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S5

Exercises on Normed Vector Spaces

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CONTENTS

Introduction

The understanding of results and notions for a student in mathematics requires solving exercises. The exercises are also meant to test the reader's understanding of the text material, and to enhance the skill in doing calculations. The goal of this collection of exercises is to help the student understand much better the basic facts which are usually presented in an introductory course in normed vector spaces, by providing various exercises, from different topics, from simple ones to, perhaps, more difficult ones, in order to help him to understand the richness of ideas and techniques which Banach spaces in general and Hilbert ones in particular offer.
These exercises are related to a course given to students in the 5^{th} semester of mathematics study. We will practice on two chapters in normed vector spaces, wich are Banach spaces in general and Hilbert spaces as a particular case of them.

For the first, we start by familiarizing the student with the norm through elementary calculations, and various spaces, while showing the topological aspect of normed spaces. Then Cauchy sequences are introduced in order to clarify the concept of completeness, and subsequently practice on Banach spaces. Next comes the continuous linear applications which define another normed vector space, among others the dual space.

The second one, we first practice the scalar product, by memorizing basic results, then Hilbert spaces with the Riesz Theorem , after which we move on to the projection theorem and finish with Fourier series by training on particular trigonometric systems.
we first set the exercises, then give the solution with many details, and finaly propose some non solved exercises left to the student, in order to push him to work and investigate.

I hope that some of the exercises herein can be of some help to the teacher of Normed vector spaces as seminar tools; and to anyone who is interested in seeing some applications of normed vector spaces.

CHAPTER 1

BANACH SPACES

Series of exercises 1

1.1 Normed spaces

Exercise 1 Let E be a normed vector space, and x, y, z, t four vectors of E. Show that,

$\|x-t\| + \|y-z\| \leq \|x-y\| + \|y-t\| + \|t-z\| + \|z-x\|$

Solution We apply the triangular inequality four times:

$$\begin{cases} \|x-t\| \leq \|x-y\| + \|y-t\| \\ \|y-z\| \leq \|y-t\| + \|t-z\| \\ \|x-t\| \leq \|x-z\| + \|z-t\| \\ \|y-z\| \leq \|y-x\| + \|x-z\| \end{cases}$$

We add these inequalities and we obtain the result by simplifying by 2:

$\|x-t\| + \|y-z\| \leq \|x-y\| + \|y-t\| + \|t-z\| + \|z-x\|$

Exercise 2

1. Let a and b be two real numbers with the property that $a \leq b+\epsilon$ for ever ϵ. Show that

$a \leq b$.

2. Let $I = [0,1]$. Show that the fellowing is a norm on $\mathcal{C}[0,1]$: $\|f\| = sup|f(x)|$.

Solution

1. Supose $a > b$. Then $a = b + \delta$ where $\delta > 0$. Set $\epsilon = \frac{1}{2}$.

   Now $a > b + \frac{1}{2} = b + \epsilon$ where $\epsilon > 0$.

   But this contradicts the hypothesis; so $a \leq b$.

2. We know thet a real continuous function on a closed intervall is bounded; so f is well defined.

   1- Since $|f'x)| \geq 0$ for every $x \in [0,1]$; $\|-f\| \geq 0$.

   also $\|f\| = 0$ iff $f(x) = 0$ for every $x \in [0,1]$ i,e; iff $f = 0$.

   2-Let $k \in \mathbb{R}$. Then

$$\|kf\| = sup|(kf)(x)| = sup|kf(x)| = sup\left(|k||f(x)|\right) = |k|sup|f(x)| = |k|\|f\|$$

   3- Now, let $\epsilon > 0$. Then $\exists x_0 \in I$ such that:

$$\begin{aligned}
\|f+g\| &= sup\left(|f(x)+g(x)|\right) \\
&\leq |f(x_0)+g(x_0)| + \epsilon \\
&\leq |f(x_0)| + |g(x_0)| + \epsilon \\
&\leq sup|f(x)| + sup|g(x)| + \epsilon \\
&= \|f\| + \|g\| + \epsilon
\end{aligned}$$

   Hence, $\|f+g\| \leq \|f\| + \|g\|$

Exercise 3

1. Show that $N : \mathbb{R}^2 \to \mathbb{R}$, defined for $u = (x,y)$ by $N(u) = \sup_{0 \leq t \leq 1} |x + ty|$ is a norm.

2. Represent the closed unit ball with center 0.

Solution In this question $u = (x, y)$ and $v = (x', y')$ are any two elements of $\mathbb{R}^2$.

- The upper bound $N(u)$ exists because it represents the maximum (reached at least for a value t_0) of the map $t \to |x + ty|$, defined and continued on $[0, 1]$.

- We obviously have the inequality $N(x, y) \geq 0$.

On the other hand $N(x, y) = 0 \Rightarrow \forall t \in [0, 1], x + ty = 0 \Rightarrow x = y = 0$ (choose $t = 0$ and $t = 1$).

- For any real λ:

$$N(\lambda u) = \sup_{0 \leq t \leq 1} |(\lambda x) + t(\lambda y)| = \sup_{0 \leq t \leq 1} |\lambda||x + ty| = |\lambda| \sup_{0 \leq t \leq 1} |x + ty| = |\lambda| N(u)$$

- For any real t of $[0, 1]$, $|(x + x') + t\,(y + y')| \leq |x + ty| + |x' + ty'| \leq N(u) + N(v)$. We can then pass to the upper bound in $|(x + x') + t\,(y + y')|$ and write:

$$N(u + v) \leq N(u) + N(v)$$

The map $u \to N(u)$ is therefore a norm on $\mathbb{R}^2$.

Let $u = (x, y)$ be any element of $\mathbb{R}^2$, and let φ be defined on $[0, 1]$ by $\varphi(t) = |x + ty|$.

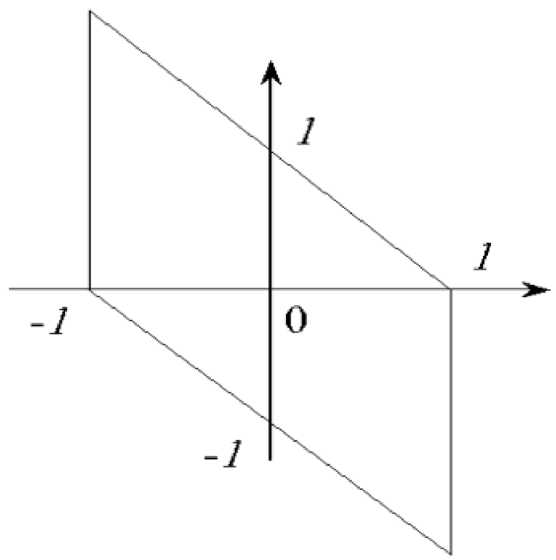
The positive map φ^2 is convex on $[0, 1]$ (its second derivative is positive or zero).

The application φ^2 therefore reaches its maximum at $t = 0$ or at $t = 1$.

The same is true for φ. We therefore have:

$$N(u) \leq 1 \Longleftrightarrow \left\{ \begin{array}{l} \varphi(0) \leq 1 \\ \varphi(1) \leq 1 \end{array} \right. \Longleftrightarrow \left\{ \begin{array}{l} |x| \leq 1 \\ |x + y| \leq 1 \end{array} \right. \Longleftrightarrow \left\{ \begin{array}{l} -1 \leq x \leq 1 \\ -x - 1 \leq y \leq -x + 1 \end{array} \right.$$

We deduce the shape of the closed unit ball:



Exercise 4 Let $\mathscr{C}^1([0,1])$ be the (real) vector space of the continuously differentiable functions $f:[0,1]\to\mathbb{R}$. We set, for $f\in\mathscr{C}^1([0,1])$: $\|f\|=\left([f(0)]^2+\int_0^1|f'(t)|^2\,dt\right)^{1/2}$.

1. Show that we thus define a norm on $\mathscr{C}^1([0,1])$.

2. We set $f_n(t)=t^n(1-t), n\geqslant 1$. Calculate $\|f_n\|$.

Solution

1. The mappings $N_0 : f\mapsto |f(0)|$ and $N_2 : f\mapsto \left(\int_0^1|f'(t)|^2\,dt\right)^{1/2}$ are semi-norms on $\mathscr{C}^1([0,1])$ (it is clear for the first; for the second, it is the composite of the norm $\|$ with the linear form $f\mapsto f'$).

   So $\|f\|=\left[N_0(f)^2+N_2(f)^2\right]^{1/2}$ defines a semi-norm on $\mathscr{C}^1([0.1])$. In fact, this is a norm because if $\|f\|=0$, on a $f(0)=0$ et $\int_0^1|f'(t)|^2\,dt=0$.

   Then f' is zero almost everywhere, therefore everywhere because it is continuous (we can of course also use the elementary result saying that the inegral of a non-zero positive continuous function is strictly positive).

   So f is constant on $[0,1]$, and since $f(0)=0, f=0$.

$$|f_n(t)-f(t)|\leqslant|f_n(0)-f(0)|+\left(\int_0^1|f_n'(u)-f'(u)|^2\,du\right)^{1/2}$$

2. We have $f_n(0) = 0$ et $f'_n(t) = t^{n-1}[n-(n+1)t]$,

   hence, squaring, $[f'_n(t)]^2 = n^2t^{2n-2} - 2n(n+1)t + (n+1)^2t^2$ and (barring a calculation error!) $\int_0^1 [f'_n(t)]^2\,dt = n\left[\frac{n}{2n-1} - \frac{n+1}{2n+1}\right]$, which gives $\|f_n\| = \sqrt{\frac{n}{4n^2-1}}$.

Non solved exercises

Exercise 1 We equip $E = R[X]$ with the norm $\|\|_\infty$ defined by:

$\forall P \in E, \|P\|_\infty = \mathrm{Sup}\left\{\left|\frac{P^{(n)}(0)}{n!}\right|, n \in \mathbb{N}\right\}$.

Check that $\|\|_\infty$ is a norm on E.

Exercise 2 Let in $\mathbb{R}^n$, $\boldsymbol{x} = (x_i)_{i=1}^n$ and $y = (y_i)_{i=1}^n \in \mathbb{R}^n$, and we define the standard:

$$\|x\|_2 = \sqrt{\sum_{i=1}^n x_i^2}$$

Using the Cauchy-Schwarz inequality,

$$\left|\sum_{i=1}^n x_i y_i\right| \leq \|x\|_2 \cdot \|y\|_2$$

Show that $\|\|_2$ is a norm.

Exercise 3 We consider in the space $\mathcal{C}([0,1],\mathbb{R})$ continuous real functions on $[0,1]$, the norms

$$\|f\|_1 = \int_0^1 |f(t)|dt \quad \text{et} \quad \|f\|_2 = \left(\int_0^1 |f(t)|^2\right)^{\frac{1}{2}}$$

1. Check that for $f \in \mathcal{C}([0,1],\mathbb{R})$ we have $\|f\|_1 \leq \|f\|_2$.

2. Consider the sequence of functions $(f_n)_n$ of $\mathcal{C}([0,1],\mathbb{R})$:

$$f_((t) = \begin{cases} 3n^2 t & \text{si } 0 \leq t \leq \frac{1}{3n} \\ n & \text{si } \frac{1}{3n} \leq t \leq \frac{2}{3n} \\ -3n^2\left(t - \frac{1}{n}\right) & \text{si } \frac{2}{3n} \leq t \leq \frac{1}{n} \\ 0 & \text{si } \frac{1}{n} \leq t \leq 1 \end{cases}$$

   a- Check that $\|f_n\|_1 \leq 1$ for each $\boldsymbol{n}$

   b- Check that $\|f_n\|_2$ tends towards infinity.

3. Are the two norms equivalent?

Exercise 4 Frobenius NormFor $A \in \mathscr{M}_n(\mathbb{R})$, we set $\|A\| = \sqrt{\operatorname{tr}(\,(^{(}AA)}$.

1. Show that it is a norm.

2. Show that: $\forall A, B \in \mathscr{M}_n(\mathbb{R}), \|AB\| \leqslant \|A\| \times \|B\|$.

Series of exercises 2

1.2 Complet Spaces

Exercise 1 Let E be a normed vector space and $(x_n)_{noN}$ a sequence of elements of E. Supose that (x_n) is a Cauchy sequence.

Show that it converges if and only if it has a convergent subsequence.

Solution A convergent sequence always admits a convergent subsequence. Conversely, if (u_n) admits a subsequence ($u_{\varphi(n)}$) which converges to ell, we set $\varepsilon > 0$. Since (u_n) is Cauchy, there exists N_1 such that

$$n, p \geq N_1 \Longrightarrow \|u_n - u_p\| \leq \varepsilon$$

We then fix n_0 such that $\varphi(n_0) > N_1$ and $\left\|u_{\varphi(n_0)} - l\right\| \leq \varepsilon$. For $n \geq \varphi(n_0) \geq N_1$, we have according to the triangular inequality:

$$\|u_n - l\| \leq \left\|u_n - u_{\varphi(n_0)}\right\| + \left\|u_{\varphi(n_0)} - l\right\| \leq 2\varepsilon$$

Exercise 2 Let be $X =]0, \infty[$. For $x, y \in X$, denote

$$\delta(x, y) = \left|\frac{1}{x} - \frac{1}{y}\right|$$

Show that δ is a distance on X. Is the metric space (X, d) complet?

Solution δ is a distance. In fact, we have:

$\delta(x, y) = \delta(y, x)$.

$\delta(x, z) = \left|\frac{1}{x} - \frac{1}{z}\right| \leq \left|\frac{1}{x} - \frac{1}{y}\right| + \left|\frac{1}{y} - \frac{1}{z}\right| = \delta(x, y) + \delta(y, z)$.

$\delta(x, y) = 0$ if and only if $x = y$.

The space (X,δ) is therefore a metric space. It is not complete. Let us indeed take the sequence $u_n = n$. This sequence (u_n) is of Cauchy. Indeed,

$$\delta\left(u_n, u_{n+p}\right) \leq \frac{1}{n} + \frac{1}{n+p} \leq \frac{2}{n}$$

This sequence is therefore a Cauchy sequence. It can not converge, because if its limit was ℓ, we would have:

$$\left|\frac{1}{n} - \frac{1}{\ell}\right| \to 0$$

which results in $1/\ell = 0$ which is of course impossible.

Exercise 3 Let E be the vectorial space of $\mathbb{R}$ valued continuous functions on $[-1,1]$. Define a norm on E by

$$\|f\|_1 = \int_{-1}^{1} |f(t)| dt$$

We want to show that E endowed with this norm is not complet. To show that we define a sequence of functions $(f_n)_{n\in\mathbb{N}^*}$ by

$$f_n(t) = \begin{cases} -1 \text{ if } -1 \leq t \leq -\frac{1}{n} \\ nt \text{ if } -\frac{1}{n} \leq t \leq \frac{1}{n} \\ 1 \text{ if } \frac{1}{n} \leq t \leq 1 \end{cases}$$

1- verify that $f_n \in E, \forall n \geq 1$. 2- Show that

$$\|f_n - f_p\| \leq \sup\left(\frac{2}{n}, \frac{2}{p}\right)$$

and deduce that $(f_n)_{n\in\mathbb{N}^*}$ is a Cauchy sequence.

3- Supose that there exists a function $f \in E$ so that (f_n) converges to f in $(E, \|\|_1)$. Then show that we have:

$$\lim_{n\to+\infty}\int_{-1}^{-a}|f_n(t)-f(t)|\,dt=0 \quad\text{ and }\quad \lim_{n\to+\infty}\int_{a}^{1}|f_n(t)-f(t)|\,dt=0$$

for all $0<\alpha<1$.

4- therefore show that

$$\lim_{n\to+\infty}\int_{-1}^{-a}|f_n(t)+1|\,dt=0 \quad\text{ and }\quad \lim_{n\to+\infty}\int_{a}^{1}|f_n(t)-1|\,dt=0$$

for all $0<\alpha<1$.

Deduce that

$$f(t)=1 \text{ for all } -1\le t<0$$

$$f(t)=-1 \text{ for all } 0<t\le 1$$

- Conclude.

Solution

1. Just check the continuity in $1/n$ and $-1/n$. But if t tends to $1/n$, then nt tends to 1, which proves continuity in $1/n$. Likewise in $-1/n$.

2. We assume $n\le p$. We then have:

$$\|f_n-f_p\|\le\int_{-1/n}^{1/n}|f_n(t)-f_p(t)|\,dt.$$

Now, f_n and f_p are always of the same sign, and therefore:

$$|f_n(t)-f_p(t)|\le|f_n(t)|=1$$

By integrating, we obtain the result.

3. The integral of a positive function over part of a segment is less than or equal to the

integral of that function over the entire segment. In particular, we have:

$$\int_{\alpha}^{1} |f(t) - f_n(t)| \leq \int_{-1}^{1} |f(t) - f_n(t)| = \|f - f_n\| \to 0.$$

Likewise for the integral between $-\alpha$ and 1.

4. It's even stronger than that! Indeed, if n is large enough for $1/n < \alpha$, we have exactly:

$$\int_{\alpha}^{1} |f_n(t) - 1| \, dt = 0$$

Now let's set $\alpha > 0$. We have, by the triangular inequality:

$$\int_{\alpha}^{1} |f(t) - 1| dt \leq \int_{\alpha}^{1} |f(t) - f_n(t)| \, dt + \int_{\alpha}^{1} |f_n(t) - 1| \, dt$$

Making n tend towards $+\infty$, we finally prove that:

$$\int_{\alpha}^{1} |f(t) - 1| dt = 0$$

Since $t \mapsto |f(t) - 1|$ is continuous on $[\alpha, 1]$ and it is a positive function, the nullity of its integral implies that this function is itself identically zero on $[\alpha, 1]$. Since α is arbitrary, we finally obtain that $f(t) = 1$ on $]0, 1]$.

The reasoning is the same on the interval $[-1, 0[$. We then obtain an absurdity, since such a function f cannot be continuous at 0. The space is not complete!

Exercise 4 Let E be a normed vector space and F be a subspace of E.

We denote by $\bar{F}$ the set of adherent points of F.

- Show that $\bar{F}$ is a subspace of E.

Solution Let x and y be two elements of $\bar{F}$. Let λ and μ be two scalars.

By definition, there exist two sequences $(x_n), (y_n)$ of F such that:

$$\lim_{n\to\infty} x_n = x \quad \textit{and} \quad \lim_{n\to\infty} y_n = y$$

. For any integer n, the vector $z_n = \lambda x_n + \mu y_n$ is an element of F.

On the other hand, the general term sequence (z_n) converges to $z = \lambda x + \mu y$.

We deduce that z is an element of $\bar{F}$, which was what had to be demonstrated.

Exercise 5 Let X be a Banach space, Y a normed vectorial space and $T : X \to Y$ a continuous linear mapping. Supose that there exists a constante $c > 0$ so that:

$$\|Tx\| \geq c\|x\| \quad \text{for all } x \in X$$

1. Show that $\mathrm{Im}(T)$ is closed in Y.

2. Show that T is an isomorphism from X to $\mathrm{Im}(T)$.

Solution

1. Let be $(y_n)_{n\geqslant 1}$ a sequence of elements of $Im(T)$ such that:

   $$y_n \to y \quad (y \in y).$$

   So,

   $$\exists\, (x_n)_{n\geqslant 1} \in X / \quad T(x_n) = y_n$$

   We have for all $p, k \geqslant 1$, $\|y_p - y_k\| = \|T(x_p) - T(x_k)\| = \|T(x_p - x_k)\| \geqslant C\, \|x_p - x_k\|_1$

   .

   On the other hand,

$$
\begin{aligned}
(y_n)_n \text{ convergente } &\Rightarrow (y_n)_n \text{ de Cauchy} \\
&\Rightarrow \forall \varepsilon > 0, \exists N \geqslant 1 : \|y_p - y_k\| \leqslant \varepsilon \quad (\forall p, k \geqslant N) \\
&\Rightarrow \varepsilon \geqslant \|y_p - y_\varepsilon\| \geqslant c\,\|x_p - x_k\| \\
&\Rightarrow \quad \|x_p - x_k\| \leq \frac{\varepsilon}{c} \quad (\forall \varepsilon > 0)
\end{aligned}
$$

So, $(x)_n$, is a Cauchy sequence in X.

Hence$(x_n)_n$ converges in X, because X is complet: $\lim x_n = x$.

-Now, since T is continuous, $\lim_n(Tx_n) = Tx$ then $\lim_n y_n = Tx$.

We have $\lim y_n = y$, so, $y = Tx$ hence $y \in \mathrm{Im}(T)$ therfore T is closed.

2. T is injective, so T is an isomorphism.

Non solved exercises

Exercise 1 In $\boldsymbol{X} =]-1, +1[$, check that $\left(1 - \frac{1}{n}\right)_n$ is a Cauchy sequence .

Is it convergent? What to deduce?

Exercise 2 Same question for the following ($\boldsymbol{x_n}$) defined by $\boldsymbol{x}_n = \boldsymbol{E}\left(2^n\sqrt{2}\right)/2^n$ In $\mathbb{Q}$ provided with the usual distance in $\mathbb{R}$.

Exercise 3 In the space $\mathcal{C}([0,1],\mathbb{R})$ provided with the norm $\|\|_1$ we consider the sequence of continuous functions

$$
f_n(t) = \begin{cases} 2^n t^n & \text{si } t \in [0, 1/2] \\ 1 & \text{si } t \in [1/2, 1] \end{cases}
$$

1. Check that the sequence is from Cauchy
2. Is the space $(\mathcal{C}([0,1],\mathbb{R}), \|\|_1)$ complete?

Exercise 4 Let X be a normed space. Prove that the following assertions are equivalent:

1. X is finite-dimensional;

2. Any linear functional on X is continuous;

3. Any linear subspace of X is closed.

Series of Exercises 3

1.3 Linear mapping

Exercise 1 We provide $E = \mathcal{M}_n(\mathbb{R})$ with one of the three usual norms: For any matrix $A = (a_{ij}):\quad \|A\|_1 = \sum |a_{ij}|,\quad \|A\|_2 = \sqrt{\sum a_{ij}^2},\quad \|A\|_\infty = \sup |a_{ij}|$.

In each case, calculate the norm of the linear map "trace".

Solution We should find the positive real minimum λ such that, for any matrix A, we have: $|\operatorname{tr}(A)| \leq \lambda \|A\|$.

- With the norm $A \to \|A\|_1$:

$$|\operatorname{tr}(A)| = \left|\sum_{i=1}^{n} a_{ii}\right| \leq \sum_{i=1}^{n} |a_{ii}| \leq \sum_{i,j}^{n} |a_{ij}| = \|A\|_1.$$ We deduce the inequality $|\operatorname{tr}(A)| \leq \|A\|_1$.

This result cannot be improved because there is equality for example if $A = I_n$.

So $\|\operatorname{tr}\| = 1$.

- With the norm $A \to \|A\|_2$: We apply Cauchy-Schwarz:

$$\begin{aligned}
\operatorname{tr}(A)^2 &= \left|\sum_{i=1}^{n} a_{ii}\right|^2 \\
&= \left|\sum_{i=1}^{n} (1 \cdot a_{ii})\right|^2 \leq \sum_{i=1}^{n} 1^2 \sum_{i=1}^{n} a_{ii}^2
\end{aligned}$$

We deduce

$$\begin{aligned}
* \operatorname{tr}(A)^2 &\leq n \sum_{i=1}^{n} a_{ii}^2 \\
&\leq n \sum_{i,j=1}^{n} a_{ij}^2.
\end{aligned}$$

In other words $|\operatorname{tr}(A)| \leq \sqrt{n}\|A\|_2$.

This inequality cannot be improved because it is an equality for example if $A = I_n$. We deduce $\|\operatorname{tr}\| = \sqrt{n}$.

- With the norm $A \to \|A\|_\infty$:

$$\begin{aligned} *|\operatorname{tr}(A)| &= \left|\sum_{i=1}^{n} a_{ii}\right| \\ &\leq \sum_{i=1}^{n} |a_{ii}| \\ &\leq n \sup\{|a_{ii}|, i = 1 \cdots n\} \\ &\leq n \sup\{|a_{ij}|, i, j = 1 \cdots n\} \end{aligned} \quad *$$

We deduce the inequality $|\operatorname{tr}(A)| \leq n\|A\|_\infty$.

This inequality cannot be improved because it is an equality for example if $A = I_n$. We deduce $\|\operatorname{tr}\| = n$.

Exercise 2 Let be E the vectorial space of $\mathbb{C}$ valued continuous functions on $[-1, 1]$, endowed with the norm sup : $\|f\|_\infty = \operatorname{Sup} |f(t)|$

$$t \in [-1, 1]$$

Let F be the vectorial space of 2π-périodique continuous functions on $\mathbb{R}$, endowed with the norm N_2 so that $N_2(f) = \frac{1}{2\pi}\sqrt{\int_{-\pi}^{\pi} |f(t)|^2 dt}$, or the norm sup $N_\infty : N_\infty(f) = \operatorname{Sup}_{t\in\mathbb{R}} f(t)$.

Let be $L : E \to F$ the mapping defined by $L(f)(t) = f(\cos t)$.

1- Show that L is well defined, is linear and injective.

2- Show that L is continuous for both of the norms N_2 and N_∞ of F, and calculate for both of them, $\|L\|_2$ and $\|L\|_\infty$.

Solution $E = \{f / f : [-1, 1] \longrightarrow \mathbb{C}\}$ and f continuous.

Assum $f \in E$. $L(f)$ is continuous on $\mathbb{R}$ because:

-f is continuous and $t \longrightarrow cost$ continuous, therefore the composition of the two applications is continuous. $L(f) \in E$.

-Moreover $t \longrightarrow cost$ is $2\pi-$periodic, so $L(f)(t)$ is $2\pi-$ periodic.

We have $L(f) \in F$ therfore L is well defined. Let's show that L is linear;

$$\forall f, g \in E, \forall \alpha, \beta \in \mathbb{K}:$$

$$L(\alpha f + \beta g) = (\alpha f + \beta g)(cost) = \alpha f(cost) + \beta g(cost)$$

$$= \alpha L(f) + \beta L(g)$$

- Now, let's show that $KerL = \{0\}$. We have:

$$L(f) = 0 \quad \Rightarrow f(\text{ cos t}) = 0 \qquad \forall t \in \mathbb{R}$$

$$\Rightarrow f(\text{ cost }) = 0 \qquad \forall t \in [0, \pi]$$

Since the function $t \longrightarrow cost$ is bijective from $[0, \pi]$ to $[-1, 1]$ then

$$f(\cos t) = 0 \quad \forall t \in [0, \pi] \quad \Leftrightarrow \quad f(u) = 0 \quad \forall u \in [-1, 1]$$

$$\Rightarrow f = 0$$

then L is injective.

2)We have:

$$L : (E, \|\|_\infty) \longrightarrow (F, N_2)$$

$$f \mapsto L(f)$$

where, $N_2(f) = \frac{1}{2\pi}\sqrt{\int_{-\pi}^{\pi} |f(t)|^2 \, dt}$

$$\begin{aligned}
N_2(L(f))^2 &= \frac{1}{4\pi^2}\int_{-\pi}^{\pi} |L(f)(t)|^2 dt \\
&= \frac{1}{4\pi^2}\int_{-\pi}^{\pi} |f(cost)|^2 dt \\
&\leq \frac{1}{4\pi^2}\int_{-\pi}^{\pi} \|f\|_\infty^2 dt \\
&= \frac{1}{2\pi}\|f\|_\infty^2
\end{aligned}$$

So $N_2(L(f)) \leq \frac{1}{\sqrt{2\pi}}\|\|_\infty$. Hence L is contiuous for the norm N_2 in F, and we have:

$$\|L\|_2 \leq \frac{1}{\sqrt{2\pi}} \cdots (1).$$

-Now, assum that $f = 1$ on $[-1, 1]$. then f is continuous, $f \in E$ and $\|f\|_\infty = 1$.

$$N_2(L(f))^2 = \frac{1}{4\pi^2} \int_{-\pi}^{\pi} 1dt = \frac{1}{2\pi}$$

so, $N_2(L(f)) = \frac{1}{\sqrt{2\pi}}$

$$N_2(L(f)) \leq \|L\|_2 \|f\|_\infty$$

$$\frac{1}{\sqrt{2\pi}} \leq \|L\|_2 \cdot 1 \quad \cdots (2)$$

from (1) and (2), $\|L\|_2 = \frac{1}{\sqrt{2\pi}}$

- For the norme $\|.\|_\infty$ we have,

$$N_\infty(L(f)) = \sup_{t\in\mathbb{R}} |f(cost)|$$

$$= \sup_{t\in[0,\pi]} |f(cost)|$$

$$= \sup_{u\in[-1,1]} |f(u)|$$

$$= \|f\|_\infty$$

So L is continuous for the norm N_∞ et $\|L\|_\infty = 1$.

Exercise 3 Consider in $L^2(\mathbb{R})$ the operator Q defined by

$$Qf(x) = xf(x),$$

with

$$D(Q) = \left\{f \in L^2(\mathbb{R}) \mid Qf \in L^2(\mathbb{R})\right\}$$

1- Show that Q is linear but not bounded, *i,e;* not continuous.

2- Show that $D(Q)$ is dense in $L^2(\mathbb{R})$.

In quantum mechanics Q is called the position operator.

Solution Let $\lambda \in \mathbb{C}$ and $f \in D(Q)$. Then trivially, $\lambda f \in D(Q)$. If $f, g \in D(Q)$, then it follows from the inequality

$$|f(x) + g(x)|^2 \leq 2\left\{2|f(x)|^2 + |g(x)|^2\right\}$$

that

$$\int_{-\infty}^{+\infty} x^2 |f(x) + g(x)|^2 dx \leq 2\int_{-\infty}^{+\infty} x^2 |f(x)|^2 dx + 2\int_{-\infty}^{+\infty} x^2 |g(x)|^2 dx < +\infty$$

and we conclude that $f + g \in D(Q)$, thus $D(Q)$ is a subspace of $L^2(\mathbb{R})$.

If $f, g \in L^2(\mathbb{R})$ and $\lambda \in \mathbb{C}$, then,

$$Q(f + \lambda g) = x \cdot \{f(x) + \lambda g(x)\} = x \cdot f(x) + \lambda \cdot x \cdot g(x) = Qf(x) + \lambda Qg(x)$$

thus

$$Q(f + \lambda g) = Qf + \lambda Qg$$

and we have proved that Q is linear.

Let $f_n = 1_{[0,n]}$. Then,

$$\|f_n\|_2^2 = \int_0^n 1 dx = n, \quad \text{thus} \quad \|f_n\|_2 = \sqrt{n}$$

and

$$\|Qf_n\|_2^2 = \int_0^n x^2 dx = \frac{n^3}{3}, \quad \text{thus} \quad \|Qf_n\|_2 = \frac{n}{\sqrt{3}} \cdot \sqrt{n} = \frac{n}{\sqrt{3}} \cdot \|f_n\|_2$$

We conclude that,

- $f_n \in D(Q)$ for every $n \in \mathbb{N}$, and - $\|Q\| \geq \frac{n}{\sqrt{3}}$ for every $n \in \mathbb{N}$, hence Q is unbounded, *i,e;* not continuous. Finally, let $f \in L^2(\mathbb{R})$, *i.e.* $\int_{-\infty}^{+\infty} |f(x)|^2 dx < +\infty$.

Then to every $\varepsilon > 0$ there is an $N \in \mathbb{N}$, such that

$$\int_{-\infty}^{-N} |f(x)|^2 dx + \int_{N}^{+\infty} |f(x)|^2 dx < \varepsilon^2$$

If we put

$$f_N(x) = \begin{cases} f(x) & \text{for } |x| \leq N \\ 0 & \text{for } |x| > N \end{cases}$$

then it follows that $\|f - f_n\|_2 < \varepsilon$.

It only remains to prove that $f_N \in D(Q)$. This follows from

$$\|Qf_N\|_2^2 = \int_{-N}^{N} x^2 |f(x)|^2 dx \leq N^2 \int_{-N}^{N} |f(x)|^2 dx \leq N^2 \int_{-\infty}^{+\infty} |f(x)|^2 dx = N^2 \|f\|_2^2$$

hence,

$$\|Qf_N\|_2 \leq N\|f\|_2 < +\infty$$

and the last claim is proved.

Exercise 4 Consider in $L^2(\mathrm{R})$ the operator P defined by

$$Pf = -i\frac{df}{dx}$$

with

$$D(P) = \left\{ f \in L^2(\mathbb{R}) \mid Pf \in L^2(\mathbb{R}) \right\}$$

1. Show that P is linear but not bounded.

2. Show that $D(P)$ is dense in $L^2(\mathbb{R})$.

In quantum mechanics P is called the momentum operator.

Solution

1. Let $f \in D(P)$ and $\lambda \in C$. Then clearly, $\lambda f \in D(P)$. If $f, g \in D(P)$, then f and g are differentiable almost everywhere, hence $f + g$ is also differentiable almost everywhere. From $f'', g' \in L^2(\mathbb{R})$, follows that also $f' + g' \in L^2(\mathbb{R})$, so $D(P)$ is a vector space. Then

clearly, $Pf(x) = -i\frac{df}{dx}(x)$ is linear. Then we shall show that P is not bounded. Let

$$f_n(x) = \begin{cases} \sin nx & \text{for } x \in [0, 2\pi] \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\|f_n\|_2^2 = \int_0^{2\pi} \sin^2 nx dx = \int_0^{2\pi} \cos^2 nx dx = \frac{1}{2}\int_0^{2\pi} \left\{\sin^2 nx + \cos^2 nx\right\} dx = \pi$$

thus $\|f_n\|_2 = \sqrt{\pi}$ for every $n \in \mathbb{N}$. It follows that

$$f_n'(x) = \begin{cases} n \cdot \cos nx & \text{for } x \in]0, 2\pi[\\ \text{not definod} & \text{for } x \in \{0, 2\pi\} \\ 0 & \text{otherwise.} \end{cases}$$

where $\{0, 2\pi\}$ clearly is a null-set. Then

$$\|Pf_n\|_2^2 = \|-if_n'(x)\|_2^2 = \int_0^{2\pi} n^2 \cdot \cos^2 nx dx = \cdots = n^2\pi = n^2 \|f_n\|_2^2$$

so $\|Pf_n\|_2 = n\|f_n\|_2$, and it follows that P is not bounded.

2. Finally, we shall show that $D(P)$ is dense in $L^2(\mathbb{R})$. Chocee any $f \in L^2(\mathbb{R})$, thus $\int_{-\infty}^{+\infty} |f(x)|^2 dx < +\infty$. There exists to every $\varepsilon > 0$ an $N \in$ N, such that

$$\int_{-\infty}^{-N} |f(x)|^2 dx + \int_N^{+\infty} |f(x)|^2 dx < \left(\frac{\varepsilon}{3}\right)^2$$

If we therefore put

$$f_N(x) = \begin{cases} f(x) & \text{for } x \leq N, \\ 0 & \text{otherwise,} \end{cases}$$

then $f \in L^2(\mathbb{R})$ and $\|f - f_N\|_2 < \frac{\varepsilon}{3}$. Furthermore, there exists a continuous function g on R, such that $g(x) = 0$ for $|x| \geq N$, and such that $\|f_N - g\|_2 < \frac{\varepsilon}{3}$. It follows from Weierstrab's Approximation Theorem that there exists a polynomial $P(x)$ with $P(-N) = P(N) = 0$, such that

$$\max_{v\in[-N,N]} |g(x) - P(x)| < \frac{\varepsilon}{3}, \frac{1}{\sqrt{2N}}$$

If we put

$$h(x) = \begin{cases} P(x) & \text{for } |x| \leq N \\ 0 & \text{otherwise.} \end{cases}$$

then

$$\|g - h\|_2^2 = \int_{-N}^{N} |g(x) - P(x)|^2 dx < \left(\frac{\varepsilon}{3}\right)^2 \cdot \frac{1}{2N} \int_{-N}^{N} dx = \left(\frac{\varepsilon}{3}\right)^2,$$

and we infer that

$$\|f - h\|_2 \leq \|f - f_N\|_2 + \|f_N - g\|_2 + \|g - h\|_2 < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

The function h is differentiable, if only $x \neq \pm N$, and since h is continuous at $\pm N$, we conclude that

$$h'(x) = \begin{cases} P'(x) & \text{for } |x| < N, \\ \text{not defined} & \text{for } x = \pm N \\ 0 & \text{otherwise} \end{cases}$$

which of course belongs to $L^2(\mathrm{R})$, because $\{-N, N\}$ is a null-set- This proves that $h \in D(P)$, and $D(P)$ is therefore dense in $L^2(\mathbb{R})$.

Exercise 5 (Continuity, null space).

Let T be a bounded linear operator. Show that:

1. $x_n \longrightarrow x$ [implies $Tx_n \longrightarrow Tx$.

2. The null space $\mathcal{N}(T)$ is closed.

Solution

1. We have, as $n \longrightarrow \infty$,

$$\|Tx_n - Tx\| = \|T(x_n - x)\| \leqq \|T\| \, \|x_n - x\| \longrightarrow 0$$

2. For every $x \in \overline{\mathcal{N}(T)}$ there is a sequence (x_n) in $\mathcal{N}(T)$ such that $x_n \longrightarrow x$.

   Hence $Tx_n \longrightarrow Tx$ by part (a) of this exercise.

   Also $Tx = 0$ since $Tx_n = 0$, so that $x \in \mathcal{N}(T)$.

   Since $x \in \overline{\mathcal{N}(T)}$ was arbitrary, $\mathcal{N}(T)$ is closed.

Non solved exercises

Exercise 1 For any $n \in \mathbb{N}$, let: $U_n : C[0,1] \rightarrow C[0,1], (U_n f)(x) = f\left(x^{1+1/n}\right)$.

1. Prove that $U_n f \rightarrow f$ in $C[0,1] \forall f \in C[0,1]$.

2. Prove that $\|U_n - I\| = 2$ for all $n \in \mathbb{N}$, and therefore U_n does not converge towards I in the norm topology, but U_n converges pointwise towards I.

Exercise 2

1. Let $\varphi(x,t) : [0,1] \times [0,1] \rightarrow [0,\infty)$ be a continuous function such that $\partial\varphi/\partial x : [0,1] \times [0,1] \rightarrow [0,\infty)$ exists and is continuous. Prove that the operator $U : C[0,1] \rightarrow C[0,1], (Uf)(x) = \int_0^1 \varphi(x,t) f(t) dt$ is linear and continuous, with $\|U\| = \int_0^1 \varphi(1,t) dt$.

2. Using (i) prove that the operator $U : C[0,1] \rightarrow C[0,1], (Uf)(x) = \int_0^1 e^{xt} f(t) dt$ is linear and continuous, and $\|U\| = e - 1$.

Exercise 3 Let (X,d) be a metric space. We denote by $\operatorname{Lip}(X)$ the set of Lipschitz functions, with real values, on X. For $f \in \operatorname{Lip}(X)$, we note $\operatorname{Lip}(f)$ the Lipschitz constant of f, namely $\operatorname{Lip}(f) = \sup\left\{\frac{|f(x)-f(y)|}{d(x,y)}; x, y \in X, x \neq y\right\}$.

1. Show that $\operatorname{Lip}(X)$ is a vector space.

2. Let $a \in X$. For $f \in \mathrm{Lip}(X)$, we set $\|f\|_{\mathrm{Lip},\ a} = |f(a)| + \mathrm{Lip}(f)$. a- Show that $\|\|_{\mathrm{Lip,a}}$ is a norm on $\mathrm{Lip}(X)$. b- Show that if b is another point of X, then the norm $\|\|_{\mathrm{Lip,b}}$ is equivalent to the norm $\|\ |_{\mathrm{Lip,a}}$.

   c- We assume that X is a norm vector space (and that d is the distance associated with the norm of X). Show that every continuous linear form on X is in $\mathrm{Lip}(X)$, and that we have $\|\varphi\|_{Lip,0} = \|\varphi\|_{X}\cdot$ for all $\varphi \in X^*$.

3. We return to the case of any metric space (X, d). Show that $\mathrm{Lip}(X)$ is complete for any norm $\|f\|_{\mathrm{Lip,a}}$.

Exercise 4 Let E, F be two evn of finite dimensions and $\varphi : E \to F$ linear. Show that φ is continuous.

Exercise 5 Show that The dual space X' of a normed space X is a Banach space (whether or not X is).

CHAPTER 2

HILBERT SPACES

Exercises series 4

2.1 Inner product

Exercise1 Prove that in a real vector space with inner product we have :

$$\langle x/y \rangle = 1/4(\|x+y\|^2 - \|x-y\|^2)$$

and in a complex vector space with inner product we have

$$\langle x/y \rangle = 1/4(\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2)$$

These are the so-called polar identities. They tell us that in a Hilbert space, the inner product is determined by the norm.

Solution Let V be a real vector space with an inner product. It follows straightforward that

$$\begin{aligned}
&\frac{1}{4}\left\{\|x+y\|^2-\|x-y\|^2\right\}=\frac{1}{4}\{\langle x+y,x+y\rangle-\langle x-y,x-y\rangle\}\\
&\quad=\frac{1}{4}\{\langle x,x\rangle+\langle y,x\rangle+\langle x,y\rangle+\langle y,y\rangle-\langle x,x\rangle+\langle y,x\rangle+\langle x,y\rangle-\langle y,y\rangle\}\\
&\quad=\frac{1}{4}\{2\langle y,x\rangle+2\langle x,y\rangle\}=\frac{1}{2}\{\langle x,y\rangle+\langle y,x\rangle\}=\langle x,y\rangle
\end{aligned}$$

and we have proved the claim concerning real vector spaces. Let V be a complex vector space with an inner product. Then we get analogously,

$$\begin{aligned}
&\frac{1}{4}\{\ \|x+y\|^2-\|x-y\|^2+i\|x+iy\|^2-i\|x-iy\|^2\Big\}\\
&=\frac{1}{4}\{\langle x+y,x+y\rangle-\langle x-y,x-y\rangle+i\langle x+iy,x+iy\rangle-i\langle x-iy,x-iy\rangle\}\\
&=\frac{1}{4}\{[\langle x,x\rangle+\langle x,y\rangle+\langle y,x\rangle+\langle y,y\rangle]-[\langle x,x\rangle-\langle x,y\rangle-\langle y,x\rangle+\langle y,y\rangle]\\
&\qquad+i[\langle x,x\rangle+\langle x,iy\rangle+\langle iy,x\rangle+\langle iy,iy\rangle]-i[\langle x,x\rangle-\langle x,iy\rangle-\langle iy,x\rangle+\langle iy,iy\rangle]\}\\
&=\frac{1}{4}\{2\langle x,y\rangle+2\langle y,x\rangle+2i\langle x,iy\rangle+2i\langle iy,x\rangle\}\\
&=\frac{1}{2}\{\langle x,y\rangle+\langle y,x\rangle+i\cdot(-i)\langle x,y\rangle+i\cdot i\langle y,x\rangle\}\\
&=\frac{1}{2}\{\langle x,y\rangle+\langle y,x\rangle+\langle x,y\rangle-\langle y,x\rangle\}\\
&=\frac{1}{2}\cdot 2\langle x,y\rangle=\langle x,y\rangle
\end{aligned}$$

and the claim is proved in the complex case.

Exercise 2 $E=\mathcal{M}_{(m,n)}(\mathbb{R})$ is the real vector space of matrices with m rows and n columns. For $a\in E, b\in E$, we put

$$\langle a,b\rangle=tr(a^t.b).$$

- Show that we define an inner product on E.

Solution $E=\mathcal{M}_{m,n}(\mathbb{R})$

$a,b\in E$, $\langle a,b\rangle=tr\left(a^t,b\right)$

At first , let's see if $tr\left(a^t,b\right)$ exists.

We have: $a_{m\times n}\rightarrow a^t_{n\times m}$

So, $a^t_{n\times m}b_{m\times n} = A_{nxn}$, then the trace of this product has a meaning, therfore the inner product is well defined.

1. $\forall a, b, c \in M_{m\times n}(\mathbb{R}), \forall \lambda, \mu \in \mathbb{C}$:

$$\begin{aligned}\langle \lambda a + \mu b, c\rangle &= tr(\lambda a + \mu b)^t \cdot c = tr\left((\lambda a)^t + (\mu b)^t \cdot c\right)\\ &= tr\left((\lambda a)^t \cdot c\right) + tr(\mu b)^t \cdot c\Big)\\ &= tr\left(\lambda a^t \cdot c\right) + tr\left(\mu b^t \cdot c\right)\\ &= \lambda tr\left(a^t \cdot c\right) + \mu tr\left(b^t \cdot c\right)\\ &= \lambda\langle a \cdot c\rangle + \mu\langle b \cdot c\rangle\end{aligned}$$

2. Since $tr\left(M^t\right) = tr(M)$, we have,

$$\begin{aligned}\langle a \cdot b\rangle = tr\left(a^t \cdot b\right) &= tr\left(b^t \cdot a\right)^t\\ &= tr\left(b^t \cdot a\right)\\ &= \langle b \cdot a\rangle\end{aligned}$$

3. $\langle a, a\rangle = tr\left(a^t \cdot a\right) = \|a^2_{2,2}\|$.

   So, $\langle a, a\rangle \geqslant 0$

4. If $\langle a, a\rangle = 0$ then $\|a^2_{2,2}\| = 0$, so $a = 0$

   We conclude that $tr\left(a^t \cdot b\right)$ is an inner product on E.

Exercise 3 Show that the sup-norm on $\mathcal{C}[a, b]$ the vector space of all $\mathbb{C}-$valued continuous functions on $[a, b]$, is not induced by an inner product.

-Recall,

$$\|f\| = max_{t\in[a,b]}|f(t)|.$$

Solution We know already that if a norm is defined by an inner product, then we the law of parallelograms holds,

$$\|f+g\|^2 + \|f-g\|^2 = 2\left\{\|f\|^2 + \|g\|^2\right\}.$$

Hence, it suffices to prove that the law of parallelograms does not hold for

$$\|f\| = \sup\{|f(t)| \mid t \in [a,b]\} \quad \text{in } C([a,b]).$$

We may assume that $[a,b] = [0,1]$. Choose $f(t) = 1$ and $g(t) = t$ for $t \in [0,1]$. Then $\|f\| = 1$ and $\|g\| = 1$, and

$$\|f+g\| = \sup_{t\in[0,1]} |1+t| = 2, \quad \|f-g\| = \sup_{t\in[0,1]} |1-t| = 1.$$

Hence,

$$\|f+g\|^2 + \|f-g\|^2 = 4 + 1 = 5$$

and,

$$2\left(\|f\|^2 + \|g\|^2\right) = 2(1+1) = 4.$$

It follows from,

$$\|f+g\|^2 + \|f-g\|^2 = 5 \neq 4 = 2\left(\|f\|^2 + \|g\|^2\right),$$

that the law of parallelograms is not satisfied, so the sup-norm is not defined by an inner product.

Exercise 4 Let H be an inner product space. Describe all pairs of vectors x, y for which

$$\|x+y\| = \|x\| + \|y\|$$

Solution We have,

$$\|x+y\|^2 = \|x\|^2 + 2\|x\| \cdot \|y\| + \|y\|^2.$$

Since

$$\|x+y\|^2 = \langle x+y, x+y\rangle = \|x\|^2 + 2\,\mathrm{Re}\langle x,y\rangle + \|y\|^2,$$

we obtain $\mathrm{Re}\langle x,y\rangle = \|x\|\cdot\|y\|$.

By the Cauchy-Schwarz inequality we now get,

$$\|x\|\cdot\|y\| = \mathrm{Re}\langle x,y\rangle \leq |\langle x,y\rangle| \leq \|x\|\cdot\|y\|$$

i.e.,

$$\mathrm{Re}\langle x,y\rangle = |\langle x,y\rangle| = \|x\|\cdot\|y\|.$$

The equality $|\langle x,y\rangle| = \|x\|\cdot\|y\|$ means that $y=0$ or that $x=\lambda y$ for some $\lambda\in\mathbb{C}$.

- In the case $x=\lambda y$ we obtain $\mathrm{Re}\langle \lambda y, y\rangle = \|\lambda y\|\cdot\|y\|$, i.e., $\mathrm{Re}\,\lambda = |\lambda|$.

Hence $x=\lambda y$ with $\lambda\geq 0$. Therfore,

$$y=0 \quad \textit{or} \quad x=\lambda y \quad \textit{with} \quad \lambda\geq 0$$

Non solved exercises

Exercise 1 On $\mathbb{R}_3[X]$ we consider the following bilinear forms. Say which ones are inner products.

$$\phi(P,Q) = \int_{-1}^{1} P(t)Q(t)dt$$
$$\phi(P,Q) = \int_{-1}^{1} P'(t)Q(t) + P'(t)Q(t)dt$$
$$\phi(P,Q) = \int_{-1}^{1} P'(t)Q'(t)dt + P(0)Q(0)$$

Exercise 2 Let $E = \{f:\mathbb{R}\to\mathbb{R}$ continue 2π-periodic $\}$.

- Show that $\langle f \mid g\rangle = \int_0^{2\pi} f(t)g(t)dt$ is an inner product on E.

Exercise 3 Let E be an Euclidean space of dimension n and $x_1,\ldots,x_p$ be vectors of E such that if $i\neq j$ then $\langle x_i \mid x_j\rangle$ <0.

- Show by induction on n that $p\leqslant n+1$.

Exercises series 5

2.2 Hilbert spaces, Riesz Theorem

Exercise 1 Let $[a, b]$ be a finite interval. Show that $L^2([a, b]) \subset L^1([a, b])$.

Solution The interval $[a, b]$ is bounded, so the constant $1 \in L^2(|a, b|)$. In fact,

$$\|1\|_2^2 = \int_0^b 1^2 dt = b - a < +\infty$$

Let $f \in L^2([a, b])$. Then we get by the Cauchy-Schwarz inequality

$$\int_a^b |f(t)| dt = \int_a^b |f(t)| \cdot 1 dt \leq \|f\|_2 \cdot \|1\|_2 = \sqrt{b-a} \cdot \|f\|_2 < +\infty$$

proving that $f \in L^1([a, b])$, and thus

$$L^2([a, b]) \leqq L^1([a, b]) \quad \text{with} \quad \|f\|_1 \leq \sqrt{b-a} \cdot \|f\|_2$$

Remark 1.5 We can find $f \in L^1([a, b])$, which does not lie in $L^2([a, b])$. An example is

$$f(x) = \begin{cases} \frac{1}{\sqrt{x}} & \text{for } x \in]0, 1] \\ 0 & \text{for } x = 0 \end{cases}$$

In fact,

$$\left\| \frac{1}{\sqrt{x}\|_1} \right\|_0^1 \frac{1}{\sqrt{x}} dx = [2\sqrt{x}]_0^1 = 2$$

hence $f \in L^1([0, 1])$. On the other hand,

$$\int_0^1 \left\{ \frac{1}{\sqrt{x}} \right\}^2 dx = \int_0^1 \frac{1}{x} dx = \infty$$

hence $f \notin L^2([0,1])$.

Exercise 2 We consider the space of sequences ℓ^p, where $p \geq 1$. Let $y \in \ell^q$, where $\frac{1}{p} + \frac{1}{q} = 1$. (If $p = 1$ then $y \in \ell^\infty$, the space of bounded sequences).

Show that

$$x \mapsto \sum_{i=1}^{\infty} x_i \bar{y}_i$$

defines an element $y^\star \in (\ell^p)^*$ with norm $\left\|y^{\star|}\right\|^\star = \|y\|_q$.

Solution If $y \in \ell^q$, where $\frac{1}{p} + \frac{1}{q} = 1$, then by Hölder's inequality

$$\left|\sum_{i=1}^{+\infty} x_i \bar{y}_i\right| \leq \|x\|_p \cdot \|y\|_q$$

for every $x \in \ell^p$, proving that the linear mapping

$$y^\star(x) = \sum_{i=1}^{+\infty} x_i \bar{y}_i$$

is bounded, $y^\star \in (y^\star)$, and that $\|y^\star\|^\star \leq \|y\|_{\boldsymbol{q}}$. Then choosing $x \in \ell^p$ by $x_i = \operatorname{sign} y_i \cdot |y_i|^{\frac{2}{p}}$, we get

$$y^\star(x) = \sum_{i=1}^{+\infty} |y_i|^{1+\frac{q}{p}} = \sum_{i=1}^{+\infty} |y_i|^{\left(\frac{1}{q}+\frac{1}{p}\right)q} = \sum_{i=1}^{+\infty} |y_i|^q = \|y\|_q^q = \|y\|_q^{q\left(\frac{1}{p}+\frac{1}{q}\right)} = \|y\|_q \cdot \|y\|_q^{\frac{q}{p}}$$

Notice that

$$\|x\|_p = \left\{\sum_{i=1}^{+\infty} |x_i|^p\right\}^{\frac{1}{p}} = \left\{\sum_{i=1}^{+\infty} |y_i|^q\right\}^{\frac{1}{p}} = \left\{\|y\|_q^q\right\}^{\frac{1}{p}} = \|y\|_q^{\frac{q}{p}},$$

from which follows that

$$y^\star(x) = |y^\star(x)| = \|y\|_q \cdot \|x\|_p$$

and we conclude that $\|y^\star\|^\star \geq \|y\|_q$. When this is combined with the previous estimate, then $\|y^\star\|^\star = \|y\|_q$, as required.

Exercise 3 (Riesz Theorem) Let E be the inner product space of complex sequences $(u_n)_{n\in\mathbb{N}}$ satisfying :

$\exists\ N \in \mathbb{N}, \forall n \geq N, \quad u_n = 0$

with the inner product $\langle u/v\rangle = \sum_{n=0}^{+\infty} u_n \overline{v_n}$.

1 - Show that the mapping $\varphi(u) : E \mapsto C$ defined by $\varphi(u) = \sum_{n=1}^{+\infty} \frac{u_n}{n}$ is a linear continuous map on E.

2 - Is there an element $a \in E$ such that for all u in E, we have $\varphi(u) = \langle u/a\rangle$?

3 - What can we deduce about E ?

Solution

1. Simply apply the Cauchy-Schwarz inequality. Indeed:

$$\left|\sum_{n=1}^{+\infty} \frac{u_n}{n}\right| \leq \left(\sum_{n=1}^{+\infty} |u_n|^2\right)^{1/2} \left(\sum_{n=1}^{+\infty} \frac{1}{n^2}\right)^{1/2}.$$

   This implies that ϕ is continuous (it is clearly linear) and that

$$\|\phi\| \leq \left(\sum_{n=1}^{+\infty} \frac{1}{n^2}\right)^{1/2}$$

   .

2. Suppose there exists such an element $a = (a_n)$. Applying ϕ to the k-th element of the canonical basis of ℓ^2, $\quad e_k = (0, \ldots, 0, 1, 0, \ldots)$, we obtain $a_k = \frac{1}{k}$ for $k \geq 1$.

   This sequence is not in E.

3. E is not complete, because the answer to the previous question goes against Riesz's representation theorem in Hilbert spaces.

Non solved exercises

Exercise 1 We denote by $l_0^2 = \{(x_n)_{n\in\mathbb{N}} \subseteq \mathbb{C} \mid x_n \neq 0$ only for a finite number of $n\}$, and define $\langle\cdot\cdot\rangle : l_0^2 \times l_0^2 \to \mathbb{C}$ by $\left\langle (x_n)_{n\in\ \mathrm{V}}, (y_n)_{n\in\mathrm{H}} \right\rangle = \sum_{n=1}^{\infty} x_n \overline{y_n}$.

Prove that $(l_0^2, \langle\cdot\cdot\cdot\rangle)$ is an inner product space but not a Hilbert space.

Exercise 2 Let H be a Hilbert space, $(e_n)_{n\in\ \mathrm{V}}$ is an orthonormal basis, and $x^* : H \to \mathbb{K}$ is a linear and continuous functional.

Prove that $y = \sum_{n=1}^{\infty} \overline{x^*\,(\epsilon_n)}\epsilon_n$ is the unique element in H with the property that $x^*(x) = \langle x, y\rangle \forall x \in H$.

Exercise series 6

2.3 Projection, Orthogonality

Exercise 1

1. Let H be a Hilbert space, and let B be the closed unit ball of H.

   (a) Show that $\forall x \in H\ B, \forall z \in B$: $\left\langle z - \frac{x}{\|x\|}, x - \frac{x}{\|x\|} \right\rangle = (\|x\| - 1)\left(\left\langle z, \frac{x}{\|x\|} \right\rangle - 1\right)$

   (b) Show that $\left\langle z, \frac{x}{\|x\|} \right\rangle - 1 \leq 0$

   (c) Deduce that $Re\left\langle z - \frac{x}{\|x\|}, x - \frac{x}{\|x\|} \right\rangle \leq 0.$

   2.Can we define a projection from H to B? If yes, then give an expression of the projection onto B, the closed unit ball.

2. We consider in the space of periodic functions $L_2[-\pi, \pi]$, the subspace $F = vect\left\{e^{-int}, ..., e^{int}\right\}.$

   (a) Find the projection of f on F

   (b) Deduce the distance from f to the subspace F.

Solution (Projection on a ball)

We denote by B the closed unit ball of H.

1. a) We have Since $x \notin B$, we have $\|x\| - 1 \geq 0$:

$$\left\langle z - \frac{x}{\|x\|}, x - \frac{x}{\|x\|} \right\rangle.$$

$$\begin{aligned}\left\langle z - \frac{x}{\|x\|}, x - \frac{x}{\|x\|} \right\rangle &= \langle z, x\rangle - \|x\| - \left\langle z, \frac{x}{\|x\|} \right\rangle + 1 \\ &= (\|x\| - 1)\left(\left\langle z, \frac{x}{\|x\|} \right\rangle - 1\right)\end{aligned}$$

b) Since $z \in B$ we have $\|z\| \leq 1$ so $\left|\left\langle z, \frac{x}{\|x\|} \right\rangle\right| \leq 1$, by the Cauchy Schwartz inequality.

c- Since $\left\langle z-\frac{x}{\|x\|}, x-\frac{x}{\|x\|}\right\rangle \leq 0$, then $Re\left\langle z-\frac{x}{\|x\|}, x-\frac{x}{\|x\|}\right\rangle \leq$ ()it's a real number)

2-We have proved that $\forall z \in B$, we have:

$\left\langle z-\frac{x}{\|x\|}, x-\frac{x}{\|x\|}\right\rangle \leq 0$

We deduce that If $x \notin B$, then, $P(x) = x/\|x\|$

2. It is easy to check that the vectors $e^{int}/2\pi$ are orthonormal. Thus

$$\operatorname{Proj} f=\sum_{m=-n}^{n}\left\langle f, e^{int}/2\pi\right\rangle e^{imt}/2\pi=\frac{1}{2\pi}\sum_{m=-n}^{n}\hat{f}(m)e^{imt},$$

where $\hat{f}(m)$ is the m-th Fourier coefficient. The distance is

$$\operatorname{dist}\left(f, \operatorname{span}\left(e^{int}\right)_{-n}^{n}\right)=\|f-\operatorname{Proj} f\|.$$

Exercise 2 Let be $H=\ell^2(\mathbb{N},\mathbb{R})$ (the real Hilbert space). We denote

$$C=\{x=(x_n)\in H; \forall n\in\mathbb{N}, x_n\geq 0\}.$$

1 – Prove that C is a closed convex set.

2 – Determine the projection on this convex C.

3 – Resume the previous question with $H=\ell^2(\mathbb{N},\mathbb{C})$

Solution

1. Simply apply the definition to show that C is convex.

   C is convex if:

$$\forall x, y\in C, \forall t\in[0,1], \quad x+(1-t)y\in c.$$

   Assum $x, y\in C$ then we have: $\forall n\in\mathbb{N}: x_n \geqslant 0$ and $y_n \geqslant 0$

   So, $tx_n+(1-t)y_n \geqslant 0$. Therfore C is convex.

   On the other hand, C is closed: if (x^p) is a sequence of C that converges to $x\in H$, and if $x_n^p \geq 0$, we clearly have by passing to the limit $x_n \geq 0$, and therefore $x \in C$.

2. Assum $x \in \ell_2$. Guess the formula for $P_C(x)$. The only way out is to make a $-$dimensional drawing and try to guess what the formula for $P_C(x)$ is. In dimension 2, C simply corresponds to the top left-hand quarter. There are 4 different cases to determine the projection of x, depending on its position in either half-plane. This is how we are led to ask $P_C(x) = (y_n)$, where $y_n = x_n$ if $x_n \geq 0$, and $y_n = 0$ otherwise. To prove that this is the projection of x on C, it is sufficient to verify that for all z of C, we have:

$$< x - y, z - y > \leq 0.$$

But,

$$< x - y, z - y > = \sum_{n \geq 0} (x_n - y_n)(z_n - y_n).$$

Two cases are possible:

- Let us say $x_n \geq 0$, and in this case $x_n - y_n = 0$.
- Lets say $x_n \leq 0$, but then $y_n = 0$, and therefore $(x_n - y_n)(z_n - y_n) \leq 0$. In all cases, we have $(x_n - y_n)(z_n - y_n) \leq 0$, which proves that

$$< x - y, z - y > \leq 0$$

.

3. Slightly adapt for the complex case. It is necessary and sufficient this time that $y = P_C(x)$ checks for all z of $\mathbb{C}$

$$Re(< x - y, z - y >) = Re\left(\sum_{n \geq 0} (x_n - y_n)\overline{(z_n - y_n)}\right) \leq 0.$$

In all cases, $z_n - y_n$ is real, and therefore

$$Re(< x - y, z - y >) = \sum_{n \geq 0} (z_n - y_n)(Re(x_n) - y_n).$$

One is then led to ask $y_n = Re\,(x_n)$ if $Re\,(x_n) \geq 0$ and $y_n = 0$ otherwise. We get a negative quantity.

Exercise 3 For all $N \in \mathbb{N}$, note by M_N the vector subspace of $\ell^2(\mathbb{N}, \mathbb{C})$ formed with sequences $(x_n)_{(n \in N)}$ such that $\sum_{n=0}^{N} x_n = 0$.

1 - Show that the mapping $(x_n)_n \mapsto \sum_{k=0}^{N} x_k$ is linear continuous from $\ell^2(\mathbb{N}, \mathbb{C})$ to $\mathbb{C}$. What can we deduce about M_N ? Conclude that $\ell^2(\mathbb{N}, \mathbb{C}) = M_N \oplus M_N^{\perp\cdot}$

2 - Let be $E = \{(y_n)_n$ such that, for all $0 \leq i \leq j \leq N$, we have $y_i = y_j$ and $y_n = 0$ for $n > N\}$

3 - Show that the orthogonal $M_N^{\perp\cdot}$ of M_N contains E.

4 - Show that $M_N^{\perp\cdot} = E$ (note that, for $0 \leq i \leq j \leq N$, the sequence (x_n)such that $x_i = 1, x_j = -1$ and $x_n = 0$ if $n \neq i$ and $n \neq j$ belongs to M_N

Solution

1. Let us note T for this application. We have

$$|T(x)| \leq \sum_{n=0}^{N} |x_n| \leq \|x\|_2 \left(\sum_{n=0}^{N} 1^2 \right)^{1/2} \leq N^{1/2} \|x\|_2$$

where the crucial point is the Cauchy-Schwarz inequality. T is therefore continuous, and M_N is a closed subspace of the Hilbert space $\ell^2(\mathbb{N}, \mathrm{C})$. We deduce the requested result.

2. (a) For any $x \in M_N$ and $y \in E$. We have,

$$\langle x, y \rangle = \sum_{k=0}^{N} x_k \overline{yk} = \overline{y0} \sum_{k=0}^{N} x_k = 0$$

(b) It is necessary to show the opposite inclusion. Let us therefore take $y \in M_N^{\perp}$, and let x be the sequence given by the statement, member of M_N, with $i = 0$ and $0 < j \leq N$.

We have,

$$\langle x, y\rangle = y_0 - y_j = 0,$$

which proves that $y_j = y_0$ for $j = 0, \dots, N$.

On the other hand, for $j > N$, we consider the sequence x such that $x_j = 1$ and $x_k = 0$ for $k \neq i$.

The inner product of y with this sequence gives $y_j = 0$, which proves that $y \in E$.

Exercise 4 Let H be a Hilbert space and $P : H \longrightarrow H$ a projector, that is to say a linear mapping such that $P^2 = P$.

1. Show that $imP = ker(Id_H - P)$ and that H is the direct algebraic sum of $kerP$ and imP.

   We assume in the following P continuous and not zero.

2. a) Show that $\|P\| \geq 1$.

   b) Show that the adjoint operator $P^\star$ is also a projector.

3. We assume in this question that P is self-adjoint.

   a) Show that $\|P\| = 1$. b) Show that P is the orthogonal projection on imP. 4) We assume in this question that $\|P\| = 1$.. a) Expand $\|x - P^\star x\|^2$, and deduce that $ker(Id_H - P) \subset ker(Id_H - P^\star)$, then that$ker(Id_H - P) = ker(IdH - P^\star)$.

   b) Show that P is self-adjoint.

Solution

1. Let $y \in imP$; then, $\exists x \in H$ such that $y = Px$.

   So, $y - Py = Px - p^2x = P(x - Px) = 0 =$; Therfore $y \in ker(Id_H - P)$.

   Conversely, if $y \in ker(Id_H - P)$, we have $y - Py = 0$, therefore $y = Py \in imP$.

As $imP = ker(Id_H - P)$, we have, if $y \in imP \cap kerP$, we have $Py = 0$ and $y - Py = 0$; therefore $y = 0$.

On the other hand, all $x \in H$ is written $x = Px + (x - Px) \in imP + kerP$ (because $P(x - Px) = Px - p^2x = 0)$. $i; e, x - Px \in kerP$ So $H = imp \oplus kerP$, algebraically.

2. For all $A, B \in \mathcal{L}(H)$, we have $\|AB\| \leq \|A\|\|B\|$; therefore $\|P\| = \|P^2\| \leq \|P\|^2$;

   Since $P \neq 0$ causes$\|P\| > 0$, we obtain..$\|P\| \geq 1$.

   b) For $x \in H$, we have $(p^*2x, y) = (P^*x, Py) = (x, p^2y) = (x, Py) = (P^*x, y)$, for all $y \in H$; so $p^*2x = P^*x$ then P^* is a projector.

3. 3) a) If P is self-adjoint, we have,

   for all $x \in H : \langle Px, x\rangle = \langle P^2x, x\rangle = \langle Px, P^\star x\rangle = \langle Px, Px\rangle = \|Px\|^2$.

   As, by the Cauchy-Schwarz inequality, we have

   $\|Px\|^2 = \langle Px, x\rangle \leq \|Px\|\|x\|$, we obtain $\|Px\| \leq \|x\|$. So $\|P\| \leq 1$.

   As we saw that $\|P\| \geq 1$, we finally have $\|P\| = 1$.

   b)Let us show that $kerP$ and imP are orthogonal. Let $x \in kerP$ and $y \in imP$; there exists $u \in H$ such that $y = Pu$. Then $\langle x, y\rangle = \langle x, Pu\rangle = \langle P^\star x, u\rangle = \langle Px, u\rangle = 0$.

   4) a) We have $\|x - P^*x\|^2 = \|x\|^2 + \|P^*x\|^2 - 2Re\,\langle x, P^*x\rangle$.

   If $x \in ker(Id_H - P)$, we have $x = Px$;

   Therefore $\langle x, P^*x\rangle = \langle Px, x\rangle = \langle x, x\rangle = \|x\|^2$.

   On the other hand,$\|P^*\| = \|P\| = 1$;

   So, $\|P^*x\|^2 \leq (\|P^*\|\|x\|)^2 = (\|P\|\|x\|)^2 = \|x\|^2$,

   Consequently, $\|x - P^*x\|^2 \leq \|x\|^2 + \|x\|^2 - 2\|x\|^2 = 0$,

   then $x - P^*x = 0$ and $x \in ker(Id_H - P^*)$.

   So $ker(Id_H - P) \subset ker(Id_H - P^*)$.

   We do the same by expanding $\|x - Px\|^2$ (or, we exchange the roles of P and P^*, which is possible because P^* is a projector of norm $\|P^*\| = 1$, and because $P^* * = P$), we obtain reverse inclusion, and therefore equality.

b) For all $x \in H$, if $Px \in ker(Id_H - P)$; then $Px \in ker(Id_H - P^*)$.

Tthat is to say that $Px - P^*Px = 0$. Therefore $P = P^*P$.

Taking the adjoint, this results in $P^* = (P^*P)^* = P^*P^{*}* = P^*P = P$.

Remarque: As the orthogonal projection on any closed subspace not reduced to 0 is of norm 1, we obtain that there is equivalence, for a non-zero projector P of a Hilbert space, between:

a)$\|P\| = 1$; b) P is self-adjoint; c) P is an orthogonal projector.

Non solved exrcises

Exercise 1

Exercise 2

Exercise 3

Exercises series 7

2.4 Fourier series

Exercise 1 Find the Fourier coefficients of the following functions:

(a)$f\,(t) = t$

(b) $f\,(t) = t^2$

(c) $\cos at$ $t \in \mathbb{R}\backslash\mathbb{Z}$ ($\mathbb{Z}$ is the set of integers)

(d) $f\,(t) = |t|$

Use the Parseval equality to prove that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$

Find $\sum_{n=1}^{\infty} \frac{1}{n^4}$

Solution (a) Since $f(t) = t$ is odd, we obtain $a_0 = 0, a_k = 0$, and for $n = 1, 2, \ldots$

$$b_n = \frac{2}{\sqrt{\pi}} \int_0^{\pi} t \sin(nt) dt = \frac{2(-1)^{n+1}\sqrt{\pi}}{n}.$$

(b) Here $b_n = 0$ since $f(t) = t^2$ is even, and

$$a_0 = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} t^2 dt = \frac{\pi^2\sqrt{2\pi}}{3}$$
$$a_n = \frac{2}{\sqrt{\pi}} \int_0^{\pi} t^3 \cos(nt) dt = (-1)^n \frac{4\sqrt{\pi}}{n^2}$$

(c) $b_n = 0$,

$$a_0 = \frac{2}{\sqrt{2\pi}} \int_0^{\pi} \cos(at) dt = \frac{\sqrt{2}}{a\sqrt{\pi}} \sin(a\pi)$$
$$a_n = \frac{2}{\sqrt{\pi}} \int_0^{\pi} \cos(nt)\cos(at) dt = \frac{1}{\sqrt{\pi}} \left[\frac{\sin(n-a)\pi}{n-a} + \frac{\sin(n+a)\pi}{n+a}\right]$$

(d) $a_0 = a_n = 0$,

$$b_n = \frac{2}{\sqrt{\pi}} \int_0^{\pi} \sin(nt) dt = \frac{2}{n\pi} \left(1 - (-1)^n\right).$$

(e) $b_n = 0$,

$$a_0 = \frac{2}{\sqrt{2\pi}} \int_0^\pi t dt = \frac{\pi\sqrt{\pi}}{\sqrt{2}},$$
$$a_n = \frac{2}{\sqrt{\pi}} \int_0^\pi t\cos(nt) dt = -\frac{2}{\sqrt{n^2\pi}} \left(1 - (-1)^n\right).$$

The Parseval equality has the form

$$\|f\|^2 = |a_0|^2 + \sum_{n=1}^{\infty} \left(|a_n|^2 + |b_n|^2\right).$$

Taking $f(t) = t$, we obtain

$$\pi \sum_{n=1}^{\infty} \frac{4}{n^2} = \int_{-\pi}^{\pi} t^2 dt,$$

which after integration gives

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

Using the Parseval equality for the function $f(t) = t^2$, we obtain

$$|a_0|^2 + \sum_{n=1}^{\infty} |a_k|^2 = \int_{-\pi}^{\pi} t^4 dt.$$

Hence

$$\frac{2\pi^5}{9} + \sum_{n=1}^{\infty} \frac{16\pi}{n^4} = \frac{2\pi^5}{5}$$

and hence $\sum_{n=1}^{\infty} 1/n^4 = \pi^4/90$.

Exercise 2 Let $f(x)$ be a ditferentiable $2\pi-$periodic function in $[-\pi, \pi]$ with derivative $f'(x) \in L_2[-\pi, \pi]$. Let f_n for $n \in \mathbb{Z}$ be the Fourier coefficients of $f(x)$ in the system $\left\{e^{inx}/\sqrt{2\pi}\right\}$.

Prove that $\sum_{n\in\mathbb{Z}} |f_n| < \infty$.

Solution For differentiable periodic functions we have

$$\begin{aligned} f_n &= \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(x) e^{-inx} dx \\ &= \frac{1}{\sqrt{2\pi}} \left(\frac{f(x)e^{-inx}}{-in} \Big|_{-\pi}^{\pi} + \frac{1}{in} \int_{-\pi}^{\pi} f'(x) e^{-inx} dx \right) \\ &= \frac{1}{in\sqrt{2\pi}} \int_{-\pi}^{\pi} f'(x) e^{-inx} dx \\ &= \frac{1}{in} \left(f'\right)_n . \end{aligned}$$

From the Cauchy-Schwarz and the Bessel inequalities we obtain

$$\begin{aligned} \sum_{\substack{n=-M \\ n\neq 0}}^{N} |f_n| &= \sum_{\substack{n=-M \\ n\neq 0}}^{N} \frac{1}{|n|} \left|\left(f'\right)_n\right| \\ &\leq \sqrt{\sum_{\substack{n=-M \\ n\neq 0}}^{N} \frac{1}{|n|^2} \sqrt{\sum_{\substack{n=-M \\ n\neq 0}}^{N} \left|\left(f'\right)_n\right|^2}} \\ &\leq \|f'\|_{L_2[-n,n]} \sqrt{\sum_{n\in\mathbb{Z}\setminus(0)} \frac{1}{|n|^2}}. \end{aligned}$$

Hence the series converges.

Exercise3 Consider the sequence of functions $f_n : \mathbb{R} \to \mathbb{C}$ given by

$$f_n(x) = \pi^{-1/2} \frac{(x-i)^n}{(x+i)^{n+1}}.$$

Prove that the family $\{f_1, f_2 \ldots\}$ is orthonormal in $L_2(\mathbb{R})$, that is,

$$\int_{-x}^{\infty} f_m(x)\overline{f_n(x)} dx = \begin{cases} 1, & m = n. \\ 0, & m \neq n. \end{cases}$$

Solution We have, $\int_{-x}^{\infty} f_m(x)\overline{f_n(x)}dx = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{(x-i)^m}{(x+i)^{m+1}} \frac{(x+i)^n}{(x-i)^{n+1}} dx = \frac{1}{\pi}\int_{-x}^{\infty} \frac{(x-i)^{(m-n)-1}}{(x+i)^{(m-n)+1}} dx$. For m and n we have the following cases: a) $m - n \geq 1$. We consider then the function $f : \mathbb{C}\setminus\{-i\} \to \mathbb{C}$ given by $f(z) = (z-i)^{(m-n)-1}/(z+i)^{(m-n)+1}$ and the domain in the complex plane delimited by Ox and the upper half of the circle of center 0 and radius R (denoted by $\Gamma(R)$). On this domain f is a holomorphic function, and therefore

$$\frac{1}{\pi}\int_{-R}^{R}\frac{(x-i)^{(m-n)-1}}{(x+i)^{(m-n)+1}}dx+\int_{\Gamma(R)}f(z)dz=0$$

for any $R>0$. But $\lim_{|z|\to\infty}|zf(z)|=0$ and therefore $\lim_{R\to\infty}\int_{Y^Y(R)}f(z)dz=0$. We obtain that

$$\int_{-\infty}^{\infty}\frac{(x-i)^{(m-n)-1}}{(x+i)^{(m-n)+1}}dx=0$$

b) $m-n\leq -1$. We obtain that

$$\begin{aligned}\int_{-\infty}^{\infty}\frac{(x-i)^{(m-n)-1}}{(x+i)^{(m-n)+1}}dx&=\int_{-\infty}^{\infty}\frac{(x+i)^{(n-m)-1}}{(x-i)^{(n-m)+1}}dx=\int_{-\infty}^{\infty}\frac{(-t+i)^{(n-m)-1}}{(-t-i)^{(n-m)+1}}dt\\&=\int_{-\infty}^{\infty}\frac{(t-i)^{(n-m)-1}}{(t+i)^{(n-m)+1}}dt=0\end{aligned}$$

by (a), since $n-m\geq 1$. c) $m-n=0$. Then

$$\int_{-\infty}^{\infty}f_m(x)\overline{f_m(x)}dx=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{1}{1+x^2}dx=\frac{1}{\pi}\arctan x\bigg|_{-\infty}^{\infty}=1$$

Exercise 4 Example 1.27 Show that the Legendre polynomials are orthogonat in $L^2([-1,1])$, and show that even normalived Legendre functions $(p_n)\,,n=0,2,4,\ldots$ is an orthonormal basis for the closed subspace of even functions in $L^2([-1,1])$. By the way, why is this subspace closed?

Solution We note that

$$P_{\mathrm{m}}(t)=\frac{1}{2^m m!}\frac{d^m}{dt^m}\left(\left(t^2-1\right)^m\right)$$

is a polynomial of degree m,

$$P_m(t) = a_0 + a_1 t + \cdots + a_m t^m$$

Then clearly the Legendre polynomials are orthogonal, if we can prove that

$$\int_{-1}^{1} t^k P_n(t) dt = 0, \qquad \text{for } k = 0, 1, \ldots, m \text{ and } m < n$$

We get by partial integration for $k \leq m < n$,

$$\begin{aligned}
\int_{-1}^{1} t^k P_n(t) dt &= \frac{1}{2^n n!} \int_{-1}^{1} t^k \frac{d^n}{dt^n} \left(\left(t^2 - 1 \right) \right) dt \\
&= \left[\frac{1}{2^n n} t^k \frac{d^{n-1}}{dt^{n-1}} \left(\left(t^2 - 1 \right)^2 \right) \right]_{-1}^{1} - \frac{k}{2^n n!} \int_{-1}^{1} t^{k-1} \frac{d^{n-1}}{dt^{n-1}} \left(\left(t^2 - 1 \right)^n \right) dt \\
&= \cdots = (-1)^k \frac{k!}{2^n n!} \int_{-1}^{1} 1 \cdot \frac{d^{n-k}}{dt^{n-k}} \left(\left(t^2 - 1 \right)^n \right) dt \\
&= (-1)^k \cdot \frac{k!}{2^n n!} \left[\frac{d^{n-k-1}}{dt^{n-k-1}} \left(\left(t^2 - 1 \right)^n \right) \right]_{-1}^{1} = 0
\end{aligned}$$

In fact, from $k < n$, follows that $n - k - 1 \geq 0$, so

$$\frac{d^{n-k-1}}{dt^{n-k-1}} \left(\left(t^2 - 1 \right)^n \right)$$

is a polynomial, which at least contains the factor $t^2 - 1$, hence the boundary values are 0 . Combining this result with Example 1.26 we obtain that the Legendre polynomials form an orthogonal system.

Denote by $U \subseteq L^2([-1,1])$ the closed subspace of all even functions. We have proved above that $\sqrt{\frac{2n+1}{2}} P_n(t)$ is an orthonormal sequence, and since we get thern from $1, t, t^2, \ldots$ by Gram-Schmidt's orthogonalizing method, they form an orthonormal basis for all of $L^2([-1,1])$. Every function from $L^2([-1,1])$ can uniquely be written as a sum of an even and an odd function. Thus the next claim will be solved if we can prove that $P_n(t)$ is an even function, when n is even, and an odd function for n odd.

Non solved exrcises

Exercise 1 Using the following identities and by integration:

$$2\cos nx\cos mx = \cos(n+m)x + \cos(n-m)x$$

$$2\sin nx\sin mx = \cos(n-m)x - \cos(n+m)x$$

$$2\cos nx\sin mx = \sin(n+m)x - \sin(n-m)x$$

Show that the sequence of functions

$$\frac{1}{\sqrt{2\pi}}, \frac{\cos x}{\sqrt{\pi}}, \frac{\sin x}{\sqrt{\pi}}, \frac{\cos 2x}{\sqrt{\pi}}, \frac{\sin 2x}{\sqrt{\pi}}, \dots$$

is a complete orthonormal system in $L^2([-\pi,\pi])$.

Exercise 2 Find the Fourier series corresponding to the function

$$f(x) = x^2, 0 < x < 2\pi$$

, where $f(x)$ has period 2π outside of the interval $(0, 2\pi)$.

Exercise 3

1. Find the Fourier coefficients corresponding to the function

$$f(x) = \begin{cases} 0 & -5 < x < 0 \\ 3 & 0 < x < 5 \end{cases} \qquad \text{Period } = 10$$

2. Write the corresponding Fourier series.

3. How should $f(x)$ be defined at $x = -5, x = 0$ and $x = 5$ in order that the Fourier series will converge to $f(x)$ for $-5 \leqq x \leqq 5$?

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