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Contents

Chapter 1

Taylor's Formula and Limited Development

The limited development $LD_n(x_0)$ is useful in many areas of mathematics and physics, including solving differential equations, performing integrations, evaluating limits, and analyzing the local behavior of a function and its polynomial approximation.

1.1 Taylor's Formula

1.1.1 Taylor's Formula

The Taylor formula allows the approximation of a function differentiable several times in the neighborhood of a point by a polynomial whose coefficients depend only derivatives of the function at this point.

Definition 1.1.1. *A continuous function on $[a, b]$ and differentiable at $x_0 \in]a, b[$ can be written in the neighborhood of x_0 as follows*

$$f(x) = f(x_0) + (x - x_0)f'(x_0) + R(x),$$

where $R(x) = \varepsilon(x)(x - x_0)$, and $\lim_{x \to x_0} \varepsilon(x) = 0$, which shows that, if f is differentiable, then f is approximated by a polynomial of degree 1 (a line).

Example 1.1.2. *Consider the function $f = e^x$, and $x_0 = 0$. f can be written as*

$$f(x) \simeq f(0) + (x-0)f'(0) = x+1$$

Taylor's formula generalizes this result by showing that n times differentiable functions can be approximated in the neighborhood of x_0 by polynomials of degree n, that is to say

$$\begin{aligned} f(x) &= \underbrace{\sum_{k=0}^{n} \frac{f^{(k)(x_0)}}{k!}(x-x_0)^k}_{p_n(x)} + R_n(x), \\ &= f(x_0) + (x-x_0)f'(x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + R_n(x), \end{aligned}$$

where $R_n(x)$ is the remainder of order n, such that

$$\begin{aligned} R_n(x) &= \frac{f^{(n+1)}(x_0)}{(n+1)!}(x-x_0)^{n+1} + \cdots \\ &= \varepsilon(x)(x-x_0)^n \qquad \textit{and} \qquad \lim_{x\to x_0} \varepsilon(x) = 0. \end{aligned}$$

1.1.2 Taylor's Theorem

Let f and $g : [a,b] \longrightarrow \mathbb{R}$ two functions satisfying the following conditions:

1. $f \in C^n([a,b])$, and $f^{(n)}$ is differentiable on $]a,b[$.
2. The function $g \in C([a,b])$ and differentiable on $]a,b[$ and $\forall x \in]a,b[$, $g'(x) \neq 0$, for $x_0 \in [a,b]$: then $\forall x \in [a,b]$, $x \neq x_0$ we have:

$$\begin{aligned} f(x) &= f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + R_n(x_0,x).....(*) \\ &= \textstyle\sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + R_n(x_0,x) \end{aligned}$$

such that $R_n(x_0,x) = \dfrac{f^{(n+1)}(c)(x-c)^n \; g(x)g(x_0)}{n! \; g'(c)}, \quad c \in]x,x_0[.....(**)$

- The expression (*) is called the Taylor formula with generalized remainder (**).
- The choice of various functions g checking the condition (**) leads to various forms of the remainder $R_n(x_0,x)$.

1.1.3 Taylor's three formulas

Notation 1.1.3. *Let $I = [a, b]$ be an interval of* $\mathbb{R}$, x_0 *be a point interior to* I, *and* $f : I \longrightarrow R$ *be a function. We fix a natural number* n.

We say that a function is of class C^n *on* I *if it is* n *times differentiable on* I, *and if its* n*-th derivative is continuous on* I.

Taylor-Lagrange

Theorem 1.1.4. *Let* f *be of class* C^{n+1} *on* I, *and* $x_0 \in [a, b]$. *For all* $x \in [a, b]$, $x \neq x_0$, *we have:*

$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}, \quad c \in]x, x_0[.$

The term $\frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}$ *is called the Lagrange remainder.*

Example 1.1.5.

1. *Consider the function* $\sin(x)$. *The* ***Taylor-Lagrange*** *formula of order* 3 *in the neighborhood of* 0 *is written*

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^4}{4!}\cos(c).$$

2. *Consider again* $x \longrightarrow e^x$. *The* ***Taylor-Lagrange*** *formula of order 4 in the neighborhood of 0 is written*

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!}\, e^c.$$

Taylor-Maclaurin

If $x_0 = 0$ in Taylor Lagrange's formula we obtain the Maclaurin formula

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \frac{f^{(n+1)}(\theta x)}{(n+1)!}x^{n+1}, \quad 0 < \theta < 1.$$

Taylor-Young

Theorem 1.1.6. *Let* $f : [a, b] \longrightarrow \mathbb{R},\ x_0 \in [a, b]$*, suppose that* $f^{(n)}(x_0)$ *exists and finitely, then* $\forall x_0 \in [a, b]$

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + o(x - x_0)^n,$$

the remainder $R_n(x_0, x) = o(x - x_0)^n$ *is called Young's remainder, which posits the following property:*

$$\lim_{x \to x_0} \frac{R_n(x_0, x)}{(x - x_0)^n} = 0.$$

By setting $\varepsilon(x) = \dfrac{R_n(x_0, x)}{(x - x_0)^n}$ *for* $x \neq x_0$ *and* $\varepsilon(x_0) = 0$ *we obtain the Young remainder in the following form:* $R_n(x_0, x) = \varepsilon(x)(x - x_0)^n$ *where* $\lim_{x \to x_0} \varepsilon(x) = 0$

If $x_0 = 0$*, we obtain the Maclaurin-Young formula*

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}x^k + x^n \varepsilon(x) \quad \text{where} \quad \lim_{x \to 0} \varepsilon(x) = 0$$

Example 1.1.7. *The Taylor-Young formula for the function* $\sin(x)$ *at order* $2n + 1$ *at 0 is written*

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n + 1)!} + x^{2n+1}\varepsilon(x).$$

Indeed, we must calculate the successive derivatives of $\sin(x)$ *at 0. We have*

$$\sin(0) = 0,\ \sin'(0) = \cos(0) = 1, \quad \sin''(0) = -\sin(0) = 0, \cdots$$

More generally, for all $k \in \mathbb{N}$ *we have*

$$\sin^{(2k)}(0) = 0, \quad \text{and} \quad \sin^{(2k+1)}(0) = (-1)^k \cos(0) = (-1)^k$$

hence the result.

1.2 Limited Development

1.2.1 Developments limited to the neighborhood of 0

Definition 1.2.1. *Let f be a function defined in the neighborhood of $x = 0$, except perhaps at 0. We say that f admits a limited development of order n in the neighborhood of 0 if there exist numbers $a_0,\ a_1, a_2, \cdots, a_n \in \mathbb{R}$ and a function ε such that for any non-zero element x of an interval I of $\mathbb{R}$:*

$$\begin{aligned} f(x) &= a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + x^n\varepsilon(x) \\ &= P_n(x) + x^n\varepsilon(x) \end{aligned},$$

such that $\lim\limits_{x\to 0}\varepsilon(x) = 0$.

Remark 1.2.2. *The polynomial $P_n(x)$ is called regular part of the limited development and $x^n\varepsilon(x)$ is remainder or complementary part.*

Example 1.2.3. *Let $f = \dfrac{1}{1-x}$. f admits $LD_n(0)$, indeed:*
Since $1 - x^{n+1} = (1-x)(1 + x + x^2 + \cdots + x^n)$, we have

$$\frac{1}{1-x} - \frac{x^{n+1}}{1-x} = \frac{1-x^{n+1}}{1-x} = \frac{(1-x)(1+x+\cdots+x^n)}{1-x} = 1 + x + \cdots + x^n,$$

where

$$\frac{1}{1-x} = 1 + x + \cdots + x^n + \frac{x^{n+1}}{1-x} = 1 + x + \cdots + x^n\frac{x}{1-x},$$

Therefore the function $f(x) = \dfrac{1}{1-x},\ \ x \neq 1$ admits a limited development of order n at $x = 0$, with $\varepsilon(x) = \dfrac{x}{1-x}$, where $\lim\limits_{x\to 0}\varepsilon(x) = 0$.

1.2.2 Properties of Limited Development

- If f admits a $LD_n(x_0)$, then $\lim\limits_{x\to x_0} f(x)$ exists, finite and is equal to a_0. This criterion is generally used to demonstrate that a function does not admit $LD_n(x_0)$.

  For example the function $\ln(x)$ does not admit $LD_n(0)$, because $\lim\limits_{x\to 0}\ln(x) = -\infty$.

- A function does not necessarily have an $LD_n(x_0)$, but if it exists, then it is unique.

- **Parity**

  - **Even function** The $LD_n(x_0)$ of an even function has a main part that contains only monomials of even degree. That is to say the coefficients $a_{2k+1} = 0$.

  - **Odd function** The $LD_n(x_0)$ of an odd function has a main part that contains only monomials of odd degree. That is to say the coefficients $a_{2k} = 0$.

- The $LD_n(x_0)$ of a polynomial of degree n is itself.

1.2.3 Obtaining Limited Development Using the Taylor-Young Formula

Theorem 1.2.4. *If f is class C^{n-1} in a neighborhood of a and if $f^{(n)}(a)$ exists, then the function f has a limited expansion in a neighborhood of a, of order n. This limited development is given by the formula of Taylor-Young*

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + (x-a)^n\varepsilon(x)$$

where $\lim_{x\to a}\varepsilon(x) + 0.$

particular case: If $f^{(n)} = (0)$ exists then f has the following limited development

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + x^n\varepsilon(x), \quad \textit{such that} \quad \lim_{x\to 0}\varepsilon(x) + 0.$$

Corollary 1.2.5. *If $f^{(n)}(0)$ exists and if f admits a limited expansion of order n*

$$f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + x^n\varepsilon(x),$$

then $f(0) = a_0, \quad \frac{f'(0)}{1!} = a_1, \quad \frac{f''(0)}{2!} = a_2, \cdots, \frac{f^{(n)}(0)}{n!} = a_n.$

1.2.4 Limited Development of usual Functions

Below, we show some very famous limited development of common function of order n, at $x = 0$ using Maclaurin's formula :

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + o(x^n)$$

$$\ln(x1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n+1}\frac{x^n}{n} + o(x^n)$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^n + o(x^n)$$

$$\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} - \cdots + (-1)^{n-1}\frac{1 \times 3 \times 5 \times \cdots \times (2n-3)}{2^n n!}x^n + o(x^n)$$

$$\frac{1}{\sqrt{1+x}} = 1 - \frac{x}{2} + \frac{3x^2}{8} - \cdots + (-1)^n\frac{1 \times 3 \times 5 \times \cdots \times (2n-1)}{2^n n!}x^n + o(x^n)$$

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \cdots + \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}x^n + o(x^n)$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + (-1)^n\frac{x^{2n}}{(2n)!} + o(x^{2n+1})$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + (-1)^n\frac{x^{2n+1}}{(2n+1)!} + o(x^{2n+2})$$

Remark 1.2.6. *We will often work at $x_0 = 0$, based on changes of variables:*

1. *If $x_0 \in \mathbb{R}^*$, we put $t = x - x_0$, and then $t \longrightarrow 0$ when $x \longrightarrow x_0$.*
2. *If $x_0 \longrightarrow \infty$, we put $t = \frac{1}{x}$, and then $t \longrightarrow 0$ when $x \longrightarrow \infty$.*

Example 1.2.7. *Find $LD_3(\frac{\pi}{4})$ for the function $x \longrightarrow \sin(x)$.*
We put $t = x - \frac{\pi}{4}$, then $t \to 0$ when $x \to \frac{\pi}{4}$. Thus, $x = t + \frac{\pi}{4}$.
So

$$\begin{aligned}
f(x) &= \sin x = \sin(t+\frac{\pi}{4}) = \sin(t)\cos(\frac{\pi}{4}) + \cos(t)\sin(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}\sin(t) + \frac{\sqrt{2}}{2}\cos(t).\\
&= \frac{\sqrt{2}}{2}\left(t - \frac{t^3}{6} + o(t^3)\right) + \frac{\sqrt{2}}{2}\left(1 - \frac{t^2}{2} + o(t^3)\right) = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}t - \frac{\sqrt{2}}{4}t^2 - \frac{\sqrt{2}}{12}t^3 + o(t^3)\ .\\
&= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right) - \frac{\sqrt{2}}{4}\left(x - \frac{\pi}{4}\right)^2 - \frac{\sqrt{2}}{12}\left(x - \frac{\pi}{4}\right)^3 + o\left(\left(x - \frac{\pi}{4}\right)^3\right)
\end{aligned}$$

Example 1.2.8. *Find $LD_n(1)$ for the function $x \longrightarrow e^x$.*

We put $t = x - 1$, then $t \to 0$ when $x \to 1$. Thus, $x = t + 1$. Thus

$$\begin{aligned}
e^x &= e\left(1 + y + \frac{y^2}{2!} + \cdots + \frac{y^n}{n!} + o(y^n)\right)\\
&= e\left(1 + (x-1) + \frac{(x-1)^2}{2!} + \cdots + \frac{(x-1)^n}{n!} + o((x-1)^n)\right)
\end{aligned}$$

1.2.5 Operation on Limited Development

- **Sum:** If f admits a $LD_n(0)$: $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + o(x^n)$,

  and g admits a $LD_n(0)$: $g(x) = b_0 + b_1x + b_2x^2 + \cdots + b_nx^n + o(x^n)$.

  Then $f + g$ admits a $LD_n(0)$, which is given by the sum of the two limited development:

$$(f+g)(x) = (a_0+b_0) + (a_1+b_1)x + (a_2+b_2)x^2 + \cdots + (a_n+b_n)x^n + o(x^n).$$

Example 1.2.9. *Find the $LD_4(0)$ of $ln(1+x) + e^x$.*

As

$$\begin{aligned}
\ln(x+1) &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + o(x^4)\\
e^x &= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + o(x^4)
\end{aligned}$$

Hence: $ln(1+x) + e^x = 1 + 2x + \frac{x^3}{2} - \frac{5x^4}{24} + o(x^4)$

- **Product:** If f admits a $LD_n(0)$: $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + o(x^n)$,

  and g admits a $LD_n(0)$: $g(x) = b_0 + b_1x + b_2x^2 + \cdots + b_nx^n + o(x^n)$.

  Then fg admits a $LD_n(0)$, obtained by keeping only the monomials of degree n or less in the product: $(a_0 + a_1x + a_2x^2 + \cdots + a_nx^n)(b_0 + b_1x + b_2x^2 + \cdots + b_nx^n)$.

  Example 1.2.10. *Find $LD_3(0)$ of $x \longrightarrow \sin(x)\cos(x)$.*

  We have

$$\begin{aligned} \cos(x) &= 1 - \frac{x^2}{2} + o(x^3) \\ \sin(x) &= x - \frac{x^3}{6} + o(x^3) \end{aligned}$$

  Then, we developing the product, only considering terms of order 3 or less:

$$\begin{aligned} \cos(x)\sin(x) &= \left(1 - \frac{x^2}{2} + o(x^3)\right)\left(x - \frac{x^3}{6} + o(x^3)\right) \\ &= x - \frac{2x^3}{3} + o(x^3) \end{aligned}$$

- **Quotient:** If f admits a $LD_n(0)$: $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + o(x^n)$,

  and g admits a $LD_n(0)$: $g(x) = b_0 + b_1x + b_2x^2 + \cdots + b_nx^n + o(x^n)$, with $b_0 \neq 0$.

  Then $\dfrac{f}{g}$ admits a $LD_n(0)$, obtained by the division according to the increasing degrees to order n of the polynomial $(a_0 + a_1x + a_2x^2 + \cdots + a_nx^n)$ by the polynomial $(b_0 + b_1x + b_2x^2 + \cdots + b_nx^n)$.

  Example 1.2.11. *Let us compute $LD_5(0)$ for $x \longrightarrow \tan(x) = \dfrac{\sin(x)}{\cos(x)}$*

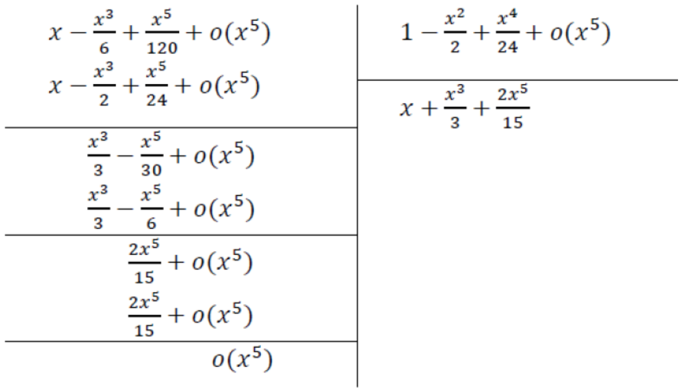
  We have

$$\begin{aligned} \sin(x) &= x - \frac{x^3}{6} + \frac{x^5}{120} + o(x^5) \\ \cos(x) &= 1 - \frac{x^2}{2} + \frac{x^4}{24} + o(x^5) \end{aligned}$$

  Thus,

$$\tan(x) = \frac{\sin(x)}{\cos(x)} = \frac{x - \frac{x^3}{6} + \frac{x^5}{120} + o(x^5)}{1 - \frac{x^2}{2} + \frac{x^4}{24} + o(x^5)}$$

  Then, we developing the division according to the increasing degrees to order 5:



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Therefore, $\tan(x) = x + \dfrac{x^3}{3} + \dfrac{2x^5}{15} + o(x^5)$

- **Composition** If f admits a $LD_n(g(0))$:

$$f(x) = a_0 + a_1(x - g(0)) + a_2(x - g(0))^2 + \cdots + a_n(x - g(0))^n + (x - g(0))^n\varepsilon(x),$$

  and g admits a $LD_n(0)$: $g(x) = b_0 + b_1x + \cdots + b_nx^n + x^n\varepsilon(x)$.

  Then, $f \circ g$ admits a $LD_n(0)$, obtained by replacing the limited development of g in that of f and keeping only the monomials of degree n or less.

Example 1.2.12. *Let us compute* $LD_3(0)$ *for* $x \longrightarrow \sin\left(\dfrac{1}{1-x} - 1\right)$.

Since,

$$\begin{aligned} \frac{1}{1-x} - 1 &= -x + x^2 - x^3 + o(x^3) \\ \sin(x) &= x - \frac{x^3}{6} + o(x^3) \end{aligned}$$

Then, we compose, only considering terms of order 3 or less:

$$\begin{aligned} \sin\left(\frac{1}{1-x} - 1\right) &= -x + x^2 - x^3 - \frac{1}{6}(-x^3) + o(x^3) \\ &= -x + x^2 - \frac{5x^3}{6} + o(x^3) \end{aligned}$$

- **Differentiability:** If $f : I \longrightarrow \mathbb{R}$ admits a $LD_{n+1}(0)$ and f is differentiated at least $n+1$ times, then f' admits a $LD_n(0)$, obtained by deriving the limited development of f.

Example 1.2.13. *compute* $LD_3(0)$ *for* $x \longrightarrow \dfrac{1}{(1-x)^2}$.
Since $\dfrac{1}{(1-x)^2} = \left(\dfrac{1}{1-x}\right)'$, *and* $\dfrac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + o(x^4)$. *Derive the* $LD_4(0)$ *of* $\dfrac{1}{1-x}$, *we obtain* $LD_3(0)$ *for* $\dfrac{1}{(1-x)^2}$:

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + o(x^3).$$

- **Integration:** If $f : I \longrightarrow \mathbb{R}$ admits a $LD_n(0)$, and f is integrable on I, then f admits a $LD_{n+1}(0)$, obtained by integrating the limited development of f. i.e: if

$$f(x) = a_0 + a_1 x + \cdots + a_n x^n + x^n \varepsilon(x), \text{ where } \lim_{x \to 0} \varepsilon(x) = 0$$

then

$$F(x) = \int_0^x f(t)dt = a_0 x + \frac{a_1}{2} x^2 + \cdots + \frac{a_n}{n+1} x^{n+1} + x^{n+1}\tau(x), \text{ where } \lim_{x \to 0} \tau(x) = 0.$$

1.2.6 Generalized LD

Let f be a function defined in the neighborhood of 0 except perhaps at 0. We suppose that f does not admit a limited expansion to the neighborhood of 0 but the function $x^\alpha f(x)$ (α positive real) admits a limited development to the neighborhood of 0 then for $\alpha \neq 0$

$$x^\alpha f(x) = a_0 + a_1 x + \cdots + a_n x^n + o(x^n)$$

Hence $f(x) = \dfrac{1}{\alpha}\left(a_0 + a_1 x + \cdots + a_n x^n + o(x^n)\right)$.
this expression is called generalized limited development in the neighborhood of 0.

Example 1.2.14. *Consider the function* $f(x) = \dfrac{1}{x - x^2}$. f *does not admit an expansion* $LD(0)$ *because* $\lim_{x \to 0} f(x) = +\infty$. *But*

$$xf(x) = x.\frac{1}{x - x^2} = \frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + o(x^n).$$

The generalized limited expansion of f *is*

$$\begin{aligned} f(x) &= \frac{1}{x}\left(1 + x + x^2 + \cdots + x^n + o(x^n)\right) \\ &= \frac{1}{x} + 1 + x + \cdots + x^{n-1} + o(x^{n-1}) \end{aligned}$$

Exercise: Find the $LD_4(0)$ of the following functions:

1. $f(x) = \dfrac{x}{\sin(x)}$.
2. $g(x) = \dfrac{1}{\cos(x)}$.
3. $h(x) = \dfrac{\ln(1+x)}{1+x}$.
4. $k(x) = e^{\cos(x)}$

Chapter 2

Integral Calculus

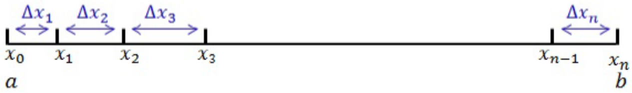
Calculus is built on two operations "dierentiation" and "integration". The integration at its most basic, allows us to analyze the area under a curve. Of course, its application and importance extend far beyond areas and it plays a central role in solving dierential equations.

2.1 The Definite Integrals

2.1.1 Partition

Definition 2.1.1. *Let f be a function defined and bounded on $[a, b]$. A set $P = (x_0, x_1, x_2, \cdots, x_n)$ is called a partition of a closed interval $[a, b]$ if for any positive integer n,*

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b.$$



Notes:

- The division of the interval $[a, b]$ by the partition P generates n subintervals:

$$[x_0, x_1], [x_1, x_2], [x_2, x_3], \cdots, [x_{n-1}, x_n]$$

.

- The length of each subinterval $[x_{k-1}, x_k]$ is $\Delta x_k = x_k - x_{k-1}$.
- The union of subintervals gives the whole interval $[a, b]$.
- The strictly positive real $\delta(P) = \max(x_k - x_{k-1})$ is the maximum length of a subintervals.

2.1.2 Definition of Darboux Sums

Definition 2.1.2. *Let f be a bounded function defined on a closed bounded interval $[a, b]$, and P is a partition of $[a, b]$. Let m_k and M_k be the infinumum and the supremum of f over $[x_{k-1}, x_k]$*

$$m_k = \inf_{x\in[x_{k-1},x_k]} f(x)$$
$$M_k = \sup_{x\in[x_{k-1},x_k]} f(x)$$

Consider the lower and upper Darboux sums of f corresponding to a partition P:

$$L(f, P) = \sum_{k=1}^{n} m_k(x_k - x_{k-1})$$

$$U(f, P) = \sum_{k=1}^{n} M_k(x_k - x_{k-1})$$

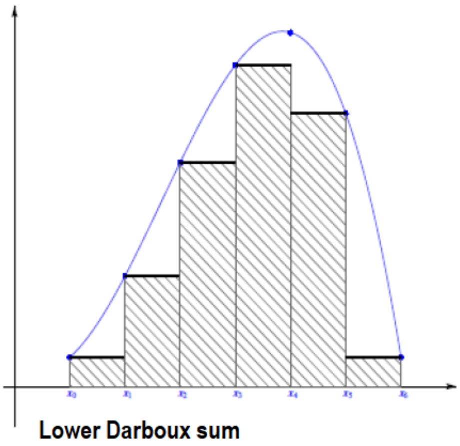
Definition 2.1.3. ***(Upper and Lower Integrals)***

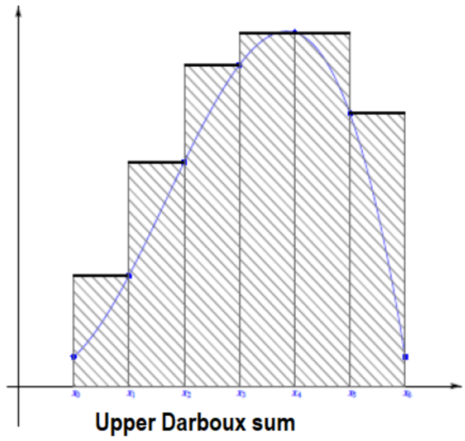
Suppose $f : [a, b] \longrightarrow \mathbb{R}$ is bounded. Then we define the lower integral of f on $[a, b]$ as:

$$\int_{*a}^{b} f(x)dx = \sup(L),\ \textit{where}\ L = \{L(f, P)|P\ \textit{ is a partition of }\ [a, b]\},$$

and the upper integral of f on $[a, b]$ as:

$$\int_{a}^{*b} f(x)dx = \inf(U),\ \textit{where}\ U = \{U(f, P)|P\ \textit{ is a partition of }\ [a, b]\}.$$





2.1.3 Properties of Darboux Sums

- $\forall P \subset [a,b], \quad U(f,P) \geq L(f,P)$.
- If $P \subset P'$, then $L(f,P) \leq L(f,P')$,and $U(f,P) \geq U(f,P')$.
- $\forall P,\ P' \subset [a,b],\ L(f,P) \leq \int_{*a}^{b} f(x)\ dx \leq \int_{a}^{b} f(x)\ dx \leq \int_{a}^{*b} f(x)\ dx \leq U(f,P')$.

2.1.4 Integrable functions, Riemann integrals

Definition 2.1.4. *Suppose $f : [a,b] \longrightarrow \mathbb{R}$ is bounded. We says that f is integrable in the sense of Riemann if:*

$$\int_{*a}^{b} f(x)\ dx = \int_{a}^{*b} f(x)\ dx.$$

The value of the lower integral and the upper integral is called the Riemann integral of f over $[a,b]$ and is denoted by $\int_a^b f(x)\ dx$.

Theorem 2.1.5. *A function f is Riemann integrable on $[a,b]$ if and only if for every $\varepsilon > 0$ there is a partition P of $[a,b]$ such that $U(f,P) - L(f,P) < \varepsilon$.*

Theorem 2.1.6.

- *Any bounded and monotone function on $[a,b]$ is integrable on $[a,b]$.*
- *Any continuous function on $[a,b]$ is integrable on $[a,b]$.*

2.1.5 Riemann Sums

Let $f : [a, b] \longrightarrow \mathbb{R}$ be a bounded function, and $P = x_0, x_1, \cdots, x_n$ a partition of $[a, b]$.

Definition 2.1.7. *The sum* $\sigma(f, P) = \sum_{k=1}^{n} f(\zeta_k)(x_k - x_{k-1})$ *where* $\zeta_k \in [x_{k-1}, x_k], \quad k = \overline{1, n}$ *is said to be the Riemann sum of* f *corresponding to* P *and the system of points* $\zeta = (\zeta_1, \zeta_2, \cdots, \zeta_n)$.

Definition 2.1.8. *We say that the number* A *is the limit of* $\sigma(f, P, \zeta)$ *when* $\delta(P)$ *tends to 0 and we write* $A = \lim\limits_{\delta(P)\to 0} \sigma(f, P, \zeta)$ *if:*

$$\forall \varepsilon > 0, \ \exists\ \tau > 0, \ \forall \sigma(f, P, \zeta), \ \delta(P) < \tau \Longrightarrow |\sigma(f, P, \zeta) - A| < \varepsilon.$$

Theorem 2.1.9. *If* $f : [a, b] \longrightarrow \mathbb{R}$ *is integrable then:*

$$A = \lim_{\delta(P)\to 0} \sigma(f, P, \zeta) = \int_a^b f(x)\ dx.$$

Particular case:

$$\int_a^b f(x)\ dx = \lim_{n\to+\infty} \frac{b-a}{n} \sum_{k=1}^{n} f\left(a + k\frac{b-a}{n}\right).$$

Example 2.1.10. *Calculate:* $I = \lim\limits_{n\to+\infty} \frac{1}{n} \sum_{k=1}^{n} (\ln(n+k) - \ln(n))$.

$$\begin{aligned} I &= \lim_{n\to+\infty} \frac{1}{n} \sum_{k=1}^{n} (\ln(n+k) - \ln(n)) \\ &= \lim_{n\to+\infty} \frac{1}{n} \sum_{k=1}^{n} \ln\left(\frac{n+k}{n}\right) \\ &= \lim_{n\to+\infty} \frac{1}{n} \sum_{k=1}^{n} \ln\left(1 + \frac{k}{n}\right) \end{aligned}$$

Therefore, $f(x) = \ln(x), \ a = 1, \ and\ b = 2.$

$$\begin{pmatrix} b - a = 1 \\ and \\ 1 + \dfrac{k}{n} = a + k\dfrac{b-a}{n} \end{pmatrix} \Longrightarrow \begin{pmatrix} b = a + 1 \\ 1 + \dfrac{k}{n} = a + k\dfrac{(a+1-a)}{n} \end{pmatrix} \quad \Longrightarrow \quad a = 1, \quad and \quad b = 2$$

So

$$\lim_{n\to+\infty} \frac{1}{n} \sum_{k=1}^{n} (\ln(n+k) - \ln(k)) = \int_1^2 \ln(x)\ dx.$$

2.1.6 Properties of the Riemann integral

Let f and g be two bounded and integrable functions on $[a, b]$, $c \in [a, b]$ and $\alpha \in \mathbb{R}$

1. $\displaystyle\int_{\alpha}^{\alpha} f(x)\ dx = 0.$
2. $\displaystyle\int_{a}^{b} f(x)\ dx = -\int_{b}^{a} f(x)\ dx.$
3. $\displaystyle\int_{a}^{b} (f(x) + g(x))\ \ dx = \int_{a}^{b} f(x)\ dx + \int_{a}^{b} g(x)\ dx.$
4. $\displaystyle\int_{a}^{b} (\alpha f(x))\ \ dx = \alpha \int_{a}^{b} f(x)\ dx.$
5. $\displaystyle\int_{a}^{b} f(x)\ dx = \int_{a}^{c} f(x)\ dx + \int_{c}^{b} f(x)\ dx.$
6. $\displaystyle\left|\int_{a}^{b} f(x)\ dx\right| \leq \int_{a}^{b} |f(x)|\ \ dx.$
7. $\displaystyle f(x) \leq g(x),\ \forall x \in [a, b] \Longrightarrow \int_{a}^{b} f(x)\ dx\ \ \leq \int_{a}^{b} g(x)\ dx.$
8. $\displaystyle f(x) \geq 0 \Longrightarrow \int_{a}^{b} f(x)\ dx \geq 0.$

   but if $f \geq 0$ and continues on $[a, b]$ and if $\displaystyle\int_{a}^{b} f(x)\ dx = 0 \Longrightarrow f(x) = 0$ on $[a, b]$.

2.1.7 Mean Theorem

If f and g are two bounded and integrable functions on $[a, b]$. If $g \geq 0$ and if $m \leq f(x) \leq M$, $\forall\ m,\ M \in \mathbb{R}$ then:

$$m\int_{a}^{b} g(x)\ dx \leq \int_{a}^{b} f(x).g(x)\ dx \leq M\int_{a}^{b} g(x)\ dx.$$

Particular case:

- If $g = 1$, we obtain:

$$m(b-a) \leq \int_{a}^{b} f(x)\ dx \leq M(b-a).$$

- First formula for the average: If f is a continuous function on $[a,b]$ and if g is a bounded integrable function, of constant sign on $[a,b]$, then there exists an element $c \in [a,b]$ such that:

$$\int_a^b f(x).g(x)\ dx = f(c)\int_a^b g(x)\ dx.$$

  If $g = 1$, we obtain the

- Second average formula: If a function f is continuous on $[a,b]$, there exists an element $c \in [a,b]$ such that:

$$\int_a^b f(x)\ dx = (b-a)f(c).$$

  The number $f(c)$ is called the average value of f over $[a,b]$.

2.1.8 Cauchy-Schwarz Inequality

If f and g are two bounded and integrable functions on $[a,b]$, we have:

$$\left(\int_a^b f(x).g(x)\ dx\right)^2 \leq \int_a^b f^2(x)\ dx\ .\int_a^b g^2(x)\ dx.$$

2.1.9 Antiderivative of a continuous function

Definition 2.1.11. *Let $f : [a,b] \longrightarrow \mathbb{R}$ a function. We say that a differentiable function $F : [a,b] \longrightarrow \mathbb{R}$ is an antiderivative of f if: $\forall x \in [a,b],\ F'(x) = f(x)$*

Example 2.1.12. *1. $F(x) = \frac{1}{3}x^3 + 5x + 2$ is an antiderivative of $f(x) = x^2 + 5$, since $F'(x) = (\frac{1}{3}x^3 + 5x + 2)' = x^2 + 5$.*

2. e^x is an antiderivative of e^x, since $(e^x)' = e^x$.

Theorem 2.1.13. *Any continuous function $f : [a,b] \longrightarrow \mathbb{R}$ admits an antiderivative. The application $x \longrightarrow F(x) = \int_a^b f(t)\ dt$ is an antiderivative of f.*

2.1.10 General method of calculating the integral

1. **Integration by parts method:**

Proposition 2.1.14. *Let f and g be two functions in class C^1 on an open interval $I=[a,b]$, we have:*

$$\int_a^b f'(x)g(x)\ dx = f(b)g(b) - f(a)g(a) - \int_a^b f(x)g'(x)\ dx.$$

Example 2.1.15. *Calculate:* $I = \int_0^1 \arctan(x)\ dx$

$$\begin{cases} f(x) = \arctan(x) \\ \text{and} \\ g'(x) = dx \end{cases} \Longrightarrow \begin{cases} f'(x) = \dfrac{1}{1+x^2} \\ \text{and} \\ g(x) = x \end{cases}$$

we have:

$$\begin{aligned} I &= \int_0^1 \arctan(x)\ dx \\ &= x.\arctan(x)|_0^1 - \int_0^1 \frac{x}{1+x^2} dx \\ &= 1.\arctan(1) - 0 - \frac{1}{2}\int_0^1 \frac{2x}{1+x^2}\ dx \\ &= \frac{\pi}{4} - \frac{1}{2}\ln\left(1+x^2\right)|_0^1 \\ &= \frac{\pi}{4} - \frac{1}{2}\ln(2). \end{aligned}$$

2. **Integration By Substitution (Change of Variables):**

Proposition 2.1.16. *Let f be a continuous function on $[a,b]$ and $g \in C^1([a,b])$, then:*

$$\int_{g(a)}^{g(b)} f(x)\ dx = \int_a^b f(g(t))g'(t)\ dt.$$

Example 2.1.17. *Calculate the integral* $I = \int_0^\pi \cos^4(x)\sin(x)\ dx$.

We set $t = \cos(x) \Longrightarrow dt = \sin(x)\ dx$, *therefore*

$$\begin{cases} if \quad x = 0 \Longrightarrow t = 1 \\ if \quad x = \pi \Longrightarrow t = -1 \end{cases}$$

so

$$\begin{aligned} I &= \int_0^{\pi} \cos^4(x)\sin(x)\ dx \\ &= \int_1^{-1} t^4(-dt) = \int_{-1}^{1} t^4\ dt \\ &= \frac{1}{5}t^5|_{-1}^{1} = \frac{2}{5} \end{aligned}$$

2.2 Indefinite integral

Definition 2.2.1. *The indefinite integral of $f(x) : I \longrightarrow \mathbb{R}$ is the collection of all antiderivatives of $f(x)$, denoted by $\int f(x)\ dx$.*
If we have: F is an antiderivative of f on I, then we write:

$$\int f(x)\ dx = F + c, \quad \forall c \in \mathbb{R}.$$

Example 2.2.2. *We have:*

- *The indefinite integral of the function $\frac{1}{x}$ on $]-\infty, 0[$ is defined by:*

$$\int \frac{dx}{x} = \ln(-x) + c_1, \quad c_1 \in \mathbb{R}$$

- *The indefinite integral of the function $\frac{1}{x}$ on $]0, +\infty[$ is defined by:*

$$\int \frac{dx}{x} = \ln(x) + c_2, \quad c_2 \in \mathbb{R}$$

$\Longrightarrow$ *The indefinite integral of the function $\frac{1}{x}$ on $\mathbb{R}^*$ is defined by:*

$$\int \frac{dx}{x} = \ln|x| + c, \quad c \in \mathbb{R}$$

2.2.1 Existence of the Indefinite Integral

Theorem 2.2.3. *Let $f : I \longrightarrow \mathbb{R}$ be a function defined on I, then we have the following implication:*

$$f \text{ is continuous on } I \Longrightarrow f \text{ admits an antiderivative on } I.$$

2.2.2 Change of variable and integration by parts in indefinite integrals

1. **Change of Variables:**

   Let I and J be two intervals and $h : J \longrightarrow I$ of class $C^1(J)$.

   Theorem 2.2.4. *Let $f \in C(I)$, then*

   $$F = \int f(x)\ dx = F \circ h = \int (f \circ h)h'(x).$$

   In other words, if F is a primitive of f then $F \circ h$ is a primitive of $(f \circ h)h'$.
   $\int f(x)\ dx = \int f(h(t))h'(t)\ dt, \quad x = h(t) \Longrightarrow dx = h'(t)dt$. *Indeed the change of variable* $x = h(t)$.

   Example 2.2.5. *Calculate:* $I = \int \cos^3(t)\ dt$.
   $\int \cos^3(t)\ dt = f(h(t))h'(t)dt$, *so*

   $$\cos^3(t)\ dt = \cos^2(t)\cos(t)\ dt = \cos^2(t)(\sin(t))'dt = (1 - \sin^2(t))(\sin(t))'dt.$$

   We set $x = \sin(t)$, *we obtain*

   $$\begin{aligned} \cos^2(t)\cos(t)dt &= (1 - x^2)\ dx \\ \int \cos^3(t)\ dt &= \int (1 - x^2)\ dx = x - \frac{x^3}{3} + c = \sin(t) - \frac{1}{3}\sin^3(t) + c. \end{aligned}$$

2. **Integration by parts:**

   Theorem 2.2.6. *Let $u,\ v : I \longrightarrow \mathbb{R}$, $u,\ v \in C^1(I)$ then:*

   $$\int u'(x)v(x)\ dx = u(x)v(x) - \int u(x)v'(x)\ dx$$

   Example 2.2.7. *(a) Compute* $\int xe^x\ dx$. *We set* $u(x) = x$, *and* $v'(x) = e^x$, *then we get:*

   $$\begin{aligned} \int xe^x\ dx &= xe^x - \int e^x\ dx \\ &= xe^x - e^x + c \\ &= e^x(x - 1) + c. \end{aligned}$$

(b) Compute $\int x\ln(x)\ dx$. *We set* $u(x) = \ln x$, *and* $v'(x) = x$, *then we get:*

$$\begin{aligned}\int x\ln x\ dx &= \frac{1}{2}x^2\ln x - \frac{1}{2}\int x^2.\frac{1}{x}\ dx\\ &= \frac{1}{2}x^2\ln x - \frac{1}{4}x^2 + c.\end{aligned}$$

2.2.3 Antiderivative of rational fractions

Any rational fraction can be written in a single way as the sum of a polynomial and a finite number of rational fractions (simple elements) of the form $\frac{A}{(x-a)^k}$, $\frac{Mx+N}{((x-\alpha)^2+\beta^2)^k}, \cdots$. We are therefore brought back to looking for antiderivatives of the polynomial and each of the simple elements there is no difficulty with integral polynomial as well as the simple elements of the first kind $\frac{A}{(x-a)^k}$.

$$\int \frac{1}{(x-a)^k}dx = -\frac{1}{(k-1)(x-a)^{k-1}} + c, \quad k > 1, \quad a \in \mathbb{R}$$

$$\int \frac{1}{(x-a)}dx = \ln|x-a| + c, \qquad a \in \mathbb{R}$$

The integration of simple elements of the second kind $\frac{Mx+N}{((x-\alpha)^2+\beta^2)^k}$

Reduce by the change of variable $x = \alpha + \beta t$ we calculate the integrals

$$I_k = \int \frac{t}{(1+t^2)^k}dt, \qquad J_k = \int \frac{1}{(1+t^2)^k}dt.$$

- The calculation of I_k is immediate in fact the change of variable $u = 1+t^2$ then $I_k = \frac{1}{2}\int \frac{du}{u^k}$ hence
  - For $k = 1$: $I_1 = \frac{1}{2}\ln(1+t^2) + c.$
  - For $k > 1$: $I_k = \frac{-1}{2(k-1)(1+t^2)^{k-1}} + c$
- For the calculation of J_k an integration by parts gives us:

$$\begin{cases} u' = dt \\ v = \dfrac{1}{(1+t^2)^k} \end{cases} \Longrightarrow \begin{cases} u = 1 \\ v' = -k\dfrac{2t(1+t^2)^{k-1}}{(1+t^2)^{2k}} = -k\dfrac{2t}{(1+t^2)^{k+1}} \end{cases}$$

so

$$\begin{aligned} J_k &= \int \frac{1}{(1+t^2)^k}\ dt \\ &= \frac{t}{(1+t^2)^k} + 2k\int \frac{t^2}{(1+t^2)^{k+1}}dt \\ &= \frac{t}{(1+t^2)^k} + 2k\int \frac{1}{(1+t^2)^k}dt - 2k\int \frac{1}{(1+t^2)^{k+1}}dt \end{aligned}$$

we have $J_{k+1} = \int \frac{1}{(1+t^2)^{k+1}}dt$, then

$$J_k = \frac{t}{(1+t^2)^k} + 2kJ_k - 2kJ_{k+1}$$

hence the recurrence relation:

$$2kJ_{k+1} = \frac{t}{(1+t^2)^k} + (2k-1)J_k.$$

The calculation comes down to that of:

$$J_1 = \int \frac{1}{1+t^2}dt = \arctan(t) + c.$$

Example 2.2.8. *Calculate* $I = \int \frac{x+3}{x^2-3x+2}\ dx.$

We have

$$\begin{aligned} \frac{x+3}{x^2-3x+2} &= \frac{x+3}{(x-2)(x-1)} = \frac{a}{(x-2)} + \frac{b}{(x-1)} \\ &= \frac{5}{(x-2)} - \frac{4}{(x-1)} \end{aligned}$$

so

$$\begin{aligned} I &= \int \frac{x+3}{x^2-3x+2}\ dx \\ &= \int \left(\frac{5}{x-2} - \frac{4}{x-1}\right) dx \\ &= \int \frac{5}{x-2}dx - \int \frac{4}{x-1}dx \\ &= 5\ln|x-2| - 4\ln|x-1| + c \end{aligned}$$

Example 2.2.9. *Calculate* $J = \int \frac{5x-1}{(x+2)^2(x^2-1)}dx$. *The decomposition into simple elements gives*

$$\begin{aligned}\frac{5x-1}{(x+2)^2(x^2+1)} &= \frac{a}{x+2} + \frac{b}{(x+2)^2} + \frac{c}{x-1} + \frac{d}{x+1} \\ &= \frac{1}{9}\left(\frac{-29}{x+2} - \frac{33}{(x+2)^2} + \frac{2}{x-1} + \frac{27}{x+1}\right)\end{aligned}$$

Now, we integrat the simple elements obtained after decomposition. From the property of indefinite integrals we have:

$$\begin{aligned}J &= \frac{1}{9}\int\left(\frac{-29}{x+2} - \frac{33}{(x+2)^2} + \frac{2}{x-1} + \frac{27}{x+1}\right)dx \\ &= \frac{1}{9}\int\frac{-29}{x+2}dx - \frac{1}{9}\int\frac{33}{(x+2)^2}dx + \frac{1}{9}\int\frac{2}{x-1}dx + \frac{1}{9}\int\frac{27}{x+1}dx \\ &= \frac{-29}{9}\ln|x+2| + \frac{11}{3}\frac{1}{x+2} + \frac{2}{9}\ln|x-1| + 3\ln|x+1| + c.\end{aligned}$$

Example 2.2.10. *Calculate* $L = \int \frac{x+1}{(x^2-2x+3)^2}dx$.

$$\begin{aligned}L &= \int\frac{x+1}{(x^2-2x+3)^2}dx \\ &= \frac{1}{2}\int\frac{2(x+1)}{(x^2-2x+3)^2}dx \\ &= \frac{1}{2}\int\frac{2x-2+4}{(x^2-2x+3)^2}dx \\ &= \frac{1}{2}\int\frac{2x-2}{(x^2-2x+3)^2}dx + 2\int\frac{1}{(x^2-2x+3)^2}dx\end{aligned}$$

we set $L_1 = \int \frac{2x-2}{(x^2-2x+3)^2}dx$, *and* $L_2 = \int \frac{1}{(x^2-2x+3)^2}dx$

$$L_1 = \int\frac{2x-2}{(x^2-2x+3)^2}dx = -\frac{1}{(x^2-2x+3)}.$$

Now calculate $L_2 = \displaystyle\int \frac{1}{(x^2-2x+3)^2}dx$.

We have $x^2-2x+3 = (x-1)^2+2 = 2\left(\frac{(x-1)^2}{2}+1\right) = 2\left(\left(\frac{x-1}{\sqrt{2}}\right)^2+1\right)$.

we set $t = \frac{x-1}{\sqrt{2}} \Longrightarrow dt = \frac{1}{\sqrt{2}}dx \Longrightarrow dx = \sqrt{2}dt$ *hence:*

$$\begin{aligned} L_2 &= \int \frac{1}{(x^2-2x+3)^2}dx = \int \frac{1}{\left(2\left(\left(\frac{x-1}{\sqrt{2}}\right)^2+1\right)\right)^2}dx \\ &= \frac{\sqrt{2}}{4}\underbrace{\int \frac{1}{(t^2+1)^2}dt}_{J_2} \end{aligned}$$

From (I), *for* $k=1$

$$2kJ_{k+1} = \frac{t}{(1+t^2)^k} + (2k-1)J_k \Longrightarrow 2J_2 = \frac{t}{(1+t^2)} + J_1 \Longrightarrow J_2 = \frac{1}{2}\frac{t}{(1+t^2)} + \frac{1}{2}J_1$$

where $J_1 = \displaystyle\int \frac{1}{1+t^2}dt = \arctan t$, *then*

$$\begin{aligned} L_2 &= \frac{\sqrt{2}}{4}\left(\frac{1}{2}\frac{t}{(1+t^2)} + \frac{1}{2}J_1\right) = \frac{\sqrt{2}}{8}\frac{t}{(1+t^2)} + \frac{\sqrt{2}}{8}J_1 \\ &= \frac{\sqrt{2}}{8}\frac{\frac{x-1}{\sqrt{2}}}{\left(1+\left(\frac{x-1}{\sqrt{2}}\right)^2\right)} + \frac{\sqrt{2}}{8}\arctan\left(\frac{x-1}{\sqrt{2}}\right). \end{aligned}$$

So

$$\begin{aligned} L &= -\frac{1}{2(x^2-2x+3)} + 2\left(\frac{x-1}{4\,(x^2-2x+3)} + \frac{\sqrt{2}}{8}\arctan\left(\frac{x-1}{\sqrt{2}}\right)\right) \\ &= -\frac{1}{2(x^2-2x+3)} + \frac{x-1}{2\,(x^2-2x+3)} + \frac{\sqrt{2}}{4}\arctan\left(\frac{x-1}{\sqrt{2}}\right) + c \\ &= \frac{x-2}{2(x^2-2x+3)} + \frac{\sqrt{2}}{4}\arctan\left(\frac{x-1}{\sqrt{2}}\right) + c. \end{aligned}$$

2.2.4 Integration of Trigonometric Functions

In the calculation of trigonometric function integrals, we distinguish the following cases:

- **Integrals of type** $\int R(\cos(x))\sin(x)dx$ **or** $\int R(\sin(x))\cos(x)dx$ **with** $R(x)$ **is a rational fraction:**

  – If we have: $I = \int R(\cos(x))\sin(x)dx$, we make a change of variable of the form

  $$t = \cos(x), \qquad \text{and} \qquad dt = -\sin(x)\ dx$$

  – If we have: $I = \int R(\sin(x))\cos(x)dx$, we make a change of variable of the form

  $$t = \sin(x), \qquad \text{and} \qquad dt = \cos(x)\ dx$$

  – If we have: $I = \int R(\tan(x))\frac{1}{\cos^2(x)}dx$, we make a change of variable of the form

  $$t = \tan(x), \qquad \text{and} \qquad dt = \frac{1}{\cos^2(x)}\ dx$$

  Example 2.2.11. *calculate* $I = \int \frac{\cos^3(x)}{\sin^2(x)}$. *We have* $\frac{\cos^3(x)}{\sin^2(x)} = \frac{1-\sin^2(x)}{\sin^2(x)}\cos(x) \Longrightarrow I$ *of type* $\int R(\sin(x))\cos(x)\ dx$. *So we perform the following change of variable:*

  $$\begin{aligned} t &= \sin(x) \Longrightarrow dt = \cos(x)dx \\ &\Longrightarrow I = \int \frac{1-t^2}{t^2}dt = \int \frac{1}{t^2}dt - \int dt \\ &\Longrightarrow I = -\frac{1}{t} - t + c, \quad c \in \mathbb{R} \end{aligned}$$

  Finally, if we replace t *by* $\sin(x)$ *we obtain:*

  $$I = -\frac{1}{\sin(x)} - \sin(x) + c, \quad c \in \mathbb{R}.$$

- **Integrals of type** $\int R(cos(x), sin(x))dx$**:**

  The following change of variable can be used for integrals of this kind. We put

  $$t = \tan\left(\frac{x}{2}\right), \quad \cos(x) = \frac{1-t^2}{1+t^2}, \quad \sin(x) = \frac{2t}{1+t^2}, \quad dx = \frac{2dt}{1+t^2}.$$

By replacing the expressions dx, $\cos(x)$ and $\sin(x)$ in the integral, we obtain an integral of type: $\int R(t)dt$ where $R(t)$ is a rational fraction. To return to the variable x in the result, replace t by $\tan\left(\frac{x}{2}\right)$.

Example 2.2.12. *Compute* $I = \int \frac{1}{1-\cos(x)}dx$. *We put:*

$$t = \tan\left(\frac{x}{2}\right), \qquad \cos(x) = \frac{1-t^2}{1+t^2}, \qquad dx = \frac{2dt}{1+t^2}$$

Replacing in I *gives us:*

$$I = \int \frac{1}{t^2}dt = -\frac{1}{t} + c, \quad c \in \mathbb{R}$$

Finally, we replace t *by* $\tan\left(\frac{x}{2}\right)$ *we find::*

$$I = -\frac{1}{\tan\left(\frac{x}{2}\right)} + c, \quad c \in \mathbb{R}$$

2.2.5 Integration of rational functions in e^x

It is always possible to bring this integration back to a rational fraction by changing the variable $t = e^x$.

Example 2.2.13. *Calculate* $I = \int \frac{1}{1+e^x}dx$.

We put

$$t = e^x \Longrightarrow dt = e^x\ dx \Longrightarrow dx = \frac{dt}{e^x} = \frac{dt}{t}.$$

so

$$\begin{aligned}
I &= \int \frac{1}{1+e^x}dx = \int \frac{1}{1+t}\frac{dt}{t} \\
&= \int \left(\frac{a}{1+t} + \frac{b}{t}\right)dt, \qquad (a=-1, \quad b=1) \\
&= \int \frac{-1}{1+t}dt + \int \frac{1}{t}dt \\
&= -\ln|1_t| + \ln|t| + c, \quad c \in \mathbb{R} \\
&= \ln\left|\frac{t}{1+t}\right| + c, \quad c \in \mathbb{R} \\
&= \ln\left|\frac{e^x}{1+e^x}\right| + c, \quad c \in \mathbb{R}
\end{aligned}$$

Chapter 3

Differential Equations

Calculus is the mathematics of change, and rates of change are expressed by derivatives. Thus, one of the most common ways to use calculus is to set up an equation containing an unknown function $y = f(x)$ and its derivative, known as a differential equation. Solving such equations often provides information about how quantities change and frequently provides insight into how and why the changes occur.

3.1 Differential equations of order n

Definition 3.1.1. *Let $n \in \mathbb{N},\ n \leq 1$, we call a differential equation of order n and of unknown function y any relation of the form*

$$y^{(n)} = f(x, y(x), y'(x), \cdots, y^{(n-1)}(x)) \tag{3.1}$$

with initial conditions

$$y(x_0) = y_0, \quad y'(x_0) = y_1, \cdots, y^{(n-1)}(x_0) = y_{n-1} \tag{3.2}$$

where f is a function defined on a part of $\mathbb{R}^{n+1}$, $(x_0, y_0, \cdots, y_{n-1})$ is vector fixed in $\mathbb{R}^{n+1}$ and the unknown is a function y of class C^n defined on an open interval of $\mathbb{R}$ containing x_0.

- *We call solution of this equation any function y of class C^n defined on an open interval containing x_0 and verifying the equations (3.1), and (3.2).*
- *The largest order of derivation is called the order of equation (3.1).*

Example 3.1.2. *1. $y' = y + x$ is a first order differential equation.*

2. $y'' - 3y' + 2y = 0$ is a second order differential equation.

3.2 First Order Differential Equations

Let $f : D \subset \mathbb{R}^2 \Longrightarrow \mathbb{R}$ and $y : I \subset \mathbb{R} \Longrightarrow \mathbb{R}$ two functions such that y is differentiable on I.

Definition 3.2.1. *We call a first order differential equation any equation of type*

$$y'(x) = f(x, y) \tag{3.3}$$

We say that (3.3) admits a solution $y_0(x)$ if $y_0' = f(x, y_0)$.

3.2.1 Equations with separable variables

A first order differential equation is said to have separate variables if it is of the form

$$f(y)y' = g(x) \tag{3.4}$$

where $f,\ g$ are two real (continuous) functions of the real variable. To solve (3.4) we use the definition

$$y' = \frac{dy}{dx}$$

hence

$$\begin{aligned} f(y)y' &= g(x) \Longrightarrow f(y)\frac{dy}{dx} = g(x) \\ &\Longrightarrow f(y)dy = g(x)dx \\ &\Longrightarrow \int f(y)dy = \int g(x)dx \\ &\Longrightarrow F(x) = G(x) + c \end{aligned}$$

where F is an antiderivative of f and G is an antiderivative of g. Moreover, if F admits an inverse function F^{-1} then

$$y(x) = F^{-1}(G(x) + c).$$

Example 3.2.2. *Let's solve the equation* $\frac{1}{y}y' = (x^2 + 1)$.

$$\begin{aligned} \frac{1}{y}y' &= (x^2+1) \Longrightarrow \frac{1}{y}\frac{dy}{dx} = (x^2+1) \\ &\Longrightarrow \int \frac{1}{y}dy = \int (x^2+1)\,dx \\ &\Longrightarrow \ln|y| = \frac{x^3}{3} + x + c, \quad c \in \mathbb{R} \\ &\Longrightarrow |y| = e^{\frac{x^3}{3}+x+c}, \quad c \in \mathbb{R} \\ &\Longrightarrow |y| = ke^{\frac{x^3}{3}+x}, \quad k = e^c \\ &\Longrightarrow y = Ke^{\frac{x^3}{3}+x}, \quad K \in \mathbb{R}^*, \quad K = \pm k. \end{aligned}$$

Example 3.2.3. $y'(x) = x^2y(x) + x^2$, *is with separable variables. Indeed, we can bring it back to the form*

$$\frac{y'(x)}{y(x)+1} = x^2$$

consequently, passing to the antiderivatives, we have

$$\ln|y+1| = \frac{x^3}{3} + c, \quad c \in \mathbb{R}$$

which implies to

$$y(x) = Ke^{\frac{x^3}{3}} - 1, \quad K = \pm e^c$$

3.2.2 Homogeneous Differential Equations

A first order differential equation is said to be homogeneous with respect to x and y if it is of the form

$$y' = f\left(\frac{y}{x}\right) \tag{3.5}$$

where f is a real (continuous) function of the real variable.

To solve (3.5) we put

$$t = \frac{y}{x}$$

so $y = xt$, and then $dy = x\ dt + t\ dx$. we replace in the equation (3.5), we obtain

$$\begin{aligned} \frac{dy}{dx} &= f\left(\frac{y}{x}\right) \Longrightarrow dy = f(t)\ dx \\ &\Longrightarrow x\ dt + t\ dx = f(t)\ dx \\ &\Longrightarrow x\ dt = (f(t) - t)\ dx \\ &\Longrightarrow \frac{dt}{f(t) - t} = \frac{dx}{x} \end{aligned}$$

this last equation is an equation with separate variables.

Example 3.2.4. *To solve the equation*

$$x^2 y' = x^2 + y^2 - xy$$

by dividing both sides of the equality by x^2, this equation can be reduced to

$$y' = 1 + \left(\frac{y}{x}\right)^2 - \frac{y}{x}$$

we put $t = \dfrac{y}{x}$, then

$$\begin{aligned}
\frac{dy}{dx} &= 1+\left(\frac{y}{x}\right)^2-\frac{y}{x}=1+t^2-t\\
&\Longrightarrow dy=(1+t^2-t)dx\\
&\Longrightarrow x\ dt+t\ dx=(1+t^2-t)\ dx\\
&\Longrightarrow x\ dt=(1+t^2-2t)\ dx\\
&\Longrightarrow \frac{dt}{1+t^2-2t}=\frac{dx}{x}\quad (t\neq 1)\\
&\Longrightarrow \int\frac{dt}{(1-t)^2}=\int\frac{dx}{x}\\
&\Longrightarrow \frac{1}{1-t}=\ln|x|+c
\end{aligned}$$

and so

$$t=1-\frac{1}{\ln|x|+c}$$

and we return to y

$$y=xt\Longrightarrow y=x-\frac{x}{\ln|x|+c}$$

and do not forget that if $u=1$ then $y=x$ is also a solution of the given equation.

Exercise 3.2.5. *Solve the differential equation:* $(2x+y)dx-(4x-y)dy=0$.

3.2.3 Linear Differential Equations

A first order differential equation is said to be linear if it has the form

$$a(x)y'+b(x)y=c(x) \tag{3.6}$$

where a, b and c are real (continuous) functions of the real variable, a being assumed not identically zero.

Equation (3.6) is called an equation with a second member, we associate it with the following equation called without a second member (or homogeneous equation)

$$a(x)y'+b(x)y=0 \qquad (H.E)$$

1. **Solving the homogeneous equation associated:**

   to solve this one just need to separate the variables

$$\begin{aligned}\frac{y'}{y} &= -\frac{b(x)}{a(x)} \Longrightarrow \frac{dy}{y} = -\frac{b(x)}{a(x)}\,dx, \quad if \quad y \neq 0\\ &\Longrightarrow \int \frac{dy}{y} = \int -\frac{b(x)}{a(x)}\,dx\\ &\Longrightarrow \ln|y| = -A(x) + c, \quad c \in \mathbb{R}\end{aligned}$$

   where $\int -\frac{b(x)}{a(x)}\,dx = -A(x) + x$. So

$$y_h(x) = Ke^{-A(x)}, \quad K = \pm e^c$$

2. **Particular solution by variation of the constant:**

   To find y_p the particular solution of (3.6) we use the method of variation of the constant.

   We pose $y_p = K(x)e^{-A(x)}$, with K a function to be determined.

   and consequently

$$y_p' = K'(x)e^{-A(x)} - A'(x)K(x)e^{-A(x)} = K'(x)e^{-A(x)} - K(x)\frac{b(x)}{a(x)}e^{-A(x)}$$

   so

$$a(x)y_p' + b(x)y(x) = c(x)$$

   implies that

$$a(x)\left(K'(x)e^{-A(x)} - K(x)\frac{b(x)}{a(x)}e^{-A(x)}\right) + b(x)K(x)e^{-A(x)} = c(x)$$

   which give

$$a(x)K'(x)e^{-A(x)} = c(x)$$

   and so $K'(x) = \frac{c(x)}{a(x)e^{-A(x)}}$, hence $K(x) = \int \frac{c(x)}{a(x)e^{-A(x)}}\,dx$. So the particular solution is:

$$y_p = e^{-A(x)} \int \frac{c(x)}{a(x)e^{-A(x)}}\ dx$$

and therefore the general solution of (3.6) is given by

$$y = y_h + y_p = Ke^{-A(x)} + K(x)e^{-A(x)}, \quad k \in \mathbb{R}.$$

Example 3.2.6. *Solve the differential equation*

$$x^2y' + 2xy = e^x$$

we solve the homogeneous equation

$$\begin{aligned} x^2y' + 2xy &= 0 \Longrightarrow \frac{y'}{y} = -\frac{2x}{x^2} \\ &\Longrightarrow \frac{dy}{y} = -\frac{2x}{x^2}\ dx \\ &\Longrightarrow \ln|y| = -\ln x^2 + c, \quad c \in \mathbb{R} \\ &\Longrightarrow y = \frac{k}{x^2}, \quad k = \pm e^c \end{aligned}$$

and so the homogemeous equation is given by

$$y_h = \frac{k}{x^2}, \quad k \in \mathbb{R}.$$

Let's look for the particular solution:

$$y_p = \frac{k(x)}{x^2}$$

therefore

$$y_p' = \frac{k'(x)x^2 - 2xk(x)}{x^4}$$

we replace in the equation $x^2y' + 2xy = e^x$ *we obtain*

$$x^2\frac{k'(x)x^2 - 2xk(x)}{x^4} + 2x\frac{k(x)}{x^2} = e^x$$

which give $k'(x) = e^x$, and so $k(x) = e^x$. hence

$$y_p = \frac{e^x}{x^2}$$

then the general solution is given by

$$y = y_h + y_p = \frac{k + e^x}{x^2}, \quad k \in \mathbb{R}.$$

Exercise 3.2.7. *Solve the differential equation on $I = \left]0, \frac{\pi}{2}\right[$*

$$y' \sin(x) - y \cos(x) = x.$$

3.2.4 Bernoulli Equations

A first order differential equation is said to be Bernoulli if it has the form

$$a(x)y' + b(x)y + c(x)y^\alpha = 0 \tag{3.7}$$

where a, b and c are real (continuous) functions of the real variable, α is a non-zero real and $\alpha \neq 1$, (if $\alpha = 1$, or $\alpha = 0$ the equation is linear).

To solve the equation (3.7) we will follow the following process

$$a(x)y' + b(x)y + c(x)y^\alpha = 0 \implies a(x)\frac{y'}{y^\alpha} + b(x)\frac{y}{y^\alpha} + c(x) = 0$$

$$\implies a(x)y'y^\alpha + b(x)y^{1-\alpha} + c(x) = 0$$

we pose $t = y^{1-\alpha}$, and then $t' = (1-\alpha)y'y^{-\alpha}$, we substitute in our equation to obtain

$$\frac{a(x)}{(1-\alpha)}t' + b(x)t + c(x) = 0$$

and this last equation is a linear equation that we know how to solve.

Example 3.2.8. *Solving the Bernoulli equation*

$$xy' - y = 2xy^2, \quad \alpha = 2 \qquad (EB)$$

(EB)$\Longrightarrow xy'y^{-2} - y^{-1} = 2x$. *We pose* $t = y^{-1} = \dfrac{1}{y} \Longrightarrow t' = -\dfrac{y'}{y^2} = -y'y^{-2}$.

so

$$(EB) \Longrightarrow -xt' - t = 2x$$

the homogeneous equation associated with (EB) is

$$\begin{aligned} -xt' - t = 0 &\Longrightarrow -xt' = t \\ &\Longrightarrow \frac{t'}{t} = -\frac{1}{x} \\ &\Longrightarrow \int \frac{t'}{t}\, dx = -\int \frac{1}{x}\, dx \\ &\Longrightarrow \ln|z| = -\ln|x| + c, \quad c \in \mathbb{R} \\ &\Longrightarrow t = e^{\ln\frac{1}{|x|}+c} = k\frac{1}{|x|}, \quad k = \pm e^c. \end{aligned}$$

The particular solution of (EB):

Note that $t = -x$ *is a particular solution of (EB), so the general solution is*

$$t = \frac{k}{|x|} - x$$

$$t = \frac{1}{y} \Longrightarrow y = \frac{1}{t} \Longrightarrow y = \frac{1}{\frac{k}{|x|} - x} = \frac{|x|}{k - x|x|}.$$

3.2.5 Riccati Equation

A first order differential equation is said to be Riccati if it is of the form

$$a(x)y' + b(x)y + c(x)y^2 = d(x) \tag{3.8}$$

where $a,\ b,\ c$ and d are real (continuous) functions of the real variable, α being assumed to be non-zero. To solve (3.8) you must first find a particular solution y_p.

$$a(x)y_p' + b(x)y_p + c(x)y_p^2 = d(x)$$

We pose $t = y - y_p \Longrightarrow y = t + y_p \Longrightarrow y' = t' + y'_p$, and $y^2 = t^2 + y_p^2 + 2ty_p$. To determine t let's replace y in the equation

$$\begin{aligned} a(x)y' + b(x)y + c(x)y^2 = d(x) &\Longrightarrow a(x)(y'_p + t') + b(x)(y_p + t) + c(x)(y_p + t)^2 = d(x) \\ &\Longrightarrow a(x)t' + [b(x) + 2c(x)y_p]\, t + c(x)t^2 = 0 \end{aligned}$$

and this last equation is a Bernoulli equation in t with $\alpha = 2$, an equation that we know how to solve.

Example 3.2.9. *Solve the equation*

$$\begin{cases} e^x y' - 2e^x y + y^2 = 1 - e^2 x \\ y_p = e^x \end{cases} \qquad (E.R)$$

we pose $t = y - y_p \Longrightarrow y = t + y_p = t + e^x \Longrightarrow y' = t' + e^x$, *and* $y^2 = t^2 + e^{2x} + 2te^x$.

Replacing in (E.R) we obtain:

$$\begin{aligned} (E.R) &\Longleftrightarrow e^{-x}(t' + e^x) - 2e^x(t + e^x) + t^2 + 2te^x + e^{2x} = 1 - e^{2x} \\ &\Longleftrightarrow e^{-x}t' + t^2 = 0 \\ &\Longleftrightarrow e^{-x}t' = -t^2 \\ &\Longleftrightarrow \frac{t'}{t^2} = -e^x \\ &\Longleftrightarrow \int \frac{t'}{t^2}\, dx = \int -e^x\, dx \\ &\Longleftrightarrow \frac{-1}{t} = -e^x + c, \quad c \in \mathbb{R} \\ &\Longleftrightarrow t = \frac{1}{e^x - c} \end{aligned}$$

Then the general solution of (E.R) is

$$y = t + e^x \Longrightarrow y = \frac{1}{e^x - c} + e^x.$$

3.3 Second Order Differential Equations with Constant Coefficients

A second order linear differential equation with constant coefficients is a differential equation of the form:

$$ay'' + by' + cy = f(x) \qquad (E)$$

where $a,\ b,\ c$ and d are real constants $a \neq 0$, and $f \in C^0(I)$ a real function of the real variable.

3.3.1 The homogeneous equation associated with (E)(without second member)

$$ay'' + by' + cy = 0 \qquad (H.E)$$

To solve (H.E) we associate the algebraic equation called characteristic equation

$$ar^2 + br + c = 0$$

- 1^{st} **case:** $\Delta = b^2 - 4ac > 0$

  In this case the characteristic equation has two real roots

  $$r_1 = \frac{-b - \sqrt{\Delta}}{2a}, \qquad \text{and} \qquad r_1 = \frac{-b + \sqrt{\Delta}}{2a}$$

  then the general solution of (H.E) is given by the formula

  $$y_h = C_1 e^{r_1 x} + C_2 e^{r_2 x}, \quad C_1,\ C_2 \in \mathbb{R}.$$

- 2^{nd} **case:** $\Delta = b^2 - 4ac = 0$

  In this case the characteristic equation has a double real root

  $$r_0 = \frac{-b}{2a}$$

  then the general solution of (H.E) is given by the formula

  $$y_h = (C_1 + C_2 x)\, e^{r_0 x}, \quad C_1,\ C_2 \in \mathbb{R}.$$

- 3^{rd} **case:** $\Delta = b^2 - 4ac < 0$

  In this case the characteristic equation has two conjugated complex roots

$$r_1 = \alpha + i\beta, \qquad \text{and} \qquad r_2 = \alpha - i\beta$$

then the general solution of (H.E) is given by the formula

$$y_0 = (C_1 \cos(\beta x) + C_2 \sin(\beta x))\, e^{\alpha x}, \quad C_1,\ C_2 \in \mathbb{R}.$$

Example 3.3.1. *To solve the equation*

$$y'' + y' + y = 0$$

we consider the characteristic equation

$$r^2 + r + 1 = 0 \Longrightarrow \Delta = -3 < 0$$

we have two complex roots conjugate:

$$r_1 = \frac{-1}{2} + i\frac{\sqrt{3}}{2}, \qquad \textit{and} \qquad \frac{-1}{2} - i\frac{\sqrt{3}}{2}$$

and the general solution is given by

$$y_h = \left(C_1 \cos \frac{\sqrt{3}}{2} x + C_2 \sin \frac{\sqrt{3}}{2} x \right) e^{\frac{-1}{2}x}, \quad C_1,\ C_2 \in \mathbb{R}.$$

3.3.2 Solving the Linear Differential Equation of Order 2 with Second Member

A linear differential equation of order 2 is of the form $ay'' + by' + cy = f(x)$ this equation with second member solving in two steps:

1. We first solve the equation without a second member and obtain y_h.
2. We solve the equation with right hand side by looking for a particular solution y_p of (E) and in this case $y = y_h + y_p$.

Proposition 3.3.2. ***(Particular solution of the equation with second member)*** *Following the expression of the second member, we will summarize the particular solutions possible in the case of the differential equation with second member:*

- *If* $f(x) = P_n(x)$, *we are looking for a particular solution of the form* $x^k P_n(x) = Q_n(x)$
  - $k = 0$ *if* r *is not solution of the characteristic equation.*
  - $k = 1$ *if* r *is a root of the characteristic equation.*
  - $k = 2$ *if* r *is a double root of the characteristic equation.*

  On only $y_p = Q_n(x)$, Q_n *is a polynomial of degree* n.

- *If* $f(x) = e^{rx}(\lambda \cos\theta x + \mu \sin\theta x)$, $\quad \theta \in \mathbb{R}^*$
  - *If* $r + i\theta$, *and* $r - i\theta$ *are not roots of (C.E), we are looking for a particular solution of the form*

  $$y_p = e^{rx}(\alpha \cos\theta x + \beta \sin\theta x).$$

  - *If* $r + i\theta$, *and* $r - i\theta$ *are roots of (C.E), then*

  $$y_p = xe^{rx}\left(\frac{\lambda}{2\theta}\sin\theta x - \frac{\mu}{2\theta}\cos\theta x\right).$$

- *If* $f(x) = e^{\alpha x}P_n(x)$, *then* $y_p = x^k e^{\alpha x} P_n(x)$
  - $k = 0$ *if* α *is not solution of the characteristic equation.*
  - $k = 1$ *if* α *is a root of the characteristic equation.*
  - $k = 2$ *if* r *is a double root of the characteristic equation.*

Exercise 3.3.3. *Solve the following differential equations*

1. $y' \sin x = y \cos x$.
2. $y^2 + (x+1)y' = 0$.
3. $xy' - \alpha y = 0, \quad \alpha \in \mathbb{R}^*$.
4. $xy^2y' = x^3 + y^3$.

5. $y'' + y = (x + 1)$.

6. $y'' + 2y' + y = e^3x$.

7. $y'' + 5y' + 6y = (x^2 + 1)$.

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