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A Collocation Method for Solving Multi-Dimensional Integral Equations

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F. Birem

الملخــــص

الهدف الرئيسي من هذه الأطروحة هو تقديم طريقة عددية مباشرة وفعّالة وسهلة التطبيق للحصول على حلول تقريبية لمشكلة غورسات للمعادلات التفاضلية الجزئية ، و معادلات فولتيرا التكاملية ذات بعدين من النوع الأول وكذا معادلات فولتيرا التكاملية ثلاثية الأبعاد. تم تطوير خوارزميات تعتمد على استخدام كثيرات حدود تايلور لحل هذه المعادلات عددياً. بالإضافة إلى ذلك، يتم تقديم تحليل دقيق للخطأ. تشمل الأطروحة أمثلة عددية لإثبات صحة وفعالية تقارب الخوارزميات .

الكلمات المفتاحية : معادلات فولتيرا التكاملية ذات بُعدين (النوع الأول والثاني)، معادلات فولتيرا التكاملية ثلاثية الأبعاد، طريقة التجميع، كثيرات حدود تايلور، تحليل الخطأ.

ABSTRACT

The primary objective of this thesis is to present a straightforward, efficient, and easily applicable numerical approach for obtaining approximate solutions to the Goursat problem in hyperbolic partial differential equations with variable coefficients, as well as for solving two-dimensional Volterra integral equations of the first kind and three-dimensional Volterra integral equations. The study develops algorithms utilizing Taylor polynomials to numerically solve these types of equations. Additionally, a comprehensive error analysis is provided. To demonstrate the accuracy and effectiveness of the proposed convergent algorithms, numerical examples are included.

Key Words: Two-dimensional Volterra integral equations of the first and second kind, Three-dimensional Volterra integral equations, Collocation method, Taylor polynomials, Error analysis.

RÉSUMÉ

L'objectif principal de cette thèse est de présenter une méthode numérique simple, efficace et facilement applicable pour obtenir des solutions approximatives au problème de Goursat dans les équations aux dérivées partielles hyperboliques à coefficients variables, ainsi que pour résoudre les équations intégrales de Volterra à deux dimensions de premier type et les équations intégrales de Volterra à trois dimensions. L'étude développe des algorithmes utilisant des polynômes de Taylor pour résoudre numériquement ces types d'équations. De plus, une analyse d'erreur approfondie est fournie. Des exemples numériques sont inclus pour démontrer la précision et l'efficacité des algorithmes convergents proposés.

Mots-clés: Équations intégrales de Volterra à deux dimensions (premier et deuxième éspèse), Équations intégrales de Volterra à trois dimensions, Méthode de collocation, Polynômes de Taylor, Analyse d'erreur.

CONTENTS

GENERAL INTRODUCTION

Volterra integral equations have long been a cornerstone in the analysis of both mathematical and physical phenomena. Named after the renowned Italian mathematician Vito Volterra, these equations are widely applied across various disciplines, including physics, biology, and engineering. While classical Volterra integral equations typically involve functions of a single variable, the increasing complexity and diversity of real-world challenges have led to the development of multi-dimensional Volterra integral equations (MDVIEs).

MDVIEs extend the classical concept to functions of multiple variables, making them particularly valuable for modeling systems that depend on more than one spatial or temporal variable. This multi-dimensional framework allows for a more nuanced and comprehensive representation of dynamic systems, capturing complex interactions across various dimensions. The study of MDVIEs encompasses both theoretical and numerical approaches. Theoretical research primarily focuses on solutions' existence, uniqueness, and stability, while numerical methods are employed to obtain efficient and accurate approximations of these complex equations. Given the inherent complexity of MDVIEs, analytical solutions are often impractical, underscoring the importance of developing effective numerical techniques. Over time, various methods have been designed to solve MDVIEs, each tailored to specific types of problems. For instance:

- Trapezoidal rule [1] is frequently generalized for multi-dimensional integrals by discretizing each independent variable. This method is extensively utilized in applications such as heat transfer and fluid dynamics, where accurate numerical integration is essential.
- Simpson's rule [25, 31], widely recognized for its higher accuracy compared to the trapezoidal rule, has been extended to multi-dimensional integrals by applying quadratic interpolation to approximate the integrand. This refinement improves the precision and efficiency of numerical integration, making it effective for computations in higher-dimensional spaces.
- Operational matrices method [44] is a direct approach that approximates functions using Bernstein multiscaling polynomials. By employing operational matrices, this method transforms integral equations into a system of algebraic equations, facilitating efficient numerical solutions.
- Galerkin method [37, 47] is an approach for solving integral equations by representing the solution as a combination of basis functions and ensuring that the residual error is minimized in a weighted manner. This method provides accurate approximations and is widely used in numerical analysis and computational mathematics.
- Collocation methods [12, 30, 54] approximate solutions by using polynomial expansions, such as Taylor or Lagrange polynomials, and solving the integral equation at specific discrete locations known as collocation points. This approach ensures high accuracy in numerical approximations and is widely applied in solving integral and differential equations.

Multi-dimensional Volterra integral equations represent a versatile class of integral equations that are widely used in various scientific, engineering, and mathematical applications [15, 17, 36, 46, 64]. Below are some real-world applications where MDIEs play a crucial role:

- **Heat conduction:** A notable real-world application of MDVIEs appears in the modeling of heat conduction in materials with memory effects [15]. Unlike classical Fourier heat conduction, which implies infinite speed of thermal propagation, materials with memory exhibit a delayed response governed by a relaxation kernel.

  In this context, the temperature distribution $u(x,t)$ in a rigid heat conductor is described by the following integro-differential equation:

  $$u(x,t) = \int_0^t K(t-\tau)\,\Delta u(x,\tau)\,d\tau + \int_0^t f(x,\tau)\,d\tau,$$

  where:

  - $x \in \mathbb{R}^3$ denotes the spatial position and t is the time.
  - $K(t-\tau)$ is the memory kernel, which accounts for hereditary effects in heat conduction.
  - $\Delta u(x,\tau)$ is the Laplacian of the temperature field, representing spatial diffusion.
  - $f(x,\tau)$ is an external source term or internal heat generation function.

  This model has been used to capture the behavior of materials with finite thermal response time, such as polymers, composite materials, and biological tissues. It allows for more accurate modeling of heat waves, phase transitions, and thermal relaxation effects. Importantly, it also enables the mathematical treatment of cases where the kernel $K(t)$ is singular at $t = 0$, reflecting real materials' sharp initial thermal responses.

  Such equations are not only theoretically significant but also vital in applications including laser heating, electronic cooling, and the design of thermal metamaterials.

- **Ecological Modeling:** Ecological systems are inherently complex and evolve over both time and space. Traditional models, such as ordinary differential equations, typically describe temporal dynamics alone and often neglect spatial heterogeneity. However, real ecosystems exhibit localized interactions, spatial patterns, and memory-dependent behaviors, which require more sophisticated mathematical tools for accurate modeling.

  A promising framework for capturing such dynamics is the use of MDIEs. These equations are particularly useful in ecological systems where the current state depends not only on the immediate conditions but also on the cumulative effects of past spatial and temporal interactions.

  One generalized form of a spatio-temporal ecological MDIE is:

  $$u(x,t) = \int_0^t \int_\Omega K(x,y,t-s)\, R\big(u(y,s)\big)\, dy\, ds + f(x,t),$$

  where:

  - $u(x,t)$ denotes a state variable such as biomass or nutrient concentration at spatial location $x \in \Omega \subset \mathbb{R}^d$ and time t.
  - $K(x,y,t-s)$ is a space-time kernel representing the influence of location y at earlier time s on location x at current time t.
  - $R(u)$ describes nonlinear ecological interactions, such as predation or competition.
  - $f(x,t)$ is an external source term or forcing function.

  In the work of Chen, Q. et al. [17], various ecological modeling paradigms are discussed, including reaction-diffusion equations, cellular automata, and individual-based models.

  Such models naturally lend themselves to MDIEs, making them powerful tools in modern ecological forecasting, conservation planning, and environmental resource management.

- **Population dynamics and infectious disease:** MDIEs naturally arise in the modeling of population dynamics and infectious disease [64], where individual movement and delayed biological responses are essential. For example:
  - In population dynamics, individuals may disperse over a region while retaining memory of past environmental conditions.
  - In infectious disease modeling, transmission may depend on contact with individuals who were infected in the past and are now in different spatial locations.

  A representative form of a model is given by the following system:

$$\begin{cases} \partial_t u(x,t) = D\partial_x^{(2)} u(x,t) - \gamma u(x,t) + f\Big(u(x,t),\, (g * h)(u(x,t))\Big), & t > 0,\ x \in \mathbb{R}, \\ u(x,t) = \psi(x,t), & t \leq 0, \end{cases}$$

  where:

  - $u(x,t)$ is the population density at location x and time t.
  - $D > 0$ is the diffusion rate, and $\gamma > 0$ is the per capita mortality rate.
  - f is a nonlinear function modeling local reactions (e.g., birth or competition).
  - $(g * h)(u(x,t))$ represents a spatio-temporal convolution, defined by:

$$(g * h)(u(x,t)) = \int_0^{\infty} \int_{-\infty}^{\infty} G(s,x,y)\, k(s)\, h(u(y,t-s))\, dy\, ds,$$

  where:

  - $G(s,x,y) = \dfrac{1}{\sqrt{4\pi D s}} \exp\left(-\dfrac{(x-y)^2}{4Ds}\right)$ is the spatial diffusion kernel.
  - $k(s)$ is a probability density function modeling memory in time, satisfying $\int_0^{\infty} k(s)\, ds = 1$.
  - $h(u)$ is a continuous nonlinear function (e.g., a density-dependent birth or infection rate).

The work by Zhao, Z. and Rong, E. [64] establishes the existence, uniqueness, and asymptotic behavior of solutions to this system, making it a rigorous and well-grounded example of an MDIE in applied mathematical biology.

The primary objective of this thesis is to introduce a straightforward yet effective numerical method for approximating solutions to the Goursat problem in hyperbolic partial differential equations with variable coefficients, as well as for solving two-dimensional Volterra integral equations of the first kind. Additionally, it extend the approach to derive solutions for three-dimensional linear and nonlinear Volterra integral equations by using Taylor collocation method. This method is particularly advantageous because it can approximate the exact solution of an integral equation by employing an appropriate function from a predefined finite-dimensional space. This method ensures the approximation satisfies the integral equation at specific collocation points. One of the key strengths of this approach is its flexibility, allowing for adjustments to both the number of subintervals and the degree of the Taylor polynomials, which in turn improves the accuracy of the results. Furthermore, the method is straightforward to implement, relying on iterative formulas rather than solving complex algebraic equations, and it provides a clear and predictable convergence rate.

This thesis is organized into four chapters, each addressing a distinct aspect of the research:

Chapter 1 lays the groundwork by introducing essential concepts, definitions, and key theorems necessary for the subsequent chapters. Topics covered include Taylor series, integral equations, the Leibniz rule, piecewise polynomial spaces, and the Taylor collocation method. Additionally, we cover some important discrete and integral inequalities that will be referenced in later chapters.

Chapter 2 focuses on the development of a collocation method, as introduced by [35], to solve the Goursat problem in hyperbolic linear partial differential equations

(2.3). The Goursat problem is converted into a linear Volterra integral equation of the second kind (2.4). We create an algorithm using Taylor polynomials to approximate the solution of the integral equation within a chosen finite-dimensional function space (2.5). The solution is constructed to satisfy the integral equation at collocation points. This chapter includes error estimates and numerical illustrations to validate the findings.

Chapter 3 broadens the scope of a numerical approach that employs Taylor polynomials, building upon the foundational work of [7, 35], to approximate the solution of nonlinear two-dimensional Volterra integral equations of the first kind (3.1) by constructing a collocation solution in space (2.5). An algorithm is created to solve these equations after they are transformed to (3.3). We prove the algorithm's convergence and validate its effectiveness with numerical examples, demonstrating the accuracy and efficiency of the proposed method.

Chapter 4 extends the collocation method, building upon previous works such as [6, 33, 34, 35] to solve three-dimensional Volterra integral equations (4.1) and (4.2) by applying Taylors theorem in three variables. In this chapter, we approximate the exact solution within a piecewise polynomial spline space (4.3). We conduct a convergence analysis to affirm the method's reliability and present comparative numerical examples to showcase the methods performance and practical applicability.

In the conclusion, we summarize this thesis's contributions, highlight potential areas for further improvement, and propose directions for future research.

CHAPTER 1

BASIC AND ESSENTIAL CONCEPTS

In this chapter, we introduce the essential concepts and foundational ideas that serve as the basis for the subsequent discussions. These preliminary notions are crucial for establishing a clear understanding of the subject and will provide the necessary definitions and theorems. By laying out these core principles, we ensure a coherent and structured progression throughout the content.

1.1 Taylor Series

The Taylor series, also known as the Taylor expansion, is a mathematical technique used to express a function as an infinite sum of terms derived from the function's derivatives at a particular point. This method is extremely useful for approximating functions that are challenging or impossible to compute directly. For many common functions, the original function and the sum of its Taylor series are identical in the near of this point. The series is named after Brook Taylor, who introduced it in 1715. A special case of the Taylor series, called the Maclaurin series, is obtained when the derivatives are evaluated at the point 0. This version was extensively studied by Colin Maclaurin in the 18*th* century.

Definition 1.1.1 *For a function φ that is differentiable up to $\eta + 1$ times at the point τ_0, the Taylor theorem provides the following approximation of $\varphi(\tau)$ near τ_0:*

$$\varphi(\tau) = \varphi(\tau_0) + \varphi^{'}(\tau_0)(\tau - \tau_0) + \frac{\varphi^{''}(\tau_0)}{2!}(\tau - \tau_0)^2 + ... + \frac{\varphi^{(\eta)}(\tau_0)}{\eta!}(\tau - \tau_0)^\eta + R_\eta(\tau),$$

where $R_\eta(\tau)$ represents the remainder term that quantifies the error incurred by truncating the series after η terms. The remainder can be written in various forms, with one common expression being the Lagrange form:

$$R_\eta(\tau) = \frac{\varphi^{(\eta+1)}(\varsigma)}{(\eta + 1)!}(\tau - \tau_0)^{\eta+1}.$$

where ς is a point between τ_0 and τ.

Lemma 1.1.1 (*Taylor's theorem for functions of two independent variables [18]*) *Let φ be p times continuously differentiable on $\mathcal{R} = [A_0, A_1] \times [B_0, B_1]$ and let $(t_0, s_0) \in \mathcal{R}$. Then for all $(t, s) \in \mathcal{R}$, we have*

$$\varphi(t,s) = \sum_{i+j=0}^{p-1} \frac{1}{i!j!}\partial_t^{(i)}\partial_s^{(j)}\varphi(t_0,s_0)(t-t_0)^i(s-s_0)^j + \sum_{i+j=p} \frac{1}{i!j!}\partial_t^{(i)}\partial_s^{(j)}\varphi(t_1,s_1)(t-t_0)^i(s-s_0)^j,$$

where

$$\begin{cases} t_1 = \theta t + (1-\theta)t_0 \in [A_0, A_1], \\ s_1 = \theta s + (1-\theta)s_0 \in [B_0, B_1], \end{cases} \qquad \theta \in (0,1).$$

1.2 Integral Equations

An integral equation is a type of equation where the unknown function $\omega(x)$ appears within the integral. Integral equations are a powerful and versatile tool in both pure and applied mathematics, playing a crucial role in solving various physical problems. Many problems related to initial and boundary conditions for ordinary differential equations and partial differential equations can be reformulated as problems of solving corresponding integral equations. A common linear form of an integral equation is expressed as:

$$\varphi(\tau)\omega(\tau) = u(\tau) + \lambda \int_{\alpha(\tau)}^{\beta(\tau)} \kappa(\tau,s)\omega(s)ds \ ,$$

and where λ is a constant parameter and $\kappa(\tau, s)$ represents the kernel. The unknown function $\omega(\tau)$ appears inside the integral. In some cases, the function $\omega(\tau)$ may appear both inside and outside of the integral. It is important to note that the functions $\varphi(\tau)$, $u(\tau)$ and $\kappa(\tau, s)$ are known functions, and the limits of integration $\alpha(\tau)$ and $\beta(\tau)$ could either be variables, constants, or a combination of both.

Integral equations (IEs) can be categorized into a wide range of types, each with distinct theoretical and numerical characteristics. One key distinction is between one-dimensional and multi-dimensional IEs, and within each of these categories, IEs can be classified as either linear or nonlinear. A further classification depends on the limits of integration, with two main types:

- If the limits of integration are constant, the equation is referred to as a Fredholm integral equation (FIE), which has the form:

$$\varphi(\tau)\omega(\tau) = u(\tau) + \lambda \int_a^b \kappa(\tau, s)\omega(s)ds \ ,$$

  where a and b are constants.

- If at least one limit is variable, the equation is known as a Volterra integral equation (VIE), with the form:

$$\varphi(\tau)\omega(\tau) = u(\tau) + \lambda \int_a^\tau \kappa(\tau, s)\omega(s)ds \ .$$

Both Fredholm and Volterra integral equations can be either homogeneous or inhomogeneous. If $u(\tau) = 0$ the equation is classified as a homogeneous FIE or VIE.
Additionally, integral equations are also classified based on how the function $\varphi(\tau)$ appears, as outlined below:

- If $\varphi(\tau) = 0$, the equation is called a first kind FIE or VIE.
- If $\varphi(\tau) = 1$, the equation is called a second kind FIE or VIE.
- If $\varphi(\tau) \neq 0$ and $\varphi(\tau) \neq 1$, the equation is referred to as a third kind FIE or VIE.

1.3 Leibniz Rule for Differentiation of Integrals

A useful approach for solving integral equations involves converting them into equivalent differential equations. The Leibniz integral rule provides a formula for differentiating under the integral sign, named after Gottfried Wilhelm Leibniz, who developed it in the 17*th* century.

Theorem 1.3.1 (*Leibniz integral rule [62]*) *Consider $\varphi(\tau, s)$ as a continuous function, and assume that $\partial_\tau \varphi$ is continuous within the domain $[\tau_0, \tau_1] \times [s_0, s_1]$, for the integral expression*

$$\Gamma(\tau) = \int_{\alpha(\tau)}^{\beta(\tau)} \varphi(\tau, t)dt,$$

the derivative of $\Gamma(\tau)$ with respect to τ is given by:

$$\Gamma'(\tau) = \varphi(\tau, \beta(\tau))\beta'(\tau) - \varphi(\tau, \alpha(\tau))\alpha'(\tau) + \int_{\alpha(\tau)}^{\beta(\tau)} \partial_\tau \varphi(\tau, t)dt.$$

Remark 1.3.1 *In situations where $\alpha(\tau) = a$ and $\beta(\tau) = b$, with a and b being constants, the Leibniz rule simplifies to the following expression:*

$$\Gamma'(\tau) = \int_a^b \partial_\tau \varphi(\tau, t)dt.$$

1.4 Piecewise Polynomial Spaces

Piecewise polynomial spaces are mathematical structures used for approximating functions across an interval by dividing the domain into smaller subintervals and defining polynomial functions on each of these subintervals. These spaces are foundational in fields like numerical analysis, the finite element method, computer-aided design, and data interpolation, where they enable a local, adaptable approach to modeling complex behavior within a domain.

Let $J_k = \{y_m : 0 = y_0 < y_1 < \cdots < y_M = b\}$ represent a grid (or mesh) on the interval $J = [0, b]$, with the stepsize defined as $k = \frac{b}{M}$.
Define the subintervals $\delta_m = [y_m, y_{m+1}]$ for $m = 0, \ldots, M - 1$.

Definition 1.4.1 *[13] For the grid J_k, the piecewise polynomial space $\mathcal{S}_\eta^{(d)}(J_k)$ is defined for* *parameters* $\eta \geq 0$ *and* $-1 \leq d \leq \eta$ *as follows:*

$$\mathcal{S}_{\eta}^{(d)}(J_k) := \{u \in C^d(J) : u|_{\delta_m} \in \pi_{\eta}, 0 \leq m \leq M-1\},$$

here, π_η *represents the space of real polynomials whose degree does not exceed* η*.*
It can be shown that $\mathcal{S}_{\eta}^{(d)}(J_k)$ *is a (real) linear vector space, and its dimension is given by:*

$$\dim \mathcal{S}_{\eta}^{(d)}(J_k) = M(\eta - d) + d + 1.$$

Remark 1.4.1 *The particular piecewise polynomial space* $\mathcal{S}_{p+d}^{(d)}(J_k)$ *with* $p \geq 1$ *and* $d \geq -1$ *where its dimension is* $Mp + d + 1$*, it may be viewed as the natural collocation space for the approximation of solutions to initial-value problems for ordinary differential equations or Volterra equations.*
The selection of the regularity degree d is determined by the number of specified initial conditions, while the factor Mp indicates that p distinct collocation points should be assigned to each of the M subintervals δ_m*. Therefore, the most suitable choice for d is as follows:*

- *For Volterra integral equations without initial conditions, we set* $d = -1$*.*
- *For first-order ordinary differential equations or Volterra integro-differential equations with one initial condition,* $d = 0$ *is used.*
- *For ordinary differential equations or Volterra integro-differential equations of order* $k \geq 2$*, which have k initial conditions, we set* $d = k - 1$*.*

The particular piecewise polynomial space $\mathcal{S}_{p-1,q-1}^{(-1)}(\Pi_{N,M})$ *of bivariate polynomial spline functions of order p (degree* $p-1$*) in t and order q (degree* $q-1$*) in s is a tensor-product space based on the univariate spline spaces* $\mathcal{S}_{p}^{(-1)}(\Pi_N)$ *and* $\mathcal{S}_{q}^{(-1)}(\Pi_M)$*. An element of this space has jumped discontinuities at the interior grid lines* $t = t_n (n = 1, \ldots, N-1)$ *and* $s = s_m (m = 1, \ldots, M-1)$*.*

1.5 Collocation Method

An effective numerical technique for approximating the solutions to differential and integral equations is the collocation method. This approach involves selecting a finite-dimensional space of potential solutions typically polynomials of a specific degree and a set of points in the domain, known as collocation points. The method aims to identify a solution that satisfies the equation at these chosen points.

The collocation method is highly adaptable and can be applied to various types of equations. For ordinary differential equations (ODEs), several notable collocation methods include: the Chebyshev collocation method, which represents solutions as Chebyshev series; the Legendre-Gauss collocation method, designed for initial value problems of second-order ODEs using Legendre-Gauss interpolation; the Bernstein collocation method, employing Bernstein polynomials to solve nonlinear ODEs; and the block hybrid collocation method, tailored to directly solve third-order ODEs. When addressing partial differential equations (PDEs), several key methods are employed. These include the sparse grid stochastic collocation method, which uses a Smolyak-type sparse grid to approximate statistical quantities associated with PDE solutions; the wavelet collocation method, based on Daubechies wavelets' autocorrelation functions for numerical solutions; and the spline-collocation method, which utilizes B-spline functions for approximation.

In the realm of integral equations, the collocation method also offers specialized techniques. The Galerkin collocation method, for example, is commonly used to solve integral equations of the first kind, while the iterated collocation method is well-suited for nonlinear Volterra integral equations of the second kind.

The effectiveness of the collocation method depends on several factors, including the selection of collocation points, the choice of basis functions, and the nature of the equation being solved. Proper attention to these factors is essential for ensuring the convergence and accuracy of the approximated solution.

A key advantage of the collocation method is its flexibility. It can be implemented without the need for meshing in some cases, such as with node-based techniques, which significantly reduces computational complexity. This ease of implementation, coupled with its ability to handle complex boundary conditions and irregular domains, makes it a powerful and versatile tool in numerical analysis. Additionally, with the careful selection of basis functions and collocation points, the method can yield highly precise approximations for a wide range of problems.

1.6 Taylor Collocation Method

The Taylor collocation method is a numerical approach used for solving various types of differential, integral, and integro-differential equations by expressing the solution as a finite Taylor series. The method involves by selecting specific points, referred to as collocation points, and determining the coefficients of the Taylor series to satisfy the equation at those points. Initially applied in the context of solving Volterra integral equations, the method has been refined and extended over the years by numerous researchers, such as:

- Karamete, A. and Sezer, M. [29] presented a spectral Taylor matrix collocation technique used for solving linear integro-differential equations by truncating the Taylor series.
- Bellour, A. and Bousselsal, M. [4, 5] developed a numerical approach using Taylor polynomials to approximate solutions to delay integral and integro-differential equations.
- Laib, H. et al. [32] applied the Taylor collocation method to a system of nonlinear Volterra delay integro-differential equations.

These developments have significantly expanded the use of the Taylor collocation method in solving a wide array of complex mathematical problems across different disciplines.

In this thesis, we intend to build upon these pioneering contributions by extending the Taylor collocation method to a broader class of Volterra integral equations, particularly focusing on the two and three dimensions. By tackling both linear and nonlinear equations, this research seeks to broaden the applicability of the Taylor collocation method and offer more efficient numerical solutions for a wider range of integral equation problems.

1.7 Comparison Theorems

In this section, we will examine several key theorems pertaining to discrete and continuous inequalities. These theorems are crucial for developing the proofs necessary to demonstrate the convergence of the approximate solution to the exact solution, which we will address in the subsequent chapters.

1.7.1 Discrete inequalities

Lemma 1.7.1 *[13] Consider the sequence $\{\kappa_j\}(j \geq 0)$, which is non-negative, and let $\{\omega_n\}$ be a sequence satisfying the conditions:*

$$\omega_0 \leq p_0 \quad \textit{and} \quad \omega_n \leq p_0 + \sum_{j=0}^{n-1} \kappa_j \omega_j, \quad n \geq 1,$$

where $p_0 \geq 0$. Then for $n \geq 1$, it follows that

$$\omega_n \leq p_0 \exp\left(\sum_{j=0}^{n-1} \kappa_j\right).$$

Lemma 1.7.2 *[52] Let $\omega(n), b(n)$ and $\epsilon(n)$ be non-negative sequences. If $\omega(n)$ satisfies*

$$\omega(n) \leq b(n) + \sum_{s=0}^{n-1} \epsilon(s)\omega(s),$$

for all $n \in \mathbb{N}$. *Then*

$$\omega(n) \leq b(n) + \sum_{s=0}^{n-1} b(s)\epsilon(s) \prod_{\sigma=s+1}^{n-1} [1 + \epsilon(\sigma)], \quad n \in \mathbb{N}.$$

Lemma 1.7.3 *[42] Let* $\omega_{n,m}$ *be a given non-negative sequence satisfies*

$$\omega_{n,m} \leq h_1\lambda_1 \sum_{\xi=0}^{n-1} \omega_{\xi,m} + h_2\lambda_2 \sum_{\rho=0}^{m-1} \omega_{n,\rho} + h_1h_2\lambda_3 \sum_{\xi=0}^{n-1} \sum_{\rho=0}^{m-1} \omega_{\xi,\rho} + \alpha, n = 0, \ldots, N; m = 0, \ldots, M,$$

such that $h_1 = \frac{A}{N}$ *and* $h_2 = \frac{B}{M}$, *where* $A, B, \lambda_1, \lambda_2, \lambda_3$ *and* α *are finite strictly positive constants independent of N and M. Then*

$$\omega_{n,m} \leq \alpha exp\left(\eta(A + B)\right),$$

where $\eta = \frac{1}{2}\left(\lambda_1 + \lambda_2 + \sqrt{(\lambda_1 + \lambda_2)^2 + 4\lambda_3}\right)$.

Lemma 1.7.4 *[51] Assume* $\omega(n, m), b(n, m)$ *and* $\epsilon(n, m)$ *are non-negative sequences, with* $b(n, m)$ *being nondecreasing in each variable n and m. Suppose* $\omega(n, m)$ *adheres to*

$$\omega(n, m) \leq b(n, m) + \sum_{s=0}^{n-1} \sum_{t=0}^{m-1} \epsilon(s, t)\omega(s, t),$$

for all $n, m \in \mathbb{N}$. *Then*

$$\omega(n, m) \leq b(n, m) \prod_{s=0}^{n-1} \left[1 + \sum_{t=0}^{m-1} \epsilon(s, t)\right], \quad n, m \in \mathbb{N}.$$

Lemma 1.7.5 *[56] Consider* $\omega(i, j, k)$ *and* $\epsilon(i, j, k)$ *as real-valued non-negative functions defined for* $(i, j, k) \in \mathbb{N}^3$, *and let* $b(i, j, k)$ *be a positive function that is nondecreasing in each of the three variables, defined for* $(i, j, k) \in \mathbb{N}^3$. *The inequality*

$$\omega(i, j, k) \leq b(i, j, k) + \sum_{\alpha=0}^{i-1} \sum_{\beta=0}^{j-1} \sum_{\gamma=0}^{k-1} \epsilon(\alpha, \beta, \gamma)\omega(\alpha, \beta, \gamma), \quad i, j, k \in \mathbb{N},$$

leads to

$$\omega(i,j,k) \le b(i,j,k) \prod_{\alpha=0}^{i-1} \left[1 + \sum_{\beta=0}^{j-1} \sum_{\gamma=0}^{k-1} \epsilon(\alpha,\beta,\gamma) \right], \quad i,j,k \in \mathbb{N}.$$

1.7.2 Integral inequalities

Lemma 1.7.6 *[40] Assume $\omega(t,s)$ and $p(t,s)$ are non-negative continuous functions defined on $\mathcal{R} = [t_0,t_1] \times [s_0,s_1]$. Suppose $p(t,s)$ is non-decreasing in each of the variables within $\mathcal{R}$ and satisfies the following inequality:*

$$\omega(t,s) \le p(t,s) + \kappa \int_{t_0}^{t} \omega(x,s)dx + \kappa \int_{s_0}^{s} \omega(t,y)dy + \kappa \int_{t_0}^{t} \int_{s_0}^{s} \omega(x,y)dydx, (t,s) \in \mathcal{R},$$

where κ is a positive constant. Then there exists a positive constant ν, such that

$$\omega(t,s) \le \nu p(t,s).$$

Lemma 1.7.7 *[8] Consider $w(t,s,r)$ as a non-negative continuous function defined on $\mathcal{R} = [t_0,t_1] \times [s_0,s_1] \times [r_0,r_1]$, with ς_1, ς_2 being non-negative constants. The inequality*

$$w(t,s,r) \le \varsigma_1 + \varsigma_2 \int_{t_0}^{t} \int_{s_0}^{s} \int_{r_0}^{r} w(x,y,z)dzdydx, \quad (t,s,r) \in \mathcal{R},$$

leads to

$$w(t,s,r) \le \varsigma_1 \exp\left(\sqrt[3]{\varsigma_2}(t+s+r) \right), \quad (t,s,r) \in \mathcal{R}.$$

CHAPTER 2

TAYLOR COLLOCATION METHOD FOR SOLVING GOURSAT PROBLEM

2.1 Introduction

Partial differential equations (PDEs) are widely utilized to solve problems in a range of fields, such as engineering, physics and finance. The form of a second-order linear hyperbolic partial differential equation is expressed as follows:

$$\alpha\partial_x^{(2)}w(x,y)+\beta\partial_x\partial_y w(x,y)+\gamma\partial_y^{(2)}w(x,y)+\delta\partial_x w(x,y)+\epsilon\partial_y w(x,y)+\varepsilon w(x,y)+\eta=0, \quad (2.1)$$

where $\beta^2-4\alpha\gamma>0$, and $\alpha,\beta,\gamma,\delta,\epsilon,\varepsilon$ and η are functions of the variables x and y. We can reduce it to the canonical form of the hyperbolic equation known as the Goursat problem

$$\partial_x\partial_y w(x,y)=\phi\left(x,y,w,\partial_x w(x,y),\partial_y w(x,y)\right). \quad (2.2)$$

The study of cosmological and ecological phenomena often involves the use of hyperbolic PDEs [63]. A number of numerical methods have been proposed for this purpose, such as the Legendre multi-wavelet Galerkin method [65, 66], the finite difference methods [11, 26], the finite element methods [2], the Taylor matrix method [14, 16], and the Chebyshev wavelet scheme [24].

This chapter provides an approximate solution for the linear Goursat problem of the second order with variable coefficients, represented as follows:

$$\begin{aligned}&\partial_x\partial_y w(x,y)+\alpha(x,y)\partial_x w(x,y)+\beta(x,y)\partial_y w(x,y)+\gamma(x,y)w(x,y)=h(x,y)\\&w(0,y)=a(y),w(x,0)=b(x),a(0)=b(0),\quad (x,y)\in\mathcal{R},\end{aligned} \quad (2.3)$$

where $\mathcal{R}=[0,A]\times[0,B]$, and α, β, γ, h are smooth functions through the domain of discussion, and which was published in the reference [10]. For the existence and uniqueness of the solution, see [27].
By integrating both sides of (2.3) with respect to y and x, we obtain

$$\begin{aligned}w(t,s)=f(t,s)+\int_0^t\kappa_1(x,s)w(x,s)dx+\int_0^s\kappa_2(t,y)w(t,y)dy\\+\int_0^t\int_0^s\kappa_3(x,y)w(x,y)dydx,\quad (t,s)\in\mathcal{R},\end{aligned} \quad (2.4)$$

where the functions f, κ_1, κ_2 and κ_3 are given smooth functions defined, respectively, on $\mathcal{R}$, $S_1 := \{(x,s) : 0 \leq x \leq t \leq A, 0 \leq s \leq B\}$, $S_2 := \{(t,y) : 0 \leq t \leq A, 0 \leq y \leq s \leq B\}$, and $S_3 := \{(x,y) : 0 \leq x \leq t \leq A, 0 \leq y \leq s \leq B\}$ by:

$$\begin{aligned}
&\kappa_1(x,s) : = -\beta(x,s), \kappa_2(t,y) := -\alpha(t,y), \kappa_3(x,y) := \partial_x\alpha(x,y) + \partial_y\beta(x,y) - \gamma(x,y),\\
&f(t,s) : = a(s) + b(t) - w(0,0) + \int_0^s \alpha(0,y)a(y)dy + \int_0^t \beta(x,0)b(x)dx\\
&\qquad + \int_0^t \int_0^s h(x,y)dydx.
\end{aligned}$$

Numerical solutions to the Goursat problem have been extensively studied. For instance, the homogeneous Goursat problem (2.3), in which the coefficients rely on a same variable, was the subject of Scott, E. J. [55]. A nonlinear trapezoidal formula based on geometric means was introduced by Evans, D. J. and Sanugi, B. B. [22]. The Runge-Kutta method was used by Day, J. T. [20] to estimate solutions for (2.2). A novel exponential finite difference approach was developed by Pandey, P. K. [53] to solve (2.2). In order to determine the quadruple solution of a Goursat problem within a triangular domain, Drignei, M. C. [21] created an algorithm.

This chapter's remaining sections are organized as follows: A Taylor polynomial is used to approximate the solution of (2.4) in each collocation point in Section 2.2. Section 2.3 examines the convergence analysis. The theoretical results are presented with numerical examples in Section 2.4. Finally, Section 2.5 provides a conclusion.

2.2 Description of the Method

We define the uniform partitions of the intervals $[0, A]$ and $[0, B]$ as follows:

$$\Pi_N = \{t_i = ih_1, i = 0, \ldots, N\}, \quad \Pi_M = \{s_j = jh_2, j = 0, \ldots, M\},$$

where the step sizes are given by $h_1 = \frac{A}{N}$ and $h_2 = \frac{B}{M}$. These partitions collectively form a grid over the domain $\mathcal{R}$:

$$\Pi_{N,M} = \Pi_N \times \Pi_M = \{(t_n, s_m), 0 \le n \le N, 0 \le m \le M\},$$

we further define: $\mathcal{R}_{n,m} := \delta_n^1 \times \delta_m^2$ for all $n = 0, \ldots, N-1; m = 0, \ldots, M-1$, where

$$\delta_n^1 = [t_n; t_{n+1}), n = 0, \ldots, N-2; \quad \delta_{N-1}^1 = [t_{N-1}, t_N],$$

$$\delta_m^2 = [s_m; s_{m+1}), m = 0, \ldots, M-2; \quad \delta_{M-1}^2 = [s_{M-1}, s_M].$$

Within each rectangle $\mathcal{R}_{n,m}$, where $n = 0, \ldots, N-1$ and $m = 0, \ldots, M-1$, we employ the Taylor polynomials $\vartheta_{n,m}(t,s)$ to approximate the solution of (2.4) in the space:

$$\mathcal{S}_{p-1}^{(-1)}(\Pi_{N,M}) = \{u : u_{n,m} = u|_{\mathcal{R}_{n,m}} \in \pi_{p-1}, n = 0, \ldots, N-1; m = 0, \ldots, M-1\}, \tag{2.5}$$

its dimension is NMp^2, with π_{p-1} denoting the set of all real polynomials of degree not exceeding $p-1$ in both variables t and s. Additionally, it is observed that the solution w of (2.4) is known at the point $(0,0)$.

2.2.1 Approximate solution in $\mathcal{R}_{0,0}$

The polynomial $\vartheta_{0,0}(t,s)$ is employed to approximate $w(t,s)$ within the rectangle $\mathcal{R}_{0,0}$, such that

$$\vartheta_{0,0}(t,s) = \sum_{i+j=0}^{p-1} \frac{1}{i!j!} \partial_t^{(i)} \partial_s^{(j)} w(0,0) t^i s^j \ ; \quad (t,s) \in \mathcal{R}_{0,0}, \tag{2.6}$$

where $\partial_t^{(i)} \partial_s^{(j)} w(0,0)$ is the exact value of $\partial_t^{(i)} \partial_s^{(j)} w$ at point $(0,0)$.

By differentiating equation (2.4) j-times with respect to s and i-times with respect to t, we get

$$
\begin{aligned}
\partial_t^{(i)}\partial_s^{(j)} w(t,s) &= \partial_t^{(i)}\partial_s^{(j)} f(t,s) \\
&+ \sum_{l=0}^{j}\sum_{\eta=0}^{i-1}\binom{j}{l}\binom{i-1}{\eta}\partial_t^{(i-1-\eta)}\left[\partial_s^{(j-l)}\kappa_1(t,s)\right]\partial_t^{(\eta)}\partial_s^{(l)} w(t,s) \\
&+ \sum_{l=0}^{j-1}\sum_{\eta=0}^{i}\binom{r}{l}\binom{i}{\eta}\partial_t^{(i-\eta)}\left[\partial_s^{(j-1-l)}\kappa_2(t,s)\right]\partial_t^{(\eta)}\partial_s^{(l)} w(t,s) \\
&+ \sum_{l=0}^{j-1}\sum_{\eta=0}^{i-1}\binom{j-1}{l}\binom{i-1}{\eta}\partial_t^{(i-1-\eta)}\left[\partial_s^{(j-1-l)}\kappa_3(t,s)\right]\partial_t^{(\eta)}\partial_s^{(l)} w(t,s). \qquad (2.7)
\end{aligned}
$$

2.2.2 Approximate solution in $\mathcal{R}_{n,m}$

The polynomials $\vartheta_{n,m}(t,s)$ are used to approximate $w(t,s)$ within the rectangles $\mathcal{R}_{n,m}$, $n = 0,\dots,N-1$; $m = 0,\dots,M-1$ and $(n,m) \neq (0,0)$, such that

$$
\vartheta_{n,m}(t,s) = \sum_{i+j=0}^{p-1}\frac{1}{i!j!}\partial_t^{(i)}\partial_s^{(j)}\hat{\vartheta}_{n,m}(t_n,s_m)(t-t_n)^i(s-s_m)^j\ ; \quad (t,s)\in\mathcal{R}_{n,m}, \qquad (2.8)
$$

where $\hat{\vartheta}_{n,m}$ is the exact solution of the integral equation:

$$
\begin{aligned}
\hat{\vartheta}_{n,m}(t,s) &= f(t,s) + \sum_{\xi=0}^{n-1}\int_{t_\xi}^{t_{\xi+1}}\kappa_1(x,s)\vartheta_{\xi,m}(x,s)dx + \int_{t_n}^{t}\kappa_1(x,s)\hat{\vartheta}_{n,m}(x,s)dx \\
&+ \sum_{\rho=0}^{m-1}\int_{s_\rho}^{s_{\rho+1}}\kappa_2(t,y)\vartheta_{n,\rho}(t,y)dy + \int_{s_m}^{s}\kappa_2(t,y)\hat{\vartheta}_{n,m}(t,y)dy \\
&+ \sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1}\int_{t_\xi}^{t_{\xi+1}}\int_{s_\rho}^{s_{\rho+1}}\kappa_3(x,y)\vartheta_{\xi,\rho}(x,y)dydx \\
&+ \sum_{\xi=0}^{n-1}\int_{t_\xi}^{t_{\xi+1}}\int_{s_m}^{s}\kappa_3(x,y)\vartheta_{\xi,m}(x,y)dydx
\end{aligned}
$$

$$+\sum_{\rho=0}^{m-1}\int_{t_n}^{t}\int_{s_\rho}^{s_{\rho+1}}\kappa_3(x,y)\vartheta_{n,\rho}(x,y)dydx$$
$$+\int_{t_n}^{t}\int_{s_m}^{s}\kappa_3(x,y)\hat{\vartheta}_{n,m}(x,y)dydx, \tag{2.9}$$

and $\partial_t^{(i)}\partial_s^{(j)}\hat{\vartheta}_{n,m}(t_n,s_m)$ is the exact value of $\partial_t^{(i)}\partial_s^{(j)}\hat{\vartheta}_{n,m}$ at point (t_n,s_m). We can derive this result by differentiating equation (2.9) j-times with respect to s and i-times with respect to t:

$$\partial_t^{(i)}\partial_s^{(j)}\hat{\vartheta}_{n,m}(t,s)=\partial_t^{(i)}\partial_s^{(j)}f(t,s)+\sum_{l=0}^{j}\sum_{\eta=0}^{i-1}\binom{j}{l}\binom{i-1}{\eta}\partial_t^{(i-1-\eta)}\left[\partial_s^{(j-l)}\kappa_1(t,s)\right]\partial_t^{(\eta)}\partial_s^{(l)}\hat{\vartheta}_{n,m}(t,s)$$
$$+\sum_{l=0}^{j-1}\sum_{\eta=0}^{i}\binom{j-1}{l}\binom{i}{\eta}\partial_t^{(i-\eta)}\left[\partial_s^{(j-1-l)}\kappa_2(t,s)\right]\partial_t^{(\eta)}\partial_s^{(l)}\hat{\vartheta}_{n,m}(t,s)$$
$$+\sum_{l=0}^{j-1}\sum_{\eta=0}^{i-1}\binom{j-1}{l}\binom{i-1}{\eta}\partial_t^{(i-1-\eta)}\left[\partial_s^{(j-1-l)}\kappa_3(t,s)\right]\partial_t^{(\eta)}\partial_s^{(l)}\hat{\vartheta}_{n,m}(t,s). \tag{2.10}$$

2.3 Error Analysis

We consider the space $L^\infty(\mathcal{R})$ with the norm

$$||\Gamma||_{L^\infty(\mathcal{R})}=\inf\{K\in\mathbb{R}:|\Gamma(t,s)|\leq K\ \text{ for a.e. } (t,s)\in\mathcal{R}\}<\infty.$$

Remark 2.3.1 *In the proofs that follow, we adopt the notation $||.||$ as a substitute for $||.||_{L^\infty(\mathcal{R})}$ to streamline the expression.*

The convergence of the suggested method is established by the following theorem.

Theorem 2.3.1 *Assume f, κ_1, κ_2 and κ_3 are p times continuously differentiable on their respective domains. Then, equations (2.6) and (2.8) uniquely determine an approximation*

$\vartheta \in \mathcal{S}_{p-1}^{(-1)}(\Pi_{N,M})$. *The associated error function* $e(t,s) = w(t,s) - \vartheta(t,s)$ *satisfies the inequality:*

$$||e||_{L^\infty(\mathcal{R})} \leq \varsigma(h_1 + h_2)^p,$$

where ς *is a finite constant independent of* h_1 *and* h_2.

Proof. The error $e(t,s)$ within the rectangles $\mathcal{R}_{n,m}, n = 0, \ldots, N-1; m = 1, \ldots, M-1$ is described by:

$$e_{n,m}(t,s) = w(t,s) - \vartheta_{n,m}(t,s).$$

The proof is organized into two steps.

Step 1. Let us define $\varepsilon_{n,m}^{i+j} = ||\partial_t^{(i)}\partial_s^{(j)}\hat{\vartheta}_{n,m}||$. It will be shown that there exists a positive constant $\varphi(p)$ such that $\varepsilon_{n,m}^{i+j} \leq \varphi(p)$, for all $n = 0, \ldots, N-1$, $m = 0, \ldots, M-1$ and $i + j = 0, \ldots, p$ where $\hat{\vartheta}_{0,0}(t,s) = w(t,s)$ for $(t,s) \in \mathcal{R}_{0,0}$.

First, we have

$$\varepsilon_{0,0}^{i+j} \leq \max\left\{\left\|\partial_t^{(i)}\partial_s^{(j)}w\right\|, i + j = 0, \ldots, p\right\} = \varphi_1(p).$$

Second, for $i + j = 1, \ldots, p$, we deduce from equation (2.10) that:

$$\varepsilon_{n,m}^{i+j} \leq \lambda_1 + \lambda_2 \sum_{l=0}^{j} \sum_{\eta=0}^{i-1} \varepsilon_{n,m}^{\eta+l} + \lambda_3 \sum_{l=0}^{j-1} \sum_{\eta=0}^{i} \varepsilon_{n,m}^{\eta+l} + \lambda_4 \sum_{l=0}^{j-1} \sum_{\eta=0}^{i-1} \varepsilon_{n,m}^{\eta+l},$$

and for $i + j = 0$, we have from (2.9)

$$\begin{aligned}
\varepsilon_{n,m}^{0+0} \leq \lambda_1 &+ \lambda_5 h_1 \sum_{\xi=0}^{n-1} \sum_{a+b}^{p-1} \varepsilon_{\xi,m}^{0+0} + \lambda_5 h_1 \varepsilon_{n,m}^{0+0} + \lambda_6 h_2 \sum_{\rho=0}^{m-1} \sum_{a+b=0}^{p-1} \varepsilon_{n,\rho}^{0+0} + \lambda_6 h_2 \varepsilon_{n,m}^{0+0} \\
&+ \lambda_7 h_1 h_2 \sum_{\xi=0}^{n-1} \sum_{\rho=0}^{m-1} \sum_{a+b=0}^{p-1} \varepsilon_{\xi,\rho}^{0+0} + \lambda_7 h_1 h_2 \sum_{\xi=0}^{n-1} \sum_{a+b=0}^{p-1} \varepsilon_{\xi,m}^{0+0} \\
&+ \lambda_7 h_1 h_2 \sum_{\rho=0}^{m-1} \sum_{a+b=0}^{p-1} \varepsilon_{n,\rho}^{0+0} + \lambda_7 h_1 h_2 \varepsilon_{n,m}^{0+0},
\end{aligned}$$

here for all $i+j=0,\dots,p$

$\lambda_1 = \max\left\{\left\|\partial_t^{(i)}\partial_s^{(j)} f\right\|\right\},$

$\lambda_2 = \max\left\{ \binom{j}{l}\binom{i-1}{\eta} \left\|\partial_t^{(i-1-\eta)}\left[\partial_s^{(j-l)}\kappa_1(t,s)\right]\right\|, \quad \eta = 0,\dots,i-1; l=0,\dots,j \right\},$

$\lambda_3 = \max\left\{ \binom{j-1}{l}\binom{i}{\eta} \left\|\partial_t^{(i-\eta)}\left[\partial_s^{(j-1-l)}\kappa_2(t,s)\right]\right\|, \quad \eta = 0,\dots,i; l=0,\dots,j-1 \right\},$

$\lambda_4 = \max\left\{ \binom{j-1}{l}\binom{i-1}{\eta} \left\|\partial_t^{(i-1-\eta)}\left[\partial_s^{(j-1-l)}\kappa_3(t,s)\right]\right\|, \quad \eta = 0,\dots,i-1; l=0,\dots,j-1 \right\},$

$\lambda_5 = \max\left\{ \frac{1}{a!b!}\left\|\kappa_1(t,s)(t-t_n)^a(s-s_m)^b\right\|, a+b=0,\dots,p-1 \right\},$

$\lambda_6 = \max\left\{ \frac{1}{a!b!}\left\|\kappa_2(t,s)(t-t_n)^a(s-s_m)^b\right\|, a+b=0,\dots,p-1 \right\},$

and

$\lambda_7 = \max\left\{ \frac{1}{a!b!}\left\|\kappa_3(t,s)(t-t_n)^a(s-s_m)^b\right\|, a+b=0,\dots,p-1 \right\},$

the constants $\lambda_i, i=1,\dots,7$ are positive and independent of N and M.
Hence, for all $i+j=0,\dots,p$

$$\begin{aligned}
\varepsilon_{n,m}^{i+j} \le{}& \lambda_1 + \lambda_5 h_1 \sum_{\xi=0}^{n-1}\sum_{a+b=0}^{p-1} \varepsilon_{\xi,m}^{a+b} + \lambda_2 \sum_{a+b=0}^{i+j-1} \varepsilon_{n,m}^{a+b} + \lambda_5 h_1 \sum_{a+b=0}^{i+j-1} \varepsilon_{n,m}^{a+b} \\
&+ \lambda_6 h_2 \sum_{\rho=0}^{m-1}\sum_{a+b=0}^{p-1} \varepsilon_{n,\rho}^{a+b} + \lambda_3 \sum_{a+b=0}^{i+j-1} \varepsilon_{n,m}^{a+b} + \lambda_6 h_2 \sum_{a+b=0}^{i+j-1} \varepsilon_{n,m}^{a+b} \\
&+ \lambda_7 h_1 h_2 \sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1}\sum_{a+b=0}^{p-1} \varepsilon_{\xi,\rho}^{a+b} + \lambda_7 h_1 h_2 \sum_{\xi=0}^{n-1}\sum_{a+b=0}^{p-1} \varepsilon_{\xi,m}^{a+b} + \lambda_7 h_1 h_2 \sum_{\rho=0}^{m-1}\sum_{a+b=0}^{p-1} \varepsilon_{n,\rho}^{a+b} \\
&+ \lambda_4 \sum_{a+b=0}^{i+j-1} \varepsilon_{n,m}^{a+b} + \lambda_7 h_1 h_2 \sum_{a+b=0}^{i+j-1} \varepsilon_{n,m}^{a+b}.
\end{aligned} \tag{2.11}$$

Consider, the sequence $\Psi_{n,m} = \max\{\varepsilon^{i+j}_{n,m}, i+j = 0,\dots,p\}, n = 0,\dots,N-1; m = 0,\dots,M-1$. Then, based on equation (2.11), we have

$$\varepsilon^{i+j}_{n,m} \leq \lambda_1 + h_1 b_1 \sum_{\xi=0}^{n-1} \Psi_{\xi,m} + h_2 b_2 \sum_{\rho=0}^{m-1} \Psi_{n,\rho} + h_1 h_2 b_3 \sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1} \Psi_{\xi,\rho} + b_4 \sum_{a+b=0}^{i+j-1} \varepsilon^{a+b}_{n,m}, \tag{2.12}$$

where $b_1 = (\lambda_5+\lambda_7 B)p^2, b_2 = (\lambda_6+\lambda_7 A)p^2, b_3 = \lambda_7 p^2$ and $b_4 = \lambda_2+\lambda_5 A+\lambda_3+\lambda_6 B+\lambda_4+\lambda_7 AB$. By applying Lemma 1.7.1 with the following notation:

$$\omega_{i+j} = \varepsilon^{i+j}_{n,m}, \quad p_0 = \lambda_1 + h_1 b_1 \sum_{\xi=0}^{n-1} \Psi_{\xi,m} + h_2 b_2 \sum_{\rho=0}^{m-1} \Psi_{n,\rho} + h_1 h_2 b_3 \sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1} \Psi_{\xi,\rho}, \quad \kappa_{a+b} = b_4,$$

we derive from (2.12)

$$\begin{aligned} \varepsilon^{i+j}_{n,m} \leq{}& \underbrace{\lambda_1 \exp(pb_4)}_{c_1} + h_1 \underbrace{b_1 \exp(pb_4)}_{c_2} \sum_{\xi=0}^{n-1} \Psi_{\xi,m} + h_2 \underbrace{b_2 \exp(pb_4)}_{c_3} \sum_{\rho=0}^{m-1} \Psi_{n,\rho} \\ &+ h_1 h_2 \underbrace{b_3 \exp(pb_4)}_{c_4} \sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1} \Psi_{\xi,\rho}, \end{aligned}$$

therefore,

$$\Psi_{n,m} \leq c_1 + h_1 c_2 \sum_{\xi=0}^{n-1} \Psi_{\xi,m} + h_2 c_3 \sum_{\rho=0}^{m-1} \Psi_{n,\rho} + h_1 h_2 c_4 \sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1} \Psi_{\xi,\rho}.$$

Lemma 1.7.3 allows us to obtain for all $n = 0,\dots,N-1; m = 0,\dots,N-1$

$$\varepsilon^{i+j}_{n,m} \leq \Psi_{n,m} \leq c_1 \exp(\eta_1(A+B)) = \varphi_2(p),$$

where $\eta_1 = \frac{1}{2}\left(c_2 + c_3 + \sqrt{(c_2+c_3)^2 + 4c_4}\right)$.
Thus, the first step is accomplished by establishing

$$\varphi(p) = \max\{\varphi_1(p), \varphi_2(p)\}.$$

Step 2. It will be shown that there exists a constant ς , which is independent of h_1 and h_2, such that

$$\|e_{n,m}\| \le \varsigma(h_1 + h_2)^p,$$

for all $n = 0, \dots, N-1$; $m = 0, \dots, M-1$.

Initially, consider $(t,s) \in \mathcal{R}_{0,0}$. Utilizing Lemma 1.1.1, we derive from (2.6) the following result:

$$|e_{0,0}(t,s)| \le \sum_{i+j=p} \frac{1}{i!j!} \left\|\partial_t^{(i)}\partial_s^{(j)} w\right\| h_1^i h_2^j,$$

hence,

$$\|e_{0,0}\| \le \alpha(p) \sum_{i+j=p} \frac{1}{i!j!} h_1^i h_2^j = \underbrace{\frac{\varphi(p)}{p!}}_{\varsigma_1} (h_1 + h_2)^p.$$

Next, consider $(t,s) \in \mathcal{R}_{n,m}$, for $n = 0, \dots, N-1$; $m = 0, \dots, M-1$, excluding the case $(n,m) = (0,0)$, we have from (2.9)

$$\begin{aligned}
|w(t,s) - \hat{\vartheta}_{n,m}(t,s)| \le{}& \sum_{\xi=0}^{n-1} h_1 \kappa \|e_{\xi,m}\| + \sum_{\rho=0}^{m-1} h_2 \kappa \|e_{n,\rho}\| \\
&+ \sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1} h_1 h_2 \kappa \|e_{\xi,\rho}\| + \sum_{\xi=0}^{n-1} h_1 h_2 \kappa \|e_{\xi,m}\| + \sum_{\rho=0}^{m-1} h_1 h_2 \kappa \|e_{n,\rho}\| \\
&+ \kappa \int_{t_n}^{t} |w(x,s) - \hat{\vartheta}_{n,m}(x,s)| dx + \kappa \int_{s_m}^{s} |w(t,y) - \hat{\vartheta}_{n,m}(t,y)| dy \\
&+ \kappa \int_{t_n}^{t}\int_{s_m}^{s} |w(x,y) - \hat{\vartheta}_{n,m}(x,y)| dy dx,
\end{aligned}$$

where $\kappa = max\{\|\kappa_i\|_{L^\infty(\mathcal{R})}, i = 1,2,3\}$.

Then, applying Lemma 1.7.6, we derive

$$\begin{aligned}
|w(t,s) - \hat{\vartheta}_{n,m}(t,s)| \le{}& \sum_{\xi=0}^{n-1} h_1 \underbrace{\kappa(1+B)\nu}_{\lambda_8} \|e_{\xi,m}\| + \sum_{\rho=0}^{m-1} h_2 \underbrace{\kappa(1+A)\nu}_{\lambda_9} \|e_{n,\rho}\| \\
&+ \sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1} h_1 h_2 \underbrace{\kappa\nu}_{\lambda_{10}} \|e_{\xi,\rho}\|,
\end{aligned}$$

this implies, through the application of Lemma 1.1.1, that

$$\begin{aligned}
\|e_{n,m}\| &\leq \|w - \hat{\vartheta}_{n,m}\| + \|\hat{\vartheta}_{n,m} - \vartheta_{n,m}\| \\
&\leq \sum_{\xi=0}^{n-1} h_1 \lambda_8 \|e_{\xi,m}\| + \sum_{\rho=0}^{m-1} h_2 \lambda_9 \|e_{n,\rho}\| + \sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1} h_1 h_2 \lambda_{10} \|e_{\xi,\rho}\| \\
&+ \sum_{i+j=p} \frac{1}{i!j!} \left\|\partial_t^{(i)} \partial_s^{(j)} \hat{\vartheta}_{n,m}\right\| h_1^i h_2^j,
\end{aligned}$$

hence,

$$\|e_{n,m}\| \leq h_1 \lambda_8 \sum_{\xi=0}^{n-1} \|e_{\xi,m}\| + h_2 \lambda_9 \sum_{\rho=0}^{m-1} \|e_{n,\rho}\| + h_1 h_2 \lambda_{10} \sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1} \|e_{\xi,\rho}\| + \frac{\varphi(p)}{p!}(h_1 + h_2)^p.$$

Thus, using Lemma 1.7.3, we derive

$$\|e_{n,m}\| \leq \underbrace{\frac{\varphi(p)}{p!} \exp\left(\eta_2 (A + B)\right)}_{\varsigma_2}(h_1 + h_2)^p,$$

where $\eta_2 = \frac{1}{2}\left(\lambda_8 + \lambda_9 + \sqrt{(\lambda_8 + \lambda_9)^2 + 4\lambda_{10}}\right)$.

Consequently, the proof is concluded by setting $\varsigma = \max\{\varsigma_1, \varsigma_2\}$. ■

2.4 Experimental Results

The numerical approach detailed in this chapter is applied to three distinct illustrative examples. For each example, we determine the absolute error $|e_{n,m}| = |w - \vartheta_{n,m}|$ for all $n = 0, \ldots, N-1$ and $m = 0, \ldots, M-1$, where w represents the exact solution and $\vartheta_{n,m}$ denotes the computed approximate solution. These error values provide a basis for a detailed comparison, enabling an assessment of the method's precision and reliability.

Example 2.4.1 *Consider the following linear and homogeneous Goursat problem [20]:*

$$3\partial_x \partial_y w(x,y) = \partial_x w(x,y) + \partial_y w(x,y) + w(x,y), \quad (x,y) \in [0,1] \times [0,1],$$

with initial conditions $w(x,0) = e^x$ *and* $w(0,y) = e^y$. *This equation can be transformed into the following linear two-dimensional Volterra integral equation:*

$$w(t,s) = \frac{2}{3}(e^t + e^s) - \frac{1}{3} + \frac{1}{3}\left(\int_0^t w(x,s)dx + \int_0^s w(t,y)dy + \int_0^t \int_0^s w(x,y)dydx\right),$$

with the exact solution given by $w(t,s) = e^{t+s}$.

Table 2.1 and Figure 2.1 present the numerical findings for $p = 3,4$ *and* $h_1 = h_2 = 0.05, 0.025$ *using the TCM.*

The results in this example confirm the theoretical results and suggest that the experimental order of convergence (EOC) is p, as shown in Table 2.2, using the formula: $EOC = \dfrac{log(e_N/e_{2N})}{log(2)}$.

Table 2.1 – Numerical outcomes of Example 2.4.1

(t,s)	$N = M = 20, p = 3$	$N = M = 20, p = 4$	$N = M = 40, p = 3$
$(0.1, 0.1)$	1.62×10^{-5}	1.57×10^{-5}	4.05×10^{-6}
$(0.2, 0.2)$	3.82×10^{-5}	3.65×10^{-5}	9.47×10^{-6}
$(0.3, 0.3)$	6.67×10^{-5}	6.37×10^{-5}	1.66×10^{-5}
$(0.4, 0.4)$	1.05×10^{-4}	9.89×10^{-5}	2.59×10^{-5}
$(0.5, 0.5)$	1.56×10^{-4}	1.44×10^{-4}	3.80×10^{-5}
$(0.6, 0.6)$	2.21×10^{-4}	2.01×10^{-4}	5.35×10^{-5}
$(0.7, 0.7)$	3.04×10^{-4}	2.74×10^{-4}	7.34×10^{-5}
$(0.8, 0.8)$	4.11×10^{-4}	3.66×10^{-4}	9.86×10^{-5}
$(0.9, 0.9)$	5.48×10^{-4}	4.82×10^{-4}	1.30×10^{-4}
$(1.0, 1.0)$	1.99×10^{-3}	6.39×10^{-4}	3.38×10^{-4}

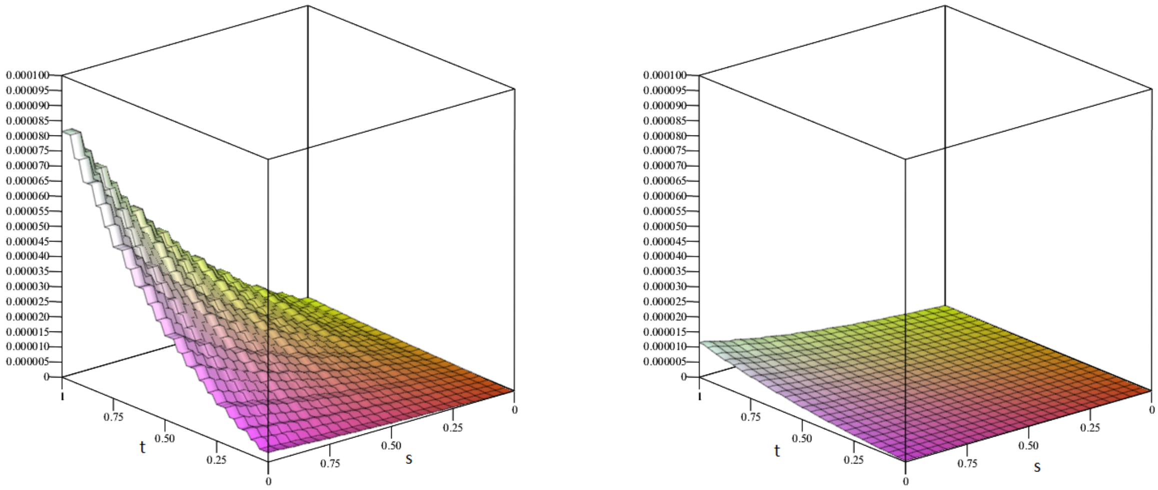
Table 2.2 – Experimental orders of convergence (EOC) for Example 2.4.1

(N, M)	$(2,2)$	$(4,4)$	$(8,8)$	$(16,16)$	$(32,32)$	$(64,64)$
$p = 2$	/	1.45	1.71	1.85	1.94	1.96
$p = 3$	/	2.40	2.69	2.84	2.92	2.96

Example 2.4.2 *Let us consider the following linear non-homogeneous Goursat problem [19]:*

$$\partial_x \partial_y w(x,y) = 4xy - x^2y^2 + w(x,y), \quad (x,y) \in [0,1] \times [0,1],$$

with initial conditions $w(x,0) = e^x$ *and* $w(0,y) = e^y$. *This equation can be transformed into the*



$h_1 = h_2 = 0.05$ $h_1 = h_2 = 0.025$

Figure 2.1 – Plot of the absolute error function for $p = 3$ of Example 2.4.1

following linear two-dimensional Volterra integral equation:

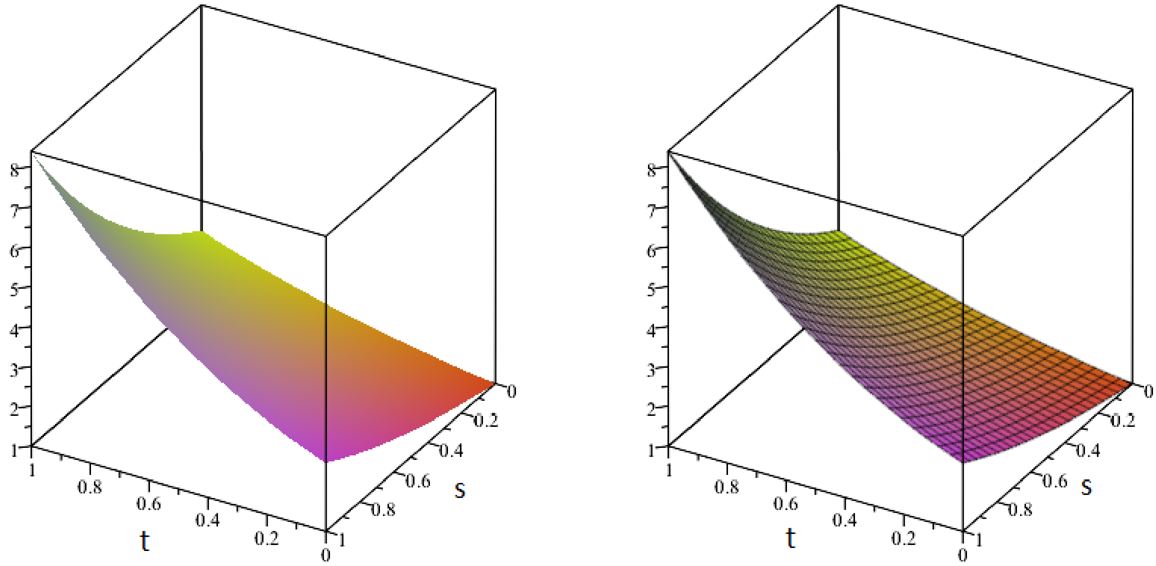
$$w(t,s) = e^t + e^s + t^2s^2 - \frac{t^3s^3}{9} + \int_0^t \int_0^s w(x,y)dydx.$$

The exact solution for this problem is $w(t,s) = t^2s^2 + e^{t+s}$.

Table 2.3 *and Figure* 2.2 *showcase the numerical findings for* $p = 3$ *and* $h_1 = h_2 = 0.05, 0.025$ *using the TCM.*

Table 2.3 – Numerical outcomes of Example 2.4.2

(t,s)	$N = M = 20, p = 3$	$N = M = 40, p = 3$
$(0.1, 0.1)$	3.64×10^{-7}	4.80×10^{-8}
$(0.2, 0.2)$	1.77×10^{-6}	2.29×10^{-7}
$(0.3, 0.3)$	4.77×10^{-6}	6.13×10^{-7}
$(0.4, 0.4)$	1.00×10^{-5}	1.28×10^{-6}
$(0.5, 0.5)$	1.83×10^{-5}	2.32×10^{-6}
$(0.6, 0.6)$	3.07×10^{-5}	3.93×10^{-6}
$(0.7, 0.7)$	4.85×10^{-5}	6.21×10^{-6}
$(0.8, 0.8)$	7.32×10^{-5}	9.36×10^{-6}
$(0.9, 0.9)$	1.06×10^{-4}	1.36×10^{-5}
$(1.0, 1.0)$	1.76×10^{-3}	2.28×10^{-4}



(a) Exact solution

(b) Approximate solution

Figure 2.2 – Numerical results of Example 2.4.2

Example 2.4.3 *Let us examine the following partial differential equation with variable coefficients:*

$$\partial_x \partial_y w(x,y) = (x+y^2)\partial_x w(x,y) + (x^2+y)\partial_y w(x,y) + xyw(x,y) + g(x,y), \quad (x,y) \in [0,1]\times[0,1],$$

with initial conditions $w(x,0) = cos(x) + e^x$ *and* $w(0,y) = cos(y) + 1 + y^2$*. This equation can be transformed into the following linear two-dimensional Volterra integral equation:*

$$w(t,s) = f(t,s) + \int_0^t (x^2+s)w(x,s)dx + \int_0^s (t+y^2)w(t,y)dy + \int_0^t \int_0^s (-2-xy)w(x,y)dydx,$$

where $f(t,s)$ *is chosen in such a way that the exact solution becomes* $w(t,s) = cos(t+s) + e^t + s^2$*. The numerical results obtained using the TCM for* $p = 3$ *and* $h_1 = h_2 = 0.05, 0.025$ *are shown in Table 2.4.*

Table 2.4 – Numerical outcomes of Example 2.4.3

(t,s)	$N = M = 20, p = 3$	$N = M = 40, p = 3$
$(0.1, 0.1)$	1.32×10^{-5}	1.66×10^{-6}
$(0.2, 0.2)$	2.84×10^{-5}	3.73×10^{-6}
$(0.3, 0.3)$	4.72×10^{-5}	6.54×10^{-6}
$(0.4, 0.4)$	7.13×10^{-5}	1.05×10^{-5}
$(0.5, 0.5)$	1.03×10^{-4}	1.62×10^{-5}
$(0.6, 0.6)$	1.46×10^{-4}	2.44×10^{-5}
$(0.7, 0.7)$	2.07×10^{-4}	3.64×10^{-5}
$(0.8, 0.8)$	2.98×10^{-4}	5.47×10^{-5}
$(0.9, 0.9)$	4.44×10^{-4}	8.47×10^{-5}
$(1.0, 1.0)$	1.04×10^{-3}	1.89×10^{-4}

2.5 Conclusion

This chapter offered a collocation method utilizing Taylor polynomials to solve a two-dimensional linear Volterra integral equation of the second kind, which is derived from converting a hyperbolic linear PDE Goursat problem. Various numerical examples were provided to demonstrate the method's accuracy and efficiency. The analysis of convergence and error was performed, showing that the numerical results aligned with theoretical estimations. These findings indicate that the method exhibits a high level of precision in convergence.

CHAPTER 3

TAYLOR COLLOCATION METHOD FOR SOLVING 2D-FIRST KIND VOLTERRA INTEGRAL EQUATIONS

3.1 Introduction

A standard form for the nonlinear 2D-VIEs of the first kind, which incorporate an unknown function ν, is as follows:

$$\int_0^t \int_0^s \kappa(t,s,x,y)G(\nu(x,y))dydx = h(t,s), \quad (t,s) \in \mathcal{R}, \tag{3.1}$$

where $\mathcal{R} = [0,A] \times [0,B]$, h and κ are smooth functions on their corresponding domains, the inverse function G is also continuous and nonlinear with respect to ν. Equation (3.1) is solved by substituting $w(x,y) = G(\nu(x,y))$, yielding the linear equation

$$\int_0^t \int_0^s \kappa(t,s,x,y)w(x,y)dydx = h(t,s), \quad (t,s) \in \mathcal{R}. \tag{3.2}$$

We differentiate equation (3.2) with respect to s and t in order to convert the first-kind Volterra integral equation (3.2) into the second-kind Volterra integral equation (3.3). This conversion process is only useful when $h(t,0) = h(0,s) = 0$ and $\kappa(t,s,t,s) \neq 0$ for $(t,s) \in \mathcal{R}$. The resulting linear 2D-VIE requires the following form:

$$\begin{aligned} w(t,s) &= f(t,s) + \int_0^t \kappa_1(t,s,x)w(x,s)dx + \int_0^s \kappa_2(t,s,y)w(t,y)dy \\ &+ \int_0^t \int_0^s \kappa_3(t,s,x,y)w(x,y)dydx, \quad (t,s) \in \mathcal{R}, \end{aligned} \tag{3.3}$$

where the functions f, κ_1, κ_2 and κ_3 are given smooth functions defined, respectively, on $\mathcal{R}$, $S_1 := \{(t,s,x) : 0 \leq x \leq t \leq A, 0 \leq s \leq B\}$, $S_2 := \{(t,s,y) : 0 \leq t \leq A, 0 \leq y \leq s \leq B\}$, and $S_3 := \{(t,s,x,y) : 0 \leq x \leq t \leq A, 0 \leq y \leq s \leq B\}$ by:

$$\kappa_1(t,s,x) := -\partial_t \kappa(t,s,x,s)/\kappa(t,s,t,s), \quad \kappa_2(t,s,y) := -\partial_s \kappa(t,s,t,y)/\kappa(t,s,t,s),$$
$$\kappa_3(t,s,x,y) := -\partial_t \partial_s \kappa(t,s,x,y)/\kappa(t,s,t,s), \quad f(t,s) := \partial_t \partial_s h(t,s)/\kappa(t,s,t,s).$$

For (3.1), the approximate solution is $G^{-1}(w(x,y)) = \nu(x,y)$, and which was published in the reference [9]. The existence and uniqueness of the solution for equation (3.1), using

$G(v(x,y)) = w(x,y)$ and equation (3.3), have been explored in works such as [42, 51].

The equation (3.1) has motivated mathematicians to develop reliable methods for its solution. In [40], a method based on applying two-dimensional block-pulse functions and a hybrid of block-pulse functions was utilized to solve nonlinear 2D-VIEs of the first kind. An Euler-type technique was discussed in [42]. The Chelyshkov polynomial strategy for solving 2D-NVIEs of the first kind was considered in [38]. In [57], the Tau technique was employed to approximate the solution of linear 2D-VIEs of the first kind. Nemati, S. et al. [49] used operational matrices of Legendre polynomials to approximate the solution of a class of nonlinear 2D-VIEs of the first kind, specifically when $G = v^n$ and n is a positive integer. In [59], a multi-step method was implemented for the first kind's numerical solution of nonlinear 2D-VIEs.

This chapter is organized as follows: Section 3.3 discusses the convergence analysis of our method, Section 3.4 shows the validity of our theoretical results through a number of numerical examples, and Section 3.5 presents our conclusions. The following section describes our method for approximating the solution of equation (3.3) using Taylor polynomials.

3.2 Description of the Method

In this section, solutions of 2D-VIE (3.3) are approximated in the space $\mathcal{S}_{p-1}^{(-1)}(\Pi_{N,M})$, as defined in (2.5), the collocation solution is formulated using Taylor polynomials over each rectangle $\mathcal{R}_{n,m}$ where $n = 0, \ldots, N-1$ and $m = 0, \ldots, M-1$. Furthermore, it is observed that $w(0,0) = f(0,0)$.

3.2.1 Approximate solution in $\mathcal{R}_{0,0}$

We approximate w within the rectangle $\mathcal{R}_{0,0}$ using the polynomial

$$\vartheta_{0,0}(t,s) = \sum_{i+j=0}^{p-1} \frac{1}{i!j!} \partial_t^{(i)} \partial_s^{(j)} w(0,0) t^i s^j \; ; \quad (t,s) \in \mathcal{R}_{0,0}, \tag{3.4}$$

by differentiating equation (3.3) j-times with respect to s and i-times with respect to t, we get

$$\begin{aligned}
&\partial_t^{(i)} \partial_s^{(j)} w(t,s) = \partial_t^{(i)} \partial_s^{(j)} f(t,s) \\
&+ \sum_{l=0}^{j} \sum_{q=0}^{i-1} \sum_{\eta=0}^{q} \binom{j}{l} \binom{q}{\eta} \partial_t^{(q-\eta)} \left[\partial_t^{(i-1-q)} \big|_{x=t} \partial_s^{(j-l)} \kappa_1(t,s,x) \right] \partial_t^{(\eta)} \partial_s^{(l)} w(t,s) \\
&+ \sum_{l=0}^{j} \binom{j}{l} \int_0^t \partial_t^{(i)} \big|_{x=t} \partial_s^{(j-l)} \kappa_1(t,s,x) \partial_s^{(l)} w(x,s) dx \\
&+ \sum_{r=0}^{j-1} \sum_{l=0}^{r} \sum_{\eta=0}^{i} \binom{r}{l} \binom{i}{\eta} \partial_t^{(i-\eta)} \left[\partial_s^{(r-l)} \left(\partial_s^{(j-1-r)} \big|_{y=s} \kappa_2(t,s,y) \right) \right] \partial_t^{(\eta)} \partial_s^{(l)} w(t,s) \\
&+ \sum_{\eta=0}^{i} \binom{i}{\eta} \int_0^s \partial_t^{(i-\eta)} \partial_s^{(j)} \kappa_2(t,s,y) \partial_t^{(\eta)} w(t,y) dy \\
&+ \sum_{r=0}^{j-1} \sum_{l=0}^{r} \sum_{q=0}^{i-1} \sum_{\eta=0}^{q} \binom{r}{l} \binom{q}{\eta} \partial_t^{(q-\eta)} \left[\partial_t^{(i-1-q)} \big|_{x=t} \left(\partial_s^{(r-l)} \left[\partial_s^{(j-1-r)} \big|_{y=s} \kappa_3(t,s,x,y) \right] \right) \right] \partial_t^{(\eta)} \partial_s^{(l)} w(t,s) \\
&+ \sum_{r=0}^{j-1} \sum_{l=0}^{r} \binom{r}{l} \int_0^t \partial_t^{(i)} \left[\partial_s^{(r-l)} [\partial_s^{(j-1-r)} \big|_{y=s} \kappa_3(t,s,x,y)] \right] \partial_s^{(l)} w(x,s) dx \\
&+ \sum_{q=0}^{i-1} \sum_{\eta=0}^{q} \binom{q}{\eta} \int_0^s \partial_t^{(q-\eta)} \left[\partial_t^{(i-1-q)} \big|_{x=t} \partial_s^{(j)} \kappa_3(t,s,x,y) \right] \partial_t^{(\eta)} w(t,y) dy \\
&+ \int_0^t \int_0^s \partial_t^{(i)} \partial_s^{(j)} \kappa_3(t,s,x,y) w(x,y) dy dx.
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\partial_t^{(i)} \partial_s^{(j)} w(0,0) = \partial_t^{(i)} \partial_s^{(j)} f(0,0) \\
&+ \sum_{l=0}^{j} \sum_{q=0}^{i-1} \sum_{\eta=0}^{q} \binom{j}{l} \binom{q}{\eta} \partial_t^{(q-\eta)} \left[\partial_t^{(i-1-q)} \big|_{x=t} \partial_s^{(j-l)} \kappa_1(t,s,x) \right]_{s=0}^{t=0} \partial_t^{(\eta)} \partial_s^{(l)} w(0,0)
\end{aligned}$$

$$+\sum_{r=0}^{j-1}\sum_{l=0}^{r}\sum_{\eta=0}^{i}\binom{r}{l}\binom{i}{\eta}\partial_t^{(i-\eta)}\left[\partial_s^{(r-l)}\left(\partial_s^{(j-1-r)}\Big|_{y=s}\kappa_2(t,s,y)\right)\right]_{s=0}^{t=0}\partial_t^{(\eta)}\partial_s^{(l)}w(0,0)$$
$$+\sum_{r=0}^{j-1}\sum_{l=0}^{r}\sum_{q=0}^{i-1}\sum_{\eta=0}^{q}\binom{r}{l}\binom{q}{\eta}\partial_t^{(q-\eta)}\left[\partial_t^{(i-1-q)}\Big|_{x=t}\left(\partial_s^{(r-l)}\left[\partial_s^{(j-1-r)}\Big|_{y=s}\kappa_3(t,s,x,y)\right]\right)\right]_{s=0}^{t=0}\partial_t^{(\eta)}\partial_s^{(l)}w(0,0).$$

3.2.2 Approximate solution in $\mathcal{R}_{n,0}$

The function w is approximated by $\vartheta_{n,0}$ within the rectangles $\mathcal{R}_{n,0}$, for $n = 1,\ldots,N-1$, using polynomials

$$\vartheta_{n,0}(t,s) = \sum_{i+j=0}^{p-1}\frac{1}{i!j!}\partial_t^{(i)}\partial_s^{(j)}\hat{\vartheta}_{n,0}(t_n,0)(t-t_n)^i s^j\ ; \quad (t,s)\in\mathcal{R}_{n,0}, \tag{3.5}$$

where $\hat{\vartheta}_{n,0}$ represents the exact solution of the following integral equation:

$$\begin{aligned}\hat{\vartheta}_{n,0}(t,s) = f(t,s) &+ \int_0^s \kappa_2(t,s,y)\hat{\vartheta}_{n,0}(t,y)dy\\ &+\sum_{\xi=0}^{n-1}\int_{t_\xi}^{t_{\xi+1}}\kappa_1(t,s,x)\vartheta_{\xi,0}(x,s)dx+\int_{t_n}^{t}\kappa_1(t,s,x)\hat{\vartheta}_{n,0}(x,s)dx\\ &+\sum_{\xi=0}^{n-1}\int_{t_\xi}^{t_{\xi+1}}\int_0^s\kappa_3(t,s,x,y)\vartheta_{\xi,0}(x,y)dydx+\int_{t_n}^{t}\int_0^s\kappa_3(t,s,x,y)\hat{\vartheta}_{n,0}(x,y)dydx.\end{aligned} \tag{3.6}$$

Differentiating equation (3.6) j-times with respect to s and i-times with respect to t, we derive

$$\partial_t^{(i)}\partial_s^{(j)}\hat{\vartheta}_{n,0}(t,s) = \partial_t^{(i)}\partial_s^{(j)}f(t,s)$$
$$+\sum_{\xi=0}^{n-1}\sum_{l=0}^{j}\binom{j}{l}\int_{t_\xi}^{t_{\xi+1}}\partial_t^{(i)}\partial_s^{(j-l)}\kappa_1(t,s,x)\partial_s^{(l)}\vartheta_{\xi,0}(x,s)dx$$

$$
\begin{aligned}
&+\sum_{l=0}^{j}\sum_{q=0}^{i-1}\sum_{\eta=0}^{q}\binom{j}{l}\binom{q}{\eta}\partial_t^{(q-\eta)}\left[\partial_t^{(i-1-q)}\Big|_{x=t}\partial_s^{(j-l)}\kappa_1(t,s,x)\right]\partial_t^{(\eta)}\partial_s^{(l)}\hat{\vartheta}_{n,0}(t,s)\\
&+\sum_{l=0}^{j}\binom{j}{l}\int_{t_n}^{t}\partial_t^{(i)}\partial_s^{(j-l)}\kappa_1(t,s,x)\partial_s^{(l)}\hat{\vartheta}_{n,0}(x,s)dx\\
&+\sum_{r=0}^{j-1}\sum_{l=0}^{r}\sum_{\eta=0}^{i}\binom{r}{l}\binom{i}{\eta}\partial_t^{(i-\eta)}\left[\partial_s^{(r-l)}[\partial_s^{(j-1-r)}\Big|_{y=s}\kappa_2(t,s,y)]\right]\partial_t^{(\eta)}\partial_s^{(l)}\hat{\vartheta}_{n,0}(t,s)\\
&+\sum_{\eta=0}^{i}\binom{i}{\eta}\int_{0}^{s}\partial_t^{(i-\eta)}\partial_s^{(j)}\kappa_2(t,s,y)\partial_t^{(\eta)}\hat{\vartheta}_{n,0}(t,y)dy\\
&+\sum_{\xi=0}^{n-1}\sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\int_{t_\xi}^{t_{\xi+1}}\partial_t^{(i)}\left[\partial_s^{(r-l)}[\partial_s^{(j-1-r)}\Big|_{y=s}\kappa_3(t,s,x,y)]\right]\partial_s^{(l)}\vartheta_{\xi,0}(x,s)dx\\
&+\sum_{\xi=0}^{n-1}\int_{t_\xi}^{t_{\xi+1}}\int_{0}^{s}\partial_t^{(i)}\partial_s^{(j)}\kappa_3(t,s,x,y)\vartheta_{\xi,0}(x,y)dydx\\
&+\sum_{r=0}^{j-1}\sum_{l=0}^{r}\sum_{q=0}^{i-1}\sum_{\eta=0}^{q}\binom{r}{l}\binom{q}{\eta}\partial_t^{(q-\eta)}\left[\partial_t^{(i-1-q)}\Big|_{x=t}\left(\partial_s^{(r-l)}\left[\partial_s^{(j-1-r)}\Big|_{y=s}\kappa_3(t,s,x,y)\right]\right)\right]\partial_t^{(\eta)}\partial_s^{(l)}\hat{\vartheta}_{n,0}(t,s)\\
&+\sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\int_{t_n}^{t}\partial_t^{(i)}\left[\partial_s^{(r-l)}[\partial_s^{(j-1-r)}\Big|_{y=s}\kappa_3(t,s,x,y)]\right]\partial_s^{(l)}\hat{\vartheta}_{n,0}(x,s)dx\\
&+\sum_{q=0}^{i-1}\sum_{\eta=0}^{q}\binom{q}{\eta}\int_{0}^{s}\partial_t^{(q-\eta)}\left[\partial_t^{(i-1-q)}\ \partial_s^{(j)}\Big|_{x=t}\kappa_3(t,s,x,y)\right]\partial_t^{(\eta)}\hat{\vartheta}_{n,0}(t,y)dy\\
&+\int_{t_n}^{t}\int_{0}^{s}\partial_t^{(i)}\partial_s^{(j)}\kappa_3(t,s,x,y)\hat{\vartheta}_{n,0}(x,y)dydx.
\end{aligned}\tag{3.7}
$$

Consequently,

$$
\begin{aligned}
\partial_t^{(i)}\partial_s^{(j)}\hat{\vartheta}_{n,0}(t_n,0) &= \partial_t^{(i)}\partial_s^{(j)}f(t_n,0)\\
&+\sum_{\xi=0}^{n-1}\sum_{l=0}^{j}\binom{j}{l}\int_{t_\xi}^{t_{\xi+1}}\left[\partial_t^{(i)}\partial_s^{(j-l)}\kappa_1(t,s,x)\right]_{s=0}^{t=t_n}\partial_s^{(l)}\vartheta_{\xi,0}(x,0)dx\\
&+\sum_{l=0}^{j}\sum_{q=0}^{i-1}\sum_{\eta=0}^{q}\binom{j}{l}\binom{q}{\eta}\partial_t^{(q-\eta)}\left[\partial_t^{(i-1-q)}\Big|_{x=t}\partial_s^{(j-l)}\kappa_1(t,s,x)\right]_{s=0}^{t=t_n}\partial_t^{(\eta)}\partial_s^{(l)}\hat{\vartheta}_{n,0}(t_n,0)
\end{aligned}
$$

$$+\sum_{r=0}^{j-1}\sum_{l=0}^{r}\sum_{\eta=0}^{i}\binom{r}{l}\binom{i}{\eta}\partial_t^{(i-\eta)}\left[\partial_s^{(r-l)}[\partial_s^{(j-1-r)}\big|_{y=s}\kappa_2(t,s,y)]\right]_{s=0}^{t=t_n}\partial_t^{(\eta)}\partial_s^{(l)}\hat{\vartheta}_{n,0}(t_n,0)$$
$$+\sum_{\xi=0}^{n-1}\sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\int_{t_\xi}^{t_{\xi+1}}\partial_t^{(i)}\left[\partial_s^{(r-l)}[\partial_s^{(j-1-r)}\big|_{y=s}\kappa_3(t,s,x,y)]\right]_{s=0}^{t=t_n}\partial_s^{(l)}\vartheta_{\xi,0}(x,0)dx$$
$$+\sum_{r=0}^{j-1}\sum_{l=0}^{r}\sum_{q=0}^{i-1}\sum_{\eta=0}^{q}\binom{r}{l}\binom{q}{\eta}\partial_t^{(q-\eta)}\left[\partial_t^{(i-1-q)}\big|_{x=t}\left(\partial_s^{(r-l)}\left[\partial_s^{(j-1-r)}\big|_{y=s}\kappa_3(t,s,x,y)\right]\right)\right]_{s=0}^{t=t_n}\partial_t^{(\eta)}\partial_s^{(l)}\hat{\vartheta}_{n,0}(t_n,0).$$

3.2.3 Approximate solution in $\mathcal{R}_{n,m}$

The function w is approximated by $\vartheta_{n,m}$ within the rectangles $\mathcal{R}_{n,m}$, for $n = 0,\ldots,N-1$ and $m = 1,\ldots,M-1$, using polynomials

$$\vartheta_{n,m}(t,s) = \sum_{i+j=0}^{p-1}\frac{1}{i!j!}\partial_t^{(i)}\partial_s^{(j)}\hat{\vartheta}_{n,m}(t_n,s_m)(t-t_n)^i(s-s_m)^j\ ;\quad (t,s)\in\mathcal{R}_{n,m}, \tag{3.8}$$

in which $\hat{\vartheta}_{n,m}$ denotes the exact solution to the integral equation provided below:

$$\begin{aligned}\hat{\vartheta}_{n,m}(t,s) = f(t,s) &+ \sum_{\xi=0}^{n-1}\int_{t_\xi}^{t_{\xi+1}}\kappa_1(t,s,x)\vartheta_{\xi,m}(x,s)dx + \int_{t_n}^{t}\kappa_1(t,s,x)\hat{\vartheta}_{n,m}(x,s)dx\\
&+\sum_{\rho=0}^{m-1}\int_{s_\rho}^{s_{\rho+1}}\kappa_2(t,s,y)\vartheta_{n,\rho}(t,y)dy + \int_{s_m}^{s}\kappa_2(t,s,y)\hat{\vartheta}_{n,m}(t,y)dy\\
&+\sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1}\int_{t_\xi}^{t_{\xi+1}}\int_{s_\rho}^{s_{\rho+1}}\kappa_3(t,s,x,y)\vartheta_{\xi,\rho}(x,y)dydx\\
&+\sum_{\xi=0}^{n-1}\int_{t_\xi}^{t_{\xi+1}}\int_{s_m}^{s}\kappa_3(t,s,x,y)\vartheta_{\xi,m}(x,y)dydx\\
&+\sum_{\rho=0}^{m-1}\int_{t_n}^{t}\int_{s_\rho}^{s_{\rho+1}}\kappa_3(t,s,x,y)\vartheta_{n,\rho}(x,y)dydx\\
&+\int_{t_n}^{t}\int_{s_m}^{s}\kappa_3(t,s,x,y)\hat{\vartheta}_{n,m}(x,y)dydx.\end{aligned} \tag{3.9}$$

By differentiating equation (3.9) j-times with respect to s and i-times with respect to t, we obtain

$$
\begin{aligned}
\partial_t^{(i)}\partial_s^{(j)}\hat{\vartheta}_{n,m}(t,s) &= \partial_t^{(i)}\partial_s^{(j)} f(t,s) + \sum_{\xi=0}^{n-1}\sum_{l=0}^{j}\binom{j}{l}\int_{t_\xi}^{t_{\xi+1}} \partial_t^{(i)}\partial_s^{(j-l)}\kappa_1(t,s,x)\partial_s^{(l)}\vartheta_{\xi,m}(x,s)dx \\
&+ \sum_{l=0}^{j}\sum_{q=0}^{i-1}\sum_{\eta=0}^{q}\binom{j}{l}\binom{q}{\eta}\partial_t^{(q-\eta)}\left[\partial_t^{(i-1-q)}\Big|_{x=t}[\partial_s^{(j-l)}\kappa_1(t,s,x)]\right]\partial_t^{(\eta)}\partial_s^{(l)}\hat{\vartheta}_{n,m}(t,s) \\
&+ \sum_{l=0}^{j}\binom{j}{l}\int_{t_n}^{t}\partial_t^{(i)}\partial_s^{(j-l)}\kappa_1(t,s,x)\partial_s^{(l)}\hat{\vartheta}_{n,m}(x,s)dx \\
&+ \sum_{\rho=0}^{m-1}\sum_{\eta=0}^{i}\binom{i}{\eta}\int_{s_\rho}^{s_{\rho+1}}\partial_t^{(i-\eta)}\partial_s^{(j)}\kappa_2(t,s,y)\partial_t^{(\eta)}\vartheta_{n,\rho}(t,y)dy \\
&+ \sum_{r=0}^{j-1}\sum_{l=0}^{r}\sum_{\eta=0}^{i}\binom{r}{l}\binom{i}{\eta}\partial_t^{(i-\eta)}\left[\partial_s^{(r-l)}[\partial_s^{(j-1-r)}\Big|_{y=s}\kappa_2(t,s,y)]\right]\partial_t^{(\eta)}\partial_s^{(l)}\hat{\vartheta}_{n,m}(t,s) \\
&+ \sum_{\eta=0}^{i}\binom{i}{\eta}\int_{s_m}^{s}\partial_t^{(i-\eta)}\partial_s^{(j)}\kappa_2(t,s,y)\partial_t^{(\eta)}\hat{\vartheta}_{n,m}(t,y)dy \\
&+ \sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1}\int_{t_\xi}^{t_{\xi+1}}\int_{s_\rho}^{s_{\rho+1}}\partial_t^{(i)}\partial_s^{(j)}\kappa_3(t,s,x,y)\vartheta_{\xi,\rho}(x,y)dydx \\
&+ \sum_{\xi=0}^{n-1}\sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\int_{t_\xi}^{t_{\xi+1}}\partial_t^{(i)}\left[\partial_s^{(r-l)}[\partial_s^{(j-1-r)}\Big|_{y=s}\kappa_3(t,s,x,y)]\right]\partial_s^{(l)}\vartheta_{\xi,m}(x,s)dx \\
&+ \sum_{\xi=0}^{n-1}\int_{t_\xi}^{t_{\xi+1}}\int_{s_m}^{s}\partial_t^{(i)}\partial_s^{(j)}\kappa_3(t,s,x,y)\vartheta_{\xi,m}(x,y)dydx \\
&+ \sum_{\rho=0}^{m-1}\sum_{q=0}^{i-1}\sum_{\eta=0}^{q}\binom{q}{\eta}\int_{s_\rho}^{s_{\rho+1}}\partial_t^{(q-\eta)}\left[\partial_t^{(i-1-q)}\Big|_{x=t}\partial_s^{(j)}\kappa_3(t,s,x,y)\right]\partial_t^{(\eta)}\vartheta_{n,\rho}(t,y)dy \\
&+ \sum_{\rho=0}^{m-1}\int_{t_n}^{t}\int_{s_\rho}^{s_{\rho+1}}\partial_t^{(i)}\partial_s^{(j)}\kappa_3(t,s,x,y)\vartheta_{n,\rho}(x,y)dydx \\
&+ \sum_{r=0}^{j-1}\sum_{l=0}^{r}\sum_{q=0}^{i-1}\sum_{\eta=0}^{q}\binom{r}{l}\binom{q}{\eta}\partial_t^{(q-\eta)}\left[\partial_t^{(i-1-q)}\Big|_{x=t}\left(\partial_s^{(r-l)}\left[\partial_s^{(j-1-r)}\Big|_{y=s}\kappa_3(t,s,x,y)\right]\right)\right]\partial_t^{(\eta)}\partial_s^{(l)}\hat{\vartheta}_{n,m}(t,s) \\
&+ \sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\int_{t_n}^{t}\partial_t^{(i)}\left[\partial_s^{(r-l)}[\partial_s^{(j-1-r)}\Big|_{y=s}\kappa_3(t,s,x,y)]\right]\partial_s^{(l)}\hat{\vartheta}_{n,m}(x,s)dx
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{q=0}^{i-1}\sum_{\eta=0}^{q}\binom{q}{\eta}\int_{s_m}^{s}\partial_t^{(q-\eta)}\left[\partial_t^{(i-1-q)}\Big|_{x=t}\partial_s^{(j)}\kappa_3(t,s,x,y)\right]\partial_t^{(\eta)}\hat{\vartheta}_{n,m}(t,y)dy\\
&+\int_{t_n}^{t}\int_{s_m}^{s}\partial_t^{(i)}\partial_s^{(j)}\kappa_3(t,s,x,y)\hat{\vartheta}_{n,m}(x,y)dydx.
\end{aligned}
\tag{3.10}
$$

Consequently,

$$
\begin{aligned}
\partial_t^{(i)}\partial_s^{(j)}\hat{\vartheta}_{n,m}(t_n,s_m) &= \partial_t^{(i)}\partial_s^{(j)}f(t_n,s_m)+\sum_{\xi=0}^{n-1}\sum_{l=0}^{j}\binom{j}{l}\int_{t_\xi}^{t_{\xi+1}}\partial_t^{(i)}\partial_s^{(j-l)}\kappa_1(t_n,s_m,x)\partial_s^{(l)}\vartheta_{\xi,m}(x,s_m)dx\\
&+\sum_{l=0}^{j}\sum_{q=0}^{i-1}\sum_{\eta=0}^{q}\binom{j}{l}\binom{q}{\eta}\partial_t^{(q-\eta)}\left[\partial_t^{(i-1-q)}\Big|_{x=t}[\partial_s^{(j-l)}\kappa_1(t,s,x)]\right]_{s=s_m}^{t=t_n}\partial_t^{(\eta)}\partial_s^{(l)}\hat{\vartheta}_{n,m}(t_n,s_m)\\
&+\sum_{\rho=0}^{m-1}\sum_{\eta=0}^{i}\binom{i}{\eta}\int_{s_\rho}^{s_{\rho+1}}\partial_t^{(i-\eta)}\partial_s^{(j)}\kappa_2(t_n,s_m,y)\partial_t^{(\eta)}\vartheta_{n,\rho}(t_n,y)dy\\
&+\sum_{r=0}^{j-1}\sum_{l=0}^{r}\sum_{\eta=0}^{i}\binom{r}{l}\binom{i}{\eta}\partial_t^{(i-\eta)}\left[\partial_s^{(r-l)}[\partial_s^{(j-1-r)}\Big|_{y=s}\kappa_2(t,s,y)]\right]_{s=s_m}^{t=t_n}\partial_t^{(\eta)}\partial_s^{(l)}\hat{\vartheta}_{n,m}(t_n,s_m)\\
&+\sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1}\int_{t_\xi}^{t_{\xi+1}}\int_{s_\rho}^{s_{\rho+1}}\partial_t^{(i)}\partial_s^{(j)}\kappa_3(t_n,s_m,x,y)\vartheta_{\xi,\rho}(x,y)dydx\\
&+\sum_{\xi=0}^{n-1}\sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\int_{t_\xi}^{t_{\xi+1}}\partial_t^{(i)}\left[\partial_s^{(r-l)}[\partial_s^{(j-1-r)}\Big|_{y=s}\kappa_3(t,s,x,y)]\right]_{s=s_m}^{t=t_n}\partial_s^{(l)}\vartheta_{\xi,m}(x,s_m)dx\\
&+\sum_{\rho=0}^{m-1}\sum_{q=0}^{i-1}\sum_{\eta=0}^{q}\binom{q}{\eta}\int_{s_\rho}^{s_{\rho+1}}\partial_t^{(q-\eta)}\left[\partial_t^{(i-1-q)}\Big|_{x=t}\partial_s^{(j)}\kappa_3(t,s,x,y)\right]_{s=s_m}^{t=t_n}\partial_t^{(\eta)}\vartheta_{n,\rho}(t_n,y)dy\\
&+\sum_{r=0}^{j-1}\sum_{l=0}^{r}\sum_{q=0}^{i-1}\sum_{\eta=0}^{q}\binom{r}{l}\binom{q}{\eta}\partial_t^{(q-\eta)}\left[\partial_t^{(i-1-q)}\Big|_{x=t}\left(\partial_s^{(r-l)}\left[\partial_s^{(j-1-r)}\Big|_{y=s}\kappa_3(t,s,x,y)\right]\right)\right]_{s=s_m}^{t=t_n}\partial_t^{(\eta)}\partial_s^{(l)}\hat{\vartheta}_{n,m}(t_n,s_m).
\end{aligned}
$$

3.3 Error Analysis

The convergence of the proposed method is confirmed by the following theorems.

Theorem 3.3.1 *If κ_1, κ_2, κ_3 and f are functions that can be differentiated p times continuously* *within their respective domains, then the equations (3.4), (3.5), (3.8) establish a unique approximation ϑ in the space $\mathcal{S}_{p-1}^{(-1)}(\Pi_{N,M})$. The associated error function, $e(t,s) = w(t,s) - \vartheta(t,s)$, adheres to the inequality:*

$$\|e\|_{L^\infty(\mathcal{R})} \leq \varsigma(h_1 + h_2)^p,$$

where ς is a finite constant that does not depend on h_1 or h_2.

Proof. The proof of the theorem will consist of two steps:

Step 1. Let us define $\varepsilon_{n,m}^{i+j} = \|\partial_t^{(i)}\partial_s^{(j)}\hat{\vartheta}_{n,m}\|$. It will be shown that there exists a positive constant $\varphi(p)$ such that $\varepsilon_{n,m}^{i+j} \leq \varphi(p)$, for all $n = 0,\ldots,N-1$, $m = 0,\ldots,M-1$ and $i + j = 0,\ldots,p$ where $\hat{\vartheta}_{0,0}(t,s) = w(t,s)$ for $(t,s) \in \mathcal{R}_{0,0}$.

First, we have

$$\varepsilon_{0,0}^{i+j} \leq \max\left\{\left\|\partial_t^{(i)}\partial_s^{(j)}w\right\|, i+j=0,\ldots,p\right\} = \varphi_1(p).$$

Second, for $i + j = 1,\ldots,p$, we obtain from (3.7)

$$\begin{aligned}
\varepsilon_{n,0}^{i+j} \leq{} & \lambda_1 + \lambda_2\sum_{\xi=0}^{n-1}\sum_{l=0}^{j}\int_{t_\xi}^{t_{\xi+1}}\sum_{a+b=0}^{p-1}\varepsilon_{\xi,0}^{a+b}dx + \lambda_3\sum_{l=0}^{j}\sum_{q=0}^{i-1}\sum_{\eta=0}^{q}\varepsilon_{n,0}^{\eta+l} \\
& + \lambda_2\sum_{l=0}^{j}\int_{t_n}^{t}\varepsilon_{n,0}^{0+l}dx + \lambda_4\sum_{r=0}^{j-1}\sum_{l=0}^{r}\sum_{\eta=0}^{i}\varepsilon_{n,0}^{\eta+l} + \lambda_5\sum_{\eta=0}^{i}\int_0^s\varepsilon_{n,0}^{\eta+0}dy \\
& + \lambda_6\sum_{\xi=0}^{n-1}\sum_{r=0}^{j-1}\sum_{l=0}^{r}\int_{t_\xi}^{t_{\xi+1}}\sum_{a+b=0}^{p-1}\varepsilon_{\xi,0}^{a+b}dx + \lambda_7\sum_{\xi=0}^{n-1}\int_{t_\xi}^{t_{\xi+1}}\int_0^s\sum_{a+b=0}^{p-1}\varepsilon_{\xi,0}^{a+b}dydx \\
& + \lambda_8\sum_{r=0}^{j-1}\sum_{l=0}^{r}\sum_{q=0}^{i-1}\sum_{\eta=0}^{q}\varepsilon_{n,0}^{\eta+l} + \lambda_6\sum_{r=0}^{j-1}\sum_{l=0}^{r}\int_{t_n}^{t}\varepsilon_{n,0}^{0+l}dx \\
& + \lambda_9\sum_{q=0}^{i-1}\sum_{\eta=0}^{q}\int_0^s\varepsilon_{n,0}^{\eta+0}dy + \lambda_7\int_{t_n}^{t}\int_0^s\varepsilon_{n,0}^{0+0}dydx,
\end{aligned}$$

and for $i + j = 0$, we have from (3.6)

$$\varepsilon_{n,0}^{0+0} \leq \lambda_1 + \lambda_2\sum_{\xi=0}^{n-1}\int_{t_\xi}^{t_{\xi+1}}\sum_{a+b=0}^{p-1}\varepsilon_{\xi,0}^{a+b}dx + \lambda_2\int_{t_n}^{t}\varepsilon_{n,0}^{0+0}dx$$

$$+ \lambda_4 \sum_{r=0}^{j-1} \sum_{l=0}^{r} \sum_{\eta=0}^{i} \varepsilon_{n,0}^{\eta+l} + \lambda_5 \int_0^s \varepsilon_{n,0}^{0+0} dy$$
$$+ \lambda_7 \sum_{\xi=0}^{n-1} \int_{t_\xi}^{t_{\xi+1}} \int_0^s \sum_{a+b=0}^{p-1} \varepsilon_{\xi,0}^{a+b} dydx + \lambda_7 \int_{t_n}^{t} \int_0^s \varepsilon_{n,0}^{0+0} dydx,$$

here for all $i + j = 0, \ldots, p$

$$\lambda_1 = \max\left\{ \left\| \partial_t^{(i)} \partial_s^{(j)} f \right\| \right\},$$

$$\lambda_2 = \max\left\{ \binom{j}{l} \frac{1}{a!(b-l)!} \left\| \partial_t^{(i)} \partial_s^{(j-l)} \kappa_1(t,s,x)(x-t_n)^a s^{b-l} \right\|, l = 0, \ldots, j; a+b = 0, \ldots, p-1 \right\},$$

$$\lambda_3 = \max\left\{ \begin{array}{c} \binom{j}{l}\binom{q}{\eta} \left\| \partial_t^{(q-\eta)} \left[\partial_t^{(i-1-q)} \Big|_{x=t} \partial_s^{(j-l)} \kappa_1(t,s,x) \right] \right\|, \eta = 0, \ldots, q; q = 0, \ldots, i-1; \\ l = 0, \ldots, j \end{array} \right\},$$

$$\lambda_4 = \max\left\{ \begin{array}{c} \binom{r}{l}\binom{i}{\eta} \left\| \partial_t^{(i-\eta)} \left[\partial_s^{(r-l)} [\partial_s^{(j-1-r)} \Big|_{y=s} \kappa_2(t,s,y)] \right] \right\|, \eta = 0, \ldots, i; l = 0, \ldots, r; \\ r = 0, \ldots, j-1 \end{array} \right\},$$

$$\lambda_5 = \max\left\{ \binom{i}{\eta} \left\| \partial_t^{(i-\eta)} \partial_s^{(j)} \kappa_2(t,s,y) \right\|, \eta = 0, \ldots, i \right\},$$

$$\lambda_6 = \max\left\{ \begin{array}{c} \binom{r}{l} \frac{1}{a!(b-l)!} \left\| \partial_t^{(i)} \left[\partial_s^{(r-l)} [\partial_s^{(j-1-r)} \Big|_{y=s} \kappa_3(t,s,x,y)] \right] (x-t_n)^a s^{b-l} \right\|, l = 0, \ldots, r; \\ r = 0, \ldots, j-1; a+b = 0, \ldots, p-1 \end{array} \right\},$$

$$\lambda_7 = \max\left\{ \frac{1}{a!b!} \left\| \partial_t^{(i)} \partial_s^{(j)} \kappa_3(t,s,x,y)(x-t_n)^a y^b \right\|, a+b = 0, \ldots, p-1 \right\},$$

$$\lambda_8 = \max\left\{ \begin{array}{c} \binom{r}{l}\binom{q}{\eta} \left\| \partial_t^{(q-\eta)} \left[\partial_t^{(i-1-q)} \Big|_{x=t} \left(\partial_s^{(r-l)} \left[\partial_s^{(j-1-r)} \Big|_{y=s} \kappa_3(t,s,x,y) \right] \right) \right] \right\|, \eta = 0, \ldots, q; \\ q = 0, \ldots, i-1; l = 0, \ldots, r; r = 0, \ldots, j-1 \end{array} \right\},$$

$$\lambda_9 = \max\left\{ \binom{q}{\eta} \left\| \partial_t^{(q-\eta)} \left[\partial_t^{(i-1-q)} \partial_s^{(j)} \Big|_{x=t} \kappa_3(t,s,x,y) \right] \right\|, \eta = 0, \ldots, q; q = 0, \ldots, i-1 \right\},$$

where the constants $\lambda_i, i = 1, \ldots, 9$ are both positive and independent of N and M. Thus, for any $i + j = 0, \ldots, p$, we have

$$
\begin{aligned}
\varepsilon_{n,0}^{i+j} \leq \lambda_1 &+ \lambda_2 h_1 p \sum_{\xi=0}^{n-1} \sum_{a+b=0}^{p-1} \varepsilon_{\xi,0}^{a+b} + \lambda_3 p \sum_{l=0}^{j} \sum_{\eta=0}^{i-1} \varepsilon_{n,0}^{\eta+l} + \lambda_2 h_1 \sum_{l=0}^{j} \varepsilon_{n,0}^{0+l} \\
&+ \lambda_4 p \sum_{l=0}^{j-1} \sum_{\eta=0}^{i} \varepsilon_{n,0}^{\eta+l} + \lambda_5 h_2 \sum_{\eta=0}^{i} \varepsilon_{n,0}^{\eta+0} \\
&+ \lambda_6 h_1 p^2 \sum_{\xi=0}^{n-1} \sum_{a+b=0}^{p-1} \varepsilon_{\xi,0}^{a+b} + \lambda_7 h_1 h_2 \sum_{\xi=0}^{n-1} \sum_{a+b=0}^{p-1} \varepsilon_{\xi,0}^{a+b} + \lambda_8 p^2 \sum_{l=0}^{j-1} \sum_{\eta=0}^{i-1} \varepsilon_{n,0}^{\eta+l} \\
&+ \lambda_6 h_1 p \sum_{l=0}^{j-1} \varepsilon_{n,0}^{0+l} + \lambda_9 h_2 p \sum_{\eta=0}^{i-1} \varepsilon_{n,0}^{\eta+0} + \lambda_7 h_1 h_2 \varepsilon_{n,0}^{0+0},
\end{aligned}
$$

which yields

$$
\varepsilon_{n,0}^{i+j} \leq \lambda_1 + b_1 h_1 \sum_{\xi=0}^{n-1} \sum_{a+b=0}^{p-1} \varepsilon_{\xi,0}^{a+b} + b_2 \sum_{a+b=0}^{i+j-1} \varepsilon_{n,0}^{a+b},
$$

where $b_1 = \lambda_2 p + \lambda_6 p^2 + \lambda_7 B$ and $b_2 = \lambda_3 p + \lambda_2 A + \lambda_4 p + \lambda_5 B + \lambda_8 p^2 + \lambda_6 A p + \lambda_9 B p + \lambda_7 AB$. Let us consider the sequence $\Psi_n = \max\{\varepsilon_{n,0}^{i+j}, i + j = 0, \ldots, p\}$, $n = 0, \ldots, N-1$, we can derive

$$
\varepsilon_{n,0}^{i+j} \leq \lambda_1 + b_1 p^2 h_1 \sum_{\xi=0}^{n-1} \Psi_\xi + b_2 \sum_{a+b=0}^{i+j-1} \varepsilon_{n,0}^{a+b},
$$

using Lemma 1.7.1, we obtain

$$
\begin{aligned}
\varepsilon_{n,0}^{i+j} &\leq \left(\lambda_1 + b_1 p^2 h_1 \sum_{\xi=0}^{n-1} \Psi_\xi \right) \exp\left(\sum_{a+b=0}^{i+j-1} b_2 \right) \\
&\leq \underbrace{\lambda_1 \exp(p^2 b_2)}_{b_3} + \underbrace{b_1 p^2 \exp(p^2 b_2)}_{b_4} h_1 \sum_{\xi=0}^{n-1} \Psi_\xi,
\end{aligned}
$$

it follows that, for all $n = 0, \dots, N-1$

$$\Psi_n \leq b_3 + b_4 h_1 \sum_{\xi=0}^{n-1} \Psi_\xi,$$

again, using Lemma 1.7.1, we obtain

$$\varepsilon_{n,0}^{i+j} \leq \Psi_n \leq \underbrace{b_3 \exp(Ab_4)}_{\varphi_2(p)}.$$

Third, by following a similar process with slight adjustments, we obtain from (3.9) and (3.10) for $i + j = 0, \dots, p$

$$\begin{aligned}
\varepsilon_{n,m}^{i+j} \leq{} & \lambda_1 + \lambda_2 h_1 p \sum_{\xi=0}^{n-1} \sum_{a+b=0}^{p-1} \varepsilon_{\xi,m}^{a+b} + \lambda_3 p \sum_{a+b=0}^{i+j-1} \varepsilon_{n,m}^{a+b} + \lambda_2 h_1 \sum_{a+b=0}^{i+j-1} \varepsilon_{n,m}^{a+b} \\
& + \lambda_5 h_2 p \sum_{\rho=0}^{m-1} \sum_{a+b=0}^{p-1} \varepsilon_{n,\rho}^{a+b} + \lambda_4 p \sum_{a+b=0}^{i+j-1} \varepsilon_{n,m}^{a+b} + \lambda_5 h_2 \sum_{a+b=0}^{i+j-1} \varepsilon_{n,m}^{a+b} \\
& + \lambda_7 h_1 h_2 \sum_{\xi=0}^{n-1} \sum_{\rho=0}^{m-1} \sum_{a+b=0}^{p-1} \varepsilon_{\xi,\rho}^{a+b} + \lambda_6 h_1 p^2 \sum_{\xi=0}^{n-1} \sum_{a+b=0}^{p-1} \varepsilon_{\xi,m}^{a+b} + \lambda_7 h_1 h_2 \sum_{\xi=0}^{n-1} \sum_{a+b=0}^{p-1} \varepsilon_{\xi,m}^{a+b} \\
& + \lambda_9 h_2 p^2 \sum_{\rho=0}^{m-1} \sum_{a+b=0}^{p-1} \varepsilon_{n,\rho}^{a+b} + \lambda_7 h_1 h_2 \sum_{\rho=0}^{m-1} \sum_{a+b=0}^{p-1} \varepsilon_{n,\rho}^{a+b} + \lambda_8 p^2 \sum_{a+b=0}^{i+j-1} \varepsilon_{n,m}^{a+b} \\
& + \lambda_6 p h_1 \sum_{a+b=0}^{i+j-1} \varepsilon_{n,m}^{a+b} + \lambda_9 p h_2 \sum_{a+b=0}^{i+j-1} \varepsilon_{n,m}^{a+b} + \lambda_7 h_1 h_2 \sum_{a+b=0}^{i+j-1} \varepsilon_{n,m}^{a+b}.
\end{aligned}$$

Considering the sequence $\Psi_{n,m} = \max\{\varepsilon_{n,m}^{i+j}, i + j = 0, \dots, p\}$, $n = 0, \dots, N-1; m = 0, \dots, M-1$, we obtain

$$\varepsilon_{n,m}^{i+j} \leq \lambda_1 + h_1 b_5 \sum_{\xi=0}^{n-1} \Psi_{\xi,m} + h_2 b_6 \sum_{\rho=0}^{m-1} \Psi_{n,\rho} + h_1 h_2 b_7 \sum_{\xi=0}^{n-1} \sum_{\rho=0}^{m-1} \Psi_{\xi,\rho} + b_8 \sum_{a+b=0}^{i+j-1} \varepsilon_{n,m}^{a+b}, \tag{3.11}$$

where $b_5 = (\lambda_2 p + \lambda_6 p^2 + \lambda_7 B)p^2$, $b_6 = (\lambda_5 p + \lambda_9 p^2 + \lambda_7 A)p^2$, $b_7 = \lambda_7 p^2$ and $b_8 = \lambda_3 p + \lambda_2 A + \lambda_4 p + \lambda_5 B + \lambda_8 p^2 + \lambda_6 pA + \lambda_9 pB + \lambda_7 AB$,
using Lemma 1.7.1 with the following notation:

$$\omega_{i+j} = \varepsilon_{n,m}^{i+j}, \quad p_0 = \lambda_1 + h_1 b_5 \sum_{\xi=0}^{n-1} \Psi_{\xi,m} + h_2 b_6 \sum_{\rho=0}^{m-1} \Psi_{n,\rho} + h_1 h_2 b_7 \sum_{\xi=0}^{n-1} \sum_{\rho=0}^{m-1} \Psi_{\xi,\rho}, \quad \kappa_{a+b} = b_8,$$

we obtain from (3.3)

$$\varepsilon_{n,m}^{i+j} \leq \underbrace{\lambda_1 \exp(p^2 b_8)}_{b_9} + h_1 \underbrace{b_5 \exp(p^2 b_8)}_{b_{10}} \sum_{\xi=0}^{n-1} \Psi_{\xi,m} + h_2 \underbrace{b_6 \exp(p^2 b_8)}_{b_{11}} \sum_{\rho=0}^{m-1} \Psi_{n,\rho}$$
$$+ h_1 h_2 \underbrace{b_7 \exp(p^2 b_8)}_{b_{12}} \sum_{\xi=0}^{n-1} \sum_{\rho=0}^{m-1} \Psi_{\xi,\rho},$$

it follows that, for all $n = 0, \ldots, N-1; m = 0, \ldots, M-1$

$$\Psi_{n,m} \leq b_9 + h_1 b_{10} \sum_{\xi=0}^{n-1} \Psi_{\xi,m} + h_2 b_{11} \sum_{\rho=0}^{m-1} \Psi_{n,\rho} + h_1 h_2 b_{12} \sum_{\xi=0}^{n-1} \sum_{\rho=0}^{m-1} \Psi_{\xi,\rho},$$

using Lemma 1.7.3, we obtain

$$\varepsilon_{n,m}^{i+j} \leq \Psi_{n,m} \leq \underbrace{b_9 \exp\left(\eta_1 (A+B)\right)}_{\varphi_3(p)},$$

where $\eta_1 = \frac{1}{2}\left(b_{10} + b_{11} + \sqrt{(b_{10} + b_{11})^2 + 4b_{12}}\right)$.
Consequently, the first step is concluded by taking

$$\varphi(p) = \max\{\varphi_1(p), \varphi_2(p), \varphi_3(p)\}.$$

Step 2. It will be shown that there exists a constant ς , which is independent of h_1 and h_2, such that

$$\|e_{n,m}\| \leq \varsigma (h_1 + h_2)^p,$$

for all $n = 0, \ldots, N-1$; $m = 0, \ldots, M-1$.

First, let $(t,s) \in \mathcal{R}_{0,0}$, by using Lemma 1.1.1, we obtain from (3.4)

$$|e_{0,0}(t,s)| \leq \sum_{i+j=p} \frac{1}{i!j!} \left\| \partial_t^{(i)} \partial_s^{(j)} w \right\| h_1^i h_2^j,$$

hence,

$$\|e_{n,m}\| \leq \varphi(p) \sum_{i+j=p} \frac{1}{i!j!} h_1^i h_2^j = \underbrace{\frac{\varphi(p)}{p!}}_{\varsigma_1} (h_1 + h_2)^p.$$

Second, let $(t,s) \in \mathcal{R}_{n,0}$, for all $n = 1, \ldots, N-1$, we have from (3.6)

$$\begin{aligned}
w(t,s) - \hat{\vartheta}_{n,0}(t,s) = &\int_0^s \kappa_2(t,s,y)(w(t,y) - \hat{\vartheta}_{n,0}(t,y))dy \\
&+ \sum_{\xi=0}^{n-1} \int_{t_\xi}^{t_{\xi+1}} \kappa_1(t,s,x) e_{\xi,0}(x,s)dt + \int_{t_n}^{t} \kappa_1(t,s,x)(w(x,s) - \hat{\vartheta}_{n,0}(x,s))dx \\
&+ \sum_{\xi=0}^{n-1} \int_{t_\xi}^{t_{\xi+1}} \int_0^s \kappa_3(t,s,x,y) e_{\xi,0}(x,y)dydx + \int_{t_n}^{t} \int_0^s \kappa_3(t,s,x,y)(w(x,y) - \hat{\vartheta}_{n,0}(x,y))dydx,
\end{aligned}$$

hence,

$$\begin{aligned}
|w(t,s) - \hat{\vartheta}_{n,0}(t,s)| \leq &\sum_{\xi=0}^{n-1} h_1 \kappa \|e_{\xi,0}\| + \sum_{\xi=0}^{n-1} h_1 h_2 \kappa \|e_{\xi,0}\| \\
&+ \kappa \int_{t_n}^{t} |w(x,s) - \hat{\vartheta}_{n,0}(x,s)| dx + \kappa \int_0^s |w(t,y) - \hat{\vartheta}_{n,0}(t,y)| dy \\
&+ \kappa \int_{t_n}^{t} \int_0^s |w(x,y) - \hat{\vartheta}_{n,0}(x,y)| dydx,
\end{aligned}$$

where $\kappa = max\{\|\kappa_i\|_{L^\infty(\mathcal{R})}, i = 1,2,3\}$, then by Lemma 1.7.6

$$\begin{aligned}
|w(t,s) - \hat{\vartheta}_{n,0}(t,s)| &\leq \left(\sum_{\xi=0}^{n-1} h_1 \kappa \|e_{\xi,0}\| + \sum_{\xi=0}^{n-1} h_1 h_2 \kappa \|e_{\xi,0}\| \right) \nu \\
&\leq \sum_{\xi=0}^{n-1} h_1 \underbrace{\kappa(1+B)\nu}_{\gamma_1} \|e_{\xi,0}\|,
\end{aligned}$$

which implies, by using Lemma 1.1.1, that

$$\begin{aligned}
\|e_{n,0}\| &\le \|w - \hat{\vartheta}_{n,0}\| + \|\hat{\vartheta}_{n,0} - \vartheta_{n,0}\| \\
&\le \sum_{\xi=0}^{n-1} h_1\gamma_1\|e_{\xi,0}\| + \sum_{i+j=p} \frac{1}{i!j!}\left\|\partial_t^{(i)}\partial_s^{(j)}\hat{\vartheta}_{n,0}\right\| h_1^i h_2^j \\
&\le \sum_{\xi=0}^{n-1} h_1\gamma_1\|e_{\xi,0}\| + \frac{\alpha(p)}{p!}(h_1+h_2)^p,
\end{aligned}$$

then, by Lemma 1.7.1, we have

$$\|e_{n,0}\| \le \underbrace{\frac{\alpha(p)}{p!}\exp(A\gamma_1)}_{\varsigma_2}(h_1+h_2)^p.$$

Third, let $(t,s) \in \mathcal{R}_{n,m}$, for all $n = 0,\ldots,N-1$; $m = 1,\ldots,M-1$, we have from (3.9)

$$\begin{aligned}
|w(t,s) - \hat{\vartheta}_{n,m}(t,s)| \le{}& \sum_{\xi=0}^{n-1} h_1\kappa\|e_{\xi,m}\| + \sum_{\rho=0}^{m-1} h_2\kappa\|e_{n,\rho}\| \\
&+ \sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1} h_1h_2\kappa\|e_{\xi,\rho}\| + \sum_{\xi=0}^{n-1} h_1h_2\kappa\|e_{\xi,m}\| + \sum_{\rho=0}^{m-1} h_1h_2\kappa\|e_{n,\rho}\| \\
&+ \kappa\int_{t_n}^{t} |w(x,s) - \hat{\vartheta}_{n,m}(x,s)|dx + \kappa\int_{s_m}^{s} |w(t,y) - \hat{\vartheta}_{n,m}(t,y)|dy \\
&+ \kappa\int_{t_n}^{t}\int_{s_m}^{s} |w(t,s) - \hat{\vartheta}_{n,m}(x,y)|dydx,
\end{aligned}$$

then by Lemma 1.7.6,

$$\begin{aligned}
|w(t,s) - \hat{\vartheta}_{n,m}(t,s)| \le{}& \sum_{\xi=0}^{n-1} h_1\underbrace{\kappa(1+h_2)\nu}_{\gamma_2}\|e_{\xi,m}\| + \sum_{\rho=0}^{m-1} h_2\underbrace{\kappa(1+h_1)\nu}_{\gamma_3}\|e_{n,\rho}\| \\
&+ \sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1} h_1h_2\underbrace{\kappa\nu}_{\gamma_4}\|e_{\xi,\rho}\|,
\end{aligned}$$

which implies, by using Lemma 1.1.1, that

$$\begin{aligned}\|e_{n,m}\| &\leq \|w - \hat{\vartheta}_{n,m}\| + \|\hat{\vartheta}_{n,m} - \vartheta_{n,m}\| \\ &\leq \sum_{\xi=0}^{n-1} h_1\gamma_2\|e_{\xi,m}\| + \sum_{\rho=0}^{m-1} h_2\gamma_3\|e_{n,\rho}\| + \sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1} h_1h_2\gamma_4\|e_{\xi,\rho}\| \\ &+ \sum_{i+j=p} \frac{1}{i!j!} \left\|\partial_t^{(i)}\partial_s^{(j)}\hat{\vartheta}_{n,m}\right\| h_1^i h_2^j,\end{aligned}$$

hence,

$$\|e_{n,m}\| \leq \sum_{\xi=0}^{n-1} h_1\gamma_2\|e_{\xi,m}\| + \sum_{\rho=0}^{m-1} h_2\gamma_3\|e_{n,\rho}\| + \sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1} h_1h_2\gamma_4\|e_{\xi,\rho}\| + \frac{\varphi(p)}{p!}(h_1+h_2)^p,$$

using Lemma 1.7.3, we obtain

$$\|e_{n,m}\| \leq \underbrace{\frac{\varphi(p)}{p!}\exp\left(\eta_2(A+B)\right)}_{\varsigma_3}(h_1+h_2)^p,$$

such that $\eta_2 = \frac{1}{2}\left(\gamma_2 + \gamma_3 + \sqrt{(\gamma_2+\gamma_3)^2 + 4\gamma_4}\right)$.

Thus, the proof is completed by taking $\varsigma = \max\{\varsigma_1, \varsigma_2, \varsigma_3\}$. ∎

Theorem 3.3.2 *Let $w = G \circ v$, where G is a continuously differentiable bijective function, and suppose that its inverse G^{-1} is Lipschitz continuous with Lipschitz constant $L > 0$ on the range of v. Let ϑ be the numerical approximation to w as constructed in Theorem 3.3.1, and define $\omega := G^{-1} \circ \vartheta$ as the corresponding approximation to v. Then the error function $E := v - \omega$ satisfies*

$$\|E\|_{L^\infty(\mathcal{R})} \leq \varsigma'(h_1+h_2)^p,$$

where ς' is a finite constant independent of h_1 and h_2 .

Proof. From the assumptions, we have

$$v(t,s) = G^{-1}(w(t,s)) \quad \text{and} \quad \omega(t,s) = G^{-1}(\vartheta(t,s)).$$

Therefore, the error function $E(t,s)$ becomes

$$E(t,s) = v(t,s) - \omega(t,s) = G^{-1}(w(t,s)) - G^{-1}(\vartheta(t,s)).$$

By the Lipschitz continuity of G^{-1}, it follows that

$$|E(t,s)| = \left|G^{-1}(w(t,s)) - G^{-1}(\vartheta(t,s))\right| \leq L \cdot |w(t,s) - \vartheta(t,s)|,$$

which implies, from Theorem 3.3.1, that

$$\begin{aligned} ||E||_{L^\infty(\mathcal{R})} &\leq L \cdot ||e||_{L^\infty(\mathcal{R})} \\ &\leq \underbrace{L \cdot \varsigma}_{\varsigma'} (h_1 + h_2)^p. \end{aligned}$$

■

3.4 Experimental Results

In this section, we report some numerical experiments that show the performances of the Taylor collocation method (TCM) when applied to some problems of the form (3.1) and (3.2). Moreover, we compare our results with other methods such as the multi-step method [59], Euler-type method (EM) and trapezoidal method (TM) [42], Chelyshkov polynomials method (2D-CPs) [38], bivariate shifted Legendre functions method [50] and two-dimensional block-pulse functions method (2D-BPFs) [41].

Example 3.4.1 *Let us first begin with an illustrative example and consider the following two-dimensional linear Volterra integral equation of the first kind:*

$$\int_0^t \int_0^s (ts+1)w(x,y)dydx = h(t,s), \quad t,s \in [0,1].$$

By differentiating both sides of this equation, we obtain

$$w(t,s) = f(t,s) - \int_0^t \frac{sw(x,s)}{ts+1}dx - \int_0^s \frac{tw(t,y)}{ts+1}dy - \int_0^t \int_0^s \frac{w(x,y)}{ts+1}dydx,$$

for $t, s \in [0,1]$, *where* $f(t,s) = \dfrac{-3t^2 + (2+3t+3ts)te^s}{2(1+ts)}$ *is chosen so that the exact solution is* $w(t,s) = te^s$.

Comparing the approximate and exact solutions is demonstrated in Table 3.1, by applying TCM on the equation above at specific points with $p = 3, 4$ *and* $(N, M) = (10, 10), (20, 20)$. *Figure 3.1: (a) and (b) display the approximate and exact solutions respectively, while (c) and (d) illustrate the function error for* $p = 3$ *and* $(N, M) = (10, 10), (20, 20)$.

Table 3.1 – Numerical results of Example 3.4.1

(t,s)	$N=M=10, p=3$	$N=M=20, p=3$	$N=M=10, p=4$
$(0.1, 0.1)$	1.73×10^{-6}	5.99×10^{-7}	2.59×10^{-6}
$(0.2, 0.2)$	1.84×10^{-5}	5.29×10^{-6}	2.20×10^{-5}
$(0.3, 0.3)$	6.23×10^{-5}	1.68×10^{-5}	7.02×10^{-5}
$(0.4, 0.4)$	1.34×10^{-4}	3.53×10^{-5}	1.47×10^{-4}
$(0.5, 0.5)$	2.28×10^{-4}	5.85×10^{-5}	2.46×10^{-4}
$(0.6, 0.6)$	3.29×10^{-4}	8.35×10^{-5}	3.52×10^{-4}
$(0.7, 0.7)$	4.27×10^{-4}	1.07×10^{-4}	4.55×10^{-4}
$(0.8, 0.8)$	5.14×10^{-4}	1.28×10^{-4}	5.46×10^{-4}
$(0.9, 0.9)$	5.85×10^{-4}	1.45×10^{-4}	6.21×10^{-4}
$(1.0, 1.0)$	1.86×10^{-3}	3.17×10^{-4}	5.93×10^{-4}

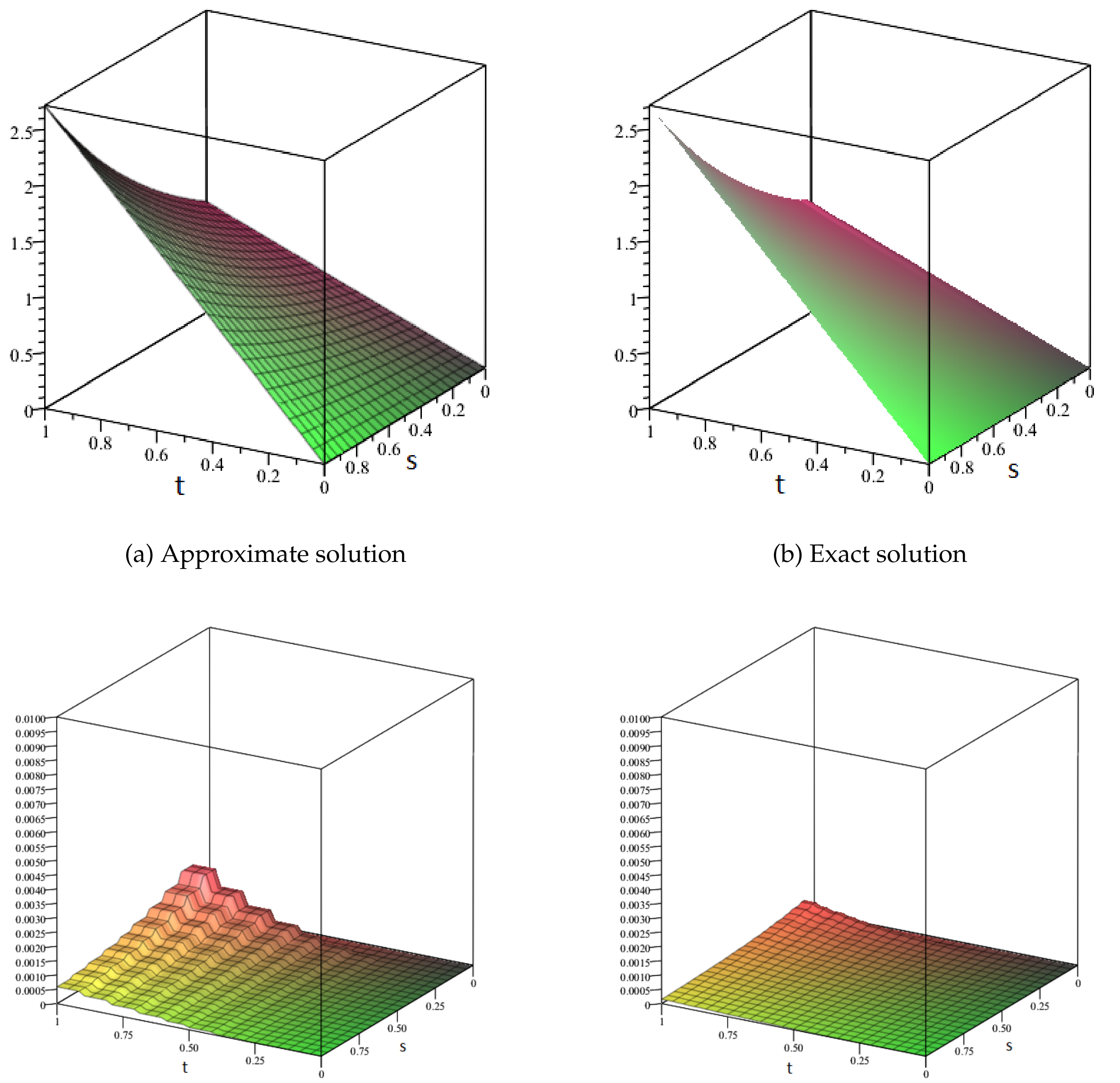
Example 3.4.2 *We selected the second example from reference [59], which is represented by the following equation:*

$$\left(\frac{t^2s^2 + 2\sin(ts) - 2ts\cos(ts)}{2s^2}\right)\sin(s) = \int_0^t \int_0^s (\sin(sx)+1)w(x,y)dydx,$$

for $t, s \in [0,1]$, *and the exact solution is* $w(t,s) = t\cos(s)$.

This equation is equivalent to the following linear 2D-VIE of the second kind:

$$w(t,s) = t\cos(s) + \frac{t^2\sin(s)\cos(ts)}{\sin(ts)+1} - \int_0^s \frac{t\cos(ts)}{\sin(ts)+1}w(t,y)dy.$$



(a) Approximate solution

(b) Exact solution

(c) Error function plot for $p = 3$, $N = M = 10$.

(d) Error function plot for $p = 3$, $N = M = 20$.

Figure 3.1 – Numerical results of Example 3.4.1

The numerical results for $p = 4$ and $N = M = 15$ of the TCM are compared with the numerical results obtained by using multi-step method [59] in Table 3.2.

The absolute error function for $p = 4$ and $N = M = 15$ are plotted in Figure 3.2.

Table 3.2 – Comparison of the absolute errors of Example 3.4.2

(t,s)	Multi-steps method	TCM
$(2^{-7},2^{-7})$	2.38×10^{-7}	2.00×10^{-12}
$(2^{-6},2^{-6})$	1.90×10^{-6}	4.00×10^{-11}
$(2^{-5},2^{-5})$	1.57×10^{-5}	1.24×10^{-9}
$(2^{-4},2^{-4})$	2.25×10^{-6}	3.97×10^{-8}
$(2^{-3},2^{-3})$	1.51×10^{-7}	1.88×10^{-7}
$(2^{-2},2^{-2})$	1.92×10^{-7}	2.66×10^{-7}
$(2^{-1},2^{-1})$	6.16×10^{-7}	8.87×10^{-8}

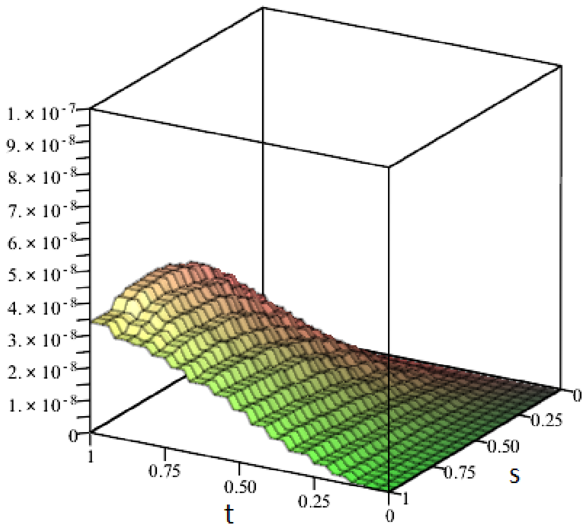


Figure 3.2 – Plot of the absolute error function for Example 3.4.2

Example 3.4.3 *Regarding the third example, we selected it from reference [42], which is described by the following equation:*

$$h(t,s) = \int_0^t \int_0^s (\sin(s+x) + \sin(t+y) + 3)w(x,y)dydx,$$

for $t,s \in [0,2]$*, where* $h(t,s)$ *is chosen so that the exact solution is* $w(t,s) = \cos(t+s)$.

The numerical results for $p = 3$ *and* $h_1 = h_2 = 0.1, 0.05$ *obtained using the present method (TCM) are compared with those derived from the Euler-type method (EM), the trapezoidal method (TM) [42], the Chelyshkov polynomials method (2D-CPs) [38], the bivariate shifted Legendre functions method [50], and the two-dimensional block-pulse functions method (2D-*

BPFs) [41], as shown in Table 3.3.

Figure 3.3 illustrates both the exact and approximate solutions for $N = M = 40$ *and* $p = 3$.

Table 3.3 – Comparison of the absolute errors for Example 3.4.3

(t, s)	TCM $h_1 = 0.05$	EM [42] $h_1 = 0.05$	TM [42] $h_1 = 0.05$	Method in [50] $M = 4$	2D-BPFs [41] $m = 32$
$(1, 1)$	8.57×10^{-7}	4.06×10^{-2}	9.80×10^{-4}	4.96×10^{-6}	6.08×10^{-2}
$(1, 2)$	2.29×10^{-5}	1.23×10^{-2}	5.47×10^{-4}	5.98×10^{-6}	4.00×10^{-3}
$(2, 1)$	2.29×10^{-5}	1.23×10^{-2}	5.47×10^{-4}	9.87×10^{-3}	4.00×10^{-3}
$(2, 2)$	4.27×10^{-5}	4.06×10^{-2}	2.03×10^{-3}	1.22×10^{-5}	4.74×10^{-2}

	TCM		2D-CPs [38]	
(t, s)	$h_1 = 0.1$	$h_1 = 0.05$	$N = 2$	$M = 4$
$(0.1, 0.1)$	2.77×10^{-7}	3.78×10^{-8}	7.41×10^{-3}	4.77×10^{-6}
$(0.2, 0.2)$	9.38×10^{-7}	1.18×10^{-7}	4.43×10^{-4}	2.10×10^{-5}
$(0.3, 0.3)$	1.63×10^{-6}	2.03×10^{-7}	5.40×10^{-3}	7.20×10^{-6}
$(0.4, 0.4)$	2.14×10^{-6}	2.67×10^{-7}	6.65×10^{-3}	8.20×10^{-6}
$(0.5, 0.5)$	2.33×10^{-6}	2.91×10^{-7}	4.48×10^{-3}	6.40×10^{-6}
$(0.6, 0.6)$	2.09×10^{-6}	2.62×10^{-7}	5.43×10^{-5}	6.90×10^{-6}
$(0.7, 0.7)$	1.28×10^{-6}	1.60×10^{-7}	4.80×10^{-3}	1.20×10^{-5}
$(0.8, 0.8)$	2.85×10^{-7}	3.66×10^{-8}	7.93×10^{-3}	5.00×10^{-6}
$(0.9, 0.9)$	2.80×10^{-6}	3.58×10^{-7}	7.08×10^{-3}	3.00×10^{-5}
$(1, 1)$	6.85×10^{-6}	8.59×10^{-7}	2.77×10^{-4}	1.80×10^{-6}

Example 3.4.4 *In the last example, let us consider the following nonlinear 2D-VIE of the first kind [41]:*

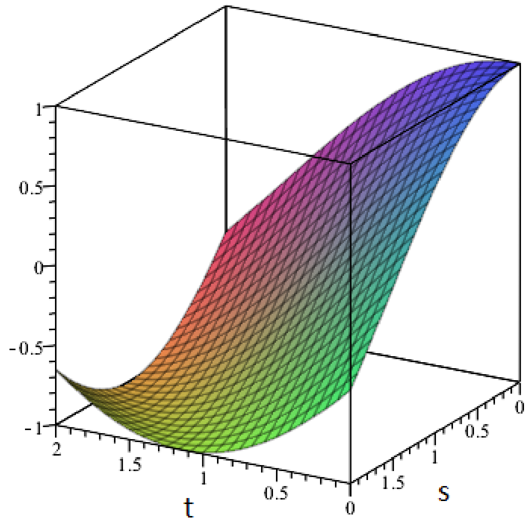
$$\frac{1}{9}(e^{t+s} - e^{t+4s} - e^{7t+s} + e^{7t+4s}) = \int_0^t \int_0^s 2e^{t+s} u^3(x, y) dy dx,$$

for $t, s \in [0, 1]$, *and the exact solution is* $u(t, s) = e^{t+2s}$.

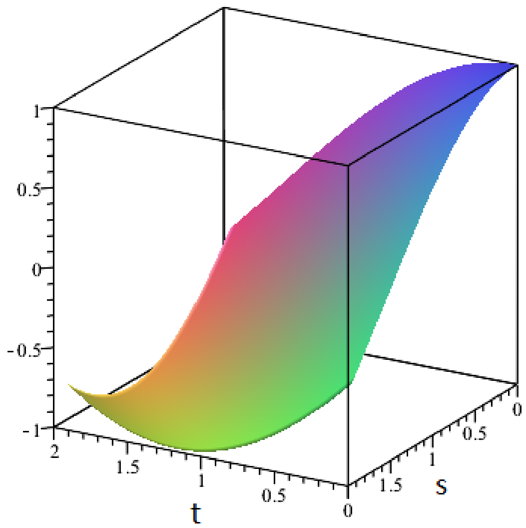
This equation is equivalent to the following linear 2D-VIE of the second kind:

$$w(t, s) = f(t, s) - \int_0^t w(x, s) dx - \int_0^s w(t, y) dy - \int_0^t \int_0^s w(x, y) dy dx,$$

where $w = u^3$. *Table 3.4 and Figure 3.4 present a comparison of the numerical results for* $p = 3$ *and* $N = M = 64$ *obtained through the TCM with those derived using the Chelyshkov polynomials method (2D-CPs) [38], the bivariate shifted Legendre functions method [50], and the two-dimensional block-pulse functions method (2D-BPFs) [41].*



(a) Approximate solution



(b) Exact solution

Figure 3.3 – Plot of the approximate and exact solution for Example 3.4.3

Table 3.4 – Comparison of the absolute errors of Example 3.4.4

$(2^{-k}, 2^{-k})$	2D-BPFs [41]	Method in [50]	2D-CPs [38]	TCM
$k=1$	1.0×10^{-1}	2.6×10^{-6}	3.5×10^{-5}	6.1×10^{-6}
$k=2$	4.6×10^{-2}	4.6×10^{-6}	2.0×10^{-6}	2.6×10^{-6}
$k=3$	2.9×10^{-2}	6.3×10^{-7}	1.5×10^{-5}	1.3×10^{-6}
$k=4$	2.3×10^{-2}	1.2×10^{-5}	1.2×10^{-5}	7.2×10^{-7}
$k=5$	2.0×10^{-2}	3.8×10^{-6}	5.9×10^{-5}	3.7×10^{-7}
$k=6$	3.1×10^{-2}	9.0×10^{-6}	9.6×10^{-5}	1.9×10^{-7}

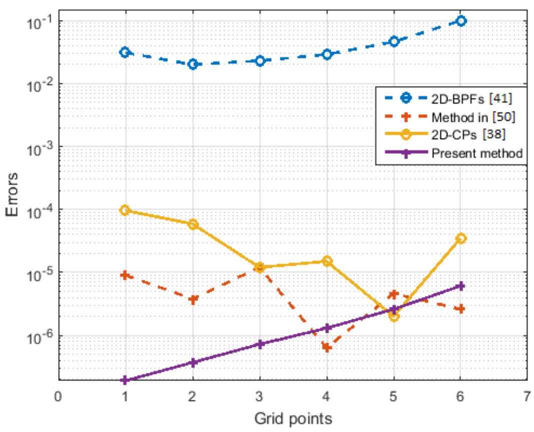


Figure 3.4 – Computational errors corresponding to different methods of Example 3.4.4

3.5 Conclusion

In this chapter, the problem described in (3.1) is converted into a linear two dimensional Volterra integral equation of the second kind, as specified by (3.3). A collocation method utilizing Taylor polynomials is formulated to solve this equation. The method's convergence and error analysis are thoroughly examined, and several numerical examples demonstrate its efficiency and precision. The numerical results align with the theoretical predictions, and comparisons with other methods are also provided.

CHAPTER 4

TAYLOR COLLOCATION METHOD FOR SOLVING 3D-VOLTERRA INTEGRAL EQUATIONS

4.1 Introduction

Multi-dimensional integral equations are crucial for modeling various phenomena in mathematics, physics, and engineering. Among these, the three-dimensional Volterra integral equations of the second kind (3D-VIEs) arise in various fields, including electromagnetic phenomena, disk problems, electrified plates, the Schrödinger equation in three-dimensional momentum space and the mathematical modeling of the spatiotemporal development of an epidemic physical, mechanical, and biological problems [28, 58, 61].
The present chapter sets out to employ the TCM for solving linear and nonlinear three-dimensional Volterra integral equations of the form:

$$w(t,s,r) = f(t,s,r) + \int_0^t \int_0^s \int_0^r \kappa(t,s,r,x,y,z)w(x,y,z)\,dz\,dy\,dx, \tag{4.1}$$

and

$$w(t,s,r) = f(t,s,r) + \int_0^t \int_0^s \int_0^r \kappa(t,s,r,x,y,z,w(x,y,z))\,dz\,dy\,dx, \tag{4.2}$$

where $(t,s,r) \in \mathcal{R}$ and the functions f and κ are sufficiently smooth, defined respectively on $\mathcal{R} := [0,A] \times [0,B] \times [0,C] \subset \mathbb{R}^3$ and $S := \{(t,s,r,x,y,z) : 0 \leq x \leq t \leq A, 0 \leq y \leq s \leq B, 0 \leq z \leq r \leq C\}$, and which was published in the reference [8]. The classical theory of Volterra illustrates the investigation into the existence and uniqueness of solutions for Equation (4.1). This study can be referenced in the literature, such as in the works [13, 60].

Numerous researchers have explored the numerical solution of 3D-VIEs, each contributing valuable insights and methodologies. For instance, Bakhshi, M. et al. [3] introduced the three-dimensional differential transform method, while Mirzaee, F. and Hadadiyan, E. [43] utilized modified block-pulse functions to solve the three-dimensional nonlinear mixed Volterra-Fredholm integral equations. Mohamed, D.S. [45] applied the shifted Chebyshev polynomial method, and Maleknejad, K. et al. [39]

employed Bernstein's approximation. Nawaz, R. et al. [48] leveraged the optimal homotopy asymptotic method, and Ghiasi, H. et al. [23] utilized operational matrix techniques with block-pulse functions.

This chapter is structured as follows: Section 4.2 outlines the approximation of the solution of (4.1) in each domain using Taylor polynomials. Section 4.3 delves into the convergence analysis of our method. We present numerical examples in Section 4.4 to validate the effectiveness of our approach. Finally, Section 4.5 concludes the chapter and summarizes our findings.

4.2 Description of the Method

We define the space of trivariate polynomial spline functions of degree (at most) $p-1$ in t, s and r as follows:

$$\mathcal{S}_{p-1}^{(-1)}(\Pi) := \{u : u_{n,m,\tau} = u|_{\mathcal{R}_{n,m,\tau}} \in \pi_{p-1}, n = 0,\ldots,N-1; m = 0,\ldots,M-1; \tau = 0,\ldots,T-1\}. \tag{4.3}$$

Its dimension is $NMTp^3$, where
$\Pi := \Pi_N \times \Pi_M \times \Pi_T = \{(t_n, s_m, r_\tau), 0 \le n \le N, 0 \le m \le M, 0 \le \tau \le T\}$, such that $\Pi_N := \{t_i = ih_1, i = 0,\ldots,N\}$, $\Pi_M := \{s_j = jh_2, j = 0,\ldots,M\}$ and $\Pi_T := \{r_k = kh_3, d = 0,\ldots,T\}$ denote, respectively, uniform partitions of the intervals $[0,A]$, $[0,B]$ and $[0,C]$, with the stepsizes are given by $h_1 = \frac{A}{N}$, $h_2 = \frac{B}{M}$ and $h_3 = \frac{C}{T}$.
Set the subintervals
$\sigma_n^1 := [t_n; t_{n+1}), n = 0,\ldots,N-2;\ \sigma_{N-1}^1 := [t_{N-1}, t_N]$,
$\sigma_m^2 := [s_m; s_{m+1}), m = 0,\ldots,M-2;\ \sigma_{M-1}^2 := [s_{M-1}, s_M]$,
$\sigma_\tau^3 := [r_\tau; r_{\tau+1}), \tau = 0,\ldots,T-2;\ \sigma_{T-1}^3 := [r_{T-1}, r_T]$,
and
$\mathcal{R}_{n,m,\tau} := \sigma_n^1 \times \sigma_m^2 \times \sigma_\tau^3, \quad n = 0,\ldots,N-1; m = 0,\ldots,M-1; \tau = 0,\ldots,T-1$. Moreover, denote by π_{p-1} the set of all real polynomials of degree not exceeding $p-1$ in t, s and r.

Note that the solutions w of equations (4.1) and (4.2) are known on part of the boundary of $\mathcal{R}$:

$w(t,0,0) = f(t,0,0)$ if $0 \leq t \leq a$,

$w(0,s,0) = f(0,s,0)$ if $0 \leq s \leq b$,

$w(0,0,r) = f(0,0,r)$ if $0 \leq r \leq c$.

Taylor collocation solution w is determined in $\mathcal{R}_{0,0,0}$, $\mathcal{R}_{n,0,0}$, $\mathcal{R}_{n,m,0}$ and $\mathcal{R}_{n,m,\tau}$ by the polynomials $\vartheta_{0,0,0}$, $\vartheta_{n,0,0}$, $\vartheta_{n,m,0}$ and $\vartheta_{n,m,\tau}$ respectively, for $n = 0,\ldots,N-1; m = 0,\ldots,M-1; \tau = 0,\ldots,T-1$, such that

$$\vartheta_{0,0,0}(t,s,r) = \sum_{i+j+k=0}^{p-1} \frac{1}{i!j!k!} \partial_t^{(i)} \partial_s^{(j)} \partial_r^{(k)} w(0,0,0) t^i s^j r^k \; ; \quad (t,s,r) \in \mathcal{R}_{0,0,0}, \tag{4.4}$$

$$\vartheta_{n,0,0}(t,s,r) = \sum_{i+j+k=0}^{p-1} \frac{1}{i!j!k!} \partial_t^{(i)} \partial_s^{(j)} \partial_r^{(k)} \hat{\vartheta}_{n,0,0}(t_n,0,0)(t-t_n)^i s^j r^k \; ; \quad (t,s,r) \in \mathcal{R}_{n,0,0}, \tag{4.5}$$

$$\vartheta_{n,m,0}(t,s,r) = \sum_{i+j+k=0}^{p-1} \frac{1}{i!j!k!} \partial_t^{(i)} \partial_s^{(j)} \partial_r^{(k)} \hat{\vartheta}_{n,m,0}(t_n,s_m,0)(t-t_n)^i (s-s_m)^j r^k \; ; \quad (t,s,r) \in \mathcal{R}_{n,m,0}, \tag{4.6}$$

and

$$\vartheta_{n,m,\tau}(t,s,r) = \sum_{i+j+k=0}^{p-1} \frac{1}{i!j!k!} \partial_t^{(i)} \partial_s^{(j)} \partial_r^{(k)} \hat{\vartheta}_{n,m,\tau}(t_n,s_m,r_\tau)(t-t_n)^i (s-s_m)^j (r-r_\tau)^k \; ; \quad (t,s,r) \in \mathcal{R}_{n,m,\tau}, \tag{4.7}$$

where the coefficients $\partial_t^{(i)} \partial_s^{(j)} \partial_r^{(k)} w(0,0,0)$, $\partial_t^{(i)} \partial_s^{(j)} \partial_r^{(k)} \hat{\vartheta}_{n,0,0}(t_n,0,0)$, $\partial_t^{(i)} \partial_s^{(j)} \partial_r^{(k)} \hat{\vartheta}_{n,m,0}(t_n,s_m,0)$ and $\partial_t^{(i)} \partial_s^{(j)} \partial_r^{(k)} \hat{\vartheta}_{n,m,\tau}(t_n,s_m,r_\tau)$ are defined differently in this section.

4.2.1 Algorithm for linear 3D-Volterra integral equations

Taylor collocation solution in $\mathcal{R}_{0,0,0}$

First, we differentiate equation (4.1) k-times with respect to r, we obtain

$$\partial_r^{(k)} w(t,s,r) = \partial_r^{(k)} f(t,s,r) + \sum_{\mu=0}^{k-1} \int_0^t \int_0^s \partial_r^{(\mu)} \left[\partial_r^{(k-1-\mu)} \Big|_{z=r} \kappa(t,s,r,x,y,z) w(x,y,r) \right] dydx$$
$$+ \int_0^t \int_0^s \int_0^r \partial_r^{(k)} \kappa(t,s,r,x,y,z) w(x,y,z) dzdydx,$$

hence,

$$\partial_r^{(k)} w(t,s,r) = \partial_r^{(k)} f(t,s,r) + \sum_{\mu=0}^{k-1} \sum_{\lambda=0}^{\mu} \binom{\mu}{\lambda} \int_0^t \int_0^s \partial_r^{(\mu-\lambda)} \left[\partial_r^{(k-1-\mu)} \Big|_{z=r} \kappa(t,s,r,x,y,z) \right]$$
$$\times \partial_r^{(\lambda)} w(x,y,r) dydx + \int_0^t \int_0^s \int_0^r \partial_r^{(k)} \kappa(t,s,r,x,y,z) w(x,y,z) dzdydx. \tag{4.8}$$

Second, we differentiate equation (4.8) j-times with respect to s, we get

$$\partial_s^{(j)} \partial_r^{(k)} w(t,s,r) = \partial_s^{(j)} \partial_r^{(k)} f(t,s,r)$$
$$+ \sum_{\mu=0}^{k-1} \sum_{\lambda=0}^{\mu} \binom{\mu}{\lambda} \sum_{q=0}^{j-1} \int_0^t \partial_s^{(q)} \left[\partial_s^{(j-1-q)} \Big|_{y=s} \left(\partial_r^{(\mu-\lambda)} \left[\partial_r^{(k-1-\mu)} \Big|_{z=r} \kappa(t,s,r,x,y,z) \right] \right) \partial_r^{(\lambda)} w(x,y,r) dx \right]$$
$$+ \sum_{\mu=0}^{k-1} \sum_{\lambda=0}^{\mu} \binom{\mu}{\lambda} \int_0^t \int_0^s \partial_s^{(j)} \left(\partial_r^{(\mu-\lambda)} \left[\partial_r^{(k-1-\mu)} \Big|_{z=r} \kappa(t,s,r,x,y,z) \right] \right) \partial_r^{(\lambda)} w(x,y,r) dydx$$
$$+ \sum_{q=0}^{j-1} \int_0^t \int_0^s \partial_s^{(q)} \left[\partial_s^{(j-1-q)} \Big|_{y=s} \left(\partial_r^{(k)} \kappa(t,s,r,x,y,z) w(x,y,z) \right) \right] dzdx$$
$$+ \int_0^t \int_0^s \int_0^r \partial_s^{(j)} \partial_r^{(k)} \kappa(t,s,r,x,y,z) w(x,y,z) dzdydx,$$

which implies,

$$\partial_s^{(j)} \partial_r^{(k)} w(t,s,r) = \partial_s^{(j)} \partial_r^{(k)} f(t,s,r) + \sum_{\mu=0}^{k-1} \sum_{\lambda=0}^{\mu} \sum_{q=0}^{j-1} \sum_{\eta=0}^{q} \binom{\mu}{\lambda} \binom{q}{\eta}$$

$$
\begin{aligned}
&\times \int_0^t \partial_s^{(q-\eta)} \left[\partial_s^{(j-1-q)}\Big|_{y=s} \left(\partial_r^{(\mu-\lambda)} \left[\partial_r^{(k-1-\mu)}\Big|_{z=r} \kappa(t,s,r,x,y,z) \right] \right) \right] \partial_s^{(\eta)} \partial_r^{(\lambda)} w(x,s,r)dx \\
&+ \sum_{\mu=0}^{k-1} \sum_{\lambda=0}^{\mu} \binom{\mu}{\lambda} \int_0^t \int_0^s \partial_s^{(j)} \left(\partial_r^{(\mu-\lambda)} \left[\partial_r^{(k-1-\mu)}\Big|_{z=r} \kappa(t,s,r,x,y,z) \right] \right) \partial_r^{(\lambda)} w(x,y,r)dydx \\
&+ \sum_{q=0}^{j-1} \sum_{\eta=0}^{q} \binom{q}{\eta} \int_0^t \int_0^s \partial_s^{(q-\eta)} \left[\partial_s^{(j-1-q)}\Big|_{y=s} \partial_r^{(k)} \kappa(t,s,r,x,y,z) \right] \partial_s^{(\eta)} w(x,s,z)dzdx \\
&+ \int_0^t \int_0^s \int_0^r \partial_s^{(j)} \partial_r^{(k)} \kappa(t,s,r,x,y,z) w(x,y,z)dzdydx.
\end{aligned} \tag{4.9}
$$

Third, we differentiate equation (4.9) i-times with respect to t, we get

$$
\begin{aligned}
&\partial_t^{(i)} \partial_s^{(j)} \partial_r^{(k)} w(t,s,r) = \partial_t^{(i)} \partial_s^{(j)} \partial_r^{(k)} f(t,s,r) + \sum_{\mu=0}^{k-1} \sum_{\lambda=0}^{\mu} \sum_{q=0}^{j-1} \sum_{\eta=0}^{q} \sum_{\alpha=0}^{i-1} \sum_{\beta=0}^{\alpha} \binom{\mu}{\lambda} \binom{q}{\eta} \binom{\alpha}{\beta} \\
&\times \partial_t^{(\alpha-\beta)} \left[\partial_t^{(i-1-\alpha)}\Big|_{x=t} \left(\partial_s^{(q-\eta)} \left[\partial_s^{(j-1-q)}\Big|_{y=s} \left(\partial_r^{(\mu-\lambda)} \left[\partial_r^{(k-1-\mu)}\Big|_{z=r} \kappa(t,s,r,x,y,z) \right] \right) \right] \right) \right] \\
&\times \partial_t^{(\beta)} \partial_s^{(\eta)} \partial_r^{(\lambda)} w(t,s,r) + \sum_{\mu=0}^{k-1} \sum_{\lambda=0}^{\mu} \sum_{q=0}^{j-1} \sum_{\eta=0}^{q} \binom{\mu}{\lambda} \binom{q}{\eta} \\
&\times \int_0^t \partial_t^{(i)} \left(\partial_s^{(q-\eta)} \left[\partial_s^{(j-1-q)}\Big|_{y=s} \left(\partial_r^{(\mu-\lambda)} \left[\partial_r^{(k-1-\mu)}\Big|_{z=r} \kappa(t,s,r,x,y,z) \right] \right) \right] \right) \partial_s^{(\eta)} \partial_r^{(\lambda)} w(x,s,r)dx \\
&+ \sum_{\mu=0}^{k-1} \sum_{\lambda=0}^{\mu} \sum_{\alpha=0}^{i-1} \sum_{\beta=0}^{\alpha} \binom{\mu}{\lambda} \binom{\alpha}{\beta} \\
&\times \int_0^s \partial_t^{(\alpha-\beta)} \left[\partial_t^{(i-1-\alpha)}\Big|_{x=t} \left(\partial_s^{(j)} \left[\partial_r^{(\mu-\lambda)} \left(\partial_r^{(k-1-\mu)}\Big|_{z=r} \kappa(t,s,r,x,y,z) \right) \right] \right) \right] \partial_t^{(\beta)} \partial_r^{(\lambda)} w(t,y,r)dy \\
&+ \sum_{\mu=0}^{k-1} \sum_{\lambda=0}^{\mu} \binom{\mu}{\lambda} \int_0^t \int_0^s \partial_t^{(i)} \left[\partial_s^{(j)} \left(\partial_r^{(\mu-\lambda)} \left[\partial_r^{(k-1-\mu)}\Big|_{z=r} \kappa(t,s,r,x,y,z) \right] \right) \right] \partial_r^{(\lambda)} w(x,y,r)dydx \\
&+ \sum_{q=0}^{j-1} \sum_{\eta=0}^{q} \sum_{\alpha=0}^{i-1} \sum_{\beta=0}^{\alpha} \binom{q}{\eta} \binom{\alpha}{\beta} \\
&\times \int_0^r \partial_t^{(\alpha-\beta)} \left[\partial_t^{(i-1-\alpha)}\Big|_{x=t} \left(\partial_s^{(q-\eta)} \left[\partial_s^{(j-1-q)}\Big|_{y=s} \partial_r^{(k)} \kappa(t,s,r,x,y,z) \right] \right) \right] \partial_t^{(\beta)} \partial_s^{(\eta)} w(t,s,z)dz \\
&+ \sum_{q=0}^{j-1} \sum_{\eta=0}^{q} \binom{q}{\eta} \int_0^t \int_0^r \partial_t^{(i)} \left(\partial_s^{(q-\eta)} \left[\partial_s^{(j-1-q)}\Big|_{y=s} \partial_r^{(k)} \kappa(t,s,r,x,y,z) \right] \right) \partial_s^{(\eta)} w(x,s,z)dzdx
\end{aligned}
$$

$$+\sum_{\alpha=0}^{i-1}\sum_{\beta=0}^{\alpha}\binom{\alpha}{\beta}\int_0^s\int_0^r \partial_t^{(\alpha-\beta)}\left[\partial_t^{(i-1-\alpha)}\Big|_{x=t}\partial_s^{(j)}\partial_r^{(k)}\kappa(t,s,r,x,y,z)\right]\partial_t^{(\beta)}w(t,y,z)dzdy$$
$$+\int_0^t\int_0^s\int_0^r \partial_t^{(i)}\partial_s^{(j)}\partial_r^{(k)}\kappa(t,s,r,x,y,z)w(x,y,z)dzdydx,$$

hence,

$$\partial_t^{(i)}\partial_s^{(j)}\partial_r^{(k)}w(0,0,0)=\partial_t^{(i)}\partial_s^{(j)}\partial_r^{(k)}f(0,0,0)+\sum_{\mu=0}^{k-1}\sum_{\lambda=0}^{\mu}\sum_{q=0}^{j-1}\sum_{\eta=0}^{q}\sum_{\alpha=0}^{i-1}\sum_{\beta=0}^{\alpha}\binom{\mu}{\lambda}\binom{q}{\eta}\binom{\alpha}{\beta}$$
$$\times\,\partial_t^{(\alpha-\beta)}\left[\partial_t^{(i-1-\alpha)}\Big|_{x=t}\left(\partial_s^{(q-\eta)}\left[\partial_s^{(j-1-q)}\Big|_{y=s}\left(\partial_r^{(\mu-\lambda)}\left[\partial_r^{(k-1-\mu)}\Big|_{z=r}\kappa(t,s,r,x,y,z)\right]\right)\right]\right)\right]_{s=r=0}^{t=0}$$
$$\times\,\partial_t^{(\beta)}\partial_s^{(\eta)}\partial_r^{(\lambda)}w(0,0,0).$$

Taylor collocation solution in $\mathcal{R}_{n,0,0}$

Initially, $\hat{\vartheta}_{n,0,0}$ is the exact solution to the integral equation that follows:

$$\hat{\vartheta}_{n,0,0}(t,s,r)=f(t,s,r)+\sum_{\xi=0}^{n-1}\int_{t_\xi}^{t_{\xi+1}}\int_0^s\int_0^r \kappa(t,s,r,x,y,z)\vartheta_{\xi,0,0}(x,y,z)dzdydx$$
$$+\int_{t_n}^t\int_0^s\int_0^r \kappa(t,s,r,x,y,z)\hat{\vartheta}_{n,0,0}(x,y,z)dzdydx. \tag{4.10}$$

First, we differentiate equation (4.10) k-times with respect to r, we get

$$\partial_r^{(k)}\hat{\vartheta}_{n,0,0}(t,s,r)=\partial_r^{(k)}f(t,s,r)$$
$$+\sum_{\xi=0}^{n-1}\sum_{\mu=0}^{k-1}\int_{t_\xi}^{t_{\xi+1}}\int_0^s \partial_r^{(\mu)}\left[\partial_r^{(k-1-\mu)}\Big|_{z=r}\kappa(t,s,r,x,y,z)\vartheta_{\xi,0,0}(x,y,r)\right]dydx$$
$$+\sum_{\xi=0}^{n-1}\int_{t_\xi}^{t_{\xi+1}}\int_0^s\int_0^r \partial_r^{(k)}\kappa(t,s,r,x,y,z)\vartheta_{\xi,0,0}(x,y,z)dzdydx$$
$$+\sum_{\mu=0}^{k-1}\int_{t_n}^t\int_0^s \partial_r^{(\mu)}\left[\partial_r^{(k-1-\mu)}\Big|_{z=r}\kappa(t,s,r,x,y,z)\hat{\vartheta}_{n,0,0}(x,y,r)\right]dydx$$

$$+\int_{t_n}^{t}\int_{0}^{s}\int_{0}^{r}\partial_r^{(k)}\kappa(t,s,r,x,y,z)\hat{\vartheta}_{n,0,0}(x,y,z)dzdydx,$$

hence,

$$\begin{aligned}\partial_r^{(k)}\hat{\vartheta}_{n,0,0}(t,s,r) &= \partial_r^{(k)}f(t,s,r)\\ &+\sum_{\xi=0}^{n-1}\sum_{\mu=0}^{k-1}\sum_{\lambda=0}^{\mu}\binom{\mu}{\lambda}\int_{t_\xi}^{t_{\xi+1}}\int_{0}^{s}\partial_r^{(\mu-\lambda)}\left[\partial_r^{(k-1-\mu)}\Big|_{z=r}\kappa(t,s,r,x,y,z)\right]\partial_r^{(\lambda)}\vartheta_{\xi,0,0}(x,y,r)dydx\\ &+\sum_{\xi=0}^{n-1}\int_{t_\xi}^{t_{\xi+1}}\int_{0}^{s}\int_{0}^{r}\partial_r^{(k)}\kappa(t,s,r,x,y,z)\vartheta_{\xi,0,0}(x,y,z)dzdydx\\ &+\sum_{\mu=0}^{k-1}\sum_{\lambda=0}^{\mu}\binom{\mu}{\lambda}\int_{t_n}^{t}\int_{0}^{s}\partial_r^{(\mu-\lambda)}\left[\partial_r^{(k-1-\mu)}\Big|_{z=r}\kappa(t,s,r,x,y,z)\right]\partial_r^{(\lambda)}\hat{\vartheta}_{n,0,0}(x,y,r)dydx\\ &+\int_{t_n}^{t}\int_{0}^{s}\int_{0}^{r}\partial_r^{(k)}\kappa(t,s,r,x,y,z)\hat{\vartheta}_{n,0,0}(x,y,z)dzdydx. \end{aligned} \tag{4.11}$$

Second, we differentiate equation (4.11) j-times with respect to s, we get

$$\begin{aligned}&\partial_s^{(j)}\partial_r^{(k)}\hat{\vartheta}_{n,0,0}(t,s,r) = \partial_s^{(j)}\partial_r^{(k)}f(t,s,r)\\ &+\sum_{\xi=0}^{n-1}\sum_{\mu=0}^{k-1}\sum_{\lambda=0}^{\mu}\sum_{q=0}^{j-1}\binom{\mu}{\lambda}\int_{t_\xi}^{t_{\xi+1}}\partial_s^{(q)}\left[\partial_s^{(j-1-q)}\Big|_{y=s}\left(\partial_r^{(\mu-\lambda)}\left[\partial_r^{(k-1-\mu)}\Big|_{z=r}\kappa(t,s,r,x,y,z)\right]\right)\partial_r^{(\lambda)}\vartheta_{\xi,0,0}(x,s,r)dx\right]\\ &+\sum_{\xi=0}^{n-1}\sum_{\mu=0}^{k-1}\sum_{\lambda=0}^{\mu}\binom{\mu}{\lambda}\int_{t_\xi}^{t_{\xi+1}}\int_{0}^{s}\partial_s^{(j)}\left(\partial_r^{(\mu-\lambda)}[\partial_r^{(k-1-\mu)}\Big|_{z=r}\kappa(t,s,r,x,y,z)]\right)\partial_r^{(\lambda)}\vartheta_{\xi,0,0}(x,y,r)dydx\\ &+\sum_{\xi=0}^{n-1}\sum_{q=0}^{j-1}\int_{t_\xi}^{t_{\xi+1}}\int_{0}^{r}\partial_s^{(q)}\left[\partial_s^{(j-1-q)}\Big|_{y=s}\partial_r^{(k)}\kappa(t,s,r,x,y,z)\vartheta_{\xi,0,0}(x,s,z)\right]dzdx\\ &+\sum_{\xi=0}^{n-1}\int_{t_\xi}^{t_{\xi+1}}\int_{0}^{s}\int_{0}^{r}\partial_s^{(j)}\partial_r^{(k)}\kappa(t,s,r,x,y,z)\vartheta_{\xi,0,0}(x,y,z)dzdydx\\ &+\sum_{\mu=0}^{k-1}\sum_{\lambda=0}^{\mu}\sum_{q=0}^{j-1}\binom{\mu}{\lambda}\int_{t_n}^{t}\partial_s^{(q)}\left[\partial_s^{(j-1-q)}\Big|_{y=s}\left(\partial_r^{(\mu-\lambda)}\left[\partial_r^{(k-1-\mu)}\Big|_{z=r}\kappa(t,s,r,x,y,z)\right]\right)\partial_r^{(\lambda)}\hat{\vartheta}_{n,0,0}(x,s,r)\right]dx\\ &+\sum_{\mu=0}^{k-1}\sum_{\lambda=0}^{\mu}\binom{\mu}{\lambda}\int_{t_n}^{t}\int_{0}^{s}\partial_s^{(j)}\left(\partial_r^{(\mu-\lambda)}[\partial_r^{(k-1-\mu)}\Big|_{z=r}\kappa(t,s,r,x,y,z)]\right)\partial_r^{(\lambda)}\hat{\vartheta}_{n,0,0}(x,y,r)dydx\end{aligned}$$

$$+\sum_{q=0}^{j-1}\int_{t_n}^{t}\int_0^r \partial_s^{(q)}\left[\partial_s^{(j-1-q)}\Big|_{y=s}\partial_r^{(k)}\kappa(t,s,r,x,y,z)\hat{\vartheta}_{n,0,0}(x,s,z)\right]dzdx$$
$$+\int_{t_n}^{t}\int_0^s\int_0^r \partial_s^{(j)}\partial_r^{(k)}\kappa(t,s,r,x,y,z)\hat{\vartheta}_{n,0,0}(x,y,z)dzdydx,$$

which implies,

$$\partial_s^{(j)}\partial_r^{(k)}\hat{\vartheta}_{n,0,0}(t,s,r)=\partial_s^{(j)}\partial_r^{(k)}f(t,s,r)+\sum_{\xi=0}^{n-1}\sum_{\mu=0}^{k-1}\sum_{\lambda=0}^{\mu}\sum_{q=0}^{j-1}\sum_{\eta=0}^{q}\binom{\mu}{\lambda}\binom{q}{\eta}$$
$$\times\int_{t_\xi}^{t_{\xi+1}}\partial_s^{(q-\eta)}\left[\partial_s^{(j-1-q)}\Big|_{y=s}\left(\partial_r^{(\mu-\lambda)}\left[\partial_r^{(k-1-\mu)}\Big|_{z=r}\kappa(t,s,r,x,y,z)\right]\right)\right]\partial_s^{(\eta)}\partial_r^{(\lambda)}\vartheta_{\xi,0,0}(x,s,r)dx$$
$$+\sum_{\xi=0}^{n-1}\sum_{\mu=0}^{k-1}\sum_{\lambda=0}^{\mu}\binom{\mu}{\lambda}\int_{t_\xi}^{t_{\xi+1}}\int_0^s\partial_s^{(j)}\left(\partial_r^{(\mu-\lambda)}[\partial_r^{(k-1-\mu)}\Big|_{z=r}\kappa(t,s,r,x,y,z)]\right)\partial_r^{(\lambda)}\vartheta_{\xi,0,0}(x,y,r)dydx$$
$$+\sum_{\xi=0}^{n-1}\sum_{q=0}^{j-1}\sum_{\eta=0}^{q}\binom{q}{\eta}\int_{t_\xi}^{t_{\xi+1}}\int_0^r\partial_s^{(q-\eta)}\left[\partial_s^{(j-1-q)}\Big|_{y=s}\partial_r^{(k)}\kappa(t,s,r,x,y,z)\right]\partial_s^{(\eta)}\vartheta_{\xi,0,0}(x,s,z)dzdx$$
$$+\sum_{\xi=0}^{n-1}\int_{t_\xi}^{t_{\xi+1}}\int_0^s\int_0^r\partial_s^{(j)}\partial_r^{(k)}\kappa(t,s,r,x,y,z)\vartheta_{\xi,0,0}(x,y,z)dzdydx$$
$$+\sum_{\mu=0}^{k-1}\sum_{\lambda=0}^{\mu}\sum_{q=0}^{j-1}\sum_{\eta=0}^{q}\binom{\mu}{\lambda}\binom{q}{\eta}$$
$$\times\int_{t_n}^{t}\partial_s^{(q-\eta)}\left[\partial_s^{(j-1-q)}\Big|_{y=s}\left(\partial_r^{(\mu-\lambda)}\left[\partial_r^{(k-1-\mu)}\Big|_{z=r}\kappa(t,s,r,x,y,z)\right]\right)\right]\partial_s^{(\eta)}\partial_r^{(\lambda)}\hat{\vartheta}_{n,0,0}(x,s,r)dx$$
$$+\sum_{\mu=0}^{k-1}\sum_{\lambda=0}^{\mu}\binom{\mu}{\lambda}\int_{t_n}^{t}\int_0^s\partial_s^{(j)}\left(\partial_r^{(\mu-\lambda)}[\partial_r^{(k-1-\mu)}\Big|_{z=r}\kappa(t,s,r,x,y,z)]\right)\partial_r^{(\lambda)}\hat{\vartheta}_{n,0,0}(x,y,r)dydx$$
$$+\sum_{q=0}^{j-1}\sum_{\eta=0}^{q}\binom{q}{\eta}\int_{t_n}^{t}\int_0^r\partial_s^{(q-\eta)}\left[\partial_s^{(j-1-q)}\Big|_{y=s}\partial_r^{(k)}\kappa(t,s,r,x,y,z)\right]\partial_s^{(\eta)}\hat{\vartheta}_{n,0,0}(x,s,z)dzdx$$
$$+\int_{t_n}^{t}\int_0^s\int_0^r\partial_s^{(j)}\partial_r^{(k)}\kappa(t,s,r,x,y,z)\hat{\vartheta}_{n,0,0}(x,y,z)dzdydx. \tag{4.12}$$

Third, Differentiating equation (4.12) i-times with respect to t yields

$$
\begin{aligned}
&\partial_t^{(i)}\partial_s^{(j)}\partial_r^{(k)}\hat{\vartheta}_{n,0,0}(t,s,r) = \partial_t^{(i)}\partial_s^{(j)}\partial_r^{(k)} f(t,s,r) + \sum_{\xi=0}^{n-1}\sum_{\mu=0}^{k-1}\sum_{\lambda=0}^{\mu}\sum_{q=0}^{j-1}\sum_{\eta=0}^{q}\binom{\mu}{\lambda}\binom{q}{\eta}\\
&\times \int_{t_\xi}^{t_{\xi+1}} \partial_t^{(i)}\left(\partial_s^{(q-\eta)}\left[\partial_s^{(j-1-q)}\Big|_{y=s}\left(\partial_r^{(\mu-\lambda)}\left[\partial_r^{(k-1-\mu)}\Big|_{z=r}\kappa(t,s,r,x,y,z)\right]\right)\right]\right)\partial_s^{(\eta)}\partial_r^{(\lambda)}\vartheta_{\xi,0,0}(x,s,r)dx\\
&+\sum_{\xi=0}^{n-1}\sum_{\mu=0}^{k-1}\sum_{\lambda=0}^{\mu}\binom{\mu}{\lambda}\int_{t_\xi}^{t_{\xi+1}}\int_0^s \partial_t^{(i)}\left(\partial_s^{(j)}\left[\partial_r^{(\mu-\lambda)}[\partial_r^{(k-1-\mu)}\Big|_{z=r}\kappa(t,s,r,x,y,z)]\right]\right)\partial_r^{(\lambda)}\vartheta_{\xi,0,0}(x,y,r)dydx\\
&+\sum_{\xi=0}^{n-1}\sum_{q=0}^{j-1}\sum_{\eta=0}^{q}\binom{q}{\eta}\int_{t_\xi}^{t_{\xi+1}}\int_0^r \partial_t^{(i)}\left(\partial_s^{(q-\eta)}\left[\partial_s^{(j-1-q)}\Big|_{y=s}\partial_r^{(k)}\kappa(t,s,r,x,y,z)\right]\right)\partial_s^{(\eta)}\vartheta_{\xi,0,0}(x,s,z)dzdx\\
&+\sum_{\xi=0}^{n-1}\int_{t_\xi}^{t_{\xi+1}}\int_0^s\int_0^r \partial_t^{(i)}\partial_s^{(j)}\partial_r^{(k)}\kappa(t,s,r,x,y,z)\vartheta_{\xi,0,0}(x,y,z)dzdydx\\
&+\sum_{\mu=0}^{k-1}\sum_{\lambda=0}^{\mu}\sum_{q=0}^{j-1}\sum_{\eta=0}^{q}\sum_{\alpha=0}^{i-1}\sum_{\beta=0}^{\alpha}\binom{\mu}{\lambda}\binom{q}{\eta}\binom{\alpha}{\beta}\\
&\times\partial_t^{(\alpha-\beta)}\left[\partial_t^{(i-1-\alpha)}\Big|_{x=t}\left(\partial_s^{(q-\eta)}\left[\partial_s^{(j-1-q)}\Big|_{y=s}\left(\partial_r^{(\mu-\lambda)}\left[\partial_r^{(k-1-\mu)}\Big|_{z=r}\kappa(t,s,r,x,y,z)\right]\right)\right]\right)\right]\partial_t^{(\beta)}\partial_s^{(\eta)}\partial_r^{(\lambda)}\hat{\vartheta}_{n,0,0}(t,s,r)\\
&+\sum_{\mu=0}^{k-1}\sum_{\lambda=0}^{\mu}\sum_{q=0}^{j-1}\sum_{\eta=0}^{q}\binom{\mu}{\lambda}\binom{q}{\eta}\\
&\times\int_{t_n}^{t}\partial_t^{(i)}\left(\partial_s^{(q-\eta)}\left[\partial_s^{(j-1-q)}\Big|_{y=s}\left(\partial_r^{(\mu-\lambda)}\left[\partial_r^{(k-1-\mu)}\Big|_{z=r}\kappa(t,s,r,x,y,z)\right]\right)\right]\right)\partial_s^{(\eta)}\partial_r^{(\lambda)}\hat{\vartheta}_{n,0,0}(x,s,r)dx\\
&+\sum_{\mu=0}^{k-1}\sum_{\lambda=0}^{\mu}\sum_{\alpha=0}^{i-1}\sum_{\beta=0}^{\alpha}\binom{\mu}{\lambda}\binom{\alpha}{\beta}\\
&\times\int_0^s\partial_t^{(\alpha-\beta)}\left[\partial_t^{(i-1-\alpha)}\Big|_{x=t}\left(\partial_s^{(j)}\left[\partial_r^{(\mu-\lambda)}[\partial_r^{(k-1-\mu)}\Big|_{z=r}\kappa(t,s,r,x,y,z)]\right]\right)\right]\partial_t^{(\beta)}\partial_r^{(\lambda)}\hat{\vartheta}_{n,0,0}(t,y,r)dy\\
&+\sum_{\mu=0}^{k-1}\sum_{\lambda=0}^{\mu}\binom{\mu}{\lambda}\int_{t_n}^{t}\int_0^s\partial_t^{(i)}\left[\partial_s^{(j)}\left(\partial_r^{(\mu-\lambda)}[\partial_r^{(k-1-\mu)}\Big|_{z=r}\kappa(t,s,r,x,y,z)]\right)\right]\partial_r^{(\lambda)}\hat{\vartheta}_{n,0,0}(x,y,r)dydx\\
&+\sum_{q=0}^{j-1}\sum_{\eta=0}^{q}\sum_{\alpha=0}^{i-1}\sum_{\beta=0}^{\alpha}\binom{q}{\eta}\binom{\alpha}{\beta}\\
&\times\int_0^r\partial_t^{(\alpha-\beta)}\left[\partial_t^{(i-1-\alpha)}\Big|_{x=t}\left(\partial_s^{(q-\eta)}\left[\partial_s^{(j-1-q)}\Big|_{y=s}\partial_r^{(k)}\kappa(t,s,r,x,y,z)\right]\right)\right]\partial_t^{(\beta)}\partial_s^{(\eta)}\hat{\vartheta}_{n,0,0}(t,s,z)dz\\
&+\sum_{q=0}^{j-1}\sum_{\eta=0}^{q}\binom{q}{\eta}\int_{t_n}^{t}\int_0^r\partial_t^{(i)}\left(\partial_s^{(q-\eta)}\left[\partial_s^{(j-1-q)}\Big|_{y=s}\partial_r^{(k)}\kappa(t,s,r,x,y,z)\right]\right)\partial_s^{(\eta)}\hat{\vartheta}_{n,0,0}(x,s,z)dzdx
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{\alpha=0}^{i-1}\sum_{\beta=0}^{\alpha}\binom{\alpha}{\beta}\int_0^s\int_0^r \partial_t^{(\alpha-\beta)}\left[\partial_t^{(i-1-\alpha)}\Big|_{x=t}\partial_s^{(j)}\partial_r^{(k)}\kappa(t,s,r,x,y,z)\right]\partial_t^{(\beta)}\hat{\vartheta}_{n,0,0}(t,y,z)dzdy\\
&+\int_{t_n}^t\int_0^s\int_0^r \partial_t^{(i)}\partial_s^{(j)}\partial_r^{(k)}\kappa(t,s,r,x,y,z)\hat{\vartheta}_{n,0,0}(x,y,z)dzdydx,
\end{aligned}
\tag{4.13}
$$

hence,

$$
\begin{aligned}
&\partial_t^{(i)}\partial_s^{(j)}\partial_r^{(k)}\hat{\vartheta}_{n,0,0}(t_n,0,0)=\partial_t^{(i)}\partial_s^{(j)}\partial_r^{(k)}f(x_n,0,0)+\sum_{\xi=0}^{n-1}\sum_{\mu=0}^{k-1}\sum_{\lambda=0}^{\mu}\sum_{q=0}^{j-1}\sum_{\eta=0}^{q}\binom{\mu}{\lambda}\binom{q}{\eta}\\
&\times\int_{t_\xi}^{t_{\xi+1}}\partial_t^{(i)}\left[\partial_s^{(q-\eta)}\left[\partial_s^{(j-1-q)}\Big|_{y=s}\left(\partial_r^{(\mu-\lambda)}\left[\partial_r^{(k-1-\mu)}\Big|_{z=r}\kappa(t,s,r,x,y,z)\right]\right)\right]\right]_{s=r=0}^{t=t_n}\\
&\times\partial_s^{(\eta)}\partial_r^{(\lambda)}\vartheta_{\xi,0,0}(x,0,0)dx+\sum_{\mu=0}^{k-1}\sum_{\lambda=0}^{\mu}\sum_{q=0}^{j-1}\sum_{\eta=0}^{q}\sum_{\alpha=0}^{i-1}\sum_{\beta=0}^{\alpha}\binom{\mu}{\lambda}\binom{q}{\eta}\binom{\alpha}{\beta}\\
&\times\partial_t^{(\alpha-\beta)}\left[\partial_t^{(i-1-\alpha)}\Big|_{x=t}\left(\partial_s^{(q-\eta)}\left[\partial_s^{(j-1-q)}\Big|_{y=s}\left(\partial_r^{(\mu-\lambda)}\left[\partial_r^{(k-1-\mu)}\Big|_{z=r}\kappa(t,s,r,x,y,z)\right]\right)\right]\right)\right]_{s=r=0}^{t=t_n}\\
&\times\partial_t^{(\beta)}\partial_s^{(\eta)}\partial_r^{(\lambda)}\hat{\vartheta}_{n,0,0}(t_n,0,0).
\end{aligned}
$$

Taylor collocation solution in $\mathcal{R}_{n,m,0}$

In the beginning, $\hat{\vartheta}_{n,m,0}$ is the exact solution to the following integral equation:

$$
\begin{aligned}
\hat{\vartheta}_{n,m,0}(t,s,r)&=f(t,s,r)+\sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1}\int_{t_\xi}^{t_{\xi+1}}\int_{s_\rho}^{s_{\rho+1}}\int_0^r\kappa(t,s,r,x,y,z)\vartheta_{\xi,\rho,0}(x,y,z)dzdydx\\
&+\sum_{\xi=0}^{n-1}\int_{t_\xi}^{t_{\xi+1}}\int_{s_m}^{s}\int_0^r\kappa(t,s,r,x,y,z)\vartheta_{\xi,m,0}(x,y,z)dzdydx\\
&+\sum_{\rho=0}^{m-1}\int_{t_n}^{t}\int_{s_\rho}^{s_{\rho+1}}\int_0^r\kappa(t,s,r,x,y,z)\vartheta_{n,\rho,0}(x,y,z)dzdydx\\
&+\int_{t_n}^{t}\int_{s_m}^{s}\int_0^r\kappa(t,s,r,x,y,z)\hat{\vartheta}_{n,m,0}(x,y,z)dzdydx.
\end{aligned}
\tag{4.14}
$$

First, we differentiate equation (4.14) k-times with respect to r, we get

$$\begin{aligned}
&\partial_r^{(k)}\hat{\vartheta}_{n,m,0}(t,s,r) = \partial_r^{(k)} f(t,s,r)\\
&+\sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1}\sum_{\mu=0}^{k-1}\sum_{\lambda=0}^{\mu}\binom{\mu}{\lambda}\int_{t_\xi}^{t_{\xi+1}}\int_{s_\rho}^{s_{\rho+1}}\partial_r^{(\mu-\lambda)}\left[\partial_r^{(k-1-\mu)}\Big|_{z=r}\kappa(t,s,r,x,y,z)\right]\partial_r^{(\lambda)}\vartheta_{\xi,\rho,0}(x,y,r)dydx\\
&+\sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1}\int_{t_\xi}^{t_{\xi+1}}\int_{s_\rho}^{s_{\rho+1}}\int_0^r\partial_r^{(k)}\kappa(t,s,r,x,y,z)\vartheta_{\xi,\rho,0}(x,y,z)dzdydx\\
&+\sum_{\xi=0}^{n-1}\sum_{\mu=0}^{k-1}\sum_{\lambda=0}^{\mu}\binom{\mu}{\lambda}\int_{t_\xi}^{t_{\xi+1}}\int_{s_m}^{s}\partial_r^{(\mu-\lambda)}\left[\partial_r^{(k-1-\mu)}\Big|_{z=r}\kappa(t,s,r,x,y,z)\right]\partial_r^{(\lambda)}\vartheta_{\xi,m,0}(x,y,r)dydx\\
&+\sum_{\xi=0}^{n-1}\int_{t_\xi}^{t_{\xi+1}}\int_{s_m}^{s}\int_0^r\partial_r^{(k)}\kappa(t,s,r,x,y,z)\vartheta_{\xi,m,0}(x,y,z)dzdydx\\
&+\sum_{\rho=0}^{m-1}\sum_{\mu=0}^{k-1}\sum_{\lambda=0}^{\mu}\binom{\mu}{\lambda}\int_{t_n}^{t}\int_{s_\rho}^{s_{\rho+1}}\partial_r^{(\mu-\lambda)}\left[\partial_r^{(k-1-\mu)}\Big|_{z=r}\kappa(t,s,r,x,y,z)\right]\partial_r^{(\lambda)}\vartheta_{n,\rho,0}(x,y,r)dydx\\
&+\sum_{\rho=0}^{m-1}\int_{t_n}^{t}\int_{s_\rho}^{s_{\rho+1}}\int_0^r\partial_r^{(k)}\kappa(t,s,r,x,y,z)\vartheta_{n,\rho,0}(x,y,z)dzdydx\\
&+\sum_{\mu=0}^{k-1}\sum_{\lambda=0}^{\mu}\binom{\mu}{\lambda}\int_{t_n}^{t}\int_{s_m}^{s}\partial_r^{(\mu-\lambda)}\left[\partial_r^{(k-1-\mu)}\Big|_{z=r}\kappa(t,s,r,x,y,z)\right]\partial_r^{(\lambda)}\hat{\vartheta}_{n,m,0}(x,y,r)dydx\\
&+\int_{t_n}^{t}\int_{s_m}^{s}\int_0^r\partial_r^{(k)}\kappa(t,s,r,x,y,z)\hat{\vartheta}_{n,m,0}(x,y,z)dzdydx.
\end{aligned}\tag{4.15}$$

Second, we differentiate equation (4.15) j-times with respect to s, we obtain

$$\begin{aligned}
&\partial_s^{(j)}\partial_r^{(k)}\hat{\vartheta}_{n,m,0}(t,s,r) = \partial_s^{(j)}\partial_r^{(k)} f(t,s,r)\\
&+\sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1}\sum_{\mu=0}^{k-1}\sum_{\lambda=0}^{\mu}\binom{\mu}{\lambda}\int_{t_\xi}^{t_{\xi+1}}\int_{s_\rho}^{s_{\rho+1}}\partial_s^{(j)}\left(\partial_r^{(\mu-\lambda)}\left[\partial_r^{(k-1-\mu)}\Big|_{z=r}\kappa(t,s,r,x,y,z)\right]\right)\partial_r^{(\lambda)}\vartheta_{\xi,\rho,0}(x,y,r)dydx\\
&+\sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1}\int_{t_\xi}^{t_{\xi+1}}\int_{s_\rho}^{s_{\rho+1}}\int_0^r\partial_s^{(j)}\partial_r^{(k)}\kappa(t,s,r,x,y,z)\vartheta_{\xi,\rho,0}(x,y,z)dzdydx\\
&+\sum_{\xi=0}^{n-1}\sum_{\mu=0}^{k-1}\sum_{\lambda=0}^{\mu}\sum_{q=0}^{j-1}\sum_{\eta=0}^{q}\binom{\mu}{\lambda}\binom{q}{\eta}
\end{aligned}$$

$$
\begin{aligned}
&\times \int_{t_\xi}^{t_{\xi+1}} \partial_s^{(q-\eta)} \left[\partial_s^{(j-1-q)} \Big|_{y=s} \left(\partial_r^{(\mu-\lambda)} \left[\partial_r^{(k-1-\mu)} \Big|_{z=r} \kappa(t,s,r,x,y,z) \right] \right) \right] \partial_s^{(\eta)} \partial_r^{(\lambda)} \vartheta_{\xi,m,0}(x,s,r)dx \\
&+ \sum_{\xi=0}^{n-1} \sum_{\mu=0}^{k-1} \sum_{\lambda=0}^{\mu} \binom{\mu}{\lambda} \int_{t_\xi}^{t_{\xi+1}} \int_{s_m}^{s} \partial_s^{(j)} \left[\partial_r^{(\mu-\lambda)} \left[\partial_r^{(k-1-\mu)} \Big|_{z=r} \kappa(t,s,r,x,y,z) \right] \right] \partial_r^{(\lambda)} \vartheta_{\xi,m,0}(x,y,r)dydx \\
&+ \sum_{\xi=0}^{n-1} \sum_{q=0}^{j-1} \sum_{\eta=0}^{q} \binom{q}{\eta} \int_{t_\xi}^{t_{\xi+1}} \int_{0}^{r} \partial_s^{(q-\eta)} \left[\partial_s^{(j-1-q)} \Big|_{y=s} \partial_r^{(k)} \kappa(t,s,r,x,y,z) \right] \partial_s^{(\eta)} \vartheta_{\xi,m,0}(x,s,z)dzdx \\
&+ \sum_{\xi=0}^{n-1} \int_{t_\xi}^{t_{\xi+1}} \int_{s_m}^{s} \int_{0}^{r} \partial_s^{(j)} \partial_r^{(k)} \kappa(t,s,r,x,y,z) \vartheta_{\xi,m,0}(x,y,z)dzdydx \\
&+ \sum_{\rho=0}^{m-1} \sum_{\mu=0}^{k-1} \sum_{\lambda=0}^{\mu} \binom{\mu}{\lambda} \int_{t_n}^{t} \int_{s_\rho}^{s_{\rho+1}} \partial_s^{(j)} \left(\partial_r^{(\mu-\lambda)} \left[\partial_r^{(k-1-\mu)} \Big|_{z=r} \kappa(t,s,r,x,y) \right] \right) \partial_r^{(\lambda)} \vartheta_{n,\rho,0}(x,y,r)dydx \\
&+ \sum_{\rho=0}^{m-1} \int_{t_n}^{t} \int_{s_\rho}^{s_{\rho+1}} \int_{0}^{r} \partial_s^{(j)} \partial_r^{(k)} \kappa(t,s,r,x,y,z) \vartheta_{n,\rho,0}(x,y,z)dzdydx \\
&+ \sum_{\mu=0}^{k-1} \sum_{\lambda=0}^{\mu} \sum_{q=0}^{j-1} \sum_{\eta=0}^{q} \binom{\mu}{\lambda} \binom{q}{\eta} \\
&\times \int_{t_n}^{t} \partial_s^{(q-\eta)} \left[\partial_s^{(j-1-q)} \Big|_{y=s} \left(\partial_r^{(\mu-\lambda)} \left[\partial_r^{(k-1-\mu)} \Big|_{z=r} \kappa(t,s,r,x,y,z) \right] \right) \right] \partial_s^{(\eta)} \partial_r^{(\lambda)} \hat{\vartheta}_{n,m,0}(x,s,r)dx \\
&+ \sum_{\mu=0}^{k-1} \sum_{\lambda=0}^{\mu} \binom{\mu}{\lambda} \int_{t_n}^{t} \int_{s_m}^{s} \partial_s^{(j)} \left(\partial_r^{(\mu-\lambda)} \left[\partial_r^{(k-1-\mu)} \Big|_{z=r} \kappa(t,s,r,x,y,z) \right] \right) \partial_r^{(\lambda)} \hat{\vartheta}_{n,m,0}(x,y,r)dydx \\
&+ \sum_{q=0}^{j-1} \sum_{\eta=0}^{q} \binom{q}{\eta} \int_{t_n}^{t} \int_{0}^{r} \partial_s^{(q-\eta)} \left[\partial_s^{(j-1-q)} \Big|_{y=s} \partial_r^{(k)} \kappa(t,s,r,x,y,z) \right] \partial_s^{(\eta)} \hat{\vartheta}_{n,m,0}(x,s,z)dzdx \\
&+ \int_{t_n}^{t} \int_{s_m}^{s} \int_{0}^{r} \partial_s^{(j)} \partial_r^{(k)} \kappa(t,s,r,x,y,z) \hat{\vartheta}_{n,m,0}(x,y,z)dzdydx.
\end{aligned} \tag{4.16}
$$

Third, we differentiate equation (4.16) i-times with respect to t, we get

$$
\begin{aligned}
&\partial_t^{(i)}\partial_s^{(j)}\partial_r^{(k)}\hat{\vartheta}_{n,m,0}(t,s,r)=\partial_t^{(i)}\partial_s^{(j)}\partial_r^{(k)}f(t,s,r)\\
&+\sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1}\sum_{\mu=0}^{k-1}\sum_{\lambda=0}^{\mu}\binom{\mu}{\lambda}\int_{t_\xi}^{t_{\xi+1}}\int_{s_\rho}^{s_{\rho+1}}\partial_t^{(i)}\left[\partial_s^{(j)}\left(\partial_r^{(\mu-\lambda)}\left[\partial_r^{(k-1-\mu)}\Big|_{z=r}\kappa(t,s,r,x,y,z)\right]\right)\right]\partial_r^{(\lambda)}\vartheta_{\xi,\rho,0}(x,y,r)dydx\\
&+\sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1}\int_{t_\xi}^{t_{\xi+1}}\int_{s_\rho}^{s_{\rho+1}}\int_0^r\partial_t^{(i)}\partial_s^{(j)}\partial_r^{(k)}\kappa(t,s,r,x,y,z)\vartheta_{\xi,\rho,0}(x,y,z)dzdydx\\
&+\sum_{\xi=0}^{n-1}\sum_{\mu=0}^{k-1}\sum_{\lambda=0}^{\mu}\sum_{q=0}^{j-1}\sum_{\eta=0}^{q}\binom{\mu}{\lambda}\binom{q}{\eta}\\
&\times\int_{t_\xi}^{t_{\xi+1}}\partial_t^{(i)}\left(\partial_s^{(q-\eta)}\left[\partial_s^{(j-1-q)}\Big|_{y=s}\left(\partial_r^{(\mu-\lambda)}\left[\partial_r^{(k-1-\mu)}\Big|_{z=r}\kappa(t,s,r,x,y,z)\right]\right)\right]\right)\partial_s^{(\eta)}\partial_r^{(\lambda)}\vartheta_{\xi,m,0}(x,s,r)dx\\
&+\sum_{\xi=0}^{n-1}\sum_{\mu=0}^{k-1}\sum_{\lambda=0}^{\mu}\binom{\mu}{\lambda}\int_{t_\xi}^{t_{\xi+1}}\int_{s_m}^{s}\partial_t^{(i)}\left(\partial_s^{(j)}\left[\partial_r^{(\mu-\lambda)}\left[\partial_r^{(k-1-\mu)}\Big|_{z=r}\kappa(t,s,r,x,y,z)\right]\right]\right)\partial_r^{(\lambda)}\vartheta_{\xi,m,0}(x,y,r)dydx\\
&+\sum_{\xi=0}^{n-1}\sum_{q=0}^{j-1}\sum_{\eta=0}^{q}\binom{q}{\eta}\int_{t_\xi}^{t_{\xi+1}}\int_0^r\partial_t^{(i)}\left(\partial_s^{(q-\eta)}\left[\partial_s^{(j-1-q)}\Big|_{y=s}\partial_r^{(k)}\kappa(t,s,r,x,y,z)\right]\right)\partial_s^{(\eta)}\vartheta_{\xi,m,0}(x,s,z)dzdx\\
&+\sum_{\xi=0}^{n-1}\int_{t_\xi}^{t_{\xi+1}}\int_{s_m}^{s}\int_0^r\partial_t^{(i)}\partial_s^{(j)}\partial_r^{(k)}\kappa(t,s,r,x,y,z)\vartheta_{\xi,m,0}(x,y,z)dzdydx\\
&+\sum_{\rho=0}^{m-1}\sum_{\mu=0}^{k-1}\sum_{\lambda=0}^{\mu}\sum_{\alpha=0}^{i-1}\sum_{\beta=0}^{\alpha}\binom{\mu}{\lambda}\binom{\alpha}{\beta}\\
&\times\int_{s_\rho}^{s_{\rho+1}}\partial_t^{(\alpha-\beta)}\left[\partial_t^{(i-1-\alpha)}\Big|_{x=t}\left(\partial_s^{(j)}\left[\partial_r^{(\mu-\lambda)}\left[\partial_r^{(k-1-\mu)}\Big|_{z=r}\kappa(t,s,r,x,y,z)\right]\right]\right)\right]\partial_t^{(\beta)}\partial_r^{(\lambda)}\vartheta_{n,\rho,0}(t,y,r)dy\\
&+\sum_{\rho=0}^{m-1}\sum_{\mu=0}^{k-1}\sum_{\lambda=0}^{\mu}\binom{\mu}{\lambda}\int_{t_n}^{t}\int_{s_\rho}^{s_{\rho+1}}\partial_t^{(i)}\left[\partial_s^{(j)}\left(\partial_r^{(\mu-\lambda)}\left[\partial_r^{(k-1-\mu)}\Big|_{z=r}\kappa(t,s,r,x,y,z)\right]\right)\right]\partial_r^{(\lambda)}\vartheta_{n,\rho,0}(x,y,r)dydx\\
&+\sum_{\rho=0}^{m-1}\sum_{\alpha=0}^{i-1}\sum_{\beta=0}^{\alpha}\binom{\alpha}{\beta}\int_{s_\rho}^{s_{\rho+1}}\int_0^r\partial_t^{(\alpha-\beta)}\left[\partial_t^{(i-1-\alpha)}\Big|_{x=t}\partial_s^{(j)}\partial_r^{(k)}\kappa(t,s,r,x,y,z)\right]\partial_t^{(\beta)}\vartheta_{n,\rho,0}(t,y,z)dzdy\\
&+\sum_{\rho=0}^{m-1}\int_{t_n}^{t}\int_{s_\rho}^{s_{\rho+1}}\int_0^r\partial_t^{(i)}\partial_s^{(j)}\partial_r^{(k)}\kappa(t,s,r,x,y,z)\vartheta_{n,\rho,0}(x,y,z)dzdydx\\
&+\sum_{\mu=0}^{k-1}\sum_{\lambda=0}^{\mu}\sum_{q=0}^{j-1}\sum_{\eta=0}^{q}\sum_{\alpha=0}^{i-1}\sum_{\beta=0}^{\alpha}\binom{\mu}{\lambda}\binom{q}{\eta}\binom{\alpha}{\beta}\\
&\times\partial_t^{(\alpha-\beta)}\left[\partial_t^{(i-1-\alpha)}\Big|_{x=t}\left(\partial_s^{(q-\eta)}\left[\partial_s^{(j-1-q)}\Big|_{y=s}\left(\partial_r^{(\mu-\lambda)}\left[\partial_r^{(k-1-\mu)}\Big|_{z=r}\kappa(t,s,r,x,y,z)\right]\right)\right]\right)\right]\partial_t^{(\beta)}\partial_s^{(\eta)}\partial_r^{(\lambda)}\hat{\vartheta}_{n,m,0}(t,s,r)
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{\mu=0}^{k-1}\sum_{\lambda=0}^{\mu}\sum_{q=0}^{j-1}\sum_{\eta=0}^{q}\binom{\mu}{\lambda}\binom{q}{\eta}\\
&\times\int_{t_n}^{t}\partial_t^{(i)}\left(\partial_s^{(q-\eta)}\left[\partial_s^{(j-1-q)}\Big|_{y=s}\left(\partial_r^{(\mu-\lambda)}\left[\partial_r^{(k-1-\mu)}\Big|_{z=r}\kappa(t,s,r,x,y,z)\right]\right)\right]\right)\partial_s^{(\eta)}\partial_r^{(\lambda)}\hat{\vartheta}_{n,m,0}(x,s,r)dx\\
&+\sum_{\mu=0}^{k-1}\sum_{\lambda=0}^{\mu}\sum_{\alpha=0}^{i-1}\sum_{\beta=0}^{\alpha}\binom{\mu}{\lambda}\binom{\alpha}{\beta}\\
&\times\int_{s_m}^{s}\partial_t^{(\alpha-\beta)}\left[\partial_t^{(i-1-\alpha)}\Big|_{x=t}\left(\partial_s^{(j)}\left[\partial_r^{(\mu-\lambda)}\left[\partial_r^{(k-1-\mu)}\Big|_{z=r}\kappa(t,s,r,x,y,z)\right]\right]\right)\right]\partial_t^{(\beta)}\partial_r^{(\lambda)}\hat{\vartheta}_{n,m,0}(t,y,r)dy\\
&+\sum_{\mu=0}^{k-1}\sum_{\lambda=0}^{\mu}\binom{\mu}{\lambda}\int_{t_n}^{t}\int_{s_m}^{s}\partial_t^{(i)}\left[\partial_s^{(j)}\left(\partial_r^{(\mu-\lambda)}\left[\partial_r^{(k-1-\mu)}\Big|_{z=r}\kappa(t,s,r,x,y,z)\right]\right)\right]\partial_r^{(\lambda)}\hat{\vartheta}_{n,m,0}(x,y,r)dydx\\
&+\sum_{q=0}^{j-1}\sum_{\eta=0}^{q}\sum_{\alpha=0}^{i-1}\sum_{\beta=0}^{\alpha}\binom{q}{\eta}\binom{\alpha}{\beta}\\
&\times\int_{0}^{r}\partial_t^{(\alpha-\beta)}\left[\partial_t^{(i-1-\alpha)}\Big|_{x=t}\left(\partial_s^{(q-\eta)}\left[\partial_s^{(j-1-q)}\Big|_{y=s}\partial_r^{(k)}\kappa(t,s,r,x,y,z)\right]\right)\right]\partial_t^{(\beta)}\partial_s^{(\eta)}\hat{\vartheta}_{n,m,0}(t,s,z)dz\\
&+\sum_{q=0}^{j-1}\sum_{\eta=0}^{q}\binom{q}{\eta}\int_{t_n}^{t}\int_{0}^{r}\partial_t^{(i)}\left(\partial_s^{(q-\eta)}\left[\partial_s^{(j-1-q)}\Big|_{y=s}\partial_r^{(k)}\kappa(t,s,r,x,y,z)\right]\right)\partial_s^{(\eta)}\hat{\vartheta}_{n,m,0}(x,s,z)dzdx\\
&+\sum_{\alpha=0}^{i-1}\sum_{\beta=0}^{\alpha}\binom{\alpha}{\beta}\int_{s_m}^{s}\int_{0}^{r}\partial_t^{(\alpha-\beta)}\left[\partial_t^{(i-1-\alpha)}\Big|_{x=t}\partial_s^{(j)}\partial_r^{(k)}\kappa(t,s,r,x,y,z)\right]\partial_t^{(\beta)}\hat{\vartheta}_{n,m,0}(t,y,z)dzdy\\
&+\int_{t_n}^{t}\int_{s_m}^{s}\int_{0}^{r}\partial_t^{(i)}\partial_s^{(j)}\partial_r^{(k)}\kappa(t,s,r,x,y,z)\hat{\vartheta}_{n,m,0}(x,y,z)dzdydx,
\end{aligned}
\tag{4.17}
$$

hence,

$$
\begin{aligned}
\partial_t^{(i)}\partial_s^{(j)}\partial_r^{(k)}\hat{\vartheta}_{n,m,0}(t_n,s_m,0)&=\partial_t^{(i)}\partial_s^{(j)}\partial_r^{(k)}f(t_n,s_m,0)+\sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1}\sum_{\mu=0}^{k-1}\sum_{\lambda=0}^{\mu}\binom{\mu}{\lambda}\\
&\times\int_{t_\xi}^{t_{\xi+1}}\int_{s_\rho}^{s_{\rho+1}}\partial_t^{(i)}\left[\partial_s^{(j)}\left(\partial_r^{(\mu-\lambda)}\left[\partial_r^{(k-1-\mu)}\Big|_{z=r}\kappa(t,s,r,x,y,z)\right]\right)\right]_{s=s_m,r=0}^{t=t_n}\\
&\times\partial_r^{(\lambda)}\vartheta_{\xi,\rho,0}(x,y,0)dydx
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{\xi=0}^{n-1}\sum_{\mu=0}^{k-1}\sum_{\lambda=0}^{\mu}\sum_{q=0}^{j-1}\sum_{\eta=0}^{q}\binom{\mu}{\lambda}\binom{q}{\eta}\\
&\times\int_{t_\xi}^{t_{\xi+1}}\partial_t^{(i)}\left[\partial_s^{(q-\eta)}\left(\partial_s^{(j-1-q)}\Big|_{y=s}\left[\partial_r^{(\mu-\lambda)}\left[\partial_r^{(k-1-\mu)}\Big|_{z=r}\kappa(t,s,r,x,y,z)\right]\right]\right)\right]_{s=s_m,r=0}^{t=t_n}\\
&\times\partial_s^{(\eta)}\partial_r^{(\lambda)}\vartheta_{\xi,m,0}(x,s_m,0)dx+\sum_{\rho=0}^{m-1}\sum_{\mu=0}^{k-1}\sum_{\lambda=0}^{\mu}\sum_{\alpha=0}^{i-1}\sum_{\beta=0}^{\alpha}\binom{\mu}{\lambda}\binom{\alpha}{\beta}\\
&\times\int_{s_\rho}^{s_{\rho+1}}\partial_t^{(\alpha-\beta)}\left[\partial_t^{(i-1-\alpha)}\Big|_{x=t}\left(\partial_s^{(j)}\left[\partial_r^{(\mu-\lambda)}\left[\partial_r^{(k-1-\mu)}\Big|_{z=r}\kappa(t,s,r,x,y,z)\right]\right]\right)\right]_{s=s_m,r=0}^{t=t_n}\\
&\times\partial_t^{(\beta)}\partial_r^{(\lambda)}\vartheta_{n,\rho,0}(t_n,y,0)dy+\sum_{\mu=0}^{k-1}\sum_{\lambda=0}^{\mu}\sum_{q=0}^{j-1}\sum_{\eta=0}^{q}\sum_{\alpha=0}^{i-1}\sum_{\beta=0}^{\alpha}\binom{\mu}{\lambda}\binom{q}{\eta}\binom{\alpha}{\beta}\\
&\times\partial_t^{(\alpha-\beta)}\left[\partial_t^{(i-1-\alpha)}\Big|_{x=t}\left(\partial_s^{(q-\eta)}\left[\partial_s^{(j-1-q)}\Big|_{y=s}\left(\partial_r^{(\mu-\lambda)}\left[\partial_r^{(k-1-\mu)}\Big|_{z=r}\kappa(t,s,r,x,y,z)\right]\right)\right]\right)\right]_{s=s_m,r=0}^{t=t_n}\\
&\times\partial_t^{(\beta)}\partial_s^{(\eta)}\partial_r^{(\lambda)}\hat{\vartheta}_{n,m,0}(t_n,s_m,0).
\end{aligned}
$$

Taylor collocation solution in $\mathcal{R}_{n,m,\tau}$

At the outset, $\hat{\vartheta}_{n,m,\tau}$ stands as the exact solution to the following integral equation:

$$
\begin{aligned}
\hat{\vartheta}_{n,m,\tau}(t,s,r)=f(t,s,r)&+\sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1}\sum_{\theta=0}^{\tau-1}\int_{t_\xi}^{t_{\xi+1}}\int_{s_\rho}^{s_{\rho+1}}\int_{r_\theta}^{r_{\theta+1}}\kappa(t,s,r,x,y,z)\vartheta_{\xi,\rho,\theta}(x,y,z)dzdydx\\
&+\sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1}\int_{t_\xi}^{t_{\xi+1}}\int_{s_\rho}^{s_{\rho+1}}\int_{r_\tau}^{r}\kappa(t,s,r,x,y,z)\vartheta_{\xi,\rho,\tau}(x,y,z)dzdydx\\
&+\sum_{\xi=0}^{n-1}\sum_{\theta=0}^{\tau-1}\int_{t_\xi}^{t_{\xi+1}}\int_{s_m}^{s}\int_{r_\theta}^{r_{\theta+1}}\kappa(t,s,r,x,y,z)\vartheta_{\xi,m,\theta}(x,y,z)dzdydx\\
&+\sum_{\xi=0}^{n-1}\int_{t_\xi}^{t_{\xi+1}}\int_{s_m}^{s}\int_{r_\tau}^{r}\kappa(t,s,r,x,y,z)\vartheta_{\xi,m,\tau}(x,y,z)dzdydx\\
&+\sum_{\rho=0}^{m-1}\sum_{\theta=0}^{\tau-1}\int_{t_n}^{t}\int_{s_\rho}^{s_{\rho+1}}\int_{r_\theta}^{r_{\theta+1}}\kappa(t,s,r,x,y,z)\vartheta_{n,\rho,\theta}(x,y,z)dzdydx\\
&+\sum_{\rho=0}^{m-1}\int_{t_n}^{t}\int_{s_\rho}^{s_{\rho+1}}\int_{r_\tau}^{r}\kappa(t,s,r,x,y,z)\vartheta_{n,\rho,\tau}(x,y,z)dzdydx
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{\theta=0}^{\tau-1}\int_{t_n}^{t}\int_{s_m}^{s}\int_{r_\theta}^{r_{\theta+1}}\kappa(t,s,r,x,y,z)\vartheta_{n,m,\theta}(x,y,z)dzdydx\\
&+\int_{t_n}^{t}\int_{s_m}^{s}\int_{r_\tau}^{r}\kappa(t,s,r,x,y,z)\hat{\vartheta}_{n,m,\tau}(x,y,z)dzdydx.
\end{aligned} \tag{4.18}
$$

By applying the same steps as before and differentiating equation (4.18) k-times with respect to r, j-times with respect to s, and i-times with respect to t, we derive:

$$
\begin{aligned}
&\partial_t^{(i)}\partial_s^{(j)}\partial_r^{(k)}\hat{\vartheta}_{n,m,\tau}(t,s,r) = \partial_t^{(i)}\partial_s^{(j)}\partial_r^{(k)}f(t,s,r)\\
&+\sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1}\sum_{\theta=0}^{\tau-1}\int_{t_\xi}^{t_{\xi+1}}\int_{s_\rho}^{s_{\rho+1}}\int_{r_\theta}^{r_{\theta+1}}\partial_t^{(i)}\partial_s^{(j)}\partial_r^{(k)}\kappa(t,s,r,x,y,z)\vartheta_{\xi,\rho,\theta}(x,y,z)dzdydx\\
&+\sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1}\sum_{\mu=0}^{k-1}\sum_{\lambda=0}^{\mu}\binom{\mu}{\lambda}\int_{t_\xi}^{t_{\xi+1}}\int_{s_\rho}^{s_{\rho+1}}\partial_t^{(i)}\left[\partial_s^{(j)}\left(\partial_r^{(\mu-\lambda)}\left[\partial_r^{(k-1-\mu)}\Big|_{z=r}\kappa(t,s,r,x,y,z)\right]\right)\right]\partial_r^{(\lambda)}\vartheta_{\xi,\rho,\tau}(x,y,r)dydx\\
&+\sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1}\int_{t_\xi}^{t_{\xi+1}}\int_{s_\rho}^{s_{\rho+1}}\int_{r_\tau}^{r}\partial_t^{(i)}\partial_s^{(j)}\partial_r^{(k)}\kappa(t,s,r,x,y,z)\vartheta_{\xi,\rho,\tau}(x,y,z)dzdydx\\
&+\sum_{\xi=0}^{n-1}\sum_{\theta=0}^{\tau-1}\sum_{q=0}^{j-1}\sum_{\eta=0}^{q}\binom{q}{\eta}\int_{t_\xi}^{t_{\xi+1}}\int_{r_\theta}^{r_{\theta+1}}\partial_t^{(i)}\left(\partial_s^{(q-\eta)}\left[\partial_s^{(j-1-q)}\Big|_{y=s}\partial_r^{(k)}\kappa(t,s,r,x,y,z)\right]\right)\partial_s^{(\eta)}\vartheta_{\xi,m,\theta}(x,s,z)dzdx\\
&+\sum_{\xi=0}^{n-1}\sum_{\theta=0}^{\tau-1}\int_{t_\xi}^{t_{\xi+1}}\int_{s_m}^{s}\int_{r_\theta}^{r_{\theta+1}}\partial_t^{(i)}\partial_s^{(j)}\partial_r^{(k)}\kappa(t,s,r,x,y,z)\vartheta_{\xi,m,\theta}(x,y,z)dzdydx\\
&+\sum_{\xi=0}^{n-1}\sum_{\mu=0}^{k-1}\sum_{\lambda=0}^{\mu}\sum_{q=0}^{j-1}\sum_{\eta=0}^{q}\binom{\mu}{\lambda}\binom{q}{\eta}\int_{t_\xi}^{t_{\xi+1}}\partial_t^{(i)}\left(\partial_s^{(q-\eta)}\left[\partial_s^{(j-1-q)}\Big|_{y=s}\left(\partial_r^{(\mu-\lambda)}\left[\partial_r^{(k-1-\mu)}\Big|_{z=r}\kappa(t,s,r,x,y,z)\right]\right)\right]\right)\\
&\times\partial_s^{(\eta)}\partial_r^{(\lambda)}\vartheta_{\xi,m,\tau}(x,s,r)dx\\
&+\sum_{\xi=0}^{n-1}\sum_{\mu=0}^{k-1}\sum_{\lambda=0}^{\mu}\binom{\mu}{\lambda}\int_{t_\xi}^{t_{\xi+1}}\int_{s_m}^{s}\partial_t^{(i)}\left[\partial_s^{(j)}\left(\partial_r^{(\mu-\lambda)}\left[\partial_r^{(k-1-\mu)}\Big|_{z=r}\kappa(t,s,r,x,y,z)\right]\right)\right]\partial_r^{(\lambda)}\vartheta_{\xi,m,\tau}(x,y,r)dydx\\
&+\sum_{\xi=0}^{n-1}\sum_{q=0}^{j-1}\sum_{\eta=0}^{q}\binom{q}{\eta}\int_{t_\xi}^{t_{\xi+1}}\int_{s_\tau}^{s}\partial_t^{(i)}\left(\partial_s^{(q-\eta)}\left[\partial_s^{(j-1-q)}\Big|_{y=s}\partial_r^{(k)}\kappa(t,s,r,x,y,z)\right]\right)\partial_s^{(\eta)}\vartheta_{\xi,m,\tau}(x,s,z)dzdx\\
&+\sum_{\xi=0}^{n-1}\int_{t_\xi}^{t_{\xi+1}}\int_{s_m}^{s}\int_{r_\tau}^{r}\partial_t^{(i)}\partial_s^{(j)}\partial_r^{(k)}\kappa(t,s,r,x,y,z)\vartheta_{\xi,m,\tau}(x,y,z)dzdydx
\end{aligned}
$$

$$+\sum_{\rho=0}^{m-1}\sum_{\theta=0}^{\tau-1}\sum_{\alpha=0}^{i-1}\sum_{\beta=0}^{\alpha}\binom{\alpha}{\beta}\int_{s_\rho}^{s_{\rho+1}}\int_{r_\theta}^{r_{\theta+1}}\partial_t^{(\alpha-\beta)}\left[\partial_t^{(i-1-\alpha)}\Big|_{x=t}\partial_s^{(j)}\partial_r^{(k)}\kappa(t,s,r,x,y,z)\right]\partial_t^{(\beta)}\vartheta_{n,\rho,\theta}(t,y,z)dzdy$$

$$+\sum_{\rho=0}^{m-1}\sum_{\theta=0}^{\tau-1}\int_{t_n}^{t}\int_{s_\rho}^{s_{\rho+1}}\int_{r_\theta}^{r_{\theta+1}}\partial_t^{(i)}\partial_s^{(j)}\partial_r^{(k)}\kappa(t,s,r,x,y,z)\vartheta_{n,\rho,\theta}(x,y,z)dzdydx$$

$$+\sum_{\rho=0}^{m-1}\sum_{\mu=0}^{k-1}\sum_{\lambda=0}^{\mu}\sum_{\alpha=0}^{i-1}\sum_{\beta=0}^{\alpha}\binom{\mu}{\lambda}\binom{\alpha}{\beta}\int_{s_\rho}^{s_{\rho+1}}\partial_t^{(\alpha-\beta)}\left[\partial_t^{(i-1-\alpha)}\Big|_{x=t}\left(\partial_s^{(j)}\left[\partial_r^{(\mu-\lambda)}\left[\partial_r^{(k-1-\mu)}\Big|_{z=r}\kappa(t,s,r,x,y,z)\right]\right]\right)\right]$$

$$\times\partial_t^{(\beta)}\partial_r^{(\lambda)}\vartheta_{n,\rho,\tau}(t,y,r)dy$$

$$+\sum_{\rho=0}^{m-1}\sum_{\mu=0}^{k-1}\sum_{\lambda=0}^{\mu}\binom{\mu}{\lambda}\int_{t_n}^{t}\int_{s_\rho}^{s_{\rho+1}}\partial_t^{(i)}\left[\partial_s^{(j)}\left(\partial_r^{(\mu-\lambda)}\left[\partial_r^{(k-1-\mu)}\Big|_{z=r}\kappa(t,s,r,x,y,z)\right]\right)\right]\partial_r^{(\lambda)}\vartheta_{n,\rho,\tau}(x,y,r)dydx$$

$$+\sum_{\rho=0}^{m-1}\sum_{\alpha=0}^{i-1}\sum_{\beta=0}^{\alpha}\binom{\alpha}{\beta}\int_{s_\rho}^{s_{\rho+1}}\int_{r_\tau}^{r}\partial_t^{(\alpha-\beta)}\left[\partial_t^{(i-1-\alpha)}\Big|_{x=t}\partial_s^{(j)}\partial_r^{(k)}\kappa(t,s,r,x,y,z)\right]\partial_t^{(\beta)}\vartheta_{n,\rho,\tau}(t,y,z)dzdy$$

$$+\sum_{\rho=0}^{m-1}\int_{t_n}^{t}\int_{s_\rho}^{s_{\rho+1}}\int_{r_\tau}^{r}\partial_t^{(i)}\partial_s^{(j)}\partial_r^{(k)}\kappa(t,s,r,x,y,z)\vartheta_{n,\rho,\tau}(x,y,z)dzdydx$$

$$+\sum_{\theta=0}^{\tau-1}\sum_{q=0}^{j-1}\sum_{\eta=0}^{q}\sum_{\alpha=0}^{i-1}\sum_{\beta=0}^{\alpha}\binom{q}{\eta}\binom{\alpha}{\beta}\int_{r_\theta}^{r_{\theta+1}}\partial_t^{(\alpha-\beta)}\left[\partial_t^{(i-1-\alpha)}\Big|_{x=t}\left(\partial_s^{(q-\eta)}\left[\partial_s^{(j-1-q)}\Big|_{y=s}\partial_r^{(k)}\kappa(t,s,r,x,y,z)\right]\right)\right]$$

$$\times\partial_t^{(\beta)}\partial_s^{(\eta)}\vartheta_{n,m,\theta}(t,s,z)dz$$

$$+\sum_{\theta=0}^{\tau-1}\sum_{q=0}^{j-1}\sum_{\eta=0}^{q}\binom{q}{\eta}\int_{t_n}^{t}\int_{r_\theta}^{r_{\theta+1}}\partial_t^{(i)}\left(\partial_s^{(q-\eta)}\left[\partial_s^{(j-1-q)}\Big|_{y=s}\partial_r^{(k)}\kappa(t,s,r,x,y,z)\right]\right)\partial_s^{(\eta)}\vartheta_{n,m,\theta}(x,s,z)dzdx$$

$$+\sum_{\theta=0}^{\tau-1}\sum_{\alpha=0}^{i-1}\sum_{\beta=0}^{\alpha}\binom{\alpha}{\beta}\int_{s_m}^{s}\int_{r_\theta}^{r_{\theta+1}}\partial_t^{(\alpha-\beta)}\left[\partial_t^{(i-1-\alpha)}\Big|_{x=t}\partial_s^{(j)}\partial_r^{(k)}\kappa(t,s,r,x,y,z)\right]\partial_t^{(\beta)}\vartheta_{n,m,\theta}(t,y,z)dzdy$$

$$+\sum_{\theta=0}^{\tau-1}\int_{t_n}^{t}\int_{s_m}^{s}\int_{r_\theta}^{r_{\theta+1}}\partial_t^{(i)}\partial_s^{(j)}\partial_r^{(k)}\kappa(t,s,r,x,y,z)\vartheta_{n,m,\theta}(x,y,z)dzdydx$$

$$+\sum_{\mu=0}^{k-1}\sum_{\lambda=0}^{\mu}\sum_{q=0}^{j-1}\sum_{\eta=0}^{q}\sum_{\alpha=0}^{i-1}\sum_{\beta=0}^{\alpha}\binom{\mu}{\lambda}\binom{q}{\eta}\binom{\alpha}{\beta}$$

$$\times\partial_t^{(\alpha-\beta)}\left[\partial_t^{(i-1-\alpha)}\Big|_{x=t}\left(\partial_s^{(q-\eta)}\left[\partial_s^{(j-1-q)}\Big|_{y=s}\left(\partial_r^{(\mu-\lambda)}\left[\partial_r^{(k-1-\mu)}\Big|_{z=r}\kappa(t,s,r,x,y,z)\right]\right)\right]\right)\right]\partial_t^{(\beta)}\partial_s^{(\eta)}\partial_r^{(\lambda)}\hat{\vartheta}_{n,m,\tau}(t,s,r)$$

$$+\sum_{\mu=0}^{k-1}\sum_{\lambda=0}^{\mu}\sum_{q=0}^{j-1}\sum_{\eta=0}^{q}\binom{\mu}{\lambda}\binom{q}{\eta}$$

$$\times\int_{t_n}^{t}\partial_t^{(i)}\left(\partial_s^{(q-\eta)}\left[\partial_s^{(j-1-q)}\Big|_{y=s}\left(\partial_r^{(\mu-\lambda)}\left[\partial_r^{(k-1-\mu)}\Big|_{z=r}\kappa(t,s,r,x,y,z)\right]\right)\right]\right)\partial_s^{(\eta)}\partial_r^{(\lambda)}\hat{\vartheta}_{n,m,\tau}(x,s,r)dx$$

(4.19)

$$
\begin{aligned}
&+\sum_{\mu=0}^{k-1}\sum_{\lambda=0}^{\mu}\sum_{\alpha=0}^{i-1}\sum_{\beta=0}^{\alpha}\binom{\mu}{\lambda}\binom{\alpha}{\beta}\int_{s_m}^{s}\partial_t^{(\alpha-\beta)}\left[\partial_t^{(i-1-\alpha)}\Big|_{x=t}\left(\partial_s^{(j)}\left[\partial_r^{(\mu-\lambda)}\left[\partial_r^{(k-1-\mu)}\Big|_{z=r}\kappa(t,s,r,x,y,z)\right]\right]\right)\right]\\
&\times\partial_t^{(\beta)}\partial_r^{(\lambda)}\hat{\vartheta}_{n,m,\tau}(t,y,r)dy\\
&+\sum_{\mu=0}^{k-1}\sum_{\lambda=0}^{\mu}\binom{\mu}{\lambda}\int_{t_n}^{t}\int_{s_m}^{s}\partial_t^{(i)}\left[\partial_s^{(j)}\left(\partial_r^{(\mu-\lambda)}\left[\partial_r^{(k-1-\mu)}\Big|_{z=r}\kappa(t,s,r,x,y,z)\right]\right)\right]\partial_r^{(\lambda)}\hat{\vartheta}_{n,m,\tau}(x,y,r)dydx\\
&+\sum_{q=0}^{j-1}\sum_{\eta=0}^{q}\sum_{\alpha=0}^{i-1}\sum_{\beta=0}^{\alpha}\binom{q}{\eta}\binom{\alpha}{\beta}\int_{r_\tau}^{r}\partial_t^{(\alpha-\beta)}\left[\partial_t^{(i-1-\alpha)}\Big|_{x=t}\left(\partial_s^{(q-\eta)}\left[\partial_s^{(j-1-q)}\Big|_{y=s}\partial_r^{(k)}\kappa(t,s,r,x,y,z)\right]\right)\right]\\
&\times\partial_t^{(\beta)}\partial_s^{(\eta)}\hat{\vartheta}_{n,m,\tau}(t,s,z)dz\\
&+\sum_{q=0}^{j-1}\sum_{\eta=0}^{q}\binom{q}{\eta}\int_{t_n}^{t}\int_{r_\tau}^{r}\partial_t^{(i)}\left(\partial_s^{(q-\eta)}\left[\partial_s^{(j-1-q)}\Big|_{y=s}\partial_r^{(k)}\kappa(t,s,r,x,y,z)\right]\right)\partial_s^{(\eta)}\hat{\vartheta}_{n,m,\tau}(x,s,z)dzdx\\
&+\sum_{\alpha=0}^{i-1}\sum_{\beta=0}^{\alpha}\binom{\alpha}{\beta}\int_{s_m}^{s}\int_{r_\tau}^{r}\partial_t^{(\alpha-\beta)}\left[\partial_t^{(i-1-\alpha)}\partial_s^{(j)}\partial_r^{(k)}\kappa(t,s,r,x,y,z)\right]\partial_t^{(\beta)}\hat{\vartheta}_{n,m,\tau}(t,y,z)dzdy\\
&+\int_{t_n}^{t}\int_{s_m}^{s}\int_{r_\tau}^{r}\partial_t^{(i)}\partial_s^{(j)}\partial_r^{(k)}\kappa(t,s,r,x,y,z)\hat{\vartheta}_{n,m,\tau}(x,y,z)dzdydx,
\end{aligned}
$$

hence,

$$
\begin{aligned}
&\partial_t^{(i)}\partial_s^{(j)}\partial_r^{(k)}\hat{\vartheta}_{n,m,\tau}(t_n,s_m,r_\tau)=\partial_t^{(i)}\partial_s^{(j)}\partial_r^{(k)}f(t_n,s_m,r_\tau)\\
&+\sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1}\sum_{\theta=0}^{\tau-1}\int_{t_\xi}^{t_{\xi+1}}\int_{s_\rho}^{s_{\rho+1}}\int_{r_\theta}^{r_{\theta+1}}\left[\partial_t^{(i)}\partial_s^{(j)}\partial_r^{(k)}\kappa(t,s,r,x,y,z)\right]_{s=s_m,r=r_\tau}^{t=t_n}\vartheta_{\xi,\rho,\theta}(x,y,z)dzdydx\\
&+\sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1}\sum_{\mu=0}^{k-1}\sum_{\lambda=0}^{\mu}\binom{\mu}{\lambda}\int_{t_\xi}^{t_{\xi+1}}\int_{s_\rho}^{s_{\rho+1}}\partial_t^{(i)}\left[\partial_s^{(j)}\left(\partial_r^{(\mu-\lambda)}\left[\partial_r^{(k-1-\mu)}\Big|_{z=r}\kappa(t,s,r,x,y,z)\right]\right)\right]_{s=s_m,r=r_\tau}^{t=t_n}\\
&\times\partial_r^{(\lambda)}\vartheta_{\xi,\rho,\tau}(x,y,r_\tau)dydx+\sum_{\xi=0}^{n-1}\sum_{\theta=0}^{\tau-1}\sum_{q=0}^{j-1}\sum_{\eta=0}^{q}\binom{q}{\eta}\\
&\times\int_{t_\xi}^{t_{\xi+1}}\int_{r_\theta}^{r_{\theta+1}}\partial_t^{(i)}\left[\partial_s^{(q-\eta)}\left[\partial_s^{(j-1-q)}\Big|_{y=s}\partial_r^{(k)}\kappa(t,s,r,x,y,z)\right]\right]_{s=s_m,r=r_\tau}^{t=t_n}\partial_s^{(\eta)}\vartheta_{\xi,m,\theta}(x,s_m,z)dzdx\\
&+\sum_{\xi=0}^{n-1}\sum_{\mu=0}^{k-1}\sum_{\lambda=0}^{\mu}\sum_{q=0}^{j-1}\sum_{\eta=0}^{q}\binom{\mu}{\lambda}\binom{q}{\eta}\\
&\times\int_{t_\xi}^{t_{\xi+1}}\partial_t^{(i)}\left[\partial_s^{(q-\eta)}\left[\partial_s^{(j-1-q)}\Big|_{y=s}\left(\partial_r^{(\mu-\lambda)}\Big|_{z=r}\left[\partial_r^{(k-1-\mu)}\kappa(t,s,r,x,y,z)\right]\right)\right]\right]_{s=s_m,r=r_\tau}^{t=t_n}
\end{aligned}
$$

$$
\begin{aligned}
&\times \partial_s^{(\eta)}\partial_r^{(\lambda)}\vartheta_{\xi,m,\tau}(x,s_m,r_\tau)dx + \sum_{\rho=0}^{m-1}\sum_{\theta=0}^{\tau-1}\sum_{\alpha=0}^{i-1}\sum_{\beta=0}^{\alpha}\binom{\alpha}{\beta}\\
&\times \int_{s_\rho}^{s_{\rho+1}}\int_{r_\theta}^{r_{\theta+1}} \partial_t^{(\alpha-\beta)}\left[\partial_t^{(i-1-\alpha)}\Big|_{x=t}\partial_s^{(j)}\partial_r^{(k)}\kappa(t,s,r,x,y,z)\right]_{s=s_m,r=r_\tau}^{t=t_n}\partial_t^{(\beta)}\vartheta_{n,\rho,\theta}(t_n,y,z)dzdy\\
&+\sum_{\rho=0}^{m-1}\sum_{\mu=0}^{k-1}\sum_{\lambda=0}^{\mu}\sum_{\alpha=0}^{i-1}\sum_{\beta=0}^{\alpha}\binom{\mu}{\lambda}\binom{\alpha}{\beta}\\
&\times \int_{s_\rho}^{s_{\rho+1}}\partial_t^{(\alpha-\beta)}\left[\partial_t^{(i-1-\alpha)}\Big|_{x=t}\left(\partial_s^{(j)}\left[\partial_r^{(\mu-\lambda)}\left[\partial_r^{(k-1-\mu)}\Big|_{z=r}\kappa(t,s,r,x,y,z)\right]\right]\right)\right]_{s=s_m,r=r_\tau}^{t=t_n}\\
&\times \partial_t^{(\beta)}\partial_r^{(\lambda)}\vartheta_{n,\rho,\tau}(t_n,y,r_\tau)dy + \sum_{\theta=0}^{\tau-1}\sum_{q=0}^{j-1}\sum_{\eta=0}^{q}\sum_{\alpha=0}^{i-1}\sum_{\beta=0}^{\alpha}\binom{q}{\eta}\binom{\alpha}{\beta}\\
&\times \int_{r_\theta}^{r_{\theta+1}}\partial_t^{(\alpha-\beta)}\left[\partial_t^{(i-1-\alpha)}\Big|_{x=t}\left(\partial_s^{(q-\eta)}\left[\partial_s^{(j-1-q)}\Big|_{y=s}\partial_r^{(k)}\kappa(t,s,r,x,y,z)\right]\right)\right]_{s=s_m,r=r_\tau}^{t=t_n}\\
&\times \partial_t^{(\beta)}\partial_s^{(\eta)}\vartheta_{n,m,\theta}(t_n,s_m,z)dz + \sum_{\mu=0}^{k-1}\sum_{\lambda=0}^{\mu}\sum_{q=0}^{j-1}\sum_{\eta=0}^{q}\sum_{\alpha=0}^{i-1}\sum_{\beta=0}^{\alpha}\binom{\mu}{\lambda}\binom{q}{\eta}\binom{\alpha}{\beta}\\
&\times \partial_t^{(\alpha-\beta)}\left[\partial_t^{(i-1-\alpha)}\Big|_{x=t}\left(\partial_s^{(q-\eta)}\left[\partial_s^{(j-1-q)}\Big|_{y=s}\left(\partial_r^{(\mu-\lambda)}\left[\partial_r^{(k-1-\mu)}\Big|_{z=r}\kappa(t,s,r,x,y,z)\right]\right)\right]\right)\right]_{s=s_m,r=r_\tau}^{t=t_n}\\
&\times \partial_t^{(\beta)}\partial_s^{(\eta)}\partial_r^{(\lambda)}\hat{\vartheta}_{n,m,\tau}(t_n,s_m,r_\tau).
\end{aligned}
$$

4.2.2 Algorithm for nonlinear 3D-Volterra integral equations

Taylor collocation solution in $\mathcal{R}_{0,0,0}$

The differentiation of (4.2) k, j and i-times in terms of r, s and t, respectively, gives

$$
\begin{aligned}
&\partial_t^{(i)}\partial_s^{(j)}\partial_r^{(k)}w(0,0,0) = \partial_t^{(i)}\partial_s^{(j)}\partial_r^{(k)}f(0,0,0) + \sum_{\mu=0}^{k-1}\sum_{q=0}^{j-1}\sum_{\alpha=0}^{i-1}\\
&\partial_t^{(\alpha)}\left[\partial_t^{(i-1-\alpha)}\Big|_{x=t}\left(\partial_s^{(q)}\left[\partial_s^{(j-1-q)}\Big|_{y=s}\left(\partial_r^{(\mu)}\left[\partial_r^{(k-1-\mu)}\Big|_{z=r}\kappa(t,s,r,x,y,z,w(x,y,z))\right]\right)\right]\right)\right]_{s=r=0}^{t=0}.
\end{aligned}
$$

Taylor collocation solution in $\mathcal{R}_{n,0,0}$

Initially, $\hat{\vartheta}_{n,0,0}$ is the exact solution to the integral equation that follows:

$$\begin{aligned}\hat{\vartheta}_{n,0,0}(t,s,r) = f(t,s,r) + \sum_{\xi=0}^{n-1}\int_{t_\xi}^{t_{\xi+1}}\int_0^s\int_0^r \kappa(t,s,r,x,y,z,\vartheta_{\xi,0,0}(x,y,z))dzdydx \\ + \int_{t_n}^{t}\int_0^s\int_0^r \kappa(t,s,r,x,y,z,\hat{\vartheta}_{n,0,0}(x,y,z))dzdydx,\end{aligned} \tag{4.20}$$

by differentiating equation (4.20) k-times with respect to r, j-times with respect to s, and i-times with respect to t, we derive

$$\begin{aligned}&\partial_t^{(i)}\partial_s^{(j)}\partial_r^{(k)}\hat{\vartheta}_{n,0,0}(t_n,0,0) = \partial_t^{(i)}\partial_s^{(j)}\partial_r^{(k)}f(x_n,0,0) + \sum_{\xi=0}^{n-1}\sum_{\mu=0}^{k-1}\sum_{q=0}^{j-1} \\ &\int_{t_\xi}^{t_{\xi+1}}\partial_t^{(i)}\left[\partial_s^{(q)}\left[\partial_s^{(j-1-q)}\Big|_{y=s}\left(\partial_r^{(\mu)}\left[\partial_r^{(k-1-\mu)}\Big|_{z=r}\kappa(t,s,r,x,y,z,\vartheta_{\xi,0,0}(x,y,z))\right]\right)\right]\right]_{s=r=0}^{t=t_n}dx \\ &+\sum_{\mu=0}^{k-1}\sum_{q=0}^{j-1}\sum_{\alpha=0}^{i-1} \\ &\partial_t^{(\alpha)}\left[\partial_t^{(i-1-\alpha)}\Big|_{x=t}\left(\partial_s^{(q)}\left[\partial_s^{(j-1-q)}\Big|_{y=s}\left(\partial_r^{(\mu)}\left[\partial_r^{(k-1-\mu)}\Big|_{z=r}\kappa(t,s,r,x,y,z,\hat{\vartheta}_{n,0,0}(x,y,z))\right]\right)\right]\right)\right]_{s=r=0}^{t=t_n}.\end{aligned}$$

Taylor collocation solution in $\mathcal{R}_{n,m,0}$

In the beginning, $\hat{\vartheta}_{n,m,0}$ is the exact solution to the following integral equation:

$$\begin{aligned}\hat{\vartheta}_{n,m,0}(t,s,r) = f(t,s,r) + \sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1}\int_{t_\xi}^{t_{\xi+1}}\int_{s_\rho}^{s_{\rho+1}}\int_0^r \kappa(t,s,r,x,y,z,\vartheta_{\xi,\rho,0}(x,y,z))dzdydx \\ + \sum_{\xi=0}^{n-1}\int_{t_\xi}^{t_{\xi+1}}\int_{s_m}^{s}\int_0^r \kappa(t,s,r,x,y,z\vartheta_{\xi,m,0}(x,y,z))dzdydx \\ + \sum_{\rho=0}^{m-1}\int_{t_n}^{t}\int_{s_\rho}^{s_{\rho+1}}\int_0^r \kappa(t,s,r,x,y,z,\vartheta_{n,\rho,0}(x,y,z))dzdydx \\ + \int_{t_n}^{t}\int_{s_m}^{s}\int_0^r \kappa(t,s,r,x,y,z,\hat{\vartheta}_{n,m,0}(x,y,z))dzdydx,\end{aligned} \tag{4.21}$$

by differentiating equation (4.21) k-times with respect to r, j-times with respect to s, and i-times with respect to t, we derive:

$$\partial_t^{(i)}\partial_s^{(j)}\partial_r^{(k)}\hat{\vartheta}_{n,m,0}(t_n,s_m,0) = \partial_t^{(i)}\partial_s^{(j)}\partial_r^{(k)}f(t_n,s_m,0) + \sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1}\sum_{\mu=0}^{k-1}$$
$$\int_{t_\xi}^{t_{\xi+1}}\int_{s_\rho}^{s_{\rho+1}} \partial_t^{(i)}\left[\partial_s^{(j)}\left(\partial_r^{(\mu)}\left[\partial_r^{(k-1-\mu)}\Big|_{z=r}\kappa(t,s,r,x,y,z,\vartheta_{\xi,\rho,0}(x,y,z))\right]\right)\right]_{s=s_m,r=0}^{t=t_n} dydx$$
$$+\sum_{\xi=0}^{n-1}\sum_{\mu=0}^{k-1}\sum_{q=0}^{j-1}$$
$$\int_{t_\xi}^{t_{\xi+1}} \partial_t^{(i)}\left[\partial_s^{(q)}\left(\partial_s^{(j-1-q)}\Big|_{y=s}\left[\partial_r^{(\mu)}\left[\partial_r^{(k-1-\mu)}\Big|_{z=r}\kappa(t,s,r,x,y,z,\vartheta_{\xi,m,0}(x,y,z))\right]\right]\right)\right]_{s=s_m,r=0}^{t=t_n} dx$$
$$+\sum_{\rho=0}^{m-1}\sum_{\mu=0}^{k-1}\sum_{\alpha=0}^{i-1}$$
$$\int_{s_\rho}^{s_{\rho+1}} \partial_t^{(\alpha)}\left[\partial_t^{(i-1-\alpha)}\Big|_{x=t}\left(\partial_s^{(j)}\left[\partial_r^{(\mu)}\left[\partial_r^{(k-1-\mu)}\Big|_{z=r}\kappa(t,s,r,x,y,z,\vartheta_{n,\rho,0}(x,y,z))\right]\right]\right)\right]_{s=s_m,r=0}^{t=t_n} dy$$
$$+\sum_{\mu=0}^{k-1}\sum_{q=0}^{j-1}\sum_{\alpha=0}^{i-1}$$
$$\partial_t^{(\alpha)}\left[\partial_t^{(i-1-\alpha)}\Big|_{x=t}\left(\partial_s^{(q)}\left[\partial_s^{(j-1-q)}\Big|_{y=s}\left(\partial_r^{(\mu)}\left[\partial_r^{(k-1-\mu)}\Big|_{z=r}\kappa(t,s,r,x,y,z,\hat{\vartheta}_{n,m,0}(x,y,z))\right]\right)\right]\right)\right]_{s=s_m,r=0}^{t=t_n}.$$

Taylor collocation solution in $\mathcal{R}_{n,m,\tau}$

Initially, $\hat{\vartheta}_{n,m,\tau}$ is the exact solution to the following integral equation:

$$\hat{\vartheta}_{n,m,\tau}(t,s,r) = f(t,s,r) + \sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1}\sum_{\theta=0}^{\tau-1}\int_{t_\xi}^{t_{\xi+1}}\int_{s_\rho}^{s_{\rho+1}}\int_{r_\theta}^{r_{\theta+1}}\kappa(t,s,r,x,y,z,\vartheta_{\xi,\rho,\theta}(x,y,z))dzdydx$$
$$+\sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1}\int_{t_\xi}^{t_{\xi+1}}\int_{s_\rho}^{s_{\rho+1}}\int_{r_\tau}^{r}\kappa(t,s,r,x,y,z,\vartheta_{\xi,\rho,\tau}(x,y,z))dzdydx$$
$$+\sum_{\xi=0}^{n-1}\sum_{\theta=0}^{\tau-1}\int_{t_\xi}^{t_{\xi+1}}\int_{s_m}^{s}\int_{r_\theta}^{r_{\theta+1}}\kappa(t,s,r,x,y,z,\vartheta_{\xi,m,\theta}(x,y,z))dzdydx$$
$$+\sum_{\xi=0}^{n-1}\int_{t_\xi}^{t_{\xi+1}}\int_{s_m}^{s}\int_{r_\tau}^{r}\kappa(t,s,r,x,y,z,\vartheta_{\xi,m,\tau}(x,y,z))dzdydx$$

(4.22)

$$+\sum_{\rho=0}^{m-1}\sum_{\theta=0}^{\tau-1}\int_{t_n}^{t}\int_{s_\rho}^{s_{\rho+1}}\int_{r_\theta}^{r_{\theta+1}}\kappa(t,s,r,x,y,z,\vartheta_{n,\rho,\theta}(x,y,z))dzdydx$$
$$+\sum_{\rho=0}^{m-1}\int_{t_n}^{t}\int_{s_\rho}^{s_{\rho+1}}\int_{r_\tau}^{r}\kappa(t,s,r,x,y,z,\vartheta_{n,\rho,\tau}(x,y,z))dzdydx$$
$$+\sum_{\theta=0}^{\tau-1}\int_{t_n}^{t}\int_{s_m}^{s}\int_{r_\theta}^{r_{\theta+1}}\kappa(t,s,r,x,y,z,\vartheta_{n,m,\theta}(x,y,z))dzdydx$$
$$+\int_{t_n}^{t}\int_{s_m}^{s}\int_{r_\tau}^{r}\kappa(t,s,r,x,y,z,\hat{\vartheta}_{n,m,\tau}(x,y,z))dzdydx,$$

by differentiating equation (4.22) k-times with respect to r, j-times with respect to s, and i-times with respect to t, we derive:

$$\partial_t^{(i)}\partial_s^{(j)}\partial_r^{(k)}\hat{\vartheta}_{n,m,\tau}(t_n,s_m,r_\tau)=\partial_t^{(i)}\partial_s^{(j)}\partial_r^{(k)}f(t_n,s_m,r_\tau)$$
$$+\sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1}\sum_{\theta=0}^{\tau-1}\int_{t_\xi}^{t_{\xi+1}}\int_{s_\rho}^{s_{\rho+1}}\int_{r_\theta}^{r_{\theta+1}}\left[\partial_t^{(i)}\partial_s^{(j)}\partial_r^{(k)}\kappa(t,s,r,x,y,z,\vartheta_{\xi,\rho,\theta}(x,y,z))\right]_{s=s_m,r=r_\tau}^{t=t_n}dzdydx$$
$$+\sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1}\sum_{\mu=0}^{k-1}\int_{t_\xi}^{t_{\xi+1}}\int_{s_\rho}^{s_{\rho+1}}\partial_t^{(i)}\left[\partial_s^{(j)}\left(\partial_r^{(\mu)}\left[\partial_r^{(k-1-\mu)}\Big|_{z=r}\kappa(t,s,r,x,y,z,\vartheta_{\xi,\rho,\tau}(x,y,z))\right]\right)\right]_{s=s_m,r=r_\tau}^{t=t_n}dydx$$
$$+\sum_{\xi=0}^{n-1}\sum_{\theta=0}^{\tau-1}\sum_{q=0}^{j-1}\int_{t_\xi}^{t_{\xi+1}}\int_{r_\theta}^{r_{\theta+1}}\partial_t^{(i)}\left[\partial_s^{(q)}\left[\partial_s^{(j-1-q)}\Big|_{y=s}\partial_r^{(k)}\kappa(t,s,r,x,y,z,\vartheta_{\xi,m,\theta}(x,y,z))\right]\right]_{s=s_m,r=r_\tau}^{t=t_n}dzdx$$
$$+\sum_{\xi=0}^{n-1}\sum_{\mu=0}^{k-1}\sum_{q=0}^{j-1}\int_{t_\xi}^{t_{\xi+1}}\partial_t^{(i)}\left[\partial_s^{(q)}\left[\partial_s^{(j-1-q)}\Big|_{y=s}\left(\partial_r^{(\mu)}\Big|_{z=r}\left[\partial_r^{(k-1-\mu)}\kappa(t,s,r,x,y,z,\vartheta_{\xi,m,\tau}(x,y,z))\right]\right)\right]\right]_{s=s_m,r=r_\tau}^{t=t_n}dx$$
$$+\sum_{\rho=0}^{m-1}\sum_{\theta=0}^{\tau-1}\sum_{\alpha=0}^{i-1}\int_{s_\rho}^{s_{\rho+1}}\int_{r_\theta}^{r_{\theta+1}}\partial_t^{(\alpha)}\left[\partial_t^{(i-1-\alpha)}\Big|_{x=t}\partial_s^{(j)}\partial_r^{(k)}\kappa(t,s,r,x,y,z,\vartheta_{n,\rho,\theta}(x,y,z))\right]_{s=s_m,r=r_\tau}^{t=t_n}dzdy$$
$$+\sum_{\rho=0}^{m-1}\sum_{\mu=0}^{k-1}\sum_{\alpha=0}^{i-1}\int_{s_\rho}^{s_{\rho+1}}\partial_t^{(\alpha)}\left[\partial_t^{(i-1-\alpha)}\Big|_{x=t}\left(\partial_s^{(j)}\left[\partial_r^{(\mu)}\left[\partial_r^{(k-1-\mu)}\Big|_{z=r}\kappa(t,s,r,x,y,z,\vartheta_{n,\rho,\tau}(x,y,z))\right]\right]\right)\right]_{s=s_m,r=r_\tau}^{t=t_n}dy$$
$$+\sum_{\theta=0}^{\tau-1}\sum_{q=0}^{j-1}\sum_{\alpha=0}^{i-1}\int_{r_\theta}^{r_{\theta+1}}\partial_t^{(\alpha)}\left[\partial_t^{(i-1-\alpha)}\Big|_{x=t}\left(\partial_s^{(q)}\left[\partial_s^{(j-1-q)}\Big|_{y=s}\partial_r^{(k)}\kappa(t,s,r,x,y,z,\vartheta_{n,m,\theta}(x,y,z))\right]\right)\right]_{s=s_m,r=r_\tau}^{t=t_n}dz$$
$$+\sum_{\mu=0}^{k-1}\sum_{q=0}^{j-1}\sum_{\alpha=0}^{i-1}\partial_t^{(\alpha)}\left[\partial_t^{(i-1-\alpha)}\Big|_{x=t}\left(\partial_s^{(q)}\left[\partial_s^{(j-1-q)}\Big|_{y=s}\left(\partial_r^{(\mu)}\left[\partial_r^{(k-1-\mu)}\Big|_{z=r}\kappa(t,s,r,x,y,z,\hat{\vartheta}_{n,m,\tau}(x,y,z))\right]\right)\right]\right)\right]_{s=s_m,r=r_\tau}^{t=t_n}.$$

4.3 Error Analysis

We consider the space $L^{\infty}(\mathcal{R})$ with the norm

$$\|\Gamma\|_{L^{\infty}(\mathcal{R})} = \inf\{K \in \mathbb{R} : |\Gamma(t,s,r)| \leq K \ \text{ for a.e. } (t,s,r) \in \mathcal{R}\} < \infty.$$

The following lemmas will be used to illustrate the convergence of the suggested approach, with an emphasis on the linear form provided in (4.1).

Lemma 4.3.1 (*Taylor's Theorem for functions of three independent variables*) *Let f be p times continuously differentiable on $\mathcal{R} = [a,b] \times [c,d] \times [e,h]$ and let $(t_0, s_0, r_0) \in \mathcal{R}$. Then for all $(t,s,r) \in \mathcal{R}$, we have*

$$\begin{aligned} f(t,s,r) = &\sum_{i+j+k=0}^{p-1} \frac{1}{i!j!k!} \partial_t^{(i)} \partial_s^{(j)} \partial_r^{(k)} f(t_0,s_0,r_0)(t-t_0)^i (s-s_0)^j (r-r_0)^k \\ &+ \sum_{i+j+k=p} \frac{1}{i!j!k!} \partial_t^{(i)} \partial_s^{(j)} \partial_r^{(k)} f(t_1,s_1,r_1)(t-t_0)^i (s-s_0)^j (r-r_0)^k, \end{aligned}$$

where,

$$\begin{cases} t_1 = \theta t + (1-\theta)t_0 \in [a,b], \\ s_1 = \theta s + (1-\theta)s_0 \in [c,d], \qquad \theta \in (0,1). \\ r_1 = \theta r + (1-\theta)r_0 \in [e,h], \end{cases}$$

Proof. let

$$F(\theta) = f\left(t(\theta), s(\theta), r(\theta)\right),$$

where $t(\theta) = t_0 + \theta(t - t_0)$, $s(\theta) = s_0 + \theta(s - s_0)$ and $r(\theta) = r_0 + \theta(r - r_0)$, the point with coordinates $(t(\theta), s(\theta), r(\theta))$ traverses the line segment joining (t_0, s_0, r_0) and (t, s, r).
By applying Taylor theorem of one variable to $F(\theta) \in C^p([0,1])$ around $\theta = 0$ up to order p, we have

$$F(1) = \sum_{n=0}^{p-1} \frac{F^{(n)}(0)}{n!} + \frac{F^{(p)}(\theta)}{p!}, \quad \theta \in (0,1),$$

where $F(1) = f(t, s, r)$ and $F(0) = f(t_0, s_0, r_0)$. We begin by calculating the derivatives of $F(\theta)$.

First derivative $F'(\theta)$ is given by:

$$\begin{aligned} F'(\theta) = \frac{d}{d\theta} f\left(t(\theta), s(\theta), r(\theta)\right) &= \partial_t f \cdot \frac{\partial t}{\partial \theta} + \partial_s f \cdot \frac{\partial s}{\partial \theta} + \partial_r f \cdot \frac{\partial r}{\partial \theta} \\ &= (t - t_0)\partial_t f + (s - s_0)\partial_s f + (r - r_0)\partial_r f. \end{aligned}$$

Now, by differentiating $F'(\theta)$ again with respect to θ, we obtain

$$\begin{aligned} F''(\theta) &= \frac{d}{d\theta}\left[(t - t_0)\partial_t f + (s - s_0)\partial_s f + (r - r_0)\partial_r f\right] \\ &= (t - t_0)\frac{d}{d\theta}\left(\partial_t f\right) + (s - s_0)\frac{d}{d\theta}\left(\partial_s f\right) + (r - r_0)\frac{d}{d\theta}\left(\partial_r f\right) \\ &= (t - t_0)\left[\partial_t^{(2)} f \cdot \frac{dt}{d\theta} + \partial_t\partial_s f \cdot \frac{ds}{d\theta} + \partial_t\partial_r f \cdot \frac{dr}{d\theta}\right] + (s - s_0)\left[\partial_s\partial_t f \cdot \frac{dt}{d\theta} + \partial_s^{(2)} f \cdot \frac{ds}{d\theta} + \partial_s\partial_r f \cdot \frac{dr}{d\theta}\right] \\ &+ (r - r_0)\left[\partial_r\partial_t f \cdot \frac{dt}{d\theta} + \partial_r\partial_s f \cdot \frac{ds}{d\theta} + \partial_r^{(2)} f \cdot \frac{dr}{d\theta}\right] \\ &= (t - t_0)^2\partial_t^{(2)} f + (t - t_0)(s - s_0)\partial_t\partial_s f + (t - t_0)(r - r_0)\partial_t\partial_r f + (t - t_0)(s - s_0)\partial_s\partial_t f \\ &+ (s - s_0)^2\partial_s^{(2)} f + (s - s_0)(r - r_0)\partial_s\partial_r f + (t - t_0)(r - r_0)\partial_r\partial_t f + (s - s_0)(r - r_0)\partial_r\partial_s f \\ &+ (r - r_0)^2\partial_r^{(2)} f \\ &= (t - t_0)^2\partial_t^{(2)} f + 2(t - t_0)(s - s_0)\partial_t\partial_s f + 2(t - t_0)(r - r_0)\partial_t\partial_r f + (s - s_0)^2\partial_s^{(2)} f \\ &+ 2(s - s_0)(r - r_0)\partial_s\partial_r f + (r - r_0)^2\partial_r^{(2)} f, \end{aligned}$$

which implies,

$$F''(\theta) = \sum_{i+j+k=2} \frac{2}{i!j!k!}(t - t_0)^i(s - s_0)^j(r - r_0)^k\partial_t^{(i)}\partial_s^{(j)}\partial_r^{(k)} f(t(\theta), s(\theta), r(\theta)).$$

The second derivative of $F(\theta)$, and more generally the nth derivative, gathers all mixed partial derivatives of order n, weighted by powers of $(t - t_0)$, $(s - s_0)$, $(r - r_0)$ and multiplied by appropriate multinomial coefficients.

In general, we find by mathematical induction that the nth derivative is given by the expression:

$$F^{(n)}(\theta) = \sum_{i+j+k=n} \frac{n!}{i!j!k!}(t-t_0)^i(s-s_0)^j(r-r_0)^k \partial_t^{(i)}\partial_s^{(j)}\partial_r^{(k)} f(t(\theta), s(\theta), r(\theta)),$$

then, we have

$$F^{(n)}(0) = \sum_{i+j+k=n} \frac{n!}{i!j!k!}(t-t_0)^i(s-s_0)^j(r-r_0)^k \partial_t^{(i)}\partial_s^{(j)}\partial_r^{(k)} f(t_0, s_0, r_0).$$

Thus,

$$\sum_{n=0}^{p-1} \frac{F^{(n)}(0)}{n!} = \sum_{i+j+k=0}^{p-1} \frac{1}{i!j!k!} \partial_t^{(i)}\partial_s^{(j)}\partial_r^{(k)} f(t_0, s_0, r_0)(t-t_0)^i(s-s_0)^j(r-r_0)^k,$$

and the Lagrange form of the remainder is:

$$\frac{F^{(p)}(\theta)}{p!} = \sum_{i+j+k=p} \frac{1}{i!j!k!} \partial_t^{(i)}\partial_s^{(j)}\partial_r^{(k)} f(t_1, s_1, r_1)(t-t_0)^i(s-s_0)^j(r-r_0)^k,$$

where $(t_1, s_1, r_1) = (t(\theta), s(\theta), r(\theta))$ with $\theta \in (0,1)$. ■

Lemma 4.3.2 *Let f and κ be p times continuously differentiable on their respective domains. Then, there exists a positive number $\varphi(p)$ such that for all $n = 0, \ldots, N-1$; $m = 0, \ldots, M-1$; $\tau = 0, \ldots, T-1$ and $i + j + k = 0, \ldots, p$, the inequality*

$$\left\| \partial_t^{(i)}\partial_s^{(j)}\partial_r^{(k)} \hat{\vartheta}_{n,m,\tau} \right\|_{L^\infty(\mathcal{R}_{n,m,\tau})} \leq \varphi(p),$$

holds, where $\hat{\vartheta}_{0,0,0}(t,s,r) = w(t,s,r)$ for $(t,s,r) \in \mathcal{R}_{0,0,0}$.

Proof. Set $\varepsilon_{n,m,\tau}^{i,j,k} = \|\partial_t^{(i)}\partial_s^{(j)}\partial_r^{(k)} \hat{\vartheta}_{n,m,\tau}\|$, the proof is organized into four steps.
Step 1. For all $i + j + k = 0, \ldots, p$, we have

$$\varepsilon_{0,0,0}^{i,j,k} \leq \max \left\{ \left\| \partial_t^{(i)}\partial_s^{(j)}\partial_r^{(k)} w \right\|, i + j + k = 0, \ldots, p \right\} = \varphi_1(p). \tag{4.23}$$

Step 2. For $i+j+k=0$, we have from (4.10)

$$\varepsilon^{i,j,k}_{n,0,0} \leq c_1 + c_5 \sum_{\xi=0}^{n-1} \sum_{a+b+c=0}^{p-1} \int_{t_\xi}^{t_{\xi+1}} \int_0^s \int_0^r \varepsilon^{a,b,c}_{\xi,0,0} dzdydx + c_5 \int_{t_n}^{t} \int_0^s \int_0^r \varepsilon^{0,0,0}_{n,0,0} dzdydx, \tag{4.24}$$

and from (4.13), we have for all $i+j+k=1,\ldots,p$

$$\begin{aligned}
\varepsilon^{i,j,k}_{n,0,0} \leq{}& c_1 + c_2 \sum_{\xi=0}^{n-1} \sum_{\mu=0}^{k-1} \sum_{\lambda=0}^{\mu} \sum_{q=0}^{j-1} \sum_{\eta=0}^{q} \int_{t_\xi}^{t_{\xi+1}} \sum_{a+b+c=0}^{p-1} \varepsilon^{a,b,c}_{\xi,0,0} dx + c_3 \sum_{\xi=0}^{n-1} \sum_{\mu=0}^{k-1} \sum_{\lambda=0}^{\mu} \int_{t_\xi}^{t_{\xi+1}} \int_0^s \sum_{a+b+c=0}^{p-1} \varepsilon^{a,b,c}_{\xi,0,0} dydx \\
&+ c_4 \sum_{\xi=0}^{n-1} \sum_{q=0}^{j-1} \sum_{\eta=0}^{q} \int_{t_\xi}^{t_{\xi+1}} \int_0^r \sum_{a+b+c=0}^{p-1} \varepsilon^{a,b,c}_{\xi,0,0} dzdx + c_5 \sum_{\xi=0}^{n-1} \int_{t_\xi}^{t_{\xi+1}} \int_0^s \int_0^r \sum_{a+b+c=0}^{p-1} \varepsilon^{a,b,c}_{\xi,0,0} dzdydx \\
&+ c_6 \sum_{\mu=0}^{k-1} \sum_{\lambda=0}^{\mu} \sum_{q=0}^{j-1} \sum_{\eta=0}^{q} \sum_{\alpha=0}^{i-1} \sum_{\beta=0}^{\alpha} \varepsilon^{\beta,\eta,\lambda}_{n,0,0} + c_2 \sum_{\mu=0}^{k-1} \sum_{\lambda=0}^{\mu} \sum_{q=0}^{j-1} \sum_{\eta=0}^{q} \int_{t_n}^{t} \varepsilon^{0,\eta,\lambda}_{n,0,0} dx \\
&+ c_7 \sum_{\mu=0}^{k-1} \sum_{\lambda=0}^{\mu} \sum_{\alpha=0}^{i-1} \sum_{\beta=0}^{\alpha} \int_0^s \varepsilon^{\beta,0,\lambda}_{n,0,0} dy + c_3 \sum_{\mu=0}^{k-1} \sum_{\lambda=0}^{\mu} \int_{t_n}^{t} \int_0^s \varepsilon^{0,0,\lambda}_{n,0,0} dydx \\
&+ c_8 \sum_{q=0}^{j-1} \sum_{\eta=0}^{q} \sum_{\alpha=0}^{i-1} \sum_{\beta=0}^{\alpha} \int_0^r \varepsilon^{\beta,\eta,0}_{n,0,0} dz + c_4 \sum_{q=0}^{j-1} \sum_{\eta=0}^{q} \int_{t_n}^{t} \int_0^r \varepsilon^{0,\eta,0}_{n,0,0} dzdx \\
&+ c_9 \sum_{\alpha=0}^{i-1} \sum_{\beta=0}^{\alpha} \int_0^s \int_0^r \varepsilon^{\beta,0,0}_{n,0,0} dzdy + c_5 \int_{t_n}^{t} \int_0^s \int_0^r \varepsilon^{0,0,0}_{n,0,0} dzdydx,
\end{aligned} \tag{4.25}$$

where for all $i+j+k=0,\ldots,p$

$$c_1 = \max\left\{ \left\| \partial_t^{(i)} \partial_s^{(j)} \partial_r^{(k)} f \right\| \right\},$$

$$c_2 = \max\left\{ \begin{array}{c} \left\| \partial_t^{(i)} \left(\partial_s^{(q-\eta)} \left[\partial_s^{(j-1-q)} \Big|_{y=s} \left(\partial_r^{(\mu-\lambda)} \left[\partial_r^{(k-1-\mu)} \Big|_{z=r} \kappa(t,s,r,x,y,z) \right] \right) \right] \right) (x-t_n)^a s^{b-\eta} r^{c-\lambda} \right\| \\ \times \binom{\mu}{\lambda} \binom{q}{\eta} \frac{1}{a!(b-\eta)!(c-\lambda)!}, \eta = 0,\ldots,q; q = 0,\ldots,j-1; \lambda = 0,\ldots,\mu; \\ \mu = 0,\ldots,k-1; a+b+c = 0,\ldots,p-1 \end{array} \right\},$$

$$c_3 = \max\left\{ \begin{array}{c} \binom{\mu}{\lambda} \frac{1}{a!b!(c-\lambda)!} \left\| \partial_t^{(i)} \left(\partial_s^{(j)} \left[\partial_r^{(\mu-\lambda)} \left[\partial_r^{(k-1-\mu)} \Big|_{z=r} \kappa(t,s,r,x,y,z) \right] \right] \right) (x-t_n)^a y^b r^{c-\lambda} \right\|, \\ \lambda = 0,\ldots,\mu; \mu = 0,\ldots,k-1; a+b+c = 0,\ldots,p-1 \end{array} \right\},$$

$$c_4 = \max\left\{ \begin{array}{c} \binom{q}{\eta}\frac{1}{a!(b-\eta)!c!}\left\|\partial_t^{(i)}\left(\partial_s^{(q-\eta)}\left[\partial_s^{(j-1-q)}\Big|_{y=s}\partial_r^{(k)}\kappa(t,s,r,x,y,z)\right]\right)(x-t_n)^a s^{b-\eta}z^c\right\|, \\ \eta=0,\ldots,q; q=0,\ldots,j-1; a+b+c=0,\ldots,p-1 \end{array}\right\},$$

$$c_5 = \max\left\{\frac{1}{a!b!c!}\left\|\partial_t^{(i)}\partial_s^{(j)}\partial_r^{(k)}\kappa(t,s,r,x,y,z)(x-t_n)^a y^b z^c\right\|, a+b+c=0,\ldots,p-1\right\},$$

$$c_6 = \max\left\{ \begin{array}{c} \binom{\mu}{\lambda}\binom{q}{\eta}\binom{\alpha}{\beta}\left\|\partial_t^{(\alpha-\beta)}\left[\partial_t^{(i-1-\alpha)}\Big|_{x=t}\left(\partial_s^{(q-\eta)}\left[\partial_s^{(j-1-q)}\Big|_{y=s}\left(\partial_r^{(\mu-\lambda)}\left[\partial_r^{(k-1-\mu)}\Big|_{z=r}\kappa(t,s,r,x,y,z)\right]\right)\right]\right)\right]\right\|, \\ \beta=0,\ldots,\alpha;\alpha=0,\ldots,i-1;\eta=0,\ldots,q;q=0,\ldots,j-1;\lambda=0,\ldots,\mu;\mu=0,\ldots,k-1 \end{array}\right\},$$

$$c_7 = \max\left\{ \begin{array}{c} \binom{\mu}{\lambda}\binom{\alpha}{\beta}\left\|\partial_t^{(\alpha-\beta)}\left[\partial^{(i-1-\alpha)}\Big|_{x=t}\left(\partial_s^{(j)}\left[\partial_r^{(\mu-\lambda)}\left[\partial_r^{(k-1-\mu)}\Big|_{z=r}\kappa(t,s,r,x,y,z)\right]\right]\right)\right]\right\|, \\ \beta=0,\ldots,\alpha;\alpha=0,\ldots,i-1;\lambda=0,\ldots,\mu;\mu=0,\ldots,k-1 \end{array}\right\},$$

$$c_8 = \max\left\{ \begin{array}{c} \binom{q}{\eta}\binom{\alpha}{\beta}\left\|\partial_t^{(\alpha-\beta)}\left[\partial_t^{(i-1-\alpha)}\Big|_{x=t}\left(\partial_s^{(q-\eta)}\left[\partial_s^{(j-1-q)}\Big|_{y=s}\partial_r^{(k)}\kappa(t,s,r,x,y,z)\right]\right)\right]\right\|, \\ \beta=0,\ldots,\alpha;\alpha=0,\ldots,i-1;\eta=0,\ldots,q;q=0,\ldots,j-1 \end{array}\right\},$$

$$c_9 = \max\left\{\binom{\alpha}{\beta}\left\|\partial_t^{(\alpha-\beta)}\left[\partial_t^{(i-1-\alpha)}\Big|_{x=t}\partial_s^{(j)}\partial_r^{(k)}\kappa(t,s,r,x,y,z)\right]\right\|, \beta=0,\ldots,\alpha;\alpha=0,\ldots,i-1\right\},$$

where the constants $c_i, i=1,\ldots,9$ are positive and independent of N, M and T.
By considering the sequence $\Psi_n = \max\{\varepsilon_{n,0,0}^{i,j,k}, i+j+k=0,\ldots,p\}$, $n=0,\ldots,N-1$,, we can combine equations (4.24) and (4.25) to obtain

$$\begin{aligned}
\varepsilon_{n,0,0}^{i,j,k} \leq\ & c_1 + c_2h_1p^7\sum_{\xi=0}^{n-1}\Psi_\xi + c_3h_1h_2p^5\sum_{\xi=0}^{n-1}\Psi_\xi + c_4h_1h_3p^5\sum_{\xi=0}^{n-1}\Psi_\xi + c_5h_1h_2h_3p^3\sum_{\xi=0}^{n-1}\Psi_\xi \\
&+ c_6p^3\sum_{\lambda=0}^{k-1}\sum_{\eta=0}^{j-1}\sum_{\beta=0}^{i-1}\varepsilon_{n,0,0}^{\beta,\eta,\lambda} + c_2h_1p^2\sum_{\lambda=0}^{k-1}\sum_{\eta=0}^{j-1}\varepsilon_{n,0,0}^{0,\eta,\lambda} + c_7h_2p^2\sum_{\lambda=0}^{k-1}\sum_{\beta=0}^{i-1}\varepsilon_{n,0,0}^{\beta,0,\lambda} \\
&+ c_3h_1h_2p\sum_{\lambda=0}^{k-1}\varepsilon_{n,0,0}^{0,0,\lambda} + c_8h_3p^2\sum_{\eta=0}^{j-1}\sum_{\beta=0}^{i-1}\varepsilon_{n,0,0}^{\beta,\eta,0} + c_4h_1h_3p\sum_{\eta=0}^{j-1}\varepsilon_{n,0,0}^{0,\eta,0} \\
&+ c_9h_2h_3p\sum_{\beta=0}^{i-1}\varepsilon_{n,0,0}^{\beta,0,0} + c_5h_1h_2h_3\varepsilon_{n,0,0}^{0,0,0}.
\end{aligned}$$

By setting $c_{10} = c_2p^7 + c_3Bp^5 + c_4Cp^5 + c_5BCp^3$ and $c_{11} = c_6p^3 + c_2Ap^2 + c_7Bp^2 + c_3ABp + c_8Cp^2 + c_4ACp + c_9BCp + c_5ABC$, we obtain

$$\varepsilon_{n,0,0}^{i,j,k} \leq c_1 + c_{10}h_1 \sum_{\xi=0}^{n-1} \Psi_\xi + c_{11} \sum_{\lambda=0}^{k-1} \sum_{\eta=0}^{j-1} \sum_{\beta=0}^{i-1} \varepsilon_{n,0,0}^{\beta,\eta,\lambda}, \tag{4.26}$$

we put for each fixed $n \in \mathbb{N}$

$$\omega(i,j,k) = \varepsilon_{n,0,0}^{i,j,k}, \quad b(i,j,k) = c_1 + c_{10}h_1 \sum_{\xi=0}^{n-1} \Psi_\xi, \quad \epsilon(\beta,\eta,\lambda) = c_{11}.$$

Then, by Lemma 1.7.5, we obtain from (4.26)

$$\begin{aligned}\varepsilon_{n,0,0}^{i,j,k} &\leq \left(c_1 + c_{10}h_1 \sum_{\xi=0}^{n-1} \Psi_\xi\right) \prod_{\lambda=0}^{k-1} \left[1 + \sum_{\eta=0}^{j-1} \sum_{\beta=0}^{i-1} c_{11}\right] \\ &\leq \underbrace{c_1 \left[1 + p^2 c_{11}\right]^p}_{c_{12}} + \underbrace{c_{10} \left[1 + p^2 c_{11}\right]^p}_{c_{13}} h_1 \sum_{\xi=0}^{n-1} \Psi_\xi,\end{aligned}$$

consequently, we obtain for all $n = 0, \ldots, N-1$

$$\Psi_n \leq c_{12} + c_{13}h_1 \sum_{\xi=0}^{n-1} \Psi_\xi,$$

therefore, by Lemma 1.7.1

$$\varepsilon_{n,0,0}^{i,j,k} \leq \Psi_n \leq c_{12} \exp(c_{13}A) = \varphi_2(p). \tag{4.27}$$

Step 3. Let the sequence $\Psi_{n,m} = \max\{\varepsilon_{n,m,0}^{i,j,k}, i+j+k = 0,\ldots,p\}$, $n = 0,\ldots,N-1; m = 0,\ldots,M-1$, by following a similar process with slight adjustments, from (4.14) and (4.17), we have, for all $n = 0,\ldots,N-1$; $m = 1,\ldots,M-1$ and $i+j+k = 0,\ldots,p$

$$\varepsilon_{n,m,0}^{i,j,k} \leq c_1 + c_3h_1h_2p^5 \sum_{\xi=0}^{n-1} \sum_{\rho=0}^{m-1} \Psi_{\xi,\rho} + c_5h_1h_2h_3p^3 \sum_{\xi=0}^{n-1} \sum_{\rho=0}^{m-1} \Psi_{\xi,\rho} + c_2h_1p^7 \sum_{\xi=0}^{n-1} \Psi_{\xi,m} + c_3h_1h_2p^5 \sum_{\xi=0}^{n-1} \Psi_{\xi,m}$$

$$+ c_4 h_1 h_3 p^5 \sum_{\xi=0}^{n-1} \Psi_{\xi,m} + c_5 h_1 h_2 h_3 p^3 \sum_{\xi=0}^{n-1} \Psi_{\xi,m} + c_7 h_2 p^7 \sum_{\rho=0}^{m-1} \Psi_{n,\rho} + c_3 h_1 h_2 p^5 \sum_{\rho=0}^{m-1} \Psi_{n,\rho}$$
$$+ c_9 h_2 h_3 p^5 \sum_{\rho=0}^{m-1} \Psi_{n,\rho} + c_5 h_1 h_2 h_3 p^3 \sum_{\rho=0}^{m-1} \Psi_{n,\rho} + c_6 p^3 \sum_{\lambda=0}^{k-1} \sum_{\eta=0}^{j-1} \sum_{\beta=0}^{i-1} \varepsilon_{n,m,0}^{\beta,\eta,\lambda} + c_2 h_1 p^2 \sum_{\lambda=0}^{k-1} \sum_{\eta=0}^{j-1} \varepsilon_{n,m,0}^{0,\eta,\lambda}$$
$$+ c_7 h_2 p^2 \sum_{\lambda=0}^{k-1} \sum_{\beta=0}^{i-1} \varepsilon_{n,m,0}^{\beta,0,\lambda} + c_3 h_1 h_2 p \sum_{\lambda=0}^{k-1} \varepsilon_{n,m,0}^{0,0,\lambda} + c_8 h_3 p^2 \sum_{\eta=0}^{j-1} \sum_{\beta=0}^{i-1} \varepsilon_{n,m,0}^{\beta,\eta,0} + c_4 h_1 h_3 p \sum_{\eta=0}^{j-1} \varepsilon_{n,m,0}^{0,\eta,0}$$
$$+ c_9 h_2 h_3 p \sum_{\beta=0}^{i-1} \varepsilon_{n,m,0}^{\beta,0,0} + c_5 h_1 h_2 h_3 \varepsilon_{n,m,0}^{0,0,0}.$$

By setting $b_1 = c_3 p^5 + c_5 C p^3$, $\quad b_2 = c_2 p^7 + c_3 B p^5 + c_4 C p^5 + c_5 B C p^3$, $b_3 = c_7 p^7 + c_3 A p^5 + c_9 C p^5 + c_5 A C p^3$ and $b_4 = c_6 p^3 + c_2 A p^2 + c_7 B p^2 + c_3 A B p + c_8 C p^2 + c_4 A C p + c_9 B C p + c_5 A B C$, we obtain

$$\varepsilon_{n,m,0}^{i,j,k} \leq c_1 + b_1 h_1 h_2 \sum_{\xi=0}^{n-1} \sum_{\rho=0}^{m-1} \Psi_{\xi,\rho} + b_2 h_1 \sum_{\xi=0}^{n-1} \Psi_{\xi,m} + b_3 h_2 \sum_{\rho=0}^{m-1} \Psi_{n,\rho} + b_4 \sum_{\lambda=0}^{k-1} \sum_{\eta=0}^{j-1} \sum_{\beta=0}^{i-1} \varepsilon_{n,m,0}^{\beta,\eta,\lambda}, \tag{4.28}$$

we put for each fixed $n, m \in \mathbb{N}$

$$\omega(i,j,k) = \varepsilon_{n,m,0}^{i,j,k}, \quad b(i,j,k) = c_1 + b_1 h_1 h_2 \sum_{\xi=0}^{n-1} \sum_{\rho=0}^{m-1} \Psi_{\xi,\rho} + b_2 h_1 \sum_{\xi=0}^{n-1} \Psi_{\xi,m} + b_3 h_2 \sum_{\rho=0}^{m-1} \Psi_{n,\rho}, \quad \epsilon(\beta,\eta,\lambda) = b_4.$$

Then, by Lemma 1.7.5, we obtain from (4.28)

$$\varepsilon_{n,m,0}^{i,j,k} \leq \left(c_1 + b_1 h_1 h_2 \sum_{\xi=0}^{n-1} \sum_{\rho=0}^{m-1} \Psi_{\xi,\rho} + b_2 h_1 \sum_{\xi=0}^{n-1} \Psi_{\xi,m} + b_3 h_2 \sum_{\rho=0}^{m-1} \Psi_{n,\rho}) \right) \prod_{\lambda=0}^{k-1} \left[1 + \sum_{\eta=0}^{j-1} \sum_{\beta=0}^{i-1} b_4 \right]$$
$$\leq \underbrace{c_1 \left[1 + p^2 b_4\right]^p}_{b_5} + \underbrace{b_1 \left[1 + p^2 b_4\right]^p}_{b_6} h_1 h_2 \sum_{\xi=0}^{n-1} \sum_{\rho=0}^{m-1} \Psi_{\xi,\rho} + \underbrace{b_2 \left[1 + p^2 b_4\right]^p}_{b_7} h_1 \sum_{\xi=0}^{n-1} \Psi_{\xi,m}$$

$$+\underbrace{b_3\left[1+p^2b_4\right]^p}_{b_8} h_2 \sum_{\rho=0}^{m-1} \Psi_{n,\rho},$$

consequently, we obtain for all $n = 0, \dots, N-1$; $m = 1, \dots, M-1$

$$\Psi_{n,m} \leq b_5 + b_6 h_1 h_2 \sum_{\xi=0}^{n-1} \sum_{\rho=0}^{m-1} \Psi_{\xi,\rho} + b_7 h_1 \sum_{\xi=0}^{n-1} \Psi_{\xi,m} + b_8 h_2 \sum_{\rho=0}^{m-1} \Psi_{n,\rho},$$

therefore, by Lemma 1.7.3, we obtain

$$\varepsilon^{i,j,k}_{n,m,0} \leq \Psi_{n,m} \leq b_5 \exp\left(\eta_1(A+B)\right) = \varphi_3(p), \tag{4.29}$$

where $\eta_1 = \frac{1}{2}\left(b_7 + b_8 + \sqrt{(b_7+b_8)^2 + 4b_6}\right)$.

Step 4. We consider the sequence $\Psi_{n,m,\tau} = \max\{\varepsilon^{i,j,k}_{n,m,\tau}, i+j+k = 0, \dots, p\}$, $n = 0, \dots, N-1$; $m = 0, \dots, M-1$; $\tau = 0, \dots, T-1$.

We have, from (4.18) and (4.19) for all $n = 0, \dots, N-1$; $m = 1, \dots, M-1$; $\tau = 0, \dots, T-1$ and $i+j+k = 0, \dots, p$

$$\begin{aligned}
\varepsilon^{i,j,k}_{n,m,\tau} \leq{}& c_1 + c_5 h_1 h_2 h_3 p^3 \sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1}\sum_{\theta=0}^{\tau-1} \Psi_{\xi,\rho,\theta} + c_3 h_1 h_2 p^5 \sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1} \Psi_{\xi,\rho,\tau} + c_5 h_1 h_2 h_3 p^3 \sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1} \Psi_{\xi,\rho,\tau} \\
&+ c_4 h_1 h_3 p^5 \sum_{\xi=0}^{n-1}\sum_{\theta=0}^{\tau-1} \Psi_{\xi,m,\theta} + c_5 h_1 h_2 h_3 p^3 \sum_{\xi=0}^{n-1}\sum_{\theta=0}^{\tau-1} \Psi_{\xi,m,\theta} + c_2 h_1 p^7 \sum_{\xi=0}^{n-1} \Psi_{\xi,m,\tau} \\
&+ c_3 h_1 h_2 p^5 \sum_{\xi=0}^{n-1} \Psi_{\xi,m,\tau} + c_4 h_1 h_3 p^5 \sum_{\xi=0}^{n-1} \Psi_{\xi,m,\tau} + c_5 h_1 h_2 h_3 p^3 \sum_{\xi=0}^{n-1} \Psi_{\xi,m,\tau} \\
&+ c_9 h_2 h_3 p^5 \sum_{\rho=0}^{m-1}\sum_{\theta=0}^{\tau-1} \Psi_{n,\rho,\theta} + c_5 h_1 h_2 h_3 p^3 \sum_{\rho=0}^{m-1}\sum_{\theta=0}^{\tau-1} \Psi_{n,\rho,\theta} + c_7 h_2 p^7 \sum_{\rho=0}^{m-1} \Psi_{n,\rho,\tau} \\
&+ c_3 h_1 h_2 p^5 \sum_{\rho=0}^{m-1} \Psi_{n,\rho,\tau} + c_9 h_2 h_3 p^5 \sum_{\rho=0}^{m-1} \Psi_{n,\rho,\tau} + c_5 h_1 h_2 h_3 p^3 \sum_{\rho=0}^{m-1} \Psi_{n,\rho,\tau} \\
&+ c_8 h_3 p^7 \sum_{\theta=0}^{\tau-1} \Psi_{n,m,\theta} + c_4 h_1 h_3 p^5 \sum_{\theta=0}^{\tau-1} \Psi_{n,m,\theta} + c_9 h_2 h_3 p^5 \sum_{\theta=0}^{\tau-1} \Psi_{n,m,\theta}
\end{aligned}$$

$$+ c_5h_1h_2h_3p^3\sum_{\theta=0}^{\tau-1}\Psi_{n,m,\theta} + c_6p^3\sum_{\lambda=0}^{k-1}\sum_{\eta=0}^{j-1}\sum_{\beta=0}^{i-1}\varepsilon_{n,m,\tau}^{\beta,\eta,\lambda} + c_2h_1p^2\sum_{\lambda=0}^{k-1}\sum_{\eta=0}^{j-1}\varepsilon_{n,m,\tau}^{0,\eta,\lambda}$$
$$+ c_7h_2p^2\sum_{\lambda=0}^{k-1}\sum_{\beta=0}^{i-1}\varepsilon_{n,m,\tau}^{\beta,0,\lambda} + c_3h_1h_2p\sum_{\lambda=0}^{k-1}\varepsilon_{n,m,\tau}^{0,0,\lambda} + c_8h_3p^2\sum_{\eta=0}^{j-1}\sum_{\beta=0}^{i-1}\varepsilon_{n,m,\tau}^{\beta,\eta,0}$$
$$+ c_4h_1h_3p\sum_{\eta=0}^{j-1}\varepsilon_{n,m,\tau}^{0,\eta,0} + c_9h_2h_3p\sum_{\beta=0}^{i-1}\varepsilon_{n,m,\tau}^{\beta,0,0} + c_5h_1h_2h_3\varepsilon_{n,m,\tau}^{0,0,0}.$$

By setting $\gamma_1 = c_5p^3$, $\quad\gamma_2 = c_3p^5 + c_5Cp^3$, $\quad\gamma_3 = c_4p^5 + c_5Bp^3$, $\quad\gamma_4 = c_9p^5 + c_5Ap^3$, $\gamma_5 = c_2p^7 + c_3Bp^5 + c_4Cp^5 + c_5BCp^3$, $\quad\gamma_6 = c_7p^7 + c_3Ap^5 + c_9Cp^5 + c_5ACp^3$, $\gamma_7 = c_8p^7 + c_4Ap^5 + c_9Bp^5 + c_5ABp^3$ and $\gamma_8 = c_6p^3 + c_2Ap^2 + c_7Bp^2 + c_3ABp + c_8Cp^2 + c_4ACp + c_9BCp + c_5ABC$, we obtain

$$\varepsilon_{n,m,\tau}^{i,j,k} \le c_1 + \gamma_1h_1h_2h_3\sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1}\sum_{\theta=0}^{\tau-1}\Psi_{\xi,\rho,\theta} + \gamma_2h_1h_2\sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1}\Psi_{\xi,\rho,\tau} + \gamma_3h_1h_3\sum_{\xi=0}^{n-1}\sum_{\theta=0}^{\tau-1}\Psi_{\xi,m,\theta}$$
$$+ \gamma_4h_2h_3\sum_{\rho=0}^{m-1}\sum_{\theta=0}^{\tau-1}\Psi_{n,\rho,\theta} + \gamma_5h_1\sum_{\xi=0}^{n-1}\Psi_{\xi,m,\tau} + \gamma_6h_2\sum_{\rho=0}^{m-1}\Psi_{n,\rho,\tau} + \gamma_7h_3\sum_{\theta=0}^{\tau-1}\Psi_{n,m,\theta}$$
$$+ \gamma_8\sum_{\lambda=0}^{k-1}\sum_{\eta=0}^{j-1}\sum_{\beta=0}^{i-1}\varepsilon_{n,m,\tau}^{\beta,\eta,\lambda},$$

we put for each fixed $n, m, \tau \in \mathbb{N}$

$$\omega(i,j,k) = \varepsilon_{n,m,\tau}^{i,j,k}, \quad \epsilon(\beta,\eta,\lambda) = \gamma_8,$$
$$b(i,j,k) = c_1 + \gamma_1h_1h_2h_3\sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1}\sum_{\theta=0}^{\tau-1}\Psi_{\xi,\rho,\theta} + \gamma_2h_1h_2\sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1}\Psi_{\xi,\rho,\tau} + \gamma_3h_1h_3\sum_{\xi=0}^{n-1}\sum_{\theta=0}^{\tau-1}\Psi_{\xi,m,\theta}$$
$$+ \gamma_4h_2h_3\sum_{\rho=0}^{m-1}\sum_{\theta=0}^{\tau-1}\Psi_{n,\rho,\theta} + \gamma_5h_1\sum_{\xi=0}^{n-1}\Psi_{\xi,m,\tau} + \gamma_6h_2\sum_{\rho=0}^{m-1}\Psi_{n,\rho,\tau} + \gamma_7h_3\sum_{\theta=0}^{\tau-1}\Psi_{n,m,\theta}.$$

Then, by Lemma 1.7.5, we get

$$\varepsilon_{n,m,\tau}^{i,j,k} \leq b(i,j,k) \prod_{\lambda=0}^{k-1} \left[1 + \sum_{\eta=0}^{j-1} \sum_{\beta=0}^{i-1} \gamma_8 \right],$$

consequently, we obtain for all $n = 0, \ldots, N-1;\ m = 1, \ldots, M-1;\ \tau = 0, \ldots, T-1$

$$\begin{aligned}\Psi_{n,m,\tau} \leq\ & c_1\gamma_9 + +\gamma_1\gamma_9 h_1 h_2 h_3 \sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1}\sum_{\theta=0}^{\tau-1} \Psi_{\xi,\rho,\theta} + \gamma_2\gamma_9 h_1 h_2 \sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1} \Psi_{\xi,\rho,\tau} + \gamma_3\gamma_9 h_1 h_3 \sum_{\xi=0}^{n-1}\sum_{\theta=0}^{\tau-1} \Psi_{\xi,m,\theta} \\ & + \gamma_4\gamma_9 h_2 h_3 \sum_{\rho=0}^{m-1}\sum_{\theta=0}^{\tau-1} \Psi_{n,\rho,\theta} + \gamma_5\gamma_9 h_1 \sum_{\xi=0}^{n-1} \Psi_{\xi,m,\tau} + \gamma_6\gamma_9 h_2 \sum_{\rho=0}^{m-1} \Psi_{n,\rho,\tau} + \gamma_7\gamma_9 h_3 \sum_{\theta=0}^{\tau-1} \Psi_{n,m,\theta},\end{aligned} \tag{4.30}$$

where $\gamma_9 = [1 + p^2\gamma_8]^p$, we put for each fixed $\tau \in \mathbb{N}$

$$\begin{aligned}&\varepsilon(n,m) = \Psi_{n,m,\tau}, \quad \beta = c_1\gamma_9 + \gamma_1\gamma_9 h_1 h_2 h_3 \sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1}\sum_{\theta=0}^{\tau-1} \Psi_{\xi,\rho,\theta} + \gamma_3\gamma_9 h_1 h_3 \sum_{\xi=0}^{n-1}\sum_{\theta=0}^{\tau-1} \Psi_{\xi,m,\theta} \\ &+ \gamma_4\gamma_9 h_2 h_3 \sum_{\rho=0}^{m-1}\sum_{\theta=0}^{\tau-1} \Psi_{n,\rho,\theta} + \gamma_7\gamma_9 h_3 \sum_{\theta=0}^{\tau-1} \Psi_{n,m,\theta},\end{aligned}$$

then, by Lemma 1.7.3, we obtain from (4.30)

$$\Psi_{n,m,\tau} \leq \beta \exp\left(\eta_2(A+B)\right),$$

where $\eta_2 = \frac{1}{2}\left((\gamma_5+\gamma_6)\gamma_9 + \sqrt{(\gamma_5+\gamma_6)^2\gamma_9^2 + 4\gamma_2\gamma_9}\right)$
consequently, we get

$$\begin{aligned}\Psi_{n,m,\tau} \leq & \left(c_1 + \gamma_1 h_1 h_2 h_3 \sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1}\sum_{\theta=0}^{\tau-1} \Psi_{\xi,\rho,\theta} + \gamma_3 h_1 h_3 \sum_{\xi=0}^{n-1}\sum_{\theta=0}^{\tau-1} \Psi_{\xi,m,\theta} \right)\gamma_{10} \\ & + \left(\gamma_4 h_2 h_3 \sum_{\rho=0}^{m-1}\sum_{\theta=0}^{\tau-1} \Psi_{n,\rho,\theta} + \gamma_7 h_3 \sum_{\theta=0}^{\tau-1} \Psi_{n,m,\theta} \right)\gamma_{10},\end{aligned}$$

where $\gamma_{10} = \gamma_9 \exp(\eta_2(A+B))$. By reapplying Lemma 1.7.3 for each fixed $m \in \mathbb{N}$, we conclude

$$\Psi_{n,m,\tau} \leq \left(c_1 + \gamma_1 h_1 h_2 h_3 \sum_{\xi=0}^{n-1} \sum_{\rho=0}^{m-1} \sum_{\theta=0}^{\tau-1} \Psi_{\xi,\rho,\theta} + \gamma_4 h_2 h_3 \sum_{\rho=0}^{m-1} \sum_{\theta=0}^{\tau-1} \Psi_{n,\rho,\theta} \right) \gamma_{11},$$

where $\gamma_{11} = \gamma_{10} \exp(\eta_3(A+C))$ and $\eta_3 = \frac{1}{2}\left(\gamma_7 + \sqrt{\gamma_7^2 + 4\gamma_3}\right)$. Applying Lemma 1.7.3 once more for each fixed $n \in \mathbb{N}$, we get

$$\Psi_{n,m,\tau} \leq \left(c_1 + \gamma_1 h_1 h_2 h_3 \sum_{\xi=0}^{n-1} \sum_{\rho=0}^{m-1} \sum_{\theta=0}^{\tau-1} \Psi_{\xi,\rho,\theta} \right) \gamma_{12},$$

where $\gamma_{12} = \gamma_{11} \exp\left(\sqrt{\gamma_4}(B+C)\right)$. Thus, by applying Lemma 1.7.5, we derive

$$\begin{aligned} \Psi_{n,m,\tau} &\leq c_1 \gamma_{12} \prod_{\xi=0}^{n-1} \left[1 + \sum_{\rho=0}^{m-1} \sum_{\theta=0}^{\tau-1} \gamma_1 \gamma_{12} h_1 h_2 h_3 \right] \\ &\leq c_1 \gamma_{12} \exp(ABC\gamma_1\gamma_{12}) = \varphi_4(p). \end{aligned} \tag{4.31}$$

Hence, by relying on (4.23), (4.27), (4.29) and (4.31), the proof of Lemma 4.3.2 is concluded by setting $\varphi(p) = \max\{\varphi_1(p), \varphi_2(p), \varphi_3(p), \varphi_4(p)\}$. ■

The upcoming theorem establishes the convergence of the proposed method.

Theorem 4.3.1 *Let f and κ be p times continuously differentiable on their respective domains. Then equations (4.4),(4.5),(4.6),(4.7) define a unique approximation $\vartheta \in \mathcal{S}_{p-1}^{(-1)}(\Pi)$, and the resulting error function $e(t,s,r) = w(t,s,r) - \vartheta(t,s,r)$ satisfies:*

$$\|e\|_{L^\infty(\mathcal{R})} \leq \varsigma(h_1 + h_2 + h_3)^p,$$

where ς is a finite constant independent of h_1, h_2 and h_3.

Proof. Define the error $e(t,s,r)$ on $\mathcal{R}_{n,m,\tau}$ by $e_{n,m,\tau}(t,s,r) = w(t,s,r) - \vartheta_{n,m,\tau}(t,s,r)$ for all $n \in \{0,\ldots,N-1\}$, $m \in \{0,\ldots,M-1\}$ and $\tau \in \{0,\ldots,T-1\}$, four steps comprise the proof.

Claim 1. Let $(t,s,r) \in \mathcal{R}_{0,0,0}$, by using Lemma 4.3.1, we obtain from (4.4)

$$|w(t,s,r) - \vartheta_{0,0,0}(t,s,r)| \leq \sum_{i+j+k=p} \frac{1}{i!j!k!} \left\|\partial_t^{(i)}\partial_s^{(j)}\partial_r^{(k)} w\right\| h_1^i h_2^j h_3^k,$$

hence, by Lemma 4.3.2, we have

$$\|e_{0,0,0}\| \leq \varphi(p) \sum_{i+j+k=p} \frac{1}{i!j!k!} h_1^i h_2^j h_3^k = \underbrace{\frac{\varphi(p)}{p!}}_{\varsigma_1} (h_1+h_2+h_3)^p.$$

Claim 2. Let $(t,s,r) \in \mathcal{R}_{n,0,0}$, $n \in \{1,\ldots,N-1\}$ we have from (4.10)

$$\begin{aligned} w(t,s,r) - \hat{\vartheta}_{n,0,0}(t,s,r) &= \sum_{\xi=0}^{n-1} \int_{t_\xi}^{t_{\xi+1}} \int_0^s \int_0^r \kappa(t,s,r,x,y,z) e_{\xi,0,0}(x,y,z) dzdydx \\ &+ \int_{t_n}^t \int_0^s \int_0^r \kappa(t,s,r,x,y,z) \left[w(x,y,z) - \hat{\vartheta}_{n,0,0}(x,y,z)\right] dzdydx, \end{aligned}$$

consequently, by setting $\kappa_1 = max\{\|\kappa\|_{L^\infty(\mathcal{R}_{n,0,0})}, n = 1,\ldots,N-1\}$, we get

$$|w(t,s,r) - \hat{\vartheta}_{n,0,0}(t,s,r)| \leq \sum_{\xi=0}^{n-1} h_1h_2h_3\kappa_1\|e_{\xi,0,0}\| + \kappa_1 \int_{t_n}^t \int_0^s \int_0^r |w(x,y,z) - \hat{\vartheta}_{n,0,0}(x,y,z)| dzdydx,$$

then, applying Lemma 1.7.7, we derive

$$|w(t,s,r) - \hat{\vartheta}_{n,0,0}(t,s,r)| \leq \sum_{\xi=0}^{n-1} h_1 \underbrace{BC\kappa_1 \exp\left(\sqrt[3]{\kappa_1}(A+B+C)\right)}_{b_1} \|e_{\xi,0,0}\|,$$

this implies, through the application of Lemma 4.3.1, that

$$\begin{aligned} \|e_{n,0,0}\| &\leq \|w - \hat{\vartheta}_{n,0,0}\| + \|\hat{\vartheta}_{n,0,0} - \vartheta_{n,0,0}\| \\ &\leq \sum_{\xi=0}^{n-1} h_1 b_1 \|e_{\xi,0,0}\| + \sum_{i+j+k=p} \frac{1}{i!j!k!} \left\|\partial_t^{(i)}\partial_s^{(j)}\partial_r^{(k)} \hat{\vartheta}_{n,0,0}\right\| h_1^i h_2^j h_3^k, \end{aligned}$$

thus, using Lemma 4.3.2, we obtain

$$\|e_{n,0,0}\| \leq \sum_{\xi=0}^{n-1} h_1 b_1 \|e_{\xi,0,0}\| + \frac{\varphi(p)}{p!}(h_1+h_2+h_3)^p,$$

applying Lemma 1.7.1, we derive

$$\|e_{n,0,0}\| \leq \underbrace{\frac{\varphi(p)}{p!}\exp(Ab_1)}_{\varsigma_2}(h_1+h_2+h_3)^p.$$

Claim 3. Let $(t,s,r) \in \mathcal{R}_{n,m,0}$, $n \in \{0,\dots,N-1\}$ and $m \in \{1,\dots,M-1\}$, we have from (4.14)

$$\begin{aligned}|w(t,s,r) - \hat{\vartheta}_{n,m,0}(t,s,r)| \leq &\sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1} h_1h_2h_3\kappa_2\|e_{\xi,\rho,0}\| + \sum_{\xi=0}^{n-1} h_1h_2h_3\kappa_2\|e_{\xi,m,0}\| + \sum_{\rho=0}^{m-1} h_1h_2h_3\kappa_2\|e_{n,\rho,0}\| \\ &+ \kappa_2\int_{t_n}^{t}\int_{s_m}^{s}\int_0^r |w(x,y,z) - \hat{\vartheta}_{n,m,0}(x,y,z)|dzdydx,\end{aligned}$$

where $\kappa_2 = max\{\|\kappa\|_{L^\infty(\mathcal{R}_{n,m,0})}, n = 0,\dots,N-1; m = 1,\dots,M-1\}$, then by Lemma 1.7.7

$$\begin{aligned}|w(t,s,r) - \hat{\vartheta}_{n,m,0}(t,s,r)| \leq &\left(\sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1}\|e_{\xi,\rho,0}\| + \sum_{\xi=0}^{n-1}\|e_{\xi,m,0}\| + \sum_{\rho=0}^{m-1}\|e_{n,\rho,0}\|\right) \\ &\times h_1h_2h_3\underbrace{\kappa_2\exp\left(\sqrt[3]{\kappa_2}(A+B+C)\right)}_{b_2},\end{aligned}$$

which implies, by using Lemma 4.3.1, that

$$\begin{aligned}\|e_{n,m,0}\| &\leq \|w - \hat{\vartheta}_{n,m,0}\| + \|\hat{\vartheta}_{n,m,0} - \vartheta_{n,m,0}\| \\ &\leq \left(\sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1}\|e_{\xi,\rho,0}\| + \sum_{\xi=0}^{n-1}\|e_{\xi,m,0}\| + \sum_{\rho=0}^{m-1}\|e_{n,\rho,0}\|\right)h_1h_2h_3b_2 \\ &+ \sum_{i+j+k=p}\frac{1}{i!j!k!}\left\|\partial_t^{(i)}\partial_s^{(j)}\partial_r^{(k)}\hat{\vartheta}_{n,m,0}\right\| h_1^i h_2^j h_3^k,\end{aligned}$$

hence, by Lemma 4.3.2, we obtain

$$\|e_{n,m,0}\| \leq h_1h_2Cb_2\sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1}\|e_{\xi,\rho,0}\| + h_1BCb_2\sum_{\xi=0}^{n-1}\|e_{\xi,m,0}\| + h_2ACb_2\sum_{\rho=0}^{m-1}\|e_{n,\rho,0}\| + \frac{\varphi(p)}{p!}(h_1+h_2+h_3)^p.$$

Moreover, by Lemma 1.7.3, we deduce that

$$\|e_{n,m,0}\| \leq \underbrace{\frac{\varphi(p)}{p!}\exp\left(\eta_4(A+B)\right)}_{\varsigma_3}(h_1+h_2+h_3)^p,$$

where $\eta_4 = \frac{1}{2}\left(BCb_2 + ACb_2 + \sqrt{(BCb_2+ACb_2)^2 + 4Cb_2}\right)$.

Claim 4. For all $n \in \{0,\dots,N-1\}$, $m \in \{0,\dots,M-1\}$ and $\tau \in \{1,\dots,T-1\}$, let $(t,s,r) \in \mathcal{R}_{n,m,\tau}$, we have from (4.18)

$$\begin{aligned}|w(t,s,r) - \hat{\vartheta}_{n,m,\tau}(t,s,r)| \leq & \sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1}\sum_{\theta=0}^{\tau-1} h_1h_2h_3\kappa_3\|e_{\xi,\rho,\theta}\| + \sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1} h_1h_2h_3\kappa_3\|e_{\xi,\rho,\tau}\| \\ & + \sum_{\xi=0}^{n-1}\sum_{\theta=0}^{\tau-1} h_1h_2h_3\kappa_3\|e_{\xi,m,\theta}\| + \sum_{\xi=0}^{n-1} h_1h_2h_3\kappa_3\|e_{\xi,m,\tau}\| \\ & + \sum_{\rho=0}^{m-1}\sum_{\theta=0}^{\tau-1} h_1h_2h_3\kappa_3\|e_{n,\rho,\theta}\| + \sum_{\rho=0}^{m-1} h_1h_2h_3\kappa_3\|e_{n,\rho,\tau}\| \\ & + \sum_{\theta=0}^{\tau-1} h_1h_2h_3\kappa_3\|e_{n,m,\theta}\| + \kappa_3\int_{t_n}^{t}\int_{s_m}^{s}\int_{r_\tau}^{r}|w(x,y,z) - \hat{\vartheta}_{n,m,\tau}(x,y,z)|dzdydx,\end{aligned}$$

where $\kappa_3 = max\{\|\kappa\|_{L^\infty(\mathcal{R}_{n,m,\tau})}, n = 0,\dots,N-1; m = 0,\dots,M-1; \tau = 1,\dots,T-1\}$, then by Lemma 1.7.7

$$|w(t,s,r) - \hat{\vartheta}_{n,m,\tau}(t,s,r)| \leq \left(\sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1}\sum_{\theta=0}^{\tau-1}\|e_{\xi,\rho,\theta}\| + \sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1}\|e_{\xi,\rho,\tau}\|\right)h_1h_2h_3b_4$$

$$
+\left(\sum_{\xi=0}^{n-1}\sum_{\theta=0}^{\tau-1}||e_{\xi,m,\theta}||+\sum_{\xi=0}^{n-1}||e_{\xi,m,\tau}||+\sum_{\rho=0}^{m-1}\sum_{\theta=0}^{\tau-1}||e_{n,\rho,\theta}||\right)h_1h_2h_3b_4
$$
$$
+\left(\sum_{\rho=0}^{m-1}||e_{n,\rho,\tau}||+\sum_{\theta=0}^{\tau-1}||e_{n,m,\theta}||\right)h_1h_2h_3b_4,
$$

where $b_4=\kappa_3 exp(\sqrt[3]{\kappa_3}(A+B+C))$, then by Lemma 4.3.1

$$
\begin{aligned}
||e_{n,m,\tau}|| &\leq ||w-\hat{\vartheta}_{n,m,\tau}||+||\hat{\vartheta}_{n,m,\tau}-\vartheta_{n,m,\tau}|| \\
&\leq \left(\sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1}\sum_{\theta=0}^{\tau-1}||e_{\xi,\rho,\theta}||+\sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1}||e_{\xi,\rho,\tau}||\right)h_1h_2h_3b_4 \\
&+\left(\sum_{\xi=0}^{n-1}\sum_{\theta=0}^{\tau-1}||e_{\xi,m,\theta}||+\sum_{\xi=0}^{n-1}||e_{\xi,m,\tau}||+\sum_{\rho=0}^{m-1}\sum_{\theta=0}^{\tau-1}||e_{n,\rho,\theta}||\right)h_1h_2h_3b_4 \\
&+\left(\sum_{\rho=0}^{m-1}||e_{n,\rho,\tau}||+\sum_{\theta=0}^{\tau-1}||e_{n,m,\theta}||\right)h_1h_2h_3b_4+\sum_{i+j+k=p}\frac{1}{i!j!k!}\left\|\partial_t^{(i)}\partial_s^{(j)}\partial_r^{(k)}\hat{\vartheta}_{n,m,\tau}\right\|h_1^ih_2^jh_3^k,
\end{aligned}
$$

hence, by Lemma 4.3.2, we obtain

$$
\begin{aligned}
||e_{n,m,\tau}|| \leq &\left(\sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1}\sum_{\theta=0}^{\tau-1}||e_{\xi,\rho,\theta}||+\sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1}||e_{\xi,\rho,\tau}||\right)h_1h_2h_3b_4 \\
&+\left(\sum_{\xi=0}^{n-1}\sum_{\theta=0}^{\tau-1}||e_{\xi,m,\theta}||+\sum_{\xi=0}^{n-1}||e_{\xi,m,\tau}||+\sum_{\rho=0}^{m-1}\sum_{\theta=0}^{\tau-1}||e_{n,\rho,\theta}||\right)h_1h_2h_3b_4 \\
&+\left(\sum_{\rho=0}^{m-1}||e_{n,\rho,\tau}||+\sum_{\theta=0}^{\tau-1}||e_{n,m,\theta}||\right)h_1h_2h_3b_4+\frac{\varphi(p)}{p!}(h_1+h_2+h_3)^p,
\end{aligned}
$$

then, by Lemma 1.7.5, we obtain

$$
||e_{n,m,\tau}|| \leq \left(\sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1}||e_{\xi,\rho,\tau}||+\sum_{\xi=0}^{n-1}\sum_{\theta=0}^{\tau-1}||e_{\xi,m,\theta}||+\sum_{\rho=0}^{m-1}\sum_{\theta=0}^{\tau-1}||e_{n,\rho,\theta}||\right)h_1h_2h_3b_4b_5
$$

$$+\left(\sum_{\xi=0}^{n-1}\|e_{\xi,m,\tau}\|+\sum_{\rho=0}^{m-1}\|e_{n,\rho,\tau}\|+\sum_{\theta=0}^{\tau-1}\|e_{n,m,\theta}\|\right)h_1h_2h_3b_4b_5+\frac{\varphi(p)}{p!}b_5(h_1+h_2+h_3)^p, \tag{4.32}$$

where $b_5 = \exp(ABCb_4)$, we put for each fixed $\tau \in \mathbb{N}$

$$\omega(n,m) = \|e_{n,m,\tau}\|, \quad \epsilon(n,m) = h_1h_2h_3b_4b_5,$$
$$b(n,m) = \left(\sum_{\xi=0}^{n-1}\sum_{\theta=0}^{\tau-1}\|e_{\xi,m,\theta}\|+\sum_{\rho=0}^{m-1}\sum_{\theta=0}^{\tau-1}\|e_{n,\rho,\theta}\|+\sum_{\xi=0}^{n-1}\|e_{\xi,m,\tau}\|+\sum_{\rho=0}^{m-1}\|e_{n,\rho,\tau}\|+\sum_{\theta=0}^{\tau-1}\|e_{n,m,\theta}\|\right)h_1h_2h_3b_4b_5$$
$$+\frac{\varphi(p)}{p!}b_5(h_1+h_2+h_3)^p.$$

Lemma 1.7.4 therefore allows us to derive from (4.32)

$$\begin{aligned}\|e_{n,m,\tau}\| &\le b(n,m)\prod_{\xi=0}^{n-1}\left[1+\sum_{\rho=0}^{m-1}\epsilon(\xi,\rho)\right]\\ &\le \left(\sum_{\xi=0}^{n-1}\sum_{\theta=0}^{\tau-1}\|e_{\xi,m,\theta}\|+\sum_{\rho=0}^{m-1}\sum_{\theta=0}^{\tau-1}\|e_{n,\rho,\theta}\|+\sum_{\rho=0}^{m-1}\|e_{n,\rho,\tau}\|+\sum_{\theta=0}^{\tau-1}\|e_{n,m,\theta}\|+\sum_{\xi=0}^{n-1}\|e_{\xi,m,\tau}\|\right)h_1h_2h_3b_4b_6\\ &\quad+\frac{\varphi(p)}{p!}b_6(h_1+h_2+h_3)^p,\end{aligned} \tag{4.33}$$

where $b_6 = b_5\exp(ABCb_4b_5)$, we put for each fixed $m \in \mathbb{N}$

$$\omega(n,\tau) = \|e_{n,m,\tau}\|, \quad \epsilon(n,\tau) = h_1h_2h_3b_4b_6,$$
$$b(n,\tau) = \left(\sum_{\rho=0}^{m-1}\sum_{\theta=0}^{\tau-1}\|e_{n,\rho,\theta}\|+\sum_{\rho=0}^{m-1}\|e_{n,\rho,\tau}\|+\sum_{\theta=0}^{\tau-1}\|e_{n,m,\theta}\|+\sum_{\xi=0}^{n-1}\|e_{\xi,m,\tau}\|\right)h_1h_2h_3b_4b_6$$
$$+\frac{\varphi(p)}{p!}b_6(h_1+h_2+h_3)^p.$$

Using Lemma 1.7.4 again, we derive from (4.33)

$$\|e_{n,m,\tau}\| \le b(n,\tau)\prod_{\xi=0}^{n-1}\left[1+\sum_{\theta=0}^{\tau-1}\epsilon(\xi,\theta)\right],$$

hence,

$$\|e_{n,m,\tau}\| \leq \left(\sum_{\rho=0}^{m-1}\sum_{\theta=0}^{\tau-1}\|e_{n,\rho,\theta}\| + \sum_{\rho=0}^{m-1}\|e_{n,\rho,\tau}\| + \sum_{\theta=0}^{\tau-1}\|e_{n,m,\theta}\| + \sum_{\xi=0}^{n-1}\|e_{\xi,m,\tau}\|\right) h_1h_2h_3b_4b_7$$
$$+ \frac{\varphi(p)}{p!}b_7(h_1+h_2+h_3)^p$$
$$\leq h_2h_3Ab_4b_7\sum_{\rho=0}^{m-1}\sum_{\theta=0}^{\tau-1}\|e_{n,\rho,\theta}\| + h_2ACb_4b_7\sum_{\rho=0}^{m-1}\|e_{n,\rho,\tau}\| + h_3ABb_4b_7\sum_{\theta=0}^{\tau-1}\|e_{n,m,\theta}\|$$
$$+ h_1BCb_4b_7\sum_{\xi=0}^{n-1}\|e_{\xi,m,\tau}\| + \frac{\varphi(p)}{p!}b_7(h_1+h_2+h_3)^p,$$

where $b_7 = b_6\exp(ABCb_4b_6)$, by using Lemma 1.7.3, for each fixed $n \in \mathbb{N}$

$$\|e_{n,m,\tau}\| \leq \left(h_1BCb_4b_7\sum_{\xi=0}^{n-1}\|e_{\xi,m,\tau}\| + \frac{\varphi(p)}{p!}b_7(h_1+h_2+h_3)^p\right)\exp\left(b_8(B+C)\right)$$
$$\leq h_1BCb_4b_9\sum_{\xi=0}^{n-1}\|e_{\xi,m,\tau}\| + \frac{\varphi(p)}{p!}b_9(h_1+h_2+h_3)^p,$$

where $b_8 = \frac{1}{2}\left(ACb_4b_7 + ABb_4b_7 + \sqrt{(ACb_4b_7 + ABb_4b_7)^2 + 4Ab_4b_7}\right)$ and $b_9 = b_7\exp\left(b_8(B+C)\right)$. In addition, Lemma 1.7.1 yields

$$\|e_{n,m,\tau}\| \leq \underbrace{\frac{\varphi(p)}{p!}b_9\exp(ABCb_4b_9)}_{\varsigma_4}(h_1+h_2+h_3)^p,$$

by taking $\varsigma = \max\{\varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4\}$, the proof is thus completed. ■

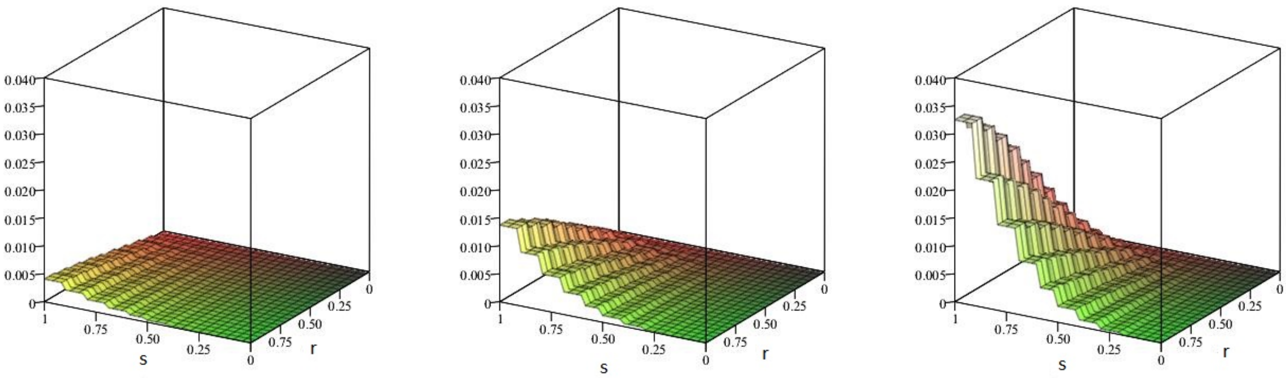
4.4 Experimental Results

This section presents several examples with analytical solutions to assess the efficiency of the method described in Section 4.2 for solving equations (4.1) and (4.2). These examples are sourced from a range of references [3, 23, 39, 43, 45, 67], enabling a comparison of the numerical outcomes.

Example 4.4.1 *Consider the following three-dimensional linear Volterra integral equation [67]:*

$$w(t,s,r) = t\cos(r) - \frac{t^3 s^3}{9}\sin(r) + \int_0^t \int_0^s \int_0^r xy^2 w(x,y,z)\,dz\,dy\,dx,$$

where $t,s,r \in [0,1]$, the exact solution can be expressed as $w(t,s,r) = t\cos(r)$. Applying the Taylor collocation method to the above integral equation, with $p = 2$ and a grid size of $(N,M,T) = (100,100,100)$, the error results are presented in Table 4.1. Furthermore, a comparison between the errors computed using the current method (TCM) and the shifted Chebyshev polynomial (SCP) [45], as well as three-dimensional block-pulse functions (3D-BPFs) [23], is provided. The plots of the error functions for $N = M = T = 10$, with fixed values of t, s and r for each plot, are shown in Figures 4.1, 4.2 and 4.3, respectively.



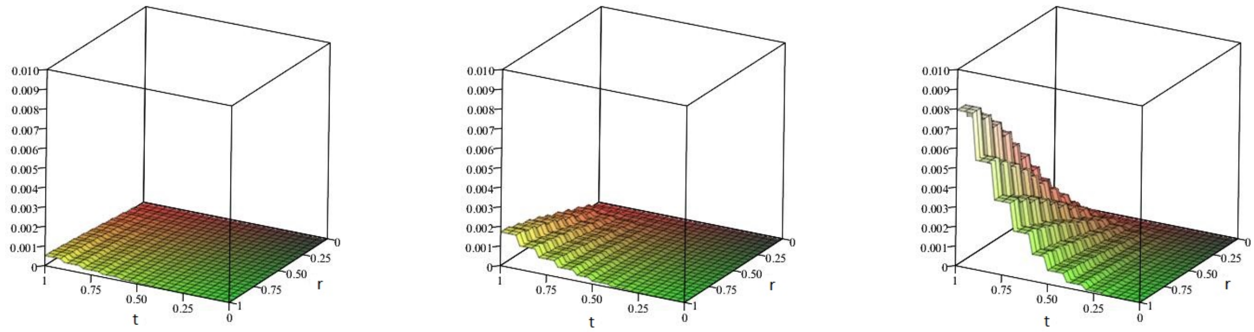
$t = 0.4$ $t = 0.6$ $t = 0.8$

Figure 4.1 – Error function plot for a fixed t in Example 4.4.1

Example 4.4.2 *Consider the given three-dimensional linear Volterra integral equation [67]:*

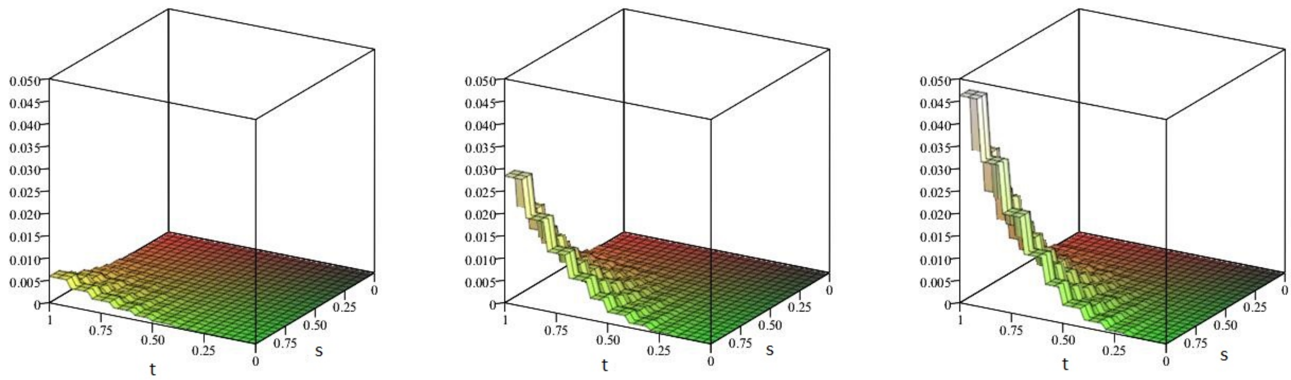
$$w(t,s,r) = e^{t+s} + (e^r - 1)(e^t + e^s - 1) + \int_0^t \int_0^s \int_0^r w(x,y,z)\,dz\,dy\,dx,$$

where $t,s,r \in [0,1]$, the exact solution can be expressed as $w(t,s,r) = e^{t+s+r}$. Table 4.2 presents the absolute errors at certain points with $p = 2$ and $N = M = T = 10$. Additionally, a comparison between the errors computed using the current method (TCM) and shifted Chebyshev polynomial (SCP) [45], is provided.



$s = 0.2$ $\quad$ $s = 0.3$ $\quad$ $s = 0.5$

Figure 4.2 – Error function plot for a fixed s in Example 4.4.1



$r = 0.1$ $\quad$ $r = 0.5$ $\quad$ $r = 0.9$

Figure 4.3 – Error function plot for a fixed r in Example 4.4.1

Example 4.4.3 *Consider the given three-dimensional linear Volterra integral equation [3]:*

$$w(t,s,r) = f(t,s,r) - 24\int_0^t\int_0^s\int_0^r t^2 s w(x,y,z)dzdydx,$$

for $t,s,r \in [0,1]$ *and* $f(t,s,r) = 4t^5s^3r + 4t^3s^3r^3 + 3t^4s^3r^2 + t^2s + sr^2 + tsr$. *This equation has an exact solution* $w(t,s,r) = t^2s + sr^2 + tsr$. *Table 4.3 presents the absolute errors at certain points with* $p = 2$ *and* $N = M = T = 10$. *Additionally, a comparison between the errors computed using the TCM and the three-dimensional block-pulse function method (3D-BF) [43], as well as the three-dimensional block-pulse functions method (3D-BPFs) [23] and three-dimensional Bernstein polynomials (3D-BPs) [39], is provided.*

Table 4.1 – Comparison of the absolute errors of Example 4.4.1

t	s	r	SCP [45]	TCM
0.1	0.1	0.1	2.29×10^{-4}	1.10×10^{-8}
0.01	0.1	0.1	2.29×10^{-5}	1.10×10^{-11}
0.01	0.01	0.1	2.29×10^{-5}	1.10×10^{-14}
0.01	0.01	0.01	2.92×10^{-6}	1.11×10^{-15}
0.001	0.01	0.01	2.92×10^{-7}	1.11×10^{-18}
0.001	0.001	0.01	2.92×10^{-7}	1.11×10^{-21}
0.001	0.001	0.001	2.99×10^{-8}	1.11×10^{-22}

t	s	r	3D-BPFs [23]	TCM
0.1	0.1	0.1	6.40×10^{-2}	1.10×10^{-8}
0.3	0.3	0.3	3.07×10^{-3}	2.39×10^{-5}
0.5	0.5	0.5	1.11×10^{-2}	8.32×10^{-4}
0.7	0.7	0.7	1.14×10^{-3}	8.42×10^{-3}
0.9	0.9	0.9	9.92×10^{-4}	4.63×10^{-2}

Table 4.2 – Comparison of the absolute errors of Example 4.4.2

t	s	r	SCP [45]	TCM
0.1	0.1	0.1	3.30×10^{-2}	1.16×10^{-3}
0.01	0.1	0.1	2.16×10^{-2}	1.11×10^{-4}
0.01	0.01	0.1	1.19×10^{-2}	1.06×10^{-5}
0.01	0.01	0.01	3.66×10^{-3}	1.01×10^{-6}
0.001	0.01	0.01	2.55×10^{-3}	1.01×10^{-7}
0.001	0.001	0.01	1.45×10^{-3}	1.00×10^{-8}
0.001	0.001	0.001	3.69×10^{-4}	1.00×10^{-9}

Example 4.4.4 *Let us now consider the three-dimensional nonlinear Volterra integral equation [67]*

$$w(t,s,r) = tsr - \frac{(tsr)^3}{27} + \int_0^t\int_0^s\int_0^r w^2(x,y,z)dzdydx,$$

which has the exact solution $w(t,s,r) = tsr$. By applying the Taylor collocation method to the above integral equation, the absolute errors for $p = 2$ and $(N,M) = (10,10)$ at some points are shown in Table 4.4.

Example 4.4.5 *Consider the three-dimensional nonlinear Volterra integral equation [60]*

$$w(t,s,r) = f(t,s,r) + \int_0^t\int_0^s\int_0^r (tsr + xy + z^2)w^2(x,y,z)dzdydx,$$

for $t,s,r \in [0,1]$ and $f(t,s,r) = t^3s^5r^3 - \frac{1}{539}t^8s^{12}r^8 - \frac{1}{672}t^8s^{12}r^7 - \frac{1}{693}t^7s^{11}r^9$. This equation has

Table 4.3 – Comparison of the absolute errors of Example 4.4.3

$(2^{-k}, 2^{-k}, 2^{-k})$	3D-BFs [43]	3D-BPs[39]	TCM
$k = 1$	8.44×10^{-2}	3.03×10^{-11}	2.14×10^{-2}
$k = 2$	2.30×10^{-2}	7.13×10^{-12}	4.19×10^{-5}
$k = 3$	7.36×10^{-3}	7.93×10^{-13}	8.19×10^{-8}
$k = 4$	2.80×10^{-3}	4.78×10^{-12}	1.60×10^{-10}
$k = 5$	4.12×10^{-4}	1.81×10^{-13}	3.12×10^{-13}
$k = 6$	4.92×10^{-4}	1.65×10^{-13}	6.10×10^{-16}

t	s	r	3D-BPFs [23]	TCM
0.1	0.1	0.1	9.30×10^{-4}	1.10×10^{-8}
0.3	0.3	0.3	2.45×10^{-3}	2.16×10^{-4}
0.5	0.5	0.5	3.63×10^{-2}	2.14×10^{-2}
0.7	0.7	0.7	1.38×10^{-2}	4.43×10^{-1}
0.9	0.9	0.9	6.78×10^{-2}	4.26×10^{-0}

Table 4.4 – Numerical results of Example 4.4.4

t	s	r	TCM
0.1	0.1	0.1	3.70×10^{-11}
0.2	0.2	0.2	1.89×10^{-8}
0.3	0.3	0.3	7.29×10^{-7}
0.4	0.4	0.4	9.70×10^{-6}
0.5	0.5	0.5	7.23×10^{-5}
0.6	0.6	0.6	3.73×10^{-4}
0.7	0.7	0.7	1.49×10^{-3}
0.8	0.8	0.8	4.97×10^{-3}
0.9	0.9	0.9	1.43×10^{-2}
1.0	1.0	1.0	5.66×10^{-2}

an exact solution $w(t, s, r) = t^3 s^5 r^3$.

The absolute errors for $p = 2$ *and* $(N, M) = (10, 10)$ *are also shown in Table 4.5.*

4.5 Conclusion

This chapter has introduced a new method for solving linear and nonlinear 3D-VIEs of the second kind by applying the TCM. Developing an algorithm utilizing Taylor polynomials enabled us to approximate solutions effectively within a finite-dimensional space. A convergence analysis was conducted to validate the efficacy of our method, affirm-

Table 4.5 – Numerical results of Example 4.4.5

t	s	r	TCM
0.1	0.1	0.1	3.11×10^{-30}
0.2	0.2	0.2	4.43×10^{-22}
0.3	0.3	0.3	2.65×10^{-17}
0.4	0.4	0.4	6.61×10^{-14}
0.5	0.5	0.5	2.87×10^{-11}
0.6	0.6	0.6	4.13×10^{-9}
0.7	0.7	0.7	2.77×10^{-7}
0.8	0.8	0.8	1.06×10^{-5}
0.9	0.9	0.9	2.67×10^{-4}
1.0	1.0	1.0	3.03×10^{-1}

ing its reliability in providing accurate solutions. Additionally, comparison examples were presented, demonstrating the method's effectiveness compared to alternative approaches. Through these comparisons, our approach proved efficient and reliable in resolving 3D-VIEs. The results of this study contribute to advancing mathematical methodologies for solving complex integral equations, offering practical implications across various disciplines.

CONCLUSION AND PERSPECTIVES

In summary, multi-dimensional Volterra integral equations significantly expand classical integral equations, allowing for the modeling of complex systems with multiple variables. Their theoretical underpinnings and practical uses remain a vibrant field of research, driving progress across various scientific and engineering fields.

In this thesis, a new numerical method was developed using the Taylor collocation method to approximate solutions of the Goursat problem within hyperbolic linear partial differential equations. This method operates in the real polynomial spline space $\mathcal{S}_{p-1}^{(-1)}(\Pi_{N,M})$ and has been proven to be convergent.

Furthermore, a numerical framework was established to approximate solutions for nonlinear two-dimensional Volterra integral equations of the first kind. By converting these problems into Volterra integral equations of the second kind, an efficient and accurate solution was proposed using Taylor polynomial approximations.

In the spline space $\mathcal{S}_{p-1}^{(-1)}(\Pi)$, approximate solutions for three-dimensional Volterra integral equations were obtained by applying Taylor's theorem in three variables. The convergence of these approximate solutions to the exact solutions was studied. This method is straightforward to implement, with the coefficients of the approximate solution determined by iterative formulas without requiring the solution of any algebraic equations. Numerous numerical examples demonstrated the method's convergence and accuracy, aligning with theoretical estimates.

This thesis has demonstrated the effectiveness of the Taylor collocation method in solving various classes of Volterra integral equations. While the proposed approach has yielded promising results, several directions remain open for further research:

- **Extension to Goursat Problems in Higher-Order Linear Hyperbolic Equations**

  A possible extension of this work involves solving higher-order linear hyperbolic equations with Goursat-type initial conditions, given by:

$$\frac{\partial^k w}{\partial x_1 \partial x_2 \ldots \partial x_k} + \sum_{i,j=1, i<j}^{k} \psi_{i,j} \frac{\partial^2 w}{\partial x_i \partial x_j} + \sum_{i=1}^{k} \psi_i \frac{\partial w}{\partial x_i} + \psi w = F,$$

  where $\psi_{i,j}, \psi_i$ for $i < j, i, j = 1, 2, \ldots, k$, and ψ and F are specified real functions. Developing a numerical collocation method to approximate the solution of such equations would be valuable for applications in wave propagation, fluid dynamics, and elasticity theory.

- **Extension to Three-Dimensional Nonlinear Delay Volterra Integral Equations**

  Another potential research direction is the application of collocation techniques to three-dimensional nonlinear delay Volterra integral equations, which are of the form:

$$w(t,s,r) = f(t,s,r) + \int_0^{t-\tau_1} \int_0^{s-\tau_2} \int_0^{r-\tau_3} \kappa(t,s,r,x,y,z,w(x,y,z))\, dz\, dy\, dx,$$

  with f and κ being sufficiently smooth functions. These equations arise in biological systems, population dynamics, and control processes with hereditary effects. Investigating numerical methods for solving such equations can improve modeling accuracy in these domains.

- **Development of Collocation Methods for Transport-Diffusion Equations**

A further research avenue is the development of collocation methods for transport-diffusion equations, given by:

$$\frac{\partial u}{\partial t} + v(t,x) \cdot \nabla u - \kappa \Delta u + \lambda u = f(t,x),$$

where:

- $u = u(t,x)$ is the unknown function representing the transported and diffused quantity.
- $v(t,x)$ is the velocity vector of the transport process.
- κ is the diffusion coefficient.
- λ is a large reaction coefficient.
- $f(t,x)$ is a given source term.
- Δ is the Laplacian operator.

Many real-world transport-diffusion problems, particularly those involving sharp gradients, turbulence, or fast reaction kinetics, require adaptive numerical strategies to efficiently capture localized behavior.

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