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Exercise Series
Mathematical Analysis 1

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Foreword

This handout is intended for students in the first year of the Bachelor's degree in Mathematics. It is composed of five chapters. The first chapter contains the body of real and complex numbers, while the second chapter is devoted to numerical sequences. The third chapter gives the functions of a real variable with a real value. The fourth chapter contains the real derivable functions. The last chapter is devoted to the study of elementary functions (cosine, sine, arc cosine, ...).

Chapter 1

Exercises in: The Field of Real and Complex Numbers

Exercise 1

Let A and B be two non-empty and bounded sets. We define:

$$-A = \{-x \mid x \in A\},\ A + B = \{x = a + b \mid a \in A, b \in B\} \text{ and}$$

$$A - B = \{x = a - b \mid a \in A, b \in B\}$$

1. Show that: $\sup(-A) = -\inf(A)$ and $\inf(-A) = -\sup(A)$.

2. Show that if for all $a \in A$ and $b \in B$ we have $a \leq b$, then $\sup(A) \leq \inf(B)$.

3. Show that $A \cup B$ is a bounded subset of $\mathbb{R}$ and:

   - $\sup(A \cup B) = \max(\sup(A), \sup(B))$.
   - $\inf(A \cup B) = \min(\inf(A), \inf(B))$. (*)

4. Show that $\sup(A) + \sup(B)$ is an upper bound of $A + B$ and:

   - $\sup(A + B) = \sup(A) + \sup(B)$.
   - $\inf(A + B) = \inf(A) + \inf(B)$.

Exercise 2

1. Show that if $r \in \mathbb{Q}$ and $x \notin \mathbb{Q}$ then $r + x \notin \mathbb{Q}$ and if $r \neq 0$ then $r.x \notin \mathbb{Q}$.

2. Show that $\sqrt{2} \notin \mathbb{Q}$.

3. Show that $\dfrac{ln3}{ln2}$ is irrational.

4. Let a and b be two positive rationals such that $\sqrt{a}$ and $\sqrt{b}$ are irrational. show that $\sqrt{a}+\sqrt{b}$ is irrational. (*)

Exercise 3

Let A and B be two subsets of $\mathbb{R}$ such that $B \subset A$. Show that:

1. A is bounded $\Longrightarrow$ B is bounded .

2. $\inf(A) \leq \inf(B)$, and $\sup(A) \geq \sup(B)$.

Exercise 4

Let $A = \{a_n \in \mathbb{R} \mid a_n = \dfrac{n+3}{\frac{n}{4}+1};\ n \in \mathbb{N}\}$ and $B = \{b_n \in \mathbb{R} \mid b_n = \frac{1}{n^2} + \dfrac{2}{n} + 4;\ n \in \mathbb{N}^*\}$.

1. Show that A and B are bounded in $\mathbb{R}$ and that $\sup(A) = \inf(B)$.

2. Determine $\sup(A)$ and $\inf(B)$.

Exercise 5

Determine the supremum (the upper bound) and infimum (the lower bound), if they exist of the following sets:

$$A = \{ax+b \mid x \in [-2,1] \quad and \quad a,b \in \mathbb{R}\},\ B = \{2 - \frac{1}{n};\ n \in \mathbb{N}\};$$

$$C = \{\sin \tfrac{2n\Pi}{7};\ n \in \mathbb{Z}\}.\ (*)$$

Exercise 6

1. Write the following numbers in the form $a+ib,(a,\ b\in\mathbb{R})$:

$$z_1=\frac{5+2i}{1-2i}\ ,\ z_2=-\frac{2}{1-i\sqrt{3}},\ z_3=\frac{2+5i}{1-i}+\frac{2-5i}{1+i}(*)$$

2. Let the complex number $z=5+12i$.

   (a) Verify that $|z|=13$.

   (b) Determine the square roots of z.

   (c) Deduce the complex solutions of the equation $(1+i)z^2+z-2-i=0$.

Exercise 7

Using complex numbers, calculate $\cos(5\theta)$ and $\sin(5\theta)$ in terms of $\cos(\theta)$ and $\sin(\theta)$.

Exercise 8

1. Calculate the modulus and the argument of $u=\dfrac{\sqrt{6}-i\sqrt{2}}{2}$ and $v=1-i$.

2. Deduce the modulus and the argement of $\dfrac{u}{v}$.

Exercise 9

(Supplementary)

1. Let z be an n^{th} root of -1, so $z^n=-1$ with $n>2$ and $z\neq-1$.

2. calculate $S_n=\sum_{k=0}^{n-1}z^{2k}=1+z^2+z^4+.....+z^{2(n-1)}$.

Remark 1.0.1. *Exercises marked with (*) are left to students.*

Chapter 2

Solutions to exercises in: The Field of Real and Complex Numbers

Solution 1

1. **a)** Let's show that: $\sup(-A) = -\inf(A)$ and $\inf(-A) = -\sup(A)$.

   we have: $\forall x \in A,\ x \geq \inf(A) \Longrightarrow -x \leq -\inf(A)$. So "$-\inf(A)$" is an upper bound of $-A$, and since $\sup(-A)$ is the smallest upper bound of $-A$ then: $\sup(-A) \leq -\inf(A) \cdots$ **(1)**.

   On the other hand: $\forall -x \in -A,\ -x \leq \sup(-A) \Longrightarrow x > -\sup(-A)$, so $-\sup(-A)$ is a lower bound of A, and since $\inf(A)$ is the greatest lower bound of A then $\inf(A) \geq -\sup(-A)$, therefore: $-\inf(A) \leq \sup(-A) \cdots$ **(2)** .

   From **(1)** and **(2)** we get: $\sup(-A) = -\inf(A)$.

   b) $\inf(-A) = -\sup(A)$:

   ▶ We have: $\forall x \in A,\ x \leq \sup(A) \Longrightarrow -x \geq -\sup(A)$, so $-\sup(A)$ is a lower bound of $-A$, since $\inf(-A)$ is the greatest lower bound of -A then $-\sup(A) \leq \inf(-A) \cdots$ **(1)**.

   On the other hand: $\forall -x \in -A,\ -x \geq \inf(-A) \Longrightarrow x \leq -\inf(-A)$, so $-\inf(-A) \geq \sup(A) \Longrightarrow \inf(-A) \leq -\sup(A) \cdots$ **(2)**.

From **(1)** and **(2)** we get: $-\sup(A) = \inf(-A)$.

2. We show that $\sup(A) \leq \inf(A)$.

   We have $\forall a \in A,\ b \in B:\ a \leq b \Longrightarrow a \leq \inf(B)$, so $\inf(B)$ is an upper bound of A, but $\sup(A)$ is the smallest upper bound of A then: $\sup(A) \leq \inf(B)$.

3. We show that $A \cup B$ is a bounded subset of $\mathbb{R}$: let $x \in A \cup B$, then: $x \in A$ or $x \in B$, therefore $\inf(A) \leq x \leq \sup(A)$ and $\inf(B) \leq x \leq \sup(B)$, So $\min(\inf(A), \inf(B)) \leq x \leq \max(\sup(A), \sup(B))$.

   a) We have: $$\begin{cases} \sup(A \cup B) & \leq & \max(\sup(A), \sup(B)) \cdots\cdots (1) \\ \inf(A \cup B) & \geq & \min(\inf(A), \inf(B)) \cdots\cdots (2) \end{cases}$$

   On the other hand we have: $A \subset A \cup B$ and $B \subset A \cup B$, so

   $$\begin{aligned} \sup(A) & \leq & \sup(A \cup B) \\ \sup(B) & \leq & \sup(A \cup B) \end{aligned}$$

   then: $\max(\sup(A), \sup(B)) \leq \sup(A \cup B) \cdots\cdots (1^*)$, so from (1) and (1^*) we get: $\sup(A \cup B) = \max(\sup(A), \sup(B))$. In the same way we show that $\inf(A \cup B) = \min(\inf(A), \inf(B))$.

4. We show that $\sup(A+B) = \sup(A)+\sup(B)$: we have $\forall x \in A:\ \inf(A) \leq x \leq \sup(A)$, and $\forall y \in B:\ \inf(B) \leq y \leq \sup(B)$, thus: $\inf(A) + \inf(B) \leq x + y \leq \sup(A)+\sup(B)$, so $\inf(A)+\inf(B)$ is a lower bound of $A+B$, but $\inf(A+B)$ is the greatest lower bound of $A+B$, then: $\inf(A+B) \geq \inf(A)+\inf(B) \cdots$ **(1)**. and also $\sup(A + B) \leq \sup(A) + \sup(B) \cdots$ **(2)**.

   On the other hand: $\forall x \in A:\ x \leq \sup(A + B) - y$, then $\sup(A + B) - y$ is an apper bound of A

$$\begin{array}{rrcl}
\Longrightarrow & \sup(A) & \leq & \sup(A+B)-y,\ \forall y\in B,\\
\Longrightarrow & y & \leq & \sup(A+B)-\sup(A),\ \forall y\in B,\\
\Longrightarrow & \sup(B) & \leq & \sup(A+B)-\sup(A),\\
\Longrightarrow & \sup(A)+\sup(B) & \leq & \sup(A+B)\cdots(1^*).
\end{array}$$

From **(1)** and (1^*) we get: $\sup(A+B)=\sup(A)+\sup(B)$. The same to show that $\inf(A+B)=\inf(A)+\inf(B)$.

Solution 2

1. **a)** We show that if $r\in\mathbb{Q}$, and $x\notin\mathbb{Q}$, then $r+x\notin\mathbb{Q}$. we suppose that $x+r\in\mathbb{Q}$, we have: $r\in\mathbb{Q}$ so $\exists\ p,\ q\in\mathbb{Z}$ such that $r=\dfrac{p}{q},\ q\neq 0$.
   And $x+r\in\mathbb{Q}\Longrightarrow\exists\ p',\ q'\in\mathbb{Z}$ such that: $x+r=\dfrac{p'}{q'},\ q'\neq 0$.
   So: $x=\dfrac{p'}{q'}-\dfrac{p}{q}=\dfrac{p'q-pq'}{q'q},\ q'q\neq 0\Longrightarrow x\in\mathbb{Q}$. This is a contradiction because $x\notin\mathbb{Q}$, then $x+r\notin\mathbb{Q}$.

   b) We show that if $x\notin\mathbb{Q}$ and $r\in\mathbb{Q}$ then $x.r\notin\mathbb{Q}$:

   We have $r\in\mathbb{Q}\Longrightarrow r=\dfrac{p}{q},\ q\neq 0$, and $p\neq 0$ $(r\neq 0)$. We assume that $x.r\in\mathbb{Q}$, then $x.r=\dfrac{p'}{q'},\ q'\neq 0\Longrightarrow x=\dfrac{p'}{q'}.\dfrac{q}{p}=\dfrac{p'q}{q'p},\ q'p\neq 0$, thus $x\in\mathbb{Q}$. Contradiction, then $x.r\notin\mathbb{Q}$.

2. We show that $\sqrt{2}\notin\mathbb{Q}$. Suppose that $\sqrt{2}\in\mathbb{Q}\Longrightarrow\exists\ p.\ q\in\mathbb{Z}$ such that $\sqrt{2}=\dfrac{p}{q},\ q\neq 0$.
   suppose that p and q are prime, then $\sqrt{2}=\dfrac{p}{q}\Longrightarrow q\sqrt{2}=p\Longrightarrow 2q^2=p^2$, therefore p^2 is even $\Longrightarrow p$ is even, then $p=2p',\ p'\in\mathbb{Z}$.

   So $2q^2=(2p')^2=4p'^2\Longrightarrow q^2=2p'^2$, therefore q^2 is even $\Longrightarrow q$ is even. Contradiction, then $\sqrt{2}\notin\mathbb{Q}$.

3. We show that $\frac{\ln 3}{\ln 2}$ is irrational. Assume that $\frac{\ln 3}{\ln 2} \in \mathbb{Q} \Longrightarrow \exists\, p,\ q \in \mathbb{Z},\ q \neq 0$ such that $\frac{\ln 3}{\ln 2} = \frac{p}{q} \Longrightarrow q \ln 3 = p \ln 2 \Longrightarrow e^{q \ln 3} = e^{p \ln 2} \Longrightarrow 3^q = 2^p$.

   - If $p = 0$, then $3^q = 2^0 = 1 \Longrightarrow q = 0$ (contradiction because $q \neq 0$).
   - If $p > 0$, then 3^q is odd and 2^p is even. (contradiction), then $\frac{\ln 3}{\ln 2} \notin \mathbb{Q}$.

Solution 3

1. We show that if A is bounded then B is bounded. A is bounded $\Longleftrightarrow \exists\, m,\ M \in \mathbb{R},\ \forall x \in A :\ m \leq x \leq M$.

   We have $B \subset A \Longleftrightarrow \forall x \in B,\ x \in A$, and A is bounded so $m \leq x \leq M$, then B is bounded.

2. **a)** Show that $\inf(A) \leq \inf(B)$. We have $B \subset A \Longrightarrow \forall x \in B :\ x \geq \inf(A)$, therefore $\inf(A)$ is an upper bound of B, then $\inf(A) \leq \inf(B)$ because $\inf(B)$ is the greatest upper bound of B.

   b) Show that $\sup(A) \geq \sup(B)$. We have $B \subset A$, then $\forall x \in B :\ \inf(A) \leq x \leq \sup(A)$, therefore $\sup(A)$ is an upper bound of B, and since $\sup(B)$ is the smallest upper bound of B then $\sup(B) \leq \sup(A)$.

Solution 4

1. $A = \left\{ a_n \in \mathbb{R} \mid a_n = \dfrac{n+3}{\frac{n}{4}+1},\ n \in \mathbb{N} \right\}$. We show that A is bounded, i.e: $\exists\, m,\ M \in \mathbb{R} |\ \forall a_n \in A :\ m \leq a_n \leq M$. We have: $\forall n \in \mathbb{N}$

$$\begin{aligned}\frac{n+3}{\frac{n}{4}+1} &= 4\left(\frac{n+3}{n+4}\right)\\ &= 4\left(\frac{n+4-1}{n+4}\right)\\ &= 4\left(1-\frac{1}{n+4}\right) = 4-\frac{4}{n+4}\end{aligned}$$

$\forall n \geq 0,\ n+4 \geq 4 \Longrightarrow \frac{1}{n+4} \leq \frac{1}{4}$, therfore $-\frac{4}{n+4} \geq -1 \Longrightarrow 4-\frac{4}{n+4} \geq 3 \Longrightarrow a_n \geq 3 \cdots\cdots$ **(1)**.

$\forall n \geq 0:\ n+4 \geq 4 > 0 \Longrightarrow \frac{1}{n+4} > 0$, so $-\frac{4}{n+4} < 0 \Longrightarrow 4-\frac{4}{n+4} < 4 \Longrightarrow a_n < 4 \cdots\cdots$ **(2)**.

Then from **(1)** and **(2)**, we get $3 \leq a_n \leq 4$. So $\inf(A) = 3$, and since $3 \in A$, then: $\inf(A) = \min(A) = 3$, $(a_0 = 3 \in A)$, and $\sup(A) = 4$.

Now let's show that $\sup(A) = 4$.

$$\sup(A) = 4 \Longleftrightarrow \begin{cases} \forall\ a_n \in A:\ a_n < 4, \\ \forall \varepsilon > 0,\ \exists\ n_\varepsilon \in \mathbb{N}:\ a_n > 4-\varepsilon. \end{cases}$$

We have: $a_n < 4$, $\forall a_n \in A$ verify: $\forall\ \varepsilon > 0$, $a_n > 4-\varepsilon \Longrightarrow 4-\frac{4}{n+4} > 4-\varepsilon \Longrightarrow \frac{4}{n+4} < \varepsilon$, therfore: $\frac{n+4}{4} > \frac{1}{\varepsilon} \Longrightarrow n+4 > \frac{4}{\varepsilon} \Longrightarrow n > \frac{4}{\varepsilon}-4$. Just take $n_\varepsilon = \left[\frac{4}{\varepsilon}-4\right]+1$, then $\sup(A) = 4$.

2. $B = \left\{b_n \in \mathbb{R} \mid b_n = \frac{1}{n^2}+\frac{2}{n}+4\right\}$. We show that B is bounded, for all $n \geq 1 \Longrightarrow \frac{2}{n} \leq 2$, and $\frac{1}{n^2} \leq 1$, therfore $\frac{2}{n}+\frac{1}{n^2} \leq 3 \Longrightarrow \frac{2}{n}+\frac{1}{n^2}+4 \leq 7$, then $b_n \leq 7 \cdots\cdots$ **(1)**.

   On the other hand: $\frac{2}{n} > 0$, and $\frac{1}{n^2} > 0$, then $\frac{2}{n}+\frac{1}{n^2} > 0 \Longrightarrow \frac{2}{n}+\frac{1}{n^2}+4 > 4$, so $b_n > 4 \cdots\cdots$ **(2)**.

   From **(1)**, and **(2)**, we get: $\forall n \in \mathbb{N},\ 4 < b_n \leq 7$, then B is bounded in $\mathbb{R}$, such that $\sup(B) = \max(B) = 7$, and $\inf(B) = 4$. Now we must to prove that $\inf(B) = 4$.

$$\inf(B)=4 \Longleftrightarrow \begin{cases} \forall\, b_n \in B,\ b_n > 4, \\ \\ \forall\, \varepsilon > 0,\ \exists\, n_\varepsilon \in \mathbb{N}^* :\ b_n < 4+\varepsilon. \end{cases}$$

We have $b_n < 4+\varepsilon \Longrightarrow \dfrac{1}{n^2}+\dfrac{2}{n}+4 < 4+\varepsilon \Longrightarrow \dfrac{1}{n^2}+\dfrac{2}{n} < \varepsilon$, also: $n^2 \geq n \Longrightarrow \dfrac{1}{n^2} \leq \dfrac{1}{n}$, and $\dfrac{1}{n^2}+\dfrac{2}{n} \leq \dfrac{3}{n}$.

We are only looking for a n_ε such that $\dfrac{3}{n} < \varepsilon$, i.e, $n > \dfrac{3}{\varepsilon}$, therfore we just take $n_\varepsilon = \left[\dfrac{3}{\varepsilon}\right]+1$, then $\inf(B)=4=\sup(A)$.

Solution 5

1. $A=\{ax+b \mid x \in [-2,1],\ a,\ b \in \mathbb{R}\}$. Assume that:

$$\begin{aligned} f:\mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longrightarrow f(x)=ax+b \end{aligned}$$

- If $a=0 \Longrightarrow f(x)=b$, then f is constant, and $A=\{b\}$ is bounded such that $\sup(A)=\inf(A)=b$.

- If $a>0 \Longrightarrow f(x)$ is increasing, so for all $-2 \leq x \leq 1$, we have: $f(-2) \leq f(x) \leq f(1) \Longrightarrow -2a+b \leq f(x) \leq a+b$, then $\forall x \in [-2,1]$, A is bounded such that: $\inf(A)=\min(A)=-2a+b$, and $\sup(A)=\max(A)=a+b$.

- If $a<0 \Longrightarrow f(x)$ is decreasing, then A is bounded and $\inf(A)=a+b$, $\sup(A)=-2a+b$.

2. $B=\left\{2-\dfrac{1}{n},\ n \in \mathbb{N}^*\right\}$. For $n=1 \Longrightarrow B=1$, and for $n \longrightarrow \infty \Longrightarrow B=2$, so $B=[1,2[$. The set of upper bounds of B is $[2,+\infty[$ therefore $\sup(B)=2$, and the set of lower bounds of B is $]-\infty,1]$, therfore $\inf(B)=1$, since $1 \in B$, then $\inf(B)=\min(B)=1$, and $\max(B)$ does not exist because $2 \notin B$.

Solution 6

1. $z_1 = \dfrac{5+2i}{1-2i} = \dfrac{(5+2i)(1+2i)}{(1-2i)(1+2i)} = \dfrac{1+12i}{5} = \dfrac{1}{5} + \dfrac{12}{5}i.$

$$z_2 = \frac{-2}{1-i\sqrt{3}} = \frac{-2(1+i\sqrt{3})}{1^2+(-\sqrt{3})^2} = \frac{-2(1+i\sqrt{3})}{4} = \frac{-1}{2} - i\frac{\sqrt{3}}{2}.$$

2. We have $z = 5 + 12i$.

   (a) $|z| = |5+12i| = \sqrt{25+144} = \sqrt{169} = 13.$

   (b) Let $w \in \mathbb{C}$ such that $w = a + ib$,

$$w^2 = z \iff a^2 - b^2 + 2abi = 5 + 12i \iff \begin{cases} a^2 - b^2 = 5, \cdots L_1 \\ 2ab = 12, \cdots L_2 \end{cases}$$

   We add the equality of the modules

$$a^2 + b^2 = \sqrt{5^2 + 12^2} = \sqrt{169} = 13, \cdots L_1$$

   $L_1 + L_2 \iff 2a^2 = 18$, then $a^2 = 9 \iff a = \pm 3$, and

   $L_1 - L_2 \iff 2b^2 = 8$, then $a^2 = 4 \iff a = \pm 2$. According to L_2: a and b have the same sign, hence the square roots of z are

   - $z_1 = 3 + 2i$
   - $z_2 = -3 - 2i$

   (c) We calculate the discriminant of the equation $\Delta = 5 + 12i = z$, and we deduce from the previous question that the equation admits two distinct complex solutions

$$\begin{aligned} w_1 &= \frac{-1+3+2i}{2(1+i)} = 1 \\ w_2 &= \frac{-1-3-2i}{2(1+i)} = \frac{-2-i}{1+i} = \frac{(-2-i)(1-i)}{(1+i)(1-i)} = \frac{-3}{2} + \frac{i}{2}. \end{aligned}$$

Solution 7

Calculate $\cos 5\theta$, and $\sin 5\theta$. We have by the Moivre's formula:

$$\cos 5\theta + i \sin 5\theta = e^{i5\theta} = (e^{i\theta})^5 = (\cos\theta + i\sin\theta)^5.$$

Using Newton's binomial formula:

$$(\cos\theta + i\sin\theta)^5 =$$

$$\cos^5\theta + 5i\cos^4\theta\sin\theta - 10\cos^3\theta\sin^2\theta - 10i\cos^2\theta\sin^3\theta + 5\cos\theta\sin^4\theta + i\sin^5\theta$$

So: $\cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$,

and $\sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$.

Solution 8

1. **a)** $|u| = \left|\dfrac{\sqrt{6} - i\sqrt{2}}{2}\right| = \dfrac{\sqrt{6+2}}{2} = \dfrac{\sqrt{8}}{2} = \dfrac{2\sqrt{2}}{2} = \sqrt{2}.$

   $u = \dfrac{\sqrt{6} - i\sqrt{2}}{2} = \sqrt{2}\left(\dfrac{\sqrt{2}\times\sqrt{3} - i\sqrt{2}}{2\sqrt{2}}\right) = \sqrt{2}\left(\dfrac{\sqrt{3} - i}{2}\right) = \sqrt{2}\, e^{-i\dfrac{\pi}{6}}.$

   Then $|u| = \sqrt{2}$, and $arg(u) = -\dfrac{\pi}{6}$.

   b) $|v| = \sqrt{1^2 + (-1)^2} = \sqrt{2}.$

   $v = \sqrt{2}\left(\dfrac{\sqrt{2}}{2} - i\dfrac{\sqrt{2}}{2}\right) = \sqrt{2}\, e^{-i\dfrac{\pi}{4}}.$

   Then $|v| = \sqrt{2}$, and $arg(v) = -\dfrac{\pi}{4}$.

2. $\dfrac{u}{v} = \dfrac{\sqrt{2}\, e^{-i\frac{\pi}{6}}}{\sqrt{2}\, e^{-i\frac{\pi}{4}}} = e^{i(-\frac{\pi}{6}+\frac{\pi}{4})} = e^{i\frac{\pi}{12}}.$

   Then: $\left|\dfrac{u}{v}\right| = 1$, and $arg(\dfrac{u}{v} = \dfrac{\pi}{12})$.

Chapter 3

Exercises in: The Numerical Sequences

Exercise 1

Show by induction that:

1. $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$.
2. $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$.

Exercise 2

Determine, by justifying your answers, if the following sequences are convergent:

1. $U_n = \frac{\cos n - 2}{n^4}, \ \forall n \in \mathbb{N}^*$.
2. $V_n = \frac{3n + 5(-1)^n}{2n+1}, \ \forall n \in \mathbb{N}$.
3. $W_n = (-1)^n (\frac{n+1}{n}), \ \forall n \in \mathbb{N}^*$.
4. $Z_n = \sqrt{2n+1} - \sqrt{2n-1}, \ \forall n \in \mathbb{N}^*$.(*)

Exercise 3

Let $(u_n)_{n\in\mathbb{N}}$ be the sequence of real numbers defined by $u_0 \in]0.1]$, and by the recurrence relation

$$u_{n+1} = \frac{u_n}{2} + \frac{(u_n)^2}{4}$$

1. Show that: $\forall n \in \mathbb{N},\ u_n > 0$.

2. Show that: $\forall n \in \mathbb{N},\ u_n \leq 1$.

3. Show that the sequence is monotonic. Deduce that the sequence is convergent.

4. Determine the limit of the sequence $(u_n)_{n\in\mathbb{N}}$.

Exercise 4

Prove that the following two sequences are adjacent

$$\forall n \in \mathbb{N},\ u_n = \sum_{k=1}^{n} \frac{1}{k^2},\ v_n = u_n + \frac{1}{n}.$$

Exercise 5

1. Let $u_n = \dfrac{E(\sqrt{n})}{n}$, for all $n \in \mathbb{N}^*$, show that

$$\lim_{n\to+\infty} u_n = 0.$$

2. Let $v_n = \dfrac{E(\sqrt{n})^2}{n}$, for all $n \in \mathbb{N}^*$, show that the sequence $(v_n)_{n\in\mathbb{N}^*}$ converges and determine its limit. (*)

Exercise 6

Calculate the following limits, if they exist, of the following sequences:

1. $u_n = \dfrac{1}{2.3} + \dfrac{1}{3.4} + \cdots + \dfrac{1}{(n+1)(n+2)}$.

2. $v_n = \frac{1}{n^2} + \frac{2}{n^2} + \cdots + \frac{n-1}{n^2}$.

3. $w_n = \frac{\ln(n+1)}{\ln n}$.

4. $z_n = \sqrt{n^2+n+1} - \sqrt{n^2-n+1}$. (*)

Exercise 7

We consider the sequence $(u_n)_{n\geq 1}$ given by: $u_n = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2}$.

1. Show that $\frac{1}{n^2} \leq \frac{1}{n-1} - \frac{1}{n}$.

2. Show that the sequence $(u_n)_{n\geq 1}$ is bounded above by 2.

3. Show that the sequence $(u_n)_{n\geq 1}$ is increasing.

4. Deduce that $(u_n)_{n\geq 1}$ is converges.

Exercise 8

We consider the sequence $(u_n)_{n\in\mathbb{N}}$ defined by $u_0 = 0$ and by the recurrence relation

$$u_{n+1} = \frac{1}{6}u_n^2 + \frac{3}{2}$$

1. Show that for all $n \in \mathbb{N}^*$, $u_n > 0$.

2. Calculate the limit of the sequence $(u_n)_{n\in\mathbb{N}}$.

3. Show that for all $n \in \mathbb{N}$, $u_n < 3$.

4. Show that the sequence is increasing, what can we conclude from this?

Exercise 9

(Supplementary)

We consider the sequence $(u_n)_{n\in\mathbb{N}^*}$ defined by

$$u_n = \frac{1}{3+|\sin(1)|\sqrt{1}} + \frac{1}{3+|\sin(2)|\sqrt{2}} + \cdots + \frac{1}{3\,|\sin(n)|\sqrt{n}}$$

Show that $\lim_{n\to+\infty} u_n = +\infty$.

Chapter 4

Solutions ti exercises in: The Numerical Sequences

Solution 1

1. $1+2+3+\cdots+n=\dfrac{n(n+1)}{2}\cdots\cdots P(n)$

   For $n=1 \Longrightarrow 1=\dfrac{1.(1+1)}{2}$ is true.

   For $n\geq 2$: assume that $P(n)$ is true, and show that $P(n+1)$ is true, this means showing that if $1+2+3+\cdots+n=\dfrac{n(n+1)}{2}$ then $1+2+3+\cdots+(n+1)=\dfrac{(n+1)(n+2)}{2}$.

   We have: $1+2+3+\cdots+n+n+1=\dfrac{n(n+1)}{2}+(n+1)=\dfrac{n(n+1)+2(n+1)}{2}=\dfrac{(n+1)(n+2)}{2}$.

   Then $P(n)$ is true, therfore $1+2+3+\cdots+n=\dfrac{n(n+1)}{2}$.

2. $1^2+2^2+3^2+\cdots+n^2=\dfrac{n(n+1)(2n+1)}{6}$.

   For $n=1 \Longrightarrow 1=\dfrac{1.(2)(3)}{6}$ is true.

   For $n\geq 2$: assume that $1^2+2^2+3^2+\cdots+n^2=\dfrac{n(n+1)(2n+1)}{6}$, and show that $1^2+2^2+3^2+\cdots+(n+1)^2=\dfrac{(n+1)(n+2)(2n+3)}{6}$.

   We have:

$$\begin{aligned}
1^2+2^2+\cdots+n^2+(n+1)^2 &= \frac{n(n+1)(2n+1)}{6}+(n+1)^2\\
&= \frac{n(n+1)(2n+1)+6(n+1)^2}{6}\\
&= \frac{(n+1)[2n^2+n+6n+6]}{6}\\
&= \frac{(n+1)[2n^2+7n+6]}{6}\\
&= \frac{(n+1)(n+2)(2n+3)}{6}.
\end{aligned}$$

Then: $1^2+2^2+3^2+\cdots+n^2=\dfrac{n(n+1)(2n+1)}{6}$.

Solution 2

1. $U_n=\dfrac{\cos n-2}{n^4}$, $\forall n\in\mathbb{N}^*$.

   For all $n\in\mathbb{N}^*$:

$$\begin{array}{rcccl}
-1 & \le & \cos n & \le 1\\
-3 & \le & \cos n-2 & \le -1\\
\dfrac{-3}{n^4} & \le & \dfrac{\cos n-2}{n^4} & \le \dfrac{-1}{n^4}
\end{array}$$

   Since $\lim\limits_{n\to\infty}\dfrac{-3}{n^4}=\lim\limits_{n\to\infty}\dfrac{-1}{n^4}=0$, then $\lim\limits_{n\to\infty}U_n=0$.

2. $V_n=\dfrac{3n+5(-1)^n}{2n+1}$, $\forall n\in\mathbb{N}$.

   For all $n\in\mathbb{N}$, we have

$$\frac{3n+5(-1)^n}{2n+1}=\frac{3n}{2n+1}+\frac{5(-1)^n}{2n+1}=\frac{3}{2(1+\frac{1}{n})}+\frac{5(-1)^n}{2n+1}.$$

   On the one hand since $\lim\limits_{n\to\infty}1+\dfrac{1}{n}=1$, then $\lim\limits_{n\to\infty}\dfrac{3}{2(1+\frac{1}{n})}=\dfrac{3}{2}$. On the other hand since $(-1)^n$ is bounded, and $\lim\limits_{n\to\infty}\dfrac{5}{2n+1}=0$. We deduce that $\lim\limits_{n\to\infty}\dfrac{5(-1)^n}{2n+1}=0$. So $\lim\limits_{n\to\infty}V_n=\dfrac{3}{2}$.

3. $W_n = (-1)^n(\dfrac{n+1}{n}),\ \forall n \in \mathbb{N}^*$.

   We have: $W_n = (-1)^n(\dfrac{n+1}{n}) = (-1)^n + \dfrac{(-1)^n}{n}$, since $(-1)^n$ is bounded and $\lim\limits_{n\to\infty} \dfrac{1}{n} = 0$, then $\lim\limits_{n\to\infty} \dfrac{(-1)^n}{n} = 0$. Also $(-1)^n$ does not admit a limits, therfore we consider the subsequences of even and odd ranks respectively $(W_{2n})_{n\in\mathbb{N}^*}$, and $(W_{2n+1})_{n\in\mathbb{N}^*}$, so for all $n \in \mathbb{N}^*$ we have:

$$\begin{aligned} W_{2n} &= (-1)^{2n} + \frac{(-1)^{2n}}{2n} = 1 + \frac{1}{2n} \underset{n\to\infty}{\longrightarrow} 1 \\ W_{2n+1} &= (-1)^{2n+1} + \frac{(-1)^{2n+1}}{2n+1} = -1 - \frac{1}{2n+1} \underset{n\to\infty}{\longrightarrow} -1. \end{aligned}$$

   So the sequence $(W_n)_{n\in\mathbb{N}^*}$ admits two subsequences that converge to different limits, and therefore it is not convergent.

Solution 3

$$\begin{cases} u_0 \in\ ]0,1], \\ u_{n+1} = \dfrac{u_n}{2} + \dfrac{(u_n)^2}{4}. \end{cases}$$

1. We show that: $\forall n \in \mathbb{N},\ u_n > 0$. (reasoning by induction)

   For $n = 0$, we have $u_0 \in\ ]0,1]$, then $u_n > 0$.

   For $n \geq 1$, we assume that $u_n > 0$ and we show that $u_{n+1} > 0$. We have $u_n > 0$, so: $\dfrac{u_n}{2} > 0$, and $\dfrac{(u_n)^2}{4} > 0$, therfore: $u_{n+1} = \dfrac{u_n}{2} + \dfrac{(u_n)^2}{4} > 0$. Then $\forall n \in \mathbb{N},\ u_n > 0$.

2. We show that: $\forall n \in \mathbb{N},\ u_n \leq 1$:

   For $n = 0$, we have $u_0 \in\ ]0,1]$, then $u_n \leq 1$.

   For $n \geq 1$, we assume that $u_n \leq 1$ and we show that $u_{n+1} \leq 1$.

   We have $0 < u_n \leq 1$, then

$$u_{n+1} = \frac{u_n}{2} + \frac{(u_n)^2}{4} \leq \frac{1}{2} + \frac{1}{4} = \frac{3}{2} \leq 1.$$

So $\forall n \in \mathbb{N},\ u_n \leq 1$.

3. We calculate:

$$u_{n+1} - u_n = \frac{u_n}{2} + \frac{(u_n)^2}{4} - u_n = -\frac{u_n}{2} + \frac{(u_n)^2}{4} = \frac{u_n}{4}(-2 + u_n).$$

Since $0 < u_n \leq 1$, we get $-2 + u_n < 0$, then $u_{n+1} - u_n < 0$. It shows that the sequence is strictly decreasing.

4. The sequence is strictly decreasing and bounded below by 0, so it converges to a limit noted l and verified

$$\begin{aligned} l = \frac{l}{2} + \frac{l^2}{4} \quad &\Longleftrightarrow \quad 0 = -\frac{l}{2} + \frac{l^2}{4} \\ &\Longleftrightarrow \quad -2l + l^2 = 0 \\ &\Longleftrightarrow \quad l(-2 + l) = 0 \end{aligned}$$

so $l = 0$ or $l = 2$. Therefore $l = 0$.

Solution 4

$\forall n \in \mathbb{N}^*$, we have: $u_n = \sum_{k=1}^{n} \frac{1}{k^2}$, and $v_n = u_n + \frac{1}{n}$, we show that $(u_n)_{n\in\mathbb{N}^*}$, and $(v_n)_{n\in\mathbb{N}^*}$ are adjacent:

1. $$\begin{aligned} u_{n+1} - u_n &= \sum_{k=1}^{n+1} \frac{1}{k^2} - \sum_{k=1}^{n} \frac{1}{k^2} \\ &= \frac{1}{(n+1)^2} > 0 \end{aligned}$$
therfore $(u_n)_{n\in\mathbb{N}^*}$ is increasing.

2. $$\begin{aligned} v_{n+1} - v_n &= u_{n+1} + \frac{1}{n+1} - u_n - \frac{1}{n} \\ &= \frac{1}{(n+1)^2} + \frac{1}{n+1} - \frac{1}{n} \\ &= \frac{n + n(n+1) - (n+1)^2}{n(n+1)^2} \\ &= \frac{-1}{n(n+1)^2} < 0 \end{aligned}$$
therfore $(v_n)_{n\in\mathbb{N}^*}$ is decreasing.

3. $\lim_{n\to\infty} u_n - v_n = \lim_{n\to\infty} u_n - u_n - \frac{1}{n} = \lim_{n\to\infty} \frac{-1}{n} = 0.$

   So $(u_n)_{n\in\mathbb{N}^*}$, and $(v_n)_{n\in\mathbb{N}^*}$ are adjacent.

Solution 5

$\forall n \in \mathbb{N}^*$ we have: $u_n = \frac{E(\sqrt{n})}{n}$, we show that $\lim_{n\to\infty} u_n = 0$.

Assume that $P = E(\sqrt{n})$, then $\forall n \in \mathbb{N}^*$ we have:

$$P \leq \sqrt{n} < P + 1 \Longrightarrow P^2 \leq n < (P+1)^2,$$

therfore: $\frac{1}{(P+1)^2} < \frac{1}{n} \leq \frac{1}{P^2} \cdots\cdots (*)$.

We multiply $(*)$ by $P = E(\sqrt{n}) > 0$ (because $n \geq 1$), we get:

$$\frac{P}{(P+1)^2} < \frac{P}{n} \leq \frac{P}{P^2} \Longrightarrow \frac{E(\sqrt{n})}{(E(\sqrt{n})+1)^2} < \frac{E(\sqrt{n})}{n} \leq \frac{1}{E(\sqrt{n})}.$$

When $n \longrightarrow +\infty$, $E(\sqrt{n}) \longrightarrow +\infty$, then $\lim_{n\to\infty} \frac{E(\sqrt{n})}{n} = 0$.

Solution 6

1. $u_n = \frac{1}{2.3} + \frac{1}{3.4} + \cdots + \frac{1}{(n+1)(n+2)}$.

$$\begin{aligned}
\lim_{n\to+\infty} u_n &= \lim_{n\to+\infty} \left[\left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n+1} - \frac{1}{n+2}\right)\right] \\
&= \lim_{n\to+\infty} \left(\frac{1}{2} - \frac{1}{n+2}\right) \\
&= \frac{1}{2}.
\end{aligned}$$

2. $v_n = \frac{1}{n^2} + \frac{2}{n^2} + \cdots + \frac{n-1}{n^2}$.

$$\begin{aligned}
\lim_{n\to+\infty} v_n &= \lim_{n\to+\infty} \frac{1}{n^2}(1 + 2 + \cdots + n - 1) \\
&= \lim_{n\to+\infty} \frac{1}{n^2}\frac{n(n-1)}{2} \\
&= \frac{1}{2}.
\end{aligned}$$

3. $w_n = \dfrac{\ln(n+1)}{\ln n}$.

$$\begin{aligned}\lim_{n\to+\infty} w_n &= \lim_{n\to+\infty} \frac{\ln\left[n(1+\frac{1}{n})\right]}{\ln\ n}\\ &= \lim_{n\to+\infty} \frac{\ln\ n + \ln\ (1+\frac{1}{n})}{n}\\ &= \lim_{n\to+\infty} 1 + \frac{\ln(1+\frac{1}{n})}{\ln\ n} = 1.\end{aligned}$$

Solution 7

$$\forall n \in \mathbb{N}^* :\ u_n = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2}.$$

1. We show that $\dfrac{1}{n^2} \leq \dfrac{1}{n-1} - \dfrac{1}{n} = \dfrac{1}{n(n-1)}$:

   we have: $\forall n \in \mathbb{N}^* :\ n \geq n-1 \Longrightarrow n^2 \geq n(n-1)$, so

$$\frac{1}{n^2} \leq \frac{1}{n(n-1)} = \frac{1}{n-1} - \frac{1}{n}.$$

2. We show that $(u_n)_{n\geq 1}$ is bounded above by 2:

   we have: $\dfrac{1}{n^2} \leq \dfrac{1}{n-1} - \dfrac{1}{n}$, then

$$\frac{1}{2^2} \leq 1 - \frac{1}{2},\ \frac{1}{3^2} \leq \frac{1}{2} - \frac{1}{3},\ \cdots,\ \frac{1}{n^2} \leq \frac{1}{n-1} - \frac{1}{n}$$

   therefore:

$$\begin{aligned}1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} &\leq 1 + 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \cdots + \frac{1}{n-1} - \frac{1}{n}\\ u_n &\leq 2 - \frac{1}{n} < 2\end{aligned}.$$

   So $(u_n)_{n\geq 1}$ is bounded above by 2.

3. We show that $(u_n)_{n\geq 1}$ is increasing:

$$\begin{aligned}u_{n+1} - u_n &= 1 + \frac{1}{2^2} + \cdots + \frac{1}{(n+1)^2} - 1 - \frac{1}{2^2} - \cdots - \frac{1}{n^2}\\ &= \frac{1}{(n+1)^2} > 0.\end{aligned}$$

Then $(u_n)_{n\geq 1}$ is increasing.

4. $(u_n)_{n\geq 1}$ is increasing and bounded above by 2, so $(u_n)_{n\geq 1}$ is convergent.

Solution 8

$$\begin{cases} u_0 = 0 \\ u_{n+1} = \dfrac{1}{6}\, u_n^2 + \dfrac{3}{2} \end{cases}$$

1. We show that $\forall n \in \mathbb{N}^*,\ u_n > 0$.

   - For $n = 1 \Longrightarrow u_1 = \dfrac{1}{6}\, u_0^2 + \dfrac{3}{2} = \dfrac{3}{2} > 0$.
   - For $n \geq 2 \Longrightarrow$, we assume that $u_n > 0$ and we prove that $u_{n+1} > 0$. We have $u_n > 0$, then $\dfrac{1}{6}u_n^2 > 0$, therefore: $\dfrac{1}{6}u_n^2 + \dfrac{3}{2} > \dfrac{3}{2} > 0$, so $u_{n+1} > 0 \Longrightarrow \forall n \in \mathbb{N}^*,\ u_n > 0$.

2. If the sequence u_n admits a limit l then:

$$\begin{aligned} l = \frac{1}{6}\, l^2 + \frac{3}{2} &\iff l^2 - 6l + 9 = 0 \\ &\iff (l-3)^2 = 0 \\ &\iff l = 3. \end{aligned}$$

3. We show that $\forall n \in \mathbb{N},\ u_n < 3$: (reasoning by induction)

   - For $n = 0$, we have $u_0 = 0 < 3$.
   - For $n \geq 1$, we assume that $u_n < 3$, and we prove that $u_{n+1} < 3$. We have

$$\begin{aligned} u_n < 3 &\Longrightarrow u_n^2 < 9 \\ &\Longrightarrow \frac{1}{6}\, u_n^2 + \frac{3}{2} < 3. \end{aligned}$$

   So $\forall n \in \mathbb{N},\ u_n < 3$.

4. $u_{n+1} - u_n = \frac{1}{6}\left(u_n - 3\right)^2 > 0$, the sequence $(u_n)_{n\in\mathbb{N}}$ is strictly increasing, and since it is bounded by 3, it therefore converges to a limit l, such that

$$l = \frac{1}{6}l^2 + \frac{3}{2} \Longrightarrow l = 3.$$

Chapter 5

Exercises in: The Real-Valued Functions of a Real Variable

Exercise 1

Determine the definition domains of the following functions:

1. $f(x) = \sqrt{\dfrac{x+1}{x-1}}$.

2. $g(x) = \sqrt{x^2 + x - 2}$.

3. $h(x) = \ln\left(\dfrac{2+x}{2-x}\right)$

4. $k(x) = \dfrac{\sin x - \cos x}{x - \pi}$.

5. $p(x) = (1+x)^{\frac{1}{x}}$.

6. $\phi(x) = \begin{cases} \dfrac{\sin x . \cos x}{x - \pi} & if \quad x \neq \pi \\ 1 & Otherwise \end{cases}$

Exercise 2

Let the function f be defined on $]-1, 1[$ by: $f(x) = \dfrac{x}{1+|x|}$.

Show that f is strictly increasing.

Exercise 3

Calculate the following limits:

1. $\lim\limits_{x\to+\infty} e^{x-\sin x}$.

2. $\lim\limits_{x\to 0} \dfrac{(\tan x)^2}{\cos(2x)-1}$.

3. $\lim\limits_{x\to 0^+} \dfrac{x}{b}\left[\frac{c}{x}\right]$.

4. $\lim\limits_{x\to 0} \dfrac{\ln(1+x^2)}{\sin^2 x}$.

5. $\lim\limits_{x\to 0} \dfrac{\sqrt{1+x}-\sqrt{1-x}}{x}$.

6. $\lim\limits_{x\to+\infty} \dfrac{x\ln x+5}{x^2+4}$.

Exercise 4

Determine the values a and b so that the functions f, and g are continuous on $\mathbb{R}$

$$f(x)=\begin{cases}\dfrac{\sin(ax)}{x}, & x<0\\ 1, & x=0\\ 2be^x-x, & x<0\end{cases},\quad g(x)=\begin{cases}\sqrt{x}-\dfrac{1}{x}, & x\geq 4\\ (x+a)^2, & x<4\end{cases}$$

Exercise 5

Are the following functions continuous at the point $x_0=0$?

$$f(x)=\begin{cases}x+\dfrac{\sqrt{x^2}}{x} & : x\neq 0\\ 0 & : x=0\end{cases},\quad g(x)=\begin{cases}1+x\cos(\dfrac{1}{x}) & : x\neq 0\\ 0 & : x=0\end{cases}.$$

Exercise 6

1. Show that the following functions are continuous over their defined domains:

$$f(x) = \frac{x^3 + 2x + 3}{x^3 + 1}, \;\; g(x) = \frac{(1+x)^n - 1}{x}.$$

2. Study the existence of extension by continuity over $\mathbb{R}$.

Exercise 7

1. Show that any periodic and non-constant function does not admit a limit in $+\infty$.

2. Let $f : [0, +\infty[\longrightarrow \mathbb{R}$ be a function such that $f(0) > 0$. We assume that $\lim\limits_{x \to +\infty} \dfrac{f(x)}{x} = a < 1$.

   Show that there exists $x_0 \in [0, +\infty[$ such that $f(x_0) = x_0$.

Chapter 6

Solutions to exercises in: Real-Valued Functions of a Real Variable

Solution 1

1. $f(x) = \sqrt{\dfrac{x+1}{x-1}}$.

   $D_f = \left\{x \in \mathbb{R} |\ \dfrac{x+1}{x-1} \geq 0,\ and\ x-1 \neq 0\right\}$

   $\dfrac{x+1}{x-1} \geq 0 \implies x \in]-\infty, -1] \cup [1, +\infty[$, and $x - 1 \neq 0 \implies x \neq 1$, so

   $D_f =]-\infty, -1] \cup]1, +\infty[$.

2. $g(x) = \sqrt{x^2 + x - 2}$.

   $D_g = \{x \in \mathbb{R} |\ x^2 + x - 2 \geq 0\} =]-\infty, -2] \cup [1, +\infty[$.

3. $h(x) = \ln\left(\dfrac{2+x}{2-x}\right)$.

   $D_h = \left\{x \in \mathbb{R} |\ \dfrac{2+x}{2-x} > 0,\ and\ 2-x \neq 0\right\}$, so $D_h =]-2, 2[$.

4. $k(x) = \dfrac{\sin x - \cos x}{x - \pi}$.

   $D_k = \{x \in \mathbb{R} |\ x \neq \pi\} =]-\infty, \pi[\cup]\pi, +\infty[$.

5. $p(x) = (1+x)^{\frac{1}{x}} = e^{\frac{1}{x}\ln(1+x)}$.

   $D_p = \{x \in \mathbb{R} |\ x \neq 0,\ and\ 1 + x > 0\} =]-1, 0[\cup]0, +\infty[$.

6. $\phi(x) = \begin{cases} \dfrac{\sin x . \cos x}{x - \pi} & if \quad x \neq \pi \\ 1 & Otherwise \end{cases}$

$D_\phi = \mathbb{R}$.

Solution 2

It is necessary to show that $x_1 < x_2 \Longrightarrow f(x_1) < f(x_2)$.

We have

$$f(x) = \begin{cases} \dfrac{x}{1+x} & if \quad x \geq 0 \\ \dfrac{x}{1-x} & if \quad x < 0 \end{cases}$$

- If $x_1 < 0 < x_2$, then it is obvious that $f(x_1) < 0 < f(x_2)$ (if one of the two is zero it is also obvious).
- If $0 < x_1 < x_2$, we note that: $f(x) = \dfrac{x}{x+1} = 1 - \dfrac{1}{1+x}$, so:

$$\begin{aligned} x_1 < x_2 \Longrightarrow \quad & x_1 + 1 < x_2 + 1 \\ \Longrightarrow \quad & \frac{-1}{x_1+1} < \frac{-1}{x_2+1} \\ \Longrightarrow \quad & 1 - \frac{1}{x_1+1} < 1 - \frac{1}{x_2+1} \end{aligned}$$

  Therefore, $f(x_1) < f(x_2)$, and f is strictly increasing.

- If $x_1 < x_2 < 0$, in the same way and take $f(x) = \dfrac{x}{1-x} = -1 + \dfrac{1}{1-x}$.

Solution 3

1. $\lim\limits_{x \to +\infty} e^{x - \sin x}$, we have:

$$\forall x \in \mathbb{R}, \qquad -1 \leq \sin x \leq 1$$
$$\Longrightarrow -1 \leq -\sin x \leq 1$$
$$\Longrightarrow x-1 \leq x-\sin x \leq x+1$$

therefore: $x - \sin x \geq x - 1 \Longrightarrow e^{x-\sin x} \geq e^{x-1}$, and because $\lim\limits_{x\to+\infty} e^{x-1} = +\infty$ then $\lim\limits_{x\to+\infty} e^{x-\sin x} = +\infty$.

2. $\lim\limits_{x\to 0} \dfrac{(\tan x)^2}{\cos(2x)-1}$.

   We have $\cos(2x) = 2\cos^2 x - 1$, then

$$\cos(2x) - 1 = 2\cos^2 x - 2 = -2(1-\cos^2 x) = -2\sin^2 x.$$

   So

$$\frac{(\tan x)^2}{\cos(2x)-1} = \frac{\frac{\sin^2 x}{\cos^2 x}}{-2\sin^2 x} = \frac{-\sin^2 x}{2\cos^2 x \sin^2 x} = \frac{-1}{2\cos^2 x}$$

   whene $x \longrightarrow 0$ then $\cos^2 x \longrightarrow 1$, therefore, $\lim\limits_{x\to 0}\dfrac{\tan^2 x}{\cos(2x)-1} = \dfrac{-1}{2}$.

3. $\lim\limits_{x\to 0^+} \dfrac{x}{b}\left[\dfrac{c}{x}\right]$. We have:

$$\left[\frac{c}{x}\right] \leq \frac{c}{x} \leq \left[\frac{c}{x}\right] + 1$$
$$\Longrightarrow \frac{x}{b}\left[\frac{c}{x}\right] \leq \frac{x}{b}\frac{c}{x} \leq \frac{x}{b}\left[\frac{c}{x}\right] + \frac{x}{b}$$
$$\Longrightarrow 0 \leq \frac{c}{b} - \frac{x}{b}\left[\frac{c}{x}\right] \leq \frac{x}{b}$$

   $\lim\limits_{x\to 0}\dfrac{x}{b} = 0 \Longrightarrow \lim\limits_{x\to 0^+}\dfrac{c}{b} - \dfrac{x}{b}\left[\dfrac{c}{x}\right] = 0$, so $\lim\limits_{x\to 0^+}\dfrac{x}{b}\left[\dfrac{c}{x}\right] = \dfrac{c}{b}$.

4. $\lim\limits_{x\to 0} \dfrac{\ln(1+x^2)}{\sin^2 x}$. We use the L'Hpital's rule, we set $f(x) = \ln(1+x^2)$, and $g(x) = \sin^2 x$, then: $f'(x) = \dfrac{2x}{1+x^2}$, and $g'(x) = 2\sin x\cos x$.

   $\dfrac{f'(x)}{g'(x)} = \dfrac{x}{\sin x}.\dfrac{1}{\cos x(1+x^2)}$, we note that $\lim\limits_{x\to 0}\dfrac{x}{\sin x} = 1$ $\left(\lim\limits_{x\to 0}\dfrac{\sin x}{x} = 1\right)$, and $\lim\limits_{x\to 0}\dfrac{1}{(1+x^2)\cos x} = 1$, so $\lim\limits_{x\to 0}\dfrac{\ln(1+x^2)}{\sin^2 x} = 1$.

5. $\lim_{x\to 0} \frac{\sqrt{1+x}-\sqrt{1-x}}{x}$. we have:

$$\begin{aligned}\lim_{x\to 0} \frac{\sqrt{1+x}-\sqrt{1-x}}{x} &= \lim_{x\to 0} \frac{(1+x)-(1-x)}{x(\sqrt{1+x}-\sqrt{1-x})} \\ &= \lim_{x\to 0} \frac{2x}{x(\sqrt{1+x}-\sqrt{1-x})} \\ &= 1.\end{aligned}$$

6. $\lim_{x\to+\infty} \frac{x\ln x+5}{x^2+4} = \lim_{x\to+\infty} \frac{x\ln x\left(1+\frac{5}{x\ln x}\right)}{x^2\left(1+\frac{4}{x^2}\right)} = \lim_{x\to+\infty} \frac{\ln x}{x}\left(\frac{1+\frac{5}{x\ln x}}{1+\frac{4}{x^2}}\right) = 0.$

Solution 4

1. We have:

$$\begin{aligned} f:\mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longrightarrow f(x) = \begin{cases} \frac{\sin ax}{x} & : x<0 \\ 1 & : x=0 \\ 2be^x - x & : x>0 \end{cases}\end{aligned}$$

we note that for $x>0$, and $x<0$ the function f is continuous. For f to be continuous on $\mathbb{R}$, it must be continuous on the right and left of 0.

we have $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} 2be^x - x = 2b = f(0) = 1$, so $b = \frac{1}{2}$.

And $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{\sin ax}{x} = a \lim_{x\to 0^-} \frac{\sin ax}{ax} = a = f(0) = 1$, so $a = 1$.

2. $g(x) = \begin{cases} \sqrt{x} - \frac{1}{x}, & x \geq 4 \\ (x+a)^2, & x<4 \end{cases}$

For the function g to be continuous on $\mathbb{R}$, it is enough to study the continuity at point 4.

$\lim_{x\to 4^+} g(x) = \lim_{x\to 4^+} \sqrt{x} - \frac{1}{x} = \frac{7}{4}.$

$\lim_{x\to 4^-} g(x) = \lim_{x\to 4^-} (x+a)^2 = (4+a)^2.$

g is continuous in 4, i.e.

$$\lim_{x\to 4^+} g(x) = \lim_{x\to 4^-} g(x) \Leftrightarrow (4+a)^2 = \frac{7}{4} \Leftrightarrow |4+a| = \frac{\sqrt{7}}{2}.$$

$$\Longleftrightarrow \begin{cases} 4+a & = \dfrac{\sqrt{7}}{2} \\ -4-a & = \dfrac{\sqrt{7}}{2} \end{cases} \Longleftrightarrow \begin{cases} a & = \dfrac{\sqrt{7}}{2} - 4 \\ a & = \dfrac{-\sqrt{7}}{2} - 4 \end{cases}$$

Solution 5

1. $f(x) = \begin{cases} x + \dfrac{\sqrt{x^2}}{x} & : \ x \neq 0 \\ 0 & : \ x = 0 \end{cases}$

   We note that the function f is continuous on $\mathbb{R}^*$, for the continuity at 0 we have:

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (x+1) = 1.$$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (x-1) = -1.$$

   $\lim_{x\to 0^+} f(x) \neq \lim_{x\to 0^-} f(x)$, so f is not continuous at 0.

2. $g(x) = \begin{cases} 1 + x\cos\left(\dfrac{1}{x}\right) & : \ x \neq 0 \\ 0 & : \ x = 0 \end{cases}$

   the function g is continuous on $\mathbb{R}^*$.

$$\lim_{x\to 0} g(x) = \lim_{x\to 0} \left(1 + x\cos\left(\frac{1}{x}\right)\right) = 1.$$

   because $\lim_{x\to 0} x\cos\left(\dfrac{1}{x}\right) = 0$ $\left(0 < \left|x\cos\left(\dfrac{1}{x}\right)\right| < |x|\right)$. Since $\lim_{x\to 0} f(x) = 1 \neq 0 = f(0)$, then g is not continuous at 0.

Solution 6

1. $f(x) = \dfrac{x^3+2x+3}{x^3+1}$, $D_f = \mathbb{R} - \{-1\}$.

   f is continuous on D_f, as f is a quotient of two continuous polynoms. We note that (-1) is a root of the numerator too so on D_f we have

$$f(x) = \frac{(x+1)(x^2-x+3)}{(x+1)(x^2-x+1)} = \frac{(x^2-x+3)}{(x^2-x+1)}$$

   so $\lim\limits_{x\to-1} f(x) = \lim\limits_{x\to-1} \dfrac{x^2-x+3}{x^2-x+1} = 3$ (exist), then f admits an extension by continuity at the point (-1) given by:

$$\tilde{f}(x) = \begin{cases} f(x) & : x \neq -1 \\ 3 & : x = -1 \end{cases}$$

2. $g(x) = \dfrac{(1+x)^n - 1}{x}$, $D_g = \mathbb{R}|\{0\}$.

   - If $n = 0$, then $g(x) = 0$, so $\lim\limits_{x\to 0} g(x) = 0$, and g admits an extension by continuity on $\mathbb{R}$ given by $\tilde{g} = 0$.

   - If $n \geq 1$, we use the Newton binomial formula

$$(1+x)^n = \sum_{k=0}^{n} C_n^k x^k \, 1^{n-k} = 1 + C_n^1 x + C_n^2 x^2 + \cdots + C_n^n x^n.$$

     such that $C_n^k = \dfrac{n!}{k!(n-k)!}$, $C_n^1 = n$, $C_n^2 = \dfrac{n(n-1)}{n}, \cdots, C_n^n = 1$.
     So $g(x) = \dfrac{1}{x}\left[C_n^1 x + C_n^2 x^2 + \cdots + C_n^n x^n\right] = C_n^1 x + C_n^2 x + \cdots + C_n^n x^{n-1}$, and $\lim\limits_{x\to 0} g(x) = C_n^1 = n$ (exist), then g admits extension by continuity on $\mathbb{R}$ given by:

$$\tilde{g}(x) = \begin{cases} g(x) = \sum\limits_{k=1}^{n} C_n^k x^{k-1} & : x \neq 0 \\ n & : x = 0 \end{cases}$$

Solution 7

1. Let $p > 0$ such that $\forall x \in \mathbb{R},\ f(x+p) = f(x)$. By induction we can show

$$\forall n \in \mathbb{N} :\ \forall x \in \mathbb{R}\ f(x+np) = f(x).$$

since f is not constant, then $\exists\ a,\ b \in \mathbb{R}$ such that $f(a) \neq f(b)$. We denote $x_n = a+np$ and $y_n = b+np$, assume that f has a limit in $+\infty$, since $x_n \longrightarrow \infty$ then $f(x_n) \longrightarrow l$, but $f(x_n) = f(a+np) = f(a)$, so $l = f(a)$.

Likewise with the sequence (y_n), $y_n \longrightarrow \infty$ then $f(y_n) \longrightarrow l$, and $f(y_n) = f(b+np) = f(b)$, so $l = f(b)$.

Because $f(a) \neq f(b)$ we get a contradiction.

2. We consider the function $g(x) = f(x) - x$ on $[0, +\infty[$. g is continuous, and $g(0) = f(0) > 0$.

$\lim\limits_{x\to+\infty} g(x) = \lim\limits_{x\to+\infty} (f(x) - x) = \lim\limits_{x\to+\infty} x \left(\frac{f(x)}{x} - 1 \right) = -\infty$. (because $\lim\limits_{x\to+\infty} \left(\frac{f(x)}{x} \right) = a$, and $a - 1 < 0$).

So $\exists\ b \in \mathbb{R}_+^*$ such that $g(b) < 0$ (also $g(x) < 0$ if $x \geq b$) on $[0, b]$. We have g is continuous and $g(0) > 0,\ g(b) < 0$, according to the intermediate value theorem: $\exists\ x_0 \in [0, b]$ such that $g(x_0) = 0$, so $f(x_0) = x_0$.

Chapter 7

Exercises in: The Differentiable Functions

Exercise 1

Study the differentiability of the function f at the point x_0 in the following cases:

1. $f(x) = \begin{cases} x^2 \cos \dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}, \; x_0 = 0.$

2. $f(x) = \begin{cases} \sin x \sin \dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}, \; x_0 = 0.$

3. $f(x) = \begin{cases} \exp(\frac{1}{x^2 - a^2}), & |x| < a \\ 0, & |x| \geq a \end{cases}, \; |x_0| = a, \; a \in \mathbb{R}_+$

Exercise 2

Let the function f be defined on $\mathbb{R}_+$ by:

$$f(x) = \begin{cases} ax^2 + bx + 1, & 0 \leq x \leq 1 \\ \sqrt{x}, & x > 1 \end{cases}$$

Determine the real numbers a and b so that f is differentiable on $\mathbb{R}_+$. Calculate $f'(x)$.

Exercise 3

1. Calculate the derivatives of the following functions:

   (a) $y_1(x) = \sqrt{\ln x + 1} + \ln(\sqrt{x} + 1)$.

   (b) $y_2(x) = \dfrac{\sqrt{\cos x}}{1 - e^x}$.

   (c) $y_3(x) = e^{\cos \sqrt{x}}$.

2. Calculate the $n - th$ derivatives of the following functions:

   (a) $y_1(x) = \ln(1 + x)$.

   (b) $y_2(x) = \dfrac{1 + x}{1 - x}$.

   (c) $y_3(x) = (x + 1)^3 e^{-x}$.

   (d) $y_4(x) = x^2 \sin 3x$.

Exercise 4

Determine the extrema of the following functions:

1. $f(x) = \sin x^2$, on $[0, \pi]$.

2. $g(x) = x^4 - x^3 + 1$, on $\mathbb{R}$.

Exercise 5

1. Can we apply Rolle's theorem to the following functions?

   (a) $f(x) = \sin^2 x$, on $[0, \pi]$.

(b) $g(x) = \dfrac{\sin x}{2x}$, on $[-\dfrac{\pi}{2}, \dfrac{\pi}{2}]$.

2. Show that $\forall x, y \in \mathbb{R}_+^*,\ 0 < x < y :\ x < \dfrac{y-x}{\ln y - \ln x} < y$

Exercise 6

Using l'Hopital's theorem, calculate the following limits:

1. $\lim\limits_{x\to 0} \dfrac{1-\cos x}{e^x - 1}$.

2. $\lim\limits_{x\to \pi} \dfrac{\sin x}{x^2 - \pi^2}$.

3. $\lim\limits_{x\to 1} \dfrac{e^{x^2+x} - e^{2x}}{\cos(\frac{\pi}{2}x)}$.

Chapter 8

Solution to exercises in: The Differentiable Functions

Solution 1

1. $f(x) = \begin{cases} x^2 \cos \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$, $x_0 = 0.$
   we have:

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{x^2 \cos \frac{1}{x}}{x - 0} = \lim_{x \to 0} x \cos \frac{1}{x} = 0$$

   because $\left(-x \leq x \cos \frac{1}{x} \leq x, \ \textit{and} \ \lim_{x \to 0} x = 0 \right)$. So the function f is differentiable in x_0 and $f'(0) = 0$.

2. $f(x) = \begin{cases} \sin x \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$, $x_0 = 0$. we have:

   $\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{\sin x \sin \frac{1}{x}}{x - 0} = \lim_{x \to 0} \frac{\sin x}{x} \sin \frac{1}{x} = \lim_{x \to 0} \sin \frac{1}{x}$, does not exist.

   $\left(\lim_{x \to 0} \frac{\sin x}{x} = 1 \right)$, therefore f is not differentiable at $x_0 = 0$.

3. $f(x) = \begin{cases} \exp(\frac{1}{x^2 - a^2}), & |x| < a \\ 0, & |x| \geq a \end{cases}$, $|x_0| = a, \ a \in \mathbb{R}_+$.
   We have

$$f(x) = \begin{cases} \exp(\frac{1}{x^2 - a^2}), & -a < x < a \\ 0, & x \in \left]-\infty, -a\right] \cup \left[a, +\infty\right[\end{cases}$$

the differentiability of f in x_0=a: $\lim\limits_{x\to a^+} \dfrac{f(x)-f(a)}{x-a} = \lim\limits_{x\to a^+} \dfrac{0-0}{x-a} = 0 = f'_r(a)$

$\lim\limits_{x\to a^-} \dfrac{f(x)-f(a)}{x-a} = \lim\limits_{x\to a^-} \dfrac{\exp(\frac{1}{x^2-a^2})-0}{x-a} = 0 = f'_l(a)$

We have $f'_r(a) = f'_l(a)$, then f is differentiable at $x_0 = a$, and $f'(a) = 0$.

The differentiability of f in $x_0 = -a$:

$\lim\limits_{x\to -a^-} = \dfrac{f(x)-f(-a)}{x+a} = \lim\limits_{x\to -a^-} \dfrac{0-0}{x+a} = 0 = f'_r(-a)$.

$\lim\limits_{x\to -a^+} = \dfrac{f(x)-f(-a)}{x+a} = \lim\limits_{x\to -a^+} \dfrac{\exp(\frac{1}{x^2-a^2})-a}{x+a} = 0 = f'_l(-a)$.

We have: $f'_r(-a) = f'_l(-a)$, then f is differentiable at $x_0 = -a$, and $f'(-a) = 0$.

Solution 2

$$f(x) = \begin{cases} ax^2+bx+1, & 0 \le x \le 1 \\ \sqrt{x}, & x > 1 \end{cases}$$

We determine a and b such that f is differentiable on $\mathbb{R}_+^*$, we have $\sqrt{x}$ is differentiable on $]0,1[$, and ax^2+bx+1 is differentiable on $]1,+\infty[$, so f is differentiable on $]0,1[\cup]1,+\infty[$.

The differentiability of f in $x_0 = 1$: $(f(1) = 1)$

$\lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1^+} \sqrt{x} = 1 = f(1)$

$\lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^-} ax^2+bx+1 = a+b+1$.

f is continuous at $x_0 = 1 \iff \lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1^-} f(x) = f(1) \Longrightarrow a+b+1 = 1 \iff a = -b$. Therefore f is continuous for $a = -b$.

$\lim\limits_{x\to 1^+} \dfrac{f(x)-f(1)}{x-1} = \lim\limits_{x\to 1^+} \dfrac{\sqrt{x}-1}{x-1} = \lim\limits_{x\to 1^+} \dfrac{\sqrt{x}-1}{(\sqrt{x}-1)(\sqrt{x}+1)} = \lim\limits_{x\to 1^+} \dfrac{1}{\sqrt{x}+1} = \dfrac{1}{2} = f'_r(1)$.

$\lim\limits_{x\to 1^-}\dfrac{f(x)-f(1)}{x-1}=\lim\limits_{x\to 1^-}\dfrac{ax^2+bx+1-1}{x-1}=\lim\limits_{x\to 1^-}\dfrac{ax^2-ax}{x-1}=\lim\limits_{x\to 1^-}\dfrac{ax(x-1)}{x-1}=a=$ $f'_l(1)$.

f is differentiable on $x_0=1 \Leftrightarrow f'_r(1)=f'_l(1)\Rightarrow a=\dfrac{1}{2}$, *and* $b=-a=-\dfrac{1}{2}$. So f is differentiable on $x_0=1$ for $a=\dfrac{1}{2}$, and $b=-\dfrac{1}{2}$.

Calculate: $f'(x)$:

$$f'(x)=\begin{cases}\dfrac{1}{2\sqrt{x}} & : \ 0<x\leq 1,\\ x-\dfrac{1}{2} & : \ x>1.\end{cases}$$

Solution 3

1. Calculate derivatives:

- $y_1(x)=\sqrt{\ln x+1}+\ln(\sqrt{x}+1)\Longrightarrow y_1'=\dfrac{1}{2x\sqrt{\ln x+1}}+\dfrac{1}{2(x+\sqrt{x})}$.
- $y_2(x)=\dfrac{\sqrt{\cos x}}{1-e^x}\Longrightarrow y_2'=\dfrac{-\sin x+\sin xe^{-x}-\sqrt{\cos x}e^{-x}}{(1-e^{-x})^2}$.
- $y_3(x)=e^{\cos\sqrt{x}}\Longrightarrow y_3'=\dfrac{-1}{2\sqrt{x}}\sin(\sqrt{x})e^{\cos\sqrt{x}}$.

2. Calculate $n-th$ derivatives:

- $y_1(x)=\ln(1+x)$

$$\begin{aligned} y_1'(x)&=\frac{1}{1+x} & y_1''(x)&=\frac{-1}{(1+x)^2}\\ y_1^{(3)}(x)&=\frac{2}{(1+x)^3} & y_1^{(4)}(x)&=-\frac{2\times 3}{(1+x)^4}\\ y_1^{(5)}(x)&=\frac{2\times 3\times 4}{(1+x)^5} & y_1^{(6)}(x)&=-\frac{2\times 3\times 4\times 5}{(1+x)^6}\ .\end{aligned}$$

$$\vdots$$

$$y_1^{(n)}(x)=\frac{(-1)^{n-1}(n-1)!}{(1+x)^n}$$

- $y_2(x)=\dfrac{1+x}{1-x}$

$$y_2' = \frac{2}{(1-x)^2} \qquad y_2'' = \frac{2\times 2}{(1-x)^3}$$
$$y_2^{(3)} = \frac{2\times 2\times 3}{(1-x)^4} \qquad y_2^{(4)} = \frac{2\times 2\times 3\times 4}{(1-x)^5}$$
$$\vdots$$
$$y_2^{(n)} = \frac{2n!}{(1-x)^{n+1}}$$

- $y_3(x) = (x+1)^3 e^{-x}$

  Assume that $g(x) = (x+1)^3$, and $f(x) = e^{-x}$, so

  $$g'(x) = 3(x+1)^2, \quad g''(x) = 6(x+1), \quad g^{(3)}(x) = 6,\ g^{(n)}(x) = 0,\ \forall n \geq 4$$

  $$f'(x) = -e^{-x}, \quad f''(x) = e^{-x}, \quad f^{(n)}(x) = (-1)^n e^{-x},\ \forall n \in \mathbb{N}$$

  then,

  $$\begin{aligned}(y_3)^{(n)} &= \sum_{k=0}^{3} C_n^k \ (e^{-x})^{(n-k)} \ (1+x)^{(k)} \\ &= \sum_{k=0}^{3} C_n^k \ (-1)^{n-k} \, e^{-x} . \frac{3!}{(3-k)!} \, (1+x)^{(3-k)}\end{aligned}$$

- $y_4(x) = x^2 \sin 3x$, according to Leibniz

  $f^{(n)} = \sum_{k=0}^{n} C_n^k \ (\sin\ 3x)^{(n-k)} \ (x^2)^{(k)}$

  $= 3^n x^2 \sin\left(3x + \frac{n\pi}{2}\right) + 2xn3^{n-1} \sin\left(3x + \frac{(n-1)\pi}{2}\right) + n\,(n-1)\,3^{n-2} \sin\left(3x + \frac{(n-2)\pi}{2}\right)$

Solution 4

Determine the extrema:

x_0 is extremum $\Longleftrightarrow f'(x_0) = 0$ and $f''(x_0) \neq 0$.

1. $f(x) = \sin x^2$, on $[0, \pi]$, $f'(x) = 2x\cos x^2$ the critical points are:

$$f'(x) = 0 \Longleftrightarrow 2x\cos x^2 = 0 \Longleftrightarrow \begin{cases} x = 0 \\ \cos x^2 = 0 \end{cases}$$

therefore, $\begin{cases} x = 0 \\ x^2 = \dfrac{\pi}{2} + k\pi \end{cases} \Longleftrightarrow \begin{cases} x = 0 \\ x = \sqrt{\dfrac{\pi}{2} + k\pi} \end{cases}$

$f''(x) = 2\cos x^2 - 4x^2 \sin x^2$, so:

- Four $x = 0$, $f''(0) = 2 > 0$, then 0 is an extremum (minimum).
- Four $x = \sqrt{\frac{\pi}{2} + k\pi}$, $f''(\sqrt{\frac{\pi}{2} + k\pi}) = -4\left(\frac{\pi}{2} + k\pi\right) \sin\left(\frac{\pi}{2} + k\pi\right) \neq 0$.
  - if k is even: $\sin\left(\frac{\pi}{2} + k\pi\right) = 1$, and $f''(\sqrt{\frac{\pi}{2} + k\pi}) = -4\left(\frac{\pi}{2} + k\pi\right) < 0$, so $\sqrt{\frac{\pi}{2} + k\pi}$ is an extremum (maximum).
  - if k is odd $\sin\left(\frac{\pi}{2} + k\pi\right) = -1$, and $f''(\sqrt{\sin\frac{\pi}{2} + k\pi}) = 4\left(\sqrt{\frac{\pi}{2} + k\pi}\right) > 0$, so $\sqrt{\frac{\pi}{2} + k\pi}$ is an extremum (minimum).

2. $g(x) = x^4 - x^3 + 1$, on $\mathbb{R}$

   $g'(x) = 4x^3 - 3x^2$, the critical points are:

$$g'(x) = 0 \Longleftrightarrow 4x^3 - 3x^2 = 0 \Longleftrightarrow x^2(4x-3) = 0 \Longleftrightarrow \begin{cases} x = 0 \\ x = \dfrac{3}{4} \end{cases}$$

   $g''(x) = 12x^2 - 6x$

   - For $x = \frac{3}{4}$, $g''(\frac{3}{4}) = \frac{9}{4} > 0$, so $\frac{3}{4}$ is an extremum (minimum).
   - For $x = 0$, $g''(0) = 0 \Longrightarrow f^{(3)}(x) = 24x - 6 \Longrightarrow f^{(3)}(0) \neq 0$, so 0 is not an extremum.

Solution 5

1. (a) $f(x) = \sin^2 x$, on $[0, \pi]$

   we have f is continunous on $\mathbb{R}$, so it is continuous on $[0, \pi]$, and differentiable on $]0, \pi[$.

$f(0) = 0$, and $f(\pi) = 0 \Longrightarrow f(0) = f(\pi)$, so we can apply Rolle's theorem on f.

(b) the same for $g(x) = \dfrac{\sin x}{2x}$, on $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

2. We show that $x < \dfrac{y-x}{\ln y - \ln x} < y$. $\forall x,\ y \in \mathbb{R}_+^*$, we apply the Mean value theorem on the function $f(t) = \ln t$ on the iterval $[x, y]$ such that $0 < x < y$.

   $f(t) = \ln t$ is continuous on $[x, y]$, and differentiable on $]x, y[$, then according to the Mean value theorem:

   $\exists\ x \in]x, y[:\ \ f'(c) = \dfrac{f(y) - f(x)}{y - x}$ *so* $\dfrac{1}{c} = \dfrac{\ln y - \ln x}{y - x} \Longrightarrow c = \dfrac{y - x}{\ln y - \ln x}$, and $c \in]x, y[$, then $x < \dfrac{y - x}{\ln y - \ln x} < y$.

Solution 6

1. $\lim\limits_{x \to 0} \dfrac{1 - \cos x}{e^x - 1} = \dfrac{0}{0}$

   $\lim\limits_{x \to 0} \dfrac{1 - \cos x}{e^x - 1} = \lim\limits_{x \to 0} \dfrac{(1 - \cos x)'}{(e^x - 1)'} = \lim\limits_{x \to 0} \dfrac{\sin x}{e^x} = 0.$

2. $\lim\limits_{x \to \pi} \dfrac{\sin x}{x^2 - \pi^2} = \dfrac{0}{0}$

   $\lim\limits_{x \to \pi} \dfrac{\sin x}{x^2 - \pi^2} = \lim\limits_{x \to \pi} \dfrac{(\sin x)'}{(x^2 - \pi^2)'} = \lim\limits_{x \to \pi} \dfrac{\cos x}{2x} = \dfrac{-1}{2\pi}.$

Chapter 9

Exercises in: Elementary Functions

Exercise 1

Consider the function f defined by

$$f(x) = x^x$$

1. On what set is this function defined and continuous?

2. Show that f is extendable by continuity on $[0, \infty[$.

3. Calculating the derivative of f wherever it is not a problem. On what set is f differentiable, what can we deduce about the graph of f at 0?

4. Study the variations of f on $[0, \infty[$. Then calculate the limit of f in ∞.

5. Sketch the graph of f.

Exercise 2

1. Let a and b be two real numbers, show that:

$$ch(a)ch(b) = \frac{1}{2}(ch(a+b) + ch(a-b)).$$

2. Show that $\forall t \in \mathbb{R}$

$$\cos(2t) = \frac{1 - \tan^2(t)}{1 + \tan^2(t)}$$

Exercise 3

Soit $a \in \mathbb{R},\ a > 0$. Solve:

$$\ln(ch(x)) = a.$$

Exercise 4

Calculate limits:

1. $\lim_{x\to+\infty} e^{-x}(ch^3(x) - sh^3(x))$.
2. $\lim_{x\to+\infty} x - \ln(ch(x))$.

Chapter 10

Solutions to exercises in: Elementary Functions

Solution 1

1. We have $f(x) = x^x = e^{x\ln x}$. so f is defined and continuous on $D_f =]0, +\infty[$.

2. We have $\lim\limits_{x\to 0^+} x\ln x = 0$, then $\lim\limits_{x\to 0^+} f(x) = e^0 = 1$.

   In other words f is extendable by continuity to 0 by $f(0) = 1$.

3. $f'(x) = \left(\ln x + x \times \frac{1}{x}\right) e^{x\ln x} = (\ln x + 1)\, e^{x\ln x}$. f is differentiable on $]0, +\infty[$.

   We have $\lim\limits_{x\to 0^+} f'(x) = -\infty$. Therefore f is not differentiable at 0 and the graph of f admits a vertical half-tangent at 0.

4. The sign of the derivative is the same as that of $\ln(x) + 1$.

$$\begin{aligned} \ln(x) + 1 = 0 &\iff \ln(x) = -1 \iff x = e^{-1} = \frac{1}{e} \\ 0 < x < \frac{1}{e} &\iff \ln(x) < \ln\left(\frac{1}{e}\right) = -1 \iff \ln(x) + 1 < 0. \end{aligned}$$

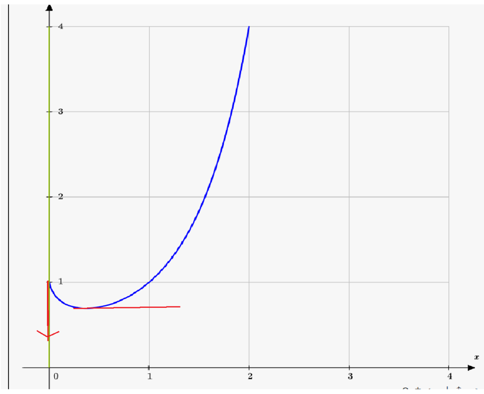
   The same for:

$$x > \frac{1}{e} \Longrightarrow \ln(x) + 1 > 0.$$

   So f is decreasing on $\left]0, \frac{1}{e}\right[$, and it is increasing on $\left]\frac{1}{e}, +\infty\right[$.

   It is clear that $\lim\limits_{x\to +\infty} f(x) = +\infty$.

5. The graphe of f



Solution 2

1. we have, in development

$$
\begin{aligned}
ch(a)\ ch(b) &= \frac{1}{4}\left(e^{a}+e^{-a}\right)\left(e^{b}+e^{-b}\right) = \frac{1}{4}\left(e^{a+b}+e^{-(a+b)}+e^{a-b}+e^{-(a-b)}\right) \\
&= \frac{1}{2}\left(ch(a+b)+ch(a-b)\right).
\end{aligned}
$$

2. We have

$$
\begin{aligned}
\frac{1-\tan^2(t)}{1+\tan^2(t)} &= \frac{1-\dfrac{\sin^2(t)}{\cos^2(t)}}{1+\dfrac{\sin^2(t)}{\cos^2(t)}} \\
&= \frac{\cos^2(t)-\sin^2(t)}{\cos^2(t)+\sin^2(t)} = \cos^2(t)-\sin^2(t) = \cos(2t).
\end{aligned}
$$

Solution 3

$$
\begin{aligned}
\ln(ch(x)) = a &\iff ch(x) = e^{a} \\
&\iff \frac{e^{x}+e^{-x}}{2} = e^{a} \iff e^{x}+\frac{1}{e^{x}} = 2e^{a}
\end{aligned}
$$

We pose: $X = e^{x}$, so

$$X + \frac{1}{X} = 2e^a \Longleftrightarrow X^2 + 1 = 2Xe^a$$
$$\Longleftrightarrow X^2 - 2Xe^a + 1 = 0$$

$\Delta = 4e^{2a} - 4 = 4\left(e^{2a} - 1\right) > 0$, the roots are

$$X_1 = \frac{2e^a - 2\sqrt{e^{2a}-1}}{2} = e^a - \sqrt{e^{2a}-1} \; and \; X_2 = e^a + \sqrt{e^{2a}-1}$$

We note that $e^{2a} > e^{2a} - 1$, hence $e^a > \sqrt{e^{2a}-1}$, which shows that $X_1 > 0$, for X_2 it's obvious. so the solutions of $\ln(ch(x)) = a$ are $x_1 = \ln\left(e^a - \sqrt{e^{2a}-1}\right)$ and $x_2 = \ln\left(e^a + \sqrt{e^{2a}-1}\right)$.

Solution 4

1. $\lim_{x\to+\infty} e^{-x}\left(ch^3(x) - sh^3(x)\right)$. We have

$$e^{-x}\left(ch^3(x) - sh^3(x)\right) = e^{-x}\left[\left(\frac{e^x + e^{-x}}{2}\right)^3 - \left(\frac{e^x - e^{-x}}{2}\right)^3\right]$$

$$= \frac{e^{-x}}{8}\left[e^{3x} + 3e^x + 3e^{-x} + e^{-3x} - \left(e^{3x} - 3e^x + 3e^{-x} - e^{-3x}\right)\right]$$

$$= \frac{e^{-x}}{8}\left(6e^x + 2e^{-3x}\right) = \frac{3}{4} + \frac{1}{4}e^{-4x}$$

so $\lim_{x\to+\infty} e^{-x}\left(ch^3(x) - sh^3(x)\right) = \lim_{x\to+\infty} \frac{3}{4} + \frac{1}{4}e^{-4x} = \frac{3}{4}$.

2.
$$\lim_{x\to+\infty} x\ln(ch(x)) = \lim_{x\to+\infty} x\ln\left(\frac{e^x + e^{-x}}{2}\right) = \lim_{x\to+\infty} x\ln\left(e^x.\frac{1+e^{-2x}}{2}\right)$$

$$= \lim_{x\to+\infty}\left[x - \ln(e^x) - \ln\left(\frac{1+e^{-2x}}{2}\right)\right] = \lim_{x\to+\infty}\left[-\ln\left(\frac{1+e^{-2x}}{2}\right)\right].$$

We have $\lim_{x\to+\infty} \frac{1+e^{-2x}}{2} = \frac{1}{2}$. Hence $\lim_{x\to+\infty} x\ln(ch(x)) = -\ln\left(\frac{1}{2}\right) = \ln(2)$.

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