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Asymptotics and periodicity of positive solutions on a nonlinear rational systems of difference equations

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ABSTRACT

This thesis explores the solutions of difference equation systems related to well-known sequences such as Bell and Jacobsthal. It focuses on the link between these sequences and the dynamic behavior of nonlinear systems. Chapter 1 presents the algebraic properties of generalized Pell and Jacobsthal sequences. Chapters 2 and 3 analyze two nonlinear systems using these sequences to derive explicit solutions and study stability. The methodology involves examining fixed points and long-term behavior. The aim is to highlight the connection between special number sequences and the qualitative dynamics of discrete systems.

Keywords: General solution, System of difference equations, $(k,h)-$Pell sequence, k-Jacobsthal sequence, Stability.

ملخص

تهدف هذه المذكرة إلى دراسة حلول أنظمة معادلات الفروق المرتبطة بمتتاليات عددية شهيرة، مثل متتاليتَي Pell و Jacobsthal ويتركز الاهتمام على استكشاف العلاقة بين هذه السلاسل العددية والسلوك الديناميكي للأنظمة غير الخطية. يتناول الفصل الأول الخصائص الجبرية للنسخ المعممة من متتاليتَي Pell و Jacobsthal أما الفصلان الثاني والثالث، فيُخصصان لدراسة نظامين غير خطيين، حيث تُستخدم هذه المتتاليات لاشتقاق حلول صريحة وتحليل خصائص الاستقرار. وتعتمد المنهجية المتبعة على تحويل هذه الأنظمة غير الخطية إلى أنظمة خطية ترتبط بشكل مباشر بهذه المتتاليات المعروفة.

الكلمات الأساسية: الحل العام، جمل معادلات الفروق، متتالية بال المعممة، متتالية جاكوبستال المعممة، الاستقرار.

RÉSUMÉ

Ce mémoire étudie les solutions des systèmes d'équations aux différences liés à des suites célèbres telles que celles de Bell et de Jacobsthal. Il met l'accent sur la relation entre ces suites et le comportement dynamique des systèmes non linéaires. Le premier chapitre présente les propriétés algébriques des suites généralisées de Pell et de Jacobsthal. Les chapitres 2 et 3 analysent deux systèmes non linéaires en utilisant ces suites pour obtenir des solutions explicites et étudier la stabilité. La méthodologie repose sur l'analyse des points d'équilibre et du comportement à long terme. L'objectif est de montrer le lien entre suites spéciales et dynamique des systèmes discrets.

Mots-clés: Solution générale, Système d'équations aux différences, Suite de $(k, h)-$Pell, Suite de k-Jacobsthal, Stabilité.

CONTENTS

INTRODUCTION

Difference equations are fundamental tools for studying discrete changes, as they describe the relationships between successive terms in a sequence. These equations are closely linked to several well-known sequences that have significant applications in various fields of mathematics and science, such as the Fibonacci sequence, the Bell sequence, and the Jacobsthal sequence. Each of these sequences can be defined or generated using specific types of difference equations that capture the recurrence rules governing the progression of terms. Understanding the connection between difference equations and such famous sequences not only simplifies their analysis but also paves the way for generalizations and new models used to solve real-world problems in computer science, cryptography, number theory, and other disciplines.

In this master's thesis, we explore the intricate relationship between special numerical sequences and the qualitative behavior of nonlinear difference equations. Our primary objective is to utilize the structural and algebraic properties of generalized sequences specifically the generalized Pell and Jacobsthal sequences to analyze and solve discrete dynamical systems. By integrating these mathematical tools, we derive explicit solutions, study equilibrium points, and assess the stability of nonlinear iterative systems.

In order to carry out this master's thesis efficiently, the work has been divided as follows:

In **Chapter 1**, we introduce and investigate fundamental numerical sequences, with particular emphasis on generalized Pell and Jacobsthal sequences. We explore their algebraic properties, convergence behavior, and stability characteristics, laying the groundwork for their application in subsequent chapters.

In **Chapter 2**, we consider a nonlinear iterative system of the form:

$$x_{n+1} = \frac{1}{2k + hy_{n-l}}, \quad y_{n+1} = \frac{1}{2k + hx_{n-l}}, \quad n \geq 0,$$

where k, h, and l are fixed parameters. We employ the sequences studied in Chapter 1 to derive explicit solutions of the system, determine its equilibrium points, and conduct a comprehensive analysis of its dynamical behavior and stability.

Chapter 3 is devoted to the analysis of a second nonlinear system given by:

$$x_{n+1} = \frac{1}{k + 2y_{n-l}}, \quad y_{n+1} = \frac{1}{k + 2x_{n-l}}, \quad n \geq 0.$$

Following a similar methodological approach, we construct exact solutions, identify equilibrium configurations, and investigate the system's long-term behavior and stability characteristics.

Through these investigations, we aim to establish a clear link between special number sequences and the qualitative behavior of nonlinear difference equations.

CHAPTER 1

GENERALIZATION OF SOME WELL-KNOWN SEQUENCES

This chapter establishes the theory by examining Pell sequences, generalized (k, h)-Pell sequences, k-Jacobsthal sequences, and (k, h)-Pell-Lucas sequences and their algebraic and numerical characteristics. We provide core definitions, recurrence relations, and closed-form solutions, highlighting characteristics which are central to the examination of recurrence systems in subsequent chapters.

1.1 Some generalization of the Pell sequence

1.1.1 Pell sequence

The Pell sequence is a sequence of integers defined by the recurrence relation:

$$P_n = 2P_{n-1} + P_{n-2}, \tag{1.1}$$

where the initial conditions are given by:

$$P_0 = 0, \quad P_1 = 1. \tag{1.2}$$

The first few terms of the Pell sequence are:

$$0, 1, 2, 5, 12, 29, 70, 169, \ldots. \tag{1.3}$$

This sequence arises in various mathematical contexts, including continued fractions and approximations of square roots. Specifically, the ratio of consecutive Pell numbers approximates $1 + \sqrt{2}$, similar to how Fibonacci numbers approximate the golden ratio.

The explicit formula for the Pell sequence is given by:

$$P_n = \frac{(1 + \sqrt{2})^n - (1 - \sqrt{2})^n}{2\sqrt{2}}. \tag{1.4}$$

The Pell sequence has applications in number theory, combinatorics, and the solution of Pell's equation:

$$x^2 - 2y^2 = \pm 1, \tag{1.5}$$

where its terms appear as solutions for x and y.

1.1.2 The (k, h)−Pell sequence

The (k, h)-Pell sequence $\{\varphi_n\}_{n \geq 0}$ is a generalization of the classical Pell sequence, defined by the recurrence relation [4]:

$$\varphi_n = 2k\varphi_{n-1} + h\varphi_{n-2}, \quad \text{for } n \geq 2, \tag{1.6}$$

with initial conditions:

$$\varphi_0 = 0, \quad \varphi_1 = 2k. \tag{1.7}$$

The characteristic equation associated with this recurrence relation is given by:

$$r^2 - 2kr - h = 0. \tag{1.8}$$

Solving for the roots, we obtain:

$$\alpha = k + \sqrt{k^2 + h}, \quad \beta = k - \sqrt{k^2 + h}. \tag{1.9}$$

Thus, the general solution for φ_n can be expressed as:

$$\varphi_n = c_1\alpha^n + c_2\beta^n. \tag{1.10}$$

By using the initial conditions, we solve for the constants c_1 and c_2:

$$c_1 + c_2 = 0, \quad c_1\alpha + c_2\beta = 2k. \tag{1.11}$$

Solving this system, we obtain:

$$c_1 = \frac{2k}{\alpha - \beta}, \quad c_2 = -\frac{2k}{\alpha - \beta}. \tag{1.12}$$

Since $\alpha - \beta = 2\sqrt{k^2 + h}$, we can write:

$$\varphi_n = \frac{k}{\sqrt{k^2 + h}}(\alpha^n - \beta^n). \tag{1.13}$$

1.1.3 Properties of the (k, h)-Pell Sequence

The (k, h)-Pell sequence $\{\varphi_n\}_{n\geq 0}$ satisfies several important properties, which can be derived using its recurrence relation and characteristic equation [4].

Binet formula

The (k,h)-Pell sequence is defined by the recurrence relation:

$$\varphi_n = 2k\varphi_{n-1} + h\varphi_{n-2}, \quad \text{for } n \geq 2, \tag{1.14}$$

with initial conditions:

$$\varphi_0 = 0, \quad \varphi_1 = 2k. \tag{1.15}$$

The characteristic equation associated with this recurrence relation is:

$$r^2 - 2kr - h = 0. \tag{1.16}$$

Solving for the roots, we obtain:

$$\alpha = k + \sqrt{k^2 + h}, \quad \beta = k - \sqrt{k^2 + h}. \tag{1.17}$$

Thus, the explicit formula for φ_n is:

$$\varphi_n = \frac{k}{\sqrt{k^2 + h}}(\alpha^n - \beta^n). \tag{1.18}$$

Summation identities

Several summation identities hold for the (k,h)-Pell sequence:

1. The sum of the first n terms:

$$\sum_{m=0}^{n-1} \varphi_m = \frac{\varphi_n + h\varphi_{n-1} - 2k}{2k + h - 1}. \tag{1.19}$$

2. The sum of the product of consecutive terms:

$$\sum_{m=0}^{n-1} \varphi_m\varphi_{m-1} = \frac{4k^2}{(k^2 + h)}\left[2k - \Lambda_3 + h\Lambda_{2n-1} - h^3\Lambda_{2n-3}\right]. \tag{1.20}$$

3. The sum of squares of the terms:

$$\sum_{m=0}^{n-1} \varphi_m^2 = \frac{4k^2}{(k^2+h)}\left[h^2\Lambda_{2n-2} - \Lambda_{2n} - \Lambda_2 + 2 - \frac{(h^n-1)}{h+1}\right]. \tag{1.21}$$

Limit of the quotient of consecutive terms

The limit of the quotient of two consecutive terms of the (k,h)-Pell sequence converges to the largest root of the characteristic equation:

$$\lim_{n\to\infty} \frac{\varphi_n}{\varphi_{n-1}} = \alpha = k + \sqrt{k^2+h}. \tag{1.22}$$

This limit plays a crucial role in the asymptotic analysis of the sequence.

Relation to the (k,h)-Pell-Lucas sequence

The (k,h)-Pell sequence and the (k,h)-Pell-Lucas sequence are related by:

$$\Lambda_n = \alpha^n + \beta^n, \quad \varphi_n = \frac{k}{\sqrt{k^2+h}}(\alpha^n - \beta^n). \tag{1.23}$$

1.1.4 The $(k,h)-$Pell-Lucas sequence

The **Pell-Lucas sequence** $\{Q_n\}$ is defined by the recurrence relation [4]:

$$Q_n = 2Q_{n-1} + Q_{n-2}, \quad \text{for } n \geq 2,$$

with the initial conditions:

$$Q_0 = 2, \quad Q_1 = 2.$$

The explicit formula for the n-th term is:

$$Q_n = 2P_n,$$

where P_n represents the n-th Pell number.

The (k, h)-Pell-Lucas sequence $\{\Lambda_n\}_{n\geq 0}$ is a generalization of the classical Pell-Lucas sequence. It is defined by the recurrence relation [4]:

$$\Lambda_n = 2k\Lambda_{n-1} + h\Lambda_{n-2}, \quad \text{for } n \geq 2, \tag{1.24}$$

with the initial conditions:

$$\Lambda_0 = 2, \quad \Lambda_1 = 2k. \tag{1.25}$$

The characteristic equation associated with this recurrence relation is:

$$r^2 - 2kr - h = 0. \tag{1.26}$$

Solving for the roots, we obtain:

$$\alpha = k + \sqrt{k^2 + h}, \quad \beta = k - \sqrt{k^2 + h}. \tag{1.27}$$

Thus, the general formula for Λ_n is given by:

$$\Lambda_n = \alpha^n + \beta^n. \tag{1.28}$$

1.1.5 Properties of the (k, h)-Pell-Lucas sequence

The (k, h)-Pell-Lucas sequence $\{\Lambda_n\}_{n\geq 0}$ satisfies several important properties, which can be derived using its recurrence relation and characteristic equation [4].

Binet formula

The (k, h)-Pell-Lucas sequence is defined by the recurrence relation:

$$\Lambda_n = 2k\Lambda_{n-1} + h\Lambda_{n-2}, \quad \text{for } n \geq 2, \tag{1.29}$$

with initial conditions:

$$\Lambda_0 = 2, \quad \Lambda_1 = 2k. \tag{1.30}$$

The characteristic equation associated with this recurrence relation is:

$$r^2 - 2kr - h = 0. \tag{1.31}$$

Solving for the roots, we obtain:

$$\alpha = k + \sqrt{k^2 + h}, \quad \beta = k - \sqrt{k^2 + h}. \tag{1.32}$$

Thus, the explicit formula for Λ_n is:

$$\Lambda_n = \alpha^n + \beta^n. \tag{1.33}$$

Summation identities

Several summation identities hold for the (k, h)-Pell-Lucas sequence:

1. The sum of the first n terms:

$$\sum_{m=0}^{n-1} \Lambda_m = \frac{\Lambda_n + h\Lambda_{n-1} + 2k - 2}{2k + h - 1}. \tag{1.34}$$

2. The sum of the product of consecutive terms:

$$\sum_{m=0}^{n-1} \Lambda_m \Lambda_{m-1} = \frac{2k - \Lambda_3 + h\Lambda_{2n-1} - h^3\Lambda_{2n-3}}{-h(1 + \Lambda_2 + h^2)}. \tag{1.35}$$

3. The sum of squares of the terms:

$$\sum_{m=0}^{n-1} \Lambda_m^2 = \frac{2 - \Lambda_2 + h^2\Lambda_{2n-2} - \Lambda_{2n}}{1 - \Lambda_2 + h^2 + 2\frac{(1-h)^n}{1+h}}. \tag{1.36}$$

Limit of the quotient of consecutive terms

The limit of the quotient of two consecutive terms of the (k,h)-Pell-Lucas sequence converges to the largest root of the characteristic equation:

$$\lim_{n\to\infty} \frac{\Lambda_n}{\Lambda_{n-1}} = \alpha = k + \sqrt{k^2 + h}. \tag{1.37}$$

This limit plays a crucial role in the asymptotic analysis of the sequence.

Relation to the (k,h)-Pell sequence

The (k,h)-Pell-Lucas sequence and the (k,h)-Pell sequence are related by:

$$\Lambda_n = \alpha^n + \beta^n, \quad \varphi_n = \frac{2k}{\sqrt{k^2 + h}}(\alpha^n - \beta^n). \tag{1.38}$$

1.2 Generalization of Jacobsthal sequence

1.2.1 The k-Jacobsthal sequence

The **k-Jacobsthal sequence** $\{J_{k,n}\}_{n\geq 1}$ for any positive real number k is defined by the recurrence relation [5]:

$$J_{k,n+1} = kJ_{k,n} + 2J_{k,n-1}, \quad \text{for } n \geq 1, \tag{1.39}$$

with the initial conditions:

$$J_{k,0} = 0, \quad J_{k,1} = 1$$

The characteristic equation associated with this recurrence relation is given by:

$$r^2 - kr - 2 = 0$$

Solving for the roots, we obtain:

$$r_1 = \frac{k + \sqrt{k^2+8}}{2}, \quad r_2 = \frac{k - \sqrt{k^2+8}}{2}.$$

Thus, the general solution for $\{J_{k,n}\}$ can be expressed as:

$$J_{k,n} = c_1 r_1^n + c_2 r_2^n.$$

By using the initial conditions, we solve for the constants c_1 and c_2:

$$c_1 + c_2 = 0, \quad c_1 r_1 + c_2 r_2 = 1. \tag{1.40}$$

Solving this system, we obtain:

$$c_1 = \frac{1}{r_1 - r_2}, \quad c_2 = \frac{-1}{r_1 - r_2}. \tag{1.41}$$

Thus, the explicit formula for $\{J_{k,n}\}$ is given by:

$$J_{k,n} = \frac{r_1^n - r_2^n}{r_1 - r_2}. \tag{1.42}$$

1.2.2 Properties of the k-Jacobsthal sequence

The k-Jacobsthal sequence $\{J_{k,n}\}_{n\geq 1}$ satisfies several important properties, which can be derived using its recurrence relation and characteristic equation [5].

Binet formula

The k-Jacobsthal sequence is defined by the recurrence relation:

$$J_{k,n+1} = kJ_{k,n} + 2J_{k,n-1}, \quad \text{for } n \geq 1, \tag{1.43}$$

with initial conditions:

$$J_{k,0} = 0, \quad J_{k,1} = 1. \tag{1.44}$$

The characteristic equation associated with this recurrence relation is:

$$r^2 - kr - 2 = 0. \tag{1.45}$$

Solving for the roots, we obtain:

$$r_1 = \frac{k + \sqrt{k^2 + 8}}{2}, \quad r_2 = \frac{k - \sqrt{k^2 + 8}}{2}. \tag{1.46}$$

Thus, the explicit formula for $\{J_{k,n}\}$ is:

$$J_{k,n} = \frac{r_1^n - r_2^n}{r_1 - r_2}. \tag{1.47}$$

Explicit formula for the general term of the k-Jacobsthal sequence

Binet's formula allows us to express the k-Jacobsthal numbers in function of the roots r_1 and r_2 of the following characteristic equation, associated to the recurrence (1.39)

$$r^2 = kr + 2.$$

Summation identities

Several summation identities hold for the k-Jacobsthal sequence:

1. Catalan's identity:

$$J_{k,n-r}J_{k,n+r} - J_{k,n}^2 = (-1)^{n+1-r}J_{k,r}^2 2^{n-r}. \tag{1.48}$$

2. D'ocagne's identity: If $m > n$ then

$$J_{k,m}J_{k,n+1} - J_{k,m+1}J_{k,n} = (-2)^n J_{k,m-n}. \tag{1.49}$$

3. Another explicit expression for calculating the general term of the$k$-Jacobsthal sequence :

$$J_{k,n} = \frac{1}{2^{n-1}} \sum_{i=0}^{\lfloor \frac{n-1}{2} |\rfloor} \binom{n}{2i+1} k^{n-1-2i}(k^2+8), \tag{1.50}$$

where $\lfloor a | \rfloor$ is the floor of a, that is $\lfloor a | \rfloor = \sup\{n \in \mathbb{N} \mid n \leq a\}$ and says the integer part of a, for $a \geq 0$

Limit of the quotient of consecutive terms

The limit of the quotient of two consecutive terms of the k-Jacobsthal sequence converges to the positive root of the corresponding characteristic equation :

$$\lim_{n\to\infty} \frac{J_{k,n}}{J_{k,n-1}} = r_1. \tag{1.51}$$

This limit plays a crucial role in the asymptotic analysis of the sequence.

CHAPTER 2

SYSTEM OF DIFFERENCE EQUATIONS LINKED TO THE (k, h)-PELL SEQUENCE

This chapter investigates a higher-order difference equation system whose solutions are expressed in terms of generalized Pell sequences. We derive closed-form solutions and analyze the stability and asymptotic behavior of the system. Through the connection of the solutions to this type of general sequences, the study provides new theoretical results on recursive systems. The results enhance the understanding of dynamic processes modeled by such difference equations.

2.1 Introduction

We propose some theoretical explanations pertaining to the representation for the solution of the system of the higher-order difference equations

$$x_{n+1} = \frac{1}{2k + hy_{n-l}}, \quad y_{n+1} = \frac{1}{2k + hx_{n-l}}, \tag{2.1}$$

with $k \in \mathbb{Z}$ and $n, l \in \mathbb{N}$.

The initial conditions $x_{-l}, x_{-l+1}, \ldots, x_0, y_{-l}, y_{-l+1}, \ldots, y_0$, are non zero real numbers such that their solution is related to a generalized Pell sequences. We also study the stability character and asymptotic behavior of this system.

We will present two lemmas so that the first lemma provides the solutions of two homogeneous second order linear autonomous difference equations, which is essential for representations the solution of system (2.1). Its proof utilizes the characteristic roots of the caracteristic polynomial $\theta^2 \pm 2k\theta - h$. On the other hand, the second lemma offers the solution of a system of second order linear autonomous difference equations, which plays a crucial role in solving the system (2.1).

2.2 Preliminary Results

In this section, we explore second-order linear difference equations within the framework of (k,h)-Pell-Lucas sequences. From our analysis, explicit solutions to two principal homogeneous equations are obtained prior to applying the results to coupled systems through variable decoupling techniques. The solutions expressed in terms of initial conditions and (k,h)-Pell-Lucas sequences reveal basic recursive forms and expose their underlying algebraic structure. This systematic approach shows the way complex systems can be mapped to solvable types through careful transformations.

lemma 2.2.1 *Consider the two homogenous second order linear autonomous differences equa-*

tions :

$$y_{n+2} - 2ky_{n+1} - hy_n = 0, \tag{2.2}$$

$$z_{n+2} + 2kz_{n+1} - hz_n = 0. \tag{2.3}$$

Then we have for all $n \in \mathbb{N}_0$*:*

$$y_n = \frac{hy_0}{2k}\varphi_{n-1} + \frac{y_1}{2k}\varphi_n,$$

$$z_n = \frac{(-1)^{n+1}}{2k}\left(-hz_0\varphi_{n-1} + z_1\varphi_n\right).$$

Proof.

As is well known, the recurrence relation

$$y_{n+2} - 2ky_{n+1} - hy_n = 0, \quad n \in \mathbb{N}_0,$$

with initial conditions $y_0, y_1 \in \mathbb{R}$, is associated with the characteristic equation:

$$\theta^2 - 2k\theta - h = 0.$$

Solving the characteristic equation, we obtain the roots

$$\alpha = k + \sqrt{k^2 + h}, \quad \beta = k - \sqrt{k^2 + h}.$$

Therefore, the general solution to the recurrence relation is given by

$$y_n = c_1\alpha^n + c_2\beta^n. \tag{2.4}$$

By using the initial conditions y_0 and y_1 with some calculations we get

$$c_1 = \frac{y_0\beta - y_1}{\beta - \alpha}, \quad c_2 = \frac{y_1 - y_0\alpha}{\beta - \alpha}.$$

By compensation in the equation (2.4) we get

$$\begin{aligned} y_n &= y_0\frac{\alpha\beta(\alpha^{n-1}-\beta^{n-1})}{-(\alpha-\beta)} + y_1\frac{\alpha^n-\beta^n}{\alpha-\beta} \\ &= \frac{hy_0}{p}(\alpha^{n-1}-\beta^{n-1}) + \frac{y_1}{p}(\alpha^n-\beta^n) \\ &= \frac{hy_0}{2k}y_{n-1} + \frac{y_1}{2k}y_n. \end{aligned}$$

By the same argument, we get

$$z_n = \frac{(-1)^{n+1}}{2k}\left(-z_0h\varphi_{n-1} + z_1\varphi_n\right).$$

■

lemma 2.2.2 *Consider the linear system of second order linear autonomous differences equations*

$$u_{n+2} - 2kv_{n+1} - hu_n = 0, \quad v_{n+2} - 2ku_{n+1} - hv_n = 0, \quad n \in \mathbb{N}_0. \tag{2.5}$$

Then

$$u_{2n} = \frac{1}{2k}\left(hu_0\varphi_{2n-1} + v_1\varphi_{2n}\right),$$

$$u_{2n+1} = \frac{1}{2k}\left(hv_0\varphi_{2n} + u_1\varphi_{2n+1}\right),$$

$$v_{2n} = \frac{1}{2k}\left(hv_0\varphi_{2n-1} + u_1\varphi_{2n}\right),$$

$$v_{2n+1} = \frac{1}{2k}\left(hu_0\varphi_{2n} + v_1\varphi_{2n+1}\right).$$

Proof.

Through the combination of addition and subtraction of equations, we get

$$u_{n+2} + v_{n+2} = 2k\left(u_{n+1} + v_{n+1}\right) + h(u_n + v_n), \tag{2.6}$$

$$u_{n+2} - v_{n+2} = -2k\left(u_{n+1} - v_{n+1}\right) + h(u_n - v_n). \tag{2.7}$$

By posing the following changes of variables

$$R_n = u_n + v_n, \quad S_{n=}u_n - v_n. \tag{2.8}$$

The equations (2.6) and (2.7) becomes

$$R_{n+2} = 2kR_{n+1} + hR_n,$$

$$S_{n+2} = -2kS_{n+1} + hS_n,$$

which are in the form of equations (2.2) and (2.3). Then it follows from Lemma (2.2.1) that

$$R_{2n} = \frac{hR_0}{2k}\varphi_{2n-1} + \frac{R_1}{2k}\varphi_{2n}, \quad R_{2n+1} = \frac{hR_0}{2k}\varphi_{2n} + \frac{R_1}{2k}\varphi_{2n+1},$$

$$S_{2n} = -\frac{1}{2k}\left(-hS_0\varphi_{2n-1} + S_1\varphi_{2n}\right), \quad S_{2n+1} = \frac{1}{2k}\left(-hS_0\varphi_{2n} + S_1\varphi_{2n+1}\right).$$

And we have from (2.8) that

$$u_n = \frac{1}{2}\left(R_n + S_n\right), \quad v_n = \frac{1}{2}\left(R_n - S_n\right),$$

$$u_{2n} = \frac{1}{2}\left(R_{2n} + S_{2n}\right), \quad u_{2n+1} = \frac{1}{2}\left(R_{2n+1} + S_{2n+1}\right),$$

$$v_{2n} = \frac{1}{2}\left(R_{2n} - S_{2n}\right), \quad v_{2n+1} = \frac{1}{2}\left(R_{2n+1} - S_{2n+1}\right).$$

By substitution, we get

$$u_{2n} = \frac{1}{2}\left(\frac{hR_0}{2k}\varphi_{2n-1} + \frac{R_1}{2k}\varphi_{2n} + \frac{hS_0}{2k}\varphi_{2n-1} - \frac{S_1}{2k}\varphi_{2n}\right),$$

$$u_{2n+1} = \frac{1}{2}\left(\frac{hR_0}{2k}\varphi_{2n} + \frac{R_1}{2k}\varphi_{2n+1} - \frac{hS_0}{2k}\varphi_{2n} + \frac{S_1}{2k}\varphi_{2n+1}\right),$$

$$v_{2n} = \frac{1}{2}\left(\frac{hR_0}{2k}\varphi_{2n-1} + \frac{R_1}{2k}\varphi_{2n} - \frac{hS_0}{2k}\varphi_{2n-1} + \frac{S_1}{2k}\varphi_{2n}\right),$$

$$v_{2n+1} = \frac{1}{2}\left(\frac{hR_0}{2k}\varphi_{2n} + \frac{R_1}{2k}\varphi_{2n+1} + \frac{hS_0}{2k}\varphi_{2n} - \frac{S_1}{2k}\varphi_{2n+1}\right).$$

So

$$u_{2n} = \frac{1}{2k}\left(hu_0\varphi_{2n-1} + v_1\varphi_{2n}\right),$$

$$u_{2n+1} = \frac{1}{2k}\left(hv_0\varphi_{2n} + u_1\varphi_{2n+1}\right),$$

$$v_{2n} = \frac{1}{2k}\left(hv_0\varphi_{2n-1} + u_1\varphi_{2n}\right),$$

$$v_{2n+1} = \frac{1}{2k}\left(hu_0\varphi_{2n} + v_1\varphi_{2n+1}\right).$$

■

2.3 Closed-form solution of system (2.1)

In this section, we begin by reformulating the original system through a valid variable change. This new variable transformation leads us to a simpler equivalent system, which we then study in detail. Through recursive relations and exploiting known information from previous lemmas, we obtain closed-form solutions of the system. We then express these solutions in an explicit form involving (k, h)-Pell-Lucas sequences. Finally, we summarize our findings in a main theorem and provide a corollary providing the complete solution to the provided system.

From (2.1) we can write

$$x_{(l+1)(n+1)-j} = \frac{1}{2k + hy_{(l+1)n-j}}, \quad y_{(l+1)(n+1)-j} = \frac{1}{2k + hx_{(l+1)n-j}}.$$

By using the following change of variables:

$$x_n^{(j)} =_{(l+1)n-j}, \quad y_n^{(j)} = y_{(l+1)n-j},$$

in system (2.1) we get

$$x_{n+1}^{(j)} = \frac{1}{2k + hy_n^j}, \quad y_{n+1}^{(j)} = \frac{1}{2k + hx_n^j}, \quad n \in \mathbb{N}_0. \tag{2.9}$$

Hence we can use the change of variables

$$x_n^{(j)} = \frac{W_n}{U_{n+1}}, \quad y_n^{(j)} = \frac{U_n}{W_{n+1}},$$

in (2.9) and obtain

$$x_{n+1}^{(j)} = \frac{W_{n+1}}{U_{n+2}}, \quad y_{n+1}^{(j)} = \frac{U_{n+1}}{W_{n+2}}.$$

So,

$$\begin{aligned}
\frac{W_{n+1}}{U_{n+2}} &= \frac{1}{2k + \frac{hU_n}{W_{n+1}}}, \\
&= \frac{1}{\frac{2kW_{n+1} + hU_n}{W_{n+1}}}, \\
&= \frac{W_{n+1}}{2kW_{n+1} + hU_n}.
\end{aligned}$$

And

$$\begin{aligned}
\frac{U_{n+1}}{W_{n+2}} &= \frac{1}{2k + \frac{hW_n}{U_{n+1}}}, \\
&= \frac{1}{\frac{2kU_{n+1} + hW_n}{U_{n+1}}}, \\
&= \frac{U_{n+1}}{2kU_{n+1} + hW_n}.
\end{aligned}$$

So the system (2.9) becomes

$$U_{n+2} = 2kW_{n+1} + hU_n,$$

$$W_{n+2} = 2kU_{n+1} + hW_n.$$

Then it follows from Lemma (2.2.2) that

$$U_{2n} = \frac{1}{2k}\left(hU_0\varphi_{2n-1} + W_1\varphi_{2n}\right),$$

$$U_{2n+1} = \frac{1}{2k}\left(hW_0\varphi_{2n} + U_1\varphi_{2n+1}\right),$$

$$W_{2n} = \frac{1}{2k}\left(hW_0\varphi_{2n-1} + U_1\varphi_{2n}\right),$$

$$W_{2n+1} = \frac{1}{2k}\left(hU_0\varphi_{2n} + W_1\varphi_{2n+1}\right).$$

So,

$$\begin{aligned} x_{2n+1}^{(j)} &= \frac{W_{2n+1}}{U_{2n+2}}, \\ &= \frac{W_1\varphi_{2n+1} + hU_0\varphi_{2n}}{hU_0\varphi_{2n+1} + W_1\varphi_{2n+2}}, \\ &= \frac{\dfrac{hU_0}{W_1}\varphi_{2n} + \varphi_{2n+1}}{\dfrac{hU_0}{W_1}\varphi_{2n+1} + \varphi_{2n+2}}, \end{aligned}$$

$$x_{2n+1}^{(j)} = \frac{hy_0^{(j)}\varphi_{2n} + \varphi_{2n+1}}{hy_0^{(j)}\varphi_{2n+1} + \varphi_{2n+2}}.$$

As well as

$$\begin{aligned} x_{2n}^{(j)} &= \frac{W_{2n}}{U_{2n+1}}, \\ &= \frac{hW_0\varphi_{2n-1} + U_1\varphi_{2n}}{hW_0\varphi_{2n} + U_1\varphi_{2n+1}}, \\ &= \frac{\dfrac{hW_0}{U_1}\varphi_{2n-1} + \varphi_{2n}}{\dfrac{hW_0}{U_1}\varphi_{2n} + \varphi_{2n+1}}, \end{aligned}$$

$$x_{2n}^{(j)} = \frac{hx_0^{(j)}\varphi_{2n-1} + \varphi_{2n}}{hx_0^{(j)}\varphi_{2n} + \varphi_{2n+1}}.$$

From the above,

$$\begin{aligned} y_{2n+1}^{(j)} &= \frac{U_{2n+1}}{W_{2n+2}}, \\ &= \frac{hW_0\varphi_{2n} + U_1\varphi_{2n+1}}{hW_0\varphi_{2n+1} + U_1\varphi_{2n+2}}, \\ &= \frac{\dfrac{hW_0}{U_1}\varphi_{2n} + \varphi_{2n+1}}{\dfrac{hW_0}{U_1}\varphi_{2n+1} + \varphi_{2n+2}}, \end{aligned}$$

$$y_{2n+1}^{(j)} = \frac{hx_0^{(j)}\varphi_{2n} + \varphi_{2n+1}}{hx_0^{(j)}\varphi_{2n+1} + \varphi_{2n+2}}.$$

Also,

$$\begin{aligned} y_{2n}^{(j)} &= \frac{U_{2n}}{W_{2n+1}}, \\ &= \frac{hU_0\varphi_{2n-1} + W_1\varphi_{2n}}{hU_0\varphi_{2n} + W_1\varphi_{2n+1}}, \\ &= \frac{\dfrac{hU_0}{W_1}\varphi_{2n-1} + \varphi_{2n}}{\dfrac{hU_0}{W_1}\varphi_{2n} + \varphi_{2n+1}}, \end{aligned}$$

$$y_{2n}^{(j)} = \frac{hy_0^{(j)}\varphi_{2n-1} + \varphi_{2n}}{hy_0^{(j)}\varphi_{2n} + \varphi_{2n+1}}.$$

Theorem 2.3.1 *Let $\{x_n^{(j)}, y_n^{(j)}\}_{n\geq 0}$ be the solution to system* (2.9) *then for $n \in \mathbb{N}$ and $j = 0, ..., l$*

$$x_{2n+1}^{(j)} = \frac{hy_0^{(j)}\varphi_{2n} + \varphi_{2n+1}}{hy_0^{(j)}\varphi_{2n+1} + \varphi_{2n+2}}, \quad x_{2n}^{(j)} = \frac{hx_0^{(j)}\varphi_{2n-1} + \varphi_{2n}}{hx_0^{(j)}\varphi_{2n} + \varphi_{2n+1}},$$

$$y_{2n+1}^{(j)} = \frac{hx_0^{(j)}\phi_{2n} + \varphi_{2n+1}}{hx_0^{(j)}\varphi_{2n+1} + \varphi_{2n+2}}, \quad y_{2n}^{(j)} = \frac{hy_0^{(j)}\varphi_{2n-1} + \varphi_{2n}}{hy_0^{(j)}\varphi_{2n} + \varphi_{2n+1}}.$$

The following corollory is our main result wich gives the explicit formula of solution to system

Corollary 2.3.1 *Let $\{x_n, y_n\}_{n\geq 0}$ be the solution to system (2.1) then for $n \in \mathbb{N}$ and $j = 0,1,...,l$*

$$x_{(l+1)(2n+1)-j} = \frac{hy_{-j}\varphi_{2n} + \varphi_{2n+1}}{hy_{-j}\varphi_{2n+1} + \varphi_{2n+2}}, \quad x_{(l+1)(2n)-j} = \frac{hx_{-j}\varphi_{2n-1} + \varphi_{2n}}{hx_{-j}\varphi_{2n} + \varphi_{2n+1}},$$

$$y_{(l+1)(2n+1)-j} = \frac{hx_{-j}\varphi_{2n} + \varphi_{2n+1}}{hx_{-j}\varphi_{2n+1} + \varphi_{2n+2}}, \quad y_{(l+1)(2n)-j} = \frac{hy_{-j}\varphi_{2n-1} + \varphi_{2n}}{hy-j\varphi_{2n} + \varphi_{2n+1}}.$$

Proof.

We have

$$x_n^{(j)} = x_{(l+1)(n)-j}, \quad j = 0,1,...,l.$$

So

$$x_{2n+1}^{(j)} = x_{(l+1)(2n+1)-j},$$

and

$$x_0^{(j)} = x_{-j}.$$

Then

$$\begin{aligned} x_{2n+1}^{(j)} &= x_{(l+1)(2n+1)-j}, \\ &= \frac{hy_{-j}\varphi_{2n} + \varphi_{2n+1}}{hy_{-j}\varphi_{2n+1} + \varphi_{2n+2}}, \end{aligned}$$

and

$$\begin{aligned} x_{2n}^{(j)} &= x_{(l+1)(2n)-j}, \\ &= \frac{hx_{-j}\varphi_{2n-1} + \varphi_{2n}}{hx_{-j}\varphi_{2n} + \varphi_{2n+1}}. \end{aligned}$$

We have

$$y_n^{(j)} = y_{(l+1)(n)-j}, j = 0,1,...,l.$$

So

$$y^{(j)}_{2n+1} = y_{(l+1)(2n+1)-j},$$

and

$$y^{j}_{0} = y_{-j}.$$

Hence

$$\begin{aligned} y^{(j)}_{2n+1} &= y_{(l+1)(2n+1)-j}, \\ &= \frac{hx_{-j}\varphi_{2n} + \varphi_{2n+1}}{hx_{-j}\varphi_{2n+1} + \varphi_{2n+2}}, \end{aligned}$$

Also

$$\begin{aligned} y^{(j)}_{2n} &= y_{(l+1)(2n+)-j}, \\ &= \frac{hy_{-j}\varphi_{2n-1} + \varphi_{2n}}{hy_{-j}\varphi_{2n} + \varphi_{2n+1}}. \end{aligned}$$

■

2.4 Global stability of the solutions to system (2.1)

In this section, we investigate the global stability of the solutions of system of non-linear difference equations (2.1). By finding the equilibrium points of the system, we apply linearization techniques and Rouche's Theorem to determine local and global asymptotic stability.

The equilibrium points of the system are given by

$$\overline{M} = (\overline{x}, \overline{y}) = \left(\frac{-k + \sqrt{k^2 + h}}{h}, \frac{-k + \sqrt{k^2 + h}}{h} \right),$$

$$\overline{M} = (\overline{x}, \overline{y}) = \left(\frac{-k - \sqrt{k^2+h}}{h}, \frac{-k - \sqrt{k^2+h}}{h} \right).$$

Theorem 2.4.1 *The equilibrium point*

$$\overline{M} = (\overline{x}, \overline{y}) = \left(\frac{-k + \sqrt{k^2+h}}{h}, \frac{-k + \sqrt{k^2+h}}{h} \right),$$

is locally asymptotically stable.

Proof.

We linearize the system around the equilibrium point $\overline{M} = (\overline{x}, \overline{y}) = \left(\frac{-k+\sqrt{k^2+h}}{h}, \frac{-k+\sqrt{k^2+h}}{h} \right)$. The resulting linear system can be written as :

$$X_{n+1} = JX_n,$$

where the state vector is defined by

$$X_n = \begin{pmatrix} x_n, & x_{n-1}, & \cdots, & x_{n-l}, & y_n, & y_{n-1}, & \cdots, & y_{n-l} \end{pmatrix}^t,$$

and the matrix J is given as

$$J = \begin{pmatrix}
0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \frac{-h}{\left(k+\sqrt{k^2+h}\right)^2} \\
1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
0 & 1 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & \frac{-h}{\left(k+\sqrt{k^2+h}\right)^2} & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0 & \cdots & 1 & 0
\end{pmatrix}.$$

The characteristic polynomial of matrix J is:

$$P(\lambda) = (-\lambda)^{2l+2} - \left(\frac{h}{(k+\sqrt{k^2+h})^2}\right)^2.$$

Let us define the functions :

$$\delta(\lambda) = (-\lambda)^{2l+2}, \quad \phi(\lambda) = \left(\frac{h}{(k+\sqrt{k^2+h})^2}\right)^2.$$

It holds that

$$|\phi(\lambda)| < |\delta(\lambda)|, \quad \text{for all} \lambda \in \mathbb{C}, \quad \text{with} |\lambda| = 1.$$

By Rouche's Theorem the function $P = \delta + \phi$ has the same number of zeros as φ within the unit disc $|\phi(\lambda)| < 1$, and since δ has a root of multiplicity $2(l+1)$ at $\lambda = 0$, then all the roots of P are in the disc $|\phi(\lambda)| < 1$. Thus, the equilibrium point is locally asymptotically stable. ■

Corollary 2.4.1 *The equilibrium point $\overline{M}$ is globaly asymptotically stable.*

Proof.

According to Theorem (2.4.1), $\overline{M}$ is globally asymptotically stable. To demonstrate global asymptotic stability, we utilize Corollary (2.3.1). Consider the following limit.

$$\lim_{n\to\infty} x_{(l+1)(2n)-j} = \lim_{n\to\infty} \frac{hx_{-j}\varphi_{2n-1} + \varphi_{2n}}{hx_{-j}\varphi_{2n} + \varphi_{2n+1}} = \lim_{n\to\infty} \frac{hx_{-j}\frac{\varphi_{2n-1}}{\varphi_{2n}} + 1}{hx_{-j} + \frac{\varphi_{2n+1}}{\varphi_{2n}}}.$$

Using the known limits

$$\lim_{n\to\infty}\left(\frac{\varphi_{2n-1}}{\varphi_{2n}}\right) = \alpha, \quad \lim_{n\to\infty}\left(\frac{\varphi_{2n+1}}{\varphi_{2n}}\right) = \frac{1}{\alpha} = \beta,$$

we get

$$\begin{aligned}
\lim_{n\to\infty} x_{(l+1)(2n)-j} &= \frac{hx_{-j}\beta+1}{hx_{-j}+\alpha}, \\
&= \frac{hx_{-j}(k-\sqrt{k^2+h})+1}{hx_{-j}+k+\sqrt{k^2+h}}, \\
&= \frac{-k+\sqrt{k^2+h}}{h}, \\
&= \bar{x}.
\end{aligned}$$

However, we have

$$\lim_{n\to\infty} x_{(l+1)(2n+1)-j} = \lim_{n\to\infty} \frac{hy_{-j}\varphi_{2n}+\varphi_{2n+1}}{hy_{-j}\varphi_{2n+1}+\varphi_{2n+2}} = \lim_{n\to\infty} \frac{hy_{-j}\frac{\varphi_{2n}}{\varphi_{2n+1}}+1}{hy_{-j}+\frac{\varphi_{2n+2}}{\varphi_{2n+1}}} = \frac{hy_{-j}\beta+1}{hy_{-j}+\alpha}.$$

Using the following two limits

$$\lim_{n\to\infty}\left(\frac{\varphi_{2n-1}}{\varphi_{2n}}\right)=\alpha, \quad \lim_{n\to\infty}\left(\frac{\varphi_{2n+1}}{\varphi_{2n}}\right)=\frac{1}{\alpha}=\beta,$$

we get

$$\lim_{n\to\infty} x_{(l+1)(2n+1)-j} = \frac{-k+\sqrt{k^2+h}}{h} = \bar{x}.$$

So,

$$\lim_{n\to\infty} x_{(l+1)(2n+1)-j} = \bar{x}.$$

Similarly, it can proven that

$$\lim_{n\to\infty} y_{(l+1)(2n+1)-j} = \bar{y}.$$

Hence, the solution converges globally to the equilibrium point

$$\lim_{n\to\infty}\left(x_{(l+1)(2n+1)-j}, y_{(l+1)(2n+1)-j}\right) = (\bar{x},\bar{y}).$$

■

2.5 Rate of convergence

In this section, we analyze the convergence rate of the solution to the equilibrium point $(\overline{x}, \overline{y})$ for the given nonlinear difference system (2.1). Linearization around the equilibrium and expression of the dynamics as

$$A_{n+1} = (M + B_n) A_n, \tag{2.10}$$

where A_n is a vector of dimensions $2l$, $M \in C^{2l\times 2l}$ is a constant matrix and $B : \mathbb{Z}^+ \to C^{2l\times 2l}$ is a matrix function satisfying

$$||B_n|| \longrightarrow 0, \quad when \quad n \to \infty, \tag{2.11}$$

where $||.||$ indicates any matrix norm which is associated with the vector norm $||.||$. We determine the convergence rate via the spectral radius of M. The parameters k, h, and delay l explicitly influence this rate, providing insights into the system's stability and asymptotic behavior.

Theorem 2.5.1 *[10] **(Perron's first Theorem)***

Suppose that condition (2.11) *holds. If A_n is a solution of* (2.10), *then either $A_n = 0$ for all largen or*

$$\rho = \lim_{n\to\infty} \frac{||A_{n+1}||}{||A_n||},$$

exists and is equal to the modulus of one of the eigenvalues of matrix M.

Theorem 2.5.2 *[10] **(Perron's second Theorem)***

Suppose that condition (2.11) *holds. If A_n is a solution of* (2.10), *then either $A_n = 0$ for all largen or*

$$\rho = \lim_{n\to\infty} (\|A_{n+1}\|)^{\frac{1}{n}},$$

exists and is equal to the modulus of one of the eigenvalues of matrix M.

Theorem 2.5.3 *[10] Let the solution $\{(x_n, y_n)\}_{n\geq -1}$ of system (2.1) converges to the equilibrium point $(\overline{x}, \overline{y})$ wich is globally asymptotically stable. So the error vector*

$$e_n = \begin{pmatrix} e_n^{(1)} \\ e_{n-1}^{(1)} \\ \vdots \\ e_{n-l}^{(1)} \\ e_n^2 \\ e_{n-1}^2 \\ \vdots \\ e_{n-l}^2 \end{pmatrix} = \begin{pmatrix} x_n - \bar{x} \\ x_{n-1} - \bar{x} \\ \vdots \\ x_{n-l} - \bar{x} \\ y_n - \bar{y} \\ y_{n-1} - \bar{y} \\ \vdots \\ y_{n-l} - \bar{y} \end{pmatrix}$$

of any solution of system (2.1) satisfies both of the following asymptotic behaviors

$$\lim_{n\to\infty} \frac{\|e_{n+1}\|}{\|e_n\|} = |\lambda_i J_F((\bar{x}, \bar{y}))|, \quad i = 1, 2, ..., l.$$

$$\lim_{n\to\infty} (\|e_{n+1}\|)^{\frac{1}{n}} = |\lambda_i J_F((\bar{x}, \bar{y}))|, \quad i = 1, 2, ..., l,$$

where $|\lambda_i J_F((\bar{x}, \bar{y}))|$ corresponds to the absolute value of one of the eigenvalues of the Jacobian matrix evaluated at the equilibrium $(\bar{x}, \bar{y})$.

Proof.

To establish the desired result, we start by formulating a system that governs the evolution of the error terms. These error termes are defined as follows :

$$\begin{cases} x_{n+1} - \bar{x} = \sum_{i=0}^{l} C_i (x_{n-i} - \bar{x}) + \sum_{i=0}^{l} D_i (y_{n-i} - \bar{y}) \quad for \quad i = 1, 2, ..., l, \\ y_{n+1} - \bar{y} = \sum_{i=0}^{l} G_i (x_{n-i} - \bar{x}) + \sum_{i=0}^{l} H_i (y_{n-i} - \bar{y}) \quad for \quad i = 1, 2, ..., l. \end{cases} \tag{2.12}$$

Set

$$e_n^{(1)} = x_n - \bar{x}, \quad e_n^{(2)} = y_n - \bar{y}$$

Then, the system (2.12) become

$$\begin{cases} e_{n+1}^{(1)} = \sum_{i=0}^{l} C_i e_{n-i}^{(1)} + \sum_{i=0}^{l} D_i e_{n-i}^{(2)} \quad for \quad i = 1,2,....,l, \\ e_{n+1}^{(2)} = \sum_{i=0}^{l} G_i e_{n-i}^{(1)} + \sum_{i=0}^{l} H_i e_{n-i}^{(2)} \quad for \quad i = 1,2,....,l, \end{cases}$$

where

$$C_i = 0 \quad i = 1,2,....,l$$

$$D_i = 0 \quad i = 1,2,....,l-1, \quad D_l = \frac{-h}{(2k + hy_{n-l})^2}$$

$$G_i = 0 \quad i = 1,2,....,l-1, \quad G_l = \frac{-h}{(2k + hx_{n-l})^2}$$

$$H_i = 0 \quad i = 1,2,....,l$$

As the system approaches equilibrium, it becomes clear that

$$\lim_{n\to\infty} C_i = 0 \quad i = 1,2,....,l$$

$$\lim_{n\to\infty} D_i = 0 \quad i = 1,2,....,l-1, \quad \lim_{n\to\infty} D_l = \frac{-h}{(2k + h\bar{y})^2}$$

$$\lim_{n\to\infty} G_i = 0 \quad i = 1,2,....,l-1, \quad \lim_{n\to\infty} G_l = \frac{-h}{(2k + h\bar{x})^2}$$

$$\lim_{n\to\infty} H_i = 0 \quad i = 1,2,....,l$$

So, that means

$$D_l = \frac{-h}{(2k + h\bar{y})^2} + \alpha_n, \quad G_l = \frac{-h}{(2k + h\bar{x})^2} + \beta_n.$$

We can now express the system in the form

$$e_{n+1} = (M + B_n)\, e_n,$$

where $e_n = \left(e_n^{(1)},\ e_{n-1}^{(1)},\ \cdots,\ e_{n-l}^{(1)},\ e_n^{(2)},\ e_{n-1}^{(2)},\ \cdots,\ e_{n-l}^{(2)}\right)^t$ and

$$B_n = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \alpha_n \\ 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \beta_n & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}.$$

$$M = J_F(\bar{x}, \bar{y}) = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \frac{-h}{\left(k+\sqrt{k^2+h}\right)^2} \\ 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \frac{-h}{\left(k+\sqrt{k^2+h}\right)^2} & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}.$$

$\|B_n\| \to 0$ when $n \to \infty$. As a result, the asymptotic error system can be expressed as

$$e_{n+1} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \frac{-h}{\left(k+\sqrt{k^2+h}\right)^2} \\ 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \frac{-h}{\left(k+\sqrt{k^2+h}\right)^2} & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 1 & 0 \end{pmatrix} \begin{pmatrix} e_n^{(1)} \\ e_{n-1}^{(1)} \\ \vdots \\ e_{n-l}^{(1)} \\ e_n^2 \\ e_{n-1}^2 \\ \vdots \\ e_{n-l}^2 \end{pmatrix},$$

and $\|B_n\| \to 0$ when $n \to \infty$. Clearly, this system corresponds to the linear approximation of equation (2.1) near $(\bar{x}, \bar{y})$ the equilibrium point. Thus, the result is a direct consequence of Perron's Theorems. ■

2.6 Numerical Examples

In this section, we studied a system of nonlinear difference equations with the use of exact numerical techniques. We performed iterative calculations and documented the findings in detailed tables and graphical plots that illustrate the dynamics of the system and its convergence towards equilibrium points. The study was conducted in an attempt to trace the evolution of variables with extremely high numerical accuracy using advanced computational techniques as a foundation for subsequent theoretical analysis and practical application.

example 2.6.1 *Let the following system of difference equations*

$$x_{n+1} = \frac{1}{2k + hy_{n-l}}, \quad y_{n+1} = \frac{1}{2k + hx_{n-l}}, \quad n \geq 0, \tag{2.13}$$

where the parameters and initial conditions are chosen as follows:

- $k = 1, h = 2$ *and* $l = 3$.
- *Initial conditions for x:* $x_0 = 0.2, \quad x_1 = 0.9, \quad x_2 = 0.1, \quad x_3 = 0.7,$
- *Initial conditions for y:* $y_0 = 0.8, \quad y_1 = 0.05, \quad y_2 = 0.6, \quad y_3 = 0.3.$

The following table presents the computed values of the sequences (x_n) *and* (y_n) *generated by the system of difference equations with the given initial conditions and parameters. The values are displayed for* $n = 1$ *to* $n = 30$.

n	1	2	3	4	5	6	7	8	9	10
x_n	0.9000	0.1000	0.7000	0.3571	0.3030	0.3226	0.3469	0.3603	0.3639	0.3644
y_n	0.0500	0.6000	0.3000	0.3125	0.4545	0.3448	0.3309	0.3262	0.3247	0.3245

n	11	12	13	14	15	16	17	18	19	20
x_n	0.3645	0.3645	0.3645	0.3645	0.3645	0.3645	0.3645	0.3645	0.3645	0.3645
y_n	0.3244	0.3244	0.3244	0.3244	0.3244	0.3244	0.3244	0.3244	0.3244	0.3244

n	21	22	23	24	25	26	27	28	29	30
x_n	0.3645	0.3645	0.3645	0.3645	0.3645	0.3645	0.3645	0.3645	0.3645	0.3645
y_n	0.3244	0.3244	0.3244	0.3244	0.3244	0.3244	0.3244	0.3244	0.3244	0.3244

Table 2.1: Computed Values of x_n and y_n for $n = 1$ to 30 from the Difference Equation System

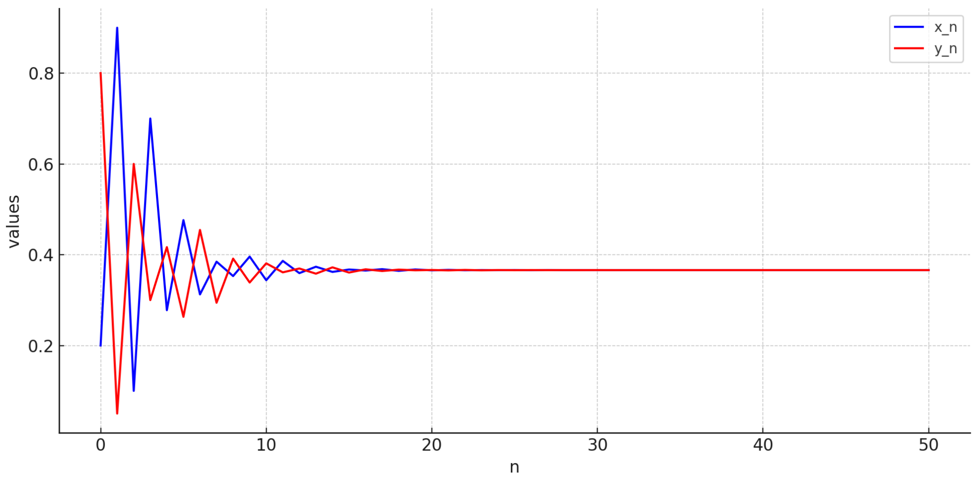

Figure 2.1: Plot of the numerical solution of the system (2.13)

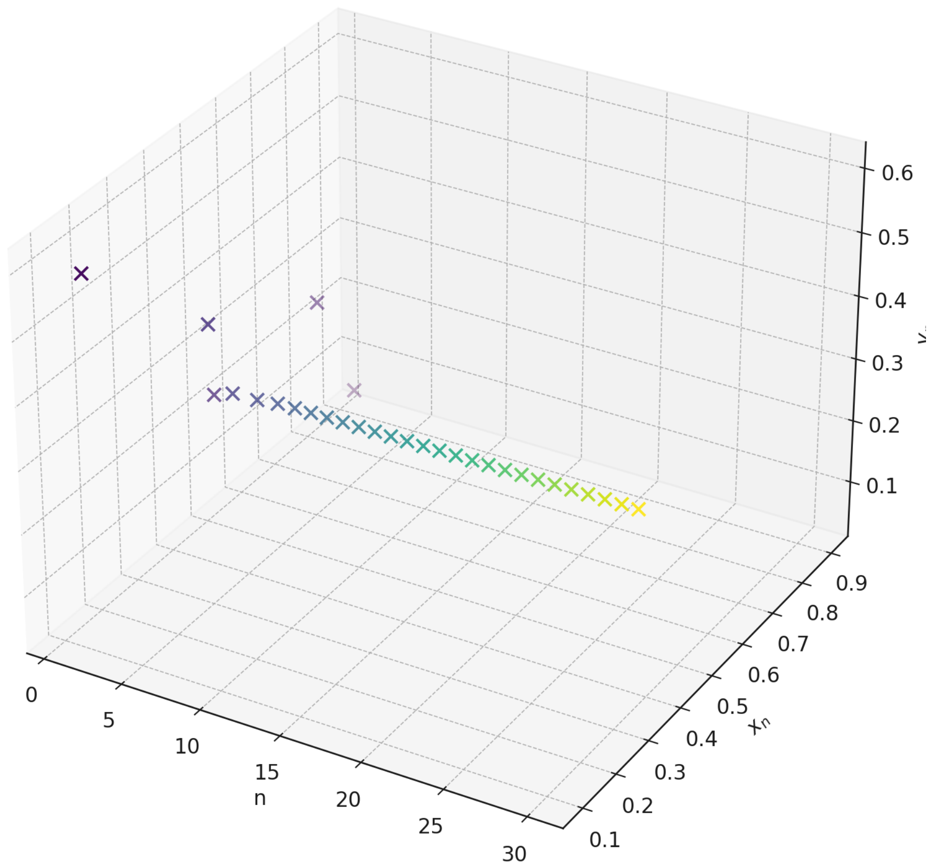

Figure 2.2: 3D phase surface plot of the system (2.13)

example 2.6.2 *Let the following system of difference equations*

$$x_{n+1} = \frac{1}{2k + hy_{n-l}}, \quad y_{n+1} = \frac{1}{2k + hx_{n-l}}, \quad n \geq 0, \tag{2.14}$$

where the parameters and initial conditions are chosen as follows:

- $k = 1, h = 2$ *and* $l = 12$.
- *Initial conditions for x:* $x_0 = 0.5488, \quad x_1 = 0.7152, \quad x_2 = 0.6028, \quad x_3 = 0.5449,$
  $x_4 = 0.4237, \quad x_5 = 0.6459, \quad x_6 = 0.4376, \quad x_7 = 0.8918,$
  $x_8 = 0.9637, \quad x_9 = 0.3834, \quad x_{10} = 0.7917, \quad x_{11} = 0.5289, \quad x_{12} = 0.5680,$
- *Initial conditions for y:* $y_0 = 0.9256, \quad y_1 = 0.0710, \quad y_2 = 0.0871, \quad y_3 = 0.0202,$

$y_4 = 0.8326, \quad y_5 = 0.7782, \quad y_6 = 0.8700, \quad y_7 = 0.9786,$

$y_8 = 0.7992, \quad y_9 = 0.4615, \quad y_{10} = 0.7805, \quad y_{11} = 0.1183, \quad y_{12} = 0.6399.$

The following table presents the computed values of the sequences (x_n) *and* (y_n) *generated by the system of difference equations with the given initial conditions and parameters. The values are displayed for* $n = 1$ *to* $n = 100$.

n	1	2	3	4	5	6	7	8	9	10
x_n	0.7152	0.6028	0.5449	0.4237	0.6459	0.4376	0.8918	0.9637	0.3834	0.7917
y_n	0.0710	0.0871	0.0202	0.8326	0.7782	0.8700	0.9786	0.7992	0.4615	0.7805

n	11	12	13	14	15	16	17	18	19	20
x_n	0.5289	0.5680	0.4668	0.4599	0.4901	0.2728	0.2812	0.2674	0.2527	0.2779
y_n	0.1183	0.6399	0.2915	0.3120	0.3236	0.3512	0.3038	0.3478	0.2643	0.2546

n	21	22	23	24	25	26	27	28	29	30
x_n	0.3421	0.2808	0.4471	0.3049	0.3871	0.3811	0.3777	0.3700	0.3835	0.3710
y_n	0.3614	0.2791	0.3270	0.3189	0.3409	0.3425	0.3355	0.3928	0.3903	0.3945

n	31	32	33	34	35	36	37	38	39	40
x_n	0.3955	0.3985	0.3673	0.3909	0.3768	0.3791	0.3729	0.3724	0.3744	0.3590
y_n	0.3991	0.3913	0.3725	0.3904	0.3455	0.3832	0.3605	0.3620	0.3629	0.3650

n	41	42	43	44	45	46	47	48	49	50
x_n	0.3596	0.3585	0.3574	0.3594	0.3643	0.3596	0.3716	0.3615	0.3675	0.3671
y_n	0.3614	0.3647	0.3583	0.3575	0.3657	0.3595	0.3632	0.3626	0.3642	0.3643

n	51	52	53	54	55	56	57	58	59	60
x_n	0.3669	0.3663	0.3673	0.3664	0.3681	0.3683	0.3661	0.3678	0.3668	0.3670
y_n	0.3638	0.3679	0.3677	0.3680	0.3684	0.3678	0.3665	0.3678	0.3645	0.3672

n	61	62	63	64	65	66	67	68	69	70
x_n	0.3665	0.3665	0.3666	0.3655	0.3656	0.3655	0.3654	0.3655	0.3659	0.3656
y_n	0.3656	0.3657	0.3658	0.3659	0.3657	0.3659	0.3655	0.3654	0.3660	0.3656

n	71	72	73	74	75	76	77	78	79	80
x_n	0.3664	0.3657	0.3661	0.3661	0.3661	0.3660	0.3661	0.3661	0.3662	0.3662
y_n	0.3658	0.3658	0.3659	0.3659	0.3659	0.3662	0.3661	0.3662	0.3662	0.3662

n	81	82	83	84	85	86	87	88	89	90
x_n	0.3660	0.3662	0.3661	0.3661	0.3661	0.3661	0.3661	0.3660	0.3660	0.3660
y_n	0.3661	0.3661	0.3659	0.3661	0.3660	0.3660	0.3660	0.3660	0.3660	0.3660

n	91	92	93	94	95	96	97	98	99	100
x_n	0.3660	0.3660	0.3660	0.3660	0.3661	0.3660	0.3660	0.3660	0.3660	0.3660
y_n	0.3660	0.3660	0.3660	0.3660	0.3660	0.3660	0.3660	0.3660	0.3660	0.3660

Table 2.2: Computed Values of x_n and y_n for $n = 1$ to 100 and $l = 12$ from the difference equation system

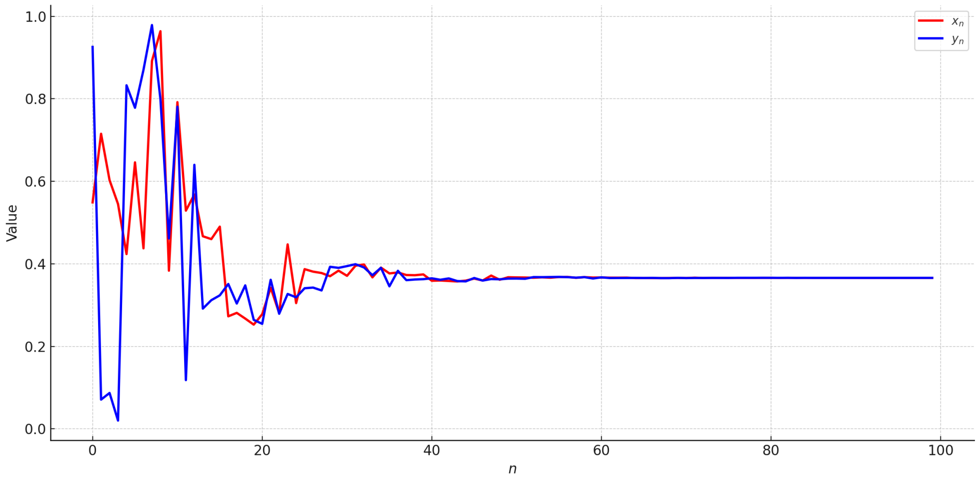

Figure 2.3: Plot of the numerical solution of the system (2.14)

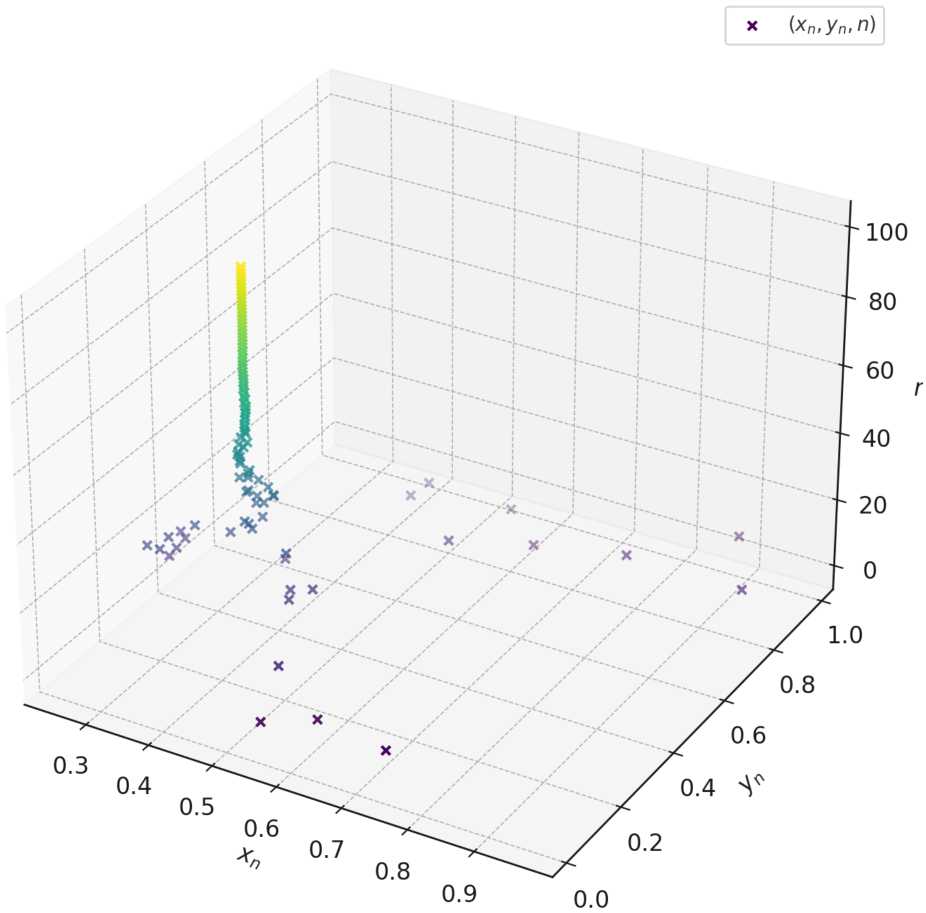

Figure 2.4: 3D phase surface plot of the system (2.14)

CHAPTER 3

SYSTEM OF DIFFERENCE EQUATIONS LINKED TO THE k-JACOBSTHAL SEQUENCE

This chapter investigates a higher-order difference equation system whose solutions are expressed in terms of $k-$Jacobsthal sequences. We derive closed-form solutions and analyze the stability and asymptotic behavior of the system. Through the connection of the solutions to this type of general sequences, the study provides new theoretical results on recursive systems. The results enhance the understanding of dynamic processes modeled by such difference equations.

3.1 Introduction

We propose some theoretical explanations pertaining to the representation for the solution of the system of the higher-order difference equations

$$x_{n+1} = \frac{1}{k + 2y_{n-l}}, \quad y_{n+1} = \frac{1}{k + 2x_{n-l}}, \tag{3.1}$$

with $k \in \mathbb{Z}$ and $n, l \in \mathbb{N}$.

The initial conditions $x_{-l}, x_{-l+1}, \ldots, x_0, y_{-l}, y_{-l+1}, \ldots, y_0$, are non zero real numbers such that their solution is related to a generalized Jacobsthal sequences. We also study the stability character and asymptotic behavior of this system.

We will present two lemmas so that the first lemma provides the solutions of two homogeneous second order linear autonomous difference equations, which is essential for representations the solution of system (3.1). Its proof utilizes the characteristic roots of the characteristic polynomial $\theta^2 \pm k\theta - 2$. On the other hand, the second lemma offers the solution of a system of second order linear autonomous difference equations, which plays a crucial role in solving the system (3.1).

3.2 Preliminary Results

In this section, we explore second-order linear difference equations within the framework of $k-$Jacobsthal sequences. From our analysis, explicit solutions to two principal homogeneous equations are obtained prior to applying the results to coupled systems through variable decoupling techniques. The solutions expressed in terms of initial conditions and $k-$Jacobsthal sequences reveal basic recursive forms and expose their underlying algebraic structure. This systematic approach shows the way complex systems can be mapped to solvable types through careful transformations.

lemma 3.2.1 *Consider the two homogeneous second order linear autonomous differences equa-*

tions :

$$B_{n+2} - kB_{k,n+1} - 2B_n = 0, \tag{3.2}$$

$$P_{n+2} + kP_{n+1} - 2P_n = 0. \tag{3.3}$$

Then we have for all $n \in \mathbb{N}_0$:

$$B_n = 2B_0 J_{k,n-1} + B_1 J_{k,n},$$

$$P_n = (-1)^n \left(2P_0 J_{k,n-1} - P_1 J_{k,n}\right).$$

Proof.

As is well-known, the recurrence relation

$$B_{n+2} - kB_{n+1} - 2B_{k,n} = 0, \quad n \in \mathbb{N}_0,$$

with inial conditions $B_0, \quad B_1 \in \mathbb{R}$, is associated with the characteristic equation:

$$\theta^2 - k\theta - 2 = 0.$$

Solving the characteristic equation, we obtain the roots:

$$r_1 = \frac{k + \sqrt{k^2+8}}{2}, \quad r_2 = \frac{k - \sqrt{k^2+8}}{2}.$$

Thus, the formulas of general solution is:

$$B_n = c_1 r_1^n + c_2 r_2^n. \tag{3.4}$$

By using the initial conditions B_0 and B_1 with some calculations we get:

$$c_1 = \frac{B_0 r_2 - B_1}{r_2 - r_1}, \quad c_2 = \frac{B_1 - r_1 B_0}{r_2 - r_1}.$$

By compensation in the equation (3.4) we get

$$\begin{aligned}
B_n &= B_0\frac{r_1^n r_2 - r_1 r_2^n}{r_2 - r_1} + B_1\frac{r_2^n - r_1^n}{r_2 - r_1},\\
&= B_0\frac{r_1 r_2\left(r_1^{n-1} - r_2^{n-1}\right)}{r_2 - r_1} + B_1\frac{r_2^n - r_1^n}{r_2 - r_1},\\
&= B_0\frac{-r_1 r_2\left(r_1^{n-1} - r_2^{n-1}\right)}{r_1 - r_2} + B_1\frac{r_1^n - r_2^n}{r_1 - r_2},\\
&= 2B_0\frac{r_1^{n-1} - r_2^{n-1}}{r_1 - r_2} + B_1\frac{r_1^n - r_2^n}{r_1 - r_2},\\
&= 2B_0 J_{k,n-1} + B_1 J_{k,n}.
\end{aligned}$$

By the same argument, we get

$$P_n = (-1)^n\left(2P_0 J_{n-1} - P_1 J_{k,n}\right).$$

■

lemma 3.2.2 *Consider the linear systam of second order linear autonomous differences equations*

$$\mu_{n+2} - k\nu_{n+1} - 2\mu_{k,n} = 0, \quad \nu_{n+2} - k\mu_{n+1} - 2\nu_{k,n} = 0, \quad n \in \mathbb{N}_0. \tag{3.5}$$

Then

$$\mu_{2n} = 2\mu_0 J_{k,2n-1} + \nu_1 J_{k,2n},$$

$$\mu_{2n+1} = 2\nu_0 J_{k,2n} + \mu_1 J_{k,2n+1},$$

$$\nu_{2n} = 2\nu_0 J_{k,2n-1} + \mu_1 J_{k,2n},$$

$$\nu_{2n+1} = 2\mu_0 J_{k,2n} + \nu_1 J_{k,2n+1}.$$

Proof.

Through the combination of addition and subtraction of equations, we get

$$\mu_{n+2} + \nu_{n+2} = k\left(\nu_{n+1} + mu_{n+1}\right) + 2\left(\mu_n + \nu_n\right), \tag{3.6}$$

$$\mu_{n+2} - \nu_{n+2} = -k\left(\mu_{n+1} - \nu_{n+1}\right) + 2\left(\mu_n - \nu_n\right). \tag{3.7}$$

By posing the following changes of variables :

$$R_n = \mu_n + \nu_n, \quad S_n = \mu_n - \nu_n. \tag{3.8}$$

The equations (3.6) and (3.7) becomes:

$$R_{n+2} = kR_{n+1} + 2R_n,$$

$$S_{n+2} = -kS_{n+1} + 2S_n,$$

wich are in the form of equations (3.2) and (3.3) Then it follows from Lemma (3.2.1) that

$$R_{2n} = 2R_0 J_{k,2n-1} + R_1 J_{k,2n}, \quad R_{2n+1} = 2R_0 J_{k,2n} + R_1 J_{k,2n+1},$$

$$S_{2n} = 2S_0 J_{k,2n-1} - S_1 J_{k,2n}, \quad S_{2n+1} = -\left(2S_0 J_{k,2n} + S_1 J_{k,2n+1}\right).$$

And we have from (3.8) that

$$\mu_n = \frac{1}{2}\left(R_n + S_n\right), \quad \nu_n = \frac{1}{2}\left(R_n - S_n\right),$$

$$\nu_{2n} = \frac{1}{2}\left(R_{2n} + S_{2n}\right), \quad \nu_{2n+1} = \frac{1}{2}\left(R_{2n+1} + S_{2n+1}\right),$$

$$\nu_{2n} = \frac{1}{2}\left(R_{2n} - S_{2n}\right), \quad \nu_{2n+1} = \frac{1}{2}\left(R_{2n+1} - S_{2n+1}\right).$$

By substitution, we get

$$\mu_{2n} = \frac{1}{2}\left(2R_0 J_{k,2n-1} + R_1 J_{k,2n} + 2S_0 J_{k,2n-1} - S_1 J_{k,2n}\right),$$

$$\mu_{2n+1} = \frac{1}{2}\left(2R_0 J_{k,2n} + R_1 J_{k,2n+1} - 2S_0 J_{k,2n} + S_1 J_{k,2n+1}\right),$$

$$\nu_{2n} = \frac{1}{2}\left(2R_0 J_{k,2n-1} + R_1 J_{k,2n} - 2S_0 J_{k,2n-1} + S_1 J_{k,2n}\right),$$

$$\nu_{2n+1} = \frac{1}{2}\left(2R_0 J_{k,2n} + R_1 J_{k,2n+1} + 2S_0 J_{k,2n} - S_1 J_{k,2n+1}\right).$$

So

$$\mu_{2n} = 2\mu_0 J_{k,2n-1} + \nu_1 J_{k,2n},$$

$$\mu_{2n+1} = 2\nu_0 J_{k,2n} + \mu_1 J_{k,2n+1},$$

$$\nu_{2n} = 2\nu_0 J_{k,2n-1} + \mu_1 J_{k,2n},$$

$$\nu_{2n+1} = 2\mu_0 J_{k,2n} + \nu_1 J_{k,2n+1}.$$

■

3.3 Closed-form solution of system (3.1)

In this section, we begin by reformulating the original system through a valid variable change. This new variable transformation leads us to a simpler equivalent system, which we then study in detail. Through recursive relations and exploiting known information from previous lemmas, we obtain closed-form solutions of the system. We then express these solutions in an explicit form involving $k-$Jacobsthal sequences. Finally, we summarize our findings in a main theorem and provide a corollary providing the complete solution to the provided system.

From (3.1) we can write

$$x_{(l+1)(n+1)-j} = \frac{1}{k + 2y_{(l+1)n-j}}, \quad y_{(l+1)(n+1)-j} = \frac{1}{k + 2x_{(l+1)n-j}}.$$

By using the following change of variables

$$x_n^{(j)} = x_{(l+1)n-j}, \quad y_n^{(j)} = y_{(l+1)n-j},$$

in system (3.1) we get

$$x_{n+1}^{(j)} = \frac{1}{k+2y_n^j}, \quad y_{n+1}^{(j)} = \frac{1}{k+2x_n^j}. \quad n \in \mathbb{N}_0. \tag{3.9}$$

Hence we can use the change of variables

$$x_n^{(j)} = \frac{W_n}{U_{n+1}}, \quad y_n^{(j)} = \frac{U_n}{W_{n+1}},$$

in (3.9) and obtain

$$x_{n+1}^{(j)} = \frac{W_{n+1}}{U_{n+2}}, \quad y_{n+1}^{(j)} = \frac{U_{n+1}}{W_{n+2}}.$$

So,

$$\begin{aligned} \frac{W_{n+1}}{U_{n+2}} &= \frac{1}{k+\frac{2U_n}{W_{n+1}}}, \\ &= \frac{1}{\frac{kW_{n+1}+2U_{k,n}}{W_{n+1}}}, \\ &= \frac{W_{n+1}}{kW_{n+1}+2U_n}. \end{aligned}$$

And

$$\begin{aligned} \frac{U_{n+1}}{Wn+2} &= \frac{1}{k+\frac{2W_n}{U_{n+1}}}, \\ &= \frac{1}{\frac{kU_{n+1}+2W_n}{U_{n+1}}}, \\ &= \frac{U_{n+1}}{kU_{n+1}+2W_n}. \end{aligned}$$

So the system (3.9) becomes:

$$U_{n+2} = kW_{n+1} + 2U_n,$$

$$W_{n+2} = kU_{n+1} + 2W_n.$$

Then it follows from lemma (3.2.2) that:

$$U_{2n} = 2U_0J_{2n-1} + W_1J_{k,2n},$$

$$U_{2n+1} = 2W_0J_{k,2n} + U_1J_{k,2n+1},$$

$$W_{2n} = 2W_0J_{k,2n-1} + U_1J_{k,2n},$$

$$W_{2n+1} = 2U_0J_{k,2n} + W_1J_{k,2n+1}.$$

So

$$\begin{aligned} x_{2n+1}^{(j)} &= \frac{W_{2n+1}}{U_{2n+2}}, \\ &= \frac{2U_0J_{2n}+W_1J_{k,2n+1}}{2U_0J_{k,2n+1}+W_1J_{k,2n+2}}, \\ &= \frac{\frac{2U_0}{W_1}J_{k,2n}+J_{k,2n+1}}{\frac{2U_0}{W_1}J_{k,2n+1}+J_{k,2n+2}}, \end{aligned}$$

$$x_{2n+1}^{(j)} = \frac{2y_0^{(j)} J_{k,2n} + J_{k,2n+1}}{2y_0^{(j)} J_{k,2n+1} + J_{k,2n+2}}.$$

As well as

$$\begin{aligned} x_{2n}^{(j)} &= \frac{W_{2n}}{U_{2n+1}}, \\ &= \frac{2W_0 J_{k,2n-1} + U_1 J_{k,2n}}{2W_0 J_{k,2n} + U_1 J_{k,2n+1}}, \\ &= \frac{\frac{2W_0}{U_1} J_{k,2n-1} + J_{k,2n}}{\frac{2W_0}{U_1} J_{k,2n} + J_{k,2n+1}}, \end{aligned}$$

$$x_{2n}^{(j)} = \frac{2x_0^{(j)} J_{k,2n-1} + J_{k,2n}}{2x_0^{(j)} J_{k,2n} + J_{k,2n+1}}.$$

And we have

$$\begin{aligned} y_{2n+1}^{(j)} &= \frac{U_{2n+1}}{W_{2n+2}}, \\ &= \frac{2W_0 J_{k,2n} + U_1 J_{2n+1}}{2W_0 J_{2n+1} + U_1 J_{k,2n+2}}, \\ &= \frac{\frac{2W_0}{U_1} J_{k,2n} + J_{k,2n+1}}{\frac{2W_0}{U_{k,1}} J_{k,2n+1} + J_{k,2n+2}}, \end{aligned}$$

$$y_{2n+1}^{(j)} = \frac{2x_0^{(j)} J_{k,2n} + J_{2n+1}}{2x_0^{(j)} J_{k,2n+1} + J_{k,2n+2}}.$$

Also

$$\begin{aligned} y_{2n}^{(j)} &= \frac{U_{2n}}{W_{2n+1}}, \\ &= \frac{2U_0 J_{k,2n-1} + W_1 J_{k,2n}}{2U_0 J_{k,2n} + W_1 J_{k,2n+1}}, \\ &= \frac{\frac{2U_0}{W_1} J_{k,2n-1} + j_{k,2n}}{\frac{2U_0}{W_{k,1}} J_{k,2n} + J_{k,2n+1}}, \end{aligned}$$

$$y_{2n}^{(j)} = \frac{2y_0^{(j)} J_{k,2n-1} + J_{k,2n}}{2y_0^{(j)} J_{k,2n} + J_{k,2n+1}}.$$

Theorem 3.3.1 *Let $\{x_n^{(j)}, y_n^{(j)}\}_{n\geq 0}$ be the solution to system* (3.9) *then for $n \in \mathbb{N}$ and $j = 0, ..., l$*

$$x_{2n+1}^{(j)} = \frac{2y_0^{(j)} J_{k,2n} + J_{k,2n+1}}{2y_0^{(j)} J_{k,2n+1} + J_{k,2n+2}}, \quad x_{2n}^{(j)} = \frac{2x_0^{(j)} J_{k,2n-1} + J_{k,2n}}{2x_0^{(j)} J_{k,2n} + J_{k,2n+1}},$$

$$y_{2n+1}^{(j)} = \frac{2x_0^{(j)} J_{k,2n} + J_{2n+1}}{2x_0^{(j)} J_{k,2n+1} + J_{k,2n+2}}, \quad y_{2n}^{(j)} = \frac{2y_0^{(j)} J_{k,2n} + J_{2n+1}}{2y_0^{(j)} J_{k,2n+1} + J_{k,2n+2}}.$$

The following corollory is our main result wich gives the explicit formula of solution to system

Corollary 3.3.1 *Let $\{x_n, y_n\}_{n\geq 0}$ be the solution to system* (3.1) *then for $n \in \mathbb{N}$ and $j = 0, 1, ..., l$.*

$$x_{(l+1)(2n+1)-j} = \frac{2y_{-j} J_{k,2n} + J_{k,2n+1}}{2y_{-j} J_{k,2n+1} + J_{k,2n+2}}, \quad x_{(l+1)(2n)-j} = \frac{2x_{-j} J_{k,2n-1} + J_{k,2n}}{2x_{-j} J_{k,2n} + J_{k,2n+1}},$$

$$y_{(l+1)(2n+1)-j} = \frac{2x_{-j} J_{k,2n} + J_{2n+1}}{2x_{-j} J_{k,2n+1} + J_{k,2n+2}}, \quad y_{(l+1)(2n)-j} = \frac{2y_{-j} J_{k,2n} + J_{2n+1}}{2y_{-j} J_{k,2n+1} + J_{k,2n+2}}.$$

Proof.

We have

$$x_n^{(j)} = x_{(l+1)(n)-j}, \quad j = 0, 1, ..., l.$$

So

$$x_{2n+1}^{(j)} = x_{(l+1)(2n+1)-j},$$

and

$$x_0^{(j)} = x_{-j}.$$

Then

$$x_{2n+1}^{(j)} = x_{(l+1)(2n+1)-j} = \frac{2y_{-j}J_{k,2n} + J_{k,2n+1}}{2y_{-j}J_{k,2n+1} + J_{k,2n+2}},$$

and

$$x_{2n}^{(j)} = x_{(l+1)(2n)-j} = \frac{2x_{-j}J_{k,2n-1} + J_{k,2n}}{2x_{-j}J_{k,2n} + J_{k,2n+1}}.$$

We have

$$y_n^{(j)} = y_{(l+1)(n)-j}, \quad j = 0, 1, ..., l.$$

So

$$y_{2n+1}^{(j)} = y_{(l+1)(2n+1)-j},$$

and

$$y_0^{(j)} = y_{-j}.$$

Hence

$$y_{2n+1}^{(j)} = y_{(l+1)(2n+1)-j} = \frac{2x_{-j}J_{k,2n} + J_{2n+1}}{2x_{-j}J_{k,2n+1} + J_{k,2n+2}},$$

Also

$$y_{2n}^{(j)} = y_{(l+1)(2n+)-j} = \frac{2y_{-j}J_{k,2n} + J_{2n+1}}{2y_{-j}J_{k,2n+1} + J_{k,2n+2}}.$$

■

3.4 Global stability of the solution to system (3.1)

In this section, we investigate the global stability of the solutions of a system of nonlinear difference equations (3.1). By finding the equilibrium points of the system, we apply linearization techniques and Rouche's Theorem to determine local and global asymptotic stability.

The equilibrium points of the system are given by :

$$\overline{M} = (\overline{x}, \overline{y}) = \left(\frac{-k + \sqrt{k^2+8}}{4}, \frac{-k + \sqrt{k^2+8}}{4} \right),$$

$$\overline{M} = (\overline{x}, \overline{y}) = \left(\frac{-k - \sqrt{k^2+8}}{4}, \frac{-k - \sqrt{k^2+8}}{4} \right).$$

Theorem 3.4.1 *The equilibrium point*

$$\overline{M} = (\overline{x}, \overline{y}) = \left(\frac{-k + \sqrt{k^2+8}}{4}, \frac{-k + \sqrt{k^2+8}}{4} \right),$$

is locally asymptotically stable.

Proof.

We linearize the system around the equilibrium point $\overline{M} = (\overline{x}, \overline{y}) = \left(\frac{-k+\sqrt{k^2+8}}{4}, \frac{-k+\sqrt{k^2+8}}{4} \right)$. The resulting linear system can be written as :

$$X_{n+1} = JX_n,$$

where the state vector is defined by

$$X_n = \left(x_n, \quad x_{n-1}, \quad \cdots, \quad x_{n-l}, \quad y_n, \quad y_{n-1}, \quad \cdots, \quad y_{n-l} \right)^t,$$

and the matrix J is given as

$$J = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \frac{-8}{\left(k+\sqrt{k^2+8}\right)^2} \\ 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \frac{-8}{\left(k+\sqrt{k^2+8}\right)^2} & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}.$$

The characteristic polynomial of matrix J is:

$$P(\lambda) = (-\lambda)^{2l+2} - \left(\frac{-8}{(k+\sqrt{k^2+8})^2}\right)^2.$$

Let us define the functions :

$$\varphi(\lambda) = (-\lambda)^{2l+2}, \quad \phi(\lambda) = \left(\frac{-8}{(k+\sqrt{k^2+8})^2}\right)^2.$$

It holds that

$$|\phi(\lambda)| < |\varphi(\lambda)|, \quad \text{for all} \lambda \in \mathbb{C}, \quad \text{for all} |\lambda| = 1.$$

By Rouche's Theorem, the function $P = \varphi + \phi$ has the same number of zeros as φ within the unit disc $|\phi(\lambda)| < 1$. since φ has a root of multiplicity $2(l+1)$ at $\lambda = 0$, then all the roots of P are in the disc $|\phi(\lambda)| < 1$. Thus, the equilibrium point is locally asymptotically stable.

■

Corollary 3.4.1 *The equilibrium point $\overline{M}$ is globally asymptotically stable.*

Proof.

According to Theorem (3.4.1), the point $\overline{M}$ is locally stable. To demonstrate global asymptotic stability, we utilize Corollary (3.3.1). Consider the following limit

$$\lim_{n\to\infty} x_{(l+1)(2n)-j} = \lim_{n\to\infty} \frac{2x_{-j}J_{k,2n-1} + J_{k,2n}}{2x_{-j}J_{k,2n} + J_{k,2n+1}} = \lim_{n\to\infty} \frac{2x_{-j}\frac{J_{k,2n-1}}{J_{k,2n}} + 1}{2x_{-j} + \frac{J_{k,2n+1}}{J_{k,2n}}}.$$

Using the known limits

$$\lim_{n\to\infty}\left(\frac{J_{k,2n-1}}{J_{k,2n}}\right) = \frac{1}{r_1} = r_2, \quad \lim_{n\to\infty}\left(\frac{J_{k,2n+1}}{J_{k,2n}}\right) = r_1,$$

we get

$$\begin{aligned}
\lim_{n\to\infty} x_{(l+1)(2n)-j} &= \frac{2x_{-j}r_2 + 1}{2x_{-j} + r_1}, \\
&= \frac{2x_{-j}(\frac{k-\sqrt{k^2+8}}{2}) + 1}{2x_{-j} + \frac{k+\sqrt{k^2+8}}{2}}, \\
&= \frac{-k + \sqrt{k^2+8}}{4}, \\
&= \overline{x}.
\end{aligned}$$

However,we have

$$\lim_{n\to\infty} x_{(l+1)(2n+1)-j} = \lim_{n\to\infty} \frac{2y_{-j}J_{k,2n} + J_{k,2n+1}}{2y_{-j}J_{k,2n+1} + J_{k,2n+2}} = \lim_{n\to\infty} \frac{2y_{-j}\frac{J_{k,2n}}{J_{k,2n+1}} + 1}{2y_{-j} + \frac{J_{k,2n+2}}{J_{k,2n+1}}} = \frac{hy_{-j}r_2 + 1}{hy_{-j} + r_1}.$$

Using the following two limits

$$\lim_{n\to\infty}\left(\frac{J_{k,2n-1}}{J_{k,2n}}\right) = \frac{1}{r_1} = r_2, \quad \lim_{n\to\infty}\left(\frac{J_{k,2n+1}}{J_{k,2n}}\right) = r_1,$$

we get

$$\lim_{n\to\infty} x_{(l+1)(2n+1)-j} = \frac{-k + \sqrt{k^2+8}}{4} = \overline{x},$$

So

$$\lim_{n\to\infty} x_{(l+1)(2n+1)-j} = \overline{x}.$$

Similarly, it can be proven that

$$\lim_{n\to\infty} y_{(l+1)(2n+1)-j} = \overline{y}.$$

Hence, the solution converges globally to the equilibrium point

$$\lim_{n\to\infty} \left(x_{(l+1)(2n+1)-j}, y_{(l+1)(2n+1)-j}\right) = (\overline{x}, \overline{y}).$$

■

3.5 Rate of convergence

In this section, we analyze the rate of convergence of solutions to the equilibrium point $(\overline{x}, \overline{y})$ for the given nonlinear difference system (3.1). By linearizing around the equilibrium and expressing the dynamics as

$$A_{n+1} = (M + Z_n)\, A_n, \tag{3.10}$$

where A_n is a $2l$-dimensional vector, $M \in C^{2l\times 2l}$ is a constant matrix and $Z : \mathbb{Z}^+ \to C^{2l\times 2l}$ is a matrix function satisfying

$$\|Z_n\| \longrightarrow 0, \quad when \quad n \to \infty, \tag{3.11}$$

where $\|.\|$ indicates any matrix norm which is associated with the vector norm $\|.\|$. We determine the convergence rate via the spectral radius of M. The parameter k, and delay l explicitly influence this rate, providing insights into the system's stability and asymptotic behavior.

Theorem 3.5.1 *[10] **(Perron's first Theorem)***

Suppose that condition (3.11) *holds. If A_n is a solution of* (3.10)*, then either $A_n = 0$ for all largen or*

$$\rho = \lim_{n\to\infty} \frac{||A_{n+1}||}{||A_n||},$$

exists and is equal to the modulus of one of the eigenvalues of matrix M.

Theorem 3.5.2 *[10]* ***(Perron's second Theorem)***

Suppose that condition (3.11) *holds. If A_n is a solution of* (3.10)*, then either $A_n = 0$ for all largen or*

$$\rho = \lim_{n\to\infty} (||A_{n+1}||)^{\frac{1}{n}},$$

exists and is equal to the modulus of one of the eigenvalues of matrix M.

Theorem 3.5.3 *[10] Let the solution $\{(x_n, y_n)\}_{n\geq -1}$ of system* (3.1) *converges to the equilibrium point $(\overline{x}, \overline{y})$ wich is globally asymptotically stable. So the error vector*

$$e_n = \begin{pmatrix} e_n^{(1)} \\ e_{n-1}^{(1)} \\ \vdots \\ e_{n-l}^{(1)} \\ e_n^2 \\ e_{n-1}^2 \\ \vdots \\ e_{n-l}^2 \end{pmatrix} = \begin{pmatrix} x_n - \bar{x} \\ x_{n-1} - \bar{x} \\ \vdots \\ x_{n-l} - \bar{x} \\ y_n - \bar{y} \\ y_{n-1} - \bar{y} \\ \vdots \\ y_{n-l} - \bar{y} \end{pmatrix}$$

of any solution of system (3.1) *satisfies both of the following asymptotic behaviors*

$$\lim_{n\to\infty} \frac{||e_{n+1}||}{||e_n||} = |\lambda_i J_F((\bar{x}, \bar{y}))|, \quad i = 1, 2, ..., l,$$

$$\lim_{n\to\infty} (||e_{n+1}||)^{\frac{1}{n}} = |\lambda_i J_F((\bar{x}, \bar{y}))|, \quad i = 1, 2, ..., l,$$

where $|\lambda_i J_F((\bar{x}, \bar{y}))|$ corresponds to the absolute value of one of the eigenvalues of the Jacobian matrix evaluated at the equilibrium $(\bar{x}, \bar{y})$.

Proof.

To establish the desired result, we start by formulating a system that governs the evolution of the error terms. These error termes are defined as follows :

$$\begin{cases} x_{n+1}-\bar{x} = \sum_{i=0}^{l} C_i(x_{n-i}-\bar{x}) + \sum_{i=0}^{l} D_i(y_{n-i}-\bar{y}) \quad for \quad i=1,2,....,l, \\ y_{n+1}-\bar{y} = \sum_{i=0}^{l} G_i(x_{n-i}-\bar{x}) + \sum_{i=0}^{l} H_i(y_{n-i}-\bar{y}) \quad for \quad i=1,2,....,l. \end{cases} \tag{3.12}$$

Set

$$e_n^{(1)} = x_n - \bar{x}, \quad e_n^{(2)} = y_n - \bar{y},$$

Then, the system (3.12) become

$$\begin{cases} e_{n+1}^{(1)} = \sum_{i=0}^{l} C_i e_{n-i}^{(1)} + \sum_{i=0}^{l} D_i e_{n-i}^{(2)} \quad for \quad i=1,2,....,l, \\ e_{n+1}^{(2)} = \sum_{i=0}^{l} G_i e_{n-i}^{(1)} + \sum_{i=0}^{l} H_i e_{n-i}^{(2)} \quad for \quad i=1,2,....,l, \end{cases}$$

where

$$C_i = 0 \quad i=1,2,....,l$$

$$D_i = 0 \quad i=1,2,....,l-1, \quad D_l = \frac{-2}{(k+2y_{n-l})^2}$$

$$G_i = 0 \quad i=1,2,....,l-1, \quad G_l = \frac{-2}{(k+2x_{n-l})^2}$$

$$H_i = 0 \quad i=1,2,....,l$$

As the system approaches equilibrium, it becomes clear that

$$\lim_{n\to\infty} C_i = 0 \quad i=1,2,....,l$$

$$\lim_{n\to\infty} D_i = 0 \quad i=1,2,....,l-1, \quad \lim_{n\to\infty} D_l = \frac{-2}{(k+2\bar{y})^2}$$

$$\lim_{n\to\infty} G_i = 0 \quad i=1,2,....,l-1, \quad \lim_{n\to\infty} G_l = \frac{-2}{(k+2\bar{x})^2}$$

$$\lim_{n\to\infty} H_i = 0 \quad i=1,2,....,l$$

So, that means

$$D_l = \frac{-2}{(k+2\bar{y})^2} + \alpha_n, \quad G_l = \frac{-2}{(k+2\bar{x})^2} + \beta_n.$$

We can now express the system in the form

$$e_{n+1} = (M + Z_n)\, e_n,$$

where $e_n = \left(e_n^{(1)},\ e_{n-1}^{(1)},\ \cdots,\ e_{n-l}^{(1)},\ e_n^{(2)},\ e_{n-1}^{(2)},\ \cdots,\ e_{n-l}^{(2)}\right)^t$ and

$$B_n = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \alpha_n \\ 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \beta_n & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}.$$

$$M = J_F(\bar{x}, \bar{y}) = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \frac{-8}{\left(k+\sqrt{k^2+8}\right)^2} \\ 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \frac{-8}{\left(k+\sqrt{k^2+8}\right)^2} & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}.$$

$\|Z_n\| \to 0$ when $n \to \infty$. As a result, the asymptotic error system can be expressed as

$$e_{n+1} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \frac{-8}{\left(k+\sqrt{k^2+8}\right)^2} \\ 1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \frac{-8}{\left(k+\sqrt{k^2+8}\right)^2} & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 1 & 0 \end{pmatrix} \begin{pmatrix} e_n^{(1)} \\ e_{n-1}^{(1)} \\ \vdots \\ e_{n-l}^{(1)} \\ e_n^2 \\ e_{n-1}^2 \\ \vdots \\ e_{n-l}^2 \end{pmatrix},$$

and $\|Z_n\| \to 0$ when $n \to \infty$. Clearly, this system coressponds to the linear approximation of equation (3.1) near $(\bar{x}, \bar{y})$ the equilibrium point. Thus, the result is a direct consequence of Perron's Theorems. ■

3.6 Numerical Examples

In this section, we studied a system of nonlinear difference equations with the use of exact numerical techniques. We performed iterative calculations and documented the findings in detailed tables and graphical plots that illustrate the dynamics of the system and its convergence towards equilibrium points. The study was conducted in an attempt to trace the evolution of variables with extremely high numerical accuracy using advanced computational techniques as a foundation for subsequent theoretical analysis and practical application.

example 3.6.1 *Let the following system of difference equations*

$$x_{n+1} = \frac{1}{k + 2y_{n-l}}, \quad y_{n+1} = \frac{1}{k + 2x_{n-l}}, \quad n \geq 0, \tag{3.13}$$

where the parameters and initial conditions are chosen as follows:

- $k = 1, l = 3$.
- *Initial conditions for x:* $x_0 = 0.1, \quad x_1 = 2.5, \quad x_2 = 0.3, \quad x_3 = 5.0,$
- *Initial conditions for y:* $y_0 = 3.0, \quad y_1 = 0.2, \quad y_2 = 1.7, \quad y_3 = 0.05.$

The following table presents the computed values of the sequences (x_n) and (y_n) generated by the system of difference equations with the given initial conditions and parameters. The values are displayed for $n = 1$ to $n = 30$.

n	1	2	3	4	5	6	7	8	9	10
x_n	2.5000	0.3000	5.0000	0.7143	0.2273	0.9091	0.7500	0.4444	0.8462	0.5484
y_n	0.2000	1.7000	0.0500	0.1667	0.6250	0.0909	0.4118	0.6875	0.3548	0.4000

n	11	12	13	14	15	16	17	18	19	20
x_n	0.4211	0.5849	0.5556	0.4857	0.5738	0.5118	0.4795	0.5204	0.5135	0.4964
y_n	0.5294	0.3714	0.4769	0.5429	0.4609	0.4737	0.5072	0.4656	0.4942	0.5105

n	21	22	23	24	25	26	27	28	29	30
x_n	0.5178	0.5029	0.4948	0.5050	0.5034	0.4991	0.5044	0.5007	0.4987	0.4987
y_n	0.4900	0.4933	0.5018	0.4913	0.4985	0.5026	0.4975	0.4983	0.5005	0.5005

Table 3.1: Computed Values of x_n and y_n for $n = 1$ to 30 from the difference equation system

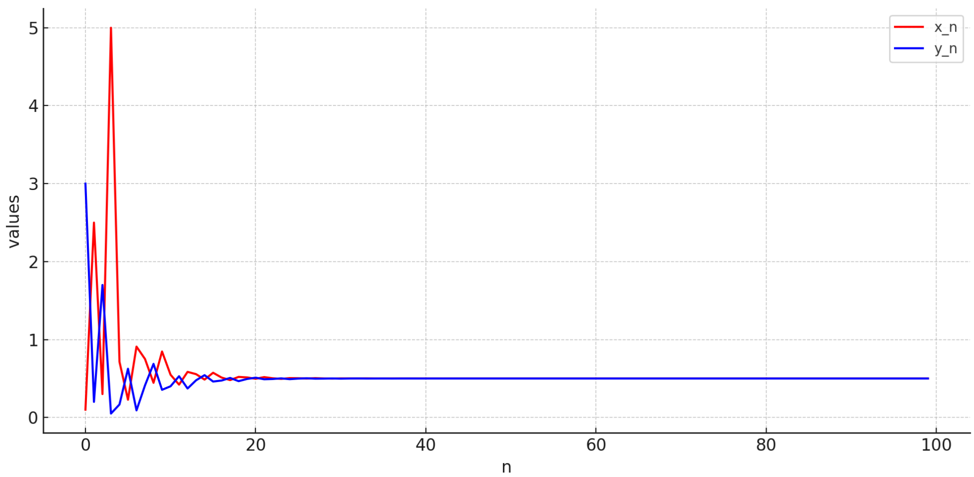

Figure 3.1: Plot of the numerical solution of the system (3.13)

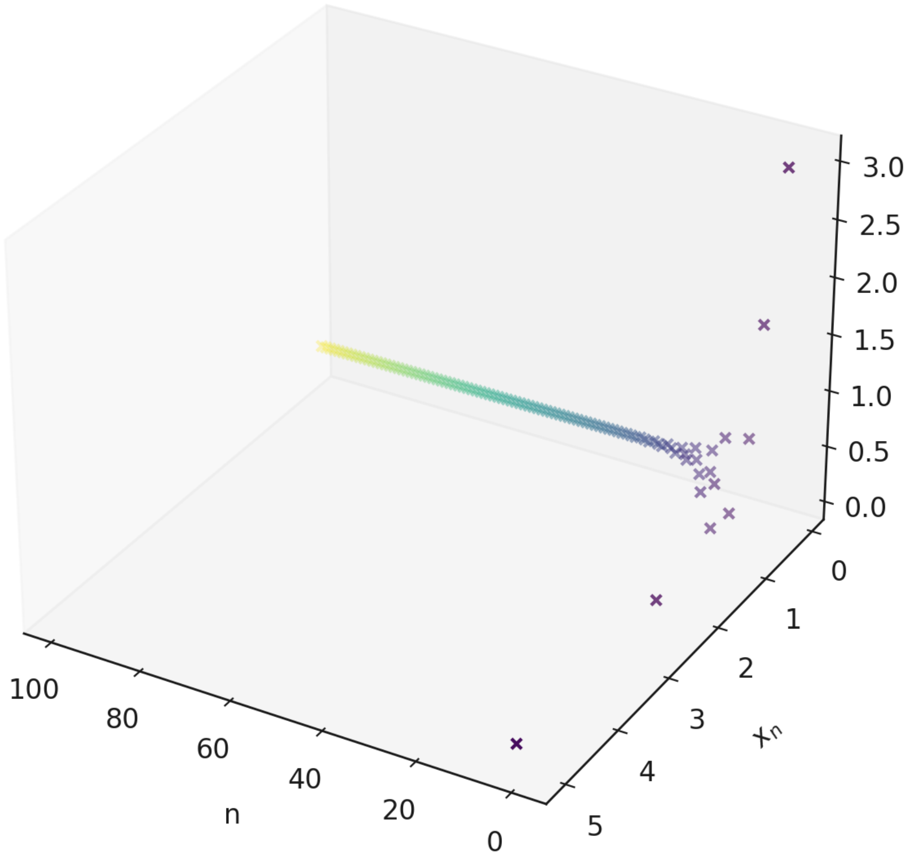

Figure 3.2: 3D phase surface plot of the system (3.13)

example 3.6.2 *Let the following system of difference equations*

$$x_{n+1} = \frac{1}{k + 2y_{n-l}}, \quad y_{n+1} = \frac{1}{k + 2x_{n-l}}, \quad n \geq 0, \tag{3.14}$$

where the parameters and initial conditions are chosen as follows:

- $k = 1, l = 12.$
- *Initial conditions for x:* $x_0 = 0.5488,\quad x_1 = 0.7152,\quad x_2 = 0.6028,\quad x_3 = 0.5449,$
  $x_4 = 0.4237,\quad x_5 = 0.6459,\quad x_6 = 0.4376,\quad x_7 = 0.8918,$
  $x_8 = 0.9637,\quad x_9 = 0.3834,\quad x_{10} = 0.7917,\quad x_{11} = 0.5289,\quad x_{12} = 0.5680,$
- *Initial conditions for y:* $y_0 = 0.9256,\quad y_1 = 0.0710,\quad y_2 = 0.0871,\quad y_3 = 0.0202,$

$y_4 = 0.8326, \quad y_5 = 0.7782, \quad y_6 = 0.8700, \quad y_7 = 0.9786,$

$y_8 = 0.7992, \quad y_9 = 0.4615, \quad y_{10} = 0.7805, \quad y_{11} = 0.1183, \quad y_{12} = 0.6399.$

The following table presents the computed values of the sequences (x_n) and (y_n) generated by the system of difference equations with the given initial conditions and parameters. The values are displayed for $n = 1$ to $n = 120$.

n	1	2	3	4	5	6	7	8	9	10
x_n	0.7152	0.6028	0.5449	0.4237	0.6459	0.4376	0.8918	0.9637	0.3834	0.7917
y_n	0.0710	0.0871	0.0202	0.8326	0.7782	0.8700	0.9786	0.7992	0.4615	0.7805

n	11	12	13	14	15	16	17	18	19	20
x_n	0.5289	0.5680	0.8757	0.8516	0.9612	0.3752	0.3912	0.3650	0.3382	0.3849
y_n	0.1183	0.6399	0.4115	0.4534	0.4785	0.5413	0.4363	0.5333	0.3592	0.3416

n	21	22	23	24	25	26	27	28	29	30
x_n	0.5200	0.3905	0.8087	0.4386	0.5486	0.5244	0.5110	0.4802	0.5340	0.4839
y_n	0.5660	0.3871	0.4860	0.4682	0.3635	0.3699	0.3422	0.5713	0.5611	0.5781

n	31	32	33	34	35	36	37	38	39	40
x_n	0.5819	0.5941	0.4690	0.5636	0.5071	0.5164	0.5791	0.5748	0.5937	0.4667
y_n	0.5965	0.5651	0.4902	0.5615	0.3821	0.5327	0.4768	0.4881	0.4946	0.5101

n	41	42	43	44	45	46	47	48	49	50
x_n	0.4712	0.4638	0.4560	0.4695	0.5050	0.4710	0.5668	0.4842	0.5119	0.5060
y_n	0.4836	0.5082	0.4621	0.4570	0.5160	0.4701	0.4965	0.4919	0.4634	0.4652

n	51	52	53	54	55	56	57	58	59	60
x_n	0.5027	0.4950	0.5084	0.4959	0.5197	0.5225	0.4921	0.5154	0.5018	0.5041
y_n	0.4572	0.5172	0.5148	0.5188	0.5230	0.5158	0.4975	0.5149	0.4687	0.5080

n	61	62	63	64	65	66	67	68	69	70
x_n	0.5190	0.5180	0.5224	0.4915	0.4927	0.4908	0.4887	0.4922	0.5012	0.4927
y_n	0.4941	0.4970	0.4986	0.5025	0.4959	0.5020	0.4904	0.4890	0.5040	0.4924

n	71	72	73	74	75	76	77	78	79	80
x_n	0.5162	0.4960	0.5029	0.5015	0.5007	0.4987	0.5021	0.4990	0.5049	0.5056
y_n	0.4991	0.4980	0.4907	0.4912	0.4891	0.5043	0.5037	0.5047	0.5057	0.5039

n	81	82	83	84	85	86	87	88	89	90
x_n	0.4980	0.5038	0.5004	0.5010	0.5047	0.5045	0.5055	0.4979	0.4982	0.4977
y_n	0.4994	0.5037	0.4920	0.5020	0.4985	0.4992	0.4997	0.5006	0.4990	0.5005

n	91	92	93	94	95	96	97	98	99	100
x_n	0.4972	0.4981	0.5003	0.4982	0.5040	0.4990	0.5007	0.5004	0.5002	0.4997
y_n	0.4976	0.4972	0.5010	0.4981	0.4998	0.4995	0.4977	0.4978	0.4972	0.5011

n	101	102	103	104	105	106	107	108	109	110
x_n	0.5005	0.4997	0.5012	0.5014	0.4995	0.5010	0.5001	0.5003	0.5012	0.5011
y_n	0.5009	0.5012	0.5014	0.5010	0.4998	0.5009	0.4980	0.5005	0.4996	0.4998

n	111	112	113	114	115	116	117	118	119	120
x_n	0.5014	0.4995	0.4995	0.4994	0.4993	0.4995	0.5001	0.4995	0.5010	0.4998
y_n	0.4999	0.5002	0.4997	0.5001	0.4994	0.4993	0.5002	0.4995	0.4999	0.4999

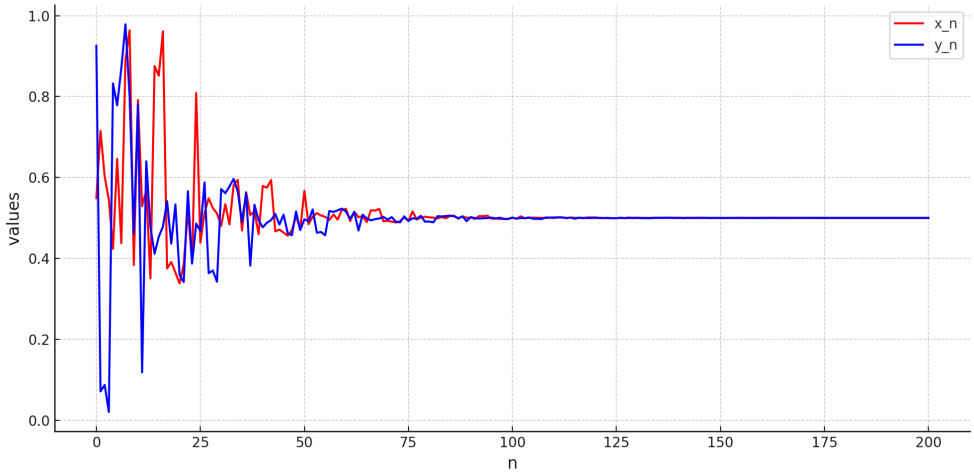

Figure 3.3: Plot of the numerical solution of the system (3.14))

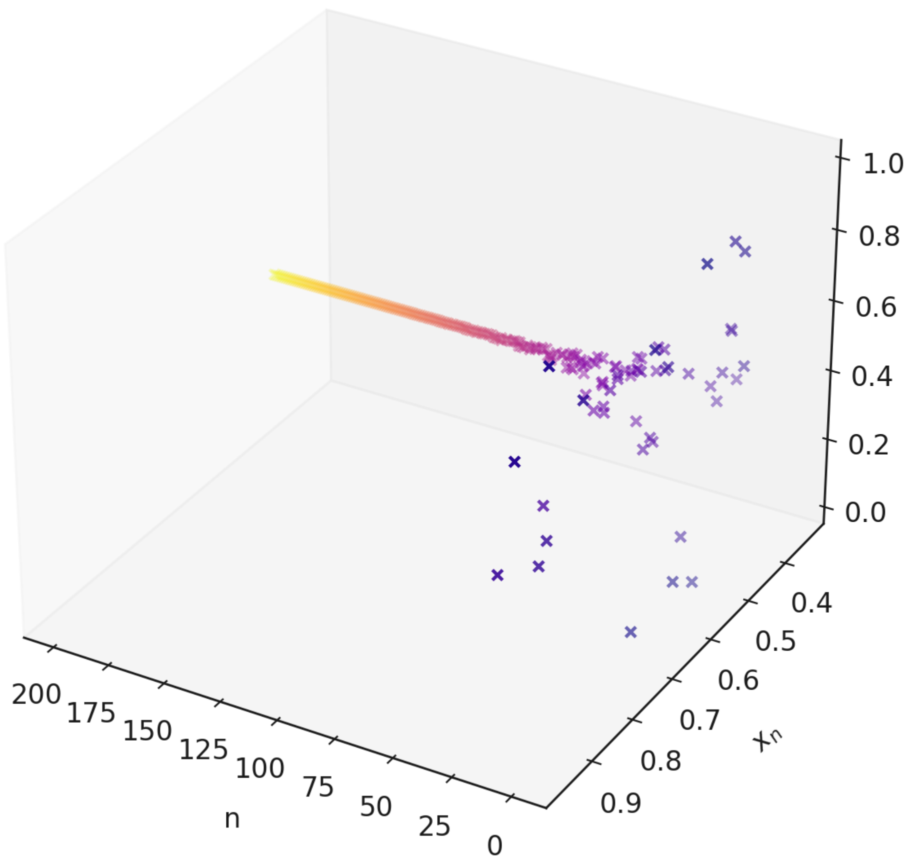

Figure 3.4: 3D phase surface plot of the system (3.14)

CONCLUSION

In conclusion, this master's thesis has highlighted the strong relationship between the solutions of nonlinear systems of difference equations and well-known numerical sequences, particularly the generalized Bell and Jacobsthal sequences. The results demonstrate that these sequences play a fundamental role in characterizing the behavior of such systems, both in deriving explicit solutions and in analyzing equilibrium and stability. The structural and algebraic properties of the sequences provide valuable insights into the dynamics of discrete systems, offering a powerful framework for understanding their long-term behavior. This connection underscores the theoretical and practical significance of using classical number sequences in the study and modeling of nonlinear difference equations across various mathematical and applied disciplines.

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