sigma}|\mathrm{h}_{\mathtt{m}}(\upsilon, \varphi(\upsilon))| + b_{\sigma}\int_{\upsilon_{\mathtt{m}}}^{\upsilon}|\mathrm{h}_{\mathtt{m}}(\Theta, \varphi(\Theta))|d\Theta + b_{\mu}\int_{0}^{\mathfrak{s}_{\mathtt{m}}}|\mathrm{g}(\Theta, \varphi(\Theta))|d\Theta,\\
&\leq |\mathcal{H}_{\mathtt{m}}^{\mu}| + a_{\sigma}\mathcal{H}_{\mathtt{m}}(\upsilon) + b_{\sigma}\int_{\upsilon_{\mathtt{m}}}^{\upsilon}\mathcal{H}_{\mathtt{m}}(\Theta)d\Theta + b_{\mu}\int_{0}^{\mathfrak{s}_{\mathtt{m}}}\varrho(\Theta)\psi(\|\varphi\|)d\Theta,\\
&\leq |\mathcal{H}_{\mathtt{m}}^{\mu}| + a_{\sigma}\mathcal{H}^{*} + b_{\sigma}\mathcal{T}\mathcal{H}^{*} + +b_{\mu}\mathcal{T}\varrho^{*}\psi(\mathrm{R}),\\
&\leq |\mathcal{H}_{\mathtt{m}}^{\mu}| + (a_{\sigma} + b_{\sigma}\mathcal{T})\mathcal{H}^{*} + b_{\mu}\mathcal{T}\varrho^{*}\psi(\mathrm{R}),\\
&\leq \mathrm{R}.
\end{aligned}$$

Thus, for $\upsilon \in \theta$ and $\varphi \in \mathcal{PC}1$, we get

$$\|\mathrm{N}(\varphi)\|_{\mathcal{PC}1} \leq \mathrm{R}.$$

This demonstrates that N transforms the ball $\mathcal{B}_{\mathrm{R}}$ into itself.

Step 2. Consider a sequence $\{\varphi_k\}_{k\in\mathbb{N}}$ $\varphi_k \to \varphi \in \mathcal{B}_{\mathrm{R}}$ and $\mathrm{N} : \mathcal{B}_{\mathrm{R}} \to \mathcal{B}_{\mathrm{R}}$, for each $\upsilon \in \theta_{\mathtt{m}}$, we derive

$$\begin{aligned}
|(\mathrm{N}\varphi_k)(\upsilon) - (\mathrm{N}\varphi)(\upsilon)| &= |C_{\mathtt{m}}^{\mu,\lambda} + A_{\mu}^{\lambda}\mathrm{g}(\upsilon, \varphi_k(\upsilon)) - \lambda B_{\mu}^{\lambda}\int_{\mathfrak{s}_{\mathtt{m}}}^{\upsilon}\varphi_k(\Theta)d\Theta + B_{\mu}^{\lambda}\int_{\mathfrak{s}_{\mathtt{m}}}^{\upsilon}\mathrm{g}(\Theta, \varphi_k(\Theta))d\Theta\\
&\quad - C_{\mathtt{m}}^{\mu,\lambda} - A_{\mu}^{\lambda}\mathrm{g}(\upsilon, \varphi(\upsilon)) + \lambda B_{\mu}^{\lambda}\int_{\mathfrak{s}_{\mathtt{m}}}^{\upsilon}\varphi(\Theta)d\Theta - B_{\mu}^{\lambda}\int_{\mathfrak{s}_{\mathtt{m}}}^{\upsilon}\mathrm{g}(\Theta, \varphi(\Theta))d\Theta|,\\
&\leq A_{\mu}^{\lambda}|\mathrm{g}(\upsilon, \varphi_k(\upsilon)) - \mathrm{g}(\upsilon, \varphi(\upsilon))| + \lambda B_{\mu}^{\lambda}\int_{\mathfrak{s}_{\mathtt{m}}}^{\upsilon}|\varphi_k(\Theta) - \varphi(\Theta)|d\Theta\\
&\quad + B_{\mu}^{\lambda}\int_{\mathfrak{s}_{\mathtt{m}}}^{\upsilon}|\mathrm{g}(\Theta, \varphi_k(\Theta)) - \mathrm{g}(\Theta, \varphi(\Theta))|d\Theta.
\end{aligned}$$

Now, for each $\upsilon \in J_{\mathtt{m}}$,

$$\begin{aligned}
|(\mathrm{N}\varphi_k)(\upsilon) - (\mathrm{N}\varphi)(\upsilon)| &= |\mathcal{H}^{\mu}_{\mathtt{m}} + a_\sigma \mathrm{h}_{\mathtt{m}}(\upsilon, \varphi_k(\upsilon)) + b_\sigma \int_{\upsilon_{\mathtt{m}}}^{\upsilon} \mathrm{h}_{\mathtt{m}}(\Theta, \varphi_k(\Theta))d\Theta - b_\mu \int_0^{\mathfrak{s}_{\mathtt{m}}} \mathrm{g}(\Theta, \varphi_k(\Theta))d\Theta \\
&- \mathcal{H}^{\mu}_{\mathtt{m}} - a_\sigma \mathrm{h}_{\mathtt{m}}(\upsilon, \varphi(\upsilon)) - b_\sigma \int_{\upsilon_{\mathtt{m}}}^{\upsilon} \mathrm{h}_{\mathtt{m}}(\Theta, \varphi(\Theta))d\Theta + b_\mu \int_0^{\mathfrak{s}_{\mathtt{m}}} \mathrm{g}(\Theta, \varphi(\Theta))d\Theta|, \\
&\leq a_\sigma |\mathrm{h}_{\mathtt{m}}(\upsilon, \varphi_k(\upsilon)) - \mathrm{h}_{\mathtt{m}}(\upsilon, \varphi(\upsilon))| \\
&\quad + b_\sigma \int_{\upsilon_{\mathtt{m}}}^{\upsilon} |\mathrm{h}_{\mathtt{m}}(\Theta, \varphi_k(\Theta)) - \mathrm{h}_{\mathtt{m}}(\Theta, \varphi(\Theta))|d\Theta \\
&\quad\quad + b_\mu \int_0^{\mathfrak{s}_{\mathtt{m}}} |\mathrm{g}(\Theta, \varphi_k(\Theta)) - \mathrm{g}(\Theta, \varphi(\Theta))|d\Theta.
\end{aligned}$$

Since $\varphi_k \to \varphi$ as $n \to \infty$, g and $\mathrm{h}_{\mathtt{m}}$ are continuous, hence Utilizing the Lebesgue dominated convergence theorem, one can obtain

$$\|(\mathrm{N}\varphi_k)(\upsilon) - \mathrm{N}(\varphi)(\upsilon)\|_{\mathcal{PC}1} \to 0 \quad \text{as} \quad n \to \infty.$$

Therefore, N is a continuous operator.

Step 3. Proving that $\mathrm{N}(\mathcal{B}_{\mathrm{R}})$ is bounded and equicontinuous.

Due to Step 1, $\mathrm{N}(\mathcal{B}_{\mathrm{R}})$ is bounded, since $\mathrm{N}(\mathcal{B}_{\mathrm{R}}) \subset \mathcal{B}_{\mathrm{R}}$.

Now, let $\nu_1, \nu_2 \in \theta_{\mathtt{m}}$, $\mathtt{m} = 0, \cdots, \mathrm{n} : \nu_1 < \nu_2$ and $\varphi \in \mathcal{B}_{\mathrm{R}}$. Then

$$\begin{aligned}
|(\mathrm{N}\varphi)(\nu_2) - (\mathrm{N}\varphi)(\nu_1)| &= |C^{\mu,\lambda}_{\mathtt{m}} + A^{\lambda}_{\mu} \mathrm{g}(\nu_2, \varphi(\nu_2)) - \lambda B^{\lambda}_{\mu} \int_{\mathfrak{s}_{\mathtt{m}}}^{\nu_2} \varphi(\Theta)d\Theta + B^{\lambda}_{\mu} \int_{\mathfrak{s}_{\mathtt{m}}}^{\nu_2} \mathrm{g}(\Theta, \varphi(\Theta))d\Theta \\
&- C^{\mu,\lambda}_{\mathtt{m}} - A^{\lambda}_{\mu} \mathrm{g}(\nu_1, \varphi(\nu_1)) + \lambda B^{\lambda}_{\mu} \int_{\mathfrak{s}_{\mathtt{m}}}^{\nu_1} \varphi(\Theta)d\Theta - B^{\lambda}_{\mu} \int_{\mathfrak{s}_{\mathtt{m}}}^{\nu_1} \mathrm{g}(\Theta, \varphi(\Theta))d\Theta|, \\
&\leq A^{\lambda}_{\mu} |\mathrm{g}(\nu_2, \varphi(\nu_2)) - \mathrm{g}(\nu_1, \varphi(\nu_1))| + \lambda B^{\lambda}_{\mu} \int_{\nu_1}^{\nu_2} |\varphi(\Theta)|d\Theta \\
&\quad + B^{\lambda}_{\mu} \int_{\nu_1}^{\nu_2} |\mathrm{g}(\Theta, \varphi(\Theta))|d\Theta, \\
&\leq A^{\lambda}_{\mu} |\mathrm{g}(\nu_2, \varphi(\nu_2)) - \mathrm{g}(\nu_1, \varphi(\nu_1))| + \lambda B^{\lambda}_{\mu} \mathrm{R}(\nu_2 - \nu_1) \\
&\quad + B^{\lambda}_{\mu} \varrho^* \psi(\mathrm{R})(\nu_2 - \nu_1).
\end{aligned}$$

Since g is continuous, the expression on the right side of the preceding inequality approaches zero as $\nu_1 \to \nu_2$.

Also, for each $\upsilon \in J_{\mathtt{m}}$, $\mathtt{m} = 0, 1, \ldots, \mathrm{n}$, $\nu_1, \nu_2 \in J_{\mathtt{m}}$, $\mathtt{m} = 0, 1, \ldots, \mathrm{n}$ with $\nu_1 < \nu_2$ and $\varphi \in \mathcal{B}_{\mathrm{R}}$, we have

$$\begin{aligned}
|(\mathrm{N}\varphi)(\nu_2) - (\mathrm{N}\varphi)(\nu_1)| &\leq a_\sigma |\mathrm{h}_{\mathtt{m}}(\nu_2, \varphi(\nu_2)) - \mathrm{h}_{\mathtt{m}}(\nu_1, \varphi(\nu_1))| + b_\sigma \int_{\nu_1}^{\nu_2} |\mathrm{h}_{\mathtt{m}}(\Theta, \varphi(\Theta))|d\Theta, \\
&\leq a_\sigma |\mathrm{h}_{\mathtt{m}}(\nu_2, \varphi(\nu_2)) - \mathrm{h}_{\mathtt{m}}(\nu_1, \varphi(\nu_1))| + b_\sigma \mathcal{H}^*(\nu_2 - \nu_1).
\end{aligned}$$

Due to the continuity of $\mathrm{h}_{\mathtt{m}}$, the right side of the above inequality approaches zero as $\nu_1 \to \nu_2$. As a result, $\mathrm{N}(\mathcal{B}_{\mathrm{R}})$ is equicontinuous and bounded.

Step 4. Proving that N is a γ-contraction.

From Step 2 and Step 3, we have $\mathrm{N} : \mathcal{B}_{\mathrm{R}} \to \mathcal{B}_{\mathrm{R}}$ is bounded, continuous and $\mathrm{N}(\mathcal{B}_{\mathrm{R}})$ is equicontinuous. We must now demonstrate that the operator N is a γ-contraction.

Let $\mathcal{D} \subset \mathcal{B}_{\mathrm{R}}$ and $\upsilon \in \theta_{\mathtt{m}}$, $\mathtt{m} = 0, 1, \ldots, \mathrm{n}$. Then

$$\begin{aligned}\eta(\mathrm{N}(\mathcal{D})(\upsilon) &\leq A_\mu^\lambda \eta(\mathrm{g}(\upsilon, \varphi(\upsilon)), \varphi \in \mathcal{D}) + \lambda B_\mu^\lambda \eta \left(\int_{\mathfrak{s}_{\mathtt{m}}}^{\upsilon} \varphi(\Theta) d\Theta, \varphi \in \mathcal{D} \right) \\ &\qquad + B_\mu^\lambda \eta \left(\int_{\mathfrak{s}_{\mathtt{m}}}^{\upsilon} \mathrm{g}\left(\Theta, \varphi(\Theta)\right) d\Theta, \varphi \in \mathcal{D} \right), \\ &\leq A_\mu^\lambda \Phi(\upsilon) \eta(\mathcal{D}) + \lambda B_\mu^\lambda \int_{\mathfrak{s}_{\mathtt{m}}}^{\upsilon} \eta(\mathcal{D}(\Theta)) d\Theta + B_\mu^\lambda \int_{\mathfrak{s}_{\mathtt{m}}}^{\upsilon} \eta(\mathrm{g}\left(\Theta, \mathcal{D}\right)) d\Theta, \\ &\leq \left((A_\mu^\lambda + B_\mu^\lambda \mathcal{T}) \Phi^* + \lambda B_\mu^\lambda \mathcal{T} \right) \eta(\mathcal{D}), \\ &\leq \gamma_1 \eta(\mathcal{D}).\end{aligned}$$

For each $\upsilon \in J_{\mathtt{m}}$, $\mathtt{m} = 0, 1, \ldots, \mathrm{n}$, we get

$$\begin{aligned}\eta(\mathrm{N}(\mathcal{D})(\upsilon) &\leq \eta\left(a_\sigma \mathrm{h}_{\mathtt{m}}(\upsilon, \varphi(\upsilon)), \varphi \in \mathcal{D}\right) + \eta \left(b_\sigma \int_{x_{\mathtt{m}}}^{\upsilon} \mathrm{h}_{\mathtt{m}}(\Theta, \varphi(\Theta)) d\Theta, \varphi \in \mathcal{D} \right) \\ &\qquad + \eta \left(b_\mu \int_0^{\mathfrak{s}_{\mathtt{m}}} \mathrm{g}(\Theta, \varphi(\Theta)) d\Theta, \varphi \in \mathcal{D} \right), \\ &\leq a_\sigma \eta(\mathrm{h}_{\mathtt{m}}(\upsilon, \mathcal{D})) + b_\sigma \int_{x_{\mathtt{m}}}^{\upsilon} \eta(\mathrm{h}_{\mathtt{m}}(\Theta, \mathcal{D}) d\Theta + b_\mu \int_0^{\mathfrak{s}_{\mathtt{m}}} \eta(\mathrm{g}(\Theta, \mathcal{D})) d\Theta, \\ &\leq \left((a_\sigma + b_\sigma \mathcal{T}) \mathcal{H}^* + b_\mu \mathcal{T} \Phi^* \right) \eta(\mathcal{D}), \\ &\leq \gamma_2 \eta(\mathcal{D}).\end{aligned}$$

Therefore, for each $\upsilon \in \theta$, we have

$$\eta(\mathrm{N}(\mathcal{D})(\upsilon) \leq \gamma \; \eta(\mathcal{D}), \quad \text{such as } \; \gamma = \max\{\gamma_1, \gamma_2\}.$$

In accordance with condition (2.7), the operator N is a γ-contraction. Following the application of Theorem 1.4, we deduce that N admits a fixed point, which is a solution to the problem (2.1). $\square$

2.3 Hyers-Ulam stability results

Definition 2.1 ([82]). *It is stated that the equation (2.1) is stable in the HU sense if there exists $c_{g,\mu,\sigma,h_m} \in \mathbb{R}_*^+$ where for any $\varepsilon > 0$ and for every solution $\mathfrak{z} \in \mathcal{PC}1(\theta, \mathbb{R})$ of the inequalities*

$$\left|{}^{CF}\mathfrak{D}^{\mu}_{\mathfrak{s}_m,\upsilon}\mathfrak{z}(\upsilon) + \lambda\mathfrak{z}(\upsilon) - g(\upsilon, \mathfrak{z}(\upsilon))\right| \leq \varepsilon, \quad \upsilon \in (\mathfrak{s}_m, \upsilon_{m+1}], \ m = 0, 1, \ldots, n, \tag{2.8}$$

and

$$\left|\mathfrak{z}(\upsilon) - p - {}^{CF}\mathfrak{I}^{\sigma}_{\upsilon_m,\upsilon}h_m(\upsilon, \mathfrak{z}(\upsilon)) + {}^{CF}\mathfrak{I}^{\mu}_{0,\mathfrak{s}_m}g\left(\mathfrak{s}_m, \mathfrak{z}\left(\mathfrak{s}_m\right)\right)\right| \leq \varepsilon, \quad \upsilon \in (\upsilon_m, \mathfrak{s}_m], m = 1, 2, \ldots n, \tag{2.9}$$

there exists a solution $\varphi \in \mathcal{PC}1(\theta, \mathbb{R})$ of (2.1) with

$$|\mathfrak{z}(\upsilon) - \varphi(\upsilon)| \leq c_{g,\mu,\sigma,h_m}\varepsilon, \quad \upsilon \in \theta.$$

Theorem 2.3. *Suppose hypotheses* $(A_1)-(A_3)$ *and condition (2.5) are fulfilled. In that case, problem (2.1) is stable in the Hyers-Ulam sense.*

Proof. Consider $\varepsilon > 0$ and $\mathfrak{z} \in \mathcal{PC}1(\theta, \mathbb{R})$ be a function satisfying the inequalities (2.8) and (2.9). Let φ represent the unique solution to the subsequent problem

$$\begin{cases} {}^{CF}\mathfrak{D}^{\mu}_{\mathfrak{s}_m,\upsilon}\varphi(\upsilon) = -\lambda\varphi(\upsilon) + g(\upsilon, \varphi(\upsilon)), \quad \upsilon \in (\mathfrak{s}_m, \upsilon_{m+1}], \ m = 0, 1, \ldots, n, \ \mu \in]0, 1[, \\ \varphi(\upsilon) = p + {}^{CF}\mathfrak{I}^{\sigma}_{\upsilon_m,\upsilon}h_m(\upsilon, \varphi(\upsilon)) - {}^{CF}\mathfrak{I}^{\mu}_{0,\mathfrak{s}_m}g\left(\mathfrak{s}_m, \varphi\left(\mathfrak{s}_m\right)\right), \ \upsilon \in (\upsilon_m, \mathfrak{s}_m], \ m = 1, 2, \ldots, n, \\ \varphi(0) = \varphi(0). \end{cases} \tag{2.10}$$

Then, it follows that

$$\varphi(\upsilon) = \begin{cases} C^{\mu,\lambda}_m + A^{\lambda}_{\mu}g(\upsilon, \varphi(\upsilon)) - \lambda B^{\lambda}_{\mu}\int_{\mathfrak{s}_m}^{\upsilon}\varphi(\Theta)d\Theta + B^{\lambda}_{\mu}\int_{\mathfrak{s}_m}^{\upsilon}g(\Theta, \varphi(\Theta))d\Theta, \ \upsilon \in \theta_m, \ m = 0, 1, \ldots, n, \\ p + {}^{CF}\mathfrak{I}^{\sigma}_{\upsilon_m,\upsilon}h_m(\upsilon, \varphi(\upsilon)) - {}^{CF}\mathfrak{I}^{\mu}_{0,\mathfrak{s}_m}g\left(\mathfrak{s}_m, \varphi\left(\mathfrak{s}_m\right)\right), \ \upsilon \in J_m, \ m = 1, 2, \ldots, n. \end{cases}$$

By integrating the inequality (2.8), we can derive

$$|\mathfrak{z}(\upsilon) - C^{\mu,\lambda}_m - A^{\lambda}_{\mu}g(\upsilon, \mathfrak{z}(\upsilon)) + \lambda B^{\lambda}_{\mu}\int_{\mathfrak{s}_m}^{\upsilon}\mathfrak{z}(\Theta)d\Theta - B^{\lambda}_{\mu}\int_{\mathfrak{s}_m}^{\upsilon}g(\Theta, \mathfrak{z}(\Theta))d\Theta| \leq (a_\mu + b_\mu\mathcal{T})\varepsilon.$$

For $\upsilon \in \theta_m, m = 0, 1, \ldots, n$, we obtain

$$\begin{aligned} |\mathfrak{z}(\upsilon) - \varphi(\upsilon)| &= \left|\mathfrak{z}(\upsilon) - C^{\mu,\lambda}_m - A^{\lambda}_{\mu}g(\upsilon, \varphi(\upsilon)) + \lambda B^{\lambda}_{\mu}\int_{\mathfrak{s}_m}^{\upsilon}\varphi(\Theta)d\Theta - B^{\lambda}_{\mu}\int_{\mathfrak{s}_m}^{\upsilon}g(\Theta, \varphi(\Theta))d\Theta\right|, \\ &\leq \left|\mathfrak{z}(\upsilon) - C^{\mu,\lambda}_m - A^{\lambda}_{\mu}g(\upsilon, \mathfrak{z}(\upsilon)) + \lambda B^{\lambda}_{\mu}\int_{\mathfrak{s}_m}^{\upsilon}\mathfrak{z}(\Theta)d\Theta - B^{\lambda}_{\mu}\int_{\mathfrak{s}_m}^{\upsilon}g(\Theta, \mathfrak{z}(\Theta))d\Theta\right| \\ &\quad + \left|A^{\lambda}_{\mu}g(\upsilon, \mathfrak{z}(\upsilon)) - A^{\lambda}_{\mu}g(\upsilon, \varphi(\upsilon))\right| \end{aligned}$$

$$
\begin{aligned}
&\quad + \left|-\lambda B_{\mu}^{\lambda}\int_{\mathfrak{s}_{\mathtt{m}}}^{\upsilon}\mathfrak{z}(\Theta)d\Theta + \lambda B_{\mu}^{\lambda}\int_{\mathfrak{s}_{\mathtt{m}}}^{\upsilon}\varphi(\Theta)d\Theta\right| \\
&\qquad + \left|B_{\mu}^{\lambda}\int_{\mathfrak{s}_{\mathtt{m}}}^{\upsilon}\mathrm{g}(\Theta,\mathfrak{z}(\Theta))d\Theta - B_{\mu}^{\lambda}\int_{\mathfrak{s}_{\mathtt{m}}}^{\upsilon}\mathrm{g}(\Theta,\varphi(\Theta))d\Theta\right|, \\
&\leq (a_{\mu}+b_{\mu}\mathcal{T})\varepsilon + A_{\mu}^{\lambda}|\mathrm{g}(\upsilon,\mathfrak{z}(\upsilon)) - \mathrm{g}(\upsilon,\varphi(\upsilon))| + \lambda B_{\mu}^{\lambda}\int_{\mathfrak{s}_{\mathtt{m}}}^{\upsilon}|\mathfrak{z}(\Theta)-\varphi(\Theta)|d\Theta \\
&\quad + B_{\mu}^{\lambda}\int_{\mathfrak{s}_{\mathtt{m}}}^{\upsilon}|\mathrm{g}(\Theta,\mathfrak{z}(\Theta)) - \mathrm{g}(\Theta,\varphi(\Theta))|d\Theta, \\
&\leq (a_{\mu}+b_{\mu}\mathcal{T})\varepsilon + ((A_{\mu}^{\lambda}+B_{\mu}^{\lambda}\mathcal{T})L_{\mathrm{g}} + \lambda B_{\mu}^{\lambda}\mathcal{T})\|\mathfrak{z}-\varphi\|_{\mathcal{PC}1},
\end{aligned}
$$

this implying that

$$
\|\mathfrak{z}-\varphi\|_{\mathcal{PC}1} \leq \left\{\frac{(a_{\mu}+b_{\mu}\mathcal{T})}{1-((A_{\mu}^{\lambda}+B_{\mu}^{\lambda}\mathcal{T})L_{\mathrm{g}}+\lambda B_{\mu}^{\lambda}\mathcal{T})}\right\}\varepsilon.
$$

Thus,

$$
|\mathfrak{z}(\upsilon)-\varphi(\upsilon)| \leq c_{\mathrm{g},\mu,\sigma,\mathrm{h}_{\mathtt{m}}}\varepsilon, \quad \text{where} \quad c_{\mathrm{g},\mu,\sigma,\mathrm{h}_{\mathtt{m}}} = \frac{(a_{\mu}+b_{\mu}\mathcal{T})}{1-((A_{\mu}^{\lambda}+B_{\mu}^{\lambda}\mathcal{T})L_{\mathrm{g}}+\lambda B_{\mu}^{\lambda}\mathcal{T})}. \tag{2.11}
$$

Now, for $\upsilon \in J_{\mathtt{m}}, \mathtt{m} = 1,2,\ldots,\mathrm{n}$, we get

$$
\begin{aligned}
|\mathfrak{z}(\upsilon)-\varphi(\upsilon)| &= \left|\mathfrak{z}(\upsilon) - p - {}^{CF}\mathfrak{I}_{\upsilon_{\mathtt{m}},\upsilon}^{\sigma}\mathrm{h}_{\mathtt{m}}(\upsilon,\varphi(\upsilon)) + {}^{CF}\mathfrak{I}_{0,\mathfrak{s}_{\mathtt{m}}}^{\mu}\mathrm{g}\left(\mathfrak{s}_{\mathtt{m}},\varphi\left(\mathfrak{s}_{\mathtt{m}}\right)\right)\right|, \\
&\leq \left|\mathfrak{z}(\upsilon) - p - {}^{CF}\mathfrak{I}_{\upsilon_{\mathtt{m}},\upsilon}^{\sigma}\mathrm{h}_{\mathtt{m}}(\upsilon,\mathfrak{z}(\upsilon)) + {}^{CF}\mathfrak{I}_{0,\mathfrak{s}_{\mathtt{m}}}^{\mu}\mathrm{g}\left(\mathfrak{s}_{\mathtt{m}},\mathfrak{z}\left(\mathfrak{s}_{\mathtt{m}}\right)\right)\right| \\
&\quad + \left|{}^{CF}\mathfrak{I}_{\upsilon_{\mathtt{m}},\upsilon}^{\sigma}\mathrm{h}_{\mathtt{m}}(\upsilon,\mathfrak{z}(\upsilon)) - {}^{CF}\mathfrak{I}_{\upsilon_{\mathtt{m}},\upsilon}^{\sigma}\mathrm{h}_{\mathtt{m}}(\upsilon,\varphi(\upsilon))\right| \\
&\qquad + \left|{}^{CF}\mathfrak{I}_{0,\mathfrak{s}_{\mathtt{m}}}^{\mu}\mathrm{g}\left(\mathfrak{s}_{\mathtt{m}},\mathfrak{z}\left(\mathfrak{s}_{\mathtt{m}}\right)\right) - {}^{CF}\mathfrak{I}_{0,\mathfrak{s}_{\mathtt{m}}}^{\mu}\mathrm{g}\left(\mathfrak{s}_{\mathtt{m}},\varphi\left(\mathfrak{s}_{\mathtt{m}}\right)\right)\right|, \\
&\leq \varepsilon + a_{\sigma}|\mathrm{h}_{\mathtt{m}}(\upsilon,\mathfrak{z}(\upsilon)) - \mathrm{h}_{\mathtt{m}}(\upsilon,\varphi(\upsilon))| + b_{\sigma}\int_{\upsilon_{\mathtt{m}}}^{\upsilon}|\mathrm{h}_{\mathtt{m}}(\Theta,\mathfrak{z}(\Theta)) - \mathrm{h}_{\mathtt{m}}(\Theta,\varphi(\Theta))| \\
&\quad + a_{\mu}|\mathrm{g}(\mathfrak{s}_{\mathtt{m}},\mathfrak{z}(\mathfrak{s}_{\mathtt{m}})) - \mathrm{g}(\mathfrak{s}_{\mathtt{m}},\varphi(\mathfrak{s}_{\mathtt{m}}))| + b_{\mu}\int_{\upsilon_{\mathtt{m}}}^{\upsilon}|\mathrm{g}(\Theta,\mathfrak{z}(\Theta)) - \mathrm{g}(\Theta,\varphi(\Theta))|d\Theta, \\
&\leq \varepsilon + \left((a_{\sigma}+b_{\sigma}\mathcal{T})L_{\mathrm{h}_{\mathtt{m}}} + (a_{\mu}+b_{\mu}\mathcal{T})L_{\mathrm{g}})\right)\|\mathfrak{z}-\varphi\|_{\mathcal{PC}1},
\end{aligned}
$$

this implies that

$$
\|\mathfrak{z}-\varphi\|_{\mathcal{PC}1} \leq \left\{\frac{1}{1-((a_{\sigma}+b_{\sigma}\mathcal{T})L_{\mathrm{h}_{\mathtt{m}}}+(a_{\mu}+b_{\mu}\mathcal{T})L_{\mathrm{g}}))}\right\}\varepsilon.
$$

Thus,

$$
|\mathfrak{z}(\upsilon)-\varphi(\upsilon)| \leq c_{\mathrm{g},\mu,\sigma,\mathrm{h}_{\mathtt{m}}}\varepsilon, \quad \text{where} \quad c_{\mathrm{g},\mu,\sigma,\mathrm{h}_{\mathtt{m}}} = \left\{\frac{1}{1-((a_{\sigma}+b_{\sigma}\mathcal{T})L_{\mathrm{h}_{\mathtt{m}}}+(a_{\mu}+b_{\mu}\mathcal{T})L_{\mathrm{g}}))}\right\}. \tag{2.12}
$$

Therefore, the relations (2.11) and (2.12) demonstrate that the problem (2.1) is stable in the Hyers-Ulam sense with respect to ε.

□

2.4 Examples

This section showcases 2 examples to illustrate all obtained outcomes. In the initial example shows that problem (2.13) admits at least one solution defined on θ, and the second one illustrate that the (2.14) is HU stable.

Example 2.1. *Let consider the subsequent problem*

$$\begin{cases} \textit{For} \quad \upsilon \in \theta_{\mathtt{m}},\ \mathtt{m} = 0,1,2,\ \mu \in]0,1[,\ \lambda > 0, \\ {}^{CF}\mathfrak{D}^{\mu}_{\mathfrak{s}_{\mathtt{m}},\upsilon}\varphi(\upsilon) = -\lambda\varphi(\upsilon) + \mathrm{g}(\upsilon,\varphi(\upsilon)), \\ \\ \textit{For} \quad \upsilon \in J_{\mathtt{m}}, \mathtt{m} = 1,2,\ \sigma \in]0,1[,\ \sigma \neq \mu, \\ \varphi(\upsilon) = p + {}^{CF}\mathfrak{I}^{\sigma}_{x_{\mathtt{m}},\upsilon}\mathrm{h}_{\mathtt{m}}(\upsilon,\varphi(\upsilon)) - {}^{CF}\mathfrak{I}^{\mu}_{0,\mathfrak{s}_{\mathtt{m}}}\mathrm{g}\left(\mathfrak{s}_{\mathtt{m}},\varphi\left(\mathfrak{s}_{\mathtt{m}}\right)\right), \\ \\ \varphi(0) = \varphi_0. \end{cases} \tag{2.13}$$

We pose $\theta = [0,1],\ \theta_0 = [0,\frac{1}{5}],\ J_1 = (\frac{1}{5},\frac{2}{5}],\ \theta_1 = (\frac{2}{5},\frac{3}{5}],\ J_2 = (\frac{3}{5},\frac{4}{5}],\ \theta_2 = (\frac{4}{5},1],$

$$\mathrm{g}(\upsilon,\varphi(\upsilon)) = \frac{1}{(A^{\lambda}_{\mu} + B^{\lambda}_{\mu}\mathcal{T})e^{\upsilon+6}}\left(\frac{1+\varphi(\upsilon)}{2+|\varphi(\upsilon)|}\right),$$

where $\upsilon \in \theta_0 \cup \theta_1 \cup \theta_2$ *and*

$$\mathrm{h}_{\mathtt{m}}(\upsilon,\varphi(\upsilon)) = \frac{1+\varphi(\upsilon)}{3e^{\upsilon+6}(a_{\sigma} + b_{\sigma}\mathcal{T})}, \quad \mathtt{m} = 1,2.$$

Next, we establish that the problem (2.1) admits at least one solution on θ. To achieve this, we will utilize Theorem 2.2. Specifically, we demonstrate the satisfaction of hypotheses $(H_1) - (H_4)$ and (A_1), and furthermore confirm that condition (2.7) is satisfied, i.e.,

$$\gamma = \max\left\{\left((A^{\lambda}_{\mu} + B^{\lambda}_{\mu}\mathcal{T})\Phi^* + \lambda B^{\lambda}_{\mu}\mathcal{T}\right), ((a_{\sigma} + b_{\sigma}\mathcal{T})\mathcal{H}^* + b_{\mu}\mathcal{T}\Phi^*)\right\} < 1.$$

It is evident that the functions $\mathrm{h}_{\mathtt{m}}$ and g are continuous. So, for each $\upsilon \in \theta_{\mathtt{m}}, \mathtt{m} = 0,1,2$, we obtain

$$|\mathrm{g}(\upsilon,\varphi(\upsilon))| \leq \frac{e^{-(\upsilon+6)}}{(A^{\lambda}_{\mu} + B^{\lambda}_{\mu}\mathcal{T})}(1+|\varphi(\upsilon)|), \text{with } \varrho(\upsilon) = \frac{e^{-(\upsilon+6)}}{(A^{\lambda}_{\mu} + B^{\lambda}_{\mu}\mathcal{T})},$$

and for each $\upsilon \in J_{\mathtt{m}}, \mathtt{m} = 1,2$, we get

$$|\mathrm{h}_{\mathtt{m}}(\upsilon,\varphi(\upsilon))| \leq \frac{e^{-(\upsilon+6)}}{3(a_{\sigma} + b_{\sigma}\mathcal{T})}(1+|\varphi(\upsilon)|), \text{with } \mathcal{H}_{\mathtt{m}}(\upsilon) = \frac{e^{-(\upsilon+6)}}{3(a_{\sigma} + b_{\sigma}\mathcal{T})}.$$

Thus, the hypotheses (A_1), (H_1) and (H_2) are fulfilled, given that

$$\Phi^* = \frac{e^{-6}}{A_\mu^\lambda + B_\mu^\lambda \mathcal{T}}, \quad \text{and} \quad \mathcal{H}^* = \frac{e^{-6}}{3(a_\sigma + b_\sigma \mathcal{T})}.$$

Now, we will examine the condition (2.7). Certainly, when $\lambda = \dfrac{(1-e^{-6})M(\mu)(2-\mu)}{(4-2e^{-6}))}$. We get

$$\begin{aligned}
\gamma &= \max\{\gamma_1, \gamma_2\}, \\
&= \max\left\{\left((A_\mu^\lambda + B_\mu^\lambda \mathcal{T})\mu^* + \lambda B_\mu^\lambda \mathcal{T}\right), ((a_\sigma + b_\sigma \mathcal{T})\mathcal{H}^* + b_\mu T\mu^*)\right\}, \\
&= \max\left\{(e^{-6} + \lambda B_\mu^\lambda), e^{-6}\left(\frac{1}{3} + \frac{b_\mu}{A_\mu^\lambda + B_\mu^\lambda}\right)\right\}, \\
&= \max\left\{\left(e^{-6} + \frac{2\mu\lambda}{2(1-\mu)\lambda + M(\mu)(2-\mu)}\right), \left(\frac{e^{-6}}{3} + e^{-6}\mu\left(1 + \frac{2(1-\mu)\lambda}{(2-\mu)M(\mu)}\right)\right\}.
\end{aligned}$$

Therefore, it is evident that γ is contingent on μ, leading to variations in its value as μ ranges from 0 to 1.

Table 2.1 systematically outlines the γ_1 and γ_2 values for $\mu \in (0,1)$ and $M(\mu) = \mu^2 - \mu + 1$. This table gives a comprehensive preview of the changes in γ_1 and γ_2 as μ undergoes variations. Remarkably, throughout all instances presented in this table, a consistent pattern emerges where γ_1 consistently exceeds γ_2, and γ_1 remains beneath 1. Table 2.2 complements our investigation, examining the progression of γ_1 and γ_2 while maintaining a constant $M(\mu) = 1$, as μ varies from 0 to 1. As a result,

$$\gamma = \gamma_1 = e^{-6} + \frac{2\mu\lambda}{2\lambda(1-\mu) + M(\mu(2-\mu))} < 1.$$

Table 2.1: γ-values for various μ within the interval $]0,1[$ and $M(\mu) = \mu^2 - \mu + 1$.

μ	λ	$M(\mu)$	γ_1	γ_2	γ
0.1	4.317	0.910	0.093	0.002	γ_1
0.2	1.887	0.840	0.169	0.002	γ_1
0.3	1.117	0.790	0.233	0.002	γ_1
0.4	0.759	0.760	0.288	0.002	γ_1
0.5	0.561	0.750	0.335	0.002	γ_1
0.6	0.442	0.760	0.377	0.002	γ_1
0.7	0.366	0.790	0.413	0.002	γ_1
0.8	0.314	0.840	0.446	0.003	γ_1
0.9	0.277	0.910	0.475	0.003	γ_1

Table 2.2: γ-values for $M(\mu) = 1$, $\mu \in]0,1[$.

$\boldsymbol{\mu}$	$\boldsymbol{\lambda}$	$\boldsymbol{\gamma_1}$	$\boldsymbol{\gamma_2}$	$\boldsymbol{\gamma}$
0.1	4.744	0.093	0.002	γ_1
0.2	2.247	0.169	0.002	γ_1
0.3	1.414	0.233	0.002	γ_1
0.4	0.998	0.288	0.002	γ_1
0.5	0.749	0.335	0.002	γ_1
0.6	0.582	0.377	0.002	γ_1
0.7	0.463	0.413	0.002	γ_1
0.8	0.374	0.446	0.003	γ_1
0.9	0.305	0.475	0.003	γ_1

Figure 2.1 visually depicts the fluctuations in γ_1 and γ_2 for $\mu \in]0,1[$, considering $M(\mu) = \mu^2 - \mu + 1$. The graph offers a enhanced comprehension regarding the manner in which γ_1 and γ_2 change with varying μ. As depicted in Figure 2.1, γ_1 persistently exceeds γ_2, γ_1 varies between 0.093 and 0.475, whereas γ_2 stays notably lower, not surpassing 3×10^{-3}. The visual evidence provided strongly reinforces our assertion that condition (2.7) is hold, thereby confirming the existence of at least one solution to problem (2.13) over θ.

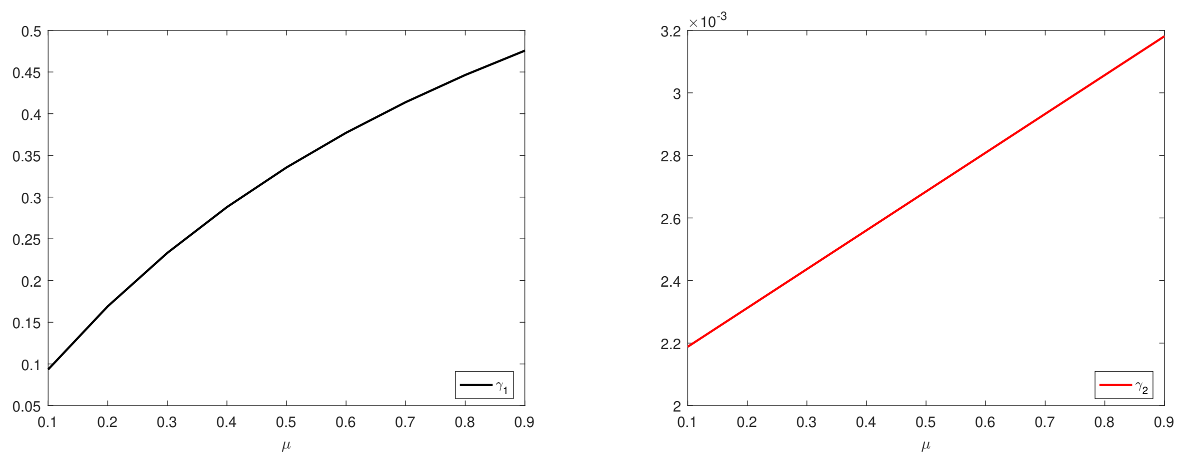

Figure 2.1: γ_1 and γ_2 variations for $M(\mu) = \mu^2 - \mu + 1$ with $\mu \in]0,1[$.

Remark 2.1. *It is evident from both Table 2.1 and Table 2.2 that identical values of γ were obtained when using either $M(\mu) = \mu^2 - \mu + 1$ or $M(\mu) = 1$.*

Example 2.2. *Let's consider the following problem*

$$\begin{cases} \textit{For} \quad \upsilon \in \theta_{\mathtt{m}},\ \mathtt{m}=0,1,2,\ \mu \in]0,1[,\ \lambda>0, \\ {}^{CF}\mathfrak{D}^{\mu}_{\mathfrak{s}_{\mathtt{m}},\upsilon}\varphi(\upsilon) = -\lambda\varphi(\upsilon) + \mathrm{g}(\upsilon,\varphi(\upsilon)), \\ \\ \textit{For} \quad \upsilon \in J_{\mathtt{m}}, \mathtt{m}=1,2,\ \sigma \in]0,1[,\ \sigma \neq \mu, \\ \varphi(\upsilon) = p + {}^{CF}\mathfrak{I}^{\sigma}_{\upsilon_{\mathtt{m}},\upsilon}\mathrm{h}_{\mathtt{m}}(\upsilon,\varphi(\upsilon)) - {}^{CF}\mathfrak{I}^{\mu}_{0,\mathfrak{s}_{\mathtt{m}}}\mathrm{g}\left(\mathfrak{s}_{\mathtt{m}},\varphi\left(\mathfrak{s}_{\mathtt{m}}\right)\right), \\ \\ \varphi(0)=\varphi_0. \end{cases} \tag{2.14}$$

And

$$\begin{cases} \textit{For} \quad \upsilon \in \theta_{\mathtt{m}},\ \mathtt{m}=0,1,2,\ \mu \in]0,1[,\ \lambda>0, \\ \left|{}^{CF}\mathfrak{D}^{\mu}_{\mathfrak{s}_{\mathtt{m}},\upsilon}\mathfrak{z}(\upsilon) + \lambda\mathfrak{z}(\upsilon) - \mathrm{g}(\upsilon,\mathfrak{z}(\upsilon))\right| \leq \varepsilon, \\ \\ \textit{For} \quad \upsilon \in J_{\mathtt{m}}, \mathtt{m}=1,2,\ \sigma \in]0,1[,\ \sigma \neq \mu, \\ \left|\mathfrak{z}(\upsilon) - p - {}^{CF}\mathfrak{I}^{\sigma}_{\upsilon_{\mathtt{m}},\upsilon}\mathrm{h}_{\mathtt{m}}(\upsilon,\mathfrak{z}(\upsilon)) + {}^{CF}\mathfrak{I}^{\mu}_{0,\mathfrak{s}_{\mathtt{m}}}\mathrm{g}\left(\mathfrak{s}_{\mathtt{m}},\mathfrak{z}\left(\mathfrak{s}_{\mathtt{m}}\right)\right)\right| \leq \varepsilon. \end{cases} \tag{2.15}$$

Consider $\theta=[0,2]$, $\theta_0=[0,\frac{2}{5}]$, $J_1=(\frac{2}{5},\frac{4}{5}]$, $\theta_1=(\frac{4}{5},\frac{6}{5}]$, $J_2=(\frac{6}{5},\frac{8}{5}]$, $\theta_2=(\frac{8}{5},2]$,

$$\mathrm{g}(\upsilon,\varphi(\upsilon)) = \frac{\upsilon^2}{10} + \frac{1}{(A^{\lambda}_{\mu}+B^{\lambda}_{\mu}\mathcal{T})}\left(\frac{\sin|\varphi(\upsilon)|}{60+\upsilon^2}\right),$$

with $\upsilon \in \theta_0 \cup \theta_1 \cup \theta_2$ *and*

$$\mathrm{h}_{\mathtt{m}}(\upsilon,\varphi(\upsilon)) = \frac{1}{(a_\sigma+b_\sigma\mathcal{T})}\frac{e^{-\upsilon}}{(1+e^{\upsilon})}\frac{|\varphi(\upsilon)|}{(1+|\varphi(\upsilon)|)}, \quad \mathtt{m}=1,2.$$

This example explores the stability in the HU sence, hence, we will employ the Theorem 2.3. Specifically, we will verify that hypotheses (A_1), (A_2), and (A_3) holds. Additionally, we will demonstrate that condition (2.5) is satisfied, i.e.,

$$\chi = \max\left\{\left((A^{\lambda}_{\mu}+B^{\lambda}_{\mu}\mathcal{T})L_{\mathrm{g}} + \lambda B^{\lambda}_{\mu}\mathcal{T}\right), \left((a_\sigma+b_\sigma\mathcal{T})L_{\mathrm{h}_{\mathtt{m}}} + (a_\mu+b_\mu\mathcal{T})L_{\mathrm{g}})\right)\right\} < 1.$$

It's evident that the functions $\mathrm{h}_{\mathtt{m}}$ and g are continuous. So for any $\upsilon \in \theta$ and $\varphi_1,\varphi_2 \in \mathbb{R}$

$$\begin{aligned} |\mathrm{g}(\upsilon,\varphi_1)-\mathrm{g}(\upsilon,\varphi_2))| &= \left|\frac{1}{(A^{\lambda}_{\mu}+B^{\lambda}_{\mu}\mathcal{T})}\frac{\sin|\varphi_1|-\sin|\varphi_2|}{60+x^2}\right|, \\ &= \frac{2}{(A^{\lambda}_{\mu}+B^{\lambda}_{\mu}\mathcal{T})(60+x^2)}\left|\sin\left(\frac{|\varphi_1|-|\varphi_2|}{2}\right)\cos\left(\frac{|\varphi_1|-|\varphi_2|}{2}\right)\right|, \\ &\leq \frac{1}{60(A^{\lambda}_{\mu}+B^{\lambda}_{\mu}\mathcal{T})}\left||\varphi_1|-|\varphi_2|\right|, \\ &\leq L_{\mathrm{g}}|\varphi_1-\varphi_2|, \quad \text{where} \quad L_{\mathrm{g}} = \frac{1}{60(A^{\lambda}_{\mu}+B^{\lambda}_{\mu}\mathcal{T})}. \end{aligned}$$

For each $\upsilon \in J_{\mathtt{m}}$, $\mathtt{m} = 1, 2$, and all $\varphi_1, \varphi_2 \in \mathbb{R}$.

$$\begin{aligned}
|\mathrm{h}_{\mathtt{m}}(\upsilon, \varphi_1) - \mathrm{h}_{\mathtt{m}}(\upsilon, \varphi_2))| &\leq \frac{e^{-\upsilon}|\varphi_1 - \varphi_2|}{(a_\sigma + b_\sigma \mathcal{T})(1 + e^\upsilon)(1 + |\varphi_1|)(1 + |\varphi_2|)}, \\
&\leq \frac{e^{-\upsilon}}{(a_\sigma + b_\sigma \mathcal{T})(1 + e^\upsilon)}|\varphi_1 - \varphi_2|, \\
&\leq L_{\mathrm{h}_{\mathtt{m}}}|\varphi_1 - \varphi_2|, \quad \text{where } \; L_{\mathrm{h}_{\mathtt{m}}} = \frac{1}{2(a_\sigma + b_\sigma \mathcal{T})}.
\end{aligned}$$

Thus, the hypotheses $(\mathrm{A}_1) - (\mathrm{A}_3)$ are satisfied.

Next, we will verify the condition (2.5). For $M(\mu) = \mu^2 - \mu + 1$, $\mathcal{T} = 2$ and $\lambda = \frac{(2-\mu)M(\mu)}{15\mu}$, we have

$$\begin{aligned}
\chi &= \max\{\chi_1, \chi_2\}, \\
&= \max\left\{\left((A_\mu^\lambda + B_\mu^\lambda \mathcal{T})L_{\mathrm{g}} + \lambda B_\mu^\lambda \mathcal{T}\right), ((a_\sigma + b_\sigma \mathcal{T})L_{\mathrm{h}_{\mathtt{m}}} + (a_\mu + b_\mu \mathcal{T})L_{\mathrm{g}})\right\}, \\
&= \max\left\{\left(\frac{1}{60} + \frac{4\lambda\mu}{M(\mu)(2-\mu) + 2\lambda(1-\mu)}\right), \left(\frac{1}{2} + \frac{2(1-\mu)\lambda + M(\mu)(2-\mu)}{M(\mu)(120 - 60\mu)}\right)\right\}.
\end{aligned}$$

It is worth noting that χ is dependent on μ, leading to variations in its value as μ ranges from 0 to 1.

Table 2.3 illustrates the variations of χ_1 and χ_2 for select values of $\mu \in]0, 1[$, let $M(\mu) = \mu^2 - \mu + 1$ and $\lambda = \frac{(2-\mu)M(\mu)}{15\mu}$. This table presents a concise summary of how the values of χ_1 and χ_2 change with varying μ. Remarkably, in all instances depicted in this table, a consistent pattern emerges where χ_1 consistently exceeds χ_2, and χ_1 remains below 1. Consequently,

$$\chi = \chi_2 = \frac{1}{2} + \frac{(2-\mu)M(\mu) + 2\lambda(1-\mu)}{60(2-\mu)M(\mu)} < 1.$$

Table 2.3: χ-values with distinct values of $\mu \in]0, 1[$.

$\boldsymbol{\mu}$	$\boldsymbol{\lambda}$	$\boldsymbol{M(\mu)}$	$\boldsymbol{\chi_1}$	$\boldsymbol{\chi_2}$	$\boldsymbol{\chi}$
0.1	1.272	0.910	0.143	0.536	χ_2
0.2	0.610	0.840	0.212	0.525	χ_2
0.3	0.391	0.790	0.265	0.521	χ_2
0.4	0.282	0.760	0.307	0.520	χ_2
0.5	0.216	0.750	0.339	0.518	χ_2
0.6	0.171	0.760	0.359	0.518	χ_2
0.7	0.137	0.790	0.364	0.517	χ_2
0.8	0.110	0.840	0.353	0.517	χ_2
0.9	0.087	0.910	0.325	0.516	χ_2

Figure 2.2 depicts the variations in χ_1 and χ_2 for $\mu \in]0,1[$, considering $M(\mu) = \mu^2 - \mu + 1$ and $\lambda = \frac{(2-\mu)M(\mu)}{15\mu}$. The graph gives a enhanced comprehension regarding the manner in which χ_1 and χ_2 vary in response to changes in μ. As depicted in the Figure 2.2, χ_2 persistently exceeds χ_1, with χ_2 not surpassing 0.516 and χ_1 not overtaking 0.325, both of which are below 1. Therefore, condition (2.5) holds. As per Theorem 2.1, the problem (2.14) has a unique solution. Based on Theorem 2.3, the problem (2.14) is stable in the HU sense. Next, we

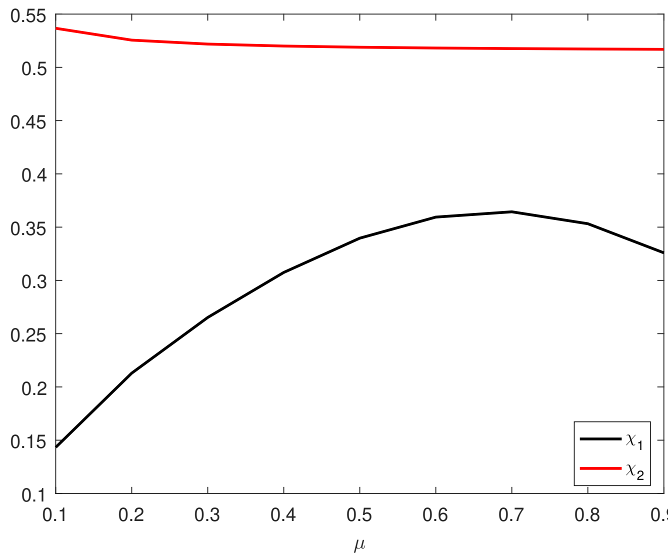

Figure 2.2: χ_1 and χ_2 -variations with distinct values of $\mu \in]0,1[$.

compute the values of the constant $c_{\mathrm{g},\mu,\sigma,\mathrm{h}_{\mathtt{m}}}$. Let $z \in \mathcal{PC}1(\theta,\mathbb{R})$ be a solution of the inequality (2.15), and φ the solution of the problem (2.14).
Let $\upsilon \in \theta_{\mathtt{m}}$, $\mathtt{m} = 0,1,2$, we obtain

$$\begin{aligned} |\mathfrak{z}(\upsilon) - \varphi(\upsilon)| &\leq c_{\mathrm{g},\mu,\sigma,\mathrm{h}_{\mathtt{m}}}\varepsilon, \\ &\leq \frac{(a_\mu + b_\mu \mathcal{T})}{1 - ((A_\mu^\lambda + B_\mu^\lambda \mathcal{T})L_{\mathrm{g}} + \lambda B_\mu^\lambda \mathcal{T})}\varepsilon, \\ &\leq \frac{a_\mu + b_\mu T}{\frac{59}{60} - \lambda B_\mu^\lambda T}\varepsilon. \end{aligned}$$

Considering $\upsilon \in J_{\mathtt{m}}, \mathtt{m} = 1,2$, we get

$$\begin{aligned} |\mathfrak{z}(\upsilon) - \varphi(\upsilon)| &\leq c_{\mathrm{g},\mu,\sigma,\mathrm{h}_{\mathtt{m}}}\varepsilon, \\ &\leq \frac{1}{1 - ((a_\sigma + b_\sigma \mathcal{T})L_{\mathrm{h}_{\mathtt{m}}} + (a_\mu + b_\mu \mathcal{T})L_{\mathrm{g}}))}\varepsilon \leq \frac{1}{\frac{1}{2} - \frac{a_\mu + b_\mu T}{60(A_\mu^\lambda + B_\mu^\lambda \mathcal{T})}}\varepsilon. \end{aligned}$$

Tables 2.4 and 2.5 detail the variation of the parameter $c_{\mathrm{g},\mu,\sigma,\mathrm{h}_{\mathtt{m}}}$ for distinct values of $\mu \in]0,1[$. These tables provide an exhaustive view on the variation of these parameters under diverse conditions. Figure 2.3 supplements the tabular information by graphically illustrating the changes in $c_{\mathrm{g},\mu,\sigma,\mathrm{h}_{\mathtt{m}}}$ across varying μ values within the similar interval. Notably, both the tables and the figure highlight that all $c_{\mathrm{g},\mu,\sigma,\mathrm{h}_{\mathtt{m}}}>0$, Thus, there exists a positive constant $c_{\mathrm{g},\mu,\sigma,\mathrm{h}_{\mathtt{m}}} > 0$, such that for any ε and for every solution $\mathfrak{z} \in \mathcal{PC}1(\theta, \mathbb{R})$ of the inequalities (2.15), there exists a solution $\varphi \in \mathcal{PC}1(\theta, \mathbb{R})$ of problem (2.14) satisfying

$$|\mathfrak{z}(\upsilon) - \varphi(\upsilon)| \leq c_{\mathrm{g},\mu,\sigma,\mathrm{h}_{\mathtt{m}}}\varepsilon, \quad \upsilon \in \theta,$$

demonstrating the HU stability of problem (2.14).

Table 2.4: Values of $c_{\mathrm{g},\mu,\sigma,\mathrm{h}_{\mathtt{m}}}$ for distinct values of $\mu \in]0,1[$, $\upsilon \in \theta_{\mathtt{m}}$, $\mathtt{m} = 0, 1, 2$.

μ	0.1	0.2	0.3	0,4	0.5	0.6	0.7	0.8	0.9
$c_{\mathrm{g},\mu,\sigma,\mathrm{h}_{\mathtt{m}}}$	0.0102	0.0375	0.0832	0.1511	0.2448	0.3684	0.5275	0.7334	1.0118

Table 2.5: Values of $c_{\mathrm{g},\mu,\sigma,\mathrm{h}_{\mathtt{m}}}$ for distinct values of $\mu \in]0,1[$, $\upsilon \in J_{\mathtt{m}}, \mathtt{m} = 1, 2$.

μ	0.1	0.2	0.3	0,4	0.5	0.6	0.7	0.8	0.9
$c_{\mathrm{g},\mu,\sigma,\mathrm{h}_{\mathtt{m}}}$	2.1510	2.0949	2.0742	2.0631	2.0566	2.0532	2.0525	2.0543	2.0592

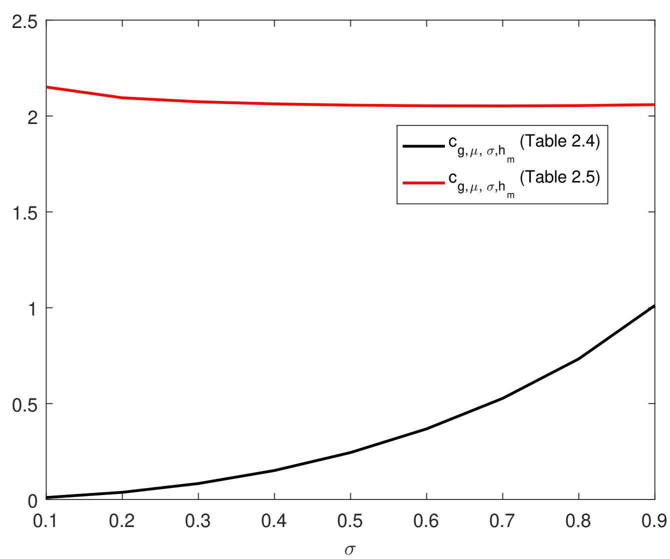

Figure 2.3: Variations of $c_{\mathrm{g},\mu,\sigma,\mathrm{h}_{\mathtt{m}}}$ from Table 2.4 and Table 2.5 with distinct values of $\mu \in]0,1[$.

2.5 Conclusion

In this Chapter, we delved into a formerly unexplored category of FDEs that encompass non-instantaneous impulses (NIIs) under CFfd. Utilizing the Banach FPT and Darbo's FPT combined with the KMN, we rigorously established the existence and uniqueness results. Additionally, our exploration offered significant insights into the stability of solutions in the HU sense, emphasizing the the dependability of systems controlled by these FDEs. Aside from our theoretical advancements, we showcased two illustrative examples that demonstrate the diversity and the importance of our findings. The examples provided did not focus solely on a particular value of μ instead, they encompassed a broad spectrum from 0.1 to 0.9. This practical examples highlights how widely applicable and adaptable our findings are. Moving ahead, our subsequent research will be devoted to find a numerical solution for these FDEs with NIIs. This marks a promising avenue for future research, permitting for an in-depth exploration of the tangible effects of our findings and their practical implications.

Chapter 3

Novel Existence Results for a Class of Fractional Integro-Differential Equations with Non-Instantaneous Impulses

3.1 Introduction

The purpose of this chapter is to undertake a exhaustive study into the existence of solutions concerning fractional integro-differential equations (FIDEs) characterized by non-instantaneous impulses under the Caputo fractional derivative. This investigation relies significantly on the application of two fundamental fixed point theorems credited to Krasnoselskii and Darbo associated with the Kuratowski's measure of noncompactness (KMNC). In order to substantiate and concretely demonstrate the practical implications of our theoretical findings, we offer the elucidation of two illustrative examples. Through these illustrative instances, we aim not only to highlight the validity and significance of our derived results but also to offer a practical lens through which to comprehend the behavior of solutions in both theoretical and applied contexts.

Inspired by the paper [51], we investigate the existence of at least one solution for the following non-instantaneous impulsive FIDEs:

$$\begin{cases} {}^{C}\mathfrak{D}^{\mu}\varphi(\upsilon) = \mathfrak{U}(\upsilon, \varphi(\upsilon)) + \displaystyle\int_0^{\upsilon} \Psi(\upsilon, r, \varphi(r))dr, \quad \upsilon \in (\delta_{\mathtt{m}}, \upsilon_{\mathtt{m}+1}], \mathtt{m} = 0, \dots n, \\ \varphi(\upsilon) = \dfrac{1}{\Gamma(\mu)} \displaystyle\int_{\upsilon_{\mathtt{m}}}^{\upsilon} (\upsilon - r)^{\mu-1} \mathbb{G}_{\mathtt{m}}(r, \varphi(r_{\mathtt{m}}^{-}))dr, \quad \upsilon \in (\upsilon_{\mathtt{m}}, \delta_{\mathtt{m}}], \mathtt{m} = 1, \dots n, \\ \alpha_1\varphi(0) + \alpha_2\varphi(\upsilon) = \eta(0). \end{cases} \tag{3.1}$$

Where ${}^{C}\mathfrak{D}^{\mu}$ is Caputo's differential operator of order $\mu \in (0,1]$, $\theta = [0,\mathcal{T}], \mathcal{T} > 0$, $\alpha_1, \alpha_2 \in \mathbb{R}$. $\theta_0 = [0, \upsilon_1]$, $\theta_{\mathtt{m}} = (\delta_{\mathtt{m}}, \upsilon_{\mathtt{m}+1}]; \mathtt{m} = 0, \ldots, n$, $J_{\mathtt{m}} = (\upsilon_{\mathtt{m}}, \delta_{\mathtt{m}}]; \mathtt{m} = 1, \ldots n$, $\mathfrak{U} : \theta \times \mathbb{R} \to \mathbb{R}, \Psi : \theta \times \theta \times \mathbb{R} \to \mathbb{R}$, $\mathbb{G}_{\mathtt{m}} : J_{\mathtt{m}} \times \mathbb{R} \to \mathbb{R}$ are continuous functions with $\mathfrak{U}(\upsilon, \varphi(\upsilon))_{\upsilon=0} = 0$. We consider the split of the interval θ with respect to $\upsilon_{\mathtt{m}}, \delta_{\mathtt{m}}$ such that $0 < \upsilon_{\mathtt{m}} < \delta_{\mathtt{m}} < \mathcal{T}$ for $\mathtt{m} = 1, 2, 3, \ldots, n$ and assume $\upsilon_{n+1} = \mathcal{T}$.

3.2 Main results

In this section, we discuss several results related to the existence of solutions for the problem (3.1). The initial outcome relies on the Krasnoselskii FPT, while the subsequent result is grounded in the Darbo FPT.
Let's consider the Banach space

$$\mathcal{PC} = \Big\{ \varphi : \theta \to \mathbb{R} : \varphi|_{J_{\mathtt{m}}}; \ \mathtt{m} = 1, \ldots, m, \ \varphi|_{\theta_{\mathtt{m}}}; \mathtt{m} = 0, \ldots, n \text{ are continuous and there exist } \varphi(\delta_{\mathtt{m}}^-), \varphi(\delta_{\mathtt{m}}^+), \varphi(\upsilon_{\mathtt{m}}^-) \text{ and } \varphi(\upsilon_{\mathtt{m}}^+) \text{ with } \varphi(\upsilon_{\mathtt{m}}^-) = \varphi(\upsilon_{\mathtt{m}}) \Big\},$$

equipped with

$$\|\varphi\|_{\mathcal{PC}} = \sup_{\upsilon \in \theta} |\varphi(\upsilon)|.$$

Lemma 3.1. *[51] Let $\mu \in (0,1]$ and $\mathbb{H}(\upsilon, \varphi(\upsilon)) \in \mathcal{C}(\theta, \mathbb{R})$. Then $\varphi \in \mathcal{PC}(\theta, \mathbb{R})$ is a solution of*

$$\begin{cases} {}^{C}\mathfrak{D}^{\mu}\varphi(\upsilon) = \mathbb{H}(\upsilon, \varphi(\upsilon)), \quad \upsilon \in \theta_{m}, m = 0, \ldots n, \\ \varphi(\upsilon) = \dfrac{1}{\Gamma(\mu)} \displaystyle\int_{\upsilon_m}^{\upsilon} (\upsilon - r)^{\mu-1} \mathbb{G}_m(r, \varphi(r_m^-)) dr, \quad \upsilon \in J_m, m = 1, \ldots n, \\ \alpha_1 \varphi(0) + \alpha_2 \varphi(\upsilon) = \eta(0), \end{cases} \tag{3.2}$$

if and only if φ verifies the following integral equation

$$\varphi(\upsilon) = \begin{cases} C - \dfrac{\mathfrak{b}}{\Gamma(\mu)} \left[\displaystyle\int_{\upsilon_m}^{\delta_m} (\delta_m - r)^{\mu-1} \mathbb{G}_m(r, \varphi(r_m^-)) dr + \int_{\upsilon_m}^{\mathcal{T}} (\mathcal{T} - r)^{\mu-1} \mathbb{H}(r, \varphi(r)) dr \right] \\ + \dfrac{1}{\Gamma(\mu)} \displaystyle\int_0^{\upsilon} (\upsilon - r)^{\mu-1} (\mathbb{H}(r, \varphi(r)) dr, \text{ if } \upsilon \in \theta_0, \\ \dfrac{1}{\Gamma(\mu)} \displaystyle\int_{\upsilon_m}^{\upsilon} (\upsilon - r)^{\mu-1} \mathbb{G}_m(r, \varphi(r_m^-)) dr, \text{ if } \upsilon \in J_m, m = 1, \ldots n, \\ \dfrac{1}{\Gamma(\mu)} \displaystyle\int_{\upsilon_m}^{\delta_m} (\delta_m - r)^{\mu-1} \mathbb{G}_m(r, \varphi(r_m^-)) dr + \dfrac{1}{\Gamma(\mu)} \int_{\upsilon_m}^{\upsilon} (\upsilon - r)^{\mu-1} \mathbb{H}(r, \varphi(r)) dr, \text{ if, } \upsilon \in \theta_m, m = 0, \ldots n. \end{cases} \tag{3.3}$$

with $C = \frac{\eta(0)}{\alpha_1}$ and $\mathfrak{b} = \frac{\alpha_2}{\alpha_1}$.

Proof. We split the proof into the subsequent cases.

Case 1: For $\upsilon \in \theta_0 = [0, \upsilon_1]$, upon applying the integral operator $\mathfrak{I}^{\mu}$ to equation (3.2), we obtain

$$\varphi(\upsilon) = \varphi(0) + \frac{1}{\Gamma(\mu)} \int_0^{\upsilon} (\upsilon - s)^{\mu-1} \mathbb{H}(r, \varphi(r)) dr. \tag{3.4}$$

Case 2: For $\upsilon \in \theta_{\mathtt{m}} = (\delta_{\mathtt{m}}, \upsilon_{\mathtt{m}+1}]$, $\mathtt{m} = 0, \ldots n$, applying the integral operator $\mathfrak{I}^{\mu}$ on (3.2), we obtain

$$\varphi(\upsilon) = \varphi(\delta_{\mathtt{m}}) + \frac{1}{\Gamma(\mu)} \int_{\upsilon_{\mathtt{m}}}^{\upsilon} (\upsilon - r)^{\mu-1} \mathbb{H}(r, \varphi(r)) dr. \tag{3.5}$$

Using the impulsive relation:

$$\varphi(\upsilon) = \frac{1}{\Gamma(\mu)} \int_{\upsilon_{\mathtt{m}}}^{\upsilon} (\upsilon - r)^{\mu-1} \mathbb{G}_{\mathtt{m}} \left(r, \varphi\left(r_{\mathtt{m}}^{-}\right)\right) dr,$$

we get

$$\varphi(\delta_{\mathtt{m}}) = \frac{1}{\Gamma(\mu)} \int_{\upsilon_{\mathtt{m}}}^{\delta_{\mathtt{m}}} (\delta_{\mathtt{m}} - r)^{\mu-1} \mathbb{G}_{\mathtt{m}} \left(r, \varphi\left(r_{\mathtt{m}}^{-}\right)\right) dr.$$

Thus,(3.5) implies

$$\varphi(\upsilon) = \frac{1}{\Gamma(\mu)} \int_{\upsilon_{\mathtt{m}}}^{\delta_{\mathtt{m}}} (\delta_{\mathtt{m}} - r)^{\mu-1} \mathbb{G}_{\mathtt{m}} \left(r, \varphi\left(r_{\mathtt{m}}^{-}\right)\right) dr + \frac{1}{\Gamma(\mu)} \int_{\upsilon_{\mathtt{m}}}^{\upsilon} (\upsilon - r)^{\mu-1} \mathbb{H}(r, \varphi(r)) dr,$$

now, using the condition $\alpha_1 \varphi(0) + \alpha_2 \varphi(\upsilon) = \eta(0)$, we obtain

$$\varphi(0) = \frac{\eta(0)}{\alpha_1} - \frac{\alpha_2}{\alpha_1 \Gamma(\mu)} \left[\int_{\upsilon_{\mathtt{m}}}^{\delta_{\mathtt{m}}} (\delta_{\mathtt{m}} - r)^{\mu-1} \mathbb{G}_{\mathtt{m}} \left(r, \varphi\left(r_{\mathtt{m}}^{-}\right)\right) dr + \int_{\upsilon_{\mathtt{m}}}^{\mathcal{T}} (T - r)^{\mu-1} \mathbb{H}(r, \varphi(r)) dr \right], \tag{3.6}$$

thus, by the help of (3.4) and (3.6), for $\upsilon \in \theta_0$, we get

$$\begin{aligned}
\varphi(\upsilon) =& \frac{\eta(0)}{\alpha_1} - \frac{\alpha_2}{\alpha_1 \Gamma(\mu)} \left[\int_{\upsilon_{\mathtt{m}}}^{\delta_{\mathtt{m}}} (\delta_{\mathtt{m}} - r)^{\mu-1} \mathbb{G}_{\mathtt{m}} \left(r, \varphi\left(r_{\mathtt{m}}^{-}\right)\right) dr + \int_{\upsilon_{\mathtt{m}}}^{\mathcal{T}} (T - r)^{\mu-1} \mathbb{H}(r, \varphi(r)) dr \right] \\
&+ \frac{1}{\Gamma(\mu)} \int_0^{\upsilon} (t - r)^{\mu-1} \mathbb{H}(r, \varphi(r)) dr, \\
&= C - \frac{\mathfrak{b}}{\Gamma(\mu)} \left[\int_{\upsilon_{\mathtt{m}}}^{\delta_{\mathtt{m}}} (\delta_{\mathtt{m}} - r)^{\mu-1} \mathbb{G}_{\mathtt{m}} \left(r, \varphi\left(r_{\mathtt{m}}^{-}\right)\right) dr + \int_{\upsilon_{\mathtt{m}}}^{\mathcal{T}} (T - r)^{\mu-1} \mathbb{H}(r, \varphi(r)) dr \right] \\
&+ \frac{1}{\Gamma(\mu)} \int_0^{\upsilon} (t - r)^{\mu-1} \mathbb{H}(r, \varphi(r)) dr.
\end{aligned}$$

Case 3: For $\upsilon \in J_{\mathtt{m}} = (\upsilon_{\mathtt{m}}, \delta_{\mathtt{m}}]$, $\mathtt{m} = 1, \ldots n$, we obtain

$$\varphi(\upsilon) = \frac{1}{\Gamma(\mu)} \int_{\upsilon_{\mathtt{m}}}^{\upsilon} (\upsilon - r)^{\mu-1} \mathbb{G}_{\mathtt{m}} \left(r, \varphi\left(r_{\mathtt{m}}^{-}\right)\right) dr.$$

This concludes the demonstration. □

Corollary 3.1. *By replacing $\mathbb{H}(\upsilon,\varphi(\upsilon))$ in the system (3.3) by $\mathfrak{U}(\upsilon,\varphi(\upsilon))+\int_0^\upsilon \Psi(\upsilon,r,\varphi(r))dr$ we obtain the following integral equation for FIDEs (3.1):*

$$\varphi(\upsilon)=\begin{cases} C-\dfrac{\mathfrak{b}}{\Gamma(\mu)}\Bigg[\displaystyle\int_{\upsilon_m}^{\delta_m}(\delta_m-r)^{\mu-1}\mathbb{G}_m(r,\varphi(r_m^-))dr+\int_{\upsilon_m}^{\mathcal{T}}(\mathcal{T}-r)^{\mu-1}(\mathfrak{U}(r,\varphi(r)) \\ +\displaystyle\int_0^s\Psi(r,\mathfrak{e},\varphi(\mathfrak{e}))d\mathfrak{e})dr\Bigg]+\dfrac{1}{\Gamma(\mu)}\displaystyle\int_0^\upsilon(\upsilon-r)^{\mu-1}(\mathfrak{U}(r,\varphi(r)) \\ +\displaystyle\int_0^s\Psi(r,\mathfrak{e},\varphi(\mathfrak{e}))d\mathfrak{e})dr,\ if\ \upsilon\in\theta_0, \\ \dfrac{1}{\Gamma(\mu)}\displaystyle\int_{\upsilon_m}^{\upsilon}(\upsilon-r)^{\mu-1}\mathbb{G}_m(r,\varphi(r_m^-))dr,\ if\ \upsilon\in J_m,\ m=1,\dots n, \\ \dfrac{1}{\Gamma(\mu)}\displaystyle\int_{\upsilon_m}^{\delta_m}(\delta_m-r)^{\mu-1}\mathbb{G}_m(r,\varphi(r_m^-))dr+\dfrac{1}{\Gamma(\mu)}\displaystyle\int_{\upsilon_m}^{\upsilon}(\upsilon-r)^{\mu-1}\mathfrak{U}(r,\varphi(r)) \\ +\displaystyle\int_0^s\Psi(r,\mathfrak{e},\varphi(\mathfrak{e}))d\mathfrak{e})dr,\ if\ \upsilon\in\theta_m,\ m=0,\dots n. \end{cases}$$

The following assumptions will be utilized subsequently.

(H_1) $\mathbb{G}_\mathtt{m}: J_\mathtt{m}\times\mathbb{R}\to\mathbb{R}, \mathtt{m}=1,2,\dots,n$ are continuous functions. There exist positive constant K_g where

$$|\mathbb{G}_\mathtt{m}(\upsilon,\varphi(\upsilon))-\mathbb{G}_\mathtt{m}(\upsilon,\mathfrak{u}(\upsilon))|\le K_g|\upsilon-\mathfrak{u}|,$$

for any $\upsilon,\mathfrak{u}\in\mathbb{R}$ and $\upsilon\in J_\mathtt{m}; \mathtt{m}=1,2,\dots,n$.

(H_2) The function $\mathfrak{U}$ is continuous, there exist function $p(\upsilon)\in L^p(\theta,\mathbb{R}_+)$ $(p>\frac{1}{\mu})$ and a nondecreasing continuous function $\Omega_1:(0,\infty]\to(0,\infty]$, where

$$|\mathfrak{U}(\upsilon,\varphi(\upsilon))|\le p(\upsilon)\Omega_1(\|\upsilon\|),\quad \upsilon\in\theta,\varphi\in\mathbb{R}.$$

(H_3)$\Psi:\theta\times\theta\times\mathbb{R}\to\mathbb{R}$ a continuous function and there exist function $q(\upsilon)\in L^p(\theta,\mathbb{R}_+)$ $(p>\frac{1}{\mu})$ and a nondecreasing continuous function $\Omega_2:(0,\infty]\to(0,\infty]$, where

$$|\Psi(\upsilon,r,\varphi(r))|\le q(r)\Omega_2(\|\upsilon\|),\quad \upsilon,r\in\theta,\varphi\in\mathbb{R}.$$

Ultimately, we place

$$p^*=\sup_{\upsilon\in\theta}p(\upsilon),\ q^*=\sup_{\upsilon\in\theta}q(\upsilon).$$

The initial outcome regarding the existence of a solution to problem (3.1) is provided through the utilization of Krasnoselskii's FPT.

Theorem 3.1. *Assume that the hypotheses (H_1)-(H_3)are satisfied, If the inequality*

$$\gamma=\max\left\{\frac{\mathfrak{b}K_gT^\mu}{\Gamma(\mu+1)},\frac{K_gT^\mu}{\Gamma(\mu+1)}\right\}<1 \tag{3.7}$$

holds, hence there exists at least one solution defined on θ for the problem (3.1).

Proof. Let transform problem (3.1) into a fixed point problem by introducing the operator $\mathcal{F}:\mathcal{PC}\to\mathcal{PC}$ with

$$\mathcal{F}\varphi(\upsilon)=\begin{cases} C-\dfrac{\mathfrak{b}}{\Gamma(\mu)}\Big[\displaystyle\int_{\upsilon_{\mathtt{m}}}^{\delta_{\mathtt{m}}}(\delta_{\mathtt{m}}-r)^{\mu-1}\mathbb{G}_{\mathtt{m}}(r,\varphi(r_{\mathtt{m}}^{-}))dr+\int_{\upsilon_{\mathtt{m}}}^{\mathcal{T}}(\mathcal{T}-r)^{\mu-1}(\mathfrak{U}(r,\varphi(r)) \\ +\displaystyle\int_0^s\Psi(r,\mathfrak{e},\varphi(\mathfrak{e}))d\mathfrak{e})dr\Big]+\dfrac{1}{\Gamma(\mu)}\int_0^{\upsilon}(\upsilon-r)^{\mu-1}(\mathfrak{U}(r,\varphi(r)) \\ +\displaystyle\int_0^s\Psi(r,\mathfrak{e},\varphi(\mathfrak{e}))d\mathfrak{e})dr,\text{if } \upsilon\in\theta_0, \\ \dfrac{1}{\Gamma(\mu)}\displaystyle\int_{\upsilon_{\mathtt{m}}}^{\upsilon}(\upsilon-r)^{\mu-1}\mathbb{G}_{\mathtt{m}}(r,\varphi(r_{\mathtt{m}}^{-}))dr,\text{if } \upsilon\in J_{\mathtt{m}},\mathtt{m}=1,\dots n, \\ \dfrac{1}{\Gamma(\mu)}\displaystyle\int_{\upsilon_{\mathtt{m}}}^{\delta_{\mathtt{m}}}(\delta_{\mathtt{m}}-r)^{\mu-1}\mathbb{G}_{\mathtt{m}}(r,\varphi(r_{\mathtt{m}}^{-}))dr+\dfrac{1}{\Gamma(\mu)}\int_{\upsilon_{\mathtt{m}}}^{\upsilon}(\upsilon-r)^{\mu-1}\mathfrak{U}(r,\varphi(r)) \\ +\displaystyle\int_0^s\Psi(r,\mathfrak{e},\varphi(\mathfrak{e}))d\mathfrak{e})dr,\text{if } \upsilon\in\theta_{\mathtt{m}},\mathtt{m}=0,\dots n. \end{cases}\tag{3.8}$$

Clearly, the fixed points of the operator $\mathcal{F}$ are solutions of the problem (3.1).
Consider the set

$$\mathcal{Q}=\{\upsilon\in\mathcal{PC}(\theta,\mathbb{R}):\|\upsilon\|_{\mathcal{Q}=\{\upsilon\in\mathcal{PC}}\leq\mathcal{N}\},\tag{3.9}$$

$$\text{with}\quad \mathcal{N}=\max\Bigg\{\frac{C\,\Gamma(\mu+1)+T^{\mu}(\mathfrak{b}L_g+(p^*\Omega_1(\mathcal{N})+Tq^*\Omega_2(\mathcal{N}))(\mathfrak{b}+1))}{\Gamma(\mu+1)},\frac{\mathcal{T}^{\mu}(L_g+(p^*\Omega_1(\mathcal{N})+Tq^*\Omega_2(\mathcal{N})))}{\Gamma(\mu+1)}\Bigg\}.\tag{3.10}$$

For $\upsilon\in\mathcal{Q}$ and $d=\sup\limits_{\upsilon\in J_{\mathtt{m}}}|\mathbb{G}_{\mathtt{m}}(\upsilon,0)|,\upsilon\in J_{\mathtt{m}};\mathtt{m}=1,2,\dots,n$, we consider

$$\begin{aligned}|\mathbb{G}_{\mathtt{m}}(\upsilon,\varphi(\upsilon))|&=|\mathbb{G}_{\mathtt{m}}(\upsilon,\varphi(\upsilon))-\mathbb{G}_{\mathtt{m}}(\upsilon,0)+\mathbb{G}_{\mathtt{m}}(\upsilon,0)|,\\ &\leq|\mathbb{G}_{\mathtt{m}}(\upsilon,\varphi(\upsilon))-\mathbb{G}_{\mathtt{m}}(\upsilon,0)|+|\mathbb{G}_{\mathtt{m}}(\upsilon,0)|,\\ &\leq K_g\|\upsilon\|_{\mathcal{PC}}+d,\\ &\leq K_g\mathcal{N}+d=L_g^*.\end{aligned}$$

Let define the operators A and B on $\mathcal{Q}$ in the following manner

$$A\varphi(\upsilon)=\begin{cases} C-\dfrac{\mathfrak{b}}{\Gamma(\mu)}\displaystyle\int_{\upsilon_{\mathtt{m}}}^{\delta_{\mathtt{m}}}(\delta_{\mathtt{m}}-r)^{\mu-1}\mathbb{G}_{\mathtt{m}}(r,\varphi(r_{\mathtt{m}}^{-}))dr,\ \text{if } \upsilon\in\theta_0, \\ \dfrac{1}{\Gamma(\mu)}\displaystyle\int_{\upsilon_{\mathtt{m}}}^{\upsilon}(\upsilon-r)^{\mu-1}\mathbb{G}_{\mathtt{m}}(r,\varphi(r_{\mathtt{m}}^{-}))dr,\text{if } \upsilon\in J_{\mathtt{m}},\mathtt{m}=1,\dots n, \\ \dfrac{1}{\Gamma(\mu)}\displaystyle\int_{\upsilon_{\mathtt{m}}}^{\delta_{\mathtt{m}}}(\delta_{\mathtt{m}}-r)^{\mu-1}\mathbb{G}_{\mathtt{m}}(r,\varphi(r_{\mathtt{m}}^{-}))dr,\ \text{if } \upsilon\in\theta_{\mathtt{m}},\mathtt{m}=0,\dots n. \end{cases}\tag{3.11}$$

and

$$B\varphi(\upsilon)=\begin{cases}\dfrac{\mathfrak{b}}{\Gamma(\mu)}\displaystyle\int_{\upsilon_{\mathtt{m}}}^{\mathcal{T}}(\mathcal{T}-r)^{\mu-1}(\mathfrak{U}(r,\varphi(r))+\int_0^s\Psi(r,\mathfrak{e},\varphi(\mathfrak{e}))d\mathfrak{e})dr\\ +\dfrac{1}{\Gamma(\mu)}\displaystyle\int_0^{\upsilon}(\upsilon-r)^{\mu-1}(\mathfrak{U}(r,\varphi(r))+\int_0^s\Psi(r,\mathfrak{e},\varphi(\mathfrak{e}))d\mathfrak{e})dr,\ \text{if } \upsilon\in\theta_0,\\ 0,\ \text{if } \upsilon\in J_{\mathtt{m}},\mathtt{m}=1,\dots n,\\ \dfrac{1}{\Gamma(\mu)}\displaystyle\int_{\upsilon_{\mathtt{m}}}^{\upsilon}(\upsilon-r)^{\mu-1}\mathfrak{U}(r,\varphi(r))+\int_0^s\Psi(r,\mathfrak{e},\varphi(\mathfrak{e}))d\mathfrak{e})dr,\ \text{if } \upsilon\in\theta_{\mathtt{m}},\mathtt{m}=0,\dots n.\end{cases} \tag{3.12}$$

So, we can write the following operator equation

$$\mathcal{F}\varphi(\upsilon)=A\varphi(\upsilon)+B\varphi(\upsilon)=\varphi(\upsilon),\ \upsilon\in\mathcal{PC}(\theta,\mathbb{R}).$$

Step 1. We prove that $A\varphi(\upsilon)+By(\upsilon)\in\mathcal{Q}$ for any $\varphi,y\in\mathcal{Q}$.
In consideration of each $\upsilon\in\theta_0$, we get

$$\begin{aligned}|A\varphi(\upsilon)+By(\upsilon)|=&\Bigg|C-\frac{\mathfrak{b}}{\Gamma(\mu)}\int_{\upsilon_{\mathtt{m}}}^{\delta_{\mathtt{m}}}(\delta_{\mathtt{m}}-r)^{\mu-1}\mathbb{G}_{\mathtt{m}}(r,\varphi(r_{\mathtt{m}}^-))dr+\frac{\mathfrak{b}}{\Gamma(\mu)}\int_{\upsilon_{\mathtt{m}}}^{\mathcal{T}}(\mathcal{T}-r)^{\mu-1}(\mathfrak{U}(r,y(r))\\&+\int_0^s\Psi(r,\mathfrak{e},y(\mathfrak{e}))d\mathfrak{e})dr+\frac{1}{\Gamma(\mu)}\int_0^{\upsilon}(\upsilon-r)^{\mu-1}(\mathfrak{U}(r,y(r))+\int_0^s\Psi(r,\mathfrak{e},y(\mathfrak{e}))d\mathfrak{e})dr\Bigg|,\\&\le C+\mathfrak{b}\Bigg[\frac{1}{\Gamma(\mu)}\int_{\upsilon_{\mathtt{m}}}^{\delta_{\mathtt{m}}}\Big|(\delta_{\mathtt{m}}-r)^{\mu-1}\Big||\mathbb{G}_{\mathtt{m}}(r,\varphi(r_{\mathtt{m}}^-))|dr\\&+\frac{1}{\Gamma(\mu)}\int_{\upsilon_{\mathtt{m}}}^{\mathcal{T}}\Big|(\mathcal{T}-r)^{\mu-1}\Big|\bigg(|\mathfrak{U}(r,y(r))|+\int_0^s|\Psi(r,\mathfrak{e},y(\mathfrak{e}))|d\mathfrak{e}\bigg)dr\Bigg]\\&+\frac{1}{\Gamma(\mu)}\int_0^{\upsilon}(\upsilon-r)^{\mu-1}(|\mathfrak{U}(r,y(r))|+\int_0^s|\Psi(r,\mathfrak{e},y(\mathfrak{e}))|d\mathfrak{e})dr,\\&\le C+\mathfrak{b}\Bigg[\frac{L_g^*}{\Gamma(\mu)}\int_{\upsilon_{\mathtt{m}}}^{\delta_{\mathtt{m}}}(\delta_{\mathtt{m}}-r)^{\mu-1}dr\\&+\frac{1}{\Gamma(\mu)}\int_{\upsilon_{\mathtt{m}}}^{\mathcal{T}}(\mathcal{T}-r)^{\mu-1}p(r)\Omega_1(\|\upsilon\|)dr+\frac{1}{\Gamma(\mu)}\int_{\upsilon_{\mathtt{m}}}^{\mathcal{T}}(\mathcal{T}-r)^{\mu-1}\int_0^s(q(\mathfrak{e})\Omega_2(\|\upsilon\|)d\mathfrak{e}dr\Bigg]\\&+\frac{1}{\Gamma(\mu)}\int_0^{\upsilon}(\upsilon-r)^{\mu-1}p(r)\Omega_1(\|\upsilon\|)dr+\frac{1}{\Gamma(\mu)}\int_0^{\upsilon}(\upsilon-r)^{\mu-1}\int_0^s(q(\mathfrak{e})\Omega_2(\|\upsilon\|)d\mathfrak{e}dr,\\&\le C+\frac{\mathfrak{b}L_g^*\mathcal{T}^{\mu}}{\Gamma(\mu+1)}+\frac{\mathcal{T}^{\mu}(p^*\Omega_1(\mathcal{N})+Tq^*\Omega_2(\mathcal{N}))(\mathfrak{b}+1)}{\Gamma(\mu+1)},\\&\le\mathcal{N}.\end{aligned}$$

For $\upsilon \in J_{\mathtt{m}},\ \mathtt{m} = 1, \ldots, n$, we possess

$$\begin{aligned}
|A\varphi(\upsilon) + By(\upsilon)| &= \left|\frac{1}{\Gamma(\mu)}\int_{\upsilon_{\mathtt{m}}}^{\upsilon}(\upsilon - r)^{\mu-1}\mathbb{G}_{\mathtt{m}}(r, \varphi(r_{\mathtt{m}}^{-}))dr + 0\right|,\\
&\leq \frac{L_g^*}{\Gamma(\mu)}\int_{\upsilon_{\mathtt{m}}}^{\upsilon}(\upsilon - r)^{\mu-1}|\mathbb{G}_{\mathtt{m}}(r, \varphi(r_{\mathtt{m}}^{-}))|dr,\\
&\leq \frac{L_g^* T^{\mu}}{\Gamma(\mu+1)},\\
&\leq \mathcal{N}.
\end{aligned}$$

Also, in consideration of every $\upsilon \in \theta_{\mathtt{m}},\ \mathtt{m} = 0, \ldots, n$, we obtain

$$\begin{aligned}
|A\varphi(\upsilon) + By(\upsilon)| &= \left|\frac{1}{\Gamma(\mu)}\int_{\upsilon_{\mathtt{m}}}^{\delta_{\mathtt{m}}}(\delta_{\mathtt{m}} - r)^{\mu-1}\mathbb{G}_{\mathtt{m}}(r, \varphi(r_{\mathtt{m}}^{-}))dr + \frac{1}{\Gamma(\mu)}\int_{\upsilon_{\mathtt{m}}}^{\upsilon}(\upsilon - r)^{\mu-1}\mathfrak{U}(r, y(r))\right.\\
&\quad \left. + \int_0^s \Psi(r, \mathfrak{e}, y(\mathfrak{e}))d\mathfrak{e})dr\right|,\\
&\leq \frac{1}{\Gamma(\mu)}\int_{\upsilon_{\mathtt{m}}}^{\delta_{\mathtt{m}}}\left|(\delta_{\mathtt{m}} - r)^{\mu-1}\right||\mathbb{G}_{\mathtt{m}}(r, \varphi(r_{\mathtt{m}}^{-}))|dr\\
&\quad + \frac{1}{\Gamma(\mu)}\int_{\upsilon_{\mathtt{m}}}^{\upsilon}\left|(\upsilon - r)^{\mu-1}\right|\left(|\mathfrak{U}(r, y(r))| + \int_0^s |\Psi(r, \mathfrak{e}, y(\mathfrak{e}))|d\mathfrak{e}\right)dr,\\
&\leq \frac{L_g^*}{\Gamma(\mu)}\int_{\upsilon_{\mathtt{m}}}^{\delta_{\mathtt{m}}}(\delta_{\mathtt{m}} - r)^{\mu-1}dr + \frac{1}{\Gamma(\mu)}\int_{\upsilon_{\mathtt{m}}}^{\upsilon}(\upsilon - r)^{\mu-1}p(r)\Omega_1(\|\upsilon\|)dr\\
&\quad + \frac{1}{\Gamma(\mu)}\int_{\upsilon_{\mathtt{m}}}^{\upsilon}(\upsilon - r)^{\mu-1}\int_0^s (q(\mathfrak{e})\Omega_2(\|\upsilon\|))d\mathfrak{e}dr,\\
&\leq \frac{L_g^* T^{\mu}}{\Gamma(\mu+1)} + \frac{\mathcal{T}^{\mu}(p^*\Omega_1(\mathcal{N}) + Tq^*\Omega_2(\mathcal{N}))}{\Gamma(\mu+1)},\\
&\leq \mathcal{N}.
\end{aligned}$$

Hence, for $\upsilon \in \theta$, we get

$$\|A\varphi + By\|_{\mathcal{PC}} \leq \mathcal{N},$$

thus $A\varphi + By \in \mathcal{Q}$.

Step 2. A is a contraction.

Given $\varphi, x \in \mathcal{Q}$, when $\upsilon \in \theta_0$, we achieve

$$\begin{aligned}
|A\varphi(\upsilon) - Ax(\upsilon)| &= \left|C - \frac{\mathfrak{b}}{\Gamma(\mu)}\int_{\upsilon_{\mathtt{m}}}^{\delta_{\mathtt{m}}}(\delta_{\mathtt{m}} - r)^{\mu-1}\mathbb{G}_{\mathtt{m}}(r, \varphi(r_{\mathtt{m}}^{-}))dr\right.\\
&\quad \left. - \left(C - \frac{\mathfrak{b}}{\Gamma(\mu)}\int_{\upsilon_{\mathtt{m}}}^{\delta_{\mathtt{m}}}(\delta_{\mathtt{m}} - r)^{\mu-1}\mathbb{G}_{\mathtt{m}}(r, x(r_{\mathtt{m}}^{-}))dr\right)\right|,\\
&\leq \frac{\mathfrak{b}}{\Gamma(\mu)}\int_{\upsilon_{\mathtt{m}}}^{\delta_{\mathtt{m}}}\left|(\delta_{\mathtt{m}} - r)^{\mu-1}\right||\mathbb{G}_{\mathtt{m}}(r, \varphi(r_{\mathtt{m}}^{-})) - \mathbb{G}_{\mathtt{m}}(r, x(r_{\mathtt{m}}^{-}))|dr,
\end{aligned}$$

$$\leq \frac{\mathfrak{b} K_g T^{\mu}}{\Gamma(\mu+1)}|\varphi - x|,$$
$$\leq \gamma_1 |\varphi - x|.$$

For $\upsilon \in J_{\mathtt{m}}$, $\mathtt{m} = 1, \ldots, n$, we obtain

$$\begin{aligned}
|A\varphi(\upsilon) - Ax(\upsilon)| &= \Bigg| \frac{1}{\Gamma(\mu)} \int_{\upsilon_{\mathtt{m}}}^{\upsilon} (\upsilon - r)^{\mu-1} \mathbb{G}_{\mathtt{m}}(r, \varphi(r_{\mathtt{m}}^{-}))dr \\
&\quad - \left(\frac{1}{\Gamma(\mu)} \int_{\upsilon_{\mathtt{m}}}^{\upsilon} (\upsilon - r)^{\mu-1} \mathbb{G}_{\mathtt{m}}(r, x(r_{\mathtt{m}}^{-}))dr \right) \Bigg|, \\
&\leq \frac{1}{\Gamma(\mu)} \int_{\upsilon_{\mathtt{m}}}^{\upsilon} \left|(\upsilon - r)^{\mu-1}\right| |\mathbb{G}_{\mathtt{m}}(r, \varphi(r_{\mathtt{m}}^{-})) - \mathbb{G}_{\mathtt{m}}(r, x(r_{\mathtt{m}}^{-}))| dr, \\
&\leq \frac{K_g T^{\mu}}{\Gamma(\mu+1)} |\varphi - x|, \\
&\leq \gamma_2 |\varphi - x|.
\end{aligned}$$

In consideration of every $\upsilon \in \theta_{\mathtt{m}}$, $\mathtt{m} = 0, \ldots, n$, we obtain

$$\begin{aligned}
|A\varphi(\upsilon) - Ax(\upsilon)| &= \Bigg| \frac{1}{\Gamma(\mu)} \int_{\upsilon_{\mathtt{m}}}^{\delta_{\mathtt{m}}} (\delta_{\mathtt{m}} - r)^{\mu-1} \mathbb{G}_{\mathtt{m}}(r, \varphi(r_{\mathtt{m}}^{-}))dr \\
&\quad - \left(\frac{1}{\Gamma(\mu)} \int_{\upsilon_{\mathtt{m}}}^{\delta_{\mathtt{m}}} (\delta_{\mathtt{m}} - r)^{\mu-1} \mathbb{G}_{\mathtt{m}}(r, x(r_{\mathtt{m}}^{-}))dr \right) \Bigg|, \\
&\leq \frac{1}{\Gamma(\mu)} \int_{\upsilon_{\mathtt{m}}}^{\delta_{\mathtt{m}}} \left|(\delta_{\mathtt{m}} - r)^{\mu-1}\right| |\mathbb{G}_{\mathtt{m}}(r, \varphi(r_{\mathtt{m}}^{-})) - \mathbb{G}_{\mathtt{m}}(r, x(r_{\mathtt{m}}^{-}))| dr, \\
&\leq \frac{K_g T^{\mu}}{\Gamma(\mu+1)} |\varphi - x|, \\
&\leq \gamma_2 |\varphi - x|.
\end{aligned}$$

Then, for each $\upsilon \in \theta$, we have

$$\|A\varphi - Ay\|_{\mathcal{PC}} \leq \gamma \|\varphi - x\|_{\mathcal{PC}}, \text{ with } \gamma = \max\{\gamma_1, \gamma_2\}.$$

Then by (H_1), the operator A is a contraction.
Step 3. We establish the continuity of B.
Let $\{\varphi_n\}_{n\in\mathbb{N}}$ be a sequence satisfying $\varphi_n \to \varphi$ in $\mathcal{PC}(\theta, \mathbb{R})$.
For $\upsilon \in \theta_0$, we get

$$\begin{aligned}
|B\varphi_n(\upsilon) - B\varphi(\upsilon)| &= \Bigg| \frac{\mathfrak{b}}{\Gamma(\mu)} \int_{\upsilon_{\mathtt{m}}}^{\mathcal{T}} (\mathcal{T} - r)^{\mu-1} (\mathfrak{U}(r, \varphi_n(r)) + \int_0^s \Psi(r, \mathfrak{e}, \varphi_n(\mathfrak{e}))d\mathfrak{e})dr \\
&\quad + \frac{1}{\Gamma(\mu)} \int_0^{\upsilon} (\upsilon - r)^{\mu-1} (\mathfrak{U}(r, \varphi_n(r)) + \int_0^s \Psi(r, \mathfrak{e}, \varphi_n(\mathfrak{e}))d\mathfrak{e})dr
\end{aligned}$$

$$
\begin{aligned}
&-\left(\frac{\mathfrak{b}}{\Gamma(\mu)}\int_{\upsilon_{\mathtt{m}}}^{\mathcal{T}}(\mathcal{T}-r)^{\mu-1}(\mathfrak{U}(r,\varphi(r))+\int_0^s\Psi(r,\mathfrak{e},\varphi(\mathfrak{e}))d\mathfrak{e})dr\right.\\
&\left.+\frac{1}{\Gamma(\mu)}\int_0^{\upsilon}(\upsilon-r)^{\mu-1}(\mathfrak{U}(r,\varphi(r))+\int_0^s\Psi(r,\mathfrak{e},\varphi(\mathfrak{e}))d\mathfrak{e})dr\right)\Bigg|,\\
&\leq\frac{\mathfrak{b}}{\Gamma(\mu)}\int_{\upsilon_{\mathtt{m}}}^{\upsilon}\left|(\mathcal{T}-r)^{\mu-1}\right|\Bigg(|\mathfrak{U}(r,\varphi_n(r))-\mathfrak{U}(r,\varphi(r))|\\
&+\int_0^s|\Psi(r,\mathfrak{e},\varphi_n(\mathfrak{e}))-\Psi(r,\mathfrak{e},\varphi(\mathfrak{e}))|d\mathfrak{e}\Bigg)dr\\
&+\frac{1}{\Gamma(\mu)}\int_0^{\upsilon}\left|(\upsilon-r)^{\mu-1}\right|\Bigg(|\mathfrak{U}(r,\varphi_n(r))-\mathfrak{U}(r,\varphi(r))|\\
&+\int_0^s|\Psi(r,\mathfrak{e},\varphi_n(\mathfrak{e}))-\Psi(r,\mathfrak{e},\varphi(\mathfrak{e}))|d\mathfrak{e}\Bigg)dr.
\end{aligned}
$$

For $\upsilon\in J_{\mathtt{m}}$, $\mathtt{m}=1,\ldots,n$, the null operator is continuous.
For $\upsilon\in\theta_{\mathtt{m}}$, $\mathtt{m}=0,\ldots,n$, we obtain

$$
\begin{aligned}
|B\varphi_n(\upsilon)-B\varphi(\upsilon)|=&\Bigg|\frac{1}{\Gamma(\mu)}\int_{\upsilon_{\mathtt{m}}}^{\upsilon}(\upsilon-r)^{\mu-1}\mathfrak{U}(r,\varphi_n(r))+\int_0^s\Psi(r,\mathfrak{e},\varphi_n(\mathfrak{e}))d\mathfrak{e}dr\\
&-\left(\frac{1}{\Gamma(\mu)}\int_{\upsilon_{\mathtt{m}}}^{\upsilon}(\upsilon-r)^{\mu-1}\mathfrak{U}(r,\varphi(r))+\int_0^s\Psi(r,\mathfrak{e},\varphi(\mathfrak{e}))d\mathfrak{e}dr\right)\Bigg|,\\
&\leq\frac{1}{\Gamma(\mu)}\int_{\upsilon_{\mathtt{m}}}^{\upsilon}\left|(\upsilon-r)^{\mu-1}\right|\Big(|\mathfrak{U}(r,\varphi_n(r))-\mathfrak{U}(r,\varphi(r))|\\
&+\int_0^s|\Psi(r,\mathfrak{e},\varphi_n(\mathfrak{e}))-\Psi(r,\mathfrak{e},\varphi(\mathfrak{e}))|d\mathfrak{e}\Big)dr.
\end{aligned}
$$

Since $\varphi_n\to\varphi$ and since $\mathfrak{U}(\upsilon,\varphi(\upsilon))$, $\Psi(\upsilon,r,\varphi(\upsilon))$ are continuous, then we obtain

$$\|B\varphi_n-B\varphi\|_{\mathcal{PC}}\to 0 \text{ as } n\to\infty.$$

Therefore, the operator B is continuous.
Step 4. We prove the compactness of B.
We first verify that B is uniformly bounded on $\mathcal{Q}$. By (3.10),(3.12) and Step1, we have

$$\|B\upsilon\|_{\mathcal{PC}}\leq\max\left\{\frac{\mathcal{T}^{\mu}(p^*\Omega_1(\mathcal{N})+Tq^*\Omega_2(\mathcal{N}))(\mathfrak{b}+1)}{\Gamma(\mu+1)},\frac{\mathcal{T}^{\mu}(p^*\Omega_1(\mathcal{N})+Tq^*\Omega_2(\mathcal{N}))}{\Gamma(\mu+1)}\right\}\leq\mathcal{N}.$$

Thus, B is uniformly bounded on $\mathcal{Q}$. Subsequently, we establish that B maps bounded sets into equicontinuous sets in $\mathcal{Q}$. We take $\upsilon\in\mathcal{Q}$ and $0<\tau_1<\tau_2<\mathcal{T}$.
Then for $\tau_1,\tau_2\in\theta_0$,

$$|B\varphi(\tau_1)-B\varphi(\tau_2)|=\Bigg|\frac{\mathfrak{b}}{\Gamma(\mu)}\int_{\upsilon_{\mathtt{m}}}^{\mathcal{T}}(\mathcal{T}-r)^{\mu-1}(\mathfrak{U}(r,\varphi(r))+\int_0^s\Psi(r,\mathfrak{e},\varphi(\mathfrak{e}))d\mathfrak{e})dr$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(\mu)}\int_0^{\tau_1}(\tau_1-r)^{\mu-1}(\mathfrak{U}(r,\varphi(r))+\int_0^s\Psi(r,\mathfrak{e},\varphi(\mathfrak{e}))d\mathfrak{e})dr\\
&-\Bigg(\frac{\mathfrak{b}}{\Gamma(\mu)}\int_{\upsilon_{\mathtt{m}}}^{\mathcal{T}}(\mathcal{T}-r)^{\mu-1}(\mathfrak{U}(r,\varphi(r))+\int_0^s\Psi(r,\mathfrak{e},\varphi(\mathfrak{e}))d\mathfrak{e})dr\\
&+\frac{1}{\Gamma(\mu)}\int_0^{\tau_2}(\tau_2-r)^{\mu-1}(\mathfrak{U}(r,\varphi(r))+\int_0^s\Psi(r,\mathfrak{e},\varphi(\mathfrak{e}))d\mathfrak{e})dr\Bigg)\Bigg|,\\
&\leq\frac{1}{\Gamma(\mu)}\Bigg[\int_0^{\tau_1}\left|(\tau_1-r)^{\mu-1}\right|\Big(|\mathfrak{U}(r,\varphi(r))|+\int_0^s|\Psi(r,\mathfrak{e},\varphi(\mathfrak{e}))|d\mathfrak{e}\Big)dr\\
&+\int_0^{\tau_2}\left|(\tau_2-r)^{\mu-1}\right|\Big(|\mathfrak{U}(r,\varphi(r))|+\int_0^s|\Psi(r,\mathfrak{e},\varphi(\mathfrak{e}))|d\mathfrak{e}\Big)dr\Bigg],\\
&\leq\frac{1}{\Gamma(\mu)}\Bigg[\int_0^{\tau_1}\left|(\tau_1-r)^{\mu-1}\right|\Bigg(p(r)\Omega_1(\|\upsilon\|)+\int_0^s\Big(q(\mathfrak{e})\Omega_2(\|\upsilon\|)\Big)d\mathfrak{e}\Bigg)dr\\
&+\int_0^{\tau_2}\left|(\tau_2-r)^{\mu-1}\right|\Bigg(p(r)\Omega_2(\|\upsilon\|)+\int_0^s\Big(q(\mathfrak{e})\Omega_2(\|\upsilon\|)\Big)d\mathfrak{e}\Bigg)dr\Bigg],\\
&\leq\frac{(p^*\Omega_1(\mathcal{N})+Tq^*\Omega_2(\mathcal{N}))}{\Gamma(\mu)}\Bigg[\int_0^{\tau_1}(\tau_1-r)^{\mu-1}-(\tau_2-r)^{\mu-1}dr+\int_{\tau_1}^{\tau_2}(\tau_2-r)^{\mu-1}dr\Bigg].
\end{aligned}
$$

For $\tau_1,\tau_2\in J_{\mathtt{m}},\ \mathtt{m}=1,\ldots,n$, the null operator is equicontinuous.
For $\tau_1,\tau_2\in\theta_{\mathtt{m}},\ \mathtt{m}=0,\ldots,n$, we obtain

$$
\begin{aligned}
|B\varphi(\tau_1)-B\varphi(\tau_2)|&=\Bigg|\frac{1}{\Gamma(\mu)}\int_{\upsilon_{\mathtt{m}}}^{\tau_1}(\tau_1-r)^{\mu-1}\mathfrak{U}(r,\varphi(r))+\int_0^s\Psi(r,\mathfrak{e},\varphi(\mathfrak{e}))d\mathfrak{e}dr\\
&-\frac{1}{\Gamma(\mu)}\int_{\upsilon_{\mathtt{m}}}^{\tau_2}(\tau_2-r)^{\mu-1}\mathfrak{U}(r,\varphi(r))+\int_0^s\Psi(r,\mathfrak{e},\varphi(\mathfrak{e}))d\mathfrak{e}dr\Bigg|,\\
&\leq\frac{1}{\Gamma(\mu)}\Bigg[\int_{\upsilon_{\mathtt{m}}}^{\tau_1}\left|(\tau_1-r)^{\mu-1}\right|\Big(|\mathfrak{U}(r,\varphi(r))|+\int_0^s|\Psi(r,\mathfrak{e},\varphi(\mathfrak{e}))|d\mathfrak{e}\Big)dr\\
&+\int_{\upsilon_{\mathtt{m}}}^{\tau_2}\left|(\tau_2-r)^{\mu-1}\right|\Big(|\mathfrak{U}(r,\varphi(r))|+\int_0^s|\Psi(r,\mathfrak{e},\varphi(\mathfrak{e}))|d\mathfrak{e}\Big)dr\Bigg],\\
&\leq\frac{1}{\Gamma(\mu)}\Bigg[\int_{\upsilon_{\mathtt{m}}}^{\tau_1}\left|(\tau_1-r)^{\mu-1}\right|\Bigg(p(r)\Omega_1(\|\upsilon\|)+\int_0^s\Big(q(\mathfrak{e})\Omega_2(\|\upsilon\|)\Big)d\mathfrak{e}\Bigg)dr\\
&+\int_{\upsilon_{\mathtt{m}}}^{\tau_2}\left|(\tau_2-r)^{\mu-1}\right|\Bigg(p(r)\Omega_1(\|\upsilon\|)+\int_0^s\Big(q(\mathfrak{e})|\Omega_2(\|\upsilon\|)\Big)d\mathfrak{e}\Bigg)dr,\\
&\leq\frac{(p^*\Omega_1(\mathcal{N})+Tq^*\Omega_2(\mathcal{N}))}{\Gamma(\mu)}\Bigg[\int_{\upsilon_{\mathtt{m}}}^{\tau_1}(\tau_1-r)^{\mu-1}-(\tau_2-r)^{\mu-1}dr+\int_{\tau_1}^{\tau_2}(\tau_2-r)^{\mu-1}dr\Bigg].
\end{aligned}
$$

note that

$$|B\varphi(\tau_1)-B\varphi(\tau_2)|\to 0 \text{ as } \tau_1\to\tau_2.$$

As a result, $B\mathcal{Q}$ is equicontinuous on θ, thereby indicating its relative compactness. According to the Arzelà-Ascoli theorem, B is compact. Applying Krasnoselskii's FPT, we deduce that $\mathcal{F}$ has a fixed point, which is the solution of problem (3.1). □

The next outcome concerning existence of solution for problem (3.1) is dependent on the Darbo's FPT.

Consider the subsequent hypotheses:

$(\mathcal{A}_1)$ Let $\mathbb{G}_{\mathtt{m}} : J_{\mathtt{m}} \times \mathbb{R} \to \mathbb{R}, \mathtt{m} = 1, 2, \ldots, n$, are continuous functions. There exist a constant $L_g > 0$ where

$$|\mathbb{G}_{\mathtt{m}}(\upsilon, \varphi(\upsilon))| \leq L_g,\ \upsilon \in J_{\mathtt{m}}; \mathtt{m} = 1, 2, \ldots, n, \upsilon \in \mathbb{R}.$$

$(\mathcal{A}_2)$ For each bounded set $G \subset \mathbb{R}$, we have

$$\eta(\mathbb{G}_{\mathtt{m}}(\upsilon, G)) \leq L_g \eta(G),\ \text{for each } \upsilon \in J_{\mathtt{m}}, \mathtt{m} = 1, \ldots, n,$$

$$\eta(\mathfrak{U}(\upsilon, G)) \leq p(\upsilon)\eta(G),\ \text{for each } \upsilon \in \theta,$$

and

$$\eta(\Psi(\upsilon, r, G)) \leq q(r)\eta(G),\ \text{for each } \upsilon, r \in \theta.$$

Theorem 3.2. *Assume that the hypotheses $(\mathcal{A}_1)$, $(\mathcal{A}_2)$,(H_2) and (H_3) hold. If*

$$\Upsilon = \max\left\{\frac{\mathcal{T}^{\mu}(\mathfrak{b}L_g + (\mathfrak{b}+1)(p^* + Tq^*))}{\Gamma(\mu+1)}, \frac{\mathcal{T}^{\mu}(L_g + (p^* + Tq^*))}{\Gamma(\mu+1)}\right\} < 1 \tag{3.13}$$

then the problem (3.1) has at least one solution on θ.

Proof. Let $\mathcal{F} : \mathcal{PC}(\theta, \mathbb{R}) \to \mathcal{PC}(\theta, \mathbb{R})$ be the operator defined by (3.8).

Step 1. We prove that $\mathcal{F}$ is continuous.

Consider a sequence $\{\varphi_n\}_{n\in\mathbb{N}}$ that fulfills $\varphi_n \to \varphi$ in $\mathcal{PC}(\theta, \mathbb{R})$.

In consideration of every $\upsilon \in \theta_0$, we have

$$\begin{aligned}
|\mathcal{F}\varphi_n(\upsilon) - \mathcal{F}\varphi(\upsilon)| = \Bigg| & C - \frac{\mathfrak{b}}{\Gamma(\mu)}\left[\int_{\upsilon_{\mathtt{m}}}^{\delta_{\mathtt{m}}} (\delta_{\mathtt{m}} - r)^{\mu-1}\mathbb{G}_{\mathtt{m}}(r, \varphi_n(r_{\mathtt{m}}^-))dr + \int_{\upsilon_{\mathtt{m}}}^{\mathcal{T}} (\mathcal{T} - r)^{\mu-1}(\mathfrak{U}(r, \varphi_n(r))\right. \\
& \left. + \int_0^s \Psi(r, \mathfrak{e}, \varphi_n(\mathfrak{e}))d\mathfrak{e})dr\right] + \frac{1}{\Gamma(\mu)}\int_0^{\upsilon} (\upsilon - r)^{\mu-1}(\mathfrak{U}(r, \varphi_n(r)) \\
& + \int_0^s \Psi(r, \mathfrak{e}, \varphi_n(\mathfrak{e}))d\mathfrak{e})dr - \Bigg(C - \frac{\mathfrak{b}}{\Gamma(\mu)}\left[\int_{\upsilon_{\mathtt{m}}}^{\delta_{\mathtt{m}}} (\delta_{\mathtt{m}} - r)^{\mu-1}\mathbb{G}_{\mathtt{m}}(r, \varphi(r_{\mathtt{m}}^-))dr\right. \\
& \left. + \int_{\upsilon_{\mathtt{m}}}^{\mathcal{T}} (\mathcal{T} - r)^{\mu-1}(\mathfrak{U}(r, \varphi(r)) + \int_0^s \Psi(r, \mathfrak{e}, \varphi(\mathfrak{e}))d\mathfrak{e})dr\right] \\
& + \frac{1}{\Gamma(\mu)}\int_0^{\upsilon} (\upsilon - r)^{\mu-1}(\mathfrak{U}(r, \varphi(r)) + \int_0^s \Psi(r, \mathfrak{e}, \varphi(\mathfrak{e}))d\mathfrak{e})dr\Bigg)\Bigg|, \\
\leq & \frac{\mathfrak{b}}{\Gamma(\mu)}\Bigg[\int_{\upsilon_{\mathtt{m}}}^{\delta_{\mathtt{m}}} \left|(\delta_{\mathtt{m}} - r)^{\mu-1}\right||\mathbb{G}_{\mathtt{m}}(r, \varphi_n(r_{\mathtt{m}}^-)) - \mathbb{G}_{\mathtt{m}}(r, \varphi(r_{\mathtt{m}}^-))|dr \\
& + \int_{\upsilon_{\mathtt{m}}}^{\mathcal{T}} \left|(\mathcal{T} - r)^{\mu-1}\right|\Big(|\mathfrak{U}(r, \varphi_n(r)) - \mathfrak{U}(r, \varphi(r))|
\end{aligned}$$

$$+\int_0^s |\Psi(r,\mathfrak{e},\varphi_n(\mathfrak{e}))-\Psi(r,\mathfrak{e},\varphi(\mathfrak{e}))|d\mathfrak{e}\Big)dr\Big]$$
$$+\frac{1}{\Gamma(\mu)}\int_0^{\upsilon}\left|(\upsilon-r)^{\mu-1}\right|\Big(|\mathfrak{U}(r,\varphi_n(r))-\mathfrak{U}(r,\varphi(r))|$$
$$+\int_0^s |\Psi(r,\mathfrak{e},\varphi_n(\mathfrak{e}))-\Psi(r,\mathfrak{e},\varphi(\mathfrak{e}))|d\mathfrak{e}\Big)dr.$$

For $\upsilon\in J_{\mathtt{m}},\ \mathtt{m}=1,\ldots,n$, we obtain

$$\begin{aligned}|\mathcal{F}\varphi_n(\upsilon)-\mathcal{F}\varphi(\upsilon)| &= \left|\frac{1}{\Gamma(\mu)}\int_{\upsilon_{\mathtt{m}}}^{\upsilon}(\upsilon-r)^{\mu-1}\mathbb{G}_{\mathtt{m}}(r,\varphi_n(r_{\mathtt{m}}^-))dr-\frac{1}{\Gamma(\mu)}\int_{\upsilon_{\mathtt{m}}}^{\upsilon}(\upsilon-r)^{\mu-1}\mathbb{G}_{\mathtt{m}}(r,\varphi(r_{\mathtt{m}}^-))dr\right|,\\
&\le \frac{1}{\Gamma(\mu)}\int_{\upsilon_{\mathtt{m}}}^{\upsilon}\left|(\upsilon-r)^{\mu-1}\right||\mathbb{G}_{\mathtt{m}}(r,\varphi_n(r_{\mathtt{m}}^-))-\mathbb{G}_{\mathtt{m}}(r,\varphi(r_{\mathtt{m}}^-))|dr.\end{aligned}$$

For $\upsilon\in\theta_{\mathtt{m}},\ \mathtt{m}=0,\ldots,n$, we obtain

$$\begin{aligned}|\mathcal{F}\varphi_n(\upsilon)-\mathcal{F}\varphi(\upsilon)| &= \Bigg|\frac{1}{\Gamma(\mu)}\int_{\upsilon_{\mathtt{m}}}^{\delta_{\mathtt{m}}}(\delta_{\mathtt{m}}-r)^{\mu-1}\mathbb{G}_{\mathtt{m}}(r,\varphi_n(r_{\mathtt{m}}^-))dr+\frac{1}{\Gamma(\mu)}\int_{\upsilon_{\mathtt{m}}}^{\upsilon}(\upsilon-r)^{\mu-1}\mathfrak{U}(r,\varphi_n(r))\\
&\quad+\int_0^s\Psi(r,\mathfrak{e},\varphi_n(\mathfrak{e}))d\mathfrak{e})dr-\Bigg(\frac{1}{\Gamma(\mu)}\int_{\upsilon_{\mathtt{m}}}^{\delta_{\mathtt{m}}}(\delta_{\mathtt{m}}-r)^{\mu-1}\mathbb{G}_{\mathtt{m}}(r,\varphi(r_{\mathtt{m}}^-))dr\\
&\quad+\frac{1}{\Gamma(\mu)}\int_{\upsilon_{\mathtt{m}}}^{\upsilon}(\upsilon-r)^{\mu-1}\mathfrak{U}(r,\varphi(r))+\int_0^s\Psi(r,\mathfrak{e},\varphi(\mathfrak{e}))d\mathfrak{e})dr\Bigg)\Bigg|,\\
&\le \frac{1}{\Gamma(\mu)}\int_{\upsilon_{\mathtt{m}}}^{\delta_{\mathtt{m}}}\left|(\delta_{\mathtt{m}}-r)^{\mu-1}\right||\mathbb{G}_{\mathtt{m}}(r,\varphi_n(r_{\mathtt{m}}^-)))-\mathbb{G}_{\mathtt{m}}(r,\varphi(r_{\mathtt{m}}^-))|dr\\
&\quad+\frac{1}{\Gamma(\mu)}\int_{\upsilon_{\mathtt{m}}}^{\upsilon}\left|(\upsilon-r)^{\mu-1}\right|\Big(|\mathfrak{U}(r,\varphi_n(r))-\mathfrak{U}(r,\varphi(r))|\\
&\quad+\int_0^s|\Psi(r,\mathfrak{e},\varphi_n(\mathfrak{e}))-\Psi(r,\mathfrak{e},\varphi(\mathfrak{e}))|d\mathfrak{e}\Big)dr.\end{aligned}$$

As $\mathfrak{U}$, Ψ, and $\mathbb{G}_{\mathtt{m}}$ are continuous, we apply the Lebesgue Dominated Convergence Theorem whene $n\to\infty$, yielding

$$\|\mathcal{F}\varphi_n-\mathcal{F}\varphi\|_{\mathcal{PC}}\to 0.$$

Thus, $\mathcal{F}$ is a continuous operator.

Step 2. The operator $\mathcal{F}$ transforms bounded sets into bounded sets in $\mathcal{PC}(\theta,\mathbb{R})$.

In consideration of every $\upsilon\in\theta_0$, we can obtain

$$\begin{aligned}|\mathcal{F}\varphi(\upsilon)| &= \Bigg|C-\frac{\mathfrak{b}}{\Gamma(\mu)}\Bigg[\int_{\upsilon_{\mathtt{m}}}^{\delta_{\mathtt{m}}}(\delta_{\mathtt{m}}-r)^{\mu-1}\mathbb{G}_{\mathtt{m}}(r,\varphi(r_{\mathtt{m}}^-))dr+\int_{\upsilon_{\mathtt{m}}}^{\mathcal{T}}(\mathcal{T}-r)^{\mu-1}(\mathfrak{U}(r,\varphi(r))\\
&\quad+\int_0^s\Psi(r,\mathfrak{e},\varphi(\mathfrak{e}))d\mathfrak{e})dr\Bigg]+\frac{1}{\Gamma(\mu)}\int_0^{\upsilon}(\upsilon-r)^{\mu-1}(\mathfrak{U}(r,\varphi(r))+\int_0^s\Psi(r,\mathfrak{e},\varphi(\mathfrak{e}))d\mathfrak{e})dr\Bigg|,\\
&\le C+\mathfrak{b}\Bigg[\frac{1}{\Gamma(\mu)}\int_{\upsilon_{\mathtt{m}}}^{\delta_{\mathtt{m}}}(\delta_{\mathtt{m}}-r)^{\mu-1}|\mathbb{G}_{\mathtt{m}}(r,\varphi(r_{\mathtt{m}}^-))|dr+\frac{1}{\Gamma(\mu)}\int_{\upsilon_{\mathtt{m}}}^{\mathcal{T}}(\mathcal{T}-r)^{\mu-1}\Big(|\mathfrak{U}(r,\varphi(r))|\end{aligned}$$

$$
\begin{aligned}
&+\int_0^s |\Psi(r,\mathfrak{e},\varphi(\mathfrak{e}))|d\mathfrak{e}\Bigg)dr\Bigg] + \frac{1}{\Gamma(\mu)}\int_0^{\upsilon}(\upsilon-r)^{\mu-1}\Bigg(|\mathfrak{U}(r,\varphi(r))| + \int_0^s |\Psi(r,\mathfrak{e},\varphi(\mathfrak{e}))|d\mathfrak{e}\Bigg)dr,\\
&\le C + \mathfrak{b}\Bigg[\frac{L_g T^{\mu}}{\Gamma(\mu+1)} + \frac{\mathcal{T}^{\mu}}{\Gamma(\mu+1)}\Bigg(p^*\Omega_1(\mathcal{N}) + Tq^*\Omega_2(\mathcal{N})\Bigg)\Bigg]\\
&+ \frac{\mathcal{T}^{\mu}}{\Gamma(\mu+1)}\Bigg(p^*\Omega_1(\mathcal{N}) + Tq^*\Omega_2(\mathcal{N})\Bigg),\\
&\le C + \frac{\mathcal{T}^{\mu}\Bigg(\mathfrak{b}L_g + (\mathfrak{b}+1)(p^*\Omega_1(\mathcal{N}) + Tq^*\Omega_2(\mathcal{N}))\Bigg)}{\Gamma(\mu+1)} = \eta_1.
\end{aligned}
$$

For $\upsilon \in J_{\mathtt{m}}$, $\mathtt{m} = 1,\dots,n$, we obtain

$$
\begin{aligned}
|\mathcal{F}\varphi(\upsilon)| &= \left|\frac{1}{\Gamma(\mu)}\int_{\upsilon_{\mathtt{m}}}^{\upsilon}(\upsilon-r)^{\mu-1}\mathbb{G}_{\mathtt{m}}(r,\varphi(r_{\mathtt{m}}^-))dr\right|,\\
&\le \frac{1}{\Gamma(\mu)}\int_{\upsilon_{\mathtt{m}}}^{\upsilon}(\upsilon-r)^{\mu-1}|\mathbb{G}_{\mathtt{m}}(r,\varphi(r_{\mathtt{m}}^-))|dr,\\
&\le \frac{L_g}{\Gamma(\mu)}\int_{\upsilon_{\mathtt{m}}}^{\upsilon}(\upsilon-r)^{\mu-1}dr,\\
&\le \frac{L_g T^{\mu}}{\Gamma(\mu+1)} = \eta_2.
\end{aligned}
$$

For $\upsilon \in \theta_{\mathtt{m}}$, $\mathtt{m} = 0,\dots,n$, we have

$$
\begin{aligned}
|\mathcal{F}\varphi(\upsilon)| &= \left|\frac{1}{\Gamma(\mu)}\int_{\upsilon_{\mathtt{m}}}^{\delta_{\mathtt{m}}}(\delta_{\mathtt{m}}-r)^{\mu-1}\mathbb{G}_{\mathtt{m}}(r,\varphi(r_{\mathtt{m}}^-))dr + \frac{1}{\Gamma(\mu)}\int_{\upsilon_{\mathtt{m}}}^{\upsilon}(\upsilon-r)^{\mu-1}\mathfrak{U}(r,\varphi(r)) + \int_0^s \Psi(r,\mathfrak{e},\varphi(\mathfrak{e}))d\mathfrak{e}dr\right|,\\
&\le \frac{1}{\Gamma(\mu)}\int_{\upsilon_{\mathtt{m}}}^{\delta_{\mathtt{m}}}(\delta_{\mathtt{m}}-r)^{\mu-1}|\mathbb{G}_{\mathtt{m}}(r,\varphi(r_{\mathtt{m}}^-))|dr + \frac{1}{\Gamma(\mu)}\int_{\upsilon_{\mathtt{m}}}^{\upsilon}(\upsilon-r)^{\mu-1}|\mathfrak{U}(r,\varphi(r))|\\
&+ \int_0^s |\Psi(r,\mathfrak{e},\varphi(\mathfrak{e}))|d\mathfrak{e})dr,\\
&\le \frac{1}{\Gamma(\mu)}\int_{\upsilon_{\mathtt{m}}}^{\delta_{\mathtt{m}}}(\delta_{\mathtt{m}}-r)^{\mu-1}L_g dr + \frac{1}{\Gamma(\mu)}\int_{\upsilon_{\mathtt{m}}}^{\upsilon}(\upsilon-r)^{\mu-1}\Bigg(p^*\Omega_1(\mathcal{N}) + \int_0^s q^*\Omega_2(\mathcal{N})d\mathfrak{e}\Bigg)dr,\\
&\le \frac{(L_g + (p^*\Omega_1(\mathcal{N}) + Tq^*\Omega_2(\mathcal{N})))T^{\mu}}{\Gamma(\mu+1)} = \eta_3.
\end{aligned}
$$

Therefore, for each $\upsilon \in \theta$,

$$\|\mathcal{F}\|_{\mathcal{PC}} \le \eta = \max\{\eta_1,\eta_3\},$$

hence $\mathcal{F}$ is bounded.

Step 3. $\mathcal{F}$ maps bounded sets into equicontinuous sets of $\mathcal{PC}(\theta,\mathbb{R})$.

Let $\upsilon \in \mathcal{Q}$ such that $\mathcal{Q}$ be bounded set defined by (3.9), and let $0 < \tau_1 < \tau_2 < T$.

For $\tau_1, \tau_2 \in \mathcal{I}_0$, we get

$$
|\mathcal{F}\varphi(\tau_1) - \mathcal{F}\varphi(\tau_2)| = \Bigg|C - \frac{\mathfrak{b}}{\Gamma(\mu)}\Bigg[\int_{\upsilon_{\mathtt{m}}}^{\delta_{\mathtt{m}}}(\delta_{\mathtt{m}}-r)^{\mu-1}\mathbb{G}_{\mathtt{m}}(r,\varphi(r_{\mathtt{m}}^-))dr + \int_{\upsilon_{\mathtt{m}}}^{\mathcal{T}}(\mathcal{T}-r)^{\mu-1}(\mathfrak{U}(r,\varphi(r))
$$

$$
\begin{aligned}
&+\int_0^s \Psi(r,\mathfrak{e},\varphi(\mathfrak{e}))d\mathfrak{e}dr\Bigg]+\frac{1}{\Gamma(\mu)}\int_0^{\tau_1}(\tau_1-r)^{\mu-1}(\mathfrak{U}(r,\varphi(r))+\int_0^s \Psi(r,\mathfrak{e},\varphi(\mathfrak{e}))d\mathfrak{e})dr\\
&-\Bigg(C-\frac{\mathfrak{b}}{\Gamma(\mu)}\Bigg[\int_{\upsilon_{\mathtt{m}}}^{\delta_{\mathtt{m}}}(\delta_{\mathtt{m}}-r)^{\mu-1}\mathbb{G}_{\mathtt{m}}(r,\varphi(r_{\mathtt{m}}^-))dr+\int_{\upsilon_{\mathtt{m}}}^{\mathcal{T}}(\mathcal{T}-r)^{\mu-1}(\mathfrak{U}(r,\varphi(r))\\
&+\int_0^s \Psi(r,\mathfrak{e},\varphi(\mathfrak{e}))d\mathfrak{e}dr\Bigg]+\frac{1}{\Gamma(\mu)}\int_0^{\tau_2}(\tau_2-r)^{\mu-1}(\mathfrak{U}(r,\varphi(r))+\int_0^s \Psi(r,\mathfrak{e},\varphi(\mathfrak{e}))d\mathfrak{e})dr\Bigg)\Bigg|,\\
&\leq\frac{1}{\Gamma(\mu)}\Bigg[\int_0^{\tau_1}\Big|(\tau_1-r)^{\mu-1}\Big|\Big(|\mathfrak{U}(r,\varphi(r))|+\int_0^s|\Psi(r,\mathfrak{e},\varphi(\mathfrak{e}))|d\mathfrak{e}\Big)dr\\
&+\int_0^{\tau_2}\Big|(\tau_2-r)^{\mu-1}\Big|\Big(|\mathfrak{U}(r,\varphi(r))|+\int_0^s|\Psi(r,\mathfrak{e},\varphi(\mathfrak{e}))|d\mathfrak{e}\Big)dr\Bigg],\\
&\leq\frac{1}{\Gamma(\mu)}\Bigg[\int_0^{\tau_1}\Big|(\tau_1-r)^{\mu-1}\Big|\Bigg(|p(r)||\varphi(r)|+\int_0^s|q(r)||\varphi(\mathfrak{e})|d\mathfrak{e}\Bigg)dr\Bigg)\\
&+\int_0^{\tau_2}\Big|(\tau_2-r)^{\mu-1}\Big|\Bigg(|p(r)||\varphi(r)|+\int_0^s|q(r)||\varphi(\mathfrak{e})|d\mathfrak{e}\Bigg)dr\Bigg],\\
&\leq\frac{(p^*+q^*T)\mathcal{N}}{\Gamma(\mu)}\Bigg[\int_0^{\tau_1}(\tau_1-r)^{\mu-1}-(\tau_2-r)^{\mu-1}dr+\int_{\tau_1}^{\tau_2}(\tau_2-r)^{\mu-1}dr\Bigg].
\end{aligned}
\tag{3.14}
$$

For $\tau_1,\tau_2\in J_{\mathtt{m}},\ \mathtt{m}=1,\dots,n$, we have

$$
\begin{aligned}
|\mathcal{F}\varphi(\tau_1)-\mathcal{F}\varphi(\tau_2)|&=\Bigg|\frac{1}{\Gamma(\mu)}\int_{\upsilon_{\mathtt{m}}}^{\tau_1}(\tau_1-r)^{\mu-1}\mathbb{G}_{\mathtt{m}}(r,\varphi(r_{\mathtt{m}}^-))dr-\Bigg(\frac{1}{\Gamma(\mu)}\int_{\upsilon_{\mathtt{m}}}^{\tau_2}(\tau_2-r)^{\mu-1}\mathbb{G}_{\mathtt{m}}(r,\varphi(r_{\mathtt{m}}^-))dr\Bigg)\Bigg|,\\
&\leq\frac{1}{\Gamma(\mu)}\Bigg[\int_{\upsilon_{\mathtt{m}}}^{\tau_1}\Big|(\tau_1-r)^{\mu-1}\Big||\mathbb{G}_{\mathtt{m}}(r,\varphi(r_{\mathtt{m}}^-))|dr+\int_{\upsilon_{\mathtt{m}}}^{\tau_2}\Big|(\tau_2-r)^{\mu-1}\Big||\mathbb{G}_{\mathtt{m}}(r,\varphi(r_{\mathtt{m}}^-))|dr\Bigg],\\
&\leq\frac{L_g}{\Gamma(\mu)}\Bigg[\int_{\upsilon_{\mathtt{m}}}^{\tau_1}(\tau_1-r)^{\mu-1}-(\tau_2-r)^{\mu-1}dr+\int_{\tau_1}^{\tau_2}(\tau_2-r)^{\mu-1}dr\Bigg].
\end{aligned}
\tag{3.15}
$$

For $\tau_1,\tau_2\in\mathcal{I}_{\mathtt{m}},\ \mathtt{m}=0,\dots,n$, we obtain

$$
\begin{aligned}
|\mathcal{F}\varphi(\tau_1)-\mathcal{F}\varphi(\tau_2)|&=\Bigg|\frac{1}{\Gamma(\mu)}\int_{\upsilon_{\mathtt{m}}}^{\delta_{\mathtt{m}}}(\delta_{\mathtt{m}}-r)^{\mu-1}\mathbb{G}_{\mathtt{m}}(r,\varphi(r_{\mathtt{m}}^-))dr+\frac{1}{\Gamma(\mu)}\int_{\upsilon_{\mathtt{m}}}^{\tau_1}(\tau_1-r)^{\mu-1}\mathfrak{U}(r,\varphi(r))\\
&+\int_0^s \Psi(r,\mathfrak{e},\varphi(\mathfrak{e}))d\mathfrak{e}dr-\Bigg(\frac{1}{\Gamma(\mu)}\int_{\upsilon_{\mathtt{m}}}^{\delta_{\mathtt{m}}}(\delta_{\mathtt{m}}-r)^{\mu-1}\mathbb{G}_{\mathtt{m}}(r,\varphi(r_{\mathtt{m}}^-))dr\\
&+\frac{1}{\Gamma(\mu)}\int_{\upsilon_{\mathtt{m}}}^{\tau_2}(\tau_2-r)^{\mu-1}\mathfrak{U}(r,\varphi(r))+\int_0^s \Psi(r,\mathfrak{e},\varphi(\mathfrak{e}))d\mathfrak{e}dr\Bigg)\Bigg|,\\
&\leq\frac{1}{\Gamma(\mu)}\Bigg[\int_{\upsilon_{\mathtt{m}}}^{\tau_1}\Big|(\tau_1-r)^{\mu-1}\Big|\Big(|\mathfrak{U}(r,\varphi(r))|+\int_0^s|\Psi(r,\mathfrak{e},\varphi(\mathfrak{e}))|d\mathfrak{e}\Big)dr\\
&+\int_{\upsilon_{\mathtt{m}}}^{\tau_2}\Big|(\tau_2-r)^{\mu-1}\Big|\Big(|\mathfrak{U}(r,\varphi(r))|+\int_0^s|\Psi(r,\mathfrak{e},\varphi(\mathfrak{e}))|d\mathfrak{e}\Big)dr\Bigg],\\
&\leq\frac{1}{\Gamma(\mu)}\Bigg[\int_{\upsilon_{\mathtt{m}}}^{\tau_1}\Big|(\tau_1-r)^{\mu-1}\Big|\Bigg(|p(r)||\varphi(r)|+\int_0^s|q(r)||\varphi(\mathfrak{e})|d\mathfrak{e}\Bigg)dr
\end{aligned}
$$

$$+\int_{\upsilon_{\mathtt{m}}}^{\tau_2}\left|(\tau_2-r)^{\mu-1}\right|\left(|p(r)||\varphi(r)|+\int_0^s|q(r)||\varphi(\mathfrak{e})|d\mathfrak{e}\right)dr\Bigg],$$
$$\leq\frac{(p^*+q^*T)\mathcal{N}}{\Gamma(\mu)}\left[\int_{\upsilon_{\mathtt{m}}}^{\tau_1}(\tau_1-r)^{\mu-1}-(\tau_2-r)^{\mu-1}dr+\int_{\tau_1}^{\tau_2}(\tau_2-r)^{\mu-1}dr\right]. \tag{3.16}$$

The right side of the inequality (3.14),(3.15) and (3.16) tends to zero as $\tau_1 \to \tau_2$. Therefore, $\mathcal{F}(\mathcal{Q})$ is equicontinuous.

Step 4. $\mathcal{F}$ is a Υ-contraction.

From the above steps, we have $\mathcal{F}:\mathcal{Q}\to\mathcal{Q}$ is continuous, bounded and $\mathrm{N}(\mathrm{B_R})$ is equicontinuous. Next, we aim to establish that the operator $\mathcal{F}$ is a Υ-contraction.

Suppose $G\subset\mathcal{Q}$ and $\upsilon\in\theta_0$. In this case,

$$\begin{aligned}
\eta(\mathcal{F}(G)(\upsilon) &= \eta((\mathcal{F}\varphi)(\upsilon),\varphi\in G),\\
&= \eta\Bigg(C-\frac{\mathfrak{b}}{\Gamma(\mu)}\Bigg[\int_{\upsilon_{\mathtt{m}}}^{\delta_{\mathtt{m}}}(\delta_{\mathtt{m}}-r)^{\mu-1}\mathbb{G}_{\mathtt{m}}(r,\varphi(r_{\mathtt{m}}^-))dr+\int_{\upsilon_{\mathtt{m}}}^{\mathcal{T}}(\mathcal{T}-r)^{\mu-1}(\mathfrak{U}(r,\varphi(r))\\
&+\int_0^s\Psi(r,\mathfrak{e},\varphi(\mathfrak{e}))d\mathfrak{e})dr\Bigg]+\frac{1}{\Gamma(\mu)}\int_0^{\upsilon}(\upsilon-r)^{\mu-1}(\mathfrak{U}(r,\varphi(r))+\int_0^s\Psi(r,\mathfrak{e},\varphi(\mathfrak{e}))d\mathfrak{e})dr,\varphi\in G\Bigg),\\
&\leq\frac{\mathfrak{b}}{\Gamma(\mu)}\eta\left(\int_{\upsilon_{\mathtt{m}}}^{\delta_{\mathtt{m}}}(\delta_{\mathtt{m}}-r)^{\mu-1}\mathbb{G}_{\mathtt{m}}(r,\varphi(r_{\mathtt{m}}^-))dr,\varphi\in G\right)+\frac{\mathfrak{b}}{\Gamma(\mu)}\eta\Bigg(\int_{\upsilon_{\mathtt{m}}}^{\mathcal{T}}(\mathcal{T}-r)^{\mu-1}\Bigg(\mathfrak{U}(r,\varphi(r))\\
&+\int_0^s\Psi(r,\mathfrak{e},\varphi(\mathfrak{e}))d\mathfrak{e}\Bigg)dr,\varphi\in G\Bigg)+\frac{1}{\Gamma(\mu)}\eta\Bigg(\int_0^{\upsilon}(\upsilon-r)^{\mu-1}\Bigg(\mathfrak{U}(r,\varphi(r))\\
&+\int_0^s\Psi(r,\mathfrak{e},\varphi(\mathfrak{e}))d\mathfrak{e}\Bigg)dr,\varphi\in G\Bigg),\\
&\leq\frac{\mathfrak{b}}{\Gamma(\mu)}\int_{\upsilon_{\mathtt{m}}}^{\delta_{\mathtt{m}}}(\delta_{\mathtt{m}}-r)^{\mu-1}\eta(\mathbb{G}_{\mathtt{m}}(r,G))dr+\frac{\mathfrak{b}}{\Gamma(\mu)}\int_{\upsilon_{\mathtt{m}}}^{\mathcal{T}}(\mathcal{T}-r)^{\mu-1}\Bigg(\eta(\mathfrak{U}(r,G))\\
&+\int_0^s\eta(\Psi(r,\mathfrak{e},G))d\mathfrak{e}\Bigg)dr+\frac{1}{\Gamma(\mu)}\int_0^{\upsilon}(\upsilon-r)^{\mu-1}\left(\eta(\mathfrak{U}(r,G))+\int_0^s\eta(\Psi(r,\mathfrak{e},G))d\mathfrak{e}\right)dr,\\
&\leq\frac{\mathcal{T}^{\mu}(\mathfrak{b}L_g+(\mathfrak{b}+1)(p^*+Tq^*))}{\Gamma(\mu+1)}\eta(\mathcal{D}),\\
&\leq\Upsilon_1\eta(\mathcal{D}).
\end{aligned}$$

For each $\upsilon\in J_{\mathtt{m}}$, $\mathtt{m}=1,\ldots,n$, we obtain

$$\begin{aligned}
\eta(\mathcal{F}(G)(\upsilon) &= \eta((\mathcal{F}\varphi)(\upsilon),\varphi\in G),\\
&= \eta\left(\frac{1}{\Gamma(\mu)}\int_{\upsilon_{\mathtt{m}}}^{\upsilon}(\upsilon-r)^{\mu-1}\mathbb{G}_{\mathtt{m}}(r,\varphi(r_{\mathtt{m}}^-))dr,\varphi\in G\right),\\
&\leq\frac{1}{\Gamma(\mu)}\eta\left(\int_{\upsilon_{\mathtt{m}}}^{\upsilon}(\upsilon-r)^{\mu-1}\mathbb{G}_{\mathtt{m}}(r,\varphi(r_{\mathtt{m}}^-))dr,\varphi\in G\right),
\end{aligned}$$

$$\leq \frac{1}{\Gamma(\mu)} \int_{\upsilon_{\mathtt{m}}}^{\upsilon} (\upsilon - r)^{\mu-1} \eta(\mathbb{G}_{\mathtt{m}}(r, G)) dr,$$
$$\leq \frac{\mathcal{T}^{\mu} L_g}{\Gamma(\mu+1)} \eta(\mathcal{D}),$$
$$\leq \Upsilon_2 \eta(\mathcal{D}).$$

In consideration of every $\upsilon \in \theta_{\mathtt{m}}$, $\mathtt{m} = 0, \ldots, n$, we get

$$\begin{aligned}
\eta(\mathcal{F}(G)(\upsilon) &= \eta((\mathcal{F}\varphi)(\upsilon), \varphi \in G), \\
&= \eta\bigg(\frac{1}{\Gamma(\mu)} \int_{\upsilon_{\mathtt{m}}}^{\delta_{\mathtt{m}}} (\delta_{\mathtt{m}} - r)^{\mu-1} \mathbb{G}_{\mathtt{m}}(r, \varphi(r_{\mathtt{m}}^{-})) dr + \frac{1}{\Gamma(\mu)} \int_{\upsilon_{\mathtt{m}}}^{\upsilon} (\upsilon - r)^{\mu-1} \mathfrak{U}(r, \varphi(r)) \\
&+ \int_0^s \Psi(r, \mathfrak{e}, \varphi(\mathfrak{e})) d\mathfrak{e}) dr, \varphi \in G\bigg), \\
&\leq \frac{1}{\Gamma(\mu)} \eta\bigg(\int_{\upsilon_{\mathtt{m}}}^{\delta_{\mathtt{m}}} (\delta_{\mathtt{m}} - r)^{\mu-1} \mathbb{G}_{\mathtt{m}}(r, \varphi(r_{\mathtt{m}}^{-})) dr, \varphi \in G\bigg) + \frac{1}{\Gamma(\mu)} \eta\bigg(\int_{\upsilon_{\mathtt{m}}}^{\upsilon} (\upsilon - r)^{\mu-1} \bigg(\mathfrak{U}(r, \varphi(r)) \\
&+ \int_0^s \Psi(r, \mathfrak{e}, \varphi(\mathfrak{e})) d\mathfrak{e} \bigg) dr, \varphi \in G\bigg), \\
&\leq \frac{1}{\Gamma(\mu)} \int_{\upsilon_{\mathtt{m}}}^{\delta_{\mathtt{m}}} (\delta_{\mathtt{m}} - r)^{\mu-1} \eta(\mathbb{G}_{\mathtt{m}}(r, G)) dr + \frac{1}{\Gamma(\mu)} \int_{\upsilon_{\mathtt{m}}}^{\mathcal{T}} (\mathcal{T} - r)^{\mu-1} \bigg(\eta(\mathfrak{U}(r, G)) \\
&+ \int_0^s \eta(\Psi(r, \mathfrak{e}, G)) d\mathfrak{e} \bigg) dr, \\
&\leq \frac{\mathcal{T}^{\mu}(L_g + (p^* + Tq^*))}{\Gamma(\mu+1)} \eta(\mathcal{D}), \\
&\leq \Upsilon_3 \eta(\mathcal{D}).
\end{aligned}$$

Therefore, for every $\upsilon \in \theta$, we get

$$\eta(\mathcal{F}(G)(\upsilon) \leq \Upsilon \eta(G), \quad \text{such as} \quad \Upsilon = \max\{\Upsilon_1, \Upsilon_3\}.$$

Based on the condition expressed in (3.13), it can be inferred that the operator $\mathcal{F}$ is a Υ-contraction. By applying Theorem 1.4, we can deduce that $\mathcal{F}$ has a fixed point which is a solution to the problem (3.1). □

3.3 Examples

In this section, we address the FIDEs as presented in(3.1) by providing two numerical examples. These examples are used to demonstrate the two results formulated in the preceding section.

Example 3.1. *Consider the following FIDEs*

$$\begin{cases} {}^{c}\mathfrak{D}^{\mu}\varphi(\upsilon) = \mathfrak{U}(\upsilon,\varphi(\upsilon)) + \int_0^{\upsilon}\Psi(\upsilon,r,\varphi(r))dr, \quad \upsilon \in (\delta_m, \upsilon_m + 1], m = 0,1,2, \\ \varphi(\upsilon) = \dfrac{1}{\Gamma(\mu)}\int_{\upsilon_m}^{\upsilon}(\upsilon - r)^{\mu-1}\mathbb{G}_m(r,\varphi(r_m^-))dr, \quad \upsilon \in (\upsilon_m, \delta_m], m = 1,2, \\ \alpha_1\varphi(0) + \alpha_2\varphi(1) = \eta(0), \end{cases} \tag{3.17}$$

with $\alpha_2 = 1$, $\alpha_1 = 4$ and $\mathcal{T} = 1$.
Set $\theta = [0,1]$, $\theta_0 = [0, \frac{1}{5}]$, $J_1 = (\frac{1}{5}, \frac{2}{5}], \theta_1 = (\frac{2}{5}, \frac{3}{5}], J_2 = (\frac{3}{5}, \frac{4}{5}], \theta_2 = (\frac{4}{5}, 1]$.
Let

$$\mathfrak{U}(\upsilon,\varphi(\upsilon)) = \frac{1}{10e^{\upsilon+3}}\left(\frac{|\varphi(\upsilon)|}{2 + |\varphi(\upsilon)|}\right), \ \upsilon \in (0,1],$$

$$\Psi(\upsilon,r,\varphi(r)) = \frac{\upsilon r^2}{5e^{r^2+2}}\sin(\varphi(r)),$$

and

$$\mathbb{G}_{\mathtt{m}}(\upsilon,\varphi(\upsilon)) = \frac{1 + \upsilon\sin(\varphi(\upsilon))}{60}, \ \mathtt{m} = 1,2.$$

We will apply Theorem 3.1 to demonstrate that assumptions $(\mathrm{H}_1) - (\mathrm{H}_3)$ met. Additionally, we will prove that condition (3.7) holds.
Clearly, the functions $\mathfrak{U}$, $\mathbb{G}_{\mathtt{m}}$ and Ψ are continuous.
In consideration of every $\upsilon \in \theta_{\mathtt{m}}, \mathtt{m} = 0,1,2$, we get

$$|\mathfrak{U}(\upsilon,\varphi(\upsilon))| \leq \frac{1}{10e^{\upsilon+3}}|\varphi(r)|,$$

where $p(\upsilon) = \dfrac{1}{10e^{\upsilon+3}}$ and $p^* = \dfrac{1}{10e^3}$.
And

$$|\Psi(\upsilon,r,\varphi(r))| \leq \frac{r^2}{5e^{r^2+2}}|\varphi(r)|,$$

where $q(r) = \dfrac{r^2}{5e^{r^2+2}}$ and $q^* = \dfrac{1}{5e^3}$.
In consideration of every $\upsilon \in J_{\mathtt{m}}, \mathtt{m} = 1,2$, and $\upsilon \in \mathbb{R}$, we get

$$\begin{aligned} |\mathbb{G}_{\mathtt{m}}(\upsilon,\varphi(\upsilon))| &\leq \frac{1}{60}(1 + |\upsilon||sin(\varphi(\upsilon))|), \\ &\leq \frac{1}{30}, \text{ where } L_g = \frac{1}{30}. \end{aligned}$$

And for $\upsilon, \mathbf{x} \in \mathbb{R}$, we get

$$\begin{aligned} |\mathbb{G}_{\mathtt{m}}(\upsilon,\varphi(\upsilon)) - \mathbb{G}_{\mathtt{m}}(\upsilon,\mathbf{x}(\upsilon))| &\leq \frac{\upsilon|\upsilon - \mathbf{x}|}{60}, \\ &\leq \frac{1}{60}|\upsilon - \mathbf{x}|, \text{ where } K_g = \frac{1}{60}. \end{aligned}$$

Thus, the hypotheses $(\mathrm{H}_1) - (\mathrm{H}_3)$ hold.

Now, we will verify the condition (3.7).

$$\begin{aligned}\gamma &= \max\{\gamma_1, \gamma_2\},\\ &= \max\left\{\frac{\mathfrak{b} K_g T^{\mu}}{\Gamma(\mu+1)}, \frac{K_g T^{\mu}}{\Gamma(\mu+1)}\right\},\\ &= \max\left\{\frac{\frac{1}{4}\times\frac{1}{60}}{\Gamma(\mu+1)}, \frac{\frac{1}{60}}{\Gamma(\mu+1)}\right\},\\ &= \frac{\frac{1}{60}}{\Gamma(\mu+1)},\\ &= \gamma_2 < 1.\end{aligned}$$

μ	γ_1	γ_2	γ
0.1	0.0044	0.0175	γ_2
0.2	0.0045	0.0182	γ_2
0.3	0.0046	0.0186	γ_2
0.4	0.0047	0.0188	γ_2
0.5	0.0047	0.0188	γ_2
0.6	0.0047	0.0187	γ_2
0.7	0.0046	0.0183	γ_2
0.8	0.0045	0.0179	γ_2
0.9	0.0043	0.0173	γ_2
1	0.0042	0.0167	γ_2

Table 3.1: Calculation of γ for different values of μ.

As we note in Figure 3.1 and Table 3.1, for $0 < \mu \leq 1$, the values of γ_2 are greater than γ_1 and both are less than 1 . We conclude that all assumptions of the Theorem 3.1 are fulfilled, implying that problem (3.17) admits at least one solution on $\theta = [0, 1]$.

Remark 3.1. *If we take $\alpha_2 > \alpha_1$ we get $\gamma_1 > \gamma_2$, so $\gamma = \max\{\gamma_1, \gamma_2\} = \gamma_1$.*

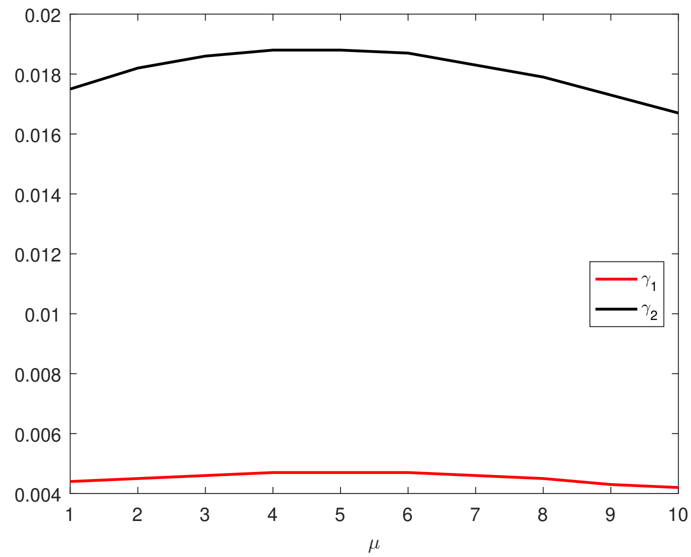

Figure 3.1: Variations of γ_1 and γ_2 for different values of μ.

Example 3.2. *Consider the FIDEs as follows:*

$$\begin{cases} {}^{c}\mathfrak{D}^{\mu}\varphi(\upsilon) = \mathfrak{U}(\upsilon, \varphi(\upsilon)) + \int_0^{\upsilon} \Psi(\upsilon, r, \varphi(r))dr, \quad \upsilon \in (\delta_{m}, \upsilon_{m}+1], m = 0,1,2, \\ \varphi(\upsilon) = \dfrac{1}{\Gamma(\mu)} \int_{\upsilon_m}^{\upsilon} (\upsilon - r)^{\mu-1} \mathbb{G}_{m}(r, \varphi(r_{m}^{-}))dr, \quad \upsilon \in (\upsilon_{m}, \delta_{m}], m = 1,2, \\ \alpha_1\varphi(0) + \alpha_2\varphi(1) = \eta(0), \end{cases} \tag{3.18}$$

with $\alpha_1 = 2.5$, $\alpha_2 = 3$ and $\mathcal{T} = 2$.
Set $\theta = [0,2]$, $\theta_0 = [0, \frac{2}{5}]$, $J_1 = (\frac{2}{5}, \frac{4}{5}]$, $\theta_1 = (\frac{4}{5}, \frac{6}{5}]$, $J_2 = (\frac{6}{5}, \frac{8}{5}]$, $\theta_2 = (\frac{8}{5}, 2]$.
Let

$$\mathfrak{U}(\upsilon, \varphi(\upsilon)) = \frac{\upsilon \sin(\varphi(\upsilon))}{8e^{\upsilon}} \left(\frac{2 + \varphi(\upsilon)}{1 + |\varphi(\upsilon)|} \right), \ \upsilon \in (0, 2],$$

$$\Psi(\upsilon, r, \varphi(r)) = \frac{r \ln(\varphi(r))}{20e^{\upsilon+r+2}},$$

and

$$\mathbb{G}_{\mathtt{m}}(\upsilon, \varphi(\upsilon)) = \frac{\cos(\varphi(\upsilon))}{14e^{\upsilon+1}}, \ \mathtt{m} = 1, 2.$$

Now, we will check that the hypotheses of the Theorem 3.2 are satisfied and also the condition (3.13).
It is evident that the functions $\mathfrak{U}$, $\mathbb{G}_{\mathtt{m}}$ and Ψ are continuous.
For each $\upsilon \in \theta_{\mathtt{m}}, \mathtt{m} = 0, 1, 2$, we have

$$|\mathfrak{U}(\upsilon, \varphi(\upsilon))| \leq \frac{\upsilon}{8e^{\upsilon}}(2 + |\varphi(r)|),$$

where $p(\upsilon) = \dfrac{\upsilon}{8e^{\upsilon}}$ and $p^* = \dfrac{1}{8e}$.
And

$$\begin{aligned}|\Psi(\upsilon, r, \varphi(r))| &\leq \frac{r}{20e^{\upsilon+r+2}}|\varphi(r)|,\\ &\leq \frac{1}{20e^{r+2}}|\varphi(r)|,\end{aligned}$$

where $q(r) = \dfrac{r}{20e^{r+2}}$ and $q^* = \dfrac{1}{20e^{3}}$.
For each $\upsilon \in J_{\mathtt{m}}, \mathtt{m} = 1, 2,,$ we get

$$\begin{aligned}|\mathbb{G}_{\mathtt{m}}(\upsilon, \varphi(\upsilon))| &\leq \frac{|\cos(\varphi(\upsilon))|}{14e^{\upsilon+1}},\\ &\leq \frac{1}{14e^{1.4}}, \text{ where } L_g = \frac{1}{14e^{1.4}}.\end{aligned}$$

Consequently, the hypotheses $(\mathcal{A}_1)$, $(\mathcal{A}_2)$,(H_2) and (H_3) are satisfied.
Next, we will verify the condition (3.13).

$$\begin{aligned}\Upsilon &= \max\{\Upsilon_1, \Upsilon_3\},\\ \Upsilon &= \max\left\{\frac{\mathcal{T}^{\mu}(\mathfrak{b}L_g+(\mathfrak{b}+1)(p^*+Tq^*))}{\Gamma(\mu+1)}, \frac{\mathcal{T}^{\mu}(L_g+(p^*+Tq^*))}{\Gamma(\mu+1)}\right\},\\ &= \max\left\{\frac{2^{\mu}(\frac{3}{2.5}\frac{1}{14e^{1.4}}+(\frac{3}{2.5}+1)(\frac{1}{8e}+\frac{2}{20e^{3}}))}{\Gamma(\mu+1)}, \frac{2^{\mu}(\frac{1}{14e^{1.4}}+(\frac{1}{8e}+\frac{2}{20e^{3}}))}{\Gamma(\mu+1)}\right\},\\ &= \frac{2^{\mu}(\frac{3}{2.5}\frac{1}{14e^{1.4}}+(\frac{3}{2.5}+1)(\frac{1}{8e}+\frac{2}{20e^{3}}))}{\Gamma(\mu+1)},\\ &= \Upsilon_1 < 1.\end{aligned}$$

μ	Υ_1	Υ_3	Υ
0.1	0.1625	0.0829	Υ_1
0.2	0.1804	0.0920	Υ_1
0.3	0.1978	0.1009	Υ_1
0.4	0.2145	0.1094	Υ_1
0.5	0.2301	0.1174	Υ_1
0.6	0.2446	0.1248	Υ_1
0.7	0.2578	0.1315	Υ_1
0.8	0.2696	0.1375	Υ_1
0.9	0.2798	0.1427	Υ_1
1	0.2884	0.1471	Υ_1

Table 3.2: Calculation of Υ for different values of μ.

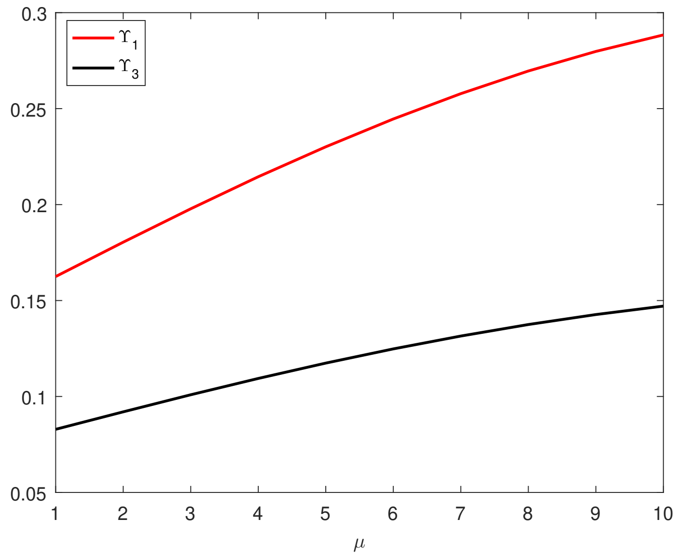

Figure 3.2: Variations of Υ_1 and Υ_3 for different values of μ.

As observed in Table 3.2 and Figure 3.2, for $0 < \mu \leq 1$, the values of Υ_1 surpass those of Υ_3, and both are below 1. Based on this, we can deduce that all the assumptions of Theorem 3.2 are checked. Consequently, the problem (3.18) have at least one solution defined on the interval $\theta = [0, 2]$.

3.4 Conclusion

This chapter explores a category of fractional integro-differential equations (FIDEs) incorporating non-instantaneous impulses (NIIs) under the Caputo fractional derivative. The main focus is on obtaining existence results for this class of equations involving NIIs. To achieve this, Krasnoselskii's and Darbo's fixed point theorems are applied. One of the novel aspects of this work is the presentation of two examples that goes beyond exploring a single specific value of μ. Instead, we consider multiple values of μ within the interval $(0, 1]$. This broader scope allows us to explore the behavior and solutions of the equations for various fractional orders, providing a more comprehensive understanding of the problem and its implications. By considering different μ values, we gain valuable insights into how the dynamics of the system change with respect to the fractional order, leading to a richer and more generalized analysis of the FIDEs with NIIs.

Chapter 4

Numerical Solution of Fredholm Fractional Integro-Differential Equations through Least Squares Approximation and Compact Combination of Shifted Chebyshev Polynomials

4.1 Introduction

This chapter delves into solving linear Fredholm fractional integro-differential equations (FIDEs) numerically under the Caputo FD. The chosen methodology involves applying the least squares method to numerically solve a category of FIDEs employing a compact combination of Shifted Chebyshev polynomials (SCP) of the first kind. The primary objective is to express the unknown function as a series of a linear combination of SCP. Subsequently, the problem is reduced to a system of linear algebraic equations, which are solved for the unknown constants associated with the approximate solution using MATLAB R2020a. The chapter concludes with the presentation of numerical examples to validate the effectiveness and suitability of this approach. Multiple numerical illustrations (Tables and Figures) are provided to demonstrate the method's efficacy comprehensively. Additionally, various comparisons are included to further elucidate the advantages and potential applications of the proposed methodology.

The purpose of our research is solving numerically the linear FIDE described by

$$ {}^{C}\mathfrak{D}^{\mu}\varphi(\upsilon) = f(\upsilon) + \int_0^1 \mathcal{K}(\upsilon,\tau)\varphi(\tau)d\tau, \quad 0 \leq \upsilon \leq 1. \tag{4.1} $$

with two cases of the initial conditions. For the first initial condition where $0 < \mu \leq 1$, we impose $\varphi(0) = 0$. Meanwhile, in the second initial conditions where $1 < \mu \leq 2$, we impose $\varphi(0) = \varphi'(0) = 0$. Functions $f(\upsilon)$ and $\mathcal{K}(\upsilon,\tau)$ are predefined, while $\varphi(\upsilon)$ denotes the unknown function to be determined.

4.2 Spectral approximation by shifted Chebyshev polynomials

4.2.1 Shifted Chebyshev polynomials

Shifted Chebyshev polynomials (SCP) represent a modified version of the Chebyshev polynomials, characterized by a shift in the interval of definition from [-1, 1] to $[0, 1]$. Similar to Chebyshev polynomials, shifted Chebyshev polynomials exhibit key properties such as orthogonality and a recursive definition. Their applications extend to fields like approximation theory, numerical analysis, and the solution of systems involving linear and nonlinear FIDEs. Noted as $\widetilde{\mathbb{T}}_{\mathtt{m}}(\upsilon)$, these polynomials are defined by the following transformation

$$ \widetilde{\mathbb{T}}_{\mathtt{m}}(\upsilon) = \mathbb{T}_{\mathtt{m}}(2\upsilon - 1), \tag{4.2} $$

here, $\mathbb{T}_{\mathtt{m}}(\upsilon)$ represents the Chebyshev polynomials of the first kind. Thus we have the polynomials:

$$ \begin{aligned} \widetilde{\mathbb{T}}_0(\upsilon) &= 1. \\ \widetilde{\mathbb{T}}_1(\upsilon) &= 2\upsilon - 1. \\ \widetilde{\mathbb{T}}_2(\upsilon) &= 8\upsilon^2 - 8\upsilon + 1. \\ \widetilde{\mathbb{T}}_3(\upsilon) &= 32\upsilon^3 - 48\upsilon^2 + 18\upsilon - 1\dots \end{aligned} $$

Using (4.2) and (1.16), we derive the recurrence relation for $\widetilde{\mathbb{T}}_{\mathtt{m}}$ with the subsequent manner:

$$ \widetilde{\mathbb{T}}_{\mathtt{m}+1}(\upsilon) = 2(2\upsilon - 1)\widetilde{\mathbb{T}}_{\mathtt{m}}(\upsilon) - \widetilde{\mathbb{T}}_{\mathtt{m}-1}(\upsilon), \quad \mathtt{m} = 1, 2, \dots, \tag{4.3} $$

where $\widetilde{\mathbb{T}}_0(\upsilon) = 1$, $\widetilde{\mathbb{T}}_1(\upsilon) = 2\upsilon - 1$.

4.2.2 Basic shifted Chebyshev polynomials properties

In this part, we embark on an exploration properties of the SCP of the first kind.

Property 4.1. *The SCP denoted as $\widetilde{\mathbb{T}}_{m}(\upsilon)$ of the first kind, fulfill:*

1.
$$\widetilde{\mathbb{T}}_{m}(0) = (-1)^{m}, \; m \geq 0. \tag{4.4}$$

2.
$$(\widetilde{\mathbb{T}}_{m})'(0) = 2(-1)^{m+1}m^2. \; m \geq 0. \tag{4.5}$$

Proof. 1. Let $P(\mathtt{m})$ be the statement: $\widetilde{\mathbb{T}}_{\mathtt{m}}(0) = (-1)^{\mathtt{m}}, \mathtt{m} \geq 0$. We shall demonstrate using mathematical induction that $P(\mathtt{m})$ holds for all $\mathtt{m} \geq 0$.

- Base case: we prove that $P(0)$ is true. For $\mathtt{m} = 0$, we get

$$\widetilde{\mathbb{T}}_0(0) = (-1)^0 = 1.$$

this result is true.

- Induction hypothesis: Assume that the statement $P(\mathtt{m})$ is true for any positive integer $\mathtt{m} = k$.

- Induction step: We will now show that $P(\mathtt{m}+1)$ is true for any positive integer $\mathtt{m} = k+1$, or $\widetilde{\mathbb{T}}_{k+1}(0) = (-1)^{k+1}$ is true.

From the recurrence relation (1.16), we have

$$\widetilde{\mathbb{T}}_{k+1}(0) = -\widetilde{\mathbb{T}}_{k}(0),$$

substituting now our statement, we obtain

$$\widetilde{\mathbb{T}}_{k+1} = -(-1)^k.$$

Now, we will use the property that $(-1) \times (-1)^k = (-1)^{k+1}$, we get

$$\widetilde{\mathbb{T}}_{k+1} = (-1) \times (-1)^k = (-1)^{k+1}.$$

So $P(k+1)$ is true. Hence by mathematical induction $P(\mathtt{m})$ is true for all $\mathtt{m} \geq 0$.

2. Let $Q(\mathtt{m})$ be the statement: $(\widetilde{\mathbb{T}}_{\mathtt{m}})'(0) = 2(-1)^{\mathtt{m}+1}\mathtt{m}^2.\; \mathtt{m} \geq 0$. We shall demonstrate using mathematical induction that $Q(\mathtt{m})$ holds for all $\mathtt{m} \geq 0$.

- Base case: we prove that $Q(0)$ is true. For $\mathtt{m} = 0$, we get $(\widetilde{\mathbb{T}}_0)'(0) = 2(-1)^{0+1} \times 0^2 = 0$. So $Q(0)$ is true since $(\widetilde{\mathbb{T}}_0)'(\upsilon) = 0$.

- Induction hypothesis: Assume that the statement $Q(\mathtt{m})$ is true for any positive integer $\mathtt{m} = k$.

- Induction step: We will now show that $Q(\mathtt{m}+1)$ is true for any positive integer $\mathtt{m} = k+1$, or $(\widetilde{\mathbb{T}}_{k+1})'(0) = 2(-1)^{k+2}(k+1)^2$ is true.

From the recurrence relation (4.3) and (4.4), we get

$$\begin{aligned}
(\widetilde{\mathbb{T}}_{k+1})'(0) &= 4\widetilde{\mathbb{T}}_k(0) - 2\widetilde{\mathbb{T}}'_k(0) - \widetilde{\mathbb{T}}'_{k-1}(0),\\
&= 4(-1)^k - 4(-1)^{k+1}k^2 - 2(-1)^k(k-1)^2,\\
&= 2(-1)^k[2 + 2k^2 - k^2 - 1 + 2k],\\
&= 2(-1)^k[k^2 + 1 + 2k],\\
&= 2(-1)^{k+2}(k+1)^2.
\end{aligned}$$

So $Q(k+1)$ is true. Hence by mathematical induction $Q(\mathtt{m})$ is true for all $\mathtt{m} \geq 0$.

□

Property 4.2. *SCP of the first kind are orthogonal over the interval* $[0,1]$ *with respect to the weight function* $\omega(\upsilon) = \frac{1}{\sqrt{\upsilon - \upsilon^2}}$*, i.e.,*

$$\int_0^1 \widetilde{\mathbb{T}}_{m_1}(\upsilon)\widetilde{\mathbb{T}}_{m_2}(\upsilon)\omega(\upsilon)\,d\upsilon = \begin{cases} 0. & if \quad m_1 \neq m_2, \\ \frac{\pi}{2}, & if \quad m_1 = m_2 \neq 0, \\ \pi, & if \quad m_1 = m_2 = 0. \end{cases}$$

Proof. To prove the orthogonality of the SCP of the first kind with respect to the weight function $\omega(\upsilon) = \frac{1}{\sqrt{\upsilon(1-\upsilon)}}$ on the interval $[0,1]$, we shall utilize the properties of these polynomials and properties of orthogonality.

First, we rewrite the integral in terms of the Chebyshev polynomials:

$$\int_0^1 \widetilde{\mathbb{T}}_{\mathtt{m}_1}(\upsilon)\widetilde{\mathbb{T}}_{\mathtt{m}_2}(\upsilon)\omega(\upsilon)\,d\upsilon = \int_0^1 \mathbb{T}_{\mathtt{m}_1}(2\upsilon-1)\mathbb{T}_{\mathtt{m}_2}(2\upsilon-1)\frac{1}{\sqrt{\upsilon(1-\upsilon)}}\,d\upsilon.$$

Let $u = 2\upsilon - 1$, then $dx = \frac{1}{2}du$ and when $\upsilon = 0$, $u = -1$ and when $\upsilon = 1$, $u = 1$. The integral becomes

$$\int_{-1}^1 \mathbb{T}_{\mathtt{m}_1}(u)\mathbb{T}_{\mathtt{m}_2}(u)\frac{1}{\sqrt{\frac{u+1}{2}\cdot\frac{1-u}{2}}}\cdot\frac{1}{2}\,du.$$

Simplify and use the orthogonality relation of the Chebyshev polynomials (1.18), we get

$$\int_{-1}^1 \mathbb{T}_{\mathtt{m}_1}(u)\mathbb{T}_{\mathtt{m}_2}(u)\frac{1}{\sqrt{1-u^2}}\,du = k_{\mathtt{m}_1}\delta_{\mathtt{m}_1\mathtt{m}_2}.$$

This means that if $\mathtt{m}_1 \neq \mathtt{m}_2$, the integral is zero. If $\mathtt{m}_1 = \mathtt{m}_2 \neq 0$, the integral is $\frac{\pi}{2}$, and if $\mathtt{m}_1 = \mathtt{m}_2 = 0$, the integral is π.
Therefore, using the relationship between $\widetilde{\mathbb{T}}_{\mathtt{m}}(x)$ and $\mathbb{T}_{\mathtt{m}}(x)$, we have shown that the SCP of the first kind are indeed orthogonal with respect to the given weight function on the interval $[0, 1]$. $\square$

4.2.3 Spectral approximation by shifted Chebyshev polynomials

Spectral approximation by orthogonal polynomials is a numerical method utilized to approximate functions or solutions to fractional differential equations by expressing them as linear combinations of these orthogonal polynomials. In our case, to determine the approximated solution for problem (4.1), we will utilize a compact combination of shifted Chebyshev Polynomials (SCP) of the first kind. This involves substituting the unknown function $\varphi(\upsilon)$ with its corresponding approximation, as given by the chosen compact combination of SCP.

$$\varphi_N(\upsilon) = \sum_{j=0}^{N-n} \beta_j \, \Phi_j(\upsilon), \quad 0 \leq \upsilon \leq 1, \tag{4.6}$$

In this context, $n \in \{1, 2\}$ denotes the number of initial conditions, β_j $(j = 0, \ldots, N-n)$ represents the the unknown coefficients to be determined, $\Phi_j(\upsilon)$ signifies a set of compact shifted Chebyshev basis functions that we will chose later. In light of this, we set

$$S_N = \operatorname{span}\{\widetilde{\mathbb{T}}_0(\upsilon), \widetilde{\mathbb{T}}_1(\upsilon), \ldots, \widetilde{\mathbb{T}}_{\mathrm{N}}(\upsilon)\},$$

here, $\{\widetilde{\mathbb{T}}_{\mathtt{m}}(\upsilon), \mathtt{m} = 0, \ldots, N\}$ are the shifted Chebyshev polynomials.

Define $V_N \subset S_N$ as the subspace where the initial conditions of problem (4.1) are hold

$$V_N = \{w \in S_N : w^{(\iota)}(0) = 0. \quad \iota = 0, 1, \ldots, n-1\}.$$

In this instance, opting for compact combinations of orthogonal polynomials, particularly compact combinations of SCP, not only enables us to exploit the properties inherent in these orthogonal polynomials but also facilitates a reduction in the the approximation space dimension from $(N+1)$ to $(N-n+1)$. Therefore, we will opt the compact shifted Chebyshev basis functions provided below

$$\Phi_{\mathtt{m}}(\upsilon) = \widetilde{\mathbb{T}}_{\mathtt{m}}(\upsilon) + \sum_{i=1}^{n} c_i^{\mathtt{m}} \, \widetilde{\mathbb{T}}_{\mathtt{m}+i}(\upsilon), \quad \mathtt{m} = 0, 1, 2, \ldots,$$

where the constants $c_i^{\mathtt{m}}$ will be selected to ensure that $\Phi_{\mathtt{m}}$ satisfies the initial conditions of the respective.

The set $\{\Phi_{\mathtt{m}}\}$ is linearly independent since it constitutes a linear combination of orthogonal SCP. Due to the dimension argument, we consequently have

$$V_N = \text{span}\{\Phi_{\mathtt{m}}(\upsilon) : \mathtt{m} = 0, 1, 2, \ldots, \mathrm{N} - \mathrm{n}\}.$$

In the lemma that follows, we determine $c_i^{\mathtt{m}}$ values where $\varphi(0) = 0$ and then where $\varphi(0) = \varphi'(0) = 0$.

Lemma 4.1. *For $n \in \{1.2\}$ and $\mathtt{m} \geq 0$,*

1. *When $n = 1$, with the condition $\varphi(0) = 0$, we consider the following:*

$$\Phi_{\mathtt{m}}(\upsilon) = \widetilde{\mathbb{T}}_{\mathtt{m}}(\upsilon) + c_1^{\mathtt{m}}\, \widetilde{\mathbb{T}}_{\mathtt{m}+1}(\upsilon),$$

to ensure $\Phi_{\mathtt{m}}(0) = 0$, it suffices to set $c_1^{\mathtt{m}} = 1$.

2. *When $n = 2$, with initial conditions $\varphi(0) = \varphi'(0) = 0$, we proceed as follows:*

$$\Phi_{\mathtt{m}}(\upsilon) = \widetilde{\mathbb{T}}_{\mathtt{m}}(\upsilon) + c_1^{\mathtt{m}}\, \widetilde{\mathbb{T}}_{\mathtt{m}+1}(\upsilon) + c_2^{\mathtt{m}}\, \widetilde{\mathbb{T}}_{\mathtt{m}+2}(\upsilon),$$

the conditions $\Phi_{\mathtt{m}}(0) = \Phi'_{\mathtt{m}}(0) = 0$ entail the following:

$$c_1^{\mathtt{m}} = \frac{4\mathtt{m} + 4}{2\mathtt{m} + 3}, \quad \textit{and} \quad c_2^{\mathtt{m}} = \frac{2\mathtt{m} + 1}{2\mathtt{m} + 3}.$$

Proof. Since $\widetilde{\mathbb{T}}_{\mathtt{m}}(0) = (-1)^{\mathtt{m}}$ and $(\widetilde{\mathbb{T}}_{\mathtt{m}})'(0) = 2(-1)^{\mathtt{m}+1}\mathtt{m}^2$, we have

1. From the boundary conditions $\Phi_{\mathtt{m}}(0) = 0$, then we find that $c_1^{\mathtt{m}}$, must satisfy

$$\Phi_{\mathtt{m}}(0) = \widetilde{\mathbb{T}}_{\mathtt{m}}(0) + c_1^{\mathtt{m}}\, \widetilde{\mathbb{T}}_{\mathtt{m}+1}(0) = 0,$$

which is equivalent to

$$(-1)^{\mathtt{m}} + c_1^{\mathtt{m}}(-1)^{\mathtt{m}+1} = 0,$$

thus, we find

$$c_1^{\mathtt{m}} = 1.$$

So

$$\Phi_{\mathtt{m}}(\upsilon) = \widetilde{\mathbb{T}}_{\mathtt{m}}(\upsilon) + \ \widetilde{\mathbb{T}}_{\mathtt{m}+1}(\upsilon).$$

2. From the boundary conditions $\Phi_{\mathtt{m}}(0) = \Phi'_{\mathtt{m}}(0) = 0$, then we find that $c_1^{\mathtt{m}}, c_2^{\mathtt{m}}$ must satisfy the system

$$\begin{cases} \Phi_{\mathtt{m}}(0) = \widetilde{\mathbb{T}}_{\mathtt{m}}(0) + c_1^{\mathtt{m}}\widetilde{\mathbb{T}}_{\mathtt{m}+1}(0) + c_2^{\mathtt{m}}\widetilde{\mathbb{T}}_{\mathtt{m}+2}(0) = 0, \\ \\ (\Phi_{\mathtt{m}})'(0) = (\widetilde{\mathbb{T}}_{\mathtt{m}})'(0) + c_1^{\mathtt{m}}(\widetilde{\mathbb{T}}_{\mathtt{m}+1})'(0) + c_2^{\mathtt{m}}(\widetilde{\mathbb{T}}_{\mathtt{m}+2})'(0) = 0, \end{cases}$$

which can be rewritten as

$$\begin{cases} (-1)^{\mathtt{m}} + c_1^{\mathtt{m}}(-1)^{\mathtt{m}+2} + c_2^{\mathtt{m}}(-1)^{\mathtt{m}+3} = 0, \\ \\ (2(-1)^{\mathtt{m}+1}\mathtt{m}^2) + c_1^{\mathtt{m}}(2(-1)^{\mathtt{m}+2}(\mathtt{m}+1)^2) + c_2^{\mathtt{m}}(2(-1)^{\mathtt{m}+3}(\mathtt{m}+2)^2) = 0, \end{cases}$$

leading to

$$\begin{cases} c_1^{\mathtt{m}} = 1 + c_2^{\mathtt{m}}, \\ \\ c_1^{\mathtt{m}} = \frac{\mathtt{m}^2}{(\mathtt{m}+1)^2} + \frac{(\mathtt{m}+2)^2}{(\mathtt{m}+1)^2} c_2^{\mathtt{m}}, \end{cases}$$

then

$$\begin{cases} c_1^{\mathtt{m}} = \frac{4\mathtt{m}+4}{2\mathtt{m}+3}, \\ \\ c_2^{\mathtt{m}} = \frac{2\mathtt{m}+1}{2\mathtt{m}+3}. \end{cases}$$

Or

$$\Phi_{\mathtt{m}}(v) = \widetilde{\mathbb{T}}_{\mathtt{m}}(v) + \frac{4\mathtt{m}+4}{2\mathtt{m}+3}\,\widetilde{\mathbb{T}}_{\mathtt{m}+1}(v) + \frac{2\mathtt{m}+1}{2\mathtt{m}+3}\,\widetilde{\mathbb{T}}_{\mathtt{m}+2}(v).$$

□

4.3 Least squares method and compact shifted Chebyshev basis functions

4.3.1 Approximation by the LSM: principle and advantages

The LSM is a mathematical technique used to find the best-fitting curve or line for a set of data points by minimizing the sum of the squares of the offsets of the points from the curve. In the context of function approximation, the method applies the principle of least squares to find the best approximation of a function by means of a weighted sum of other functions. In other words, within the interval $a < \mathrm{t} < \mathrm{b}$, the objective of the LSM is to identify a readily computable function $\varphi_n(\mathrm{t})$ that offers a reasonably precise estimation of a more complex function $\varphi(\mathrm{t})$. This approximation is described by

$$\varphi_n(\mathrm{t}) = \mathrm{c}_0\phi_0 + \mathrm{c}_1\phi_1 + \cdots + \mathrm{c}_\mathrm{n}\phi_\mathrm{n}, \tag{4.7}$$

here, $\phi_0, \ldots, \phi_n$ represent independent elements of a linear vector space with a finite dimension of $N+1$ of functions defined on the point set $\mathrm{t} = \mathrm{t}_0, \mathrm{t}_1, \ldots, \mathrm{t}_\mathrm{N}$.

Typically, we seek an approximating function $\varphi_n(\mathrm{t})$ that includes a set of parameters c_r (where $r = 0, 1, \ldots, n$, and $n < N$). The conditions for achieving a minimum,

$$\frac{\partial \mathcal{N}}{\partial c_r} = 0, \quad r = 0, 1, \ldots, n,$$

provide a system of algebraic equations for determining these parameters. Here, the function $\mathcal{N}$ is defined as

$$\mathcal{N}_n = \int_a^b w(\upsilon)\{\varphi(\upsilon) - \varphi_n(\upsilon)\}^2 d\upsilon,$$

where in $a <$ t $<$ b, the weight function $w(\mathrm{t})$ is positive.

The FIDEs within the range of $0 \leq$ t ≤ 1 could be effectively solved using the LSM, a two-step process. In the initial phase, a linear combination of basis functions, including as orthogonal polynomials, is utilized to approximate the solution. Minimizing the residual error between the solution and the FIDEs via the LSM determines the coefficients of the basis functions. In the subsequent stage, numerical techniques, like iterative methods or matrix inversion, are applied to solve the resulting system of linear equations and get the basis functions coefficients. Subsequently, the solution of the FIDEs is derived by substituting the coefficients into the approximation formula. Preferably, this method is chosen over numerical approaches due to its versatility, minimal computational and memory demands, precision, adaptability and better approximation error. Due to the favorable properties of this method, we intend to employ it in addressing fractional integro-differential equations. Specifically, we will utilize compact combinations of orthogonal polynomials as our chosen basis functions, selected through the spectral method. Selecting an optimal basis is crucial in spectral approaches to ensure a streamlined system. To accomplish this, we will select a condensed compact combination of SCP of the first kind as basis functions. Utilizing these basis functions allows us to effectively reduce the approximation space dimension from $(N + 1)$ to $(N + 1 - n)$, with N denoting the degree of SCP.

4.3.2 The application of the LSM

In this part, we shall introduce the LSM to numerically solving the FIDE (4.1) by using a compact shifted Chebyshev basis function developed in Section 4.2. By substituting (4.6) into (4.1), we derive

$${}^{C}\mathfrak{D}^{\mu}\left(\sum_{j=0}^{N-n} \beta_j \; \Phi_j(\upsilon)\right) \simeq f(\upsilon) + \int_0^1 \mathcal{K}(\upsilon, \tau) \sum_{j=0}^{N-n} \beta_j \; \Phi_j(\tau) \, d\tau.$$

As a result, the following residual equation is obtained

$$\mathcal{R}(\beta_0, \beta_1, \ldots, \beta_{N-n}, \upsilon) = \sum_{j=0}^{N-n} \beta_j \, {}^{C}\mathfrak{D}^{\mu}\Phi_j(\upsilon) - f(\upsilon) - \int_0^1 \mathcal{K}(\upsilon, \tau) \sum_{j=0}^{N-n} \beta_j \, \Phi_j(\tau) \, d\tau. \tag{4.8}$$

Let

$$\mathcal{N}(\beta_0, \beta_1, \ldots, \beta_{N-n}) = \int_0^1 (\mathcal{R}(\beta_0, \beta_1, \ldots, \beta_{N-n}, \upsilon))^2 \, d\upsilon. \tag{4.9}$$

So

$$\mathcal{N}(\beta_0, \beta_1, \ldots, \beta_{N-n}) = \int_0^1 \left(\sum_{j=0}^{N-n} \beta_j \, {}^{C}\mathfrak{D}^{\mu}\Phi_j(\upsilon) - f(\upsilon) - \int_0^1 \mathcal{K}(\upsilon, \tau) \sum_{j=0}^{N-n} \beta_j \Phi_j(\tau) \, d\tau \right)^2 d\upsilon. \tag{4.10}$$

Finding the optimal approximated solution of the FIDE (4.1) by applying LSM is analogous to obtain the values of $\{\beta_j\},\ j = 0, 1, \ldots, N-n$ which realize the minimum of $\mathcal{N}$. To attain this minimum, we set forth

$$\frac{\partial \mathcal{N}}{\partial \beta_i} = 0, \quad i = 0, 1, \ldots, N-n, \tag{4.11}$$

by applying (4.11) to (4.10), we find

$$\begin{aligned} \int_0^1 \Bigg(\sum_{j=0}^{N-n} \beta_j \, {}^{C}\mathfrak{D}^{\mu}\Phi_j(\upsilon) - f(\upsilon) \quad & - \int_0^1 \mathcal{K}(\upsilon, \tau) \sum_{j=0}^{N-n} \beta_j \Phi_j(\tau) d\tau \Bigg) \\ & \times \left({}^{C}\mathfrak{D}^{\mu}\Phi_i(\upsilon) - \int_0^1 \mathcal{K}(\upsilon, \tau)\Phi_i(\tau) d\tau \right) d\upsilon = 0. \end{aligned}$$

or

$$\begin{gathered} \int_0^1 \left(\sum_{j=0}^{N-n} \beta_j \, {}^{C}\mathfrak{D}^{\mu}\Phi_j(\upsilon) - \int_0^1 \mathcal{K}(\upsilon, \tau) \sum_{j=0}^{N-n} \beta_j \Phi_j(\tau) d\tau \right) \times \left({}^{C}\mathfrak{D}^{\mu}\Phi_i(\upsilon) - \int_0^1 \mathcal{K}(\upsilon, \tau)\Phi_i(\tau) d\tau \right) d\upsilon \\ = \\ \int_0^1 f(\upsilon) \left({}^{C}\mathfrak{D}^{\mu}\Phi_i(\upsilon) - \int_0^1 \mathcal{K}(\upsilon, \tau)\Phi_i(\tau) d\tau \right) d\upsilon, \forall i = 0, \ldots, N-n. \end{gathered} \tag{4.12}$$

By solving the above equation for $i = 0, 1, \ldots, N-n$, we create a system of $(N+1-n)$ linear equations with $(N+1-n)$ unknown coefficients β_j. This system is subsequently reformulated in matrix form:

$$A\beta = B,$$

here, A represents a square matrix of order $(N+1-n)$, while β and B are column vectors of order $(N+1-n)$.

For all $0 \leq i, j \leq N-n$, we put

$$\begin{cases} A_{ji} = \int_0^1 \left({}^C\mathfrak{D}^\mu \Phi_j(\upsilon) - \int_0^1 \mathcal{K}(\upsilon,\tau)\Phi_j(\tau)d\tau \right) \left({}^C\mathfrak{D}^\mu \Phi_i(\upsilon) - \int_0^1 \mathcal{K}(\upsilon,\tau)\Phi_i(\tau)d\tau \right) d\upsilon, \\ \\ \beta = (\beta_0, \beta_1, \ldots, \beta_{N-n})^T, \\ \\ B_i = \int_0^1 f(\upsilon) \left({}^C\mathfrak{D}^\mu \Phi_i(\upsilon) - \int_0^1 \mathcal{K}(\upsilon,\tau)\Phi_i(\tau)d\tau \right) d\upsilon. \end{cases}$$

Hence, the approximate solution of (4.1) is obtained by solving the derived system by determining the unknown coefficients β_j.

4.4 Examples

In this section, we demonstrate the aforementioned findings through multiple numerical examples of linear FIDE, implemented using MATLAB R2020a.

Error estimation is provided to highlight precision and effectiveness of our method (OM). We define the following utilized absolute error (AE):

$$AE(\upsilon) = |\varphi(\upsilon) - \varphi_N(\upsilon)|, \; 0 \leq \upsilon \leq 1.$$

Example 4.1. *Consider the following FIDE*

$$\begin{aligned} {}^C\mathfrak{D}^\mu \varphi(\upsilon) \; &= \tfrac{1}{\sqrt{\pi}} \left(\tfrac{8}{3}\upsilon^{3/2} - 2\upsilon^{1/2} \right) + \tfrac{1}{12}\upsilon + \int_0^1 \upsilon\tau \, \varphi(\tau)\, d\tau, \quad 0 \leq \upsilon \leq 1. \\ \varphi(0) \; &= 0. \end{aligned} \tag{4.13}$$

For $\mu = \frac{1}{2}$, the analytical solution is given by $\varphi(\upsilon) = \upsilon^2 - \upsilon$.

By employing OM with $\mu = \frac{1}{2}$ for $N = 3$ and $N = 4$, we derived the AE as presented in Table 4.1 and compared them with the errors for the analogous problem outlined in [37]. It is clear from the comparison that OM consistently provides superior outcomes. Figure 4.1 visually illustrates the convergence of the approximated solution towards the analytical solution for μ values ranging from $\frac{1}{8}$ to $\frac{1}{2}$ on the left side, and from $\frac{1}{2}$ to $\frac{7}{8}$ on the right side, all for $N = 3$. Figure 4.1 visually depicts the convergence of the approximated solution to the analytical solution when μ takes respectively the values $\frac{1}{8}$, $\frac{1}{4}$, $\frac{3}{8}$, $\frac{1}{2}$ on the left, and $\frac{1}{2}$, $\frac{5}{8}$, $\frac{3}{4}$, $\frac{7}{8}$ on the right, all for $N = 3$.

υ	$N=3$, [37]	$N=3$, (OM)	$N=4$, [37]	$N=4$, (OM)
0.1	$1.734723\ \times10^{-17}$	0.000000	5.925164×10^{-16}	0.000000
0.2	5.551115×10^{-17}	0.000000	7.112366×10^{-16}	0.000000
0.3	8.326672×10^{-17}	0.000000	5.724587×10^{-16}	0.000000
0.4	1.110223×10^{-16}	0.000000	3.053113×10^{-16}	0.000000
0.5	1.387778×10^{-16}	0.000000	8.326672×10^{-17}	0.000000
0.6	1.665334×10^{-16}	0.000000	6.938893×10^{-17}	0.000000
0.7	1.665334×10^{-16}	0.000000	9.714451×10^{-17}	2.77555×10^{-17}
0.8	1.110223×10^{-16}	0.000000	6.938893×10^{-17}	2.77555×10^{-17}
0.9	6.938893×10^{-17}	0.000000	2.775557×10^{-17}	2.77555×10^{-17}
1.0	2.225073×10^{-308}	0.000000	0.0000000	2.98990×10^{-17}

Table 4.1: Absolute error with $\mu = \frac{1}{2}$ for Example 4.1.

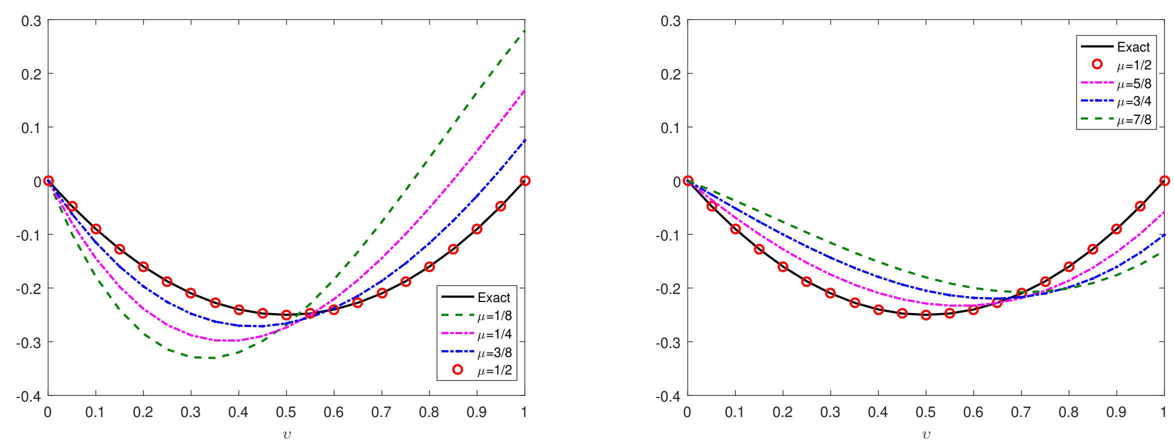

Figure 4.1: Approximated solution and analytical solution for Example 4.1 with different μ values for $N = 3$.

Example 4.2. *Let's consider the linear FIDE below*

$$\begin{aligned} {}^{C}\mathfrak{D}^{\mu}\varphi(\upsilon) &= -\tfrac{3}{91\pi}\upsilon^{1/6}\,\Gamma(5/6)(-91+216\upsilon^2)+(5-2\mathrm{e})\upsilon+\int_0^1 \upsilon e^{\tau}\,\varphi(\tau)\,d\tau, \quad 0\le\upsilon\le 1,\\ \varphi(0) &= 0. \end{aligned} \tag{4.14}$$

For $\mu = \frac{5}{6}$, the analytical solution is given by $\varphi(\upsilon) = \upsilon - \upsilon^3$.

We calculated the AE in Table 4.2 using our method for $\mu = \frac{5}{6}$ with $N = 3$ and $N = 4$, and compared them to a similar problem in [37]. Our results demonstrate significant improvement. Figure 4.2 illustrates the convergence of the approximated solution to the analytical solution for various values of μ, including $\frac{1}{7}$, $\frac{1}{3}$, $\frac{1}{2}$, and $\frac{5}{6}$ on the left, and $\frac{1}{2}$, $\frac{5}{6}$, $\frac{9}{10}$, and $\frac{99}{100}$ on the right, all for $N=3$.

υ	$N=3$, [37]	$N=3$, (OM)	$N=4$, [37]	$N=4$, (OM)
0.1	4.510281×10^{-17}	0.000000	1.105886×10^{-16}	1.38777×10^{-17}
0.2	8.326672×10^{-17}	0.000000	1.301042×10^{-16}	2.77555×10^{-17}
0.3	1.110223×10^{-16}	0.000000	1.110223×10^{-16}	0.000000
0.4	1.110223×10^{-16}	0.000000	6.938893×10^{-17}	0.000000
0.5	5.551115×10^{-17}	0.000000	0.0000000	0.000000
0.6	5.551115×10^{-17}	0.000000	2.775557×10^{-17}	0.000000
0.7	5.551115×10^{-17}	0.000000	8.326672×10^{-17}	0.000000
0.8	1.665334×10^{-16}	0.000000	8.326672×10^{-17}	1.66533×10^{-16}
0.9	1.387778×10^{-16}	0.000000	2.775557×10^{-17}	2.77555×10^{-17}
1.0	2.220446×10^{-17}	0.000000	0.0000000	7.67540×10^{-17}

Table 4.2: Comparison between the AE of OM and the approach in [37] for the Example 4.2 with $N=3$ and $N=4$ where $\mu=\frac{5}{6}$.

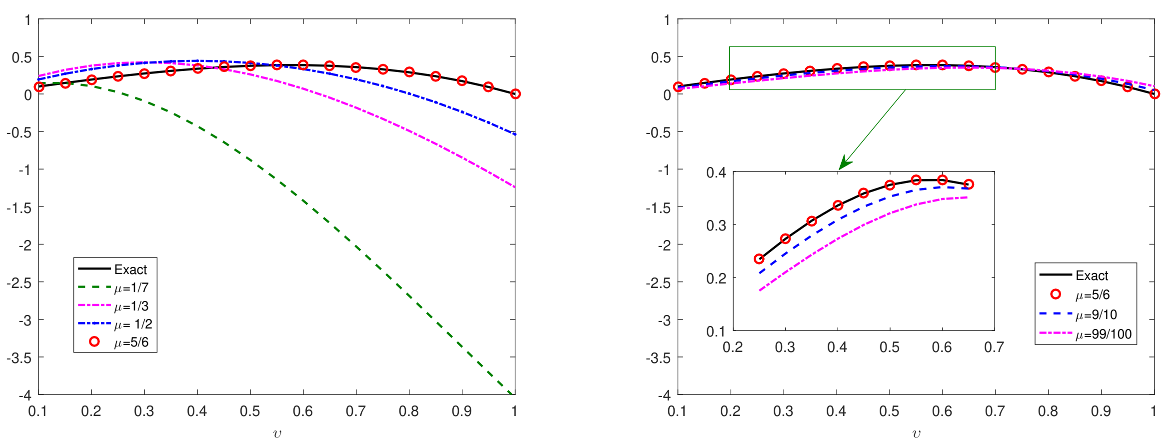

Figure 4.2: Approximated solution and analytical solution with various μ values and $N=3$ for Example 4.2.

Example 4.3. *Let's consider the following linear FIDE*

$$\begin{aligned} {}^{C}\mathfrak{D}^{\mu}\varphi(\upsilon) &= \tfrac{3\sqrt{3}\,\Gamma(2/3)}{\pi}\upsilon^{1/3} - \tfrac{1}{5}\upsilon^{2} - \tfrac{1}{4}\upsilon + \int_{0}^{1}(\upsilon\tau + \upsilon^{2}\tau^{2})\varphi(\tau)\,d\tau, \quad 0 \le \upsilon \le 1. \\ \varphi(0) &= \varphi^{'}(0) = 0. \end{aligned} \tag{4.15}$$

For $\mu = \frac{5}{3}$, the analytical solution is given by $\varphi(\upsilon) = \upsilon^2$.

Applynig our method, we calculated the AE for $\mu = \frac{5}{3}$ with $N = 3$ and $N = 4$, as presented in Table 4.3. Comparing these errors with those from a similar problem in [37], we achieved exceptional results with errors nearly approaching zero. Figure 4.3 demonstrates the convergence of the approximated solution to the analytical solution for various values of μ; specifically, $\frac{6}{5}$, $\frac{4}{3}$, $\frac{3}{2}$, and $\frac{5}{3}$ on the left, and $\frac{5}{3}$, $\frac{18}{10}$, and $\frac{19}{10}$ on the right, all for $N = 3$.

υ	$N = 3$, [37]	$N = 3$, (OM)	$N = 4$, [37]	$N = 4$, (OM)
0.1	5.486062×10^{-17}	0.000000	1.775516×10^{-17}	1.73472×10^{-18}
0.2	8.500145×10^{-17}	0.000000	1.693523×10^{-16}	6.93889×10^{-18}
0.3	9.714451×10^{-17}	0.000000	7.285838×10^{-17}	0.000000
0.4	9.714451×10^{-17}	0.000000	3.816391×10^{-17}	0.000000
0.5	1.249000×10^{-16}	0.000000	1.110223×10^{-16}	0.000000
0.6	8.326672×10^{-17}	0.000000	1.110223×10^{-16}	0.000000
0.7	1.110223×10^{-16}	0.000000	2.775557×10^{-17}	0.000000
0.8	1.110223×10^{-16}	0.000000	0.000000	0.000000
0.9	2.220446×10^{-16}	0.000000	1.110223×10^{-16}	0.000000
1.0	1.110223×10^{-16}	0.000000	0.000000	0.000000

Table 4.3: Absolute error with $\mu = \frac{5}{3}$ for Example 4.3.

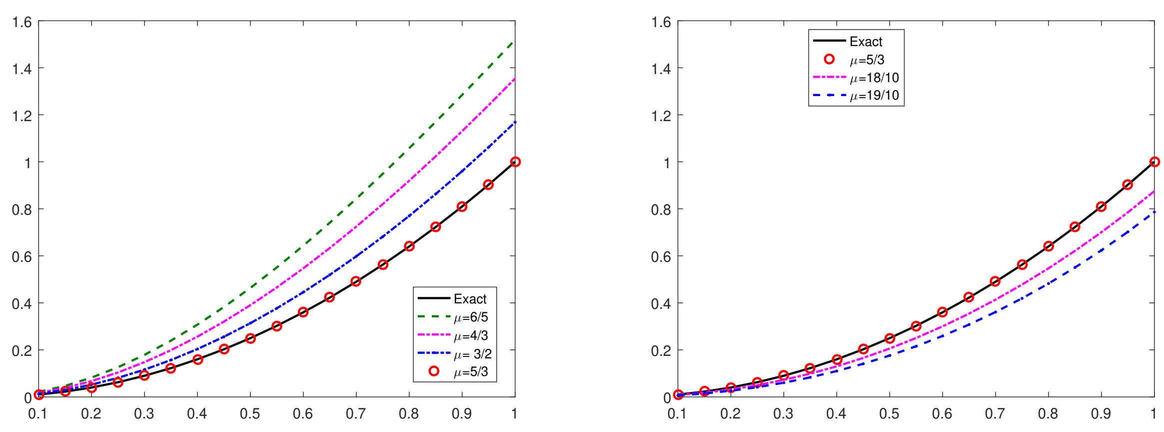

Figure 4.3: Analytical and approximated solutions with various values of μ and $N = 3$ for Example 4.3 .

Example 4.4. *Let's examine the following Fredholm integro-differential equation*

$$\begin{aligned} {}^{C}\mathfrak{D}^{\mu}\varphi(\upsilon) &= 2\upsilon - 3\upsilon^2 + \tfrac{1}{30} - \int_0^1 \varphi(\tau)\, d\tau, \quad 0 \le \upsilon \le 1. \\ \varphi(0) &= \varphi^{'}(0) = 0. \end{aligned} \tag{4.16}$$

For $\mu = 2$, the analytical solution is given by $\varphi(\upsilon) = \frac{1}{3}\upsilon^3 - \frac{1}{4}\upsilon^4$.

When applying our method with $\mu = 2$ and $N = 4$, we obtained an AE equal to 0, representing the optimal outcomes. Figure 4.4 illustrates the convergence of the approximated solution to the analytical solution for μ values of $\frac{5}{4}$, $\frac{3}{2}$, $\frac{7}{4}$, and 2.

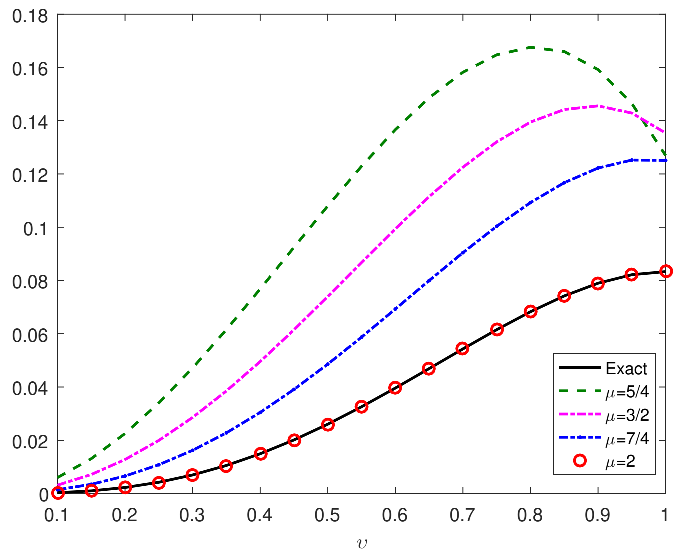

Figure 4.4: Analytical and approximated solutions for Example 4.4 with different values of μ and $N = 4$.

Example 4.5. *Let's consider the following linear FIDE*

$$\begin{aligned} {}^{C}\mathfrak{D}^{\mu}\varphi(\upsilon) &= 3 + 6\upsilon + \int_0^1 \upsilon\tau\ \varphi(\tau)\, d\tau, \quad 0 \le \upsilon \le 1, \\ \varphi(0) &= 0. \end{aligned} \tag{4.17}$$

For $\mu = 1$, the analytical solution is given by $\varphi(\upsilon) = 3\upsilon + 4\upsilon^2$.

Applying our method with $\mu = 1$ for both $N = 3$ and $N = 4$, we achieved an AE equal to 0, indicating a highly accurate approximation. The Figure 4.5 illustrates the convergence of the approximated solution towards the analytical solution as μ takes the values $\frac{1}{4}$, $\frac{1}{2}$, $\frac{3}{4}$, 1, for $N = 3$.

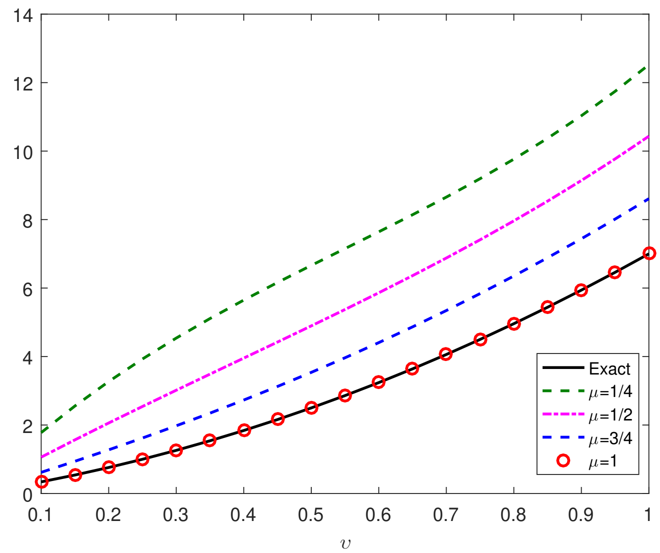

Figure 4.5: Approximated and analytical solutions with various μ values and $N = 3$ for Example 4.5.

4.5 Conclusion

In this chapter, we introduced a numerical approach to solve a linear Fredholm fractional integro-differential equations (FIDEs) with the objective of achieving higher solution accuracy. Our novel approach involves combining the LSM with spectral approximation using a compact combination of SCP of the first kind, rather than relying solely on SCP. This methodology transformed FIDEs into a system of linear algebraic equations resulting in an outstanding approximation with zero AE and optimal outcomes. To confirm the credibility of the theoretical finding, we showcased pertinent examples and manipulated the order of the FD, observing the solution's behavior utilizing MATLAB R2020a. Additionally, we compared the majority of examples with prior works, specifically referencing [37], for comprehensive evaluation. Importantly, this method extends not solely to the similar category of linear FIDEs but also to other FDEs involving various fractional derivatives, including Atangana-Baleanu FD and the Caputo-Fabrizio FD etc. In conclusion, our method not only provides enhanced accuracy in solving linear Fredholm FIDEs but also demonstrates applicability to a broader range of fractional differential equations. Future research could explore its implementation in more complex problem domains or investigate its performance with nonlinear equations. Overall, our approach presents a promising avenue for advancing numerical techniques in fractional calculus.

CONCLUSION AND PERSPECTIVE

In the course of this thesis, our investigation has delved into uncharted territories within the field of FDEs, specifically focusing on the existence, uniqueness and stability of solutions as well as numerical resolution. Initially, we discussed the existence and uniqueness of solutions for a novel category of FDEs that encompass NIIs under CFfd employing the Banach fixed point theorem (FPT) and Darbo's FPT combined with the Kuratowski's measure of noncompactness and also we examined the HU stability of solutions.

Simultaneously, another chapter of thesis focused on exploring the existence and uniqueness outcomes for a category of FIDEs with NIIs under the Caputo fractional derivative employing both of Krasnoselskii's FPT and Darbo's FPT combined with the KMNC.

Beyond theoretical contributions, our focus shifts in the last chapter to numerically solving a linear Fredholm FIDEs under the Caputo derivative. Our innovative approach integrates the LSM with spectral approximation, employing a compact combination of SCP of the first kind.

In summary, we expect that the results outlined in this thesis will contribute significantly to advancing the field of fractional differential equations, paving the way for new avenues of scientific inquiry. Our future endeavors will focus on extending the implications of this thesis by delving into fractional differential inclusions and equations. This exploration will encompass NIIs and various types of fractional derivatives, addressing both theoretical and numerical aspects, incorporating different numerical methods, diverse fixed point theorems and stability types, seeking conditions that yield optimal results. Furthermore, our investigations may encompass the introduction of infinite delay and state-dependent delay within the frameworks of Banach or Frechet spaces.

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