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# On Poincaré conjecture and Gregory Perlman famous theorem


**Presented by:** - Bara Imane


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# DIDICATION

To those who planted in me the seeds of ambition, and nurtured them with sacrifice and love, To those who stayed up countless nights for my sake, To those whose prayers were the secret behind my success...

To my dear parents,

I dedicate to you the fruit of my effort and a cornerstone of my future. All my gratitude and love are yours.

To my esteemed professors, who never hesitated to share their knowledge and guidance, I offer you my sincere thanks and deep appreciation.

And to my friends and siblings, You are my support and my cherished companions, All my affection and respect to you.

# ABSCRAT

In this graduation thesis were titled *"On Poincaré Conjecture and Gregory Perelman's Famous Theorem,"* we addressed fundamental concepts in algebraic topology and category theory. The first section covered **homotopy theory**, examining the homotopy of continuous maps, contractible spaces, and a detailed analysis of the fundamental group and Poincaré group. The second section focused on **category theory**, exploring dual categories, subcategories, and specialized categories such as the category of pointed topological spaces and the category of chain complexes, while emphasizing the role of homotopical functors in connecting these mathematical structures. Together, these concepts provided the theoretical framework essential for understanding the methods used to solve complex topological problems.

**Keywords: Homotopy- categories- functions- paths**

# RÉSUMÉ

Dans ce mémoire intitulé *« Sur la conjecture de Poincaré et la démonstration célèbre de Grigori Perelman »*, nous explorons des concepts fondamentaux en topologie algébrique et en théorie des catégories. La première partie développe des aspects clés de la théorie de l'homotopie, incluant les homotopies d'applications continues, les espaces contractiles, ainsi qu'une étude approfondie des groupes fondamentaux et des groupes de Poincaré. La seconde partie examine des structures catégoriques, analysant les catégories duales, les sous-catégories, ainsi que des catégories spécialisées comme celles des espaces topologiques pointés et des complexes de chaînes, en accordant une attention particulière aux foncteurs homotopiques qui relient ces cadres mathématiques. Ensemble, ces constructions théoriques forment le socle essentiel des travaux révolutionnaires de Perelman sur la conjecture de Poincaré et les problèmes topologiques associés.
**Mots clés: L'homotopie- Les categories- Les fonctions- Les chemins**

# ملخص

في هذه المذكرة التي بعنوان " On Poincaré Conjecture and Gregory Perelman's Famous Theorem" تطرقنا إلى مفاهيم أساسية في الطوبولوجيا الجبرية و نظرية الفئات . تناول القسم الأول نظرية التماثل (Homotopy Theory) متضمنا دراسة التماثل المستمر للخرائط و الفضاءات القابلة للانكماش، مع تحليل مفصل للزمرة الأساسية و مجموعة بوانكاريه. أما القسم الثاني فقد ركز على نظرية الفئات، (Category Theory) حيث استعرضنا الفئات المزدوجة، الفئات الفرعية، و فئات خاصة مثل فئة الفضاءات الطوبولوجية المؤشرة و فئة السلاسل المركبة، مع التأكيد على دور الدوال التماثلية في الربط بين هذه البنى الرياضية. قدمت هذه المفاهيم معا الإطار النظري الضروري لفهم الأساليب المستخدمة في حل المسائل الطوبولوجية المعقدة.

**الكلمات المفتاحية :** التماثل التجانسي _الفئات_ الدوال_ المسارات.

# INTRODUCTION

The Poincaré Conjecture is a mathematical problem proposed by Henri Poincaré in 1904, concerning the shape of three-dimensional space. In 2003, Gregory Perelman proved it using advanced geometric ideas and rejected the major awards offered to him, stating that he was not interested in money or fame.

The first chapter, titled "Topological Algebra," covers several fundamental topics, beginning with categories and functors, and then moving on to the concept of the dual category. The chapter also discusses other concepts such as subcategory, functors, and various categories. It further addresses the chain complex, followed by the category of pointed topological spaces and applications between them. The chapter concludes with a section on the category of directed sets.

In the second chapter discusses homotopic functor, beginning with the concept of homotopy between continuous maps, and delving into the construction of category related to this concept. It then focuses on contractible spaces as a special case of homotopy, followed by a presentation of relative homotopy and the essential notions of retract and deformation, which play a key role in the study of topological spaces.

Chapter three addresses the topic of Homotopy of Paths. It begins with a study of the properties of path homotopy in section, which also includes operations on paths in subsection. The chapter then moves on to the fundamental group and the Poincaré group in section, where the homotopy group of order 1 is discussed in subsection, followed by continuous applications and fundamental groups in subsection. Finally, the chapter concludes with section, which discusses homotopy of continuous maps and the Poincaré group.

# CHAPTER 1

# CATEGORIES AND FUNCTORS

In this chapter we introduce the notion of categories and functors which provides a unifying language for understandding and solving different areas of mathimatics.

**Definition:** We say that $C$ is a category if it consists of:

1. A collection of objects, denoted by Obj($C$).

2. A collection of morphisms, denoted by Mor($C$).

3. For any two objects $X$ and $Y$ in $C$, there is a subcollection $\mathrm{Mor}_C(X, Y) \subset \mathrm{Mor}(C)$ consisting of morphisms from $X$ to $Y$.

4. A composition law for morphisms, denoted by "$\circ$":

For every morphism $f \in \mathrm{Mor}_C(X, Y)$ and $g \in \mathrm{Mor}_C(Y, Z)$, there exists a morphism $h \in \mathrm{Mor}_C(X, Z)$ such that: $h = g \circ f$.

The composition law must satisfy the following properties:

1.**Associativity:** For all $f \in \mathrm{Mor}_C(X, Y)$, $g \in \mathrm{Mor}_C(Y, Z)$, and $h \in \mathrm{Mor}_C(Z, T)$, we have $(h \circ g) \circ f = h \circ (g \circ f)$.

2.**Existence of an identity morphism:** For every object $X$ in $C$, there exists an identity morphism $\mathrm{Id}_X \in \mathrm{Mor}_C(X, X)$ such that for any $g \in \mathrm{Mor}_C(X, Y)$ and $h \in$

$\text{Mor}_C(Y, X)$ we have: $\begin{cases} g \circ \text{Id}_X = g \\ \text{Id}_X \circ h = h \end{cases}$

## 1.1 Fondation of mathematics

Categories arose from three events:

- Crisis of foundations in mathematics.
- Accumulation in mathematics revealed by three paradoxes:
  - Russell's paradox (Whitehead),
  - The librarian's paradox,
  - The liar's paradox.
- Truth is relative; it depends on the perspective where one emphasizes certain advantages that are not necessarily valid in all contexts.

Thus, we must redefine the concepts on which mathematics is based, both by rethinking the mathematical terms and by creating new theories.
**Some properties**:
If $C$ is a category, then:
1. A morphism $f \in Mor_c(X, Y)$ such that: ( $f : X \to Y$ or $X \xrightarrow{f} Y$) is called an **arrow**.
2. If $f : X \to Y$ is an arrow, $X$ is called the **source** of the arrow $f$ and $Y$ is called the **target**.
3. $\text{Id}_X \in Mor_c(X, X)$ is called the **identity arrow** of $X$ or morphism identity of X.
**Definition**:
. A category $C$ is said to be small if obj C is a set.
. A category $C$ is said **Connected** if $\text{Mor}(X, Y) \neq \emptyset \ \forall X, Y \in \text{Obj}(C)$.
. A category $C$ is said **Discrete** if $\text{Mor}(X, Y) = \emptyset$ .
**Some example**:

**Category of sets and applications "Set"**:

. Obj(Set) = the collection of sets.

. Mor(Set) = the collection of functions between sets.

. If $X, Y \in$ Obj(Set), that is, $X$ and $Y$ are two sets $Mor_{set}(X, Y)$ = the collection of functions from $X$ to $Y$ .

**Remarks**:

1. A set is a collection, but a collection is not necessarily a set.

2. "$\circ$" is the **composition of application** their are exists and well-defined bacause "composition of two application is also application".

3. It is also associative.

4. If $X$ is a set, the identity map denoted by $\mathrm{Id}_X$ is the identity morphism associated to $X$.

5. This category represents the theory of sets, and is called the category of sets and mapping.

**Category of groups and homomorphisms groupes "$\mathcal{G}$"**

. Obj($\mathcal{G}$) = the collection of groups.

. Mor($\mathcal{G}$) = the collection of group homomorphisms.

. If $G, G'$ are groups, then $Mor_{\mathcal{G}}(G, G') \equiv$ the collection of homomorphisms from $G$ to $G'$.

. $'' \circ ''$ The composition of homomorphisms it is well defined because "the composition of two homomorphisms is again a homomorphism".

. It is associative.

. If $G$ is a group, $\mathrm{Id}_G$ is the automorphism of $G$ in $G$.

. $\mathcal{G}$ is the domain of the theory of groups. It is called the category of groups and group homomorphisms.

**Category of topological spaces and continuous maps**

. Obj($Top$) = Collection of topological spaces.

. Mor($Top$) = Collection of continuous maps.

. If $X$ and $Y$ are two topological spaces, then $Mor_Top(X, Y)$ is the collection of continuous mapping from $X$ to $Y$.

. "$\circ$" The composition of continuous functions it is correctly defined because "the composition of two continuous maps is continuous".

. It is associative.

. If $X$ is a topological space then $Id_X : X \to X$ is a continuous application satisfying:

$f \circ Id_X = f, \quad \forall f : X \to Y.$

$Id_X \circ g = g, \quad \forall g : Y \to X.$

Then Top is a category called the category of topological spaces and continuous maps.

**Category of mesurable sets and mesurable funtions**

. **Obj (Mes)**: Collection of mesurable sets.

. **Mor (Mes)**: Collection of mesurable functions.

. If $X$ and $Y$ are two measurable sets, then $\mathrm{Mor}_{\mathrm{Mes}}(X, Y)$ = collection of measurable functions fro $Y$.

. $\circ$ : Composition of mesurable functions(composition of two mesurable functions is mesurable function).

. It is associative.

. If $X$ is a mesurable set, then

$\mathrm{Id}_X : X \to X$ mesurable function, (identity map).

Mes is the domain of probability theory and randon variables.

**Definition:** An object $X$ of a category $C$ is called an *initial object* if and only if $\mathrm{Mor}(X, Y) = \{f\}$ (a singleton) for every object $Y$ in $C$.

**Definition:** A morphism $f \in Mor_C(X, Y)$ of a category $C$ is called isomorphism if and only if there exists a morphism $g \in \mathrm{Mor}(Y, X)$ such that:

$$f \circ g = \mathrm{Id}_Y \to Mor(Y, Y) \quad \text{and} \quad g \circ f = \mathrm{Id}_X \to Mor(X, X).$$

In this case, we say that $X$ and $Y$ are isomorphic objects. **Examples:**

1) The category **Set** $\equiv$ two objects are isomorphic $\Leftrightarrow \begin{cases} \exists f \in \mathrm{Mor}_{\mathrm{Set}}(X, Y) \quad \textit{map} \\ \text{and} \\ \exists g \in \mathrm{Mor}_{\mathrm{Set}}(Y, X) \quad \textit{map} \end{cases}$

with $\begin{cases} f \circ g = \mathrm{Id}_Y \\ \text{and} \\ g \circ f = \mathrm{Id}_X \end{cases}$

$\Leftrightarrow$ $X$ and $Y$ two bijectives sets $\Leftrightarrow$ the cardinality of X = the cardinality of Y.

2) $\mathcal{G}$ then $G$ is isomorphic to $G'$ $\Leftrightarrow$ G and G′ are isomorphic.

3) Top then $X \cong X'$ (topological space) $\Leftrightarrow$ $X$ and $X'$ are **homeomorphic**.

Indeed, if $X, X' \in \mathrm{Obj}(Top) \Leftrightarrow X$ and $X'$ are topological spaces.

If $X$ is isomorphic to $X'$, then:

$$(X \cong X') \Leftrightarrow \begin{cases} \exists f \in \mathrm{Mor}_{Top}(X, X')\ (f \text{ continuous}) \\ \exists g \in \mathrm{Mor}_{Top}(X', X)\ (g \text{ continuous}) \end{cases} \quad \text{such that: } \begin{cases} f \circ g = \mathrm{id}_{X'} \\ g \circ f = \mathrm{id}_X \end{cases}$$

$\Rightarrow$ $f$ and $g$ are continuous bijections $\Rightarrow$ $f$ and $g$ are homeomorphisms ($f = g^{-1}$)

**Definition:** A diagram is said to be commutative if both paths with the same start and end points are equal.

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ & \searrow^{h} & \downarrow g \\ & & C \end{array} \quad \text{we have } g \circ f = h$$

And :

$$\begin{array}{ccc} A & \xrightarrow{f_1} & B \\ \downarrow g_2 & & \downarrow g_1 \\ C & \xrightarrow{f_2} & D \end{array} \quad \text{we have } g_2 \circ f_1 = f_2 \circ g_1$$

**Definition:** Let $f \in \mathrm{Mor}_C(A, B)$. $f$ is a monomorphism if for all :

$g_1, g_2 : X \rightarrow A \quad we \quad have: \quad f \circ g_1 = f \circ g_2 f \Rightarrow g_1 = g_2$

$$\mathrm{X} \xrightarrow{g_1} \mathrm{A} \xrightarrow{f} \mathrm{B}$$
$$\mathrm{X} \xrightarrow{g_2} \mathrm{A} \xrightarrow{f} \mathrm{B}$$

**Definition:** Let $f \in \mathrm{Mor}_C(A, B)$. $f$ is an epimorphism if for all :

$h_1, h_2 : B \rightarrow X \quad h_1 \circ f = h_2 \circ f \Rightarrow h_1 = h_2$

$$A \xrightarrow{f} B \xrightarrow{h_1} X$$
$$A \xrightarrow{f} B \xrightarrow{h_2} X$$

**Remark:** There are categories in which monomorphisms and epimorphisms are the injective and surjective morphisms respectively as the category set.

## 1.2 Dual Category

**Principle of reversal arrow**

**Definition:** We called the *dual category* associated with a category $C$ is the category denoted by $C^*$ in which:

1. $\text{Obj}(C^*) \subseteq \text{Obj}(C)$
2. $\text{Mor}(C^*) \subseteq \text{Mor}(C)$

i.e., if $X, Y \in \text{Obj}(C^*)$ (so necessarily $X, Y \in \text{Obj}(C)$), then:

$\text{Mor}_{C^*}(X, Y) = \text{Mor}_C(Y, X)$

3. The composition law in $C^*$ is the same as in $C$
4. The identity morphism of an object $X$ in $C^*$ is the same identity morphism of $X$ in $C$

**Definition:** An object $X$ in a category $C$ is called a $2^{nd}$*order object* (category) if and only if: $\text{Mor}_C(Y, X) = \{f\}$ (singleton) for every $Y \in \text{Obj}(C)$.

**Proposition:** An object $X$ in a category $C$ is of $2^{nd}$order

$\Leftrightarrow$ $X$ in the dual category is of $1^{st}$ order

► $\text{Mor}_{C^*}(X, Y) = \text{Mor}_C(Y, X) = \{.\}, \quad \forall Y \in \text{Obj}(C)$

**Theorem:** In a category $C$, two objects of $1^{st}$ order are always isomorphic.

**Proof 1.2.1** *Let $T, W$ be two universals objects of $1^{st}$ order so by definition :*

*$Mor_C(T, X) = \{\Gamma_X\}$ (singleton) $\forall X \in Obj\ C$*

*And $Mor_C(W, Y) = \{\zeta_Y\}$ (singleton) $\forall Y \in Obj\ C$*

*so*

. *If* $X \cong W$, $Mor(T, W) = \{f\}$

. *If* $Y = T$, *so* $Mor(W, T) = \{g\}$

$C$ *category, so:*

$\exists f \circ g \in Mor(W, W) = \{.\}$

$g \circ f \in Mor(T, T) = \{.\}$

*where* $Id_W \in Mor(W, W)$ *and* $Id_T \in Mor(T, T)$

*So block gives that* $W$ *and* $T$ $1^{st}$*order*

$\Rightarrow Mor_C(W, W) = Id_W and Mor_C(T, T) = Id_T$

$\Rightarrow f \circ g = Id_W$ *and* $g \circ f = Id_T \Rightarrow T \cong W$

## 1.3 Subcategories

**Definition:** We say that $C'$ is a subcategory of a category $C$ if and only if :

1. $C'$ is a category

2. obj($C'$) $\subseteq$ obj($C$)

3. $\text{Mor}_{C'} \subseteq \text{Mor}_C$

*i.e.* $if \quad X, Y \in \text{obj}C'(so \quad necessarly \quad X, Y \in \text{obj}C) then : \text{Mor}_{C'}(X, Y) \subseteq \text{Mor}_C(X, Y)$

4.The composition in $C'$ is the same as in $C$,

5. The identity morphism of any object $X \in C'$ is the same as in $C$

**Definition:** A subcategory $C'$ of a category $C$ is said to be full if:

$$\text{Mor}_{C'}(X, Y) = \text{Mor}_C(X, Y)$$

**Examples:**

1. **Set**$_f$ the category of finite sets:

. Obj set$_f$ : finite sets.

. Mor set$_f$ : applications between finite sets.

. "o" the Composition of application.

Thus, Set$_f$ is a category which according to the definition is a subcategory of Set.

**2.** *Similarly:* **Set**$_{\mathbf{d}}$

. Obj Set$_d$ = countable sets

. Mor $\text{Set}_d$ = maps between sets

. "$\circ$" is the composition of functions

$\text{Set}_d$ is a subcategory of Set

**Remark: $\mathbf{Set_f}$** and **$\mathbf{Set_d}$** are full subcategories.

## 1.4 Functors

Functors are a generalization of the notion of functions in set theory.
If we consider a category as a mathematical domain, a functor is then a tool that serves as a bridge from one area of mathematics to another.

**Examples:**

If there exists a functor $F : \text{Top} \to G$
that is if we can construct functors from Top to G, then we can transport a problem from Top (topology) to G (algebraic problems).
This concept is what is called Algebraic Topology.

•$\mathcal{G} \xrightarrow{F} A\mathcal{G}$: Functors from G to AG. In this case, the functor F can transport problems of algebras into the category of commutative algebras.

**Definition:** A *covariant functor* F from a category $\mathcal{C}$ to another category $\mathcal{C}'$ is defined as follows, and we write F : $\mathcal{C} \to \mathcal{C}'$
if the following conditions are satisfied:

1) $\text{Obj}(\mathcal{C}) \to \text{Obj}(\mathcal{C}')$

$X \to F(x)$

2) $F : \text{Mor}(\mathcal{C}) \to \text{Mor}(\mathcal{C}')$

that is, if $X, Y \in \text{Obj}(\mathcal{C})$ and $f \in \text{Mor}_{\mathcal{C}}(X, Y)$, then

$F(f) \in \text{Mor}_{\mathcal{C}'}(F(X), F(Y))$

that is $F : \text{Mor}_{\mathcal{C}}(X, Y) \to \text{Mor}_{\mathcal{C}'}(F(X), F(Y)), \forall X, Y \in \text{Obj}(\mathcal{C})$

$(X \xrightarrow{f} Y) \rightsquigarrow (F(x) \xrightarrow{F(f)} F(y))$

3) $F(f \circ g) = F(f) \circ F(g)$, for all $f, g \in \text{Mor}_{\mathcal{C}}$ where $f \circ g$ is exist.

4) $F(Id_X) = Id_{F(x)}$ for all $X \in \text{Obj}(C)$

**Example:**

Construct the functor:

$F : G \longrightarrow \mathcal{A}_G$

$\text{obj}(G) \xrightarrow{f} \text{obj}(\mathcal{A}_G)$

$G \mapsto F(G) = \frac{G}{[G,G]}$

$f : G \to L \in \text{Mor}_G(G, L) \mapsto F(f) : \left(\frac{G}{[G,G]} \longrightarrow \frac{L}{[L,L]}\right)$

Let $\bar{x} \in \frac{G}{[G,G]}$, then:

$F(f)(\bar{x}) = f(x'), \quad x' \in x$

Is $F$ covariant ?

answer :

• Let us show that $f$ is well defined.

a) Let $x', x'' \in \bar{x} \Longleftrightarrow \bar{x}' = \bar{x}''$

$\Longleftrightarrow x'Rx''(mod[G, G])$

$\Longleftrightarrow x'x^{-1''} \in [G, G]$

$\Longleftrightarrow \ f(x'x^{-1''}) \in f([G, G])$

(because G the category of groups) $\Rightarrow f(x')f(x'')^{-1} \in f([G, G]) = f(G')$

where $f : G \longrightarrow L$ so $f(G') \subseteq L'$

hence $f(x').f(x'')^{-1} \in L = [L, L] \Rightarrow f(x').f(x'')^{-1} \in [L, L]$

$\Rightarrow \overline{f(x')} = \overline{f(x'')}$

$\Rightarrow F(f(\bar{x}')) = F(f(\bar{x}''))$

• F is it a homomorphism ?

Let $\bar{x}, \bar{y} \in G/_[G, G]f$. Consider:

$$\begin{aligned}
F(f)(\bar{x}\bar{y}) &= F(f)(\overline{xy}) \\
&= \overline{f(xy)} \\
&= \overline{f(x).f(y)} \\
&= \overline{f(x)} \cdot \overline{f(y)} \\
&= F(f)(\bar{x}) \cdot F(f)(\bar{y})
\end{aligned}$$

Let $f$ and $g$ be two morphisms of $G$:

$G \xrightarrow{f} L, \quad L \xrightarrow{g} k \Longrightarrow g \circ f : L \to k$

Consider : $G/_{G'} \xrightarrow{F(f)} L/_{L'}, \quad and \quad L/_{L'} \xrightarrow{F(g)} k/_{k'} \Longrightarrow F(g) \circ F(f) : G/_{G'} \to k/_{k'}$

$F(g \circ f)(\bar{x}) =? \quad where \quad \bar{x} \in G/_{G'}$

$$\begin{aligned} F(g \circ f)(\bar{x}) &= \overline{g(f(x))} \\ &= F(g)(\overline{f(x)}) \\ &= F(g)(F(f)(\bar{x})) \\ &= (F(g) \circ F(f))(\bar{x}) \end{aligned}$$

Therefore: $\quad F(g \circ f) = F(g) \circ F(f)$

Let $G$ and $\mathrm{Id}_G : G \to G$ $F(\mathrm{Id}_G) = \mathrm{Id}_{F(G)}$

$G/_{G'} \xrightarrow{\mathrm{Id}_{G/_{G'}}} G/_{G'}$

Let $\bar{x} \in G/G'$, then: $F(\mathrm{Id}_G)(\bar{x}) = \mathrm{Id}_{\bar{G}}(x) = \bar{x} = \mathrm{Id}_{G/G'}(\bar{x})$

**Example:**

**Reminder:**

1. A topological space $X$ is said to be *locally compact* if every point $x \in X$ has a compact neighborhood.

2. A compact Hausdorff space $X^*$ is called the compactification of a topological space $X \subset X^*$if and only if

$X^* = X \cup \{*\}, \quad$ and $\quad \bar{X} = X^* \quad$ (X is dense in $X^*$)

It is called the point at infinity of $X$.

**Theorem:** A space $X \in \mathcal{E}$ admits a compactification if and only if it is Hausdorff and locally compact.

Let $X \in \mathcal{E}$, Hausdorff and locally compact.

Then it admits C.A $\Leftrightarrow \exists * \in X^* = X \cup \{*\}$, such that $X^*$ is compact and Hausdorff, and X is dense in $X^*$.

3. If $X, Y$ are two tobological space, Hausdorff, compact, and if $f \in Mor_{Top}(X, Y)$, then we have:

$X^* = X \cup \{*\} \quad and \quad Y^{\perp} = Y \cup \{\perp\}$

We can construct:

$$X^* \xrightarrow{\tilde{f}} Y^* \quad \text{where } \tilde{f}(x) = \begin{cases} f(x), & \text{if } x \neq * \\ *, & \text{if } x = * \end{cases}$$

**Problem:** Under what conditions is $f$ continuous if $f$ is continuous ?

**Class Proper continuous maps**.

**Definition:** A continuous map $f : X \to Y$ is said to be proper if the preimage of every compact subset of $Y$ is a compact subset of $X$.

**Theorem:** $\tilde{f} : X^* \to Y^*$ is continuous if and only if $f$ is proper!

Let $TopLoc$ = the category of locally compact Hausdorff spaces and proper continuous maps.

Let $Top_C \equiv$ the category of compact spaces and continuous maps.

($TopLoc$ and $Top_C$ are subcategories of $Top$)

Consider:

$F : TopLoc \to Top_C$

$\text{obj}(Top_{Loc}) \to \text{obj}(Top_C)$

$X \rightsquigarrow X^* = F(X)$

$\text{Mor}_{TopLoc}(X, Y) \mapsto \text{Mor}_{Top_C}(F(X), F(Y))$

$(X \xrightarrow{f} Y) \rightsquigarrow X^* \xrightarrow{F(f)=\tilde{f}} Y^*$ (the proper continuous map)

We have:

$$\begin{matrix} X \xrightarrow{f} Y \quad , \quad Y \xrightarrow{g} Z \\ X^* \xrightarrow{\tilde{f}} Y^* \quad and \quad Y^{\perp} \xrightarrow{\tilde{g}} Z^T \end{matrix} \quad \Longrightarrow F(g \circ f)(x) = g \tilde{\circ} f(x) = g \circ f(x) = F(g) \circ F(f)(x)$$

$F(\text{Id}_X)(x) = \text{Id}_{X^*}(x)$

**Theorem:** If $F$ is a covariant functor from a category $C$ to another category $C'$, then if $f$ is an isomorphism in $C$, we can affirm that the morphism $F(f)$ in $C'$ is also an isomorphism.

**Consequence:** Therefore, if $X$ and $Y$ are two isomorphic objects in the category $C$, then the image objects $F(X)$ and $F(Y)$ are also isomorphic in $C'$.

**Proof 1.4.1** *Let $F : C \to C'$ be a covariant functor and suppose that $f \in Mor_C(X, Y)$ is an isomorphism. Then there exists a morphism $g \in Mor_C(Y, X)$ such that: $f \circ g =$*

$Id_Y$ *and* $g \circ f = Id_X$

$$\text{Therefore: } \begin{cases} F(f \circ g) = F(Id_Y) \\ \text{and} \\ F(g \circ f) = F(Id_X) \end{cases}$$

$$\text{So: } \begin{cases} F(f) \circ F(g) = Id_{F(X)} \\ \text{and} \\ F(g) \circ F(f) = Id_{F(Y)} \end{cases} \Rightarrow F(f) \text{ and } F(g) \text{ are isomorphisms in } \mathbf{C}' \text{ inverse to each other.}$$

Note that this theorem (consequence) is of great importance in mathematics.
Indeed; $F$ is a functor from a category, for example
Top: the category of topological spaces and continuous maps, into a category $\mathcal{G}$, which is the category of groups and group homomorphisms.
One wishes to address in Top one of the fundamental problems, which lies in the topological classification of topological spaces: let $X$ and $V$ be two topological spaces (i.e.,two objects of Top), are they homeomorphic? Do they have the same topological type?

$$\text{For that:} \begin{cases} \textit{If X and Y have the same topological type, then they are homeomorphic.} \\ \textit{If X and Y are not homeomorphic, one of the ways is to show that all continuous maps from X in Y are not homeomorphisms.} \end{cases}$$

Here the functor comes into play: for that, we consider $F(X)$ and $F(Y)$ in $\mathcal{G}$, and if $F(X)$ and $F(Y)$ (groups) are no longer isomorphic, then $X$ and $Y$ in Top cannot be homeomorphic.
This principle is the basis and foundation of algebraic topology.

**Example:**

**Problem:**

Are $\mathbb{R}^1$ and $\mathbb{R}^2$ in **Top** of the same topological type? $\Longleftrightarrow$ Is the line homeomorphic to the plane? There exists a classical proof based on advanced analysis tools using the notion of functors to solve this problem.

To do so, we will introduce the following mathematical concepts for the constructions:

**1. Convex companionship:**

Let $X$ be a topological space, We introduce on $X$ a linear relation $\mathcal{R}$ defined as follows Two elements $x, y \in X$ are in relation modulo $\mathcal{R}$ (that is, $x \equiv y \mod \mathcal{R}$) if there exists a continuous function: $w : [0,1] \to X \quad \text{such that} \quad w(0) = x \quad \text{and} \quad w(1) = y$

The relation R is relation of equivalent

**2. Free Group:**

**Theorem:**

If X is a given set whose elements are of any nature, we can always construct a group on X the group generated by this set is called the **free group** generated by X denoted $F(X)$ moreover we have the following property:

*Any function* $f : X \to G$ (where G is a group) *can be uniquely extended to a homomorphism* $f_{[*]} : F(X) \to G$ *such that* $(f_*|_X = f)$.

**Remark:**

If $f : X \to Y$ is a morphism in **Top**, then f is a continuous map in X in space topoligic Y.

then: $\begin{cases} if \quad x = y(R) \\ in \quad X \end{cases} \Rightarrow \begin{cases} f(x) = f(y)(R) \\ in \quad Y \end{cases}$

**Further solution:**

$\mathbb{R}^1 \cong \mathbb{R}^2$ (since $\mathbb{R}^1$ is isomorphic by $\mathbb{R}^2$)

Construction: $G : \text{Top} \to G$

$G : \text{obj(Top)} \to \text{obj(G)}$

$X \mapsto F(X/_R) := G(X)$ free group generated by the connected components of $X$

$G : \text{Mor}_{\text{Top}}(X, Y) \to \text{Mor}_G(F(X), F(Y))$

$(X \xrightarrow{f} Y) \Rightarrow F(X|R) \xrightarrow{G(f)} F(Y|R)$

where $X \xrightarrow{f} Y$ is continuous

$X|R \xrightarrow{\tilde{f}} Y|R \subset F(Y|R)$

$F(X|R) \xrightarrow{\text{f unique hom.}} F(Y|R)$

Let $G(f) = F_x \in \text{Mor}(G)$

We observe that $G$ is a covariant functor from Top to $G$

Assume $\mathbb{R}^1 \cong \mathbb{R}^2$ in $\mathcal{T}\wr\surd'$ so: $\mathbb{R}^2 \setminus \{0\} \cong \mathbb{R}^2 \setminus \{0\}$)(homeomorphism)

Then: $G(\mathbb{R}^1 \setminus \{0\}) \cong G(\mathbb{R}^2 \setminus \{0\})$ (functor principle)

$F(\mathbb{R}^2 \setminus \{0\}, \mathbb{R}) = F(\mathbb{R}^2 \setminus \{0\}, \mathbb{R})$ in $G$ Impossible!

Because $\mathbb{R}^2 \setminus \{0\}$ has 1 connected component

and $\mathbb{R}^2 \setminus \{0\} = \mathbb{R}^2 \setminus \{0\}$ has 2 connected component.

Then $\mathbb{R}^2$and $\mathbb{R}^1$ are not homeomorphic **Contravariant Functor:**

Let $C, C'$ be two given categories. A **contravariant functor** is the data of a relation:

$F : C \to C'$ or

$F : \text{Obj}(C) \to \text{Obj}(C')$

$X \mapsto F(X)$

$F : \text{Mor}(C) \to \text{Mor}(C')$

such that:

Let $x, y \in \text{obj}\, C$ and $f \in \text{Mor}(x, y)$ then:

$F(f) \in \text{Mor}_C(F(y), F(x))$ Moreover:

a) If $f, g$ are two morphisms in $C$ where $fog$ then : $F(f \circ g) = F(g) \circ F(f)$

b) If $\text{Id}_x$ is the identity morphism of the object $x$ in $C$, then: $f(\text{Id}_x) = \text{Id}_{f(x)}$

**Example:**

1) Category of $K$-vector spaces and linear maps, denoted $\text{Vect}_K$.

a) Objects of $\text{Vect}_K$: vector spaces defined over the same commutative field $K$.

b) Morphisms in $\text{Vect}_K$ = linear maps.

c) Composition of linear maps.

d) $Id_X$ identity automorphism $\text{Id}_X$ if $X \in K$-vector spaces.

## 1.5 Various types of categories

### 1.5.1 Category of Categories

This category is denoted by **Cat**.

Objects: Collection of categories. Morphisms: Covariant functors (or contravariant functors).

We define the composition of functors as follows:

This composition is defined whenever we have two functors:

F G $F : C \to C'$, $\quad G \circ F : C \to C''$

where,

$G \circ F : \mathrm{Obj}(C) \xrightarrow{F} \mathrm{Obj}(C' \xrightarrow{G} \mathrm{Obj}(C')$

$X \mapsto F(X) \mapsto G(F(X))$

$G \circ F : \mathrm{Mor}_C(X, Y) \xrightarrow{F} \mathrm{Mor}_{C'}(F(X), F(Y)) \xrightarrow{G} \mathrm{Mor}_{C''}(G(F(X)), G(F(Y)))$

$f : X \xrightarrow{f} Y \mapsto F(f) : F(X) \to F(Y) \mapsto G(F(f)) : G(F(X)) \xrightarrow{F(f)} G(F(Y)) \quad G(F(x)) \xrightarrow{G(F(f))} G(F(y))$

Therefore: $G(F(f)) = (G \circ F)(f)$

We can easily verify associativity (standard composition): $H \circ (G \circ F) = (H \circ G) \circ F$

And since $C$ is a category, we define the identity morphism.

1) $\mathrm{Id}_C : C \to C$ defined by:

$\mathrm{Id}_C : \mathrm{Obj}(C) \longrightarrow \mathrm{Obj}(C)$

$x \longmapsto x$

2)$\mathrm{Id}_C : \mathrm{Mor}(C) \to \mathrm{Mor}(C)$

$\mathrm{Id}_C : \mathrm{Mor}(x, y) \longrightarrow \mathrm{Mor}(x, y)$

$(x \xrightarrow{f} y) \longmapsto (x \xrightarrow{f} y)$

Thus we have the category of categories and covariant functors (contravariant functors).

### 1.5.2 Category of differential groups and their homomorphisms

We denote the category of differential groups by $\mathcal{G}d$.

**Definition (1):**

A *differential group* is a pair $(G, d)$ where:

• $G$ is an abelian group

• $d : G \to G$ is an endomorphism (i.e., $d$ is a homomorphism from $G$ to $G$) such that $d^2 = d \circ d = 0$, where $0 : G \to G$ is the zero map, $x \mapsto 0$.

**Definition (2):**

A *homomorphism between differential groups* $(G, d)$ and $(G', d')$ is the data of a group homomorphism $f : G \to G'$ that respects the differentials $d$ and $d'$, i.e., the following diagram must be commutative:

$$\begin{array}{ccc} G & \xrightarrow{f} & G' \\ {\scriptstyle d}\downarrow & & \downarrow{\scriptstyle d'} \\ G & \xrightarrow{f} & G' \end{array} \quad \Leftrightarrow \quad f \circ d = d' \circ f$$

**Example:**

Let $\mathcal{G}_d$ be defined by:

▸ $\mathrm{Obj}(\mathcal{G}_d)$: the collection of differential groups.

▸ $\mathrm{Hom}_{\mathcal{G}_d}$: the collection of homomorphisms of differential groups.

▸ Composition: composition of group homomorphisms.

▸ The identity morphism of a $(G, d)$ is $\mathrm{id}_G$: the identity map of $G$.

**Show that $\mathcal{G}_d$ is a category.**

" $\circ$ " definit:

Let $(G, d)$, $(G', d')$, and $(G'', d'')$ be differential groups.

$$\begin{array}{ccc} G & \xrightarrow{f} & G' \\ {\scriptstyle d}\downarrow & & \downarrow{\scriptstyle d'} \\ G & \xrightarrow{f} & G' \end{array} \quad \Leftrightarrow \quad f \circ d = d' \circ f$$

$$\begin{array}{ccc} G' & \xrightarrow{f} & G'' \\ \Big\downarrow{\scriptstyle d} & & \Big\downarrow{\scriptstyle d''} \\ G' & \xrightarrow{f} & G'' \end{array} \quad \Leftrightarrow \quad g \circ d' = d'' \circ g$$

$$\begin{array}{ccc} G & \xrightarrow{f} & G' \\ {\scriptstyle d}\Big\downarrow & & \Big\downarrow{\scriptstyle d'} \\ G & \xrightarrow{f} & G' \end{array} \qquad \begin{array}{ccc} G' & \xrightarrow{g} & G'' \\ {\scriptstyle d'}\Big\downarrow & & \Big\downarrow{\scriptstyle d''} \\ G' & \xrightarrow{g} & G'' \end{array}$$

Then we get:

$$d'' \circ g \circ f = g \circ d' \circ f = g \circ f \circ d$$

**Associativity:**

$$\begin{array}{ccccccc} G & \xrightarrow{f} & G' & \xrightarrow{g} & G'' & \xrightarrow{h} & G''' \\ {\scriptstyle d}\Big\downarrow & & \Big\downarrow{\scriptstyle d'} & & \Big\downarrow{\scriptstyle d''} & & \Big\downarrow{\scriptstyle d'''} \\ G & \xrightarrow{f} & G' & \xrightarrow{g} & G'' & \xrightarrow{h} & G''' \end{array}$$

We have:

$$(h \circ g) \circ f = h \circ (g \circ f) \quad \text{by definition}$$

**Identity:**

$$\begin{array}{ccc} G & \xrightarrow{\mathrm{id}_G} & G \\ {\scriptstyle d}\Big\downarrow & & \Big\downarrow{\scriptstyle d} \\ G & \xrightarrow{\mathrm{id}_G} & G \end{array}$$

Then:

$$\mathrm{id}_G \circ d = d = d \circ \mathrm{id}_G$$

Therefore, $\mathcal{G}_d$ is a category.

### 1.5.3 Category of graded groups and homomorphisms of any degree

*Note:* $\mathcal{G}_g$

**Definition 1:** A graded group $\mathcal{G}$ is defined as a collection of abelian groups indexed by integers, i.e., $\mathcal{G} = \{G_k\}_{k\in\mathbb{Z}}$.

**Definition 2:**

A homomorphism of degree $p$ (integer) between two graded groups $\{G_k\}_{k\in\mathbb{Z}}$ and $\{G'_k\}_{k\in\mathbb{Z}}$ is given by a family of homomorphisms indexed over integers:

$f_k : G_k \to G'_{k+p}, \quad \forall k \in \mathbb{Z}$ If $\left(G_k \xrightarrow{f_k} G'_{k+p}\right)_{k\in\mathbb{Z}}$, we say that $f$ is a homomorphism of degree 0.

**Definition of Gg:**

►Objects of Gg: Collection of graded groups.

►Morphisms of Gg: Homomorphisms of any degree.

►Composition: Composition of homomorphisms of degree ⇒ defined application composition.

►Identity morphism of a graded group $\{G_k\}_{k\in\mathbb{Z}}$ is:

$\{Id_{G_k}\}_{k\in\mathbb{Z}} = Id_{\{G_k\}_{k\in\mathbb{Z}}}$

$Id_{G_k} : G_k \longrightarrow G_k$ (identity map)

**Proposition:**

Gg is a category called the **category of graded groups and homomorphisms of arbitrary degree**, where:( $G_k \xrightarrow{f_k} G'_{k+p}, \quad G'_k \xrightarrow{g_k} G''_{k+q} \Rightarrow (g \circ f)_k : G_k \to G''_{k+p+q}$)

### 1.5.4 Category of differential graded groups of the Same degree $d$ and homomorphisms of arbitrary degree, noted DG

**Definition 1:**

A differential graded group of degree $d$ is defined as a family of graded differential groups over the integers $\{G_k, D_k\}_{k\in\mathbb{Z}}$ such that:

$d_k : G_k \to G_{k+d}$

$G_k \xrightarrow{d_k} G_{k+d} \xrightarrow{d_{k+d}} G_{k+2d} \quad and \quad d_{k+d} \circ d_k = 0$

**Definition 2:** A homomorphism of degree $p$ between two differential graded groups of the same degree $d$

$(G_k, d_k)_{k\in\mathbb{Z}}$ and $(G'_k, d'_k)_{k\in\mathbb{Z}}$

is defined as a family of homomorphisms (indexed by the integers) that respects the differentials.

i.e Here, we must have for all $k \in \mathbb{Z}$ the following commutative diagrams:

$$\begin{array}{ccc} G_k & \xrightarrow{f_k} & G'_k \\ {\scriptstyle d_k}\downarrow & & \downarrow{\scriptstyle d'_k} \\ G_{k+d} & \xrightarrow[f_{k+d}]{} & G'_{(k+d)+d} \end{array} \qquad \Leftrightarrow \qquad d'_{k+p} \circ f_k = f_{k+d} \circ d_k$$

**Definition of the category DG:**

► Obj DG: A collection of differential graded groups having the same degree.

► Mor DG: A collection of homomorphisms of differential graded groups of arbitrary degree.

► Composition: composition of group homomorphisms

► The identity morphism of $\{(G_k, d_k)\}_{k\in\mathbb{Z}}$:

$\text{Id}\,\{(G_k, d_k)\}_{k\in\mathbb{Z}} = \{\text{Id}_{G_k}\}_{k\in\mathbb{Z}}$

**Proposition:**

DG is a category ≡ the category of differential graded groups and homomorphisms of graded groups of arbitrary degree.

### 1.5.5 Category of Complex Chains

**Definition:** A *complex chain* is defined as any graded differential group of degree $(-1)$.

Thus, if $\{G_p, d_p\}_{p\in\mathbb{Z}}$ is a chain complex, it can be represented by the following chain:

$\cdots \xrightarrow{d_{k+1}} G_p \xrightarrow{d_k} G_{k-1} \xrightarrow{d_{k-1}} \cdots$

$d_k \circ d_{k+1} = 0 \quad \forall k \in \mathbb{Z}$

**Definition:** A family of homomorphisms $\{f_k\}_{k\in\mathbb{Z}}$ is called a *morphism of chain complexes* between:

$\{G_k, d_k\}_{k\in\mathbb{Z}}$ and $\{G'_k, d'_k\}_{k\in\mathbb{Z}}$.

1) The morphism is of degree zero.

2) The following diagram commutes:

$$\begin{array}{ccccccc} \cdots & \xrightarrow{d_{k+1}} & G_p & \xrightarrow{d_k} & G_{k-1} & \xrightarrow{d_{k-1}} & \cdots \\ & & \downarrow f_k & & \downarrow f_{k-1} & & \\ \cdots & \xrightarrow{d'_{k+1}} & G'_p & \xrightarrow{d'_k} & G'_{k-1} & \xrightarrow{d'_{k-1}} & \cdots \end{array}$$

**Proposition:**

Complex chains and homomorphisms of complex chains is a category that is a subcategory of DG.

### 1.5.6 Category of pointed topological spaces and maps between pointed topological spaces

**Definition:**

A pointed topological space is a pair $(X, x_0)$ where X is a topological space and $x_0 \in X$ is a fixed point.

**Definition:**

A continuous map between two pointed topological spaces $(X, x_0)$ and $(Y, y_0)$ is a continuous function $f : X \to Y$ such that:

$f(x_0) = y_0$ That is:

$$f : (X, x_0) \to (Y, y_0) \iff \begin{cases} f : X \to Y \text{ is continuous} \\ f(x_0) = y_0 \end{cases}$$

**Definition :** $\widehat{\dot{\text{T}}}$op:

- Obj $\dot{\widehat{\text{Top}}}$: collection of pointed topological spaces.
- Mor $\dot{\widehat{\text{Top}}}$: collection of continuous maps between pointed topological spaces.

**Composition:** the composition of the morphisms.
**Identity morphism** of $(X, x_0)$:

$$\text{Id}_{(X,x_0)} : (X, x_0) \longrightarrow (X, x_0)$$

is defined by:

$$\text{Id}_{(X,x_0)} = \text{Id}_X$$

**Proposition**
$\dot{\widehat{\textbf{Top}}}$ is a category called the *category of pointed topological spaces and continuous maps between them*.

### 1.5.7 Category of direct spectrum

Let $A$ be a set equipped with a partial order relation, i.e., a relation denoted by $\leq$ that is reflexive, anti-symmetric, and transitive.
**Definition:**
We say that $A$ is directed to the right if and only if for all $\alpha, \beta \in A$, there exists $\gamma \in A$ such that $\alpha \leq \gamma$ and $\beta \leq \gamma$.
**Definition:** A family of the type $(X_\alpha, \zeta_{\alpha\beta}, A)_{\alpha\leq\beta}$ *and* $\zeta_{\alpha\beta} \in \text{Mor}_{\text{Set}}(X_\alpha, X_\beta)$ $\forall \alpha, \beta \in R$ is said to be direct spectrum if it satisfies the following conditions:
1) $\zeta_{\alpha\alpha} : X_\alpha \to X_\alpha$ verify $\zeta_{\alpha\alpha} = \text{Id}_\alpha : X_\alpha \to X_\alpha$, i.e., identity morphism.
2) For $\alpha \leq \beta \leq \gamma$, we have:
$X_\alpha \xrightarrow{\zeta_{\alpha\beta}} X_\beta \xrightarrow{\zeta_\beta^\gamma} X_\alpha$
and we must verify that $\zeta_\alpha\gamma = \zeta_\beta\gamma \circ \zeta_\alpha\beta$

Suppose that: $((X_\alpha, \zeta_\alpha\beta, A)_{\alpha\leq\beta}, \quad \alpha, \beta \in \mathbb{R}$ and $(Y_\alpha, Y_{\alpha\beta}, A)_{\alpha\leq\beta}, \quad \alpha, \beta \in \mathbb{R}$

Given two direct systems indexed over the same directed set $A$.

**Definition:**

A family of morphisms $(f_\alpha)_{\alpha\in A}$ indexed over $A$ is called a morphism of the direct spectrum $(X_\alpha, Y_{\alpha\beta}, A)_{\alpha\leq\beta}, \quad \alpha, \beta \in \mathbb{R}$ into the direct spectrum $(Y_\alpha, \psi_{\alpha\beta}, A)_{\alpha\leq\beta}, \quad \alpha, \beta \in \mathbb{R}$ if:

1) $f_\alpha : X_\alpha \to Y_\alpha, \quad \forall\alpha \in A$

2) For all $\alpha, \beta \in A$ such that $\alpha \leq \beta$, the following diagram commutes:

$X_\alpha \quad \xrightarrow{f_\alpha} \quad Y_\alpha$

$\downarrow \zeta_{\alpha\beta} \qquad\qquad \downarrow \psi_{\alpha\beta}$ $(\alpha \leq \beta \quad \forall\alpha, \beta \in A)$ is commutative

$X_\beta \quad \xrightarrow{f_\beta} \quad Y_\beta$

**Proposition:**

The family of direct spectrums fors a category $\text{Spect}_A(\textbf{Set})$ is defined by:

• A collection of direct systems indexed over $A$ of the form:

$(X_\alpha, \zeta_{\alpha\beta}, A)_{\alpha\leq\beta,\, \alpha,\beta\in A}$

where $X_\alpha \in \text{Obj}(\textbf{Set})$ and $\zeta_{\alpha\beta} \in \text{Mor}(\textbf{Set})$.

• A collection of morphisms between direct systems constitutes a category is called the category of direct systems of sets indexed over $A$ and of morphisms between direct systems.

• Let $(f_\alpha)_{\alpha\in A}$ be a morphism from the direct system $(X_\alpha, \zeta_{\alpha\beta}, A) \to (Y_\alpha, \zeta_{\alpha\beta}, A)$, and $(g_\alpha)_{\alpha\in A}$ a morphism from $(Y_\alpha, \zeta_{\alpha\beta}, A) \to (Z_\alpha, \zeta_{\alpha\beta}, A)$.

Two morphisms of direct systems over $A$ are defined with their composition as follows: $X_\alpha \xrightarrow{f_\alpha} Y_\alpha \xrightarrow{g_\alpha} Z_\alpha$

We define the composition $(g_\alpha)_{\alpha\in A} \circ (f_\alpha)_{\alpha\in A}$ as: $(g_\alpha \circ f_\alpha)_{\alpha\in A}$

**Proof 1.5.1** *Let us prove that the composition thus defined is a morphism of the direct system over A:*

$(X_\alpha, \zeta_{\alpha\beta}, A) \to (Z_\alpha, \psi_{\alpha\beta}, A)$

*i.e., we must verify the commutativity of the following diagram:*

$$\begin{array}{ccc} X_\alpha & \xrightarrow{g_\alpha \circ f_\alpha} & Z_\alpha \\ \downarrow \zeta_{\alpha\beta} & & \downarrow \psi_{\alpha\beta} \quad \text{for } \alpha, \beta \in A,\ \alpha \leq \beta \\ X_\beta & \xrightarrow{g_\beta \circ f_\beta} & Z_\beta \end{array}$$

*Indeed, for $\alpha \leq \beta$ in A, we have the following commutative diagrams:*

$$\begin{array}{ccc} X_\alpha & \xrightarrow{f_\alpha} & Y_\alpha \\ {\scriptstyle\varphi_{\alpha\beta}}\downarrow & & \downarrow{\scriptstyle\zeta_{\alpha\beta}} \\ X_\beta & \xrightarrow[f_\beta]{} & Y_\beta \end{array} \qquad (1)$$

$$\begin{array}{ccc} Y_\alpha & \xrightarrow{g_\alpha} & Z_\alpha \\ {\scriptstyle\zeta_{\alpha\beta}}\downarrow & & \downarrow{\scriptstyle\psi_{\alpha\beta}} \\ Y_\beta & \xrightarrow[g_\beta]{} & Z_\beta \end{array} \qquad (2)$$

*From (1) and (2), we deduce the commutativity of:*

$$\begin{array}{ccccc} X_\alpha & \xrightarrow{f_\alpha} & Y_\alpha & \xrightarrow{g_\alpha} & Z_\alpha \\ {\scriptstyle\zeta_{\alpha\beta}}\downarrow & & \downarrow{\scriptstyle\psi_{\alpha\beta}} & & \downarrow{\scriptstyle\varphi_{\alpha\beta}} \\ X_\beta & \xrightarrow[f_\beta]{} & Y_\beta & \xrightarrow[g_\beta]{} & Z_\beta \end{array} \qquad (3)$$

*From the proposition on the commutativity of pasted diagrams, we deduce that the diagram (4) is commutative.*

***Consequence:***

*The composition above is correctly defined.*

*2) Existence of identity morphisms.*

*Let $(X_\alpha, \varphi_{\alpha\beta}, A)$ be a direct system over A Let us consider*

$(\mathrm{Id}_\alpha)_{\alpha\in A} = (\mathrm{Id}_{X_\alpha})_{\alpha\in A}$

*where*

$\mathrm{Id}_{X_\alpha} = \mathrm{Id}_\alpha : X_\alpha \to X_\alpha.$

*Then we have, for $\alpha \leq \beta$ the following diagrams:*

$$\begin{array}{ccc} X_\alpha & \xrightarrow{Id_\alpha} & X_\alpha \\ {\scriptstyle \varphi_{\alpha\beta}}\downarrow & & \downarrow{\scriptstyle \varphi_{\alpha\beta}} \\ X_\beta & \xrightarrow[Id_\beta]{} & X_\beta \end{array}$$

*From the commutativity of diagram (*), we deduce that*

$(Id_\alpha)_{\alpha\in A} = (Id_{X_\alpha})_{\alpha\in A}$ *is a morphism of the system*

• $(X_\alpha, \zeta_{\alpha\beta}, A) \longrightarrow (X_\alpha, \zeta_{\alpha\beta}, A)$

*and we consider:* $(Id_\alpha)_{\alpha\in A} \circ (f_\alpha)_{\alpha\in A}$ :

$(A_\alpha, \Psi_{\alpha\beta}, A) \longrightarrow (X_\alpha, f_{\alpha\beta}, A)$

$\searrow \qquad\qquad \swarrow Id_\alpha$

$(X_\alpha, \varphi_{\alpha\beta})$

*Moreover, by construction:*

$(Id_\alpha)_{\alpha\in A} \circ (\beta_\alpha)_{\alpha\in A} = (Id_{X_\alpha} \circ \chi_\alpha)_{\alpha\in A}$

*That is, if* $(g_\alpha)_{\alpha\in A}$

$(X_\alpha, \varphi_{\alpha\beta}, A) \longrightarrow (\beta_\alpha, \psi_{\alpha\beta}, A)$

*Then we can define*

$(X_\alpha, \varphi_{\alpha\beta}, A) \xrightarrow{Id_\alpha} (X_\alpha, \varphi_{\alpha\beta}, A)$

$\searrow \qquad\qquad \swarrow g_\alpha$

$(B_\alpha, \psi_{\alpha\beta}, A)$

*so:*

$(g_\alpha)_{\alpha\in A} \circ (Id_\alpha)_{\alpha\in A} = (g_\alpha \circ Id_\alpha)_{\alpha\in A}$

# CHAPTER 2

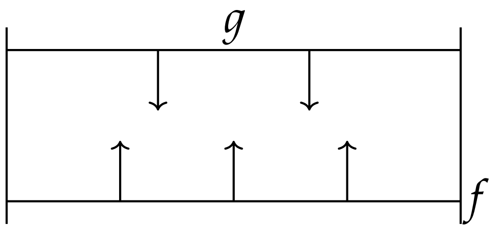
# HOMOTOPIC FUNCTOR

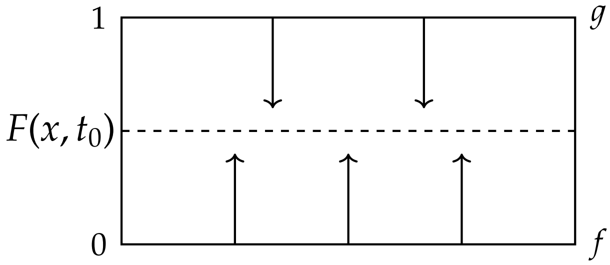
Homotopy studies the problem of continuous deformations of maps. It appears to classify topological spaces more weakly than homotopy.

## 2.1 Homotopy of continuous maps

We will study the deformations of continuous applications. let us consider Top the category of topological spaces and continuous applications: let be X and Y two objects of Top.

**Definition 2.1.1** *Two morphisms $f$, $g \in MorTop(X,Y)$ are said to be homotopic if and only if we can continuously deform $f$ to obtain $g$ or deform $g$ to obtain $f$.*



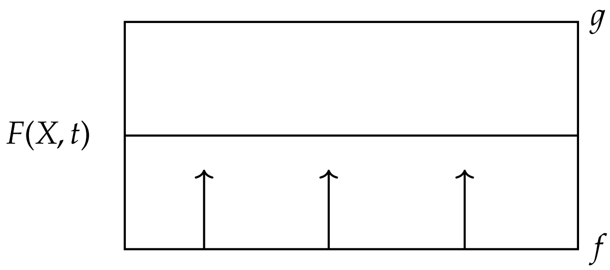

**Definition 2.1.2** *We say that a morphism $f \in Mor_{Top}(X,Y)$ is homotopic to a morphism $g \in Mor_{Top}(X,Y)$ and we denote $f \sim g$ (ie:f homotopic to g) if and only if it exists a continuous application:*

$$F : X \times [0,1] \to Y \text{ with } \begin{cases} F(x,0) = f(x) \\ F(x,1) = g(x) \end{cases} \quad \forall x \in X$$

**Theorem 2.1.1** *the relation "homotope"=homotopy relation,is an equivalence relation in $Mor_{Top}(X,Y)$.*

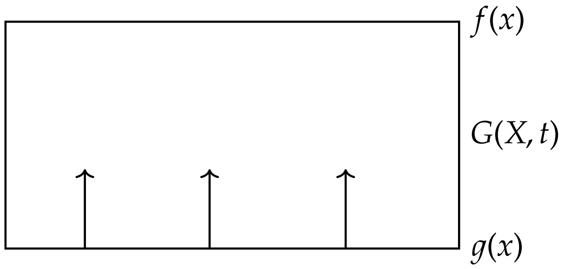
**Proof 2.1.1** *Reflixivity: Let $X, Y \in obj_{Top}$ be and we cosider the homotopy relation on $Mor_{Top}(X,Y)$ for $f \in Mor_{Top}(X,Y)$ consider:*
*We have $F : X \times [0,1] \to Y$ given by: $F(X,t) = f(x)\ \forall t \in [0,1]$*
*F is continuous and $F(X,0) = F(X,1) = f(x)$ so $f \sim f$*
*symmetry: Let $f, g \in Mor_{Top}(X,Y)$ suppose that:*

$$f \sim g \iff \exists \quad F : X \times [0,1] \to Y \text{ continuous with :} \begin{cases} F(x,0) = f(x) \\ F(x,1) = g(x) \end{cases}$$




*Consider:G:$X \times [0,1] \to Y$ where $G(x,t) = F(x, 1-t)$ for $x \in X, t \in [0,1]$*

*We have that G is contunuos and in addition* $\begin{cases} G(x,0) = F(x,1) = g(x) \\ G(x,1) = F(x,0) = f(x) \end{cases}$



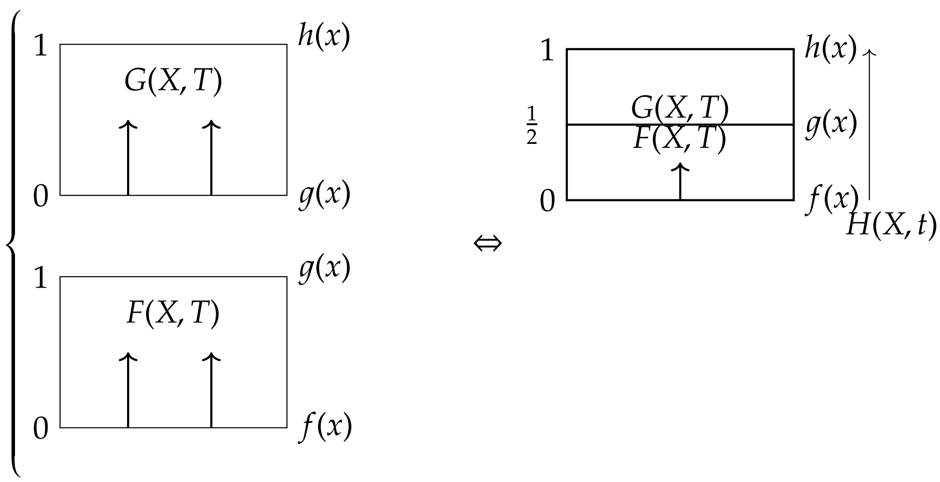

*By the previous results we conclude that* $g \sim f$ *and consequently the relation of homotopy is symmetric*

<u>*transitivity:*</u> *Let* $f, g, h \in Mor_Top(X, Y)$ *suppose the following :*

$f \sim g \iff \exists F : X \times [0,1] \to Y$ *continuous with:* $\begin{cases} F(X,0) = f(x) \\ F(X,1) = g(x) \end{cases}$

$g \sim h \iff \exists G : X \times [0,1] \to Y$ *continuous with :* $\begin{cases} G(X,0) = g(x) \\ G(X,1) = h(x) \end{cases}$

*Consider the diagram and the relation H :*




$H : X \times [0,1] \to Y$ *where* $H(X,t) = \begin{cases} F(X,2t) & 0 \leq t \leq \frac{1}{2} \\ F(X,2t) & \frac{1}{2} \leq t \leq 1 \end{cases}$

*F and G are continuous on the intervals* $[0, \frac{1}{2}]$*and* $[\frac{1}{2}, 1]$*respectively, more over if* $t = \frac{1}{2}$

$F(X, 1) = G(X, 0) = g(x)$ *consequently H is continuous and we have :*

$$\begin{cases} H(X, 0) = F(X, 0) = f(x) \\ H(X, 1) = G(X, 1) = h(x) \end{cases}$$

*Then* $f \sim g$ *From this result we deduce that the homotopic relation is transitive*

**Conclusion 2.1.1** . *The homotopy is equivalence relation* $Mor_{Top}(X, Y)$ *is .*

**Consequence 2.1.1** . *For any pair of objects X,Y of Top the homotopy relation do a partition the set of morphisms of source X and goal Y into classes called homotopy classes. Thus the homotopy relation on* $Mor_top(X, Y)$ *classifies the morphisms which can be defined continuously on each other.*

Notation

• If $F \in Mor_{Top}(X, Y)$ the homotopy classes of f are: $\begin{cases} [f] \\ \quad\quad or \\ \bar{f} \end{cases}$

• The set $Mor_{Top}(X, Y)/_{\sim}$ which is the set of homotopic classes is denoted by [X,Y]

Thus $[X, Y]$ =$\{[f]\ f \in Mor_{Top}(X, Y)? \quad f \sim g, \quad g \in Mor_{Top}\}$.

### 2.1.1 construction of a category related to homotopy

Let Top be the homotopy relation defined above and let us consider the following classes :

1) Class of objects of Top =topological space .

2) If X,Y are two isomorphic topological spaces, a morphism from X to Y will be an element of [X, Y] in other words we consider as morphism the homotopy classes of the morphisms of Top .

**Lemma 2.1.1** *The definition of the 2 classes above determines a category noted [Top] and associated with the homotopy relation.*

*In order to prove that [Top] is a category, we must clearly show a morphism composition law for each object of [Top]*

**Remark 2.1.1** *If X and Y are two topological spaces, a morphism from X to Y in the class of morphisms of [Top] is a set of morphisms that are all homotopy of* $Mor_{Top}$
*We mean that:* $Mor_{Top}(X, Y) = [X, Y] = Mor_{Top}(X, Y)/\sim$
*Or equivalently: A morphism of [Top] is a set of contunuos applications homotopic between them!!, morover:*
*Let X,Y,Z be three topological spaces:*
$[f] : X \to Y$ *and* $[g] : Y \to Z$ *two morphisms, consider the relation*
$[g]o[f] : X \to Y \to Z$ *or equivently* $[g]o[f] : X \to Z$ *where* $[g]o[f] = [g'of']$ *where* $g' \in [g], f' \in [f]$
• *Let us show that this composition is correctly defined that it is independent of the choices of morphisms in the homotopic classes.*
*For this purpose we consider* $y', y'' \in [g]$ *and* $f', f'' \in [f] \Longleftrightarrow f' \sim f''$ *and* $g' \sim g''$ *so:*

$F : X \times [0, 1] \to Y$ *contunuos with :* $\begin{cases} F(X, 0) = f'(x) \\ F(X, 1) = f''(x) \end{cases} \quad \forall x \in X$

*and* $\exists G : Y \times [0, 1] \to Z$ *continuous*
*consider :* $H(X, t) : X \times [0, 1] \to Y \times [0, 1] \to Z$ *such that* $H(X, t) = G(F(x, t), t)$ $\forall x \in X \forall t \in [0, 1]$.
*H is correctly given in fact if* $(x, t) \in X \times [0, 1] \Longrightarrow (F(x, t), t) \in Y \in [0, 1]$
*so:* $G(F(x, t), t) \in Z$ *this one is contunuos and:*

$$\begin{cases} H(x, 0) = G(F(x, 0), 0) = G'(f'(x), 0) \\ \qquad\qquad\qquad\qquad\quad = g'(f'(x)) \\ \qquad\qquad\qquad\qquad\quad = (g'of')(x) \\ H(x, 1) = G(F(x, 1), 1) = G(f''(x), 1) \\ \qquad\qquad\qquad\qquad\quad = g''(f''(x)) \\ \qquad\qquad\qquad\qquad\quad = (g''of'')(x) \end{cases}$$

**Conclusion 2.1.2** . $[g'of'] = [g''of''] = [gof]$

*Therefore to simplify writing and understanding we take :* $[g]o[f] = [gof]$

$$\bullet \textit{ It is associative} \Longrightarrow \begin{cases} ([g]o[f])oh = [gof]oh = [(gof)oh] \\ \qquad\qquad\qquad\qquad\qquad = [go(foh)] \\ \qquad\qquad\qquad\qquad\qquad = [g]o[foh] \\ \qquad\qquad\qquad\qquad\qquad = [g]o([f]o[h]) \end{cases}$$

• *Existence of the identity morphism :*

*Let X be a topology space consider :*

$[Id_X] : X \to X$ *then* $[Id_X]o[f] = [Id_X of] = [f]$ *and* $[g]o[Id_X] = [goId_X] = [g]$.

*until now we have compared the topology spaces and we know that:*

• *two objects X , Y (e.T) of the category Top are isomorphic*

$$\Longleftrightarrow \begin{cases} \exists f : X \to Y \quad \textit{a contunuos application} \\ \exists g : Y \to X \quad \textit{a contunuos application} \end{cases} \textit{with} \begin{cases} fog = Id_Y \\ gof = Id_X \end{cases}$$

**Definition 2.1.3** *The topological spaces X and Y are said to be of the same homotopic type or homotopic or homotopically equivalent if and only if :*

$$\begin{cases} f : X \to Y \textit{ contunuos}(morphismofTop) \\ g : Y \to X \textit{ contunuos} \end{cases} \textit{with} \begin{cases} fog \sim Id_Y \\ gof \sim Id_X \end{cases}$$

**Remark 2.1.2** *.It is said to be a weak classification compared to the topological classification because : homeomorphi* $\Leftrightarrow$ *homotopic*

$$\bullet \textit{ If} \begin{cases} fog \sim Id_Y \\ gof \sim Id_X \end{cases} \textit{so} \begin{cases} [fog] = [Id_X] \\ [gof] = [Id_X] \end{cases} \textit{so} \begin{cases} [f]o[g] = [Id_Y] \\ [g]o[f] = [Id_X] \end{cases}$$

*We then say that f is the homotopic inverse of g where g is the homotopic inverse of f , we also say that f and g are homotopically invertible.*

**Proposition 2.1.1** *.The homotopic relation in the class of topological spaces is an equivalence relation which thus classifies topological spaces by homotopy.*

*1) Reflixivity : Let X be a topological space then* $Id_X : X \to X$ *such that* $Id_X \circ Id_X = Id_X$ *then* $X \sim X$

*2) symmetric : Let X and Y be two homotopic topological spaces $X \sim Y$ i.e.*

$$\begin{cases} \exists f : X \to Y \quad contunuos \\ \exists g : Y \to X \quad contunuos \end{cases} and \begin{cases} fog \sim Id_Y \\ gof \sim Id_X \end{cases}$$

$$\Longleftrightarrow \begin{cases} \exists g : Y \to X \\ \exists f : X \to Y \end{cases} and \begin{cases} gof = Id_X \\ fog = Id_Y \end{cases} \Rightarrow Y \sim X.$$

*3)transitivity : Let X ,Y , Z be three objects of the category Top and suppose that $X \sim Y$ and $Y \sim Z$ that is X and Y homotopic and Y and Z homotopic therefore*

$$\begin{cases} \exists f : X \to Y \quad contunuos \\ \exists g : Y \to X \quad contunuos \end{cases} and \begin{cases} fog \sim Id_Y \\ gof \sim Id_X \end{cases}$$

$$\begin{cases} \exists \alpha : Y \to z \quad contunuos \\ \exists \beta Z \to Y \quad contunuos \end{cases} and \begin{cases} \alpha o \beta = Id_Z \\ \beta o \alpha = Id_Y \end{cases}$$

*Then :*

$$\begin{cases} [\alpha]o[f] : X \to Y \to Z \quad \Longrightarrow \exists [\alpha]o[f] : X \to Z \\ [g]o[\beta] : Z \to Y \to X \quad \Longrightarrow \exists [g]o[\beta] : Z \to X \end{cases}$$

*Letś consider their composition :*

$(\alpha o f)o(go\beta) : Z \to Z.$

$(\alpha o f)o(go\beta) = \alpha o(fog)o\beta \sim \alpha \sim Id_Y o\beta = \alpha o\beta \sim Id_Z$

*On the other hand :*

$(go\beta)o(\alpha o f) : X \to X.$

$(go\beta)o(\alpha o f) = go(\beta o\alpha)of \sim goId_Y of = gof \sim Id_X$

*then $X \sim Z$*

**consequence 2.1.1** . *This relation is equivalence relation*

## 2.2 Contractible space

**Definition 2.2.1** *A topological space X is said to be contractile it has the same homotopy type as a point if and only if X is homotopic to a singleton.*

**Remark 2.2.1** *If X is a set not reduced to a point, topologically it cannot be homeomorphic that is of the same topological type as a point however they can be homotopic*

**Proposition 2.2.1** *A topological space X is contractible if and only if the application $Id_X$ is homotopic to a constant application .*

1)Necessary condition : Suppose that X is contractible so by definition there exists

$$\begin{cases} f : X \to x_0 \\ g : x_0 \to X \end{cases} \text{such as} \begin{cases} f \circ g \sim Id_{x_0} \\ g \circ f \sim Id_X \end{cases}$$

then :

$$\begin{cases} g \circ f : X \xrightarrow{f} x_0 \xrightarrow{g} X \\ X \to x_0 \to g(x_0) \end{cases}$$

So $gof$ is a constant homotopic application to $Id_X$ that is :
$IdX \cong gof = \varepsilon_{g(x_0)}$ where $\varepsilon_{g(x_0)} : X \to X$ for $\varepsilon_{g(x_0)} = g(x_0)\ \forall x \in X$

2) sufficient condition : Suppose that $Id_X : X \to X$ homotopic to a constant application $\varepsilon_{x_0} : X \to X$, consider the application :

$$\begin{cases} X \xrightarrow{f} x_0 \\ x \to x_0 \end{cases} \text{and} \begin{cases} x_0 \xrightarrow{g} X \\ x_0 \to g(x_0) \end{cases}$$

So $fog(x) = x_0 = Id_{x_0}(x)$ that is $fog \backsimeq Id_{x_0}$
and $gof(x) = g(x_0) = \varepsilon_{x_0}(x)$ that is $gof = \varepsilon_{x_0} \backsimeq Id_X$ .
$X$ and $x_0$ are the same homotopic type $\Rightarrow$ X is contractile.

**Proposition 2.2.2** *A topological space X is contractile if and only if every continuous application $f : X \to Y$ is homotopic to a constant application .*

1) Necessary condition : Suppose that X is contractile and $f : X \to Y$ is an arbitrarity given continuous application , from the previous proposition we know that there exists a constant application $\varepsilon_{x_0} : X \to X$ which is homotopic to $Id_x : X \to X$ we thus have the following homotopics :

$$\begin{cases} f \sim f \\ and \\ Id_X \sim \varepsilon_{X_0} \end{cases} \tag{2.1}$$

So from the contrability of the composition of the application and the homotopy relation we deduced from (1,1) : $foId_X \sim fo\varepsilon x_0$ from which $f \sim \varepsilon_{f(x_0)}$ where $\varepsilon_{f(x_0)} : X \to Y$ is the constant application where $\varepsilon_{f(x_0)} = f(x_0)$ .
2) sufficient condition : Suppose that is contractible if and only if continuous application is homotopic$f : X \to Y$ , in particular $Id_X : X \to X$ it will be homotopic to a constant application so X is contractible .

**Proposition 2.2.3** *A topological space X is contractible if and only if any continuous application $f : Y \to X$ is homotopic to a constant application .*

1) Necessary condition :
Same as previous

$$\begin{cases} f \sim f \\ Id_X \sim \varepsilon'_{X_0} \end{cases} \text{so} \begin{cases} Id_X of \sim \varepsilon'_{X_0} of \\ f \sim \varepsilon'_{X_0} \end{cases}$$

2) sufficient condition : this is true for all $f : Y \to X$ in particular for $Id_x : X \to X$.

**Consequence 2.2.1** *A topological space X is contractible if and only if $f, g : Y \to X$ two continuous application are homotopic .*

1) Necessary condition : We have the following homotopics :

$$\begin{cases} f \sim f \\ Id_X \sim \varepsilon_{X_0} \end{cases} \Rightarrow \mathrm{f} \cong \varepsilon'_{X_0}$$

and $\Rightarrow f \sim g$

$$\begin{cases} g \sim g \\ Id_X \sim \varepsilon_{X_0} \end{cases} \Rightarrow g \cong \varepsilon'_{X_0}$$

2) sufficient condition : If for all pairs of continuous applications $f, g : Y \to X$ are homotopic in particular , if one of them is a constant application , thus from the preceding proposition we deduce that X is contractible .

### 2.2.1 Relative Homotopy

Let X and Y be two topological spaces, then

**Definition 2.2.2** *We say that two morphisms $f, g \in Mor_{Top}(X, Y)$ are homotopic relatively to $A \subseteq X$ if and only if :*
*1) $f(a) = g(a)\ \forall a \in A$*
*2) There exists : $F : X \times [0, 1] \to Y$ continuous with :*

$$\begin{cases} F(x, 0) = f(x) \\ F(x, 1) = g(x) \\ F(a, t) = f(a) = g(a) \forall (a, t) \in A \times [0, 1] \end{cases}$$

**Proposition 2.2.4** *The relative homotopy relation is an equivalence relation .*
*Let $X, Y \in obj_{Top}$ and $A \subset X$ consider the relative homotopy modulo A (compared to A).*
*1) Reflixitivity : Let $f : X \to Y$ then obviously $f(a) = f(a) \forall a \in A$ and $F : X \times [0, 1] \to Y$*

*where* $F(X,t) = f(x)$ *shows that f is homotopic to f modulo A.*

*2) symmetric : Let* $f, g : X \to Y$ *where f is homotopic to g relative to A therefore :*

*(1)* $f(a) = g(a),\ \forall a \in A$

*(2)* $\exists F : X \times [0,1] \to Y$ *with :* $\begin{cases} F(X,0) = f(x) \\ F(X,1) = g(x) \\ F(a,t) = f(a),\ \forall (a,t) \in A \times [0,1] \end{cases}$

*(3) consider* $G : X \times [0,1] \to Y$ *where* $G(x,t) = F(x, 1-t)$ *so :*

*G is continuous and :* $\begin{cases} G(X,0) = F(x,1) = g(x) \\ G(X,1) = F(x,0) = f(x) \\ G(a,t) = F(a, 1-t) = f(a) = g(a) \end{cases}$

$\Longleftrightarrow$ *g is relatively homotopic to f modulo A.*

*3) transitivity : Let* $f, g, h \in Mot_{Top}(X,Y)$ *with* $f(a) = g(a) = h(a),\ \forall a \in A$ *and suppose that :*

*(1) f is relatively homotopic to g modulo A.*

*(2) g is relatively homotopic to h modulo A.*

*There exists so :*

*(1)* $F : X \times [0,1] \to Y$ *continuous* $\begin{cases} F(X,0) = f(x) \\ F(X,1) = g(x) \\ F(a,t) = f(a),\ \forall (a,t) \in A \times [0,1] \end{cases}$

*(2)*$G : X \times [0,1] \to Y$ *continuous* $\begin{cases} G(X,0) = g(x) \\ G(X,1) = h(x) \\ G(a,t) = g(a),\ \forall (a,t) \in A \times [0,1] \end{cases}$

*(3) consider :* $H : X \times [0,1] \to Y$ *continuous*

*H(X,t) =* $\begin{cases} f(X, 2t) & 0 \le t \le \frac{1}{2} \\ G(X, 2t-1) & \frac{1}{2} \le t \le 1 \end{cases}$

*H is continuous*

$$\begin{cases} H(X,0) = F(X,1) = f(x) \\ H(X,1) = G(X,1) = h(x) \\ H(a,t) = f(a,2t) \qquad 0 \leq t \leq \frac{1}{2} \\ \qquad\quad = G(a,2t-1) \qquad \frac{1}{2} \leq t \leq 1 \end{cases} \Rightarrow f(a) = g(a) = h(a)$$

*Thus* $H(a,t) = f(a) = g(a) = h(a) \forall (a,t) \in A \times [0,1]$
*So f is relatively homotopic to h modulo A .*

**Consequence 2.2.2** *The relative homotopy relation (modulo A) partitions* $Mor_{Too}(X,Y)$ *into equivalence classes called relative homotopic classes thus we have :* $g \in [f]$ *(homotopic relative modulo A)* $\Longleftrightarrow$ *f relatively homotopic to g (modulo A) and we denote :* $f \sim g$ *(mod A).*

### 2.2.2 Retract and Deformations

## Restriction and application

Let $f : X \to Y$ be a given continuous application and $X_0 \subset X$ .

**Definition 2.2.3** *We called the restriction of f to* $X_0$*, denoted by:*

$$f|_{X_0}(x_0) = f(x_0), \quad \forall x_0 \in X_0$$

*In this case, f is called the extension of* $f|_{X_0}$ *on X. The inverse problem, which is the problem of the existence of such an extension, is one of the major problems in mathematics. This problem does not have a global solution up to now, meaning there is no general answer for the existence of extensions in all posed problems.*

*The problem of extension existence is stated as follows:*

*Let* $f : X_0 \to Y$ *be a continuous function where* $X_0 \subset X$*. Does there exist a continuous* *function* $\bar{f} : X \to Y$ *such that:*

$$\bar{f}|_{X_0} = f$$

*The problem of existence reduces to the existence of the following diagram:*

$$\begin{array}{ccc} X_0 & \xrightarrow{\tilde{i}} & X \\ {\scriptstyle f}\downarrow & \swarrow {\scriptstyle \bar{f}?} & \\ Y & & \end{array}$$

$\exists ? \bar{f} : X \to Y | \bar{f} \circ i = f$ *such that* $i$ *is the canonical injection.*
*An important class where the problem of the existence of a continuous extension has a solution (answer) is the class of retracts.*

## Characterization of the class of retracts in Top

**Definition 2.2.4** *A subset* $X_0$ *of a topological space* $X$ *is called a retract of* $X$ *if and only if there exists a continuous map*

$$T : X \to X_0 \quad \text{such that} \quad T \circ i = Id_{X_0}$$

*where* $i_{X_0} : X_0 \to X$ *is the canonical injection* $(X_0 \subset X)$.

**Definition 2.2.5** *(equivalent definition)*
$X_0$ *is a retract of* $X$ *if and only if there exists a commutative diagram in Top such that*

$$\begin{array}{ccc} X_0 & \xrightarrow{i} & X \\ & {\scriptstyle Id_{X_0}} \searrow & \downarrow {\scriptstyle T} \\ & & X_0 \end{array}$$

*We observe that the diagram of the retract resembles that of the existence of an extension.*

**theorem 2.2.1** *A sub space* $X_0$ *of a topological space* $X$ *is a retract of* $X$ *if and only if* *every continuous map $f : X_0 \to X$ admits a continuous extension to X.*

*1)Necessary condition : Suppose $X_0$ is a retract of X and let $f : X_0 \to X$ be a continuous map.*

*We know there exists a continuous map $T : X \to X_0$ with $T \circ \hat{i} = Id_{X_0}$.*

*Consider then:*

$$f \circ T : X \to X_0 \to X$$

*This is a continuous map and it's an extension of f because for $X_0 \to x_0$ we have:*

$$\hat{f}(X_0) = f \circ T(X_0) = f \circ T \circ \hat{i}(X_0) = f \circ Id_{X_0}(X_0) = f(X_0)$$

*2)Sufficient condition : Suppose every continuous map admits an extension. Consider $Id_{X_0} : X_0 \to X_0 \subset X$, which is continuous and therefore admits an extension $T : X \to X_0$. From the extension existence diagram, we deduce $T \circ i = Id_{X_0}$, so $X_0$ is a retract of X*

**Remark 2.2.2** *If $X_0$ is a retract of X, then the map $T : X \to X_0$ satisfying $T \circ i = Id_{X_0}$ is called a retraction .*

**Definition 2.2.6** *We say that a topological sub space $X_0$ of a topological space X is a* ***slightly retract*** *of X if there exists a continuous map $T : X \to X_0$ with $T \circ i \sim Id_{X_0}$ ($T \circ i$ homotopic to $Id_{X_0}$).*

**Remark 2.2.3** *Every retract is strongly a retract family - equality implies homotopy.*

*The notion below is a generalization of the retract concept where equality becomes homotopy. We can obtain other mathematical concepts by generalizing these notions.*

*We complete the generalization by using relative homotopy.*

**Definition 2.2.7** *A topological sub space $X_0$ of a topological space X is called a* ***strongly***

***deformation retract*** *of X if there exists a retraction* $T : X \to X_0$ *such that* $i \circ T \sim Id_X$ *and* $T \circ i \sim Id_{X_0}$ *(mod* $X_0$*). The property* $T \circ i \sim Id_{X_0}$ *(mod* $X_0$*) has a meaning:*

*(Recall:* $f \sim g$ *(mod A))* $\begin{cases} f(a) = g(a) \\ \ldots \end{cases}$ *This makes sense because:* $T \circ i(x_0) = Id_{X_0}(x_0), \forall x_0 \in X_0$

## Homotopy in $\widehat{\dot{\text{Top}}}$

Let$\widehat{\dot{\text{Top}}}$ be the category of pointed topological spaces and continuous maps.

**Definition 2.2.8** *Two morphisms* $f, g \in Mor_{\dot{\mathbf{Top}}}[(X, x_0), (Y, y_0)]$ *are said to be homotopic if and only if there exists a continuous map:*

$$F : (X \times [0,1], \{x_0\} \times [0,1]) \longrightarrow (Y, y_0) \quad \text{such that:} \begin{cases} F(x, 0) = f(x) \\ F(x, 1) = g(x) \\ F(x_0, t) = y_0 \quad \forall t \in [0, 1] \end{cases}$$

*The last equality means that* $F(x_0, [0, 1]) = y_0$.

**Proposition 2.2.5** *The homotopy relation in* $Mor_{\widehat{\dot{Top}}}[(X, x_0), (Y, y_0)]$ *is an equivalence relation.*

*It partitions* $Mor_{\dot{\mathbf{Top}}}[(X, x_0), (Y, y_0)]$ *into homotopy classes, denoted by:*

$$Mor_{\widehat{\dot{Top}}}[(X, x_0), (Y, y_0)]/\sim \quad [(X, x_0), (Y, y_0)]$$

*We can think of inverting* $\mathcal{T}$ *and i to consider:* $i \circ \mathcal{T}$

*that is,* $\exists \mathcal{T} : X \longrightarrow X_0$ *continuous such that:* $\hat{i} \circ \mathcal{T} \sim Id_X$ *(1) that is,* $\exists \mathcal{T} : X \longrightarrow X_0$ *continuous such that:* $\hat{i} \circ \mathcal{T} \sim Id_X$*... (1)*

*if we want to generalize from (1) that is there exists $\mathcal{T} : X \longrightarrow X_0$ continuous with $\hat{i} \circ \mathcal{T} = Id_X$.*

*This case is trivial because the equality: $i \circ \mathcal{T} = Id_X \Rightarrow i$ bijection $\Rightarrow i = Id_X \Rightarrow X_0 = X$.*

*This generalization does not lead to an important particular class of topological objects.*

*However, the second mathematical consideration is the class of deformations.*

**Definition 2.2.9** *We say that a topological subspace $X_0$ of a topological space $X$ is a deformation of $X$ if there exists a continuous map $d : X \longrightarrow X_0$ such that $i \circ d \sim Id_X$.*

## Generalization

**Definition 2.2.10** *A topological subspace $X_0$ of a topological space $X$ is called a deformation retract of $X$ if there exists a retraction $\mathcal{T}$ such that:*

$$\hat{i} \circ \mathcal{T} \sim Id_X \Longleftrightarrow \exists\, \mathcal{T} : X \longrightarrow X_0 \begin{cases} \mathcal{T} \circ i = Id_{X_0} \\ i \circ \mathcal{T} \sim Id_X \end{cases}$$

**Definition 2.2.11** *A topological subspace $X_0$ of a topological space $X$ is called a weak deformation retract if and only if there exists a continuous map $d : X \longrightarrow X_0$ such that:*

$$\begin{cases} d \circ i \sim Id_{X_0} \\ and \\ i \circ \underline{d} \sim Id_X \end{cases} .$$

# CHAPTER 3

# HOMOTOPY OF PATHS

Let X be a topological space and $x_0, y_0 \in X$

**Definition 3.0.1** *We call a path of X with origin $x_0$ and extremity $y_0$ any continuos application :* $\begin{cases} \mu : [0,1] \to X \\ \mu(0) = x_0 \quad \text{and} \quad \mu(1) = y_0 \end{cases}$

*Two paths $\mu$ and $\lambda$ of X which have the same origin and the same extremity are said to be homotopic if and only if*

$$\exists F : [0,1] \times [0,1] \to X \text{ continuos : } \begin{cases} F(t,0) &= \mu(t) \\ F(t,1) &= \lambda(t) \\ F(0,\tau) &= x_0 \\ F(1,\tau) &= y_0 \end{cases} \quad \forall t \in [0,1], \quad \forall \tau \in [0,1]$$

**Remark 3.0.1** *We can redefined the homotopy of paths using relative homotopy .*
*Indeed, two paths $\mu$ and $\lambda$ with origin $x_0$ and extremity $y_0$ are homotopic $\lambda \iff \mu \sim \lambda$ (relative [0,1])*

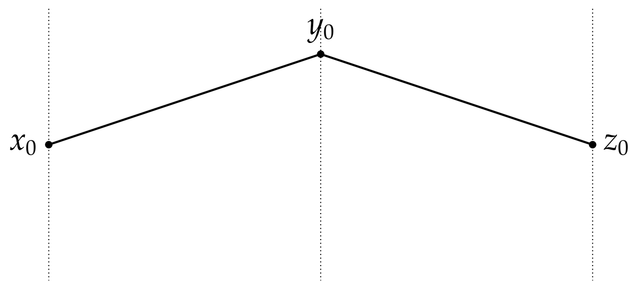
$$\text{In fact the continuous : } \begin{cases} F(0,\tau) & = x_0 \Longleftrightarrow F(0,\tau) & = \mu(0) = \lambda(0) = x_0 \\ F(1,\tau) & = y_0 \Longleftrightarrow F(1,\tau) & = \mu(1) = \lambda(1) = y_0 \end{cases}$$

## 3.1 Properties of homotopy of paths

### 3.1.1 Operation on paths

#### Composition of paths (concatenation of paths)

Let $\mu$ be a path from X with origin $x_0$ and extrimity $y_0$ and $\lambda$ is a path from X with origin $y_0$ and extrimity $z_0 \Longleftrightarrow \begin{cases} \mu & : [0,1] \to X \text{ continuous } \mu(0) = x_0, \mu(1) = y_0 \\ \lambda & : [0,1] \to X \text{ continuous } \lambda(0) = y_0, \lambda(1) = z_0 \end{cases}$




We call concatenated paths $\mu$ and $\lambda$ the path noted :

$\mu.\lambda : [0,1] \to X$ with : $(\mu.\lambda)(t)$: $\begin{cases} \mu(2t) & 0 \le t \le \frac{1}{2} \\ \lambda(2t-1) & \frac{1}{2} \le t \le 1 \end{cases}$

The composed $(\mu.\lambda)$ has the origin $x_0 = (\mu.\lambda)(0) = \mu(0)$ and extrimity $z_0 = (\mu.\lambda)(1) = \lambda(1)$

**Remark 3.1.1** *Due to concatenation we have introduced a composition law in the set of paths and this composition is not always defined this law operates in the set* $\bigcup_{x,y} C(X, \{x, y\})$ *which is the set of all paths of X.*

## Properties of concatenation

**Proposition 3.1.1** *Let $\mu \in C(X, \{x_0, y_0\}), \lambda \in C(X, \{y_0, z_0\}), \varphi \in C(X, \{z_0, l_0\})$ then we have the paths $(\mu.\lambda).\varphi \in C(X, \{x_0, l_0\})$ and $\mu.(\lambda.\varphi`) \in C(X, \{x_0, l_0\})$ these two paths are not equal however:* $(\mu.\lambda).\varphi \cong \mu.(\lambda.\varphi`)$

**Remark 3.1.2** *In $\mathbb{R}$ all intervals of type [a,b] are homotopic to [0,1] the passage applications is f:[a,b]→[0,1] such that $f(x) = \frac{x-a}{b-a}$*
*Let us show that $(\mu.\lambda).\varphi$ and $\mu.(\lambda.\varphi`)$ are two homotopic paths .*

Let $t \in [0,1]$ then : $(\mu.\lambda).\varphi(t) : \begin{cases} (\mu.\lambda)(2t) & 0 \le t \le \frac{1}{2} \\ \varphi(2t-1) & \frac{1}{2} \le t \le 1 \end{cases}$

$$: \begin{cases} \mu(2 \times 2t) & 0 \le t \le \frac{1}{4} \\ \lambda(2 \times 2t - 1) & \frac{1}{4} \le t \le \frac{1}{2} \\ \varphi(2t-1) & \frac{1}{2} \le t \le 1 \end{cases}$$

$$: \begin{cases} \mu(4t) & 0 \le t \le \frac{1}{4} \\ \lambda(4t - 1) & \frac{1}{4} \le t \le \frac{1}{2} \\ \varphi(2t-1) & \frac{1}{2} \le t \le 1 \end{cases}$$

$$\mu.(\lambda).\varphi)(t) : \begin{cases} \mu(2t) & 0 \le t \le \frac{1}{2} \\ (\lambda).\varphi)(2t-1) & \frac{1}{2} \le t \le 1 \end{cases}$$

$$: \begin{cases} \mu(2t) & 0 \le t \le \frac{1}{2} \\ \lambda(2 \times (2t-1)) & \frac{1}{2} \le t \le \frac{3}{4} \\ \varphi(2 \times (2t-1) - 1) & \frac{3}{4} \le t \le 1 \end{cases}$$

$$: \begin{cases} \mu(2t) & 0 \le t \le \frac{1}{4} \\ \lambda(4t - 1) & \frac{1}{2} \le t \le \frac{3}{4} \\ \varphi(4t-3) & \frac{3}{4} \le t \le 1 \end{cases}$$

So : $$(\mu.\lambda).\varphi(t) \neq \mu.(\lambda.\varphi)(t)$$

Let's prove that $\mu.(\lambda.\varphi)(t) \sim (\mu.\lambda).\varphi(t)$

Consider the square of homotopy

$$F : [0,1]\times[0,1]\to X \text{ where : } F(t,\tau) = \begin{cases} \mu(\frac{4t}{1+\tau}) & 0 \leq t \leq \frac{1+\tau}{4} \\ \lambda(4t-\tau-1) & \frac{1+\tau}{4} \leq t \leq \frac{2+\tau}{4} \\ \varphi(\frac{4t-\tau-2}{2-\tau}) & \frac{1+\tau}{4} \leq t \leq 1 \end{cases}$$

**Proposition 3.1.2** *Let* $\mu \in C(X,\{x,y\})$ *and* $\varepsilon_x \in C(X,\{x,x\})$*and* $\varepsilon_y \in C(X,\{y,y\})$ *Then,*

$$\begin{cases} \mu.\varepsilon_y \neq \mu \\ \varepsilon_x.\mu \neq \mu \end{cases} \quad but \begin{cases} \mu.\varepsilon_y \sim \mu \\ \varepsilon_x.\mu \sim \mu \end{cases}$$

*Let's calculate the paths :* $(\mu.\varepsilon_y)(t) = \begin{cases} \mu(2t) & 0 \leq t \leq \frac{1}{2} \\ \varepsilon_y(2t-1) & \frac{1}{2} \leq t \leq 1 \end{cases}$

$$= \begin{cases} \mu(2t) & 0 \leq t \leq \frac{1}{2} \\ y & \frac{1}{2} \leq t \leq 1 \end{cases}$$

*Let us show that they are homotopic :*

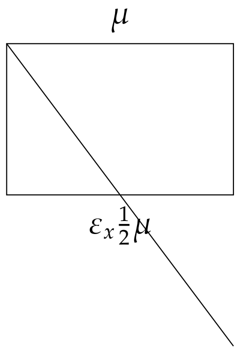
$$F(t,\tau) = \begin{cases} \mu(\frac{2t}{1+\tau}) & 0 \leq t \leq \frac{1+\tau}{2} \\ \varepsilon_y(2t-1-\tau) & \frac{1+\tau}{2} \leq t \leq 1 \end{cases}$$

$$= \begin{cases} \mu(\frac{2t}{1+\tau}) & 0 \leq t \leq \frac{1+\tau}{2} \\ y & \frac{1+\tau}{2} \leq t \leq 1 \end{cases}$$

$$F(t,0) = \begin{cases} \mu(\frac{2t}{1+0}) & 0 \leq t \leq \frac{1}{2} \\ y & \frac{1}{2} \leq t \leq 1 \end{cases} \quad =(\mu.\varepsilon_y)(t)$$

$$F(t,1) = \begin{cases} \mu(\frac{2t}{2}) & 0 \leq t \leq \frac{1+1}{2} \\ y & \frac{1+1}{2} \leq t \leq 1 \end{cases} = \begin{cases} \mu(t) & 0 \leq t \leq 1 \\ y & t = 1 \end{cases}$$

*As before :* $\varepsilon_x.\mu \sim \mu$

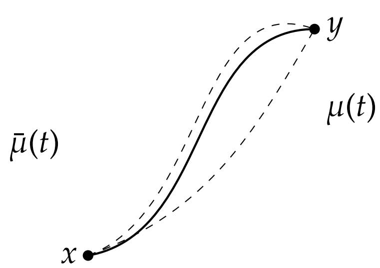



**Definition 3.1.1** *Let $\mu \in\subset (X, \{x, y\})$ be the inverse path of $\mu$ the path $\bar{\mu} \in\subset (X, \{x, y\})$ where :*

$$\bar{\mu} : [0, 1] \to X \text{ with } \bar{\mu}(t) = \mu(1 - t)$$

**Proposition 3.1.3** *If $\mu \in C(X, \{x, y\})$ then we have the paths :*

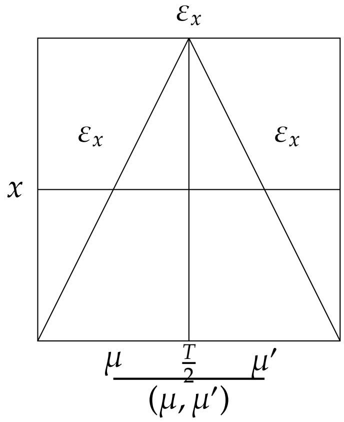
$$\begin{cases} \mu.\bar{\mu} \neq \varepsilon_x \\ \bar{\mu}.\mu \neq \varepsilon\varepsilon_y \end{cases} \quad but \begin{cases} \mu.\bar{\mu} \sim \varepsilon_x \\ \bar{\mu}.\mu \sim \varepsilon\varepsilon_y \end{cases}$$




*Let us calculate for $t \in [0, 1]$ :* $(\mu.\bar{\mu})(t) = \begin{cases} \mu(2t) & 0 \leq t \leq \frac{1}{2} \\ \bar{\mu}(2t - 1) & \frac{1}{2} \leq t \leq 1 \end{cases} \neq \varepsilon_x$

*Let us show that $\mu.\bar{\mu} \sim \varepsilon_x$ for this we consider :*

*Then consider the application :* $F(t, \tau) = \begin{cases} x & 0 \leq t \leq \frac{T}{2} \\ \mu(2t - T) & \frac{T}{2} \leq t \leq \frac{1}{2} \\ \mu(2 - 2t - T) & \frac{1}{2} \leq t \leq \frac{2-T}{2} \\ x & \frac{2-T}{2} \leq t \leq 1 \end{cases}$




*Since for the variable is (1-t) , we must find it's interval when* $t \in [\frac{1}{2}, \frac{2-\tau}{2}]$

$$\begin{cases} \mu\left(2, \frac{t}{2} - T\right) = \mu(0) = x \\ \mu\left(2, \frac{1}{2} - t\right) = \mu(1, t) \\ \mu\left(2 - 2, \frac{t}{2} - t\right) = \mu(1, t) \\ \mu\left(2 - 2, t - t\right) = \mu(0) = x \end{cases} \quad \textit{similarly, verify that :} \begin{cases} F(t, 0) = (\mu, \bar{\mu})(t) \\ F(t, 1) = \varepsilon_x(t) = x \end{cases}$$

*Then,* $\bar{\mu}.\mu \sim \varepsilon_y$.

**Remark 3.1.3** *By analyzing the properties of concatenation and homotopy of paths , we observe that operation of path compostion (concatenation) satisfies properties similar to internal operations such as associativity , neutrality (∃) , and symmetry . For this reason , we say that* $\bigcup_{x,y \in X} C(X, \{x, y\})$ *equipped with the operation of concatenation is a groupoid, that is : the operation of concatenation does not always exist for 2 paths , however if it is a path , we have the properties :*

*1)* $(\mu \cdot \lambda)\varphi \sim \mu \cdot (\lambda \cdot \varphi)$

*2)* $\mu \cdot \varepsilon_y \sim \mu$

*3)* $\mu \cdot \bar{\mu} \sim \varepsilon_x$

## 3.2 Fundamental group , Poincaré group

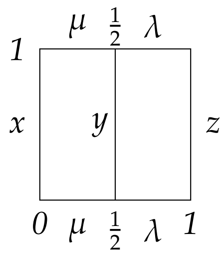
### 3.2.1 Homotopy group of order 1

**Proposition 3.2.1** *Let $\mu.\bar{\mu} \in C(X, \{x, y\})$ and $\lambda.\lambda' \in C(X, \{y, z\})$ then if :*

$$\begin{cases} \mu \sim \mu' \\ and \qquad\qquad then \quad \mu.\lambda \sim \mu'.\lambda' \\ \lambda \sim \lambda' \end{cases}$$

$$\textit{And then : } \mu \sim \mu' \Longleftrightarrow F : [0,1] \times [0,1] \to X \quad continuous \begin{cases} F(t,0) = \mu(t) \\ F(t,1) = \mu'(t) \\ F(0,\tau) = x \\ F(1,\tau) = y \end{cases}$$

$$\lambda \sim \lambda' \iff \exists G : [0,1] \times [0,1] \to X \quad contunuos \begin{cases} G(t,0) = \lambda(t) \\ G(t,1) = \lambda'(t) \\ G(0,\tau) = y \\ G(1,\tau) = z \end{cases}$$




$$\textit{Consider : } H : [0,1] \times [0,1] \to X \textit{ continuous : } H(t,\tau) = \begin{cases} F(2t,\tau) & 0 \le t \le \frac{1}{2} \\ G(2t-1,\tau) & \frac{1}{2} \le t \le 1 \end{cases}$$

*We check that for $t = \frac{1}{2}$ we have y over more,*

$$H(t,\tau) = \begin{cases} F(2t,0) & 0 \le t \le \frac{1}{2} \\ G(2t-1,0) & \frac{1}{2} \le t \le 1 \end{cases}$$

$$= \begin{cases} \mu(2t) & 0 \leq t \leq \frac{1}{2} \\ \lambda(2t-1) & \frac{1}{2} \leq t \leq 1 \end{cases}$$

*similarly :* $H(t,1) = (\mu' \cdot \lambda')(t)$

*Then consider the set L(X,x) which is* $C(X,\{x,y\})$ *where* $x = y$ *so L(X,x) is the set of paths of X of origin confused with the extremity which is x an element of L(X,x) is called a loop based at x*

**Definition 3.2.1** *We call a loop based at x of X any continuous mapping:*

$$\delta : [0,1] \to X \quad with \quad \delta(0) = \delta(1) = x$$

**Remark 3.2.1** *In* $L(X,x)$*, we can always perform the concatenation operation morover the concatenation of two elements of* $L(X,x)$ *is an element of* $L(X,x)$*.*

**consequence 3.2.1** $L(X,x)$ *equipped with the concatenation law is magma i.e Concatenation is an internal composition law in* $L(X,x)$*. It satisfies the properties of concatenation of paths considered in* $L(X,x)$*. The set of homotopy classes of loops* $L(X,x)/\sim$ *the set homotopy classes of the loops then if* $\delta, \mu \in L(X,x)$ *then their homotopy class* $[\delta].[\mu] = [\delta.\mu] \in L(X,x)/\sim$ *is a well-defined operation in* $L(X,x)/\sim$ *because* $\begin{cases} \delta' \sim \delta \\ \mu' \sim \mu \end{cases}$
$\Longrightarrow \delta'.\mu' \sim \delta.\mu \Rightarrow [\delta'.\mu'] = [\delta.\mu]$ *So* $(L(X,x)/\sim, .)$ *is a magma therefore :*
*1)* $(\delta.\mu).\varphi \sim \delta.(\mu.\delta) \Longleftrightarrow ([\delta].[\mu]).([\varphi]) = [\delta]([\mu].[\varphi])$
*2)* $\delta.\varepsilon_x \sim \varepsilon.\delta \Longleftrightarrow [\delta][\varepsilon_x] = [\delta] = [\varepsilon_x].[\delta] \Longleftrightarrow [\varepsilon_x]$
*3)* $\delta.\delta' \sim \varepsilon_x \sim \delta.\delta' \Longleftrightarrow [\delta].[\delta'] = [\varepsilon_x] = [\delta'].[\delta] \Longleftrightarrow [\delta'] = [\delta]^{-1}$

**conclusion 3.2.1** $(L(X,x)/\sim, .)$ *is a group we note it :* $\pi_1(X,x)$ *called fundamental group of X or poincaré group of X base or homotopy group of order 1 of X based at x*

**Remark 3.2.2** *In* $L(X.x_o)$ *we can consider* $\delta.\mu$ *,* $\mu.\delta$ *in general* $\delta.\mu \nsim \mu.\delta$ *then:* $\pi_1(X.x)$ *in general is not abelian Homotopy was introduced by Poincaré for the classification of*

*surfaces (two-dimensional varieties). Due to the homotopy group, we have a completed classification of two-dimensional varieties, which led in* 1904 *to Poincaré posing the problem that any compact three-dimensional variety without boundary which has a trivial fundamental group has the same type of homotopy as* $S^3$*?*
*This problem is called the "Poincaré conjecture" and its resolution in dimension* 3 *was awarded* 1 *million of Dolars in* 2003*.*
*Gregory Perelman solved this problem and received the FILDS medal in 2006, at the International Conference of the Stek Institute of Lebingiad in Madrid*

### 3.2.2 Continuous application and fundamental groups

Let $f$ continuous application such that $f \in Mor_{\widehat{\text{Top}}}((X, x_0), (Y, y_0))$ then if $\delta \in L(X, x_0)$ we have the following composition:
$f \circ \delta : [0, 1] \xrightarrow{\delta} X \xrightarrow{f} Y \Rightarrow f \circ \delta : [0, 1] \to Y$ is an application morover,

$$\begin{cases} f \circ \delta(0) = f(x_0) = y_0 \\ f \circ \delta(1) = f(x_0) = y_0 \end{cases} \quad \text{then } f \circ \delta \in L(Y, y_0) \quad \text{hence}$$

$$f \circ \delta \text{ is a Y loop based } y_0$$

It is also observed from the studied prepositions of homotopy that if $\delta, \delta' \in L(X, x_0)/\delta \sim \delta'$ hence we have the following result: $f \circ \delta \sim f \circ \delta'$

**Proposition 3.2.2** *Every continuous application* $f : (X, x_0) \to (Y, y_0)$ *induces an application* $\pi_1(f) : \pi(X, x_0) \to \pi(Y, y_0)$ *given by :* $\pi_1(f)([\delta]) = [f \circ \delta], \quad \forall[\delta] \in \pi_1(X, x_0)$ *moreover* $\pi_1(f)$ *is a group hmeomorphism .*

***Proof* 3.2.1** *Let us prove that* $\pi_1(f)$ *is a group hmeomorphism:*
*Indeed if* $[\delta], [\mu] \in \pi_1(X, x_0)$ *then* $\pi_1(f)([\delta].[\mu]) =_p i_1(f)([\delta.\mu]) = [f \circ (\delta.\mu)]$

$$\textit{Let's evaluate : } f \circ (\delta \cdot \mu) : [0, 1] \to Y \quad (\delta \cdot \mu)(t) = \begin{cases} \delta(2t) & si\ 0 \leq t \leq \frac{1}{2} \\ \mu(2t - 1) & si\ \frac{1}{2} \leq t \leq 1 \end{cases}$$

$$\Rightarrow f \circ (\delta \cdot \mu) = \begin{cases} f(\delta(2t)) & si\ 0 \leq t \leq \frac{1}{2} \\ f(\mu(2t-1)) & si\ \frac{1}{2} \leq t \leq 1 \end{cases}$$

*Thus* $f \circ (\delta \cdot \mu) = (f \circ \delta) \cdot (f \circ \mu) = \begin{cases} f \circ \delta(2t) & si\ 0 \leq t \leq \frac{1}{2} \\ f \circ \mu(2t-1) & si\ \frac{1}{2} \leq t \leq 1 \end{cases}$ *Thus :* $\Pi_1(f)([\delta] \cdot [\mu]) = [(f \circ \delta) \cdot (f \circ \mu)] = [f \circ \delta] \cdot [f \circ \mu] = \Pi_1(f)([\delta]) \cdot \Pi_1(f)([\mu])$

**Proposition 3.2.3** *The following assertions, are verified:*

1. $f : (X, x_0) \to (Y, y_0), \quad g : (Y, y_0) \to (Z, z_0)$ *then :*

$$g \circ f : (X, x_0) \to (Y, y_0) \to (Z, z_0)$$

   *and we have also :*

$$\Pi_1(g \circ f) = \Pi_1(g) \circ \Pi_1(f)$$

2. *If* $Id : (X, x_0) \to (X, x_0)$ *identity morphism then :*

$$\Pi_1(Id_{\{X,x_0\}}) = Id_{\Pi_1(X,x_0)}$$

***1)*** *If* $(g \circ f) : (X, x_0) \to (Z, z_0)$*, on a :*

$$\Pi_1(g \circ f) : \Pi_1(X, x_0) \to \Pi_1(Z, z_0)$$

*Let* $[\delta] \in \Pi_1(X, x_0)$ *then :*

$$\Pi_1(g \circ f)([\delta]) = [(g \circ f) \circ \delta] = [g \circ (f \circ \delta)] = \Pi_1(g)([f \circ \delta]) = \Pi_1(g)(\Pi_1(f)([\delta])) = \Pi_1(g) \circ \Pi_1(f)([\delta])$$

*2)*

$$\Pi_1(Id_{\{X,x_0\}}) : \Pi_1(X, x_0) \to \Pi_1(X, x_0)$$

*If* $[\delta] \in \Pi_1(X, x_0)$ *we have :*

$$\Pi_1(Id_{\{X,x_0\}})([\delta]) = [Id_{\{X,x_0\}} \circ \delta] = [\delta] = Id_{\Pi_1(X,x_0)}$$

**Consequence 3.2.1** *The relation :* $\hat{\dot{Top}} \xrightarrow{T_1}$ **Cg** *given by :*

1. $(X, x_0) \in obj_{\hat{Top}} \Rightarrow \Pi_1(X, x_0) \in obj_G$

2. $f \in Mor_{\hat{Top}}((X, x_0), (Y, y_0)) \Rightarrow \Pi_1(f) \in Mor_G(\Pi_1(X, x_0), \Pi_1(Y, y_0))$ *it is covariant functor*

$\Pi_1$ *is called* ***a homotpic functor of order 1****.*

**Theorem 3.2.1** *Classification:*
*If* $(X, x)$ *and* $(Y, y)$ *are two pointed topological spaces homotopic, then their Poincaré groups* $\Pi_1(X, x)$ *and* $\Pi_1(Y, y)$ *are isomorphic groups*
*Indeed, the image of isomorphic objects by a functor are isomorphic that is*
*(X,x) ≅ (Y,y) in* $\hat{\dot{Top}}$ *if and only if homeomorphic* $\Rightarrow \Pi_1(X, x) \cong \Pi_1(Y, y)$ *in* $Cg$

**Consequence 3.2.2** *If* $(X, x)$ *,* $(Y, y)$ *are two topological spaces such that:* $\Pi_1(X, x)$ *is not isomorphic to* $\Pi_1(Y, y)$*, then:* $(X, x)$ *and* $(Y, y)$ *are not of the same homotopy type*

**Poincaré Conjecture 1904**
$X$ is a compact 3-dimensional manifold (*Top*) without boundary such that:
$\Pi_1(X)$ is trivial $\Rightarrow X$ is of the same homotopy type as $S^3$
**Answer: 2002–2008 by Gregory Perelman**

$X$ is a 3-dimensional manifold without boundary $\Leftrightarrow \forall x \in X$ there exists an open neighborhood homeomorphic to an open subset of $\mathbb{R}^3$

## 3.3 Homotopy of Continuous Maps and the Poincaré Group

We know that if $f : (X, x) \to (Y, y)$ is a continuous map, then it induces

$$\Pi_1(f) : \Pi_1(X, x) \to \Pi_1(Y, y)$$

Let $g : (X, x) \to (Y, y)$; it also induces

$$\Pi_1(g) : \Pi_1(X, x) \to \Pi_1(Y, y)$$

**Theorem 3.3.1** *Under the above assumptions: if* $f \sim g$*, then* $\Pi_1(f) = \Pi_1(g)$ *Two homotopic maps induce the same homomorphism on the fundamental group*
*Indeed, let* $[\delta] \in \Pi_1(X, x)$*, then*

$$\Pi_1(f)([\delta]) = [f \circ \delta]$$

*Since* $f \sim g$*, we have* $f \circ \delta \sim g \circ \delta$*, so*

$$[f \circ \delta] = [g \circ \delta] \Rightarrow \Pi_1(f) = \Pi_1(g)$$

**Consequence 3.3.1** *(Homotopy Criterion) If* $\Pi_1(f) \neq \Pi_1(g) \Rightarrow f \nsim g$

**Theorem 3.3.2** *If* $(X, x)$ *and* $(Y, y)$ *are two homotopy equivalent pointed topological spaces, then:*

$$\Pi_1(X, x) \cong \Pi_1(Y, y)$$

**Proof 3.3.1** *Let* $(X, x)$ *and* $(Y, y)$ *be homotopically equivalent if and only if:*

$$\begin{cases} \exists f : (X, x) \to (Y, y) & \textit{continuous} \\ \exists g : (Y, y) \to (X, x) & \textit{continuous} \end{cases}$$

*with:*

$$\begin{cases} g \circ f \sim Id_{(X,x)} \Rightarrow \Pi_1(g \circ f) = \Pi_1(Id_{(X,x)}) \\ f \circ g \sim Id_{(Y,y)} \Rightarrow \Pi_1(f \circ g) = \Pi_1(Id_{(Y,y)}) \end{cases}$$

$$\Rightarrow \begin{cases} \Pi_1(g) \circ \Pi_1(f) = Id_{\Pi_1(X,x)} \\ \Pi_1(f) \circ \Pi_1(g) = Id_{\Pi_1(Y,y)} \end{cases} \Rightarrow \Pi_1(X,x) \cong \Pi_1(Y,y) \quad \textit{are inverses of each other}$$

**Consequence 2.3.2.** If $\Pi_1(X,x) \ncong \Pi_1(Y,y)$, then $(X,x)$ and $(Y,y)$ are not homotopic.

## 2.3.1 Change of Base and Poincaré Group

Let $X$ be a topological space and $x, y$ two given points of $X$. Then we have the pointed topological spaces $(X,x)$ and $(X,y)$, and the Poincaré groups $\Pi_1(X,x)$ and $\Pi_1(X,y)$.

#### Problem

Is there a relation between them?

**Theorem 2.3.3.** If there exists a path $\omega$ in $X$ connecting $x$ to $y$, then the groups $\Pi_1(X,x)$ and $\Pi_1(Y,y)$ are isomorphic

**Proof 3.3.2** *Let us consider:* $L_1(X,x)$ *and* $L_1(Y,y)$ *the loops based at* $x$

$$L_1(X,x) = \{\delta : [0,1] \to X \mid \delta(0) = \delta(1) = x\}$$

$$L_1(Y,y) = \{\mu : [0,1] \to Y \mid \mu(0) = \mu(1) = y\}$$

*We want to go from a loop in* $L_1(X,x)$ *to a loop in* $L_1(Y,y)$*, so we consider:*

$(\bar{\omega}\cdot\delta)\cdot\omega$ *where* $\bar{\omega}\cdot(\delta\cdot\omega)$ *because* $(\bar{\omega}\cdot\delta)\cdot\omega \sim \bar{\omega}\cdot(\delta\cdot\omega)$ *in* $\Pi_1(Y,y)$ *Where:*

$[(\bar{\omega}\cdot\delta)\cdot\omega] = [\bar{\omega}\cdot(\delta\cdot\omega)] = [\bar{\omega}\cdot\delta\cdot\omega]$

*denoted as* $[\bar{\omega}\cdot\delta\cdot\omega]$.

*We consider the isomorphism:*

*Let* $\Pi_1(X,x) \xrightarrow{\varphi_\omega} \Pi_1(Y,y)$ *defined by:*

$\varphi_\omega([\delta]) = [\bar{\omega}\cdot\delta\cdot\omega], \quad \forall[\delta]\in\Pi_1(X,x)$

*This mapping is well-defined because if* $\delta' \in [\delta]$ *then:* $\delta \sim \delta' \Rightarrow \bar{\omega}\cdot\delta\cdot\omega \sim \bar{\omega}\cdot\delta'\cdot\omega$ *i.e* $[\bar{\omega}\cdot\delta\cdot\omega] = [\bar{\omega}\cdot\delta'\cdot\omega]$

*1)*$\varphi_\omega$ *Bijective ?*

$\varphi_{\bar{\omega}} : \Pi_1(Y,y) \to \Pi_1(X,x)$

*then for* $[\delta]\in\Pi_1(X,x)$ *we have:*

$[\mu] \to [\omega.\mu.\bar{\omega}]$

$\varphi_{\overline{\omega}} \circ \varphi_\omega([\delta]) = \varphi_{\overline{\omega}}([\overline{\omega}\cdot\delta\cdot\omega]) = [\omega\cdot\overline{\omega}\cdot\delta\cdot\omega\cdot\overline{\omega}] = [\varepsilon_x\cdot\delta\cdot\varepsilon_x] = [\delta] = Id_{\Pi_1(X,x)}([\delta])$ *(since* $\omega\cdot\overline{\omega} = \varepsilon_x$*)*

*Similarly:*

$[\mu]\in\Pi(Y,y) : \varphi_\omega\circ\varphi_{\overline{\omega}}([\mu]) = \varphi_\omega([\omega\cdot\mu\cdot\overline{\omega}]) = [\overline{\omega}\cdot\omega\cdot\delta\cdot\overline{\omega}\cdot\omega] = [\varepsilon_y\cdot\delta\cdot\varepsilon_y] = [\mu] = Id_{\Pi_1(Y,y)}([\mu])$

**Conclusion 3.3.1**

$(\varphi_\omega)^{-1} = \varphi_{\overline{\omega}}$*, i.e.,* $\varphi_\omega$ *is a bijectivity*

2)$\varphi_\omega$ is a homeomorphism : Let $[\delta],[\mu]\in\Pi(X,x)$ we have:

$$\varphi_\omega : \Pi_1(X,x) \to \Pi_1(Y,y) : \begin{cases} \varphi_\omega([\delta]\cdot[\mu]) = \varphi_\omega([\delta\cdot\mu]) = [\bar{\omega}\cdot\delta\cdot\mu\cdot\omega] \\ \qquad\qquad = [(\bar{\omega}\cdot\delta\cdot\omega)\cdot(\bar{\omega}\cdot\mu\cdot\omega)] \\ \qquad\qquad = \varphi_\omega([\delta])\cdot\varphi_\omega([\mu]) \end{cases}$$

**Consequence 2.3.3.** $\varphi_\omega$ is an isomorphism

**Proposition 2.3.1.** If $\omega' \in [\omega]$, then: $\varphi_{\omega'} = \varphi_\omega$, that is the isomorphism is independent of the homotopy class of the path from $x$ to $y$.

Indeed: if $[\delta] \in \Pi_1(X,x)$ and $\omega' \in [\omega]$, then $\bar{\omega}\cdot\delta\cdot\omega \sim \bar{\omega}'\cdot\delta\cdot\omega'$ because if

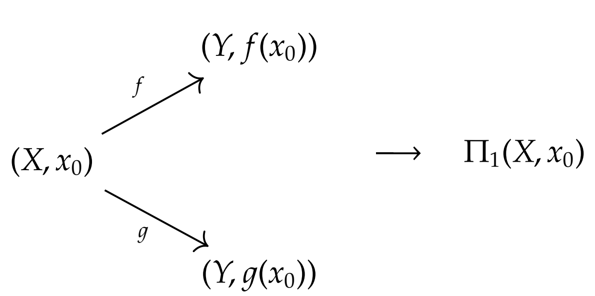
$$\omega \sim \omega' \Rightarrow \bar{\omega} \sim \bar{\omega}'$$

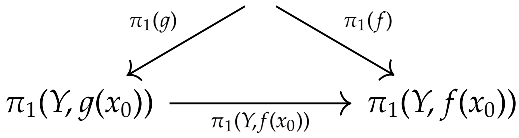
$\varphi_{\omega'}[\delta] = \varphi_\omega[\delta]$, thats why the isomorphism $\varphi_{\omega'} = \varphi_\omega$

**Consequence 2.3.4.** If $X$ is a path-connected topological space, then all the Poincaré groups of $X$ are isomorphic,

$$\Pi_1(X,x) = \Pi_1(X,y)\ \forall x,y \in X$$

Indeed: if $X$ is path-connected $\Rightarrow \exists$ a path linking $x$ and $y$ in $X$, so from the previous proposition: $\Pi_1(X,x) \backsimeq \Pi_1(Y,y)$ Thus: if $X$ is path-connected, we can omit the base point and denote $\Pi_1(X)$






# problem

Does there exist a path $\omega$ connecting $f(x_0)$ to $g(x_0)$ such that : the diagram is commutative

**Theorem 2.3.4.** Under the above assumptions, if $f$ and $g$ are homotopic, then there exists a path $\omega$ connecting $f(x_0)$ to $g(x_0)$ such that the diagram is commutative.

**Proof 3.3.3** *Since $f \sim g$ for $x_0 \in X$ the elements $f(x_0)$ and $g(x_0)$ are in the same connected component.*

*Let us now consider the path $F_{x_0} : [0,1] \to Y$, $F_{x_0}(t) = F(x_0, t)$*

$$F(x_0, t) : X \times [0,1] \to Y : \begin{cases} lF(x_0, 0) = f(x_0) \\ F(x_0, 1) = g(x_0) \end{cases}$$

*F continuous ?*

$$t = \frac{1+T}{4} \Rightarrow \begin{cases} f\left(\frac{4 \cdot \frac{1+T}{4}}{1+T}\right) = f(1) = y \\ g\left(\frac{4 \cdot \frac{1+T}{4}}{1+T} - 1 \cdot T\right) = g(0) = y \end{cases}$$

$$t = \frac{2+T}{4} \Rightarrow \begin{cases} g\left(4 \cdot \frac{2+T}{4} - 1 - T\right) = g(2 + T - 1 - T) = g(1) = z \\ h\left(\frac{4 \cdot \frac{2+T}{4} - 2 - T}{2-T}\right) = h(0) = v \end{cases}$$

$T = 0$

$$F(t,0) = \begin{cases} f(4t) & 0 \leq t \leq \frac{1}{4} \\ g(4t-1) & \frac{1}{4} \leq t \leq \frac{1}{2} \\ h(2t-1) & \frac{1}{2} \leq t \leq 1 \end{cases} = (f \cdot g) \cdot h(t)$$

$T = 1$

$$F(t,1) = \begin{cases} f(2t) & 0 \leq t \leq \frac{1}{4} \\ g(4t-2) & \frac{1}{4} \leq t \leq \frac{1}{2} \\ h(4t-3) & \frac{1}{2} \leq t \leq 1 \end{cases} \qquad = f \cdot (g \cdot h)(t)$$

# CONCLUSION

The resolution of the Poincaré Conjecture by Grigori Perelman stands as one of the most remarkable achievements in modern mathematics. It not only solved a century-old mystery about the nature of three-dimensional spaces but also demonstrated the power of deep insight, perseverance, and humility. Perelman's work reshaped geometric understanding and left a lasting legacy?proving that true discovery often transcends awards and recognition.

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