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Dynamical analysis of a satellite chaotic system model with fractional-order derivative

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اهداء

الحمد لله الذي وفقني و أعانني على انجاز هذا العمل

اهدي ثمرة جهدي ونجاحي الى:

الى النور الذي أضاء لي دربي من أحمل اسمه بكل فخر، من دعمني بلا حدود وأعطاني بلا مقابل، الى من علمني ان الدنيا كفاح وسلاحها العلم والمعرفة، الى من غرس في روحي مكارم الأخلاق، معنى الحب، داعمي الأول، سندي، قوتي وملاذي بعد الله

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لكم جميعا أهدي تخرجي ... فهو منكم واليكم.

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ملخص

في هذا العمل من هذه المذكرة، سوف نقوم بدراسة سلوكيات ديناميكيات نظام الأقمار الصناعية الفوضوي دو رتبة اشتقاق كسرية . سنقوم بعرض صور الجادب الفوضوي ونقاط التوازن والمقاييس التبددية وأسس ليابونوف. سنؤكد من خلال هذه الأساليب على وجود الفوضى وتوفر نظرة ثاقبة للسلوك الديناميكي للنظام المقترح.

بالإضافة إلى ذلك، سنقوم بمراقبة الفوضى لهدا النظام و ضمان استقرار يته باستخدام الية التحكم بالتغذية الراجعة . وأخيرًا، سنقوم بدراسة المزامنة الكاملة لنظامين قمريين صناعيين متطابقين جزئيًا باستخدام تقنية التحكم التكيفي. سيتم كدلك عرض المحاكاة .الرقمية لاختبار وإثبات صحة طرق التحكم والمزامنة المقترحة

الكلمات المفتاحية: نظام الأقمار الصناعية الفوضوي، التحكم بالتغذية الراجعة، التحكم التكيفي دالة ليابونوف، المشتق الكسري.

ABSTRACT

In this work, we present the behaviour dynamics of a fractional-order chaotic satellite system, including phase portraits, equilibrium points, dissipative measurements and Lyapunov exponents. These tools confirm the presence of chaos and provide insight into the dynamic behavior of the system under consideration. Furthermore, we construct a feedback control to ensure the stability of this system. Finally, we present the complete synchronization of two identical fractional-order satellite systems using an adaptive control technique. Numerical simulations are used to test and validate the proposed control and synchronization methods.

Keywords: Chaotic satellite system, feedback control,adaptive control, Lyapunov function, fractional derivative.

RÉSUMÉ

Dans ce travail de ce mémoire, on va étudier le comportement dynamique d'un système satellitaire chaotique d'ordre fractionnaire à savoir les portraits de phases, les points d'équilibre, les mesures dissipatives et les exposants de Lyapunov . Ces méthodes confirment la présence de chaos et offrent un aperçu du comportement dynamique du système considéré. De plus, nous construisons un feedback contrôle pour garantir la stabilité de ce système. En fin, nous présentons la synchronisation complète de deux systèmes de satellites identiques fractionnaire grâce à la technique de contrôle adaptative. Des simulations numériques sont illustrés pour tester et validité les méthodes de contrôle et de la synchronisation proposées.

Mots clés : Système chaotique satellitaire, feedback contrôle, contrôle adaptative Fonction de Lyapunov, dérivée fractionnaire.

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GENERAL INTRODUCTION

Recently, study on the dynamics of fractional differential systems has attracted interest of many researchers. Many dynamical systems have been described using fractional order differential equations, which often exhibit chaotic or hyper-chaotic behavior such as : Chua circuit [1], Duffing system [2], Chen system [3], Lü system [4] and Rössler system [5].

Chaos control and synchronization in fractional-order systems start to attract increasing attentiondue to its many practical applications in several fields such as secure communication and control processing [6, 7].

Satellite systems play a very important role in the development of space technologies and scientific research, telecommunications and their applications in civil and military states.

Recently, the study of chaotic satellite systems has attracted considerable interest with notable contributions from researchers [8, 9, 10, 11, 12]. For example, in ref [13], Kumar et all studied synchronization of chaotic satellite systems with fractional derivatives analysis using feedback active control techniques

In this work, we will mainly focus on the research paper of Kumar et all [13], we will present the chaotic behavior of a fractional-order satellite system. First, we present the mathematical model that describes this system and its characteristic tools, such as the chaotic attractor, equilibrium points, dissipativity and Lyapunov exponents.

In addition, we use feedback and adaptive control techniques to control chaos and synchronize these systems.
This work is structured as follows:

The first chapter presents some basic definitions of dynamical systems.

The second chapter is devoted to fractional calculus.

In the third chapter, we present the description of the fractional satellite system and its chaotic behavior, the control of chaos in this system, and finally the synchronization of two identical satellite systems using numerical simulations.

Finally, a general conclusion is presented at the end of this work.

CHAPTER 1

CHAOTIC DYNAMICAL SYSTEMS

In this chapter, we recall some general concepts regarding chaotic or hyperchaotic phenomena that appear in deterministic dynamical systems. We will begin by defining the concept of dynamical systems, then discuss other mathematical notions, including attractors, the basin of attraction, stationary points, the Poincaré section, and some stability concepts. Finally, we will present the main characteristics of chaotic or hyperchaotic behavior

1.1 Introduction to the Theory of Dynamical Systems

There are several systems studied in sciences such as physics, chemistry, biology, etc., which are deterministic systems because their evolution over time can be calculated and is entirely determined by their state at a given moment.

For example, we can mention population growth, the evolution of reactant concentration in a chemical reaction, etc. The description of these systems is done using numerical quantities, often called variables or observables.

Thus, a dynamical system can be defined as a description of a physical phenomenon that evolves over time (continuous systems) or concerning another variable (discrete system).

This description requires a graphical representation: each state of the system is associated with a vector x in a vector space called phase space.

From a mathematical perspective, dynamical systems are classified into two categories:

- Continuous dynamical systems.
- Discrete dynamical systems.

1.1.1 Continuous Dynamical Systems

A dynamical system has two aspects: its state and its dynamics, that is, its evolution over time. A continuous dynamical system is defined as any system of first-order differential equations given by

$$\dot{x} = \frac{dx}{dt} = F(x, t, v), \quad x \in \mathbb{U} \subseteq \mathbb{R}^n, \quad v \in \mathbb{V} \subseteq \mathbb{R}^p. \tag{1.1}$$

The system (1.1) is called a dynamic system. U is the phase space, $\mathbb{R}^p$ is the parameter space and x is called the state vector.

Example 1.1.1 *The Lotka-Volterra equations describe the interaction between a prey population x and a predator population y:*

$$\begin{cases} \dfrac{dx}{dt} = \alpha x - \beta xy \\ \dfrac{dy}{dt} = \delta xy - \gamma y. \end{cases}$$

where:

- *x represents the prey population,*
- *y represents the predator population,*
- *α is the natural growth rate of the prey,*
- *β is the predation rate,*

- *γ is the natural death rate of the predator,*

- *δ is the efficiency of converting consumed prey into predator reproduction.*

These equations lead to oscillatory dynamics, where the populations of both species fluctuate over time.

1.1.2 Descret dynamical systems

A discrete dynamic system is defined as any system of recurrent algebraic equations given by:

$$X_{k+1} = F(X_k, \mu), \quad X_k \in \mathbb{U} \subseteq \mathbb{R}^n, \tag{1.2}$$

where F is the recurrence function, $X_k \in \mathbb{U} \subseteq \mathbb{R}^n$ is the state vector at time t_k, μ is the parameter vector, and $k \in \mathbb{N}$.

Example 1.1.2 *A discrete-time version of the Lotka-Volterra predator-prey model is given by the recurrence equations:*

$$\begin{cases} x_{n+1} = x_n(1 + \alpha - \beta y_n), \\ y_{n+1} = y_n(1 - \gamma + \delta x_n). \end{cases}$$

where:

- *x_n represents the prey population at time step n,*

- *y_n represents the predator population at time step n,*

- *α is the natural growth rate of the prey,*

- *β is the rate at which predators consume prey,*

- *γ is the natural death rate of the predators,*

- *δ is the efficiency of converting consumed prey into predator reproduction.*

1.1.3 Autonomous or non-autonomous systems

Definition 1.1.1 *A differential system is said to be autonomous if its evolution law does not depend on time.*

Example 1.1.3 *As an utonomous system, we take the following system:*

$$\begin{cases} \dfrac{dx}{dt} = y, \\ \dfrac{dy}{dt} = -x, \\ x(t_0) = 1, y(t_0). = 0 \end{cases}$$

On the other hand, the differential system is non-autonomous if its evolution law depends on time

Example 1.1.4 *The following system:*

$$\begin{cases} \dfrac{dx}{dt} = y + t, \\ \dfrac{dy}{dt} = -x, \\ x(t_0) = 1, y(t_0) = 0. \end{cases}$$

is a non-autonomous system.

1.2 Flow

Consider the autonomous dynamical system:

$$\frac{dx}{dt} = f(x) \quad x \in \mathbb{U} \subseteq R^n. \tag{1.3}$$

The flow of the differential system (1.3) is called the one-parameter family of mappings $\left\{\phi_t\right\}_{t\in\mathbb{R}}$ from $\mathbb{U}$ into itself, defined by:

$$\phi_t(x_0) = \phi_t(t, x_0) = x(t, x_0). \tag{1.4}$$

where $t \in \mathbb{R}$ and $x(t, x_0)$ is the unique solution of the Cauchy problem.
The following theorem states that the solutions of a differential system depend differentiably on the initial conditions and that the flow is a one-parameter group of diffeomorphisms.

Theorem 1.2.1 *The mapping ϕ_t is differentiable on $\mathbb{U}$. The flow $\left\{\phi_t\right\}_{t\in\mathbb{R}}$ satisfies the following properties:*

1. $\phi_0 = Id$, (Id is the identity of $\mathbb{U}$).

2. $\phi_{t_1} \circ \phi_{t_2} = \phi_{t_1+t_2}$, For all $t_1, t_2 \in \mathbb{R}$ (Composition law).

From the previous theorem, we deduce that for each $t \in \mathbb{R}$, ϕ_t is a diffeomorphism of $\mathbb{U}$ and that:

$$(\phi_t)^{-1} = \phi_{-t}. \tag{1.5}$$

1.3 Attractors and basin of attraction

1.3.1 Attractors

An attractor is a geometric object which all trajectories of points in the phase space converge. Mathematically, the set A is saud to be an attractor if:

1. A is a compact and invariant set under the flow ϕ_t, i.e; $\phi_t(A) = A$, for all t.

2. For every neighborhood U of A, there exists a neighborhood V of A such that every solution $X(t, X_0) = \phi_t(X_0)$ will remain in U if $X_0 \in V$

3. $\cap\phi_t(V) = A$, for $t \geq 0$.

4. There exists a dense orbit in A.

1.3.2 Different Types of Attractors

There are two types of attractors: regular attractors and strange (chaotic) attractors.

1. Regular attractors: Regular attractors characterize the evolution of non-chaotic systems and can be of three types:

   - Fixed point: It is the simplest attractor.
   - Periodic limit cycle: It is a closed trajectory that attracts all nearby orbits.
   - Quasi-periodic: represents motions resulting from two or more frequencies, sometimes referred to as a 'torus'

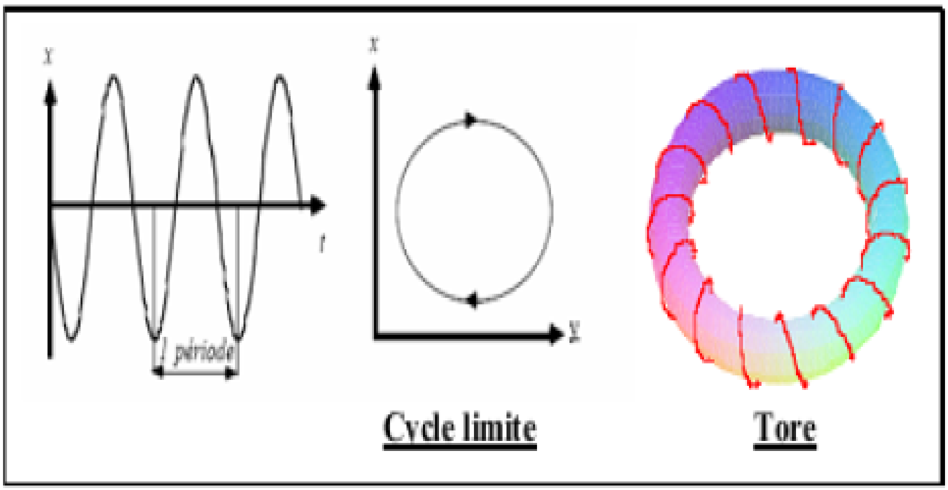

Figure 1.1: Fixed point, Periodic limit cycle and Torus.

2. Strange attractors: Strange attractors are complex geometric structures that characterize the evolution of chaotic systems.

   A chaotic attractor is characterized by:
   1. Sensitivity to initial conditions : Two initially close trajectories on the attractor always diverge over time, indicating chaotic behavior.
   2. Fractal and non-integer dimension : The attractor has a fractal structure, which explains why it is called "strange."
   3. Zero volume in phase space : The attractor occupies no volume in the phase space.

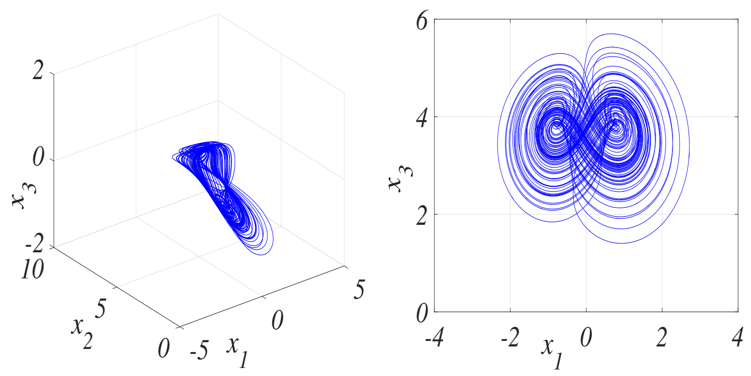

Figure 1.2: Chaotic attractors of Finance system.

1.3.3 Basin of attraction

Definition 1.3.1 *The basin of attraction $B(A)$ of an attractor A is the set of points whose trajectories asymptotically converge to A; i.e;*

$$B(A) = \bigcup_{t<0} \varphi_t(V). \tag{1.6}$$

1.4 Equilibrium points

In general, nonlinear differential systems cannot be solved explicitly. Therefore, we perform a qualitative analysis of their solutions. This study begins with finding the equilibrium points of differential equation (1.1). At these equilibrium points, the velocity is zero.

$$\dot{x} = 0. \tag{1.7}$$

The equilibrium points, which we denote as x_{eq}satisfy the following equation:

$$f(x_{eq}) = 0. \tag{1.8}$$

In the phase space, an equilibrium point is represented by a single point. Its value is determined by the chosen initial condition. Moreover, for different initial conditions, we may find multiple equilibrium points. Additionally, these points can be either stable or unstable, depending on whether nearby trajectories converge or diverge.

Example 1.4.1 *Consider the following differential equation:*

$$\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + 2x = 0.$$

We rewrite this as a ***first-order system*** *by defining* $y = \dfrac{dx}{dt}$.
So

$$\begin{cases} \dfrac{dx}{dt} = y \\ \dfrac{dy}{dt} = -2x - 3y. \end{cases}$$

An equilibrium point $E(x_{eq}, y_{eq})$ *satisfies:*

$$\begin{cases} y = 0 \\ -2x - 3y = 0. \end{cases}$$

The only equilibrium point is $E(0,0)$.

1.5 Stability of equilibrium points

The notion of stability in a dynamical system characterizes the behavior of its trajectories near equilibrium points. Analyzing the stability of a dynamical system allows us to study the evolution of its state trajectory when the initial state is very close to an equilibrium point. Lyapunov's stability theory applies to any differential equation. This concept implies that the solution of a differential equation, when initialized near an equilibrium point, always remains sufficiently close to it.

Definition 1.5.1 *(Stability) The equilibrium point* x_{eq} *of system* (1.1) *is stable in the sense of Lyapunov if and only if:*

$$\forall \epsilon > 0, \exists \delta > 0, \textit{such that } \|x_0 - x_{eq}\| < \delta \Rightarrow \|x(t; x_0) - x_{eq}\| < \epsilon, \quad \forall t \geq t_0. \tag{1.9}$$

Definition 1.5.2 *(Attractivity) The equilibrium point* x_{eq} *of system* (1.1) *is attractive if and* *only if:*

$$\forall \epsilon > 0, \exists \delta > 0, \text{ such that } \|x_0 - x_{eq}\| < \delta \Rightarrow \lim_{t \to +\infty} \|x(t, x_0) - x_{eq}\| = 0. \tag{1.10}$$

Definition 1.5.3 *(Asymptotic stability) The equilibrium point x_{eq} of system* (1.1) *is asymptotically stable if it is both stable and attractive.*

Definition 1.5.4 *(Exponential stability) The equilibrium point x_{eq} of system* (1.1) *is exponentially stable if there exist two strictly positive constants a and b, and if there exists $t_0 > 0$ such that:*

$$\|x(t; x_0) - x_{eq}\| < ae^{-bt} \text{such that } t \geq t_0 \geq 0. \tag{1.11}$$

Example 1.5.1 *According to the example (1.4.1), we have* (0.0) *is the only equlibrem point. To study stability, we use the Jacobian matrix of the system.*

$$\begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

The eigenvalues of the jacobien matrix are given by

$$\lambda_1 = -1, \ \lambda_2 = -2. \tag{1.12}$$

Both eigenvalues are negatives ($\lambda_1, \lambda_2 < 0$).
Therefore, $E(0, 0)$ is an asymptotically stable equilibrium point.

Remark 1.5.1 *Note that using the previous definitions to establish the stability of* (1.1) *in the neighborhood of its equilibrium point requires the explicit solution of equation* (1.1)*, which is often very difficult in most cases. Therefore, the following two Lyapunov methods allow us to overcome this obstacle.)*

1.5.1 Direct method

Lyapunov's direct method allows us to locally analyze the stability of system (1.1) without explicitly solving it. The stability problem is then reduced to finding such a function (called a Lyapunov function), which provides information about the stability of the system. The following theorem summarizes this method.

Theorem 1.5.2 *The equilibrium point x_{eq} of system* (1.1) *is said to be stable (respectively asymptotically stable) in the sense of Lyapunov if there exists a neighborhood D of x_{eq}) and a function $V : D \to R$ (called a Lyapunov function) of class C^1 satisfying the following properties:*

1. $V(x_{eq}) = 0$*, and* $V(x) > V(x_{eq})$*, for all* $x \neq x_{eq}$ *in* D

2. $\dot{V}(x) \leq 0$ *(respectively* $\dot{V}(x) < 0$*), for all* $x \neq x_{eq}$*.*

Example 1.5.2 *We analyze the stability of the following dynamical system:*

$$\begin{cases} \dfrac{dx}{dt} = -x + xy, \\ \dfrac{dy}{dt} = -y - x^2. \end{cases}$$

Consider the Lyapubov function:

$$V(x, y) = \frac{1}{2}x^2 + \frac{1}{2}y^2.$$

This function is positive definite because:

- $V(0, 0) = 0$*,*
- $V(x, y) > 0$ *for all* $(x, y) \neq (0, 0)$*.*

On the other hand

$$\dot{V}(x, y) = \frac{\partial V}{\partial x}\frac{dx}{dt} + \frac{\partial V}{\partial y}\frac{dy}{dt}.$$

Thus

$$\dot{V}(x, y) = -x^2 - y^2.$$

Since $\dot{V}(x, y))$ *is strictly negative for all* $(x, y) \neq (0, 0)$*, we conclude that the origin* $(0, 0)$ *is asymptotically stable.*

1.5.2 Indirect method

Lyapunov's second method is based on examining the linearization in the neighborhood of the equilibrium point x_{eq} of system (1.1). More precisely, the eigenvalues x_{eq} of the

Jacobian matrix associated with this equilibrium point are analyzed. Linearization consists of assuming:

$$x = x_{eq} + \delta x, \tag{1.13}$$

Where δx is a small perturbation applied in the neighborhood of the equilibrium point x_{eq}.
We then have:

$$\dot{x} = \dot{x}_{eq} + \dot{\delta} x. \tag{1.14}$$

The system (1.1) then becomes:

$$\dot{x}_{eq} + \dot{\delta} x = f(x_{eq} + \delta x), \tag{1.15}$$

Using a first-order Taylor expansion of f in the neighborhood of x_{eq} we obtain:

$$f(x_{eq} + \delta x) = f(x_{eq}) + J_f(x_{eq})(x - x_{eq}), \tag{1.16}$$

Where J_f represents the Jacobian matrix of f. Thus:

$$\dot{\delta x} = J_f(x_{eq})\delta x. \tag{1.17}$$

This equation shows the evolution of the perturbation δx in the vicinity of the equilibrium point. We therefore have the following theorem:

Theorem 1.5.3 *(Hartman-Grobman Theorem) [14] If the Jacobian matrix $J_f(x_{eq})$ has no zero or purely imaginary eigenvalues, then there exists a homeomorphism that maps the trajectories of the nonlinear flow to those of the linear flow in a certain neighborhood of x_{eq} .*

Remark 1.5.4 *This theorem allows us to relate the dynamics of the nonlinear system* (1.1) *to the dynamics of the linearized system* (1.17) .

The extension of the theorem on linearization is then:

Theorem 1.5.5 *Let x_{eq} be an equilibrium point of system* (1.1).

1. *If all the eigenvalues of the Jacobian matrix $J_f(x_{eq})$ have strictly negative real parts, then x_{eq} is said to be exponentially stable.*

2. *If the Jacobian matrix has at least one eigenvalue with a strictly positive real part, the equilibrium point x_{eq} is said to be unstable.*

Example 1.5.3 *We analyze the stability of the following dynamical system:*

$$\begin{cases} \dfrac{dx}{dt} = -x + xy, \\ \dfrac{dy}{dt} = -y - x^2. \end{cases}$$

We find the equilibrium points by solving:

$$\begin{cases} -x + xy = 0 \\ -y - x^2 = 0 \end{cases}$$

The only real equilibrium point is $E(0,0)$. The jacobian matrix at the equilibrium point E is given by

$$J(0,0) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

The eigenvalues of the matrix are:

$$\lambda_1 = -1, \quad \lambda_2 = -1$$

Since both eigenvalues are negative real numbers, the system is asymptotically stable at the equilibrium point $(0,0)$.

1.6 The chaos

The objective of this section is to provide a better understanding of the characteristics that allow us to recognize chaotic behavior. As a first approach, we present the main characteristics of the chaotic behavior of a deterministic dynamic system. We also mention some scenarios of transition to chaos. Finally, we conclude this section with an application on chaotic systems.

1.6.1 Main characteristics of chaotic behavior

Below, we recall some characteristics that help us better understand the key features of a chaotic system.

Non-linearity

Non-linearity is one of the fundamental characteristics of chaotic systems. In fact, any linear system cannot be chaotic.

Determinism

A chaotic system is deterministic (rather than probabilistic), meaning that it follows laws that completely describe its motion. The concept of determinism thus implies the ability to predict the future state of a phenomenon based on a past event. However, in random phenomena, it is impossible to predict the trajectories of any given particle.

Random Aspects

If the motion is random, the system's points fill the phase space randomly: no structure appears. When the motion is chaotic, the points may initially appear random. However, upon observing the system for a sufficiently long time, one notices that the points trace a particular shape. Figure 1.3 illustrates the random aspect of a chaotic Rössler system.

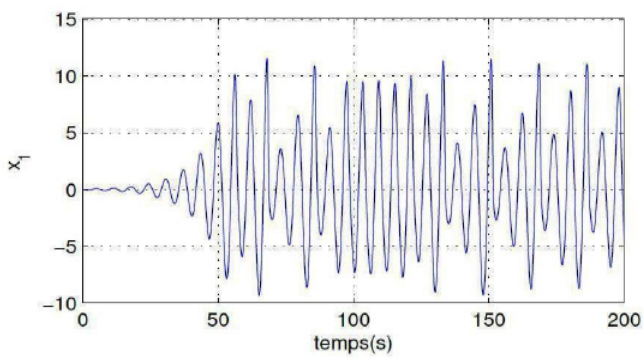

Figure 1.3: Random aspect of the Rössler system.

Chaotic attractors

The particular geometric figure that represents the attractor of a chaotic system in phase space over time is called a chaotic attractor. This attractor emerges through two simultaneous operations: stretching, which is responsible for sensitivity to initial conditions and instability, and folding, which gives the system its "strange" nature. Furthermore, an attractor is considered chaotic when its dimension is fractal. Due to this unique fractal property, these attractors are classified as strange (chaotic) attractors. They serve as the signature of chaos, allowing us to identify and authenticate chaotic behavior.

Sensitivity to initialcConditions

Chaotic systems are extremely sensitive to initial conditions. Even very small perturbations in the initial state of a system can eventually lead to a completely different final state.

Lyapunov Exponents

The rate of divergence of two initially close trajectories can be analyzed using Lyapunov exponents, which help characterize the nature of the detected chaos.The Lyapunov exponent measures the rate at which two trajectories diverge over time. As an example, consider a four-dimensional dynamical system with Lyapunov exponents $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ satisfying the condition: $\lambda_1 > \lambda_2 > \lambda_3 > \lambda_3$.

- **If** $\lambda_4 < \lambda_3 < \lambda_2 < \lambda_1 < 0$: The system is an **asymptotically stable equilibrium point**.
- **If** $\lambda_1 = 0$ and $\lambda_4 < \lambda_3 < \lambda_2 < 0$: The system has a **stable limit cycle**.
- **If** $\lambda_1 = \lambda_2 = 0$ and $\lambda_4 < \lambda_3 < 0$: The system has a **stable torus**.
- **If** $\lambda_1 > 0$, $\lambda_2 = 0$, $\lambda_3 < \lambda_4 < 0$, and $\sum_{i=1}^{4} \lambda_i < 0$: The system is **chaotic**.
- **If** $\lambda_1, \lambda_2 > 0$, $\lambda_3 = 0$, $\lambda_4 < 0$, and $\sum_{i=1}^{4} \lambda_i < 0$: The system is **hyperchaotic**.

Remark 1.6.1 • *A negative Lyapunov exponent in a given direction indicates that trajectories converge, leading to a loss of information about initial conditions. The orbit thus becomes attractive towards a periodic orbit or a fixed point, characterizing dissipative systems, which exhibit **asymptotic stability**.*

- *A positive Lyapunov exponent in a given direction indicates that the divergence between two nearby trajectories increases exponentially over time. As a result, the trajectories diverge, and the orbit becomes chaotic. Intuitively, this represents sensitivity to initial conditions.*

- *A zero Lyapunov exponent means that trajectories originating from different initial conditions maintain a constant separation, neither converging nor diverging. In this case, the system is considered conservative.*

Remark 1.6.2 *There are several algorithms for computing Lyapunov exponents, one of the most well-known being Wolf's algorithm [15]. This algorithm allows us to calculate Lyapunov exponents by effectively measuring the divergence of two trajectories in response to an introduced perturbation.*

Bifurcation diagram

A bifurcation is said to occur when such a qualitative change in the structure of a system happens due to the quantitative variation of one of its parameters (which is called the bifurcation value). The graphs that represent these bifurcations are called bifurcation diagrams.Thus, the bifurcation diagram is a very important tool for evaluating the possible behaviors of a system depending on the bifurcation values. Figure 1.6 illustrates the bifurcation diagram of the logistic map, defined on the segment $[0,1]$ by:

$$\frac{dx}{dt} = rx(1-x), \tag{1.18}$$

where $r \in [0,4]$ is a control parameter.

According to Figure 1.4, we can observe three different states of the system depending on the value of the parameter r: stable regime, a periodic regime with multiple states and finally a chaotic regime.

1.6.2 Route to Chaos

There are several scenarios that describe the transition from a fixed point to chaos. In all cases, the evolution from the fixed point to chaos is not gradual but marked by

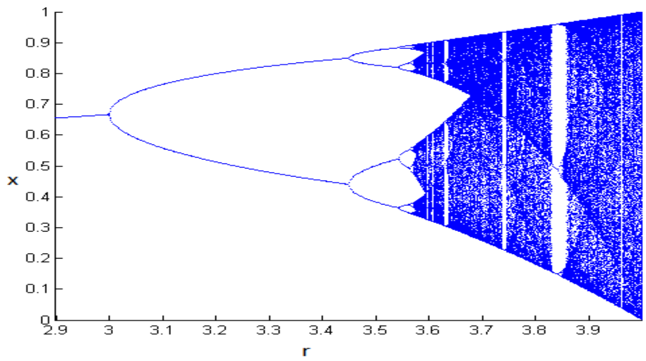

Figure 1.4: Bifurcation diagram of the logistic map.

discontinuous changes called "bifurcations." A bifurcation signifies the sudden shift from one dynamic regime to another, qualitatively different. Three main scenarios of transition to chaos can be identified:

Period doubling

This transition scenario is undoubtedly the most well-known. As the control parameter of the experiment increases, the frequency of the periodic regime doubles, then quadruples, then multiplies by 8, 16, and so on. These doublings occur at increasingly shorter intervals, leading to an accumulation point where, hypothetically, the frequency would become infinite. At this moment, the system enters a chaotic state.

Quasi-periodicity

In a dynamical system with periodic behavior at a single frequency, if we change a parameter, a second frequency appears. If the ratio between the two frequencies is rational, the behavior remains periodic. However, if the ratio is irrational, the behavior becomes quasi-periodic. By changing the parameter again, a third frequency appears, and this process continues until the system eventually reaches chaos.

Intermittency

This transition scenario via intermittency is characterized by the erratic appearance of chaotic bursts in a system that otherwise oscillates regularly. The system remains in a periodic or nearly periodic state for a certain period, maintaining a degree of "regularity," before suddenly destabilizing, leading to a kind of chaotic explosion.

1.7 Example of chaotic system

As an example of a continuous chaotic system, we consider the Lorenz system, given by the following equations:

$$\begin{cases} \dot{x} = \sigma(y - x), \\ \dot{y} = rx - y - xz, \\ \dot{z} = xy - bz. \end{cases} \tag{1.19}$$

where, x, y, and z are the state variables of the system, σ, b and r are positif constants,

1.7.1 Chaotic attractor

The system (1.19) exhibits chaotic behaviour, as shown in Figure(1.5), where the system's parameters are as follows

$$\sigma = 10\; b = \frac{8}{3}\; and\; r = 28,$$

and the initial values of the Lorenz system (1.19) are selected as

$$x(0) = y(0) = z(0) = 50 \tag{1.20}$$

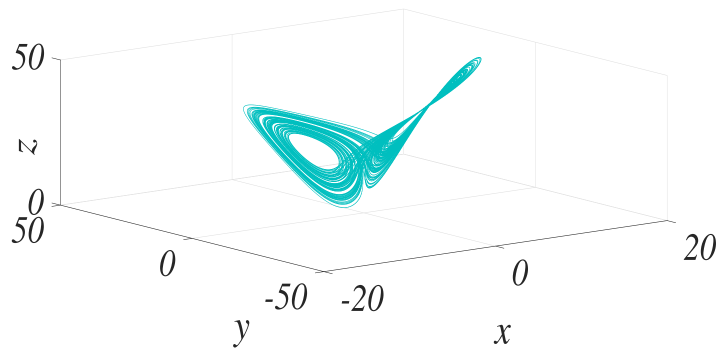

Figure 1.5: Chaotic attractor of the Lorenz system (1.19).

1.7.2 Equilibrium points

The equilibrium point (x, y, z) satisfies the condition

$$\dot{x} = \dot{y} = \dot{z} = 0.$$

For $r < 1$, there is only one equilibrium point $E_0(0, 0, 0)$.
For $r > 1$, there are two other points

$$E_1\left(\sqrt{b(r-1)},\ \sqrt{b(r-1)}, r-1\right); \quad E_2\left(-\sqrt{b(r-1)}, -\sqrt{b(r-1)}, r-1\right).$$

The Jacobian matrix of the system is given by

$$J(x, y, z) = \begin{bmatrix} \sigma & \sigma & 0 \\ r-z & -1 & -x \\ y & x & -b \end{bmatrix}$$

At the point $E_0(0, 0, 0)$, the jacobian matrix is:

$$J = \begin{bmatrix} \sigma & \sigma & 0 \\ r & 1 & 0 \\ 0 & 0 & -b \end{bmatrix}$$

So, the eigenvalues are

$$\begin{aligned} \xi_1 &= -b \\ \xi_2 &= \frac{-\sigma - 1 - \sqrt{(\sigma-1)^2 + 4r\sigma}}{2} \\ \xi_3 &= \frac{-\sigma - 1 + \sqrt{(\sigma-1)^2 + 4r\sigma}}{2}. \end{aligned}$$

We substitute the values of $\sigma = 10$ and $b = \frac{8}{3}$, we obtain:

$$\begin{aligned}
\xi_1 &= -\frac{8}{3} \\
\xi_2 &= \frac{1}{2}\left(\sqrt{40r + 81} - 11\right) \\
\xi_3 &= \frac{1}{2}\left(-\sqrt{40r + 81} - 11\right).
\end{aligned}$$

1. For example, when $r < 1$; the three real roots are negative, $\xi_3 < \xi_2 < \xi_1 < 0$. Therefore the equilibrium is a stable node.

For $r > 1$, One of the eigenvalues is positive ξ_3 : the equilibrium is therefore point.

CHAPTER 2

SOME CONCEPTS ABOUT FRACTIONNAL CALCULUS

In this chapter, some basic concepts related to fractional differentiation are presented.

2.1 Spicial functions

2.1.1 Gamma function

The Euler's gamma function [16] is one of the basic functions in the fractional calculus that generalize the factorial ($m!$) from integer to take non-integer and also complex numbers.

Definition 2.1.1 *For all* $\zeta \in \mathbb{C}$ *where* $Re(\zeta) > 0$ *Euler's Gamma function noted by* $\Gamma(\zeta)$ *is defined by:*

$$\Gamma(\zeta) = \int_0^\infty t^{\zeta-1} e^{-t}\, dt. \tag{2.1}$$

Remark 2.1.1 *We take* $Re(\zeta) > 0$ *due to the integral in* (2.1) *is absolutely converge on the complex half-plan.*

2.1.2 Beta function

Sometimes we encounter values that are combinations of the values of the gamma function. It is preferable to use the beta function instead of the complicated combination.

Definition 2.1.2 *The function β [16] is defined by:*

$$\beta(\zeta,\theta)=\int_0^1 t^{\zeta-1}(1-t)^{\theta-1}dt,\quad Re(\zeta)>0, Re(\theta)>0. \tag{2.2}$$

1. The Beta function has the symmetric property, i.e: $\beta(\zeta,\theta)=\beta(\theta,\zeta)$.

2. There is a link between the two Euler functions, Gamma and Beta. This link is given by:

$$\beta(\zeta,\theta)=\frac{\Gamma(\zeta)\Gamma(\theta)}{\Gamma(\zeta+\theta)},\quad \forall\zeta,\theta\neq -1,-2,-3,\ldots \tag{2.3}$$

2.1.3 Mittag-Leffler functions

The exponential function $\zeta\mapsto e^{\zeta}$ play an important role in resolution of ordinary differential equations. As a consequence, we need to generalize trigonometric and exponential functions and use them in the process of resolution, this is the main role of Mittag-Leffler function. This function is used to give an explicit expression of the solution. Also, this function has been introduced by Mittag-Leffler and it is considered as a generalization of one parameter of the exponential function.

Definition 2.1.3 *The function of Mittag-Leffler [16] of one parameter is defined by*

$$E_{\alpha}(\zeta)=\sum_{k=0}^{\infty}\frac{\zeta^k}{\Gamma(\alpha k+1)},\quad \zeta\in\mathbb{C},\quad Re(\alpha)>0. \tag{2.4}$$

Indeed, for $\alpha=1$, we have:

$$E_1(\zeta)=e^{\zeta}, \tag{2.5}$$

and for $\alpha=1$, we have:

$$E_2(\zeta)=\cosh(\sqrt{\zeta}). \tag{2.6}$$

Definition 2.1.4 *The mettag-Leffler function of two parameters is given by*

$$E_{\alpha,\beta}(\zeta) = \sum_{k=0}^{\infty} \frac{\zeta^k}{\Gamma(\alpha k + \beta)}, \quad \zeta, \beta \in \mathbb{C}, \quad Re(\alpha) > 0. \tag{2.7}$$

For $\beta = 1$, we get

$$E_{\alpha,1}(\zeta) = \sum_{k=0}^{\infty} \frac{\zeta^k}{\Gamma(\alpha k + 1)} = E_{\alpha}(\zeta), \quad \zeta \in \mathbb{C}, \quad Re(\alpha) > 0. \tag{2.8}$$

Some particular cases can be result as follows

$$\begin{aligned}
E_{1,1}(\zeta) &= \sum_{k=0}^{\infty} \frac{\zeta^k}{\Gamma(k+1)} = \sum_{k=0}^{\infty} \frac{\zeta^k}{k!} = e^{\zeta}, \qquad (2.9)\\
E_{1,2}(\zeta) &= \sum_{k=0}^{\infty} \frac{\zeta^k}{\Gamma(k+2)} = \sum_{k=0}^{\infty} \frac{\zeta^k}{(k+1)!} = \frac{1}{\zeta}\sum_{k=0}^{\infty} \frac{\zeta^{k+1}}{(k+1)!} = \frac{e^{\zeta}-1}{\zeta},\\
E_{1,m}(\zeta) &= \frac{1}{\zeta^{m-1}}\left[e^{\zeta} - \sum_{k=0}^{m-2} \frac{\zeta^k}{k!}\right].
\end{aligned}$$

and

$$\begin{aligned}
E_{2,1}(\zeta^2) &= \sum_{k=0}^{\infty} \frac{\zeta^{2k}}{\Gamma(2k+1)} = \sum_{k=0}^{\infty} \frac{\zeta^{2k}}{(2k)!} = \cosh(\zeta), \qquad (2.10)\\
E_{2,2}(\zeta) &= \sinh(\sqrt{\zeta}\zeta),\\
E_{2,2}(\zeta^2) &= \sum_{k=0}^{\infty} \frac{\zeta^{2k}}{\Gamma(2k+2)} = \sum_{k=0}^{\infty} \frac{1}{\zeta}\frac{\zeta^{2k+1}}{(2k+1)!} = \frac{\sinh(\zeta)}{\zeta}.
\end{aligned}$$

2.2 The Riemann-Liouville fractional derivatives

We begin by the Riemann-Liouville integral operator [17]. The Riemann-Liouville integral operator ${}_a^{RL}D_t^{-\alpha}$ is an extension of Cauchy's integral :

$$f^{(-n)}(t) = \frac{1}{\Gamma(n)} \int_a^t (t-\tau)^{n-1} f(\tau)\, d\tau, \tag{2.11}$$

By remplacing the integer n by $\alpha > 0$, we get

$$ {}_aD_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau. \tag{2.12} $$

In (2.11) the integer n must satisfy the condition $n \geq 1$. The corresponding condition for α is weaker. For the existence of the integral in (2.12), we must have $\alpha > 0$.

Example 2.2.1 *Let consider the power function*

$$ f(t) = (t-a)^{\beta}, \tag{2.13} $$

where β is a real number.
We have

$$ {}_aD_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} (\tau - a)^{\beta} d\tau. \tag{2.14} $$

By using the variable change $\tau = a + x(t-a)$, and then using the definition of the beta function, we obtain

$$ \begin{aligned} \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} (\tau-a)^{\beta} d\tau &= (t-a)^{\alpha+\beta} \frac{1}{\Gamma(\alpha)} \int_0^1 (1-x)^{\alpha-1} x^{\beta} dx \\ &= (t-a)^{\alpha+\beta} \frac{1}{\Gamma(\alpha)} \beta(\alpha, \beta+1) \\ &= (t-a)^{\alpha+\beta} \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}. \end{aligned} \tag{2.15} $$

For $f(t) = K$, we get

$$ {}_aD_t^{-\alpha} K = \frac{(t-a)^{\alpha} K}{\Gamma(\alpha+1)}. \tag{2.16} $$

2.2.1 The Riemann-Liouville fractional derivative

Let $\alpha \in \mathbb{R}^{+}$ and $n = \lceil \alpha \rceil$ ($\lceil \alpha \rceil = \min\{z \in \mathbb{Z} : z \geq \alpha\}$). The Riemann-Liouville fractional derivative of a function $f(t)$ is defined by

$$\begin{aligned} {}_{a}^{RL}D_{t}^{\alpha} f(t) &= \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t (t-\tau)^{n-\alpha-1} f(\tau) d\tau \\ &= \frac{d^n}{dt^n} ({}_{a}D_{t}^{-(n-\alpha)} f(t)). \end{aligned} \tag{2.17}$$

If $\alpha = n-1$, then we obtain a conventional integer order derivative of order $n-1$:

$${}_{a}D_{t}^{n-1} f(t) = \frac{d^n}{dt^n} ({}_{a}D_{t}^{-1} f(t)) = f^{(n-1)}(t). \tag{2.18}$$

Example 2.2.2 *Let consider the function $f(t) = (t-a)^{\beta}$, $n-1 \leq \alpha < n$.*
We have

$${}_{a}^{RL}D_{t}^{\alpha} f(t) = \frac{d^n}{dt^n} ({}_{a}D_{t}^{-(n-\alpha)} f(t)). \tag{2.19}$$

Thus

$$\begin{aligned} {}_{a}^{RL}D_{t}^{\alpha} f(t) &= \frac{d^n}{dt^n} \left((t-a)^{n-\alpha+\beta} \frac{\Gamma(\beta+1)}{\Gamma(n+\beta-\alpha+1)} \right) \\ &= \frac{\Gamma(\beta+1)}{\Gamma(-\alpha+\beta+1)} (t-a)^{\beta-\alpha}. \end{aligned} \tag{2.20}$$

For $f(t) = K$, we get

$${}_{a}^{RL}D_{t}^{\alpha} K = (t-a)^{-\alpha} \frac{K}{\Gamma(1-\alpha)}. \tag{2.21}$$

2.2.2 Caputo's fractional derivative

It was proposed by M. Caputo in 1967.

Definition 2.2.1 *Let $f \in C^n([a,b])$, $\alpha > 0$. The Caputo's fractional derivative of the function f [18] is defined by*

$${}_{a}^{C}D_{t}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(\tau) d\tau}{(t-\tau)^{\alpha-n+1}} = {}_{a}D_{t}^{-(n-\alpha)} \left(\frac{d^n}{dt^n} f(t) \right), \tag{2.22}$$

Where $n-1<\alpha<n$ and $t>a$.

Lemma 2.2.1 *Let $\alpha \geq 0$ and $n=\lceil\alpha\rceil$. Assume that f is a function, such that both ${}_a^C D_t^\alpha$ and ${}_a^{RL} D_t^\alpha$ exist. Then*

$$ {}_a^C D_t^\alpha f(t) = {}_a^{RL} D_t^\alpha f(t) - \sum_{k=0}^{n-1} f^{(k)}(a) \frac{(t-a)^{-\alpha+k}}{\Gamma(-\alpha+k+1)}. \tag{2.23} $$

Example 2.2.3 *Let consider the function $f(t)=(t-a)^\beta$, such that $\beta>n$ and $n=\lceil\alpha\rceil$. We have*

$$ {}_a^C D_t^\alpha f(t) = {}_a^{RL} D_t^\alpha f(t) - \sum_{k=0}^{n-1} f^{(k)}(a) \frac{(t-a)^{k-\alpha}}{\Gamma(k-\alpha+1)}, \tag{2.24} $$

and

$$ f^{(k)}(a)=0; \quad \forall k=0,1,\ldots,n-1, \tag{2.25} $$

then

$$ {}_a^C D_t^\alpha (t-a)^\beta = {}_a^{RL} D_t^\alpha (t-a)^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(t-a)^{\beta-\alpha}. \tag{2.26} $$

If $\beta=0,1,\ldots,n-1$, then

$$ {}_a^C D_t^\alpha (t-a)^\beta = 0. \tag{2.27} $$

Lemma 2.2.2 *Let $\alpha>0$. We have*

1. If $f \in C([a,b])$, then

$$ {}_a^C D_t^\alpha ({}_a D_t^{-\alpha} f(t)) = f(t). \tag{2.28} $$

2. If $f \in C^n([a,b])$, then

$$ {}_a D_t^{-\alpha} ({}_a^C D_t^\alpha f(t)) = f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(t-a)^k. \tag{2.29} $$

In particular, if $0<\alpha\leq 1$ and $f\in C([a,b])$, then

$$ {}_a D_t^{-\alpha} ({}_a^C D_t^\alpha f(t)) = f(t) - f(a). \tag{2.30} $$

Some properties of the Caputo's derivative

Let $n-1\leq\alpha<n$ and $f\in C^{n+1}([a,b])$:

Propertie 1: The fractional derivative of a constant function in the sense of Caputo is zero.

Proof. Let $f(t) = c$. Since $f^{(n)} = 0$, we have:

$$ {}_a^C D_t^\alpha c = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}}\, d\tau = 0. \tag{2.31}$$

■ **Propertie 2:** The fractional derivative of the function $f(t) = (t-a)^\beta$ is given by:

$$ {}_a^C D_t^\alpha (t-a)^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(t-a)^{\beta-\alpha}, for \beta > \alpha. \tag{2.32}$$

Proof. It is reminded that the fractional derivative in the sense of Caputo and that of Riemann-Liouville are related by the formula:

$$ {}_a^C D_t^\alpha f(t) = {}_a^R D_t^\alpha f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{\Gamma(k-\alpha+1)}(t-a)^{k-\alpha}. \tag{2.33}$$

If $\alpha > \beta$, we have: $f^{(k)}(a) = 0$, for all $k = 0, 1, \ldots, n-1$, which implies:

$$ {}_a^C D_t^\alpha (t-a)^\beta = {}_a^R D_t^\alpha (t-a)^\beta. \tag{2.34}$$

Let's now calculate ${}_a^R D_t^\alpha t^\beta$.

Let $n-1 < \alpha \leq n$, then for all $\beta > \alpha$, we have:

$$\begin{aligned} {}_a D_t^{-(n-\alpha)} f(t) &= \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f(\tau)\, d\tau \\ &= \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} \tau^\beta\, d\tau. \end{aligned} \tag{2.35}$$

Using the change of variables: $\tau = xt$, equation (2.36) becomes:

$$\begin{aligned}
{}_aD_t^{-(n-\alpha)} f(t) &= \frac{1}{\Gamma(n-\alpha)} \int_0^1 (1-x)^{n-\alpha-1} t^{n-\alpha-1} x^{\beta} t^{\beta+1} dx \\
&= \frac{t^{n-\alpha+\beta}}{\Gamma(n-\alpha)} \int_0^1 (1-x)^{n-\alpha-1} x^{\beta} dx \\
&= \frac{t^{n-\alpha+\beta}}{\Gamma(n-\alpha)} \beta(n-\alpha;\beta+1) \\
&= t^{n-\alpha+\beta} \frac{\Gamma(\beta+1)}{\Gamma(n-\alpha+\beta+1)}.
\end{aligned}$$

By applying relation

$$\begin{aligned}
{}_a^RD_t^{\alpha} f(t) &= \frac{d^n}{dt^n}\left({}_a^RD_t^{-(n-\alpha)} f(t)\right) \\
&= \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-\tau)^{n-\alpha-1} f(\tau)\, d\tau,
\end{aligned}$$

We get:

$$\begin{aligned}
{}_a^RD_t^{\alpha} f(t) &= \frac{d^n}{dt^n}\left({}_aD_t^{-(n-\alpha)} f(t)\right) \\
&= {}_a^RD_t^{\alpha} (t-a)^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} (t-a)^{\beta-\alpha}.
\end{aligned} \tag{2.36}$$

Finally

$$ {}_a^CD_t^{\alpha} (t-a)^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} (t-a)^{\beta-\alpha}. \tag{2.37}$$

In particular, if $\beta = 2$ and $a = 0$, the fractional derivative in the Caputo sense of $f(t) = t^2$ is given by:

$$ {}_0^CD^{\alpha} t^2 = \frac{2}{\Gamma(3-\alpha)} t^{2-\alpha}. \tag{2.38}$$

■ **Propertie 3:** For every $m \in \mathbb{N}^*$ and $n-1 < \alpha \leq n$, we have:

$$ {}_a^CD_t^{\alpha}({}_aD_t^m f(t)) = {}_a^CD_t^{\alpha+m} f(t). \tag{2.39}$$

Proof. By definition, we have:

$$\begin{aligned}
{}_a^C D_t^{\alpha}\left({}_a D_t^m f(t)\right) &= {}_a D_t^{-(n-\alpha)}{}_a D_t^n\left({}_a D_t^m f(t)\right)\\
&= {}_a D_t^{-(n-\alpha)}{}_a D_t^{n+m} f(t)\\
&= {}_a^C D_t^{m+\alpha} f(t).
\end{aligned}$$

■

2.2.3 Grünwald-Letnikov fractional derivative

As well known, the classical derivatives can be expressed as differential quotients, i.e., as limits of difference quotients. For example, the n-th order derivative of a function $f(t) \in C^n([a,b])$ is defined by

$$f^{(n)}(t) = \frac{d^n f}{dt^n} = \lim_{h\to 0} \frac{1}{h^n} \sum_{k=0}^{n} (-1)^k \binom{n}{k} f(t-kh), \tag{2.40}$$

where

$$\binom{n}{k} = \frac{n(n-1)(n-2)\dots(n-k+1)}{k!},$$

is the usual notation for the binomial coefficients.

The previous equality may be used to define a fractional derivative of Grünwald-Letnikov by replacing n by $\alpha \in \mathbb{R}^+$:

$${}_a^{GL} D_t^{\alpha} f(t) = \lim_{h\to 0} \frac{1}{h^\alpha} \sum_{k=0}^{n} (-1)^k \binom{\alpha}{k} f(t-kh).$$

Since $\alpha \in \mathbb{R}^+$, the binomial coefficient is given by

$$\binom{\alpha}{k} = \frac{\Gamma(\alpha+1)}{k!\Gamma(\alpha-k+1)}.$$

Hence,

$${}_a^{GL} D_t^{\alpha} f(t) = \lim_{h\to 0} \frac{1}{h^\alpha} \sum_{k=0}^{\frac{t-a}{h}} (-1)^k \frac{\Gamma(\alpha+1)}{k!\Gamma(\alpha-k+1)} f(t-kh). \tag{2.41}$$

where $nh = t - a$

2.3 Fractional differential equations in the Caputo sense

Consider the following initial value problem:

$$\begin{cases} {}^{C}D^{\alpha}y(t) = f(t, y(t)), \\ {}^{C}D^{j}y(0) = y^{(j)}(0), \quad j = 0, 1, \ldots, n-1. \end{cases} \tag{2.42}$$

where ${}^{C}D_t^{\alpha}$ denotes the Caputo derivative operator, and $n - 1 < \alpha \leq n$. The following theorem allows us to assert the existence and uniqueness of the solution to the initial value problem.

Theorem 2.3.1 *[19]*
Let $K > 0$, $h^* > 0$, and $y(0i) \in \mathbb{R}$ for $i = 0, 1, \ldots, n-1$. Let $f : G = [0, h^*] \times \mathbb{R} \to \mathbb{R}$ be a continuous function that satisfies the Lipschitz condition with respect to y.

$$\left| f(t, y_1) - f(t, y_2) \right| < L \left| y_1 - y_2 \right|, \tag{2.43}$$

and let:

$$h = \min \left\{ h^*, \left(\frac{K\Gamma(\alpha + 1)}{M} \right)^{\frac{1}{\alpha}} \right\}, \tag{2.44}$$

thus

$$M = \sup_{t,z \in G} \left| f(t, z) \right|, \tag{2.45}$$

Then, the problem admits a unique solution $y \in C[0, h]$.

Theorem 2.3.2 *[19] Under the assumptions of the previous theorem, the initial value problem* (2.42) *is equivalent to the Volterra integral equation:*

$$y(t) = \sum_{k=0}^{n-1} \frac{t^k}{k!} y_0^{(k)} + \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha - 1} f(\tau, y(\tau))\, d\tau. \tag{2.46}$$

2.4 Fractional chaotic systems

Chaos can not be realized in continuous autonomous dynamic systems of order less than three. The model of a chaotic system can be reorganized into three differential equations containing fractional derivatives. Now, let us consider a nonlinear system with a fractional derivative.

$$D^{\alpha}x = f(x), \tag{2.47}$$

where, $x \in \mathbb{R}^n$, $0 < \alpha < 1$ and D^{α} is the Caputo fractional derivative operator.
The equilibrium points of the system (2.47) are calculated by solving the following equation:

$$D^{\alpha}x = 0, \tag{2.48}$$

2.4.1 Stability theorem of linear fractional system

In the theory of stability for linear systems of integer order, it is well known that a system is stable if all the roots of the characteristic polynomial have strictly negative real parts, i.e., they are located on the left half of the complex plane. However, in the case of fractional linear systems, the definition of stability is strictly different from that of classical systems. Indeed, in fractional systems, it is possible to have roots in the right half of the complex plane and still be stable. The following theorems allow us to assert the necessary and sufficient conditions for the stability of fractional systems.

Theorem 2.4.1 *[20, 21]*
Consider the fractional nonlinear system described by the following model:

$$\begin{cases} D^{\alpha}x &= f(x), \\ x(0) &= x_0. \end{cases} \tag{2.49}$$

where $x \in \mathbb{R}^n$, $0 < \alpha < 1$, and $f \in \mathbb{R}^n$ is a continuous nonlinear function. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of the Jacobian matrix $\dfrac{\partial f}{\partial x}$ associated with f at the equilibrium point. Then, *system* (2.49) *is asymptotically stable if and only if:*

$$|\arg(\lambda_i)| > \alpha\frac{\pi}{2} \quad \textit{for all } i = 1, 2, \ldots, n. \tag{2.50}$$

Remark 2.4.2 *If the system under study is linear, the stability conditions in the previous theorems remain true, simply by replacing the Jacobian matrix of f with its linear part.*

Example 2.4.1 *Consider the following fractional nonlinear system:*

$$\begin{cases} D^{\alpha}x = -x + 3y, \\ D^{\alpha}y = -x + 2y - xy^2, \end{cases} \tag{2.51}$$

where D^{α} is the Caputo derivative operator.
The equilibrium points of the system satisfie

$$\begin{cases} -x + 3y = 0, \\ -x + 2y - xy^2 = 0. \end{cases} \tag{2.52}$$

The only equilibrium point of the system is $E(0, 0)$.
The jacobian matrix of the system, associated of E is given by:

$$\begin{pmatrix} -1 & 3 \\ -1 & 2 \end{pmatrix} \tag{2.53}$$

The corresponding characteristic polynomial is:

$$P(\lambda) = \lambda^2 - \lambda + 1. \tag{2.54}$$

The eigenvalues of the jacobian matrix of the system are given by:

$$\lambda_1 = \frac{1}{2} + \frac{\sqrt{3}}{2}i \quad \textit{and} \quad \lambda_2 = \frac{1}{2} - \frac{\sqrt{3}}{2}i. \tag{2.55}$$

It is easy to see that in the integer case, the equilibrium point E is unstable. However, in the *fractional cases, when $\alpha = 0.6$, we get*

$$|\arg(\lambda_1, \lambda_2)| = 1.04 > \alpha\frac{\pi}{2} = 1.02. \tag{2.56}$$

which ensure the stability of the equilibrium point E.

Theorem 2.4.3 *[22] (Fractional Lyapunov)*
If there exists a positive Lyapunov function $V(x)$, such that

$$D^{\alpha}(V(x)) < 0, \quad \forall t \geq t_0, \tag{2.57}$$

then the solution of the system

$$D^{\alpha}x(t) = f(x(t)), \tag{2.58}$$

is asymptotically stable.

Lemma 2.4.1 *[22]*
Let $x(t) \in \mathbb{R}$ be a differentiable function in the sense of Caputo. Then, for all $t > t_0$,

$$\frac{1}{2}D^{\alpha}x^2(t) \leq x(t)D^{\alpha}x(t), \quad \alpha \in (0,1). \tag{2.59}$$

Remark 2.4.4 *The previous lemma remains true if $x(t) \in \mathbb{R}^n$, and in this case:*

$$\frac{1}{2}D^{\alpha}x^T(t)x(t) \leq x^T(t)D^{\alpha}x(t), \quad \forall\alpha \in (0,1). \tag{2.60}$$

Example 2.4.2 *Consider the following fractional system:*

$$\begin{cases} D^{\alpha}x = -\sqrt{2}x + y^2 + yz, \\ D^{\alpha}y = -y(1+x) + z^2, \\ D^{\alpha}z = -z(1+y) - xy. \end{cases} \tag{2.61}$$

Also consider the following Lyapunov function:

$$V(x,y,z) = \frac{1}{2}\left(x^2 + y^2 + z^2\right). \tag{2.62}$$

When Lemma 2.7.1 is used, the fractional derivative of the Lyapunov function (2.62) *satisfies:*

$$
\begin{aligned}
D^{\alpha}V(x,y,z) &= \frac{1}{2}D^{\alpha}x^2 + \frac{1}{2}D^{\alpha}y^2 + \frac{1}{2}D^{\alpha}z^2 \qquad (2.63)\\
&\leq xD^{\alpha}x + yD^{\alpha}y + zD^{\alpha}z\\
&= x(-\sqrt{2}x + y^2 + yz)\\
&+ y(-y - xy + z^2)\\
&+ z(-z - yz - xy)\\
&\leq -\sqrt{2}x^2 - y^2 - z^2 < 0, \textit{ for all } (x,y,z) \neq (0,0,0).
\end{aligned}
$$

which shows the asymptotic stability of the system (2.61),

CHAPTER 3

DYNAMICAL ANALYSIS OF A SATELLITE CHAOTIC SYSTEMS MODEL

In this chapter, we discuss some behavior dynamics related to satellite chaotic systems model with fractional-order derivative.

3.1 Description of satellite systems

The attitude dynamics of the satellite are described in the inertial coordinate system as [23, 24]:

$$\dot{T} = T_a + T_b + T_c. \tag{3.1}$$

The total momentum acting on the satellite is T. The flywheel torques, gravitational torques, and disturbance torques are represented by T_a, T_b and T_c, respectively. The overall momentum T is calculated as follows:

$$T = I\omega. \tag{3.2}$$

where ω is the angular velocity and I is the inertia matrix. The total momentum derivatives T are represented as

$$\dot{T} = I\dot{\omega} + \omega \times I\omega. \tag{3.3}$$

The cross-product of the vectors is denoted by the symbol $\times$. When we combine these equations, we get

$$I\dot{\omega} + \omega \times I\omega = T_a + T_b + T_c. \tag{3.4}$$

Let $I = diag(I_x, I_y, I_z)$. So

$$T_a = \begin{bmatrix} T_{ax} \\ T_{ay} \\ T_{az} \end{bmatrix}, \quad T_b = \begin{bmatrix} T_{bx} \\ T_{by} \\ T_{bz} \end{bmatrix} \quad and \quad T_c = \begin{bmatrix} T_{cx} \\ T_{cy} \\ T_{cz} \end{bmatrix}$$

The satellite system [23, 24, 25] is written as.

$$\begin{cases} I_x\dot{\omega}_x = \omega_y\omega_z(I_y - I_z) + h_x + u_x, \\ I_y\dot{\omega}_y = \omega_x\omega_z(I_z - I_x) + h_y + u_y, \\ I_z\dot{\omega}_z = \omega_x\omega_y(I_x - I_y) + h_z + u_z. \end{cases} \tag{3.5}$$

where

$$\left[h_x = T_{ax} + T_{bx} + T_{cx}\right], \quad \left[h_y = T_{ay} + T_{by} + T_{cy}\right], \quad \left[h_z = T_{az} + T_{bz} + T_{cz}\right] \tag{3.6}$$

Here, u_x, u_y and u_z are three control torques, while h_x, h_y and h_z are perturbing disturbance torques.

We suppose that $I_x > I_y > I_z = 1.I_x = 3, I_y = 2$ and $I_z = 1$.

The perturbing torques [12] is defined as

$$\begin{pmatrix} h_x \\ h_y \\ h_z \end{pmatrix} = \begin{pmatrix} -1.2 & 0 & \frac{\sqrt{6}}{2} \\ 0 & 0.35 & 0 \\ -\sqrt{6} & 0 & -0.4 \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} \tag{3.7}$$

The following is the formula for the chaotic satellite system:

$$\begin{cases} \dot{x} = \sigma_x yz - \frac{1.2}{I_x}x + \frac{\sqrt{6}}{2I_x}z, \\ \dot{y} = \sigma_y xz + \frac{0.35}{I_y}y, \\ \dot{z} = \sigma_z xy - \frac{\sqrt{6}}{I_z}x - \frac{0.4}{I_z}z, \end{cases} \tag{3.8}$$

where $\sigma_x = \frac{I_y - I_z}{I_x} = \frac{1}{3}$, $\sigma_y = \frac{I_z - I_x}{I_y} = -1$ and $\sigma_z = \frac{I_x - I_y}{I_z} = 1$.
So, the $3D$ chaotic satellite system becomes

$$\begin{cases} \dot{x} = \frac{1}{3}yz - ax + \sqrt{\frac{1}{6}}z, \\ \dot{y} = -xz + by, \\ \dot{z} = xy - \sqrt{6}x - cz. \end{cases} \tag{3.9}$$

where $a = 0.4, b = 0.175$ and $c = 0.4$.
The $3D$ fractional derivative satellite system model is represented as [13]

$$\begin{cases} D^\alpha x = \frac{1}{3}yz - ax + \sqrt{\frac{1}{6}}z, \\ D^\alpha y = -xz + by, \\ D^\alpha z = xy - \sqrt{6}x - cz. \end{cases} \tag{3.10}$$

where D^α is the Caputo differentiel operator and $\alpha \in]0, 1]$ is the fractional-order.

3.1.1 Attractors of the chaotic satellite system

When the initial values of the fractional-order chaotic sarelitte system (3.10) are selected as

$$x(0) = y(0) = x_3(0) = 0.01 \tag{3.11}$$

and the parameter values are taken as

$$a = 0.4, b = 0.175 \text{ and } c = 0.4, \tag{3.12}$$

the chaotic attractors of the system (3.10) are represented in Figure 3.1 in various coordinate planes.

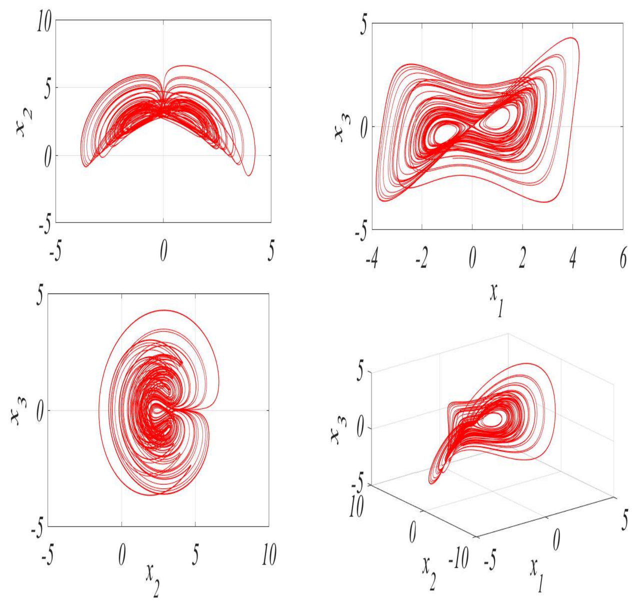

Figure 3.1: Different attractors of the fractional-order chaotic satellite system (3.10).

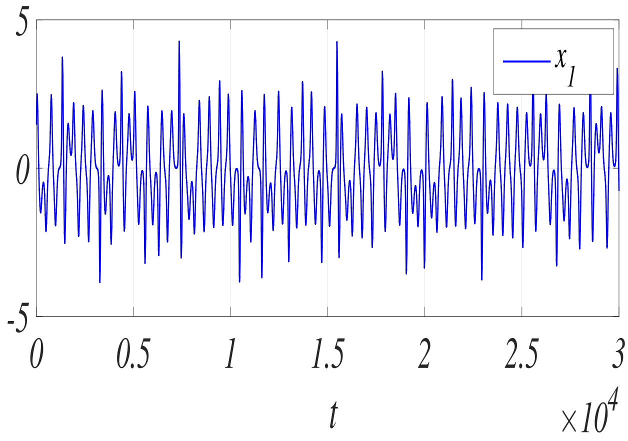

Figure 3.2: Temporel evolution of the state x_1.

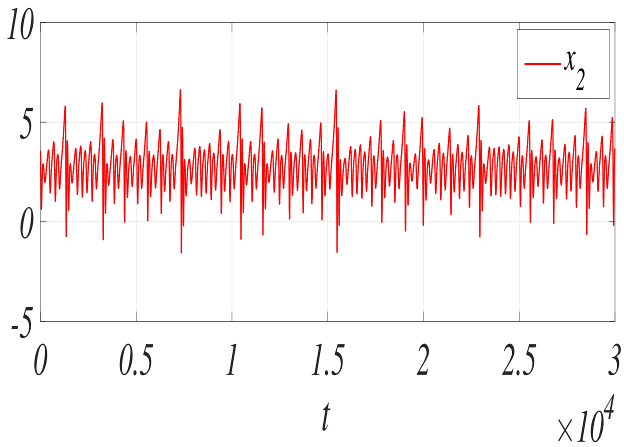

Figure 3.3: Temporel evolution of the state x_2.

3.1.2 Equilibrium points

By solving the following system of equations

$$\begin{cases} \dfrac{1}{3}yz - ax + \frac{1}{\sqrt{6}}z = 0, \\ -xz + by = 0, \\ xy - \sqrt{6}x - cz = 0, \end{cases} \tag{3.13}$$

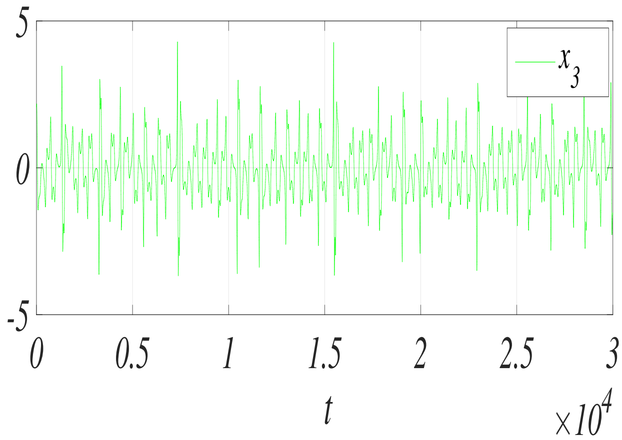

Figure 3.4: Temporel evolution of the state x_3.

the equilibrium points of the satellite system (3.10) can be found as:

$$P_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, P_1 = \begin{bmatrix} 1.1910 \\ 2.5766 \\ 0.3785 \end{bmatrix}, P_2 = \begin{bmatrix} 0.1582 \\ -1.3641 \\ -1.5086 \end{bmatrix}, P_3 = \begin{bmatrix} -0.1582 \\ -1.3641 \\ 1.5086 \end{bmatrix} \text{ and } P_4 = \begin{bmatrix} -1.1910 \\ 2.5766 \\ -0.3785 \end{bmatrix} \tag{3.14}$$

The Jacobian matrix of the system is

$$J(X) = \begin{bmatrix} -a & 0.33 * z & 0.33y + \frac{1}{\sqrt{6}} \\ -z & b & -x \\ (y - \sqrt{6}) & x & -c. \end{bmatrix} \tag{3.15}$$

At the equilibrium point P_0, the eigenvalues of system are

$$\lambda_1 = -0.4 + 0.99i, \ \lambda_2 = -0.4 + 0.99i \text{ and } \lambda_3 = 0.175. \tag{3.16}$$

Since, $\left|\arg(\lambda_3)\right| = 0 < \alpha\frac{\pi}{2}$, for all $\alpha \in (0, 1]$, the, the equilibrium point P_0 of the system is unstable point.

3.1.3 Lyapunov Exponents

In this section, we assume that the parameters b, c remain constant and a is varied in $[0, 0.5]$. By using Wolf algorithm [15], the Lyapunov exponents spectrum of system (3.10) with $b = 0.175$ and $c = 0.4$ is represented in Figure 3.5.

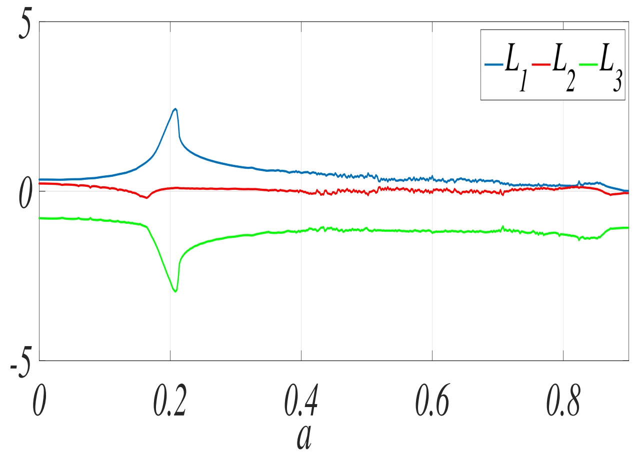

Figure 3.5: The three lyapunov exponents of the system (3.10) versus a.

In particular, for the parameter values are taken as in case (3.1.1), the values of Lyapunov exponents of non-linear system (3.10) are given by

$$\begin{cases} L_1 = 0.557, \\ L_2 = 0, \\ L_3 = -1.168. \end{cases} \tag{3.17}$$

Since $L_1 + L_2 + L_3 < 0$ and $L_1 > 0$, then the proposed system is dissipative and chaotic.

The Kaplan-Yorke dimension of the chaotic system is obtained as:

$$\begin{aligned} D_{KY} &= 2 + \frac{L_1 + L_2}{|L_3|} \\ &= 2.476, \end{aligned}$$

which is a fractal dimension.

3.1.4 Dissipative system

The system (3.10) can be written in vector notation as

$$D^{\alpha}X = F(x) \tag{3.18}$$

where, $X(t) = (x, y, z)$ and

$$F(X) = F(x, y, z) = \begin{bmatrix} F_1(x, y, z) \\ F_2(x, y, z) \\ F_3(x, y, z) \end{bmatrix} = \begin{bmatrix} \frac{1}{3}yz - ax + \sqrt{16}\,z \\ -xz + by \\ xy - \sqrt{6}\,x - cz \end{bmatrix} \tag{3.19}$$

where $a = 0.4, b = 0.175 and c = 0.4.$. We assume any smooth boundary area $\Lambda(t) \in \mathbb{R}^3$, with $\Lambda(t) = \Theta_t(\Lambda)$ where Θ_t displayed the flow of F.
Let $V(t)$ be the volume of $\Lambda(t)$. We get through Liouville's theorem

$$\dot{V}(t) = \int_{\Lambda(t)} (\nabla \cdot F)\, dx\, dy\, dz. \tag{3.20}$$

The divergence of the satellite system (3.20) is calculated as:

$$\nabla \cdot f = \left[\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right] = -a + b - c = -0.625. \tag{3.21}$$

From (3.20) and (3.21), the first derivatives ordinary differential equation can be written as follows:

$$\dot{V}(t) = -0.625V(t). \tag{3.22}$$

By integrating the equation (3.22), we get

$$V(t) = V(0)\exp(-0.625t). \tag{3.23}$$

So, when $t \to \infty$, $V(t) \to 0$, which is demonstrates the dissipative character of the satellite system (3.23).

3.2 Control of fractional-order chaotic satellite system

Consider the controlled chaotic system as follows

$$\begin{cases} D^\alpha x &= \frac{1}{3}yz - ax + \dfrac{1}{\sqrt{6}}z, \\ D^\alpha y &= -xz + by - ky, \\ D^\alpha z &= xy - \sqrt{6}x - cz. \end{cases} \tag{3.24}$$

where k is the positive feedback control gains. We have the following result.

Theorem 3.2.1 *The controlled chaotic system* (3.24) *is asymptotically stable if* $b < k$

Proof. Defne the Lyapunov functional as

$$V(x,y,z) = \frac{1}{2}(6x^2 + 3y^2 + z^2). \tag{3.25}$$

We have

$$\begin{aligned} D^\alpha V &\leq (6xD^\alpha x + 3yD^\alpha y + zD^\alpha z) \\ &= 6[\frac{1}{3}xyz - ax^2 + \frac{1}{\sqrt{6}}xz] + 3[-xyz + by^2 - ky^2] + [xyz - \sqrt{6}xz - cz^2] \\ &= -6ax^2 + 3by^2 - 3ky^2 - cz^2 \\ &= -6ax^2 - y^2(-3b + 3k) - cz^2 < 0. \end{aligned}$$

According to Theorem 2.4.3, it can be remarked that the controlled system (3.24) asymptotically converges to 0, which means that the controlled system is stable. ■ The Adams–Bashforth–Moulton method [26] is used to solve this system. The initial conditions of the system are taken as as $x(0) = 0.1$, $y(0) = 0.1$ and $z(0) = 0.2$. To ensure the stability of the controlled system, we take $k = 0.7$. The fractional-order is selected as $\alpha = 0.97$.

Figure 3.6 shows the time-history of the controlled chaotic system (3.24). Obviously, all solutions of the this system approach to the origine values, which confirm that the controlled chaotic system (3.24) is asymptotically stable.

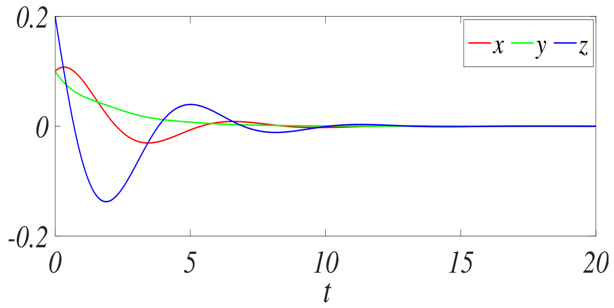

Figure 3.6: Time-history of the controlled satellite system (3.24).

3.3 Adaptive synchronization of fractional-order chaotic satellite systems

In this section, we construct an adaptive synchronizer for global synchronization of fractional-order chaotic satellite systems. The adaptive synchronizer design is carried out using Lyapunov stability theory.
As the master system, we take the chaotic of fractional-order chaotic satellite system.

$$\begin{cases} D^{\alpha}x_1 = \dfrac{1}{3}x_2x_3 - ax_1 + \dfrac{1}{\sqrt{6}}x_1x_3, \\ D^{\alpha}x_2 = -x_1x_3 + bx_2, \\ D^{\alpha}x_3 = x_1x_2 - \sqrt{6}x_1 - cx_3. \end{cases} \tag{3.26}$$

Where x_1, x_2, x_3 are state variables and a, b, c are unknown, constant, parameters.
As the slave system, we take also the fractional-order chaotic satellite system.

$$\begin{cases} D^{\alpha}y_1 = \dfrac{1}{3}y_2y_3 - ay_1 + \dfrac{1}{\sqrt{6}}y_1y_3 + U_1, \\ D^{\alpha}y_2 = -y_1y_3 + by_2 + U_2, \\ D^{\alpha}y_3 = y_1y_2 - \sqrt{6}y_1 - cy_3 + U_3. \end{cases} \tag{3.27}$$

Where y_1, y_2, y_3 are state variables and U_1, U_2, U_3 are adaptive controls. The complete synchronization error between the systems (3.26) and (3.27) is defined as:

$$\begin{cases} e_1 = y_1 - x_1, \\ e_2 = y_2 - x_2, \\ e_3 = y_3 - x_3. \end{cases} \tag{3.28}$$

The error dynamics is easily obtained as:

$$\begin{cases} D^\alpha e_1 = -ae_1 + \frac{1}{3}(y_2 y_3 - x_2 x_3) + \frac{1}{\sqrt{6}}(y_1 y_3 - x_1 x_3) + U_1, \\ D^\alpha e_2 = be_2 - (y_1 y_3 - x_1 x_3) + U_2, \\ D^\alpha e_3 = -\sqrt{6}e_1 - ce_3 + y_1 y_2 - x_1 x_2 + U_3. \end{cases} \tag{3.29}$$

We consider the adaptive controller defined by

$$\begin{cases} U_1 = \hat{a}e_1 - \frac{1}{3}(y_2 y_3 - x_2 x_3) - \frac{1}{\sqrt{6}}(y_1 y_3 - x_1 x_3) - K_1 e_1, \\ U_2 = -\hat{b}e_2 + (y_1 y_3 - x_1 x_3) - K_2 e_2, \\ U_3 = \sqrt{6}e_1 + \hat{c}e_3 - y_1 y_2 + x_1 x_2 - K_3 e_3, \end{cases} \tag{3.30}$$

where K_1, K_2, K_3 are positive gain constants, and $\hat{a}, \hat{b}, \hat{c}$ are estimates of the unknown parameters a, b, c respectively.
Substituting (3.30) into (3.29), we get

$$\begin{cases} D^\alpha e_1 = -(a - \hat{a})e_1 - K_1 e_1, \\ D^\alpha e_2 = (b - \hat{b})e_2 - K_2 e_2, \\ D^\alpha e_3 = -(c - \hat{c})e_3 - K_3 e_3. \end{cases} \tag{3.31}$$

The parameter estimation errors are defined by

$$\begin{cases} e_a = a - \hat{a}, \\ e_b = b - \hat{b}, \\ e_c = c - \hat{c}. \end{cases} \tag{3.32}$$

Substituting (3.32) into the error dynamics (3.31), we get

$$\begin{cases} D^\alpha e_1 = -e_a e_1 - K_1 e_1, \\ D^\alpha e_2 = e_b e_2 - K_2 e_2, \\ D^\alpha e_3 = -e_c e_3 - K_3 e_3. \end{cases} \tag{3.33}$$

Differentiating (3.32) with respect to t, we get

$$\begin{cases} D^\alpha e_a = -D^\alpha \hat{a}, \\ D^\alpha e_b = -D^\alpha \hat{b}, \\ D^\alpha e_c = -D^\alpha \hat{c}. \end{cases} \tag{3.34}$$

Consider the quadratic Lyapunov function defined by

$$V = \frac{1}{2}(e_1^2 + e_2^2 + e_3^2 + e_a^2 + e_b^2 + e_c^2), \tag{3.35}$$

Which is positive definite on R^6.

Differentiating V along the trajectories of (3.33) and (3.34), we get

$$\begin{aligned} D^\alpha V &\leq e_1 D^\alpha e_1 + e_2 D^\alpha e_2 + e_3 D^\alpha e_3 + e_a D^\alpha e_a + e_b D^\alpha e_b + e_c D^\alpha e_c \qquad (3.36) \\ &= -e_a e_1^2 - k_1 e_1^2 + e_b e_2^2 - k_2 e_2^2 - e_c e_3^2 - k_3 e_3^2 - e_a D^\alpha \hat{a} - e_b D^\alpha \hat{b} - e_c D^\alpha \hat{c} \\ &= -[k_1 e_1^2 + k_2 e_2^2 + k_3 e_3^2] + e_a[-e_1^2 - D^\alpha \hat{a}] + e_b[e_2^2 - D^\alpha \hat{b}] + e_c[-e_3^2 - D^\alpha \hat{c}]. \end{aligned}$$

In view of (3.37), we take the parameter update law as:

$$\begin{cases} D^{\alpha}\hat{a} = -e_1^2, \\ D^{\alpha}\hat{b} = e_2^2, \\ D^{\alpha}\hat{c} = -e_3^2. \end{cases} \tag{3.37}$$

Next, we shall establish the main result of this section.

Theorem 3.3.1 *The systems* (3.26) *and* (3.27) *are globally and exponentially synchronized by the adaptive control law* (3.30) *and the parameter update law* (3.37), *for all initial conditions, where k_1, k_2, k_3, are positive gain constants.*

Proof. We prove this result using Lyapunov stability theory.
For this purpose, we consider the quadratic Lyapunov function V defined by (3.37), which is positive definite on $\mathbb{R}^6$.
Substituting the parameter update law (3.37) into (3.37), we obtain the fractional derivative of V as:

$$D^{\alpha}V - \leq k_1e_1^2 - k_2e_2^2 - k_3e_3^2 < 0. \tag{3.38}$$

According to Theorem 2.4.3, it can be remarked that the error system (3.33) asymptotically converges to 0, which means that the synchronization between the systems (3.26) and (3.27) is achieved. ■

In the numerical simulation, we take the initial conditions of the master system as

$$x_1(0) = 0.1,\ x_2(0) = 0.1 \text{ and } x_3(0) = 0.1,$$

the initial conditions of the slave system are taken as

$$y_1(0) = 0.2,\ y_2(0) = 0.3 \text{ and } y_3(0) = -0.1.$$

thus the initial conditions of the error system are taken as

$$(e_1(0), e_2(0), e_3(0))^T = (0.1, 0.2, -0.2)^T.$$

We take the gain constants as

$$K_i = 0.5, \text{ for all } i = 1, 2, 3.$$

Regarding the initial values for the parameter estimates, we take

$$\left(\hat{a}(0),\ \hat{b}(0),\ \hat{b}(0)\right) = (0.3,\ 0.2,\ 0.3)\,.$$

We take also the fractional-order $\alpha = 0.97$. Figure 3.7 shows the synchronization error between the master system (3.26) and the slave system (3.27).
From the Figure 3.7 , one can see that all solutions of the error system (3.33) approachs to zero, which shows that the synchronization between the systems is achieved.

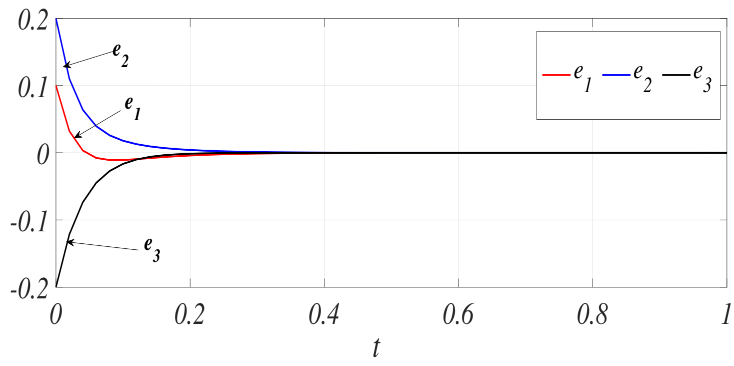

Figure 3.7: Temporal evolution of the synchronization error (3.33).

GENERAL CONCLUSION

In this work, we analyzed the chaotic behavior of fractional-order satellite systems. Specifically, we used phase portrait analysis, equilibrium point determination, and calculation of the Jacobian matrix eigenvalues at these points, as well as dissipability and Lyapunov exponents to study the chaotic behavior of this system. We also implemented a feedback control approach to control chaos in this system to ensure its asymptotic stability. We also successfully synchronized two identical fractional-order satellite systems using an adaptive control technique.

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