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Heisenberg uncertainty principles for the generalized wavelet transform

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بسم الله الرحمان

الرحيم والصلاة

والسلام على نبينا

محمد أشرف

المرسلين

إهداء

قال تعالى: " قُلِ اعْمَلُوا فَسَيَرَى اللَّهُ عَمَلَكُمْ وَرَسُولُهُ وَالْمُؤْمِنُونَ"

لم تكن الرحلة قصيرة ولا ينبغي لها أن تكون

لم يكن الحلم قريبا ولا الطريق كان محفوفا بالتسهيلات،

لكنني فعلتها ونِلتها.

الحمد لله حَمدًا شُكرا وامتنانا، الذي بفضله ها أنا اليوم أنظر

إلى حلمٍ طال انتظاره، وقد أصبح واقعا أفتخر به.

إلى من بلغ الرسالة، وأدى الأمانة.. ونصح الأمة.. إلى نبي الرحمة ونور العالمين

" سيدنا محمد صلى الله عليه وسلم "

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Contents

ملخص

الهدف من هذه المذكرة هو تعريف ودراسة التحويل بالمويجات لهانكل. حيث نقوم بإثبات التحليل التوافقي الكامل المرتبط بهذا التحويل، ونبرهن بشكل خاص صيغة بلانشريل، وخاصية التعامد، وصيغة إعادة البناء.

كما يتمثل هدف رئيسي آخر لهذا العمل في إثبات مبادئ عدم اليقين من نوع هايزنبرغ بالنسبة للتحويل بالمويجات لهانكل.

الكلمات المفتاحية :التحليل التوافقي، تحويل هانكل، التحويل بالمويجات، مبدأ عدم اليقين لهايزنبرغ.

Abstract

Our objective in this thesis is to define and study the Hankel wavelet transform. We will prove all the harmonic analysis associated to this transform, in particular a Plancherel's formula, an orthogonality property and a reconstruction formula. Another main purpose of this thesis is to prove the Heisenberg-type uncertainty principles for the Hankel wavelet transform.

Keywords: Harmonic analysis, Hankel transform, wavelet transform, Heisenberg uncertainty principle.

Résumé

L'objectif de cette mémoire est de définir et d'étudier la transformation en ondelettes de Hankel. On établit toute l'analyse harmonique associée à cette transformation, en particulier on démontre une formule de Plancherel, une propriété d'orthogonalité et une formule de reconstruction.
Un autre objectif principal de ce mémoire est de démontrer les principes d'incertitude de type Heisenberg pour la transformée en ondelettes de Hankel.

Mots clés: Analyse harmonique, Transformation de Hankel, Transformation en ondelettes, Principe d'incertitude de Heisenberg.

General Introduction

Many non-stationnary signals as seismic signal, genomic signal, electrocardiograms, and speech are gaining more attentions as they intervene in the real life. So, during the last decades, many methods of determining local spectra have been investigated. In fact, in signal theory, the Fourier transform of a given signal was firstly introduced by Joseph Fourier in 1822, defined for an integrable function f (stable signal) by

$$\hat{f}(\lambda) = \int_{\mathbb{R}} f(x) e^{-i\lambda x} \frac{dx}{\sqrt{2\pi}}; \quad \forall \lambda \in \mathbb{R},$$

represents the set of frequencies that compose the signal with their respective amplitudes that called the spectrum of the signal.
One of the major problems with the Fourier Transform consists of the fact that the frequency representation is global and does not give any temporal localization.

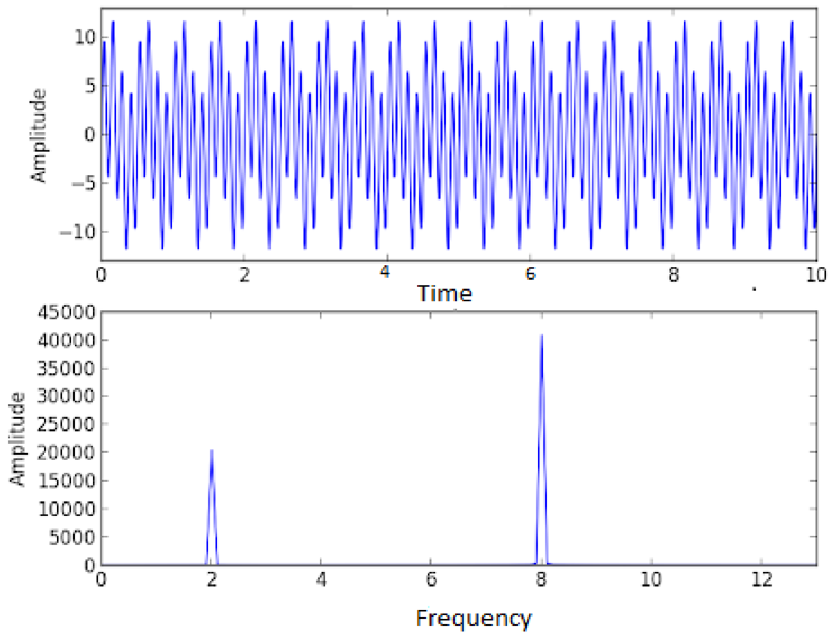


Figure 1: Loss of temporal localization of the Fourier transform

The notion of time-frequency representations was therefore introduced in order to overcome this problem, the basic idea concerning the time-frequency analysis is to introduce into the Fourier analysis, which is a purely spectral analysis, a notion of spatial or temporal locality by replacing the analyzed function f with the product of f by a function ψ suitably chosen having good localization properties, then we apply the Fourier transform to them.

The most famous time-frequency representation was introduced by Denis Gabor [10] called the Short-time Fourier transform (STFT).

Let us consider a non-zero function $\psi \in L^2(\mathbb{R})$ called window. Then, for every $f \in L^2(\mathbb{R})$, the Short-time Fourier transform of f is defined by

$$V_\psi(f)(a,r) = \int_{\mathbb{R}} f(x)\overline{\psi(x-a)}e^{-irx}\frac{dx}{\sqrt{2\pi}}; \quad a, r \in \mathbb{R}.$$

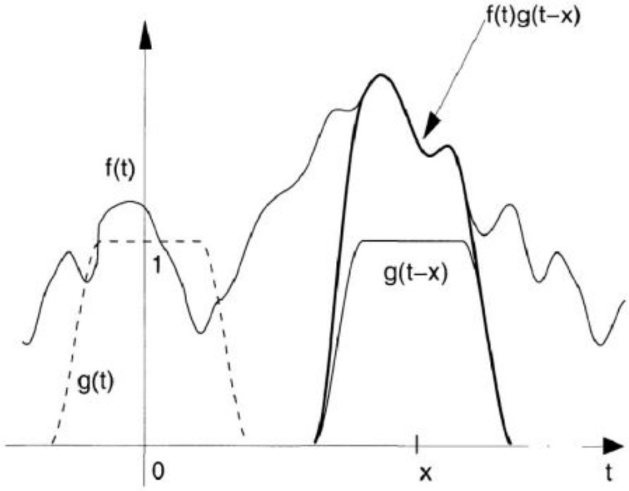


Figure 2: The Short-time Fourier transform

Example of two musical notes played one after the other: time-frequency analysis makes it possible to find both the frequencies (the notes) and the temporal information (the order in which they are played).

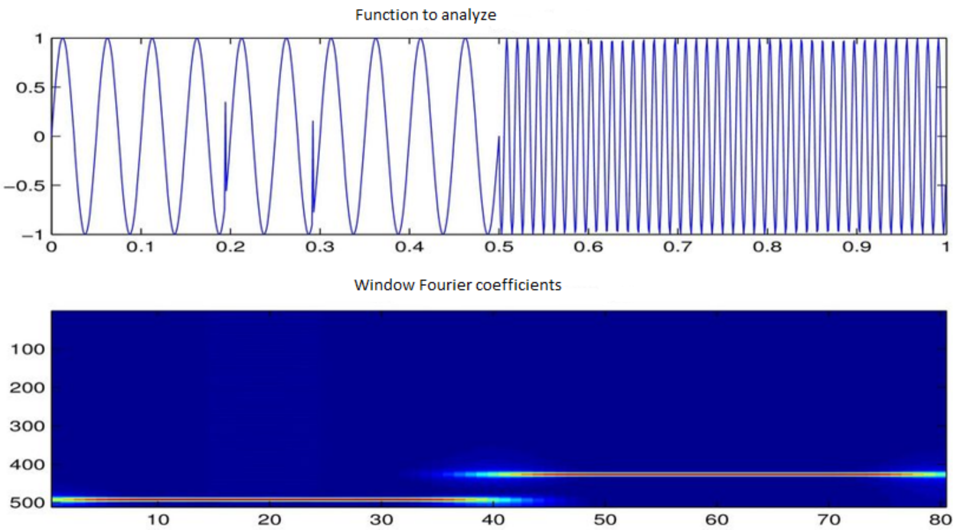


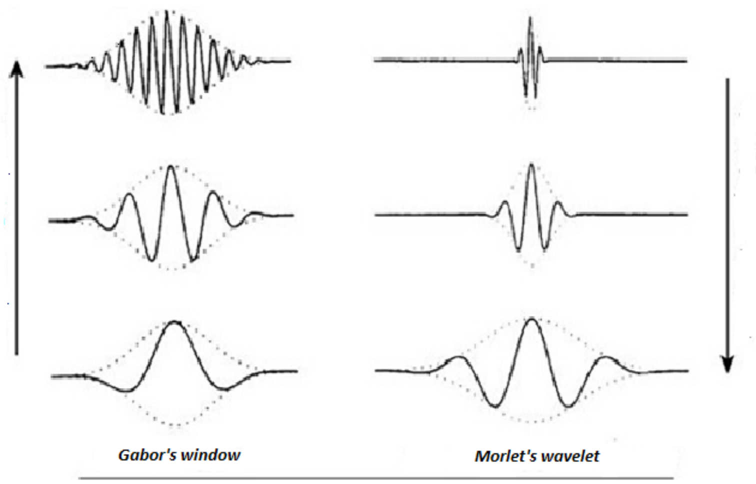
Figure 3: Time-frequency localization

But quickly, this transform showed many disadvantages like its inability to detect low frequencies and poor time resolution of high frequency events due to the fixed width of the window function this means that the short-time Fourier transform supposes a certain stationary of the signal and it might be unsuitable to non-stationary signals.
In contrast with the STFT, the wavelet transform (WT), introduced by Morlet [14] proposed to use a window of size depending on the analyzed frequency but with a fixed number of oscillations.
A non-zero function $\psi \in L^2(\mathbb{R})$ is said to be a mother wavelet if

$$\int_0^{+\infty} |\hat{\psi}(a)|^2 \frac{da}{a} < +\infty.$$

The wavelet transform W_ψ with respect to the mother ψ is defined on $L^2(\mathbb{R})$ by

$$W_\psi(f)(a,r) = \frac{1}{\sqrt{a}} \int_{\mathbb{R}} f(x)\overline{\psi\Big(\frac{x-r}{a}\Big)} \frac{dx}{\sqrt{2\pi}}; \quad a, r \in \mathbb{R}_+^* \times \mathbb{R},$$



The wavelet transform analyzes function with respect to position and scale that is why “wavelet analysis” has recently drawn a great deal of attention from mathematical community in various disciplines (see [1, 21, 23, 26]). It is creating a common link between mathematicians, physicians and electrical engineers. Hence, the wavelet transform emerged as an important tool in signal and image processing, and have many applications in several research areas, such as signal and image processing, time series analysis, geophysics, medicine (see [4, 5, 6, 7, 8, 20, 22]). This is what motivated us to work on the wavelet theory.

Recently, many authors have been interested to extend the classical wavelet transform in different settings like the Dunkl [13], the Jacobi [27] and the Hankel settings [16]

The Hankel transform also known as the Fourier-Bessel transform arises as a generalization of the Fourier transform of a radial integrable function in the euclidean space $\mathbb{R}^d$. More precisely, let $f \in L^1(\mathbb{R}^d)$ it is well known that if $f(x) = F(\|x\|)$ is radial function on $\mathbb{R}^d$, then $\hat{f}$ is also radial on $\mathbb{R}^d$ and we have

$$\forall \lambda \in \mathbb{R}^d, \quad \hat{f}(\lambda) = \int_0^{+\infty} F(x) j_{\frac{d}{2}-1}(\|\lambda\| x) \frac{x^{d-1}}{2^{\frac{d}{2}-1}\Gamma(\frac{d}{2})} dx = \mathscr{H}_{\frac{d}{2}-1}(F)(\|\lambda\|),$$

where $j_{\frac{d}{2}-1}$ is the modified Bessel function of index $\frac{d}{2}-1$ and $\mathscr{H}_{\frac{d}{2}-1}$ is the Hankel transform of index $\frac{d}{2}-1$.

As the harmonic analysis associated to the Hankel transform has shown remarkable development, it is a natural question to ask whether there exists the equivalent of the theory of time-frequency analysis for the wavelet transform in the Hankel setting. In fact, many results for the Hankel wavelet transform have been established in particular, let ψ be an admissible Hankel wavelet in $L^2(d\nu_\alpha)$. Then, we have:

• **Plancherel's formula**: Let ψ be an admissible window function in $L^2(d\nu_\alpha)$, then we have

$$\int_0^{+\infty} \int_0^{+\infty} |T_\psi^\alpha(f)(a,r)|^2 \, d\mu_\alpha(a,r) = C_\psi \int_0^{+\infty} |f(r)|^2 \, d\nu_\alpha(r)$$

• **Parseval's formula**: Let ψ be an admissible Hankel wavelet in $L^2(d\nu_\alpha)$. Then, for all f and g in $L^2(d\nu_\alpha)$, we have

$$\int_0^{+\infty} f(r)\overline{g(r)}\, d\nu_\alpha(r) = \frac{1}{C_\psi}\int_0^{+\infty}\int_0^{+\infty} T_\psi^\alpha(f)(a,r)\overline{T_\psi^\alpha(g)(a,r)}\, d\mu_\alpha(a,r).$$

• **Reconstruction formula**:Let ψ be an admissible Hankel wavelet in $L^2(d\nu_\alpha)$ such that $|\psi|$ is an admissible window function. Then, for every $f \in L^2(d\nu_\alpha)$, we have

$$f(\cdot) = \frac{1}{C_\psi}\int_0^{+\infty}\int_0^{+\infty} T_\psi^\alpha(f)(a,r)\, \psi_{a,r}^\alpha(\cdot)\, d\mu_\alpha(a,r),$$

weakly in $L^2(d\nu_\alpha)$.

The term uncertainty principle first appeared in 1927 in quantum mechanics, when Werner Heisenberg demonstrated that there is a fundamental limit to the precision with which two complementary properties of a fast-moving particle can be measured.
Heisenberg thus showed that it is impossible to determine both the position and the momentum of a quantum particle simultaneously and with arbitrary precision.

> *"The more precisely the position of a particle is determined, the less precisely its momentum is known, and vice versa."*

The theoretical formulation of this principle was established in 1928, linking the standard deviation of position Δx and the standard deviation of momentum Δp through the following inequality:

$$\Delta x \cdot \Delta p \geq \frac{h}{4\pi}$$

where h is Planck's constant.
The uncertainty principles in harmonic analysis state that a function f and its

Fourier transform $\widehat{f}$ cannot be simultaneously sharply localized, actually many mathematical formulations of this general fact have been proved, for more details we refer the reader to [9, 17]. The most famous of them is the following Heisenberg-Pauli-Weyl sharp inequality [18]. It states that for all square integrable function f on $\mathbb{R}$ with respect to the Lebesgue measure, we have

$$\left(\int_{\mathbb{R}} x^2|f(x)|^2dx\right)\left(\int_{\mathbb{R}} \xi^2|\widehat{f}(\xi)|^2d\xi\right) \geq \frac{1}{4}\left(\int_{\mathbb{R}} |f(x)|^2dx\right)^2.$$

Recently, many works have been devoted to establish the Heisenberg–Pauli–Weyl inequality in different setting and for various transforms, in [25, 40] the authors established Heisenberg-type inequalities, respectively on Chébli-Trimèche hypergroups and on the Heisenberg groups. Later, in [11] Jaming and Gohbber have proved a Heisenberg uncertainty principle for a family of integral transform including in particular, usual Fourier transform, the Hankel transform, the Dunkl transform,etc.

Nowadays, new uncertainty principles involving time–frequency representations such the Gabor and wavelet transforms have been formulated with different approaches [1, 2, 3, 12].

This thesis is arranged as follows:

In the first chapter, we give a brief background of some harmonic analysis results related to the Hankel transform .

In the second chapter, we define and study the Hankel wavelet transform.

The third chapter is devoted to prove the Heisenberg-type uncertainty principles for the Hankel wavelet transform.

Chapter 1

Hankel transform

The Hankel transform also called Fourier-Bessel transform is integral transformation whose kernel is Bessel function. When we are dealing with problems that show circular symmetry, the Hankel transform may be very useful. For example, the Hankel transform is the two-dimensional Fourier transform of a circularly symmetric function. Moreover, the Hankel transform came for the first time by studying the Fourier transform of radial functions and has been generalized later in the general case.

In this chapter, we summarize some harmonic analysis tools related to the Hankel transform that we shall use later (for more details, one can see [15, 19, 33, 37].

Notations

We denote by

- ν_α is the measure defined on $[0, +\infty[$ by

$$d\nu_\alpha(r) = \frac{r^{2\alpha+1}}{2^\alpha \Gamma(\alpha+1)}\, dr.$$

- $L^p(d\nu_\alpha)$, $p \in [1, +\infty]$, is the space of measurable functions f on $[0, +\infty[$ such that

$$\|f\|_{p,\nu_\alpha} = \begin{cases} \left(\int_0^{+\infty} |f(r)|^p \, d\nu_\alpha(r)\right)^{1/p} < +\infty, & \text{if } 1 \leq p < +\infty, \\ \text{ess sup}_{r\in[0,+\infty[} |f(r)| < +\infty, & \text{if } p = +\infty. \end{cases}$$

- $\langle \cdot \mid \cdot \rangle_{\nu_\alpha}$ the inner product on $L^2(d\nu_\alpha)$ defined by

$$\langle w \mid z \rangle_{\nu_\alpha} = \int_0^{+\infty} w(r)\overline{z(r)}\, d\nu_\alpha(r).$$

- $C^*(\mathbb{R})$: the space of even continuous functions on $\mathbb{R}$.

1.1 Bessel operator

In this section, we define the Bessel operator ℓ_α, the modified Bessel function j_α, and we give some related results. We also define the translation operator, the convolution product related to the Bessel operator, and we recall some known inequalities which can be useful throughout this manuscript.

Let ℓ_α be the Bessel operator defined on $]0,+\infty[$ by

$$\begin{aligned}\ell_\alpha &= \frac{d^2}{dr^2}+\frac{2\alpha+1}{r}\frac{d}{dr}\\ &= \frac{1}{r^{2\alpha+1}}\frac{d}{dr}\left(r^{2\alpha+1}\frac{d}{dr}\right)\end{aligned}$$

.

Proposition 1.1.1. *For all $\lambda\in\mathbb{C}$, the following problem:*

$$\begin{cases}\ell_\alpha(u)(r) = -\lambda^2 u(r),\\ u(0)=1,\\ u'(0)=0,\end{cases}$$

.

admits a unique solution given by the modified Bessel function $j_\alpha(\lambda\cdot)$, where

$$j_\alpha(r) = 2^\alpha\Gamma(\alpha+1)\frac{J_\alpha(r)}{r^\alpha} = \Gamma(\alpha+1)\sum_{k=0}^{+\infty}\frac{(-1)^k}{k!\,\Gamma(\alpha+k+1)}\left(\frac{r}{2}\right)^{2k}, \tag{1.1}$$

and J_α is the Bessel function of the first kind and index α [24, 38]. .

Proof. Let $\lambda\in\mathbb{C}$. Then, we have

$$\ell_\alpha(j_\alpha(\lambda r)) = j''_\alpha(\lambda r)+\frac{2\alpha+1}{r}j'_\alpha(\lambda r).$$

Since

$$\begin{aligned} j'_\alpha(\lambda r) &= \Gamma(\alpha+1)\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\Gamma(\alpha+k+1)}\left(\lambda r^2\right)^{2k} \\ &= \Gamma(\alpha+1)\sum_{k=1}^{\infty}\frac{(-1)^k\lambda^k}{k!\Gamma(\alpha+k+1)}\left(\lambda r^2\right)^{2k-1} \\ &= \lambda\Gamma(\alpha+1)\sum_{k=0}^{\infty}\frac{(-1)^{k+1}(k+1)}{(k+1)!\Gamma(\alpha+k+2)}\left(\lambda r^2\right)^{2k+1} \\ &= -\lambda^2\left(\frac{r^2}{2}\right)\Gamma(\alpha+2)\frac{\alpha+1}{\Gamma(\alpha+1)}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\Gamma(\alpha+k+2)}\left(\lambda r^2\right)^{2k} \\ &= -\lambda^2 r^2\frac{\alpha+1}{\Gamma(\alpha+1)}j_{\alpha+1}(\lambda r), \end{aligned} \tag{1.2}$$

and

$$\begin{aligned} j''_\alpha(\lambda r) &= -\frac{\lambda^2 r}{2(\alpha+1)}j_{\alpha+1}(\lambda r) \\ &= -\frac{\lambda^2}{2(\alpha+1)}j_{\alpha+1}(\lambda r) - \frac{\lambda^2 r}{2(\alpha+1)}j'_{\alpha+1}(\lambda r) \\ &= -\frac{\lambda^2}{2(\alpha+1)}j_{\alpha+1}(\lambda r) + \frac{\lambda^4 r^2}{4(\alpha+1)(\alpha+2)}j_{\alpha+2}(\lambda r). \end{aligned} \tag{1.3}$$

Then, by relations (1.2) and (1.3), we have.

$$\begin{aligned} \ell_\alpha\Big(j_\alpha(\lambda r)\Big) &= -\frac{\lambda^2}{2(\alpha+1)}j_{\alpha+1}(\lambda r) + \frac{\lambda^4 r^2}{4(\alpha+1)(\alpha+2)}j_{\alpha+2}(\lambda r) - \frac{\lambda^2(2\alpha+1)}{2(\alpha+1)}j_{\alpha+1}(\lambda r) \\ &= -\lambda^2\left(j_{\alpha+1}(\lambda r) - \frac{\lambda^2 r^2}{4(\alpha+1)(\alpha+2)}j_{\alpha+2}(\lambda r)\right) \end{aligned}$$

Using the fact that (see [69])

$$J_{\alpha+1}(r) + J_{\alpha-1}(r) = \frac{2\alpha}{r}J_\alpha(r)$$

thus, we get

$$j_{\alpha+1}(r) = \frac{r^2}{4(\alpha+1)(\alpha+2)} j_{\alpha+2}(r) + j_\alpha(r)$$

Furthermore, $j_\alpha(0) = 1$ and $j'_\alpha(0) = 0$. The proof is complete.

■

Proposition 1.1.2. *The function j_α has the following integral representation formula, for all $r \in \mathbb{R}$*

$$j_\alpha(r) = \begin{cases} \frac{2\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma\left(\alpha+\frac{1}{2}\right)} \int_0^1 (1-t^2)^{\alpha-1/2} \cos(tr)\, dt, & \text{if } \alpha > -\frac{1}{2}, \\ \cos(r), & \text{if } \alpha = -\frac{1}{2}. \end{cases}$$

Proof. 1) For $\alpha = -\frac{1}{2}$, we get

$$\begin{aligned} j_{-\frac{1}{2}}(r) &= \Gamma\left(\frac{1}{2}\right) \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma\left(k+\frac{1}{2}\right)} \left(\frac{r}{2}\right)^{2k} \\ &= \sqrt{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+1)\Gamma\left(k+\frac{1}{2}\right)} \left(\frac{r}{2}\right)^{2k} \\ &= \sqrt{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{2k-1}\Gamma(k)\Gamma\left(k+\frac{1}{2}\right)} r^{2k} \frac{1}{2k} \end{aligned}$$

Using the fact that

$$2^{2k-1}\Gamma(k)\Gamma\left(k+\frac{1}{2}\right) = \sqrt{\pi}\Gamma(2k),$$

then we obtain

$$\begin{aligned} j_{-\frac{1}{2}}(r) &= \sum_{k=0}^{\infty} \frac{(-1)^k}{2k\Gamma(2k)} r^{2k} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(2k+1)} r^{2k} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k r^{2k}}{(2k)!} = \cos(r). \end{aligned}$$

2) For $\alpha > -\frac{1}{2}$, we have

$$\begin{aligned} \int_0^1 (1-t^2)^{\alpha-\frac{1}{2}} \cos(tr)\, dt &= \int_0^1 (1-t^2)^{\alpha-\frac{1}{2}} \left(\sum_{k=0}^{\infty} \frac{(-1)^k (tr)^{2k}}{(2k)!} \right) dt \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k r^{2k}}{(2k)!} \int_0^1 (1-t^2)^{\alpha-\frac{1}{2}} t^{2k}\, dt. \end{aligned}$$

By the change of variable $u = 1-t^2$, we get

$$\begin{aligned} \int_0^1 (1-t^2)^{\alpha-\frac{1}{2}} \cos(tr)\, dt &= \frac{1}{2} \sum_{k=0}^{\infty} \frac{(-1)^k}{2k\Gamma(2k)} r^{2k} \int_0^1 u^{\alpha-\frac{1}{2}} (1-u)^{k-\frac{1}{2}}\, du \\ &= \frac{\sqrt{\pi}}{2} \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{2k-1}\Gamma(k)\Gamma\left(k+\frac{1}{2}\right)} r^{2k} B\left(\alpha+\frac{1}{2}, k+\frac{1}{2}\right) \\ &= \frac{\sqrt{\pi}\Gamma\left(\alpha+\frac{1}{2}\right)}{2} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(\alpha+k+1)} \left(\frac{r}{2}\right)^{2k} \\ &= \frac{\sqrt{\pi}\Gamma\left(\alpha+\frac{1}{2}\right)}{2\Gamma(\alpha+1)} j_\alpha(r). \end{aligned}$$

■

Remark 1.1.1. *The function j_α is bounded, for all $n \in \mathbb{N}$ and $r \in \mathbb{R}$, and we have*

$$\left| j_\alpha^{(n)}(r) \right| \leq 1 \tag{1.4}$$

We have also the following product formula satisfied by j_α for all $r, s \in \mathbb{R}^+$:

$$j_\alpha(r)j_\alpha(s) = \begin{cases} \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma\left(\alpha+\frac{1}{2}\right)} \int_0^\pi j_\alpha\left(\sqrt{r^2+s^2+2rs\cos\theta}\right)(\sin\theta)^{2\alpha}\, d\theta, & \text{if } \alpha > -\frac{1}{2}, \\ \frac{j_{-\frac{1}{2}}(r+s)+j_{-\frac{1}{2}}(|r-s|)}{2}, & \text{if } \alpha = -\frac{1}{2}. \end{cases} \tag{1.5}$$

1.1.1 Translation operator associated to the Bessel operator

Definition 1.1.1. *We define the **Hankel translation operator** τ_r^α, for $r \in [0, +\infty[$, and for all $f \in C^*(\mathbb{R})$, by*

$$\tau_r^\alpha(f)(s) = \begin{cases} \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma\left(\alpha+\frac{1}{2}\right)} \int_0^\pi f\left(\sqrt{r^2+s^2+2rs\cos\theta}\right)(\sin\theta)^{2\alpha}\, d\theta, & \text{if } \alpha > -\frac{1}{2}, \\ \frac{f(r+s)+f(|r-s|)}{2}, & \text{if } \alpha = -\frac{1}{2}. \end{cases} \tag{1.6}$$

Theorem 1.1.1. *Let $\alpha > -\frac{1}{2}$ and $f \in C^*(\mathbb{R})$. Then, for all $r, s \in]0, +\infty[$, the operator τ_r^α can also be written as:*

$$\tau_r^\alpha(f)(s) = \int_0^{+\infty} f(u)\,\omega_\alpha(u,r,s)\,d\nu_\alpha(u) \tag{1.7}$$

where ω_α is the Hankel kernel given by

$$\omega_\alpha(u,r,s) = \begin{cases} \frac{\Gamma^2(\alpha+1)}{\sqrt{\pi}\,\Gamma\left(\alpha+\frac{1}{2}\right)} \frac{\left[u^2-(r-s)^2\right]^{\alpha-\frac{1}{2}}\left[(r+s)^2-u^2\right]^{\alpha-\frac{1}{2}}}{(urs)^{2\alpha}}, & \text{if } |r-s| < u < r+s, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. According to Definition 1.1.1, we have for every $\alpha > -\frac{1}{2}$,

$$\tau_r^\alpha(f)(s) = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma\left(\alpha+\frac{1}{2}\right)} \int_0^\pi f\left(\sqrt{r^2+s^2+2rs\cos\theta}\right)(\sin\theta)^{2\alpha}\,d\theta$$

We set $u = \sqrt{r^2+s^2+2rs\cos\theta}$, we obtain

$$\begin{aligned}
\tau_r^\alpha(f)(s) &= \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma\left(\alpha+\frac{1}{2}\right)} \int_{|r-s|}^{r+s} f(u)\frac{[u^2-(r-s)^2]^\alpha}{(2rs)^{2\alpha}}\,\frac{u}{rs}\,\frac{[u^2-(r-s)^2]^{-\frac{1}{2}}\,[(r+s)^2-u^2]^{-\frac{1}{2}}}{(2rs)^{-1}}\,du \\
&= \frac{\Gamma(\alpha+1)}{2^{2\alpha-1}\sqrt{\pi}\Gamma\left(\alpha+\frac{1}{2}\right)} \int_{|r-s|}^{r+s} uf(u)\frac{[u^2-(r-s)^2]^{\alpha-\frac{1}{2}}\,[(r+s)^2-u^2]^{\alpha-\frac{1}{2}}}{(rs)^{2\alpha}}\,du \\
&= \frac{\Gamma^2(\alpha+1)}{2^{2\alpha-1}\sqrt{\pi}\Gamma\left(\alpha+\frac{1}{2}\right)} \int_{|r-s|}^{r+s} f(u)\frac{[u^2-(r-s)^2]^{\alpha-\frac{1}{2}}[(r+s)^2-u^2]^{\alpha-\frac{1}{2}}}{(rs)^{2\alpha}}\,\frac{u^{2\alpha+1}}{2^\alpha\Gamma(\alpha+1)}\,du \\
&= \int_0^{+\infty} f(u)\,\omega_\alpha(u,r,s)\,d\nu_\alpha(u)
\end{aligned}$$

■

kernel ω_α is symmetric in the variables u, r, s and we have

$$\int_0^{+\infty} \omega_\alpha(u,r,s)\,d\nu_\alpha(u) = 1. \tag{1.8}$$

Proposition 1.1.3. *For every $f \in L^1(d\nu_\alpha)$ and for $r \in [0,+\infty[$, the function $\tau_r^\alpha(f)$ belongs to $L^1(d\nu_\alpha)$ and we have*

$$\int_0^{+\infty} \tau_r^\alpha(f)(s)\,d\nu_\alpha(s) = \int_0^{+\infty} f(u)\,d\nu_\alpha(u). \tag{1.9}$$

Proof. From relation (1.8) and using Fubini–Tonelli's theorem, we obtain that for f $\in L^1(d\nu_\alpha)$ and for r $\in [0,+\infty[$,

$$\begin{aligned}
\int_0^{+\infty} \left|\tau_r^\alpha(f)(s)\right| d\nu_\alpha(s) &= \int_0^{+\infty} \left|\int_0^{+\infty} f(u)\,\omega_\alpha(u,r,s)\,d\nu_\alpha(u)\right| d\nu_\alpha(s) \\
&\leq \int_0^{+\infty} |f(u)| \left(\int_0^{+\infty} \omega_\alpha(u,r,s)\,d\nu_\alpha(s)\right) d\nu_\alpha(u) \\
&= \|f\|_{1,\nu_\alpha} < +\infty.
\end{aligned}$$

This shows that $\tau_r^\alpha(f)$ belongs to $L^1(d\nu_\alpha)$ and

$$\begin{aligned}
\int_0^{+\infty} \tau_r^\alpha(f)(s)\,d\nu_\alpha(s) &= \int_0^{+\infty}\int_0^{+\infty} f(u)\,\omega_\alpha(u,r,s)\,d\nu_\alpha(u)\,d\nu_\alpha(s) \\
&= \int_0^{+\infty} f(u) \left(\int_0^{+\infty} \omega_\alpha(u,r,s)\,d\nu_\alpha(s)\right) d\nu_\alpha(u) \\
&= \int_0^{+\infty} f(u)\,d\nu_\alpha(u).
\end{aligned}$$

■

Corollary 1.1.1. *$\forall r, s \in [0,+\infty[$ and $\forall \lambda \in \mathbb{C}$, we have*

$$\tau_r^\alpha\Big(j_\alpha(\lambda .)\Big)(s) = j_\alpha(\lambda r)\,j_\alpha(\lambda s). \tag{1.10}$$

Proof. Let r, s $\in [0,+\infty[$. Then, we get

$$\begin{aligned}
\tau_r^\alpha\Big(j_\alpha(\lambda .)\Big)(s) &= \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma\left(\alpha+\frac{1}{2}\right)} \int_0^\pi j_\alpha\left(\lambda\sqrt{r^2+s^2+2rs\cos\theta}\right)(\sin\theta)^{2\alpha}\,d\theta \\
&= \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma\left(\alpha+\frac{1}{2}\right)} \int_0^\pi j_\alpha\left(\sqrt{(\lambda r)^2+(\lambda s)^2+2(\lambda r)(\lambda s)\cos\theta}\right)(\sin\theta)^{2\alpha}\,d\theta \\
&= j_\alpha(\lambda r)\,j_\alpha(\lambda s).
\end{aligned}$$

■

Proposition 1.1.4. *For every $f \in L^p(d\nu_\alpha)$, $p \in [1, +\infty]$ and for every $r \in [0, +\infty[$, the function $\tau_r^\alpha(f)$ belongs to $L^p(d\nu_\alpha)$ and we have*

$$\|\tau_r^\alpha(f)\|_{p,\nu_\alpha} \leq \|f\|_{p,\nu_\alpha}. \tag{1.11}$$

Proof. Let $f \in L^p(d\nu_\alpha)$, $p \in [1, +\infty]$.

- If $p = +\infty$, then for all r, s $\in [0, +\infty[$, we have

$$\tau_r^\alpha(f)(s) = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma\left(\alpha+\frac{1}{2}\right)} \int_0^\pi f\left(\sqrt{r^2+s^2+2rs\cos\theta}\right)(\sin\theta)^{2\alpha}\,d\theta$$

Then,

$$\begin{aligned}
\left|\tau_r^\alpha(f)(s)\right| &\leq \|f\|_{\infty,\nu_\alpha} \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma\left(\alpha+\frac{1}{2}\right)} \int_0^\pi (\sin\theta)^{2\alpha}\,d\theta \\
&= \|f\|_{\infty,\nu_\alpha} \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\,\Gamma\left(\alpha+\frac{1}{2}\right)} B\left(\alpha+\frac{1}{2}, \frac{1}{2}\right) \\
&= \|f\|_{\infty,\nu_\alpha}.
\end{aligned}$$

This shows that the function $\tau_r^\alpha(f)$ belongs to $L^\infty(d\nu_\alpha)$, and

$$\|\tau_r^\alpha(f)\|_{\infty,\nu_\alpha} \leq \|f\|_{\infty,\nu_\alpha}.$$

- If $p = 1$, we know that

$$\tau_r^\alpha(f)(s) = \int_0^{+\infty} f(u)\,\omega_\alpha(u,r,s)\,d\nu_\alpha(u).$$

According to Fubini-Tonelli's theorem and by relation (1.8), we have

$$\begin{aligned}
\|\tau_r^\alpha(f)\|_{1,\nu_\alpha} &\leq \int_0^{+\infty} |f(u)| \left(\int_0^{+\infty} \omega_\alpha(u,r,s)\, d\nu_\alpha(s) \right) d\nu_\alpha(u) \\
&= \int_0^{+\infty} |f(u)|\, d\nu_\alpha(u) \\
&= \|f\|_{1,\nu_\alpha}.
\end{aligned}$$

- If $p \in (1,+\infty)$ and q is the conjugate exponent of p. According to Hölder's inequality and relations (1.7) and (1.8), we obtain

$$\begin{aligned}
\left|\tau_r^\alpha(f)(s)\right| &\leq \int_0^{+\infty} |f(u)|\, \omega_\alpha(u,r,s)^{\frac{1}{p}} \omega_\alpha(u,r,s)^{\frac{1}{q}}\, d\nu_\alpha(u). \\
&\leq \left(\int_0^{+\infty} |f(u)|^p \omega_\alpha(u,r,s)\, d\nu_\alpha(u) \right)^{\frac{1}{p}} \left(\int_0^{+\infty} \omega_\alpha(u,r,s)\, d\nu_\alpha(u) \right)^{\frac{1}{q}}. \\
&= \left(\int_0^{+\infty} |f(u)|^p \omega_\alpha(u,r,s)\, d\nu_\alpha(u) \right)^{\frac{1}{p}}.
\end{aligned}$$

Now, using Fubini-Tonelli's theorem, we have

$$\begin{aligned}
\|\tau_r^\alpha(f)\|_{p,\nu_\alpha}^p &\leq \int_0^{+\infty} \int_0^{+\infty} |f(u)|^p\, \omega_\alpha(u,r,s)\, d\nu_\alpha(s)\, d\nu_\alpha(u) \\
&= \int_0^{+\infty} |f(u)|^p \left(\int_0^{+\infty} \omega_\alpha(u,r,s)\, d\nu_\alpha(s) \right) d\nu_\alpha(u) \\
&= \|f\|_{p,\nu_\alpha}^p.
\end{aligned}$$

■

1.1.2 Convolution product for the Bessel operator

Definition 1.1.2. *The convolution product of $f, g \in L^1(d\nu_\alpha)$ is defined by*

$$\begin{aligned} f * g(r) &= \int_0^{+\infty} \tau_r^\alpha(f)(s)\, g(s)\, d\nu_\alpha(s), \\ &= \int_0^{+\infty} f(s)\, \tau_r^\alpha(g)(s)\, d\nu_\alpha(s). \end{aligned}$$

Theorem 1.1.2. *For all $f, g \in L^1(d\nu_\alpha)$, $f * g \in L^1(d\nu_\alpha)$ and we have*

$$\|f * g\|_{1,\nu_\alpha} \leq \|f\|_{1,\nu_\alpha} \|g\|_{1,\nu_\alpha}.$$

Proof. According to Fubini-Tonnelli's theorem and relation (1.11), we have for every $f, g \in L^1(d\nu_\alpha)$

$$\begin{aligned} \int_0^{+\infty} |f * g(r)|\, d\nu_\alpha(r) &\leq \int_0^{+\infty} \int_0^{+\infty} |\tau_r^\alpha(f)(s)|\, |g(s)|\, d\nu_\alpha(s)\, d\nu_\alpha(r) \\ &= \int_0^{+\infty} |g(s)| \left(\int_0^{+\infty} |\tau_s^\alpha(f)(r)|\, d\nu_\alpha(r) \right) d\nu_\alpha(s) \\ &= \int_0^{+\infty} |g(s)|\, \|\tau_s^\alpha f\|_{1,\nu_\alpha}\, d\nu_\alpha(s) \\ &\leq \|f\|_{1,\nu_\alpha} \int_0^{+\infty} |g(s)|\, d\nu_\alpha(s) \end{aligned}$$

Then, the function $f * g \in L^1(d\nu_\alpha)$ and

$$\|f * g\|_{1,\nu_\alpha} \leq \|f\|_{1,\nu_\alpha} \|g\|_{1,\nu_\alpha}.$$

∎

Theorem 1.1.3. *For all $f \in L^1(d\nu_\alpha)$, $g \in L^p(d\nu_\alpha)$ such that $p \in [1, +\infty[$, $f * g \in L^p(d\nu_\alpha)$. Furthermore,*

$$\|f * g\|_{p,\nu_\alpha} \leq \|f\|_{1,\nu_\alpha} \|g\|_{p,\nu_\alpha}.$$

Proof. Let $f \in L^1(d\nu_\alpha)$, $g \in L^p(d\nu_\alpha)$, $p \in [1, +\infty[$ and let q be the conjugate exponent of p. So, from Hölder's inequality, we obtain

$$\begin{aligned} |f * g(r)| &\leq \int_0^{+\infty} |f(s)|\, |\tau_r^\alpha(g)(s)|\, d\nu_\alpha(s) \\ &= \int_0^{+\infty} |f(s)|^{1/q}\, |f(s)|^{1/p}\, |\tau_r^\alpha(g)(s)|\, d\nu_\alpha(s) \\ &\leq \|f\|_{1/q,\nu_\alpha} \left(\int_0^{+\infty} |f(s)|\, |\tau_r^\alpha(g)(s)|^p\, d\nu_\alpha(s) \right)^{1/p}. \end{aligned}$$

Using Fubini-Tonnelli's theorem and relation (1.11), we get

$$\begin{aligned} \|f * g\|_{p,\nu_\alpha}^p &\leq \|f\|_{1,\nu_\alpha}^{p/q} \int_0^{+\infty} |f(s)| \left(\int_0^{+\infty} |\tau_s^\alpha(g)(r)|^p\, d\nu_\alpha(r) \right) d\nu_\alpha(s) \\ &= \|f\|_{1,\nu_\alpha}^{p/q} \int_0^{+\infty} |f(s)|\, \|\tau_s^\alpha(g)\|_{p,\nu_\alpha}^p\, d\nu_\alpha(s) \\ &\leq \|f\|_{1,\nu_\alpha}^{p/q+1} \|g\|_{p,\nu_\alpha}^p \\ &= \|f\|_{1,\nu_\alpha}^p \|g\|_{p,\nu_\alpha}^p. \end{aligned}$$

■

Theorem 1.1.4. *For all $f \in L^p(d\nu_\alpha)$, $g \in L^q(d\nu_\alpha)$ and for all $p, q, r \in [1, +\infty]$ such that $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$, the function $f * g$ belongs to the space $L^r(d\nu_\alpha)$ and we have the following Young's inequality.*

$$\|f * g\|_{r,\nu_\alpha} \leq \|f\|_{p,\nu_\alpha} \|g\|_{q,\nu_\alpha}. \tag{1.12}$$

1.2 Hankel transform

The main aim of this second part of the first chapter is to introduce some notations, properties, definitions and basic facts that will be used throughout this work.

Definition 1.2.1. *The Hankel transform $\mathscr{H}_\alpha$ is defined on $L^1(d\nu_\alpha)$ by*

$$\mathscr{H}_\alpha(f)(\lambda) = \int_0^{+\infty} f(r) j_\alpha(\lambda r)\, d\nu_\alpha(r), \quad \forall \lambda \in \mathbb{R}$$

where j_α is the Bessel function given by (1.1).

Proposition 1.2.1.

1. *The Hankel transform $\mathscr{H}_\alpha$ is linear, continuous from $L^1(d\nu_\alpha)$ onto $L^\infty(d\nu_\alpha)$, and*

$$\|\mathscr{H}_\alpha\| = \sup_{\|f\|_{1,\nu_\alpha} \leq 1} \|\mathscr{H}_\alpha(f)\|_{\infty,\nu_\alpha} = 1.$$

2. *For all $f, g \in L^1(d\nu_\alpha)$, we obtain the following transfer formula*

$$\int_0^{+\infty} \mathscr{H}_\alpha(f)(\lambda)\, g(\lambda)\, d\nu_\alpha(\lambda) = \int_0^{+\infty} f(r)\, \mathscr{H}_\alpha(g)(r)\, d\nu_\alpha(r). \tag{1.13}$$

Proof. 1. Let $f, g \in L^1(d\nu_\alpha)$. Then, from Definition 1.2.1, we get:

$$\mathscr{H}_\alpha(\alpha f + \beta g)(\lambda) = \alpha\, \mathscr{H}_\alpha(f)(\lambda) + \beta\, \mathscr{H}_\alpha(g)(\lambda), \quad \forall \lambda \in \mathbb{R}.$$

and by relation (1.4), we have

$$\begin{aligned} |\mathscr{H}_\alpha(f)(\lambda)| &\leq \int_0^{+\infty} |f(r)|\,|j_\alpha(\lambda r)|\, d\nu_\alpha(r) \\ &\leq \int_0^{+\infty} |f(r)|\, d\nu_\alpha(r) \\ &= \|f\|_{1,\nu_\alpha}. \end{aligned}$$

So, the Hankel transform $\mathscr{H}_\alpha$ is linear, continuous from $L^1(d\nu_\alpha)$ onto $L^\infty(d\nu_\alpha)$, and

$$\|\mathscr{H}_\alpha\| = \sup_{\|f\|_{1,\nu_\alpha} \leq 1} \|\mathscr{H}_\alpha(f)\|_{\infty,\nu_\alpha} \leq 1.$$

Let the function f defined by

$$f(r) = r^{2\alpha+1} e^{-r^2}$$

It is clear that $f \in L^1(d\nu_\alpha)$ and $\|f\|_{1,\nu_\alpha} = 1$. Then, for every $\lambda \in \mathbb{R}$, we get

$$\begin{aligned} \mathscr{H}_\alpha(f)(\lambda) &= 2^{\alpha+1} \int_0^\infty e^{-r^2} j_\alpha(\lambda r) d\nu_\alpha(r) \\ &= 2 \sum_{k=0}^\infty \frac{(-1)^k}{k!\,\Gamma(\alpha+k+1)} \left(\frac{\lambda^2}{4}\right)^k \int_0^\infty e^{-r^2} r^{2\alpha+2k+1}\, dr \\ &= \sum_{k=0}^\infty \frac{(-1)^k}{k!\,\Gamma(\alpha+k+1)} \left(\frac{\lambda^2}{4}\right)^k \int_0^\infty e^{-t} t^{\alpha+k}\, dt \\ &= \sum_{k=0}^\infty \frac{(-1)^k}{k!} \left(\frac{\lambda^2}{4}\right)^k \\ &= e^{-\frac{\lambda^2}{4}}. \end{aligned}$$

Thus

$$\left\|\mathscr{H}_\alpha(f)\right\|_{\infty,\nu_\alpha} = 1 = \left\|f\right\|_{1,\nu_\alpha}$$

Hense, we obtain

$$\|\mathscr{H}_\alpha\| = 1.$$

2. Let $f, g \in L^1(d\nu_\alpha)$. Using Fubini–Tonelli's theorem, we get

$$\begin{aligned}\int_0^\infty \mathscr{H}_\alpha(f)(\lambda)\, g(\lambda)\, d\nu_\alpha(\lambda) &= \int_0^\infty \left(\int_0^\infty f(r)\, j_\alpha(\lambda r)\, d\nu_\alpha(r)\right) g(\lambda)\, d\nu_\alpha(\lambda)\\ &= \int_0^\infty f(r)\left(\int_0^\infty g(\lambda)\, j_\alpha(\lambda r)\, d\nu_\alpha(\lambda)\right) d\nu_\alpha(r)\\ &= \int_0^\infty f(r)\, \mathscr{H}_\alpha(g)(r)\, d\nu_\alpha(r).\end{aligned}$$

■

Proposition 1.2.2. 1. *For every $f \in L^1(d\nu_\alpha)$ and $r \in [0, +\infty[$, we have*

$$\mathscr{H}_\alpha(\tau_r^\alpha(f))(\lambda) = j_\alpha(\lambda r)\, \mathscr{H}_\alpha(f)(\lambda), \quad \forall \lambda \in \mathbb{R}. \tag{1.14}$$

2. *For every $f \in L^1(d\nu_\alpha)$, we have*

$$\mathscr{H}_\alpha(f * g)(\lambda) = \mathscr{H}_\alpha(f)(\lambda) \cdot \mathscr{H}_\alpha(g)(\lambda), \quad \forall \lambda \in \mathbb{R}. \tag{1.15}$$

Proof. 1. Let $f \in L^1(d\nu_\alpha)$. Then, from Proposition 1.1.4, the function $\tau_r^\alpha(f) \in L^1(d\nu_\alpha)$. By using Fubini–Tonelli's theorem and relation (1.10), we have for every $\lambda \in \mathbb{R}$:

$$\begin{aligned}
\mathscr{H}_\alpha(\tau_r^\alpha(f))(\lambda) &= \int_0^\infty \tau_r^\alpha(f)(s)\, j_\alpha(\lambda s)\, d\nu_\alpha(s) \\
&= \int_0^\infty \left(\int_0^\infty f(u)\, \omega_\alpha(u,r,s)\, d\nu_\alpha(u)\right) j_\alpha(\lambda s)\, d\nu_\alpha(s) \\
&= \int_0^\infty f(u) \left(\int_0^\infty j_\alpha(\lambda s)\, \omega_\alpha(u,r,s)\, d\nu_\alpha(s)\right) d\nu_\alpha(u) \\
&= \int_0^\infty f(u)\, \tau_r^\alpha(j_\alpha(\lambda \cdot))(u)\, d\nu_\alpha(u) \\
&= \int_0^\infty f(u)\, j_\alpha(\lambda r)\, j_\alpha(\lambda u)\, d\nu_\alpha(u) \\
&= j_\alpha(\lambda r)\, \mathscr{H}_\alpha(f)(\lambda).
\end{aligned}$$

2. From Theorem 1.1.2, for every $f, g \in L^1(d\nu_\alpha)$, the function $f * g \in L^1(d\nu_\alpha)$, and by using Fubini–Tonelli's theorem, we obtain

$$\begin{aligned}
\mathscr{H}_\alpha(f * g)(\lambda) &= \int_0^\infty (f * g)(r)\, j_\alpha(\lambda r)\, d\nu_\alpha(r) \\
&= \int_0^\infty \left(\int_0^\infty \tau_r^\alpha(f)(s)\, g(s)\, d\nu_\alpha(s)\right) j_\alpha(\lambda r)\, d\nu_\alpha(r) \\
&= \int_0^\infty g(s) \left(\int_0^\infty \tau_s^\alpha(f)(r)\, j_\alpha(\lambda r)\, d\nu_\alpha(r)\right) d\nu_\alpha(s) \\
&= \int_0^\infty g(s)\, \mathscr{H}_\alpha(\tau_s^\alpha(f))(\lambda)\, d\nu_\alpha(s) \\
&= \int_0^\infty g(s)\, j_\alpha(\lambda s)\, \mathscr{H}_\alpha(f)(\lambda)\, d\nu_\alpha(s) \\
&= \mathscr{H}_\alpha(f)(\lambda) \mathscr{H}_\alpha(g)(\lambda).
\end{aligned}$$

■

Theorem 1.2.1. *1. (Inversion formula for the Hankel transform)*

Let $f \in L^1(d\nu_\alpha)$ such that $\mathscr{H}_\alpha(f) \in L^1(d\nu_\alpha)$, then we have

$$f(r) = \int_0^{+\infty} \mathscr{H}_\alpha(f)(\lambda)\, j_\alpha(\lambda r)\, d\nu_\alpha(\lambda) = \mathscr{H}_\alpha(\mathscr{H}_\alpha(f))(r) \quad a.e. \tag{1.16}$$

2. (Plancherel's formula)

The Hankel transform $\mathscr{H}_\beta$ can be extended to an isometric isomorphism from $L^2(d\nu_\beta)$ onto itself, and we have

$$\|\mathscr{H}_\alpha(f)\|_{2;\nu_\alpha} = \|f\|_{2;\nu_\alpha}. \tag{1.17}$$

3. (Parseval's formula)

For all $f, g \in L^2(d\nu_\beta)$, we have

$$\int_0^{+\infty} f(r)\overline{g(r)}\, d\nu_\alpha(r) = \int_0^{+\infty} \mathscr{H}_\alpha(f)(\lambda)\overline{\mathscr{H}_\alpha(g)(\lambda)}\, d\nu_\alpha(\lambda). \tag{1.18}$$

Proposition 1.2.3. *1. For every $f \in L^2(d\nu_\alpha)$ and $r \in [0, +\infty[$, we have*

$$\mathscr{H}_\alpha\left(\tau_r^\alpha(f)\right)(\lambda) = j_\alpha(\lambda r)\mathscr{H}_\alpha(f)(\lambda), \quad \forall \lambda \in \mathbb{R} \tag{1.19}$$

2. *For every $f \in L^1(d\nu_\alpha)$ and $g \in L^2(d\nu_\alpha)$, the function $f * g$ belongs to $L^2(d\nu_\alpha)$ and we have*

$$\mathscr{H}_\alpha(f * g) = \mathscr{H}_\alpha(f)\mathscr{H}_\alpha(g) \tag{1.20}$$

3. *Let $f, g \in L^2(d\nu_\alpha)$. Then $f * g \in L^2(d\nu_\alpha)$, if and only if $\mathscr{H}_\alpha(f)\mathscr{H}_\alpha(g) \in L^2(d\nu_\alpha)$ and we have*

$$\mathscr{H}_\alpha(f * g) = \mathscr{H}_\alpha(f)\mathscr{H}_\alpha(g),$$

Moreover,

$$\int_0^{+\infty} |f * g(r)|^2 \, d\nu_\alpha(r) = \int_0^{+\infty} |\mathscr{H}_\alpha(f)(\lambda)|^2 |\mathscr{H}_\alpha(g)(\lambda)|^2 \, d\nu_\alpha(\lambda). \tag{1.21}$$

where both integrals are finite or infinite.

Remark 1.2.1. *For every $f, g \in L^2(d\nu_\alpha)$ and $r \in [0, +\infty[$, we have*

$$\tau_r^\alpha(f * g) = \tau_r^\alpha(f) * g = f * \tau_r^\alpha(g) \tag{1.22}$$

Chapter 2

Hankel Wavelet transform

The Wavelet Transform is a mathematical technique used to analyze signals at multiple scales (or resolutions). Unlike the Fourier Transform, which only gives frequency information, the Wavelet Transform provides both time and frequency localization. This makes it especially useful for analyzing signals that change over time.

In wavelet transform we used different window size for different frequency components. Low scale (small window size or small time scale) is used for higher frequencies and higher scale (large window size or large time scale) is used for low frequencies.

Our investigation in this chapter is to define and study the Hankel wavelet transform and we establish several basic properties for this transform. We also prove that $T^{\alpha}_{\psi}(L^2(d\nu_{\alpha}))$ is a reproducing kernel Hilbert space with kernel function defined by

$$k_{\psi}\left((a,r);(a',r')\right) = \frac{1}{C_{\psi}} T^{\alpha}_{\psi}(\psi_{a',r'})(a,r), \quad (a,r),(a',r') \in \mathbb{R}^*_+ \times \mathbb{R}_+;$$

where $\psi^{\alpha}_{a',r'}$ is the family given by relation (2.8), and C_{ψ} is the admissibility condition for the Hankel wavelet transform given by (2.10).

In the following, we denote by

- μ_α the measure defined on $\mathbb{R}_+^* \times \mathbb{R}_+$ by

$$d\mu_\alpha(a,r) = d\nu_\alpha(a)\, d\nu_\alpha(r). \tag{2.1}$$

- $L^p(d\mu_\alpha)$, $1 \leq p \leq +\infty$, the Lebesgue space on $\mathbb{R}_+^* \times \mathbb{R}_+$, with respect to the measure μ_α with the L^p-norm denoted by $\|\cdot\|_{p,\mu_\alpha}$.

- $\langle \cdot \mid \cdot \rangle_{\mu_\alpha}$ be the inner product on $L^2(d\mu_\alpha)$ defined by

$$\langle f \mid g \rangle_{\mu_\alpha} = \int_0^{+\infty} \int_0^{+\infty} f(a,r)\, \overline{g(a,r)}\, d\mu_\alpha(a,r)$$

2.1 Dilation operator

For every $a \in \mathbb{R}_+^*$, the dilation operator D_a^α is defined for every measurable function ψ on $\mathbb{R}_+$ by

$$D_a^\alpha(\psi)(r) = a^{\alpha+1}\psi(ar), \quad \forall r \in [0,+\infty[.$$

Then, we have the following properties:

Properties 2.1.1. 1. *For every* $\psi \in L^2(d\nu_\alpha)$*,*

$$\|D_a^\alpha(\psi)\|_{2,\nu_\alpha} = \|\psi\|_{2,\nu_\alpha}. \tag{2.2}$$

2. *For all* $\psi, \phi \in L^2(d\nu_\alpha)$*,*

$$\langle D_a^\alpha(\psi) \mid \phi\rangle_{\nu_\alpha} = \langle \psi \mid D_{\frac{1}{a}}^\alpha(\phi)\rangle_{\nu_\alpha}. \tag{2.3}$$

3. *For every* $\psi \in L^2(d\nu_\alpha)$*,*

$$|D_a^\alpha(\psi)|^2 = a^{\alpha+1} D_a^\alpha |\psi|^2, \tag{2.4}$$

and

$$\sqrt{D_a^\alpha(|\psi|)} = a^{-\alpha+1} D_a^\alpha\left(\sqrt{|\psi|}\right). \tag{2.5}$$

4. *For every* $\psi \in L^2(d\nu_\alpha)$*,*

$$\tau_r^\alpha D_a^\alpha(\psi) = D_a^\alpha \tau_{ar}^\alpha(\psi). \tag{2.6}$$

5. *For every* $\psi \in L^2(d\nu_\alpha)$*,*

$$\mathscr{H}_\alpha(D_a^\alpha(\psi)) = D_{\frac{1}{a}}^\alpha(\mathscr{H}_\alpha(\psi)). \tag{2.7}$$

Proof. 1. For every ψ in $L^2(d\nu_\alpha)$, we have

$$\begin{aligned}
\|D_a^\alpha(\psi)\|_{2,\nu_\alpha}^2 &= \int_0^{+\infty} |D_a^\alpha(\psi)(r)|^2 \, d\nu_\alpha(r) \\
&= a^{2\alpha+2} \int_0^{+\infty} |\psi(ar)|^2 \, d\nu_\alpha(r) \\
&= \int_0^{+\infty} |\psi(s)|^2 \, d\nu_\alpha(s) \\
&= \|\psi\|_{2,\nu_\alpha}^2 .
\end{aligned}$$

2. For every ψ, ϕ in $L^2(d\nu_\alpha)$, we get

$$\begin{aligned}
\left\langle D_a^\alpha(\psi) \mid \phi \right\rangle_{v_\alpha} &= \int_0^{+\infty} D_a^\alpha(\psi)(r)\overline{\phi(r)} \, dv_\alpha(r) \\
&= a^{\alpha+1} \int_0^{+\infty} \psi(ar)\overline{\phi(r)} \, dv_\alpha(r) \\
&= \int_0^{+\infty} \psi(s)\overline{\left(\frac{1}{a^{\alpha+1}}\phi\left(\frac{s}{a}\right)\right)} \, dv_\alpha(s) \\
&= \left\langle \psi \mid D_{\frac{1}{a}}^\alpha(\phi) \right\rangle_{v_\alpha}
\end{aligned}$$

3. For every ψ in $L^2(d\nu_\alpha)$, we obtain

$$\begin{aligned}
|D_a^\alpha(\psi)(r)|^2 &= |a^{\alpha+1}\psi(ar)|^2 \\
&= a^{2\alpha+2}|\psi(ar)|^2 \\
&= a^{\alpha+1} D_a^\alpha(|\psi|^2)(r),
\end{aligned}$$

and

$$\begin{aligned}
\sqrt{D_a^\alpha(|\psi|)(r)} &= \sqrt{a^{\alpha+1}|\psi(ar)|} \\
&= a^{\frac{\alpha+1}{2}} \sqrt{|\psi(ar)|} \\
&= a^{-\frac{\alpha+1}{2}} D_a^\alpha\left(\sqrt{|\psi|}\right)(r).
\end{aligned}$$

4. Let $\psi \in L^2(d\nu_\alpha)$. So, by Definition (1.1.1), we have

$$\begin{aligned}
\tau_r^\alpha(D_a^\alpha(\psi))(s) &= \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma\left(\alpha+\frac{1}{2}\right)}\int_0^\pi D_a^\alpha(\psi)\left(\sqrt{r^2+s^2+2rs\cos\theta}\right)(\sin\theta)^{2\alpha}\,d\theta \\
&= a^{\alpha+1}\frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma\left(\alpha+\frac{1}{2}\right)}\int_0^\pi \psi\left(\sqrt{(ar)^2+(as)^2+2(ar)(as)\cos\theta}\right)(\sin\theta)^{2\alpha}\,d\theta \\
&= a^{\alpha+1}\tau_{ar}^\alpha(\psi)(as) \\
&= D_a^\alpha\left(\tau_{ar}^\alpha(\psi)\right)(s).
\end{aligned}$$

5. Let $\psi \in L^2(d\nu_\alpha)$, then

$$\begin{aligned}
\mathscr{H}_\alpha(D_a^\alpha(\psi))(\lambda) &= \int_0^{+\infty} D_a^\alpha(\psi)(r)j_\alpha(\lambda r)\,d\nu_\alpha(r) \\
&= a^{\alpha+1}\int_0^{+\infty}\psi(ar)j_\alpha(\lambda r)\,d\nu_\alpha(r) \\
&= \frac{1}{a^{\alpha+1}}\int_0^{+\infty}\psi(s)j_\alpha\left(\frac{\lambda}{a}s\right)d\nu_\alpha(s) \\
&= \frac{1}{a^{\alpha+1}}\mathscr{H}_\alpha(\psi)\left(\frac{\lambda}{a}\right) \\
&= D_{\frac{1}{a}}^\alpha(\mathscr{H}_\alpha(\psi))(\lambda).
\end{aligned}$$

■

2.2 The Continuous Hankel Wavelet transform

The main aim of this part is to define the Hankel wavelet transform T_ψ^α and to prove a Plancherel's formula and a reconstruction formula for this transform. We also prove that the function $T_\psi^\alpha(f)$ belongs to $L^p(d\mu_\alpha)$, $p \in [2,+\infty]$ for every $f \in L^2(d\nu_\alpha)$.

Definition 2.2.1. *For every $\psi \in L^2(d\nu_\alpha)$, the family $\psi_{a,r}^\alpha$, $(a,r) \in \mathbb{R}_+^* \times \mathbb{R}_+$, is defined by*

$$\psi_{a,r}^\alpha(s) = \tau_r^\alpha\left(D_a^\alpha(\psi)\right)(s). \tag{2.8}$$

By relations (1.11) and (2.2), we have

$$\|\psi^{\alpha}_{a,r}\|_{2,\nu_\alpha} \leq \|\psi\|_{2,\nu_\alpha}. \tag{2.9}$$

Definition 2.2.2. *A nonzero function $\psi \in L^2(d\nu_\alpha)$ is said to be an admissible Hankel wavelet if*

$$0 < C_\psi = c_\alpha \int_0^{+\infty} |\mathscr{H}_\alpha(\psi)(a)|^2 \frac{da}{a} < +\infty. \tag{2.10}$$

where

$$c_\alpha = \frac{1}{2^\alpha \Gamma(\alpha+1)}.$$

Definition 2.2.3. *Let ψ be an admissible Hankel wavelet. The continuous Hankel wavelet transform T^α_ψ is defined in $L^2(d\mu_\alpha)$ by*

$$T^\alpha_\psi(f)(a,r) = \int_0^{+\infty} f(s)\, \overline{\psi^\alpha_{a,r}(s)}\, d\mu_\alpha(s); \quad (a,r) \in \mathbb{R}^*_+ \times \mathbb{R}_+,$$

where $\psi^\alpha_{a,r}$ is given by relation (2.8).

The continuous Hankel wavelet transform can also be written as

$$T^\alpha_\psi(f)(a,r) = f \star D^\alpha_a(\overline{\psi})(r) \tag{2.11}$$

$$= \langle f \mid \psi^\alpha_{a,r} \rangle_{\mu_\alpha} \tag{2.12}$$

Proposition 2.2.1. *Let ψ be an admissible Hankel wavelet. Then, the continuous Hankel wavelet transform T_ψ^α is a bounded linear operator from $L^2(d\nu_\alpha)$ onto $L^\infty(d\mu_\alpha)$ and we have*

$$||T_\psi^\alpha(f)||_{\infty,\mu_\alpha} \leq ||f||_{2,\nu_\alpha}||\psi||_{2,\nu_\alpha}. \tag{2.13}$$

Proof. Let $\psi \in L^2(d\nu_\alpha)$ be an admissible Hankel wavelet. Then, from Cauchy-Schwarz's inequality and relations (2.9), (2.12) , we obtain

$$\begin{aligned} |T_\psi^\alpha(f)(a,r)| &= |\langle f|\psi_{a,r}^\alpha\rangle_{\nu_\alpha}| \\ &\leq ||\psi_{a,r}^\alpha||_{2,\nu_\alpha}||f||_{2,\nu_\alpha} \\ &\leq ||\psi||_{2,\nu_\alpha}||f||_{2,\nu_\alpha}. \end{aligned}$$

Then

$$||T_\psi^\alpha(f)||_{\infty,\mu_\alpha} \leq ||\psi||_{2,\nu_\alpha}||f||_{2,\nu_\alpha}.$$

∎

The Hankel wavelet transform T_ψ^α satisfies the following properties:

Theorem 2.2.1. *(Plancherel's formula) Let ψ be an admissible window function in $L^2(d\nu_\alpha)$, then we have*

$$\int_0^{+\infty}\int_0^{+\infty} |T_\psi^\alpha(f)(a,r)|^2\, d\mu_\alpha(a,r) = C_\psi \int_0^{+\infty} |f(r)|^2\, d\nu_\alpha(r) \tag{2.14}$$

Proof. From relations (1.21), (2.11) and using Fubini-Tonelli's theorem, we get

$$\begin{aligned}\int_0^{+\infty}\int_0^{+\infty} & |T^\alpha_\psi(f)(a,r)|^2\,d\mu_\alpha(a,r)\\ &= \int_0^{+\infty}\int_0^{+\infty} |f * D^\alpha_a(\psi)(r)|^2\,d\nu_\alpha(a)\,d\nu_\alpha(r)\\ &= \int_0^{+\infty}\int_0^{+\infty} |\mathscr{H}_\alpha(f)(\lambda)|^2|\mathscr{H}_\alpha(D^\alpha_a(\psi))(\lambda)|^2\,d\nu_\alpha(a)\,d\nu_\alpha(\lambda)\\ &= \int_0^{+\infty} |\mathscr{H}_\alpha(f)(\lambda)|^2\left(\int_0^{+\infty} |\mathscr{H}_\alpha(D^\alpha_a(\psi))(\lambda)|^2\,d\nu_\alpha(a)\right)d\nu_\alpha(\lambda). \quad (2.15)\end{aligned}$$

Now, using relations (2.4) and (2.7), we get

$$\begin{aligned}\int_0^{+\infty}\left|\mathscr{H}_\alpha\big(D^\alpha_a(\psi)\big)(\lambda)\right|^2 d\nu_\alpha(a) &= \int_0^{+\infty}\left|D^\alpha_{\frac{1}{a}}(\mathscr{H}_\alpha(\psi))(\lambda)\right|^2 d\nu_\alpha(a)\\ &= \int_0^{+\infty}\frac{1}{a^{\alpha+1}}D^\alpha_{\frac{1}{a}}\big(|\mathscr{H}_\alpha(\psi)|^2\big)(\lambda)d\nu_\alpha(a)\\ &= \int_0^{+\infty}\frac{1}{a^{2\alpha+2}}\left|\mathscr{H}_\alpha(\psi)\right|^2\left(\frac{\lambda}{a}\right)d\nu_\alpha(a)\\ &= c_\alpha\int_0^{+\infty}\left|\mathscr{H}_\alpha(\psi)\right|^2(a)\frac{da}{a}.\end{aligned}$$

Then, we get

$$\int_0^{+\infty}\left|\mathscr{H}_\alpha\big(D^\alpha_a(\psi)\big)(\lambda)\right|^2 d\nu_\alpha(a) = C_\psi. \quad (2.16)$$

Then, from Plancherel's formula for the Hankel transform $\mathcal{H}_\alpha$ and by combining relations (2.15) and (1.16), we obtain

$$\begin{aligned}\|T^\alpha_\psi(f)\|_{2,\mu_\alpha} &= \sqrt{C_\psi}\,\|\mathscr{H}_\alpha(f)\|_{2,\nu_\alpha}\\ &= \sqrt{C_\psi}\,\|f\|_{2,\nu_\alpha}.\end{aligned}$$

■

Corollary 2.2.1. *(Parseval's formula) Let ψ be an admissible Hankel wavelet in $L^2(d\nu_\alpha)$. Then, for all f and g in $L^2(d\nu_\alpha)$, we have*

$$\int_0^{+\infty} f(r)\overline{g(r)}\,d\nu_\alpha(r) = \frac{1}{C_\psi}\int_0^{+\infty}\int_0^{+\infty} T_\psi^\alpha(f)(a,r)\overline{T_\psi^\alpha(g)(a,r)}\,d\mu_\alpha(a,r). \quad (2.17)$$

Proof. Using Polarization identity and Plancherel's formula for the Hankel wavelet transform (2.14), we have

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty} T_\psi^\alpha(f)(a,r)\overline{T_\psi^\alpha(g)(a,r)}\,d\mu_\alpha(a,r) = \langle T_\psi^\alpha(f)|T_\psi^\alpha(g)\rangle_{\mu_\alpha}\\
=\ & \frac{1}{4}\left(\|T_\psi^\alpha(f)+T_\psi^\alpha(g)\|_{2,\mu_\alpha}^2 - \|T_\psi^\alpha(f)-T_\psi^\alpha(g)\|_{2,\mu_\alpha}^2 + \|T_\psi^\alpha(f)+iT_\psi^\alpha(g)\|_{2,\mu_\alpha}^2 - \|T_\psi^\alpha(f)-iT_\psi^\alpha(g)\|_{2,\mu_\alpha}^2\right)\\
=\ & \frac{1}{4}\left(\|T_\psi^\alpha(f+g)\|_{2,\mu_\alpha}^2 - \|T_\psi^\alpha(f-g)\|_{2,\mu_\alpha}^2 + \|T_\psi^\alpha(f+ig)\|_{2,\mu_\alpha}^2 - \|T_\psi^\alpha(f-ig)\|_{2,\mu_\alpha}^2\right)\\
=\ & C_\psi\left(\frac{1}{4}\left(\|f+g\|_{2,\nu_\alpha}^2 - \|f-g\|_{2,\nu_\alpha}^2 + \|f+ig\|_{2,\nu_\alpha}^2 - \|f-ig\|_{2,\nu_\alpha}^2\right)\right)\\
=\ & C_\psi\langle f|g\rangle_{\nu_\alpha}\\
=\ & C_\psi\int_0^{+\infty} f(r)\overline{g(r)}\,d\nu_\alpha(r)
\end{aligned}$$

■

Theorem 2.2.2. *(Reconstruction Formula) Let ψ be an admissible Hankel wavelet in $L^2(d\nu_\alpha)$ such that $|\psi|$ is an admissible window function. Then, for every $f \in L^2(d\nu_\alpha)$, we have*

$$f(\cdot) = \frac{1}{C_\psi}\int_0^{+\infty}\int_0^{+\infty} T_\psi^\alpha(f)(a,r)\,\psi_{a,r}^\alpha(\cdot)\,d\mu_\alpha(a,r),$$

weakly in $L^2(d\nu_\alpha)$.

Proof. From Corollary 2.2.1 and Fubini-Tonelli's theorem, we have for all g in

$L^2(d\nu_\alpha)$

$$\begin{aligned}
\langle f|g\rangle_{\nu_\alpha} &= \int_0^{+\infty} f(s)\overline{g(s)}\,d\nu_\alpha(s) \\
&= \frac{1}{C_\psi}\int_0^{+\infty}\int_0^{+\infty} T_\psi^\alpha(f)(a,r)\overline{T_\psi^\alpha(g)(a,r)}\,d\mu_\alpha(a,r) \\
&= \frac{1}{C_\psi}\int_0^{+\infty}\int_0^{+\infty} T_\psi^\alpha(f)(a,r)\,\langle \psi_{a,r}^\alpha|g\rangle_{\nu_\alpha}\,d\mu_\alpha(a,r).
\end{aligned}$$

which gives the result. ■

Remark 2.2.1. *Using the fact that $T_\psi^\alpha(f)$ belongs to $L^2(\mu_\alpha)$ and for almost $a \in \mathbb{R}_+^*$, the function $r \longmapsto T_\psi^\alpha(f)(a,r) = f * D_a^\alpha(\overline{\psi})(r)$ belongs to $L^2(d\nu_\alpha)$, we get from relations (1.21) and (2.11),*

$$\mathscr{H}_\alpha\Big(T_\psi^\alpha(f)(a,.)\Big)(\lambda) = \frac{1}{a^{\alpha+1}}\mathscr{H}_\alpha(f)(\lambda)\mathscr{H}_\alpha(\overline{\psi})\Big(\frac{\lambda}{a}\Big). \tag{2.18}$$

Proposition 2.2.2. *Let ψ be an admissible Hankel wavelet in $L^2(d\nu_\alpha)$. For every function f in $L^2(d\nu_\alpha)$, we have*

1. *For $r_0 \geq 0$,*

$$T_\psi^\alpha\left(\tau_{r_0}^\alpha(f)\right)(a,r) = \tau_{r_0}^\alpha\left(T_\psi^\alpha(f)(a,\cdot)\right)(r), \quad (a,r) \in \mathbb{R}_+^* \times \mathbb{R}_+. \tag{2.19}$$

2. *For $\lambda > 0$, we have*

$$T_\psi^\alpha(D_\lambda^\alpha(f)(a,r) = T_\psi^\alpha(f)\left(\frac{a}{\lambda},\lambda r\right), \quad (a,r) \in \mathbb{R}_+^* \times \mathbb{R}_+. \tag{2.20}$$

Proof. 1. From relations (1.22) and (2.11), we have

$$\begin{aligned} T_\psi^\alpha(\tau_r^\alpha(f))(a,r) &= \tau_{r_0}^\alpha(f) * D_a^\alpha(\bar{\psi})(r) \\ &= \tau_{r_0}^\alpha\left(f * D_a^\alpha(\bar{\psi})\right)(r) \\ &= \tau_{r_0}^\alpha(T_\psi^\alpha(f)(a,\cdot))(r) \end{aligned}$$

2. Using relations (2.3), (2.6), and (2.12), we have

$$\begin{aligned} T_\psi^\alpha(D_\lambda^\alpha f)(a,r) &= \langle D_\lambda^\alpha(f) \mid \tau_r^\alpha D_a^\alpha(\psi)\rangle_{\mu_\alpha} \\ &= \langle f \mid D_{\frac{1}{\lambda}}^\alpha \tau_r^\alpha D_a^\alpha(\psi)\rangle_{\mu_\alpha} \\ &= \langle f \mid \tau_{\lambda r}^\alpha D_{\frac{a}{\lambda}}^\alpha(\psi)\rangle_{\mu_\alpha} \\ &= T_\psi^\alpha(f)\left(\frac{a}{\lambda}, \lambda r\right), \quad (a,r) \in \mathbb{R}_+^* \times \mathbb{R}_+. \end{aligned}$$

■

Theorem 2.2.3. *Let ψ be an admissible Hankel wavelet in $L^2(d\nu_\alpha)$. For every $f \in L^2(d\nu_\alpha)$, the function $T_\psi^\alpha(f)$ belongs to $L^p(d\mu_\alpha)$, $p \in [2,+\infty]$ and we have*

$$\|T_\psi^\alpha(f)\|_{p,\mu_\alpha} \le C_\psi^{\frac{1}{p}} \|f\|_{2,\nu_\alpha} \|\psi\|_{2,\nu_\alpha}^{1-\frac{2}{p}}.$$

Proof. For $p = 2$. The Plancherel's formula for the Hankel wavelet transform (2.14) gives

$$\|T_\psi^\alpha(f)\|_{2,\mu_\alpha} = C_\psi^{\frac{1}{2}} \|f\|_{2,\nu_\alpha}.$$

For $p = +\infty$ and by relation (2.13), we have

$$\|T_\psi^\alpha(f)\|_{\infty,\mu_\alpha} \le \|f\|_{2,\nu_\alpha} \|\psi\|_{2,\nu_\alpha}.$$

From Riesz-Thorin's interpolation Theorem [36], we get for every $p \in [2, +\infty]$

$$\begin{aligned} \|T^{\alpha}_{\psi}(f)\|_{p,\mu_\alpha} &\leq \|T^{\alpha}_{\psi}(f)\|^{1-\frac{2}{p}}_{\infty,\mu_\alpha}\|T^{\alpha}_{\psi}(f)\|^{\frac{2}{p}}_{2,\mu_\alpha} \\ &\leq C^{\frac{1}{p}}_{\psi}\|f\|_{2,\nu_\alpha}\|\psi\|^{1-\frac{2}{p}}_{2,\nu_\alpha}. \end{aligned}$$

■

2.3 Reproducing kernel Hilbert space $T^{\alpha}_{\psi}(L^2(d\nu_\alpha))$

In this section, we prove that $T^{\alpha}_{\psi}(L^2(d\nu_\alpha))$ is a reproducing kernel Hilbert space with kernel function defined by (2.21) [32].

Definition 2.3.1. *(Reproducing kernel)*
Let H be a Hilbert space of functions defined from an arbitrary set X into $\mathbb{C}$ issued with the inner product $\langle\cdot \mid \cdot\rangle_H$. Let k be a function defined from $X \times X$ into $\mathbb{C}$, we say that k is a reproducing kernel for H, if

1. *For every $y \in X$, the function $x \mapsto k(x, y) \in H$.*

2. *For every $f \in H$ and for every $y \in X$, $f(y) = \langle f \mid k(\cdot, y)\rangle_H$.*

Definition 2.3.2. *(Reproducing kernel Hilbert space) A reproducing kernel Hilbert space is a Hilbert space H with a reproducing kernel whose span is dense in H.*

Proposition 2.3.1. *(Reproducing kernel) Let ψ be an admissible Hankel wavelet in $L^2(d\nu_\alpha)$ and $f \in L^2(d\nu_\alpha)$. Then, $T^\alpha_\psi(L^2(d\nu_\alpha))$ is a reproducing kernel Hilbert space with kernel function*

$$K_\psi((a',r');(a,r)) = \frac{1}{C_\psi} T^\alpha_\psi(\psi_{a',r'})(a,r) = \frac{1}{C_\psi}\langle \psi^\alpha_{a,r} \mid \psi^\alpha_{a',r'}\rangle_{\nu_\alpha} = \frac{1}{C_\psi}\int_0^{+\infty} \psi_{a',r'}(r)\,\overline{\psi_{a,r}(r)}\,d\mu_\alpha(r), = \frac{1}{C_\psi}\,\psi_{a',r'} \star \overline{D_a(\psi)(r)} \quad (2.21)$$

Moreover, the kernel k_ψ is pointwise bounded and

$$|K_\psi((a',r');(a,r))| \leq \frac{\|\psi\|^2_{2,\nu_\alpha}}{C_\psi}, \quad \forall (a',r');(a,r) \in \mathbb{R}^*_+ \times \mathbb{R}_+.$$

Proof. For $F \in T^\alpha_\psi(L^2(d\nu_\alpha))$, there exists a function $f \in L^2(d\nu_\alpha)$ such that

$$F(a,r) = T^\alpha_\psi(f)(a,r).$$

Then, from Corollary 2.2.1, we have

$$F(a,r) = T^\alpha_\psi(f)(a,r) = \langle f \mid \psi^\alpha_{a,r}\rangle_{\nu_\alpha} = \frac{1}{C_\psi}\langle T^\alpha_\psi(f) \mid T^\alpha_\psi(\psi^\alpha_{a,r})\rangle_{\mu_\alpha} = \langle T^\alpha_\psi(f) \mid \frac{1}{C_\psi} T^\alpha_\psi(\psi^\alpha_{a,r})\rangle_{\mu_\alpha} = \langle T^\alpha_\psi(f) \mid k_\psi((a,r);(\cdot,\cdot))\rangle_{\mu_\alpha}.$$

This shows that $k_\psi((a,r);(a',r')) = \frac{1}{C_\psi} T^\alpha_\psi(\psi^\alpha_{a,r})(a',r')$ is a reproducing kernel of the Hilbert space $T^\alpha_\psi(L^2(d\nu_\alpha))$.

Finally, for all $(a, r), (a', r') \in \mathbb{R}_+^* \times \mathbb{R}_+$, we have from Cauchy–Schwarz inequality and relation (2.9) that

$$|k_\psi((a, r); (a', r'))| = \frac{1}{C_\psi} \left| \langle \psi^\alpha_{a,r} \mid \psi^\alpha_{a',r'} \rangle_{\nu_\alpha} \right| \leq \frac{\|\psi\|^2_{2,\nu_\alpha}}{C_\psi}.$$

■

Chapter 3

Heisenberg-type uncertainty principle for T_ψ^α

The term uncertainty principle appeared in 1927 by Werner Heisenberg [18] play an important role in harmonic analysis. For the Hankel wavelet transform several uncertainty principles are proved.
The goal of this chapter is to prove a Heisenberg-type inequalities for this transform.

3.1 Heisenberg uncertainty principle for $\mathscr{H}_\alpha$

The Heisenberg-type uncertainty principle for the Hankel transform has been proved by Rösler and Voit [31], it states that for every function $f \in L^2(d\nu_\alpha)$

$$||rf||_{2,\nu_\alpha}||\lambda\mathscr{H}_\alpha(f)||_{2,\nu_\alpha} \geq (\alpha+1)||f||^2_{2,\nu_\alpha}. \tag{3.1}$$

In [25], Ma extended the previous inequality to a general form of the Heisenberg's uncertainty principle: for $s, t > 0$, there exists a constant $C > 0$ such that for every $f \in L^2(d\nu_\alpha)$, we have

$$||r^s f||^{\frac{t}{s+t}}_{2,\nu_\alpha}||\lambda^t\mathscr{H}_\alpha(f)||^{\frac{s}{s+t}}_{2,\nu_\alpha} \geq C||f||_{2,\nu_\alpha}. \tag{3.2}$$

Later, in his paper [35], Soltani gave explicitly the constant C in the case $s \geq 1$ and $t \geq 1$. More precisely, he established the following theorem (a similar result had been first given by Rassias [30] for the classical Fourier transform).

Theorem 3.1.1. *Assume $s \geq 1$ and $t \geq 1$, then for every function $f \in L^2(d\nu_\alpha)$,*

$$\|r^s f\|_{2,\nu_\alpha}^{\frac{t}{s+t}} \|\lambda^t \mathscr{H}_\alpha(f)\|_{2,\nu_\alpha}^{\frac{s}{s+t}} \geq (\alpha+1)^{\frac{st}{s+t}} \|f\|_{2,\nu_\alpha}, \tag{3.3}$$

with equality if and only if $s = t = 1$ and $f(r) = de^{-br^2/2}$ for some $d \in \mathbb{C}$ and $b > 0$.

3.2 Heisenberg uncertainty principle for T_ψ^α

In [39], the author has established Heisenberg-type inequalities for the usual wavelet transform. These results are inspired from [34]. With a more general setting, we investigate in the following similar Heisenberg-type inequalities for the Hankel wavelet transform.

Proposition 3.2.1. *Assume $s, t > 0$ and let ψ be an admissible Hankel wavelet in $L^2(d\nu_\alpha)$. Then, there exists a constant $C = C(\alpha, s, t) > 0$, such that for every function $f \in L^2(d\nu_\alpha)$.*

$$\|r^s T_\psi^\alpha(f)\|_{2,\mu_\alpha}^{\frac{t}{s+t}} \cdot \|\lambda^t \mathscr{H}_\alpha(f)\|_{2,\nu_\alpha}^{\frac{s}{s+t}} \geq C\left(\sqrt{C_\psi}\right)^{\frac{t}{s+t}} \|f\|_{2,\nu_\alpha}, \tag{3.4}$$

In particular, if $s, t \geq 1$ the constant C is given by $C = (\alpha+1)^{\frac{st}{s+t}}$.

Proof. Let us assume the non-trivial case that both integrals on the left-hand side of (3.4) are finite. Applying the Heisenberg-type inequality (3.2) for the Hankel transform to the function $r \mapsto T_\psi(f)(a, r)$, we get for all $a \in \mathbb{R}_+^*$,

$$\|r^s T_\psi^\alpha(f)\|_{2,\nu_\alpha}^{\frac{t}{s+t}} \|\lambda^t \mathscr{H}_\alpha(T_\psi^\alpha(f))\|_{2,\nu_\alpha}^{\frac{s}{s+t}} \geq C\|T_\psi^\alpha(f)\|_{2,\nu_\alpha}.$$

So

$$\left(\int_0^{+\infty} r^{2s}\left|T^{\alpha}_{\psi}(f)(a,r)\right|^2 d\nu_\alpha(r)\right)^{\frac{t}{s+t}}\left(\int_0^{+\infty}\lambda^{2t}\left|\mathscr{H}_\alpha\left(T^{\alpha}_{\psi}(f)(a,\cdot)\right)(\lambda)\right|^2 d\nu_\alpha(\lambda)\right)^{\frac{s}{s+t}} \geq C^2\int_0^{+\infty}\left|T^{\alpha}_{\psi}(f)(a,r)\right|^2 d\nu_\alpha(r). \quad (3.5)$$

Thus, integrating the relation (3.5) over a, and applying Hölder's inequality and Plancherel's theorem for the Hankel wavelet transform given by the relation (2.14), we obtain

$$\begin{aligned}&\left(\int_0^{+\infty}\int_0^{+\infty} r^{2s}\left|T^{\alpha}_{\psi}(f)(a,r)\right|^2 d\nu_\alpha(a)d\nu_\alpha(r)\right)^{\frac{t}{s+t}}\left(\int_0^{+\infty}\int_0^{+\infty}\lambda^{2t}\left|\mathscr{H}_\alpha\left(T^{\alpha}_{\psi}(f)(a,\cdot)\right)(\lambda)\right|^2 d\nu_\alpha(a)d\nu_\alpha(\lambda)\right)^{\frac{s}{s+t}}\\&\geq C^2\int_0^{+\infty}\int_0^{+\infty}\left|T^{\alpha}_{\psi}(f)(a,r)\right|^2 d\nu_\alpha(a)d\nu_\alpha(r)\\&= C^2\|T^{\alpha}_{\psi}(f)\|^2_{2,\mu_\alpha} = C^2 C_\psi\|f\|^2_{2,\nu_\alpha}. \qquad (3.6)\end{aligned}$$

On the other hand, by the relation (2.18) and the admissibility condition (2.10), we get

$$\begin{aligned}&\int_0^{+\infty}\int_0^{+\infty}\lambda^{2s}\left|\mathscr{H}_\alpha\left(T^{\alpha}_{\psi}(f)(a,\cdot)\right)(\lambda)\right|^2 d\nu_\alpha(a)d\nu_\alpha(\lambda)\\ =& \int_0^{+\infty}\lambda^{2s}\left|\mathscr{H}_\alpha(f)(\lambda)\right|^2\left(c_\alpha\int_0^{+\infty}\left|\mathscr{H}_\alpha(\psi)\left(\tfrac{\lambda}{a}\right)\right|^2\frac{da}{a}\right)d\nu_\alpha(\lambda)\\ =& C_\psi\|\lambda^s\mathscr{H}_\alpha(f)\|^2_{2,\nu_\alpha}. \qquad (3.7)\end{aligned}$$

Then, the result follows by replacing the last equality (3.7) into (3.6).

For the case $s,t\geq 1$, we apply the Heisenberg-type uncertainty principle given by relation (3.3) to the function $r\mapsto T^{\alpha}_{\psi}(f)(a,r)$, and the remainder of the proof is the same. ■

Proposition 3.2.2. *Assume $s,t>0$ and let ψ be an admissible Hankel wavelet in $L^2(d\nu_\alpha)$. Then, there exists a constant $C=C(\alpha,s,t)$ such that for every function $f\in L^2(d\nu_\alpha)$*

$$\|r^s f\|_{2,\nu_\alpha}^{\frac{t}{s+t}} \quad \|a^t T_\psi^\alpha(f)\|_{2,\mu_\alpha}^{\frac{s}{s+t}} \geq C\left(\sqrt{c_\alpha \mathcal{M}\left(|\mathscr{H}_\alpha(\psi)|^2\right)(2t)}\right)^{\frac{s}{s+t}} \|f\|_{2,\nu_\alpha}, \tag{3.8}$$

where $\mathcal{M}$ denotes the classical Mellin transform defined by

$$\mathcal{M}(f)(t)=\int_0^{+\infty}\frac{f(x)}{x^{t+1}}\,dx.$$

In particular, if $s,t\geq 1$, then $C=(\alpha+1)^{\frac{st}{s+t}}$ and we have equality if and only if $s=t=1$ and $f(r)=de^{-br^2/2}$ for some $d\in\mathbb{C}$ and $b>0$.

Proof. Let us assume the non-trivial case that both integrals on the left-hand side of (3.8) are finite.

Using Fubini's theorem, relation (2.18) and Plancherel's theorem for the Hankel transform (1.17), we have

$$\begin{aligned}
\|a^t T_\psi^\alpha(f)\|_{2,\mu_\alpha}^2 &= \int_0^{+\infty}\int_0^{+\infty} a^{2t}|T_\psi^\alpha(f)(a,r)|^2 d\nu_\alpha(a)d\nu_\alpha(r)\\
&= \int_0^{+\infty} a^{2t}\left(\int_0^{+\infty}\left|T_\psi^\alpha(f)(a,r)\right|^2 d\nu_\alpha(r)\right)d\nu_\alpha(a)\\
&= \int_0^{+\infty} a^{2t}\left(\int_0^{+\infty}\left|\mathscr{H}_\alpha\left(T_\psi^\alpha(f)(a,.)\right)(\lambda)\right|^2 d\nu_\alpha(\lambda)\right)d\nu_\alpha(a)\\
&= c_\alpha\int_0^{+\infty}|\mathscr{H}_\alpha(f)(\lambda)|^2\left(\int_0^{+\infty} a^{2t}\left|\mathscr{H}_\alpha(\psi)\left(\frac{\lambda}{a}\right)\right|^2\frac{da}{a}\right)d\nu_\alpha(\lambda).
\end{aligned}$$

By a change of variables in the inner integral, we get

$$\begin{aligned}
\|a^t T_\psi^\alpha(f)\|_{2,\mu_\alpha}^2 &= c_\alpha\int_0^{+\infty}\lambda^{2t}|\mathscr{H}_\alpha(f)(\lambda)|^2\left(\int_0^{+\infty}\left|\mathscr{H}_\alpha(\psi)(a)\right|^2\frac{da}{a^{2t+1}}\right)d\nu_\alpha(\lambda)\\
&= c_\alpha\ \mathcal{M}\left(|\mathscr{H}_\alpha(\psi)|^2\right)(2t)\ \|\lambda^t\mathscr{H}_\alpha(f)\|_{2,\nu_\alpha}^2.
\end{aligned} \tag{3.9}$$

Thus,

$$\|r^s f\|_{2,\nu_\alpha}^{\frac{t}{s+t}} \|a^t T_\psi^\alpha(f)\|_{2,\mu_\alpha}^{\frac{s}{s+t}} = \left(\sqrt{c_\alpha \mathcal{M}\left(|\mathscr{H}_\alpha(\psi)|^2\right)(2t)}\right)^{\frac{s}{s+t}} \|r^s f\|_{2,\nu_\alpha}^{\frac{t}{s+t}} \|\lambda^t \mathscr{H}_\alpha(f)\|_{2,\nu_\alpha}^{\frac{s}{s+t}}.$$

We get the result by applying the Heisenberg-type inequality for the Hankel transform (3.2).

For the case $s, t \geq 1$, we apply the Heisenberg-type inequality for the Hankel transform given by the relation(3.3). ■

In [12], the authors have proved Heisenberg-type uncertainty principle for the windowed Hankel transform involving time and frequency variables. In this part, we investigate similar results for T_ψ^α with a different approach and more general settings.

Theorem 3.2.1. *Let $s, t > 0$ and ψ be an admissible Hankel wavelet in $L^2(d\nu_\alpha)$. Then, there exists a constant $C = C(\alpha, s, t) > 0$ such that for every function $f \in L^2(d\nu_\alpha)$, we have*

$$\|r^s T_\psi^\alpha(f)\|_{2,\mu_\alpha}^{\frac{t}{s+t}} \|a^t T_\psi^\alpha(f)\|_{2,\mu_\alpha}^{\frac{s}{s+t}} \geq C\left(\sqrt{c_\alpha \mathcal{M}(|\mathscr{H}_\alpha(\psi)|^2)(2t)}\right)^{\frac{s}{s+t}} \left(\sqrt{C_\psi}\right)^{\frac{t}{s+t}} \|f\|_{2,\nu_\alpha}, \quad (3.10)$$

Moreover, if $s, t \geq 1$, then the constant C is given by $C = (\alpha+1)^{\frac{st}{s+t}}$.

Proof. From the relation (3.9),

$$\|r^s T_\psi^\alpha(f)\|_{2,\mu_\alpha}^{\frac{t}{s+t}} \|a^t T_\psi^\alpha(f)\|_{2,\mu_\alpha}^{\frac{s}{s+t}} = \left(\sqrt{c_\alpha \mathcal{M}(|\mathscr{H}_\alpha(\psi)|^2)(2t)}\right)^{\frac{s}{s+t}} \|r^s T_\psi^\alpha(f)\|_{2,\mu_\alpha}^{\frac{t}{s+t}} \|\lambda^t \mathscr{H}_\alpha(f)\|_{2,\nu_\alpha}^{\frac{s}{s+t}}.$$

The result follows from the relation (3.4). ■

In the following Corollary, we give Heisenberg-type uncertainty inequality involving a single condition on the simultaneous time–frequency behaviour.

Corollary 3.2.1. *Assume $s,t>0$ and let ψ be anadmissible Hankel wavelet in $L^2(d\nu_\alpha)$. Then, there exists a constant $C=C(\alpha,s,t)>0$ such that for every function $f\in L^2(d\nu_\alpha)$, we get*

$$\left\| |(a,r)|^s T_\psi^\alpha(f)\right\|_{2,\mu_\alpha}^{\frac{t}{s+t}} \left\| |(a,r)|^t T_\psi^\alpha(f)\right\|_{2,\mu_\alpha}^{\frac{s}{s+t}} \geq C\left(\sqrt{c_\alpha \mathcal{M}(|\mathscr{H}_\alpha(\psi)|^2)(2t)}\right)^{\frac{s}{s+t}} \left(\sqrt{C_\psi}\right)^{\frac{t}{s+t}} \|f\|_{2,\nu_\alpha} \quad (3.11)$$

Moreover, if $s,t\geq 1$, the constant C is given by $C=(\alpha+1)^{\frac{st}{s+t}}$.

Proof. The result follows from Theorem 3.2.1 and the fact that

$$\||(a,r)|^s T_\psi^\alpha(f)\|_{2,\mu_\alpha}^{\frac{t}{s+t}} \quad \||(a,r)|^t T_\psi^\alpha(f)\|_{2,\mu_\alpha}^{\frac{s}{s+t}} \geq \|r^s T_\psi^\alpha(f)\|_{2,\mu_\alpha}^{\frac{t}{s+t}} \quad \|a^t T_\psi^\alpha(f)\|_{2,\mu_\alpha}^{\frac{s}{s+t}}.$$

■

As a consequence of this corollary, we have the following local-type uncertainty principle.[28, 29]

Corollary 3.2.2. *Let $s>0$ and ψ be an admissible Hankel wavelet in $L^2(d\nu_\alpha)$. Then, there exists a constant $C=C(s,\alpha)>0$ such that for every subset Σ of $\mathbb{R}_+^*\times\mathbb{R}_+$ with finite measure $0<\mu_\alpha(\Sigma)<+\infty$ and for every function $f\in L^2(d\nu_\alpha)$, we have*

$$\|\chi_\Sigma T_\psi^\alpha(f)\|_{2,\mu_\alpha} \leq \frac{\sqrt{\mu_\alpha(\Sigma)}\,\|\psi\|_{2,\nu_\alpha}}{C\left(c_\alpha \mathcal{M}(|\mathscr{H}_\alpha(\psi)|^2)(2s)C_\psi\right)^{\frac{1}{4}}} \||(a,r)|^s T_\psi^\alpha(f)\|_{2,\mu_\alpha}$$

Moreover, if $s\geq 1$, the constant C is given by $C=(\alpha+1)^{\frac{s}{2}}$.

Proof. From the relation (2.13), we have

$$\begin{aligned} \|\chi_\Sigma T_\psi^\alpha(f)\|_{2,\mu_\alpha} &\leq \sqrt{\mu_\alpha(\Sigma)}\|T_\psi^\alpha(f)\|_{\infty,\mu_\alpha} \\ &\leq \sqrt{\mu_\alpha(\Sigma)}\|\psi\|_{2,\nu_\alpha}\|f\|_{2,\nu_\alpha}. \end{aligned}$$

According to Corollary 3.2.1 with $s = t$, we get

$$\|f\|_{2,\nu_\alpha} \leq \frac{\||(a,r)|^s T_\psi^\alpha(f)\|_{2,\mu_\alpha}}{C\Big(c_\alpha \mathcal{M}(|\mathscr{H}_\alpha(\psi)|^2)(2s)C_\psi\Big)^{\frac{1}{4}}},$$

thus,

$$\|\chi_\Sigma T_\psi^\alpha(f)\|_{2,\mu_\alpha} \leq \frac{\sqrt{\mu_\alpha(\Sigma)}\|\psi\|_{2,\nu_\alpha}}{C\Big(c_\alpha \mathcal{M}(|\mathscr{H}_\alpha(\psi)|^2)(2s)C_\psi\Big)^{\frac{1}{4}}}\||(a,r)|^s T_\psi^\alpha(f)\|_{2,\mu_\alpha}.$$

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