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Mathematical Analysis 1

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ANALYSIS 1

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Contents

1

Field of Real Numbers

We recall the usual notation for sets of numbers:

$\mathbb{N} = \{0, 1, 2, \cdot, n, \cdots\}$: is the set of natural numbers.

$\mathbb{Z} = \{\cdots, -2, -1, 0, 1, 2, \cdots\}$: is the set of relative integers.

$\mathbb{Q} = \{\frac{p}{q},\ p \in \mathbb{Z},\ q \in \mathbb{N}^*\}$: is the set of rationals.

$\mathbb{D} = \{r = \frac{p}{10^k} \in \mathbb{Q},\ p \in \mathbb{Z},\ k \in \mathbb{N}\}$: is the set of decimal numbers.

$\mathbb{R}$: the set of real numbers.

The sets without 0 are respectively denoted by $\mathbb{N}^*,\ \mathbb{Z}^*,\ \mathbb{Q}^*,\ \mathbb{D}^*,\ \mathbb{R}^*$.

Remark 1.0.1. $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{D} \subset \mathbb{Q} \subset \mathbb{R}$.

1.1 Properties of the real numbers

The set of real numbers $\mathbb{R}$ has the following two operations:

$$\forall x,\ y \in \mathbb{R} : \quad (x, y) \longrightarrow x + y \quad (Addition)$$
$$\forall x,\ y \in \mathbb{R} : \quad (x, y) \longrightarrow x \times y \quad (Multiplication)$$

with an ordering relation $(x \leq y)$ or $(y \leq x)$ satisfying the following axioms :

1. **Axiom 1:** $\mathbb{R}$ is a commutative field. For all $x,\ y,\ z \in \mathbb{R}$

- $(x + y) + z = x + (y + z)$ (Associative Law for Addition).
- $x + y = y + x$ (Commutative Law for Addition).
- $x + 0 = 0$ (Identity Law for Addition).
- $x + (-x) = 0$ (Inverses Law for Addition).
- $(xy)z = x(yz)$ (Associative Law for Multiplication).
- $xy = yx$ (Commutative Law for Multiplication).
- $x.1 = x$ (Identity Law for Multiplication).
- If $x \neq 0$, then $xx^{-1} = 1$ (Inverses Law for Multiplication).
- $x(y + z) = xy + xz$ (Distributivity).

2. **Axiom 2:** $\mathbb{R}$ is a totally ordered field. For all $x,\ y,\ z \in \mathbb{R}$
   - $x \leq x$ (Reflexive Law).
   - If $x \leq y$ and $y \leq x$, then $x = y$ (Antisymmetric Law).
   - If $x \leq y$ and $y \leq z$, then $x \leq z$ (Transitive Law).
   - If $x \leq y$, then $x + z \leq y + z$ (Addition Law for Order).
   - If $x \leq y$ and $z > 0$, then $xz \leq yz$ (Multiplication Law for Order).

3. **Axiom 3:**
   - For every non-empty subset A of $\mathbb{R}$ and bounded above, has an upper bound that we denote by $\sup(A)$.
   - For every non-empty subset A of $\mathbb{R}$ and bounded below, has a lower bound that we denote by $\inf(A)$.

Remark 1.1.1. *Let A be a non-empty subset of $\mathbb{R}$, then:*

- $A = \{x \in \mathbb{R} \mid x \in A\}$.
- $-A = \{x \in \mathbb{R} \mid -x \in A\}$.

Proposition 1.1.1. ***Newton's binomial formula***
Let $x,\ y \in \mathbb{R}$*, and* $n \in \mathbb{N}^*$*, we have*

$$(x+y)^n = \sum_{k=0}^{n} C_n^k\, x^k\, y^{n-k},\ \text{where } C_n^k = \frac{n!}{k!(n-k)!},\ 1! = 1 \text{ and } 0! = 1.$$

1.2 Intervals in $\mathbb{R}$

We now define various types of intervals in the real numbers. you most likely encountered the use of intervals in previous mathematics courses, for example, precalculus and calculus, but their importance might not have been evident in those courses. By contrast, the various types of intervals play a fundamental role in real analysis.

Definition 1.2.1.

- *An* ***open bounded interval*** *is a set of the form*

$$(a,b) = \{x \in \mathbb{R} \mid a < x < b\},$$

where $a,\ b \in \mathbb{R}$ *and* $a \leq b$.

- *A* ***closed bounded interval*** *is a set of the form*

$$[a,b] = \{x \in \mathbb{R} \mid a \leq x \leq b\},$$

where $a,\ b \in \mathbb{R}$ *and* $a \leq b$.

- *A* ***half-open interval*** *is a set of the form*

$$[a,b) = \{x \in \mathbb{R} \mid a \leq x < b\}, \quad \text{or} \quad (a,b] = \{x \in \mathbb{R} \mid a < x \leq b\},$$

where $a,\ b \in \mathbb{R}$ *and* $a \leq b$.

- *An* ***open unbounded interval*** *is a set of the form*

$$(a,\infty) = \{x \in \mathbb{R} \mid a < x\}, \quad \text{or} \quad (-\infty,b) = \{x \in \mathbb{R} \mid x < b\}, \quad \text{or} \quad (-\infty,\infty) = \mathbb{R},$$

where $a,\ b \in \mathbb{R}$ *and* $a \leq b$.

- *A* ***closed unbounded interval*** *is a set of the form*

$$[a, \infty) = \{x \in \mathbb{R} \mid a \le x\}, \qquad or \qquad (-\infty, b] = \{x \in \mathbb{R} \mid x \le b\},$$

where $a,\ b \in \mathbb{R}$ and $a \le b$.

Notation:
$\mathbb{R}_+^* = \{x \in \mathbb{R},\ x > 0\}$, $\mathbb{R}_-^* = \{x \in \mathbb{R},\ x < 0\}$, $\mathbb{R}^* = \mathbb{R} - \{0\}$,
$\mathbb{R}_+ = \{x \in \mathbb{R},\ x \ge 0\}$, $\mathbb{R}_- = \{x \in \mathbb{R},\ x \le 0\}$.

1.3 Completed number line $\overline{\mathbb{R}}$: (Extension of $\mathbb{R}$)

Definition 1.3.1. *We denote by $\overline{\mathbb{R}}$ the set $\mathbb{R} \cup \{-\infty,\ +\infty\}$. This set is called a completed number line.*

Order relation in $\mathbb{R}$

$\mathbb{R}$ is provided with a total order $\le$ extending that of $\mathbb{R}$ and further defined by:

$$\forall x \in \mathbb{R},\ -\infty \le x \le +\infty, \qquad \text{(in fact } -\infty < x < +\infty \text{).}$$

operations in $\overline{\mathbb{R}}$

Similarly, the laws + and . of $\mathbb{R}$ are "extended" (always commutatively) by posing

1) $(+\infty) + (+\infty) = (+\infty)$; $(-\infty) + (-\infty) = (-\infty)$.
2) $\forall x \in \mathbb{R},\ x + (+\infty) = +\infty$; $x + (-\infty) = -\infty$.
3) $(+\infty)(+\infty) = +\infty$; $(-\infty)(-\infty) = +\infty$; $(+\infty)(-\infty) = -\infty$.
4) $\forall x \in \mathbb{R}_-^*,\ x(+\infty) = -\infty$; $x(-\infty) = +\infty$.
5) $\forall x \in \mathbb{R}_+^*,\ x(+\infty) = +\infty$; $x(-\infty) = -\infty$.

Indeterminate forms

The following expressions are called indeterminate forms:

$$(+\infty) + (-\infty);\ 0(-\infty);\ 0(+\infty);\ \frac{\infty}{\infty};\ \frac{0}{0};\ 1^{\infty},\ 0^{0},\ \infty^{0}.$$

1.4 Archimedean property

$\mathbb{R}$ R satisfies the following Archimedean property:

$$\forall x \in \mathbb{R},\ \exists\, n \in \mathbb{N} \text{ such that}:\ n > x$$

In other words the set $\mathbb{N}$ is not bounded in $\mathbb{R}$.

1.5 Rational and irrational numbers

Definition 1.5.1. *The set of rational numbers, denoted $\mathbb{Q}$, is defined by*

$$\mathbb{Q} = \left\{x \in \mathbb{R} \mid x = \tfrac{p}{q} \text{ for some } p,\ q \in \mathbb{Z} \text{ such that } q \neq 0\right\}$$

The elements of $\mathbb{R}|\mathbb{Q}$ are called irrational numbers.

1.6 Density of $\mathbb{Q}$ in $\mathbb{R}$

Theorem 1.6.1. *Between every two distinct real numbers $x,\ y$ there exists a rational number q, i.e.:*

$$\forall x, y \in \mathbb{R},\ x < y \Longrightarrow \exists\, q \in \mathbb{Q} \text{ such that } x < q < y$$

Proof 1. *Let $(x,\ y) \in \mathbb{R}^2$, assume that $x < y$. We can introduce $n \in \mathbb{N}^*$ such that $ny - nx > 1$ (take for example $n = 1 + [\frac{1}{x-y}]$). So, $n > \frac{1}{y-x}$.*

Since $ny - nx > 1$*, there exists* $p \in \mathbb{Z}$ *such that* $nx < p < ny$ *(for example* $p = [nx] + 1$*, since* $[nx] \leq nx < \underbrace{[nx] + 1}_{p} \leq nx + 1 < ny$ *). So* $x < \frac{p}{n} < y$*, and* $\frac{p}{n} = q \in \mathbb{Q}$*.*

1.7 Absolute value

Definition 1.7.1. *The absolute value of a real x, denoted by* $|x|$*, is defined as follows:*

$$|x| = \begin{cases} x & : \ x > 0 \\ 0 & : \ x = 0 \\ -x & : \ x < 0 \end{cases}$$

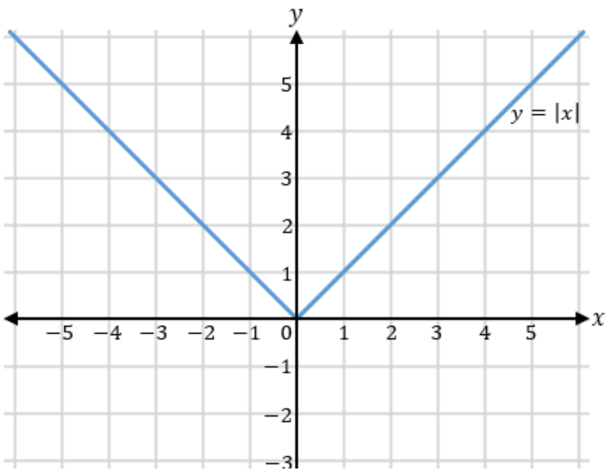


Figure 1.1: graphical presentation of $y = |x|$

Absolute Value Properties

The absolute value verify the following properties:

1. $\forall x \in \mathbb{R} : \ |x| \geq 0$
2. $\forall x, \ y \in \mathbb{R} : \ |x.y| = |x|.|y|$
3. $\forall x, \ y \in \mathbb{R} : \ |x + y| \leq |x| + |y|$

4. $\forall x,\ y \in \mathbb{R}:\ ||x| - |y|| \leq |x + y|$

5. $\forall \varepsilon > 0,\ \forall x \in \mathbb{R}:\ |x - a| < \varepsilon \Leftrightarrow a - \varepsilon \leq x \leq \varepsilon + a$

1.8 Integer part of a real number)

Definition 1.8.1. *Let $x \in \mathbb{R}$, there exists a relative integer denoted $E(x)$ such that: $E(x) \leq x \leq E(x) + 1$.*

Is the greatest integer less than or equal to x.

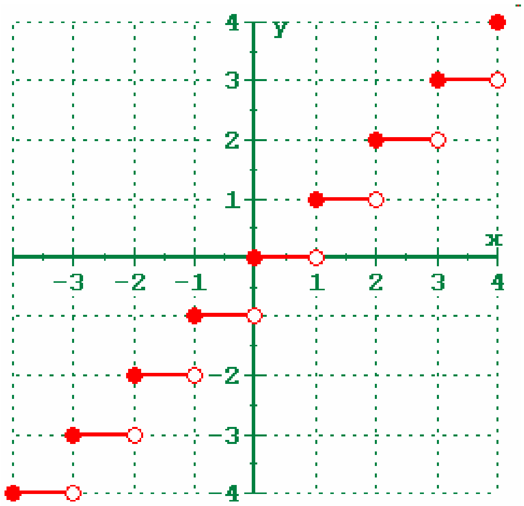


Figure 1.2: graphical presentation of $y = E(x)$

Example 1.8.1.

1) $E(0.3) = 0$, $(0 \leq 0.3 \leq 0 + 1 = 1)$.
2) $E(3.3) = 3$, $(3 \leq 3.3 \leq 3 + 1 = 4)$.
3) $E(-4) = -4$, $E(5) = 5$.
4) $E(-1.5) = -2$, $(-2 \leq -1.5 \leq -2 + 1 = -1)$.

Properties

1. the integer part is an increasing map.

2. $\forall x \in \mathbb{R},\ x = E(x) \Leftrightarrow x \in \mathbb{Z}$.

3. $\forall (x, n) \in (\mathbb{R} \times \mathbb{Z}) : E(x + n) = E(x) + n$.

1.9 Order in $\mathbb{R}$

In order to distinguish the real numbers from all other ordered fields, we will need one additional axiom, to which we now turn. This axiom uses the concepts of upper bounds and least upper bounds, while we are at it, we will also define the related concepts of lower bounds and greatest lower bounds.

Upper and lower bounds of a set)

Definition 1.9.1. *Let A be a non-empty subset of $\mathbb{R}$, we say that:*

1. *The set A is* ***bounded form above*** *if there is some $M \in \mathbb{R}$ such that $x \leq M$ for all $x \in A$. The number M is called an* ***upper bound*** *of A.*

2. *The set A is* ***bounded from below*** *if there is some $m \in \mathbb{R}$ such that $x \geq m$ for all $x \in A$. The number m is called a* ***lower bound*** *of A.*

3. *The set A is* ***bounded*** *iff there exists m and M such that: for any $x \in A,\ m \leq x \leq M$.*

4. *Let $M \in \mathbb{R}$. The number M is the* ***least upper bound*** *(also called a* ***supremum****) of A if M is an upper bound of A, and if $M \leq M'$ for all upper bounds M' of A.*

5. *Let $m \in \mathbb{R}$. The number m is the* ***greatest lower bound*** *(also called an* ***infimum****) of A if m is a lower bound of A, and if $m \geq m'$ for all lower bounds m' of A.*

Maximum, Minimum

Definition 1.9.2. *Let A be a non-empty subset of $\mathbb{R}$, we say that:*

1. *$M \in \mathbb{R}$ is a maximum of A and we denote* $\max A$ *if $M \in A$ and M is an upper bound of A.*

2. *$m \in \mathbb{R}$ is a minimum of A and we denote* $\min A$ *if $m \in A$ and m is a lower bound of A.*

Example 1.9.1. 1. *Let* $A =]0, 1[$, *A is bounded from above by* 1 *and bounded from below by* 0.

   - *The set of upper bounds is* $[1, +\infty[$, *this one admits the smallest upper bound which is* $1 \notin A$. *So* $\sup(A) = 1$ *and* $\max(A)$ *does not exist.*
   - $]-\infty, 0]$ *is the set of lower bounds, this one admits the largest of the lower bounds which is* $0 \notin A$. *So* $\inf(A) = 0$ *and* $\min(A)$ *does not exist.*

2. *Let* $B = \left\{x \in \mathbb{Z} : \; x^2 \leq 49\right\} = \{-7, -6, -5, \cdots, 5, 6, 7\}$.

   - *The set of upper bounds is:* $M = [7, +\infty[$ *and* $7 \in B$. *So* $\sup(B) = \max(B) = 7$.
   - *The set of lower bounds is:* $m =]-\infty, -7]$ *and* $-7 \in B$. *So* $\inf(B) = \min(B) = -7$.

3. $C =]-\infty, 1]$. *So C is bounded above by* $[1, \infty[$, *and not bounded below. Then,* $\max(C) = \sup(C) = 1$ *and* $\inf(C)$, $\min(C)$ *do not exist.*

1.10 The upper and lower bounds characterization

Proposition 1.10.1. *Let A be a non empty subset of* $\mathbb{R}$.

1. *If $M \in \mathbb{R}$ is an upper bound of A, then:*

$$M = \sup(A) \Leftrightarrow \begin{cases} \forall x \in A : \; x \leq M, \\ \forall \varepsilon > 0, \; \exists \, x^* \in A : \; M - \varepsilon < x^*. \end{cases}$$

2. *If $m \in \mathbb{R}$ is a lower bound of A, then:*

$$m = \inf(A) \Leftrightarrow \begin{cases} \forall x \in A : \; x \geq m, \\ \forall \varepsilon > 0, \; \exists \, x^* \in A : \; x^* < m + \varepsilon. \end{cases}$$

Exercise 1.10.1. *Let* $A = \left\{x_n = \frac{1}{2} + \frac{n}{2n+1}, \; n \in \mathbb{N}\right\}$.

1. *Show that:* $\forall x_n \in A, \; \frac{1}{2} \leq x_n < 1$.

2. *Find* $\sup(A)$, *and* $\inf(A)$.

3. *Show that:* $\sup(A) = 1$.

Solution:

1. *We show that* $\forall x_n \in A,\ \frac{1}{2} \leq x_n < 1$. *We have* $\forall n \in \mathbb{N} :\ x_n = \frac{1}{2} + \frac{n}{2n+1}$. *So*

$$\begin{aligned} \forall n \in \mathbb{N} :\ 0 \leq 2n < 2n+1 \quad &\Rightarrow \quad 0 \leq \tfrac{2n}{2n+1} < 1, \\ &\Rightarrow \quad 0 \leq \tfrac{1}{2}.\tfrac{2n}{2n+1} < \tfrac{1}{2}, \\ &\Rightarrow \quad \tfrac{1}{2} \leq \tfrac{1}{2} + \tfrac{n}{2n+1} < 1. \end{aligned}$$

   So: $\forall n \in \mathbb{N} :\ \frac{1}{2} \leq x_n < 1$.

2. *We have* $\frac{1}{2} \leq x_n < 1$, *then A is bounded, i.e:* $\inf(A)$ *and* $\sup(A)$ *are exists.*
   $\frac{1}{2}$ *is a lower bound of A, and* $\frac{1}{2} \in A \Rightarrow \min(A) = \inf(A) = \frac{1}{2}$. *And* 1 *is the smallest upper bound of A, so* $\sup(A) = 1$.

3. *Let's show that:* $\sup(A) = 1$
   We use the characteristic property of the upper bound.

$$\sup(A) = 1 \Longleftrightarrow \begin{cases} 1 \ \textit{ is an upper bound of } A, \\ \forall \varepsilon > 0,\ \exists\ x_n \in A\ (n \in \mathbb{N}),\ x_n > 1 - \varepsilon. \end{cases}$$

   Assume that: $x_n = \frac{1}{2} + \frac{n}{2n+1} > 1 - \varepsilon$, *and find n as a function of* ε.

$$\begin{aligned} x_n = \tfrac{1}{2} + \tfrac{n}{2n+1} > 1 - \varepsilon \quad &\Rightarrow \quad -\tfrac{1}{2} + \tfrac{n}{2n+1} > -\varepsilon, \\ &\Rightarrow \quad \tfrac{1}{2} - \tfrac{n}{2n+1} < \varepsilon, \\ &\Rightarrow \quad \tfrac{1}{2(2n+1)} < \varepsilon, \\ &\Rightarrow \quad 2n + 1 > \tfrac{1}{2\varepsilon}, \\ &\Rightarrow \quad n > \tfrac{1}{4\varepsilon} - \tfrac{1}{2}. \end{aligned}$$

   So $\exists\ n = E(\frac{1}{4\varepsilon} - \frac{1}{2}) + 1$, *thus* $\sup(A) = 1$.

2

Field of Complex Numbers

We know that the square of a real number is always non-negative e.g. $(4)^2 = 16$ and $(-4)^2 = 16$. Therefore, the square root of 16 is (± 4). What about the square root of a negative number? It is clear that a negative number can not have a real square root. So we need to extend the system of real numbers to a system in which we can find out the square roots of negative numbers. Euler (1707 - 1783) was the first mathematician to introduce the symbol i (iota) for the positive square root of -1 i.e., $i = \sqrt{-1}$.

2.1 Definitions and notations

Definition 2.1.1. *A number which can be written in the form $a + ib$, where a, b are real numbers and $i = \sqrt{-1}$ is called a **complex number**.*

- *If $z = a + ib$ is the complex number, then a and b are called **real** and **imaginary** parts, respectively, of the complex number and written as $Re(z) = a$, $Im(z) = b$.*

- *We denote the set of all complex numbers by $\mathbb{C}$.*

- *Order relations "greater than" and "less than" are not defined for complex numbers.*

- *If the imaginary part of a complex number is zero, then the complex number is known as purely real number and if the real part is zero, then it is called purely imaginary number. For example,* 2 *is a purely real number because its imaginary part is zero and* $3i$ *is a purely imaginary number because its real part is zero.*

- *Two complex numbers* $z_1 = a + ib$ *and* $z_2 = c + id$ *are said to be equal if* $a = c$ *and* $b = d$*.*

2.2 The complex plane

Just as real numbers can be visualized as points on a line, complex numbers can be visualized as points in a plane: plot $z = a + ib$ at the point (a, b).

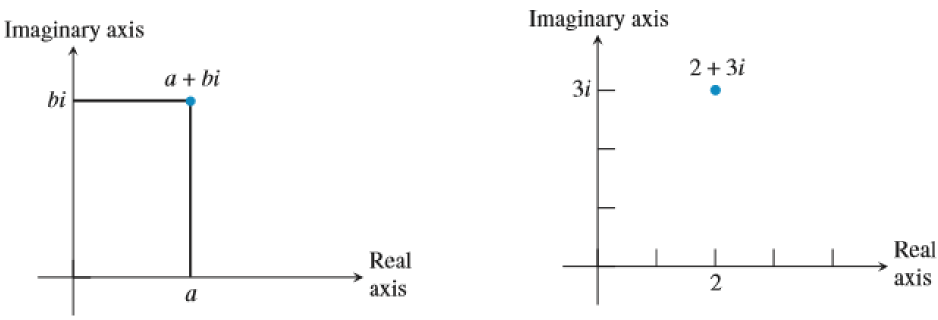


Figure 2.1: Plotting points in the complex plane

2.3 Operations on complex numbers

Addition

- Let $z_1 = a + ib$ and $z_2 = c + id$ be two complex numbers then $z_1 + z_2 = (a + c) + i(b + d)$.

- Addition of complex numbers is commutative i.e. $(z_1 + z_2 = z_2 + z_1)$, and it is also associative i.e. $((z_1 + z_2) + z_3 = z_1 + (z_2 + z_3))$.

- The identity element for addition is 0 ($\forall z = a + ib \in \mathbb{C} : \exists\, 0 = 0 + 0i \in \mathbb{C}$ *such that* $z + 0 = 0 + z = z$).

- The additive inverse of z is $-z$ ($\forall z = a+ib \in \mathbb{C}: \ \exists -z = -a-ib \in \mathbb{C}$ *such that* $z+(-z) = (-z)+z = 0$).

Multiplication

- Let $z_1 = a+ib$ and $z_2 = c+id$ be two complex numbers then $z_1.z_2 = (ac-bd)+i(ad+bc)$.
- Multiplication of complex numbers is commutative i.e. ($z_1.z_2 = z_2.z_1$), and it is also associative i.e. ($(z_1.z_2).z_3 = z_1.(z_2.z_3)$).
- The identity element for multiplication is 1 ($\forall z \in \mathbb{C}, \ \exists\, 1 = 1+i0 \in \mathbb{C}$ *such that* $z.1 = 1.z = z$).
- The multiplicative inverse of z is $\frac{1}{z}$.
- For complex numbers, multiplication is distributive over addition.

Division

Let $z_1 = a+ib$ and $z_2(\neq 0) = c+id$. Then

$$z_1 \div z_2 = \frac{a+ib}{c+id} = \frac{(ac+bd)}{c^2+d^2} + i\frac{(bc-ad)}{c^2+d^2}$$

Conjugate of a complex number

Definition 2.3.1. *In complex numbers, we define something called the conjugate of a complex number which is given by $\overline{z} = a-ib$. The conjugate is therefore simply a change the sign of the imaginary part, i.e.,* ($Re(\overline{z}) = Re(z)$, *and* $Im(\overline{z}) = -Im(z)$).

For example, if $z_1 = 3+2i$ then $\overline{z_1} = 3-2i$, if $z_2 = -4-i$ then $\overline{z_2} = -4+i$, if $z_3 = 5-3i$ then $\overline{z_3} = 5+3i$.

properties:

1. $\overline{\overline{z}} = z$.

2. $z + \overline{z} = 2Re(z),\ z - \overline{z} = 2iIm(z)$.

3. $z = \overline{z} \Longleftrightarrow z$ *is purely real.*

4. $z + \overline{z} = 0 \Longleftrightarrow z$ *is purely imaginary.*

5. $z.\overline{z} = \{Re(z)\}^2 + \{Im(z)\}^2$.

6. $(\overline{z_1 + z_2}) = \overline{z_1} + \overline{z_2},\ (\overline{z_1 - z_2}) = \overline{z_1} - \overline{z_2}$.

7. $(\overline{z_1.z_2}) = \overline{z_1}.\overline{z_2},\ (\overline{\frac{z_1}{z_2}}) = \frac{(\overline{z_1})}{(\overline{z_2})},\ (\overline{z_2} \neq 0)$.

Modulus of a complex number

Definition 2.3.2. *For any complex number $z = a + ib$, the real number $r = |z|$, defined by:*

$$r = |z| = \sqrt{a^2 + b^2}$$

is called the modulus of z.

Properties:

1. $|z^2| = z \times \overline{z},\ |\overline{z}| = |z|,\ |z_1.z_2| = |z_1|.|z_2|$.

2. $|z| = 0 \Leftrightarrow z = 0$, i.e., $Re(z) = 0$, and $Im(z) = 0$.

3. $|z_1 + z_2| \leq |z_1| + |z_2|$, (Triangle inequality).

4. $|\frac{z_1}{z_2}| = \frac{|z_1|}{|z_2|},\ z_2 \neq 0$.

5. $|\frac{1}{z}| = \frac{1}{|z|},\ z \neq 0$.

6. $|Re(z)| \leq |z|$, and $|Im(z)| \leq |z|$.

Proof 2. *(of Triangle inequality)*

$$\begin{aligned}
|z_1 + z_2|^2 &= (z_1 + z_2)(\overline{z_1 + z_2}) \\
&= z_1\overline{z_1} + z_2\overline{z_2} + z_1\overline{z_2} + \overline{z_1}z_2 \\
&= |z_1|^2 + |z_2|^2 + 2Re(z_1z_2) \\
&\leq |z_1|^2 + |z_2|^2 + 2|z_1z_2| \\
&\leq (|z_1| + |z_2|)^2
\end{aligned}$$

Argument

Definition 2.3.3. *For any $z \in \mathbb{C}$, a number $\theta \in \mathbb{R}$ such that $z = |z|(\cos\theta + i\sin\theta)$ is called an argument of z and denoted by $\theta = arg(z) = \tan^{-1}\frac{b}{a}$.*
the relationship connecting r and θ to a and b is: $a = r\cos\theta$ and $b = r\sin\theta$. i.e.,

$$arg\,(z) = \begin{cases} \cos\theta &= \frac{a}{r}, \\ \sin\theta &= \frac{b}{r}. \end{cases}$$

properties:

1. $arg\,(z_1.z_2) = arg\,(z_1) + arg\,(z_2)$.
2. $arg\,(z^n) = n\,arg\,(z)$.
3. $arg\,(\frac{1}{z}) = -\,arg\,(z)$.
4. $arg\,(\overline{z}) = -\,arg\,(z)$.
5. $arg\,(\frac{z_1}{z_2}) = arg\,(z_1) - arg\,(z_2)$.

2.4 Trigonometric form

Let $z = a + ib$, $r = |z| = \sqrt{a^2 + b^2}$, and $\theta = arg\,(z)$. We have $a = r\,\cos\theta$, $b = r\,\sin\theta$, so:

$$z = a + ib = r\cos\theta + ir\sin\theta = r\,(\cos\theta + i\sin\theta) = r\,e^{i\theta}.$$

This is the trigonometric form of z. This representation is very useful for the multiplication and division of complex numbers:

- $z_1 \times z_2 = r_1 e^{i\theta_1} \times r_2 e^{i\theta_2} = r_1.r_2 e^{i(\theta_1+\theta_2)}$.
- $\frac{z_1}{z_2} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)}$.

2.5 Inverse Euler formula

Euler's formula gives a complex exponential in terms of sines and cosines. We can turn this around to get the inverse Euler formulas.
Euler's formula says:

$$e^{it} = \cos(t) + i\sin(t) \quad \text{and} \quad e^{-it} = \cos(t) - i\sin(t).$$

By adding and subtracting we get:

$$\cos(t) = \frac{e^{it}+e^{-it}}{2} \quad \text{and} \quad \sin(t) = \frac{e^{it}-e^{-it}}{2i}.$$

2.6 Moivre's formula

For positive integers n we have the **Moivre's formula**:

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$$

Proof 3. *This is a simple consequence of Euler's formula:*

$$(\cos\theta + i\sin\theta)^n = (e^{i\theta})^n = e^{in\theta} = \cos(n\theta) + i\sin(n\theta).$$

Application:
By developing the Moivre formula using the Newton binomial formula:

$$(\cos\theta + i\sin\theta)^n = \sum_{k=0}^{n} C_n^k (\cos\theta)^{n-k}(i\sin\theta)^k.$$

Where $C_n^k = \frac{n!}{k!(n-k)!}$, $C_n^n = \frac{n!}{n!(0)!} = 1$, *and* $C_n^0 = \frac{n!}{0!(n)!} = 1$.
We have
$(\cos\theta + i\sin\theta)^n = C_n^0(\cos\theta)^n(i\sin\theta)^0 + C_n^1(\cos\theta)^{n-1}(i\sin\theta)^1 + \cdots + C_n^k(\cos\theta)^{n-k}(i\sin\theta)^k + \cdots + C_n^n(\cos\theta)^0(i\sin\theta)^n$.
So, we get
The real part:

$(\cos n\theta) = (\cos\theta)^n - C_n^2(\cos\theta)^{n-2}(\sin\theta)^2 + C_n^4(\cos\theta)^{n-4}(\sin\theta)^4 + \cdots$

and

The imaginary part:

$(\sin n\theta) = C_n^1(\cos\theta)^{n-1}(\sin\theta)^1 - C_n^3(\cos\theta)^{n-3}(\sin\theta)^3 + \cdots$

Example 2.6.1. *For $n = 3$:*

$$\begin{aligned}(\cos(\theta) + i\sin(\theta))^3 &= \textstyle\sum_{k=0}^{3} C_3^k(\cos(\theta))^{3-k}(i\sin(\theta))^k \\ &= \cos^3(\theta) + 3i\cos^2(\theta)\sin(\theta) - 3\cos(\theta)\sin^2(\theta) - i\sin^3(\theta).\end{aligned}$$

By identifying the real and imaginary parts, we deduce that:

$$\cos(3\theta) = \cos^3(\theta) - 3\cos(\theta)\sin^2(\theta), \quad and \quad \sin(3\theta) = 3\cos^2(\theta)\sin(\theta) - \sin^3(\theta).$$

2.7 n-th root of a Complex Number

Definition 2.7.1. *A complex number w is an $n-th$ root of z if:*

$$w^n = z.$$

We use the Moivre's Theorem to develop a general formula for finding the $n-th$ roots of a nonzero complex number. Suppose that $w = \rho(\cos(\theta') + i\sin(\theta'))$ is an $n-th$ root of $z = r(\cos(\theta) + i\sin(\theta))$. Then

$$\begin{cases} w^n = z \\ \rho^n e^{i\,n\,\theta'} = re^{i\theta} \end{cases} \Longrightarrow \begin{cases} \rho^n = r \\ n\theta' = \theta + 2k\pi,\ 0 \le k \le n-1. \end{cases}$$

So

$$\begin{cases} \rho = \sqrt[n]{r} \\ \theta' = \frac{\theta+2k\pi}{n},\ 0 \le k \le n-1 \end{cases}$$

thus, if $z = r(\cos(\theta) + i\sin(\theta))$, then the n distinct complex numbers

$$\sqrt[n]{r}\left(\cos\frac{\theta+2k\pi}{n} + i\sin\frac{\theta+2k\pi}{n}\right),\ 0 \le k \le n-1$$

are the $n-th$ roots of the complex number z.

Particular case:

If $z = 1$, the $n - th$ roots of 1 are

$$\cos(\tfrac{2k\pi}{n}) + i\sin(\tfrac{2k\pi}{n}),\ 0 \leq k \leq n-1.$$

Exercise 2.7.1. *Write the following numbers in algebraic form:*

1. $z_1 = \left(\dfrac{1+i}{3+2i}\right)^2.$
2. $z_2 = \dfrac{-2+i}{i} + (1-2i)^2 + \dfrac{2i}{3-i}.$

Solution:

1. $z_1 = \dfrac{(1+i)^2}{(3+2i)^2} = \dfrac{2i}{5+12i} = \dfrac{2i(5-12i)}{169} = \dfrac{24}{169} + \dfrac{10}{169}i.$
2. $z_2 = 2i + 1 - 3 - 4i + \dfrac{2i(3+i)}{10} = -\dfrac{11}{5} - \dfrac{7}{5}i.$

Exercise 2.7.2. *Calculate* $\left(-\dfrac{1}{2} - i\dfrac{\sqrt{3}}{2}\right)^{2003}$.

Solution:

$$\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)^{2003} = \left(e^{i\frac{4\pi}{3}}\right)^{2003} = e^{i\frac{8012\pi}{3}} = e^{i2670\pi}\, e^{i\frac{2\pi}{3}} = e^{i\frac{2\pi}{3}} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}.$$

Exercise 2.7.3. *Using exponential notation, find the formulas:*

$$\begin{aligned}\cos(\theta+\theta') &= \cos\theta\cos\theta' - \sin\theta\sin\theta'.\\ \sin(\theta+\theta') &= \sin\theta\cos\theta' - \cos\theta\sin\theta'.\end{aligned}$$

Solution:

We have $e^{i(\theta+\theta')} = e^{i\theta}e^{i\theta'} = (\cos\theta + i\sin\theta)(\cos\theta' + i\sin\theta')$, hence $e^{i(\theta+\theta')} = \cos\theta\cos\theta' - \sin\theta\sin\theta' + i(\cos\theta\sin\theta' + \sin\theta\cos\theta')$ by taking the real parts and the imaginary parts, we obtain the results.

3

The Numerical Sequences

3.1 The general concept of a sequence

3.1.1 *Definition*

Definition 3.1.1. *A sequence of real numbers is a real-valued function whose domain is the set of natural numbers $\mathbb{N}$ to the real numbers $\mathbb{R}$ i.e:*

$$\begin{aligned} u: \quad \mathbb{N} &\longrightarrow \mathbb{R}, \\ n &\longmapsto u_n. \end{aligned}$$

The elements of a sequence are called **the terms**. The $n-th$ term u_n or $u(n)$ is called **the general term** of the sequence.

Example 3.1.1. *1.* $(\sqrt{n})_{n\geq 0}$ *is the sequence of terms:* $0, 1, \sqrt{2}, \sqrt{3}, \cdots$.

2. $((-1)^n)_{n\geq 0}$ *is the sequence of terms that are alternated:* $+1, -1, +1, -1, \cdots$.

3.1.2 *Explicit definition*

By an explicit definition of the general term of the sequence (u_n) i.e.: Express u_n in terms of n. For example: $u_n = 3n + 1,\ v_n = \sin(n\pi/6),\ w_n = (1/2)^n$.

3.1.3 *Definition by recurrence*

By a recurrence formula, i.e. a relationship that links any term in the sequence to the one that precedes it. In this case, to calculate u_n, you need to calculate all the terms that precede it. For example

$$\begin{cases} u_0 &= 1, \\ u_{n+1} &= 2u_n + 3,\ n \in \mathbb{N}. \end{cases}$$

3.2 Qualitative features of sequences

3.2.1 *Monotonicity*

Definition 3.2.1. *A sequence u_n is called* ***increasing*** *(or* ***strictly increasing****) if $u_n \le u_{n+1}$ (or $u_n < u_{n+1}$), for all $n \in \mathbb{N}$.*
Similarly a sequence u_n is ***decreasing*** *(or* ***strictly decreasing****) if $u_n \ge u_{n+1}$ (or $u_n > u_{n+1}$), for all $n \in \mathbb{N}$.*
If a sequence is increasing (or strictly increasing), decreasing (or strictly decreasing), it is said to be ***monotonic*** *(or* ***strictly monotonic****).*

Example 3.2.1. *The sequence $u_n = \frac{2^n-1}{2^n}$ which starts*

$$\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \dots$$

is increasing. On the other hand, the sequence $v_n = \frac{n+1}{n}$ which starts

$$\frac{2}{1}, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \dots$$

is decreasing.

3.2.2 *Boundedness*

Definition 3.2.2. *Let $(u_n)_{n\in\mathbb{N}}$ be a real sequence.*

- *A sequence $(u_n)_{n\in\mathbb{N}}$ is* ***bounded from above*** *if $\exists M \in \mathbb{R},\ \forall\ n,\ u_n \leq M$.*
- *A sequence $(u_n)_{n\in\mathbb{N}}$ is* ***bounded from below*** *if $\exists m \in \mathbb{R},\ \forall\ n,\ u_n \geq m$.*
- *A sequence $(u_n)_{n\in\mathbb{N}}$ is bounded iff: it is bounded from above and bounded from below which means: $\exists M \in \mathbb{R}_+,\ \forall\ n,\ |u_n| \leq M$*

Remark 3.2.1. *If a sequence $\{u_n\}_{n=0}^{\infty}$ is increasing, then it is bounded from below by u_0, and if it is decreasing, then it is bounded from above by u_0.*

Theorem 3.2.1. *If the sequence (u_n) is bounded and monotonic, then $\lim_{n\to\infty} u_n$ exists.*

Proof 4. *Suppose that (u_n) is increasing sequence, and $\sup\limits_{n\in\mathbb{N}} u_n = M$. then for given $\varepsilon > 0$, there exists n_0 such that $M - \varepsilon \leq u_{n_0}$. Since (u_n) is increasing, we have $u_{n_0} \leq u_n$ for all $n \geq n_0$. This implies that*

$$M - \varepsilon \leq u_n \leq M \leq M + \varepsilon,\ \forall n \geq n_0.$$

That is $u_n \longrightarrow M$. For decreasing sequences we have $u_n \longrightarrow m$ such that $m = \inf\limits_{n\in\mathbb{N}} u_n$ and its proof is similar.

3.3 Convergent Sequences

Definition 3.3.1. *We say that the sequence u_n converges to the scalar l iff*

$$\forall\varepsilon > 0,\ \exists\ n_0 \in \mathbb{N} :\ \forall n \geq n_0 :\ |u_n - l| < \varepsilon.$$

In this case we write $\lim_{n\to\infty} u_n = l,\ (l\ finite)$. If there is no finite value l so that $\lim_{n\to\infty} u_n = l$, then we say that the limit does not exist, or equivalently that the sequence diverges.

Remark 3.3.1. *Any open interval with center l contains all the terms of the sequence from a certain rank.*

Example 3.3.1. *1.* $u_n = \left(\frac{3}{4}\right)^n$.

$\lim_{n\to+\infty} u_n = \lim_{n\to+\infty} \left(\frac{3}{4}\right)^n = \lim_{n\to+\infty} e^{n\ln\left(\frac{3}{4}\right)} = 0$. *So* (u_n) *converges to* 0.

2. $v_n = (-1)^n$. v_n *is a divergent sequence.*

3. $w_n = \sin(n)$. *The limit of* w_n *does not exist, so* w_n *is divergent.*

Example 3.3.2. *Consider:*

- The sequence $u_n = \dfrac{n}{n+1}$ converges to 1

Using the definition of convergence, we show that $\lim\limits_{n\to+\infty} u_n = 1$
Let $\varepsilon > 0$ we have:

$$\begin{aligned} &|u_n - 1| \leq \varepsilon \\ \Leftrightarrow\ &|\frac{n}{n+1} - 1| \leq \varepsilon \\ \Leftrightarrow\ &|\frac{n}{n+1} - 1| \leq \varepsilon \\ \Leftrightarrow\ &|1 - \frac{1}{n+1} - 1| \leq \varepsilon \\ \Leftrightarrow\ &\frac{1}{n+1} \leq \varepsilon \\ \Leftrightarrow\ &\frac{1}{\varepsilon} - 1 \leq n \end{aligned}$$

By setting $n_0 = \lfloor\frac{1}{\varepsilon}\rfloor > \frac{1}{\varepsilon} - 1$, we obtain :
$\forall\varepsilon > 0, \exists n_0 \in \mathbb{N}\,(n_0 = \lfloor\frac{1}{\varepsilon}\rfloor), \forall n \in \mathbb{N}; n \geq n_0 \implies |u_n - 1| \leq \varepsilon$
$\implies (u_n)_{n\in\mathbb{N}}$ converges to $l = 1$

Proposition 3.3.1. *If the sequence* a_n *is convergent then it has a unique limit.*

Proof 5. *Assume that* $\lim_{n\to+\infty} u_n = l$, *and* $\lim_{n\to+\infty} u_n = l'$, *we need to show that* $l = l'$.

- $\lim_{n\to+\infty} u_n = l \iff \forall\varepsilon > 0,\ \exists\, n_0 \in \mathbb{N} : \forall\, n \geq n_0 :\ |u_n - l| < \frac{\varepsilon}{2}$.

and

- $\lim_{n\to+\infty} u_n = l' \iff \forall\varepsilon > 0,\ \exists\, n_1 \in \mathbb{N} : \forall\, n \geq n_1 :\ |u_n - l'| < \frac{\varepsilon}{2}.$

We have $|l - l'| = |l - u_n + u_n - l'| \leq |l - u_n| + |u_n - l'| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. *So* $\forall\varepsilon > 0 :\ |l - l'| < \varepsilon$, *then* $l = l'$.

Proposition 3.3.2. *If the sequence u_n converges to l, then $|u_n|$ converges to $|l|$.*

Proposition 3.3.3. *any convergent sequence is bounded.*

Proof 6. *Suppose a sequence (u_n) converges to u. Then, for $\varepsilon = 1$, there exist N such that*

$$|u_n - u| \leq 1,\ \forall n \geq N.$$

This implies $|u_n| \leq |u| + 1$ for all $n \geq N$. If we let

$$M = \max\{|u_1|,\ |u_2|, \cdots, |u_{N-1}|\},$$

then $|u_n| \leq M + |u| + 1$ for all n. Hence (u_n) is a bounded sequence.

Remark 3.3.2.

- *If $(u_n)_{n\in\mathbb{N}}$ is increasing and bounded from above, then $(u_n)_{n\in\mathbb{N}}$ converges to $l = \sup u_n$.*
- *If $(u_n)_{n\in\mathbb{N}}$ is decreasing and bounded from below, then $(u_n)_{n\in\mathbb{N}}$ is converges to $l = \inf u_n$.*

3.4 The usual rules of limits

If (u_n) and (v_n) are convergent sequences to l and l' respectively, and α is any real constant then:

1) $\lim_{n\to+\infty}(u_n + v_n) = l + l'$,	5) $\lim_{n\to+\infty} \frac{1}{u_n} = \frac{1}{l},\ l \neq 0$,
2) $\lim_{n\to+\infty}(u_n \times v_n) = l \times l'$,	6) $if\ u_n \leq v_n,\ then\ l \leq l'$,
3) $\lim_{n\to+\infty}(\alpha u_n) = \alpha l$,	7) $if\ l = l',\ and\ u_n \leq w_n \leq v_n,\ then\ \lim_{n\to+\infty} w_n = l$.

3.5 Adjacent sequences

Definition 3.5.1. *We say that two real sequences (u_n), and (v_n) are adjacent if they satisfy the following properties:*

1. (u_n) *is increasing, and* (v_n) *is decreasing,*
2. $\lim_{n\to\infty}(u_n - v_n) = 0.$

Theorem 3.5.1. *If (u_n) and (v_n) are adjacent sequences, then they converge to the same limit.*

Proof. We assume that (u_n) is increasing and (v_n) is decreasing. Let $w_n = u_n - v_n$, then

$$\begin{aligned} w_{n+1} - w_n &= u_{n+1} - v_{n+1} - u_n + v_n, \\ &= (u_{n+1} - u_n) - (v_{n+1} - v_1), \\ &\geq 0. \end{aligned}$$

and $\lim_{n\to\infty} w_n = \lim_{n\to\infty}(u_n - v_n) = 0$. Since (w_n) is an increasing sequence and $\lim_{n\to\infty} w_n = 0$, then $\forall n \in \mathbb{N}: \ w_n \leq 0 \Rightarrow u_n \leq v_n$.

Therefore, $\forall n \in \mathbb{N}: \ u_0 \leq u_n \leq v_n \leq v_0$. the sequence (u_n) is convergent since it is increasing and bounded from above by v_0, also the sequence (v_n) is convergent, and since $\lim_{n\to\infty}(u_n - v_n) = 0$ we deduce that $\lim_{n\to\infty} u_n = \lim_{n\to\infty} v_n$. □

Exercise 3.5.1. *Show that the two sequences (u_n) and (v_n) are adjacent:*

- $u_n = 1 + \frac{1}{n!}$, *and* $v_n = \frac{n}{n+1}$.
- $u_n = \sum_{k=1}^{n} \frac{1}{k^2}$ *and* $v_n = u_n + \frac{2}{n+1}$.

3.6 Subsequences

It is useful to sometimes consider only some terms of a sequence. A subsequence of $\{u_n\}_{n=1}^{\infty}$ is a sequence that contains only some of the numbers from $\{u_n\}_{n=1}^{\infty}$ in the same order.

Definition 3.6.1. *The sequence* $(u_{\phi(n)})_{n\in\mathbb{N}}$ *is a subsequence of the sequence* $(u_n)_{n\in\mathbb{N}}$ *if* $\phi : \mathbb{N} \longrightarrow \mathbb{N}$ *is a strictly increasing sequence of natural numbers.*

Example 3.6.1. *Consider the sequence*

$$u_n = \left(\frac{1}{n}\right)_{n=1}^{\infty} = \left\{1, \frac{1}{2}, \frac{1}{3}, \cdots, \frac{1}{n}, \cdots\right\},$$

then letting $n_k = 2k$ *yields the subsequence*

$$u_{2k} = \left(\frac{1}{2k}\right)_{k=1}^{\infty} = \left\{\frac{1}{2}, \frac{1}{4}, \cdots, \frac{1}{2k}, \cdots\right\},$$

and letting $n_k = 2k + 1$ *yields the subsequence*

$$u_{2k+1} = \left(\frac{1}{2k+1}\right)_{k=1}^{\infty} = \left\{\frac{1}{3}, \frac{1}{5}, \cdots, \frac{1}{2k+1}, \cdots\right\}.$$

Proposition 3.6.1. *If* $\{u_n\}_{n=1}^{\infty}$ *is a convergent sequence, then every subsequence* $\{u_{n_i}\}_{i=1}^{\infty}$ *is also convergent, and*

$$\lim_{n\to+\infty} u_n = \lim_{i\to+\infty} u_{n_i}.$$

Proof 7. *Let* u_{n_i} *denote a subsequence of* u_n. *Note that* $n_i \geq i$ *for all* i. *This easy to prove by induction: in fact,* $n_1 \geq 1$ *and* $n_i \geq i$ *implies that* $n_{i+1} > n_i \geq i$ *and hence* $n_{i+1} \geq i + 1$.
Let $\lim u_n = u$, *and let* $\varepsilon > 0$. *There exists* N *so that* $n > N$ *implies* $|u_n - u| < \varepsilon$. *Now*

$$i > N \Longrightarrow n_i > N \Longrightarrow \left|u_{n_i} - u\right| < \varepsilon.$$

therefore $\lim\limits_{i\to\infty} u_{n_i} = u$.

Corollary 3.6.1. *Let* (u_n) *be a sequence, if it admits a divergent subsequence, or if it admits two subsequences converging to distinct limits, then* (u_n) *is diverges.*

Theorem 3.6.1. ***(Bolzano-Weierstrass)***
Every bounded sequence has a convergent subsequence.

To prove the Bolzano-Weierstrass theorem, we will first need two lemmas.

Lemma 3.6.1. *All bounded monotone sequences converge.*

Proof 8. *Let (u_n) be a bounded, nondecreasing sequence. Let U denote the set $u_n,\ n \in \mathbb{N}$. Then let $b = \sup\ U$ (the supremum of U).*

Choose some $\varepsilon > 0$. Then there is a corresponding N such that $u_N > b - \varepsilon$. Since (u_n) is nondecreasing, for all $n > N$, $u_n > b - \varepsilon$. But (u_n) is bounded, so we have $b - \varepsilon < u_n \leq b$. But this implies $|u_n - b| < \varepsilon$, so $\lim u_n = b$.

(The proof for nonincreasing sequences is analogous.)

Lemma 3.6.2. *Every sequence has a monotonic subsequence.*

Proof 9. *First a definition: call the nth term of a sequence dominant if it is greater than every term following it. For the proof, note that a sequence (u_n) may have finitely many or infinitely many dominant terms.*

First we suppose that (u_n) has infinitely many dominant terms. Form a subsequence (u_{n_k}) solely of dominant terms of (u_n). Then $u_{n_{k+1}} < u_{n_k}k$ by definition of ŞdominantŤ, hence (u_{n_k}) is a decreasing (monotone) subsequence of (u_n).

For the second case, assume that our sequence (u_n) has only finitely many dominant terms. Select n_1 such that n_1 is beyond the last dominant term. But since n_1 is not dominant, there must be some $m > n_1$ such that $u_m > u_{n_1}$. Select this m and call it n_2. However, n_2 is still not dominant, so there must be an $n_3 > n_2$ with $u_{n_3} > u_{n_2}$, and so on, inductively. The resulting sequence $u_1,\ u_2,\ u_3, \cdots$ is monotonic (nondecreasing).

Proof 10. *(**of Bolzano-Weierstrass**)*

The proof of the Bolzano-Weierstrass theorem is now simple: let (u_n) be a bounded sequence. By Lemma (3.6.2) it has a monotonic subsequence. By Lemma (3.6.1), the subsequence converges.

3.7 Cauchy Sequences

Definition 3.7.1. *A real sequence (u_n) is called a Cauchy sequence if for every $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that $\forall m, n \in \mathbb{N}$, if $m, n \geq N$ then*

$$|u_n - u_m| \leq \varepsilon.$$

Proposition 3.7.1. *If a sequence is Cauchy, then it is bounded.*

Proof 11. *we have a Cauchy sequence:*

$$\forall \varepsilon > 0,\ \exists N \ s.t\ \forall\ n,\ m > N,\ |u_n - u_m| < \varepsilon.$$

we want to prove: this sequence is bounded: $\forall\ n,\ |u_n| < C$. *Note that* $|u_n| = |u_n - u_m + u_m| \leq |u_n - u_m| + |u_m|$ *by the Triangle Inequality set* $\varepsilon = 1$, *because this sequence is Cauchy,* $\exists\ N$ *such that* $\forall\ m,\ n > N,\ |u_n - u_m| < 1$. *Set* $m = N + 1$. *Combined with our initial note, we can rewrite the following:* $|u_n| < 1 + |u_{N+1}|$, *and this is true for* $\forall\ n > N$.
This bounds all the terms beyond the Nth. Looking at the terms before the Nth term, we can find the maximum of them and note that this bounds that part of the sequence:

$$|u_n| < \max\left(|u_1|,\ |u_2|, \cdots, |u_N|\right)$$

and this is true for $n \leq N$. *By choosing the maximum of either* $1 + |u_{N+1}|$ *or the maximum of the aforementioned set, we can find our C which bounds all the terms in the sequence. We have shown the sequence is bounded.*

Proposition 3.7.2. $(u_n)_{n\in\mathbb{N}}$ *is a Cauchy sequence* $\Leftrightarrow$ $(u_n)_{n\in\mathbb{N}}$ *is convergent.*

Proof 12. *Suppose* (x_n) *is a convergent sequence, and* $\lim(x_n) = x$. *Let* $\varepsilon > 0$. *We can find* $N \in \mathbb{N}$ *such that for all* $n \geq N$, $|x_n - x| < \frac{\varepsilon}{2}$. *Therefore, by the triangle inequality, for all* $m,\ n \geq N$, $|x_m - x_n| \leq |x_m - x| + |x - x_n| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. *So* (x_n) *is Cauchy.*
Conversely, suppose (x_n) *is Cauchy. Let* $\varepsilon > 0$. *By a result proved in class,* (x_n) *is bounded. By Bolzano-Weierstrass, it has a convergent subsequence* (x_{n_k}) *with* $\lim(x_{n_k}) = x$ *for some x. We can find* $K \in \mathbb{N}$ *such that for all* $k \geq K$, $\left|x_{n_k} - x\right| < \frac{\varepsilon}{2}$. *We can also find M such that for all* $m,\ n \geq M$, $|x_m - x_n| < \frac{\varepsilon}{2}$. *Let* $N = \sup\ K, M$. *Then since* $n_k \geq k$ *for all k, if* $k \geq N$, *we have that* $k,\ n_k \geq M$ *and* $n_k \geq K$. *Therefore, for all* $k \geq N$, $|x_n - x| \leq \left|x_n - x_{n_k}\right| + \left|x_{n_k} - x\right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ *by the Triangle Inequality. Therefore,* (x_n) *is Cauchy.*

3.8 Arithmetic sequences

3.8.1 *Definition*

A simple way to generate a sequence is to start with a number u_0, and add to it a fixed constant r, over and over again. This type of sequence is called an **arithmetic sequence**.

Definition 3.8.1. *the sequence (u_n) is an arithmetic sequence with first term u_0 and common difference r if and only if for any integer $n \in \mathbb{N}$ we have*

$$u_{n+1} = u_n + r, \qquad (u_n = u_0 + n.r).$$

More generally: $u_n = u_p + (n - p).r$.

3.8.2 *Sum of n terms*

For the arithmetic sequence

$$S_n = u_0 + u_1 + \cdots + u_{n-1} = n.\tfrac{u_0+u_{n-1}}{2}.$$

3.9 Geometric sequences

3.9.1 *Definition*

Another simple way of generating a sequence is to start with a number v_0 and repeatedly multiply it by a fixed nonzero constant q. This type of sequence is called a geometric sequence.

Definition 3.9.1. *the sequence (v_n) is a geometric sequence with first term v_0 and common ratio $q \in \mathbb{R}^*$ if and only if for any integer $n \in \mathbb{N}$ we have*

$$v_{n+1} = q.v_n, \qquad (v_n = v_0.q^n).$$

More generally: $v_n = v_p.q^{n-p}$.

3.9.2 *Sum of* n *terms*

For a geometric sequence, if $S_n = 1 + q + q^2 + \cdots + q^n$, then

$$S_n = \begin{cases} n+1 & si \quad q = 1, \\ \frac{1-q^{n+1}}{1-q} & si \quad q \neq 1. \end{cases}$$

Exercise 3.9.1. *Let $(a_n)_n$ be a sequence defined by:*

$$\begin{cases} a_1 = \sqrt{2}, \\ a_{n+1} = \sqrt{a_n + 2}, \; for \; n \geq 1. \end{cases}$$

1. *Prove that $a_n < 2$ for all $n \in \mathbb{N}$.*
2. *Prove that $\{a_n\}$ is an increasing sequence.*
3. *Prove that $\lim\limits_{n\to\infty} a_n = 2$.*

Solution:

1. Clearly, $a_1 < 2$. Suppose that $a_k < 2$ for $k \in \mathbb{N}$. Then

$$a_{k+1} = \sqrt{2 + a_k} < \sqrt{2+2} = 2.$$

   By induction, $a_n < 2$ for all $n \in \mathbb{N}$.

2. Clearly, $a_1 = \sqrt{2} < \sqrt{2 + \sqrt{2}} = a_2$. Suppose that $a_k < a_{k+1} fork \in \mathbb{N}$. Then

$$a_k + 2 < a_{k+1} + 2$$

   which implies

$$\sqrt{a_k + 2} < \sqrt{a_{k+1} + 2}.$$

   Thus, $a_{k+1} < a_{k+2}$. By induction, $a_n < a_{n+1}$ for all $n \in \mathbb{N}$. Therefore, $\{a_n\}$ is an increasing sequence.

3. By the monotone convergence theorem, $\lim_{n\to\infty} a_n$ exists. Let $l = \lim_{n\to\infty} a_n$, since $a_{n+1} = \sqrt{2+a_n}$ and $\lim_{n\to\infty} a_{n+1} = l$, we have

$$l = \sqrt{2+l}, \text{ or } l^2 = 2 + l.$$

Solving this quadratic equation yields $l = -1$ or $l = 2$. Therefore, $\lim_{n\to\infty} a_n = 2$.

Exercise 3.9.2. *Let a and b be two positive real numbers with $a < b$. Define $a_1 = a$, $b_1 = b$, and*

$$a_{n+1} = \sqrt{a_n b_n}, \text{ and } b_{n+1} = \frac{a_n + b_n}{2}, \text{ for } n \geq 1.$$

Show that $\{a_n\}$ and $\{b_n\}$ are convergent to the same limit.

Solution:

Observe that

$$b_{n+1} = \frac{a_n + b_n}{2} \geq \sqrt{a_n b_n} = a_{n+1} \text{ for all } n \in \mathbb{N}.$$

Thus

$$a_{n+1} = \sqrt{a_n b_n} \geq \sqrt{a_n a_n} = a_n \text{ for all } n \in \mathbb{N}.$$

Hence

$$b_{n+1} = \frac{a_n + b_n}{2} \leq \frac{b_n + b_n}{2} = b_n \text{ for all } n \in \mathbb{N}.$$

It follows that $\{a_n\}$ is monotone increasing and bounded above by b_1, and $\{b_n\}$ is decreasing and bounded below by a_1. Let $x = \lim_{n\to\infty} a_n$, and $y = \lim_{n\to\infty} b_n$. Then

$$x = \sqrt{xy} \text{ and } y = \frac{x+y}{2}.$$

Therefore, $x = y$.

4

Real-Valued Functions of a Real Variable

4.1 Basics

4.1.1 *Definition*

Definition 4.1.1. *Let $D \subseteq \mathbb{R}$. A function f from D into $\mathbb{R}$ is a rule which associates with each $x \in D$ one and only one $y \in \mathbb{R}$. We denote*

$$\begin{aligned} f : D &\longrightarrow \mathbb{R}, \\ x &\longmapsto f(x). \end{aligned}$$

D is called the domain of the function. If $x \in D$, then the element $y \in \mathbb{R}$ which is associated with x is called the value of f at x or ***the image*** *of x under f, y is denoted by $f(x)$.*

4.1.2 *Graph of a function*

Definition 4.1.2. *Each couple $(x, f(x))$ corresponds to a point in the $xy-$plane. The set of all these points forms a curve called the graph of the function f.*

$$G_f = \{(x, y) |\ x \in D,\ y = f(x)\}.$$

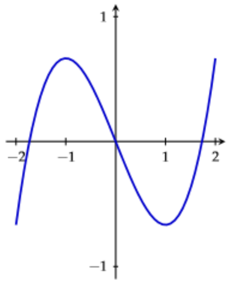


Figure 4.1: Graph of function $f(x) = 1/3x^3 - x$ in interval $[-2, 2]$.

4.1.3 *Operations on functions*

Arithmetic

Let $f, g : D \longrightarrow \mathbb{R}$ be tow functions, then:

1. $(f \pm g)(x) = f(x) \pm g(x), \ \forall x \in D,$
2. $(f.g)(x) = f(x).g(x), \ \forall x \in D,$
3. $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, \ g(x) \neq 0, \ \forall x \in D,$
4. $(\lambda.f)(x) = \lambda.f(x), \ \forall x \in D, \ \lambda \in \mathbb{R}.$

Composition

Let $f : D \longrightarrow \mathbb{R}$ and let $g : E \longrightarrow \mathbb{R}$, if $f(D) \subseteq E$, then g composed with f is the function $g \circ f : D \longrightarrow \mathbb{R}$ defined by $g \circ f = g[f(x)]$.

Restriction

We say that g is a restriction of the function f if:

$$g(x) = f(x) \text{ and } D(g) \subseteq D(f).$$

Example 4.1.1. $f(x) = \ln|x|$, *and* $g(x) = \ln x$, $\forall x \in]0, +\infty[$: $g(x) = f(x)$, *and* $D(g) \subseteq D(f)$.

4.1.4 *Bounded functions*

Definition 4.1.3. *Let $f : D \longrightarrow \mathbb{R}$ be a function, then:*

- *We say that f is* ***bounded from below*** *on its domain $D(f)$ if*

$$\forall x \in D(f),\ \exists\, m \in \mathbb{R} :\ m \leq f(x).$$

- *We say that f is* ***bounded from above*** *on its domain $D(f)$ if*

$$\forall x \in D(f),\ \exists\, M \in \mathbb{R} :\ f(x) \geq M.$$

- *Function is* ***bounded*** *if it is bounded from below and above.*

Definition 4.1.4. *Let $f, g : D \longrightarrow \mathbb{R}$ be two functions, then:*

- $f \geq g$ *si* $\forall x \in D :\ f(x) \geq g(x)$.
- $f \geq 0$ *si* $\forall x \in D :\ f(x) \geq 0$.
- $f > 0$ *si* $\forall x \in D :\ f(x) > 0$.
- *f is said to be constant over D if $\exists a \in \mathbb{R},\ \forall x \in D :\ f(x) = a$.*
- *f is said to be zero over D if $\forall x \in D :\ f(x) = 0$.*

4.1.5 *Monotone functions*

Definition 4.1.5. *Consider $f : D(f) \subseteq \mathbb{R} \longrightarrow \mathbb{R}$. For all $x, y \in D$, we have:*

- *f is* ***increasing*** *(or* ***strictly increasing****) over D if: $x \leq y \Rightarrow f(x) \leq f(y)$, (or $x < y \Rightarrow f(x) < f(y)$).*
- *f is* ***decreasing*** *(or* ***strictly decreasing****) over D if: $x \leq y \Rightarrow f(x) \geq f(y)$, (or $x < y \Rightarrow f(x) > f(y)$).*

- *f is **monotone** (or **strictly monotone**) over D if f is increasing or decreasing (strictly increasing or strictly decreasing).*

Proposition 4.1.1. *A sum of two increasing (decreasing) functions is an increasing (decreasing) function.*

Proof 13. *By induction on $N \geq 1$, for any reals $a_1, a_2, \cdots, a_N, b_1, b_2, \cdots, b_N$ with $a_i < b_1$ for all $i = 1, \cdots, N$, we have:*

$$\sum_{i=1}^{N} a_i < \sum_{i=1}^{N} b_i.$$

Assume first that the f_i are all monotone increasing (and that this means strictly). In any case we assume that they're all "the same kind of monotone".
Given reals x, y with $x < y$, letting $a_i = f_i(x)$, and $b_i = f_i(y)$, we have $a_i < b_i$ for all i, so:

$$g(x) = \sum_{i=1}^{N} a_i < \sum_{i=1}^{N} b_i = g(y),$$

so g is monotone increasing too. Similarly if the f_i are monotone decreasing.

Corollary 4.1.1. *If f is strictly monotone on D, then f is injective.*
Indeed:

$$\begin{pmatrix} x \neq y \\ x < y \end{pmatrix} \Longrightarrow \begin{pmatrix} f(x) < f(y) \\ or \\ f(x) > f(y) \end{pmatrix} \Longrightarrow f(x) \neq f(y).$$

Example 4.1.2. *Consider the function $f = 2x + 1$. We have*

$$\forall\ x,\ y \in \mathbb{R},\ x < y \Longrightarrow 2x < 2y \Longrightarrow 2x + 1 < 2y + 1 \Longrightarrow f(x) < f(y)$$

so f is strictly increasing then f is injective.

4.1.6 *Even and odd functions*

Definition 4.1.6. • *We say that function $f : D(f) \longrightarrow \mathbb{R}$ is **even** if*

$$\forall x \in D(f): \; f(-x) = f(x).$$

- *We say that function $f : D(f) \longrightarrow \mathbb{R}$ is **odd** if*

$$\forall x \in D(f): \; f(-x) = -f(x).$$

Remark 4.1.1. *1. Graph of an even function is symmetric with, respect to the y axis.*

2. Graph of an odd function is symmetric with, respect to the origin.

3. Domain of an even or odd function is always symmetric with respect to the origin.

4.1.7 *Periodic functions*

Definition 4.1.7. *A function $f : D(f) \longrightarrow \mathbb{R}$ is called **periodic** if $\exists\, T \in \mathbb{R}_+^*$ such that:*

1. $x \in D(f) \Rightarrow x \pm T \in D(f)$,
2. $x \in D(f) : f(x \pm T) = f(x)$.

Number T is called a period of f.

4.2 Limits of Functions

4.2.1 *Definition*

Definition 4.2.1. *A set $U \subset \mathbb{R}$ is a neighborhood of a point $x \in \mathbb{R}$ if:*

$$]x-\delta, x+\delta[\subset U,$$

for some $\delta > 0$. The open interval $]x-\delta, x+\delta[$ is called a $\delta-$neighborhood of x.

Example 4.2.1. *If $a < x < b$ then the closed interval $[a, b]$ is a neighborhood of x, since it contains the interval $]x-\delta, x+\delta[$ for sufficiently small $\delta > 0$. On the other hand, $[a, b]$ is not a neighborhood of the endpoints a, b since no open interval about a or b is contained in $[a, b]$.*

Definition 4.2.2. *Let f be a function defined in the neighborhood of x_0 except perhaps at x_0. A number $l \in \mathbb{R}$ is the limit of f at x_0 if:*

$$\forall \varepsilon > 0, \ \exists\, \delta > 0, \ \forall x \neq x_0 : \ |x - x_0| < \delta \Rightarrow |f(x) - l| < \varepsilon.$$

Notation: $\lim_{x\to x_0} f(x) = l$.

Example 4.2.2. *Let*

$$\begin{aligned} f : \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longrightarrow 5x - 3 \end{aligned}$$

Show that $\lim_{x\to 1} f(x) = 2$.
By definition: $\forall \varepsilon > 0, \ \exists\, \delta > 0, \ \forall x \neq 1 : \ |x - 1| < \delta \Rightarrow |f(x) - l| < \varepsilon$. *So we have:*

$$\forall \varepsilon > 0, \ |5x - 3 - 2| < \varepsilon \Rightarrow |5x - 5| < \varepsilon \Rightarrow 5\,|x - 1| < \varepsilon.$$

Then: $|x - 1| < \frac{\varepsilon}{5}$, *so* $\exists\, \delta = \frac{\varepsilon}{5} > 0$ *such that* $\lim_{x\to 1} f(x) = 2$.

4.2.2 *Right and left limits*

Definition 4.2.3. *Let f be a function defined in the neighborhood of x_0.*

- *We say that f has a limit l to the right of x_0 if:*

$$\forall \varepsilon > 0,\ \exists\, \delta > 0,\ \forall x_0 < x < x_0 + \delta \Rightarrow |f(x) - l| < \varepsilon.$$

We write $\lim_{x \to x_0^+} f(x) = \lim_{x \overset{>}{\longrightarrow} x_0} f(x) = l.$

- *We say that f has a limit l to the left of x_0 if:*

$$\forall \varepsilon > 0,\ \exists\, \delta > 0,\ \forall x_0 - \delta < x < x_0 \Rightarrow |f(x) - l| < \varepsilon.$$

We write $\lim_{x \to x_0^-} f(x) = \lim_{x \overset{<}{\longrightarrow} x_0} f(x) = l.$

- *If f admits a limit at the point x_0 then:*

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0^+} f(x) = \lim_{x \to x_0^-} f(x) = l.$$

Example 4.2.3. *Consider the integer part function at the point $x = 2$.*

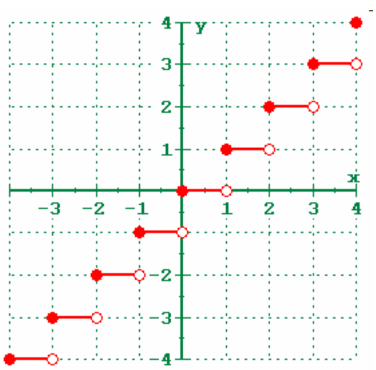


Figure 4.2: Graph of function $f(x) = E(x)$.

- *Since $x \in]2, 3[$, we have: $E(x) = 2$, and* $\lim_{x \to 2^+} E(x) = 2.$

- *Since $x \in]1, 2[$, we have: $E(x) = 1$, and* $\lim_{x \to 2^-} E(x) = 1.$

Since these two limits are different, we deduce that the function $f(x) = E(x)$ has no limit at $x = 2$.

Theorem 4.2.1. *If $\lim_{x\to x_0} f(x)$ exists, then it is unique. That is, f can have only one limit at x_0.*

Proof 14. *We assume that f has two different limits at point x_0; l and l' ($l \neq l'$). We have*

$$\lim_{x\to x_0} f(x) = l \iff \forall\varepsilon > 0,\ \exists\,\delta_1 > 0,\ \forall\ x \neq x_0,\ |x - x_0| < \delta_1 \Longrightarrow |f(x) - l| < \frac{\varepsilon}{2}$$

$$\lim_{x\to x_0} f(x) = l' \iff \forall\varepsilon > 0,\ \exists\,\delta_2 > 0,\ \forall\ x \neq x_0,\ |x - x_0| < \delta_2 \Longrightarrow |f(x) - l'| < \frac{\varepsilon}{2}$$

We pose $\delta = \min(\delta_1,\ \delta_2)$, and $\varepsilon < |l - l'|$, then

$$\forall\ \varepsilon > 0,\ \exists\,\delta > 0,\ \forall\ x \neq x_0,\ |x - x_0| < \delta \Longrightarrow \begin{cases} |f(x) - l| < \dfrac{\varepsilon}{2} \\ \text{and} \\ |f(x) - l'| < \dfrac{\varepsilon}{2} \end{cases}$$

we have

$$\begin{aligned} |l - l'| &= |l - l' + f(x) - f(x)| \\ &\leq |f(x) - l| + |f(x) - l'|\ . \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

Hence the contradiction with $\varepsilon < |l - l'|$. So $l = l'$.

Proposition 4.2.1. *If $\lim_{x\to x_0} f(x) = l$, and $\lim_{x\to x_0} g(x) = l'$, $l,\ l' \in \mathbb{R}$, then:*

1. $\lim_{x\to x_0}(\lambda.f)(x) = \lambda.\lim_{x\to x_0} f(x) = \lambda.l,\ \forall\lambda \in \mathbb{R}$.
2. $\lim_{x\to x_0}(f + g)(x) = l + l'$, *and* $\lim_{x\to x_0}(f \times g)(x) = l \times l'$.
3. *If* $l \neq 0$, *then* $\lim_{x\to x_0}\left(\frac{1}{f(x)}\right) = \frac{1}{l}$.
4. $\lim_{x\to x_0} g \circ f = l'$.
5. $\lim_{x\to x_0}\left(\frac{f(x)}{g(x)}\right) = \frac{l}{l'},\ l' \neq 0$.
6. $\lim_{x\to x_0} |f(x)| = |l|$.
7. *If* $f \leq g$, *then* $l \leq l'$.
8. *If* $f(x) \leq g(x) \leq h(x)$, *and* $\lim_{x\to x_0} f(x) = \lim_{x\to x_0} h(x) = l \in \mathbb{R}$, *then* $\lim_{x\to x_0} g(x) = l$.

4.2.3 *Relationship with limits of sequences*

Let $f : D \subset \mathbb{R} \longrightarrow \mathbb{R}$, and $x_0 \in \mathbb{R}$ so we have:

$\lim\limits_{x \to x_0} f(x) = l \Longleftrightarrow \forall$ a sequence (x_n) of D, $x_n \neq x_0$, and $\lim\limits_{n \to \infty} x_n = x_0 \Longrightarrow \lim\limits_{n \to \infty} f(x_n) = l.$

4.2.4 *Infinite limits*

Definition 4.2.4. *(Limits as $x \longrightarrow \pm\infty$)*

- $\lim_{x \to +\infty} f(x) = l \Leftrightarrow \forall \varepsilon > 0,\ \exists A > 0,\ \forall x \in \mathbb{R} :\ x > A \Rightarrow |f(x) - l| < \varepsilon.$
- $\lim_{x \to -\infty} f(x) = l \Leftrightarrow \forall \varepsilon > 0,\ \exists A > 0,\ \forall x \in \mathbb{R} :\ x < -A \Rightarrow |f(x) - l| < \varepsilon.$
- $\lim_{x \to +\infty} f(x) = +\infty$ *(resp:* $\lim_{x \to +\infty} f(x) = -\infty$*)* $\Leftrightarrow \forall A > 0,\ \exists B > 0,\ \forall x \in \mathbb{R} :\ x > B \Rightarrow f(x) > A$, *(resp:* $\forall A > 0,\ \exists B > 0,\ \forall x \in \mathbb{R} :\ x > B \Rightarrow f(x) < -A$*).*
- $\lim_{x \to -\infty} f(x) = +\infty$ *(resp:* $\lim_{x \to -\infty} f(x) = -\infty$*)* $\Leftrightarrow \forall A > 0,\ \exists B > 0,\ \forall x \in \mathbb{R} :\ x < -B \Rightarrow f(x) > A$, *(resp:* $\forall A > 0,\ \exists B > 0,\ \forall x \in \mathbb{R} :\ x < -B \Rightarrow f(x) < -A$*).*

4.2.5 *Indeterminate forms*

When the limits are not finite, the previous results remain true whenever the operations on the limits make sense.
In the case where we cannot calculate, we say that we are in the presence of an indeterminate form. If $x \longrightarrow x_0$.

1. $f(x) \longrightarrow +\infty$ and $g(x) \longrightarrow -\infty$ then $f + g$ is in the indeterminate form $+\infty - \infty$.
2. $f(x) \longrightarrow 0$ and $g(x) \longrightarrow$ then $\dfrac{f}{g}$ is in the indeterminate form $\dfrac{0}{0}$.
3. $f(x) \longrightarrow \infty$ and $g(x) \longrightarrow \infty$ then $\dfrac{f}{g}$ is in the indeterminate form $\dfrac{\infty}{\infty}$.
4. $f(x) \longrightarrow \infty$ and $g(x) \longrightarrow 0$ then $f \times g$ is in the indeterminate form $\infty \times 0$.

There are other cases of indeterminate forms of type: $1^\infty,\ 0^\infty,\ \infty^0$.

4.3 Continuous Functions

4.3.1 *Continuity at a point*

Definition 4.3.1. *Let $f : I \longrightarrow \mathbb{R}$, where $I \subset \mathbb{R}$, and suppose that $x_0 \in I$. Then f is continuous at x_0 if:*

$$\forall \varepsilon > 0,\ \exists\, \delta > 0,\ \forall x \in I :\ |x - x_0| < \delta \Longrightarrow |f(x) - f(x_0)| < \varepsilon.$$

In another word: $\lim_{x \to x_0} f(x) = f(x_0)$.

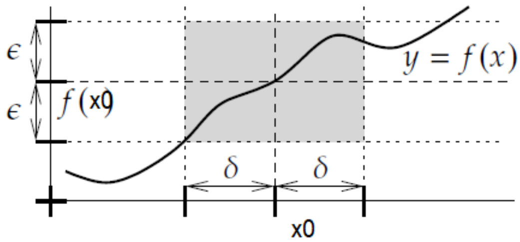


Figure 4.3: For $|x - x_0| < \delta$, the graph of $f(x)$ should be within the gray region.

A function $f : I \longrightarrow \mathbb{R}$ is continuous on a set $J \subset I$ if it is continuous at every point in J, and continuous if it is continuous at every point of its domain I.

4.3.2 *Left and right continuity*

Definition 4.3.2. *Let $f : I \longrightarrow \mathbb{R}$, we say that:*

- *f is continuous on the right of $x_0 \in I$ if:* $\lim_{x \overset{>}{\longrightarrow} x_0} f(x) = f(x_0)$.
- *f is continuous on the left of $x_0 \in I$ if:* $\lim_{x \overset{<}{\longrightarrow} x_0} f(x) = f(x_0)$.
- *f is continuous on $x_0 \in I$ if:* $\lim_{x \overset{>}{\longrightarrow} x_0} f(x) = \lim_{x \overset{<}{\longrightarrow} x_0} f(x) = f(x_0)$.

Example 4.3.1. *Let*

$$f : \mathbb{R}_+^* \longrightarrow \mathbb{R}_+$$
$$x \longrightarrow f(x) = \sqrt{x}$$

We show that f is continuous at every point $x_0 \in \mathbb{R}_+^*$, i.e.

$$\forall \varepsilon > 0,\ \exists\, \delta > 0,\ \forall x \in \mathbb{R}_+^* :\ |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon,$$

then, $\forall \varepsilon > 0$ we have:

$$|f(x) - f(x_0)| < \varepsilon \ \Rightarrow\ \left|\sqrt{x} - \sqrt{x_0}\right| < \varepsilon$$

$$\Rightarrow\ \left|\frac{x - x_0}{\sqrt{x} + \sqrt{x_0}}\right| < \varepsilon$$

$$\Rightarrow\ \frac{|x - x_0|}{\sqrt{x} - \sqrt{x_0}} < \varepsilon \Rightarrow |x - x_0| < \varepsilon\left(\sqrt{x} - \sqrt{x_0}\right).$$

So $\exists\, \delta = \varepsilon\left(\sqrt{x} - \sqrt{x_0}\right)$ such that: $|f(x) - f(x_0)| < \varepsilon$, then f is continous at x_0.

4.3.3 *Properties of continuous functions*

Theorem 4.3.1. *If $f, g : I \longrightarrow \mathbb{R}$ are continuous function at $x_0 \in I$ and $k \in \mathbb{R}$, then $k.f$, $f + g$, and $f.g$ are continuous at x_0. Moreover, if $g(x_0) \neq 0$ then f/g is continuous at x_0.*

Theorem 4.3.2. *Let $f : I \longrightarrow \mathbb{R}$ and $g : J \longrightarrow \mathbb{R}$ where $f(I) \subset J$. If f is continuous at $x_0 \in I$ and g is continuous at $f(x_0) \in J$, then $g \circ f : I \longrightarrow \mathbb{R}$ is continuous at x_0.*

Proof 15. *Fix $\varepsilon > 0$. Since g is continuous at $b = f(x_0)$,*

$$\exists\, \delta > 0,\ \forall\, y \in J :\ |y - b| < \delta \Longrightarrow |g(y) - g(b)| < \varepsilon.$$

Fix this $\delta > 0$. From the continuity of f at x_0,

$$\exists\, \gamma > 0,\ \forall\, x \in I :\ |x - x_0| < \gamma \Longrightarrow |f(x) - f(x_0)| < \delta.$$

From the above, it follows that

$$\forall\, \varepsilon > 0,\ \exists \gamma > 0,\ \forall\, x \in I:\ |x - x_0| < \gamma \Longrightarrow |g(f(x)) - g(f(x_0))| < \varepsilon.$$

This proves continuity of $g \circ f$ at x_0.

Proposition 4.3.1. *Let $f : I \longrightarrow \mathbb{R}$ and $x_0 \in I$, then:*

f is continuous at $x_0 \Longrightarrow$ for any sequence (u_n) that converges to x_0, the sequence $(f(x_0))$ converges to $f(x_0)$.

4.3.4 *Continuous extension to a point*

Definition 4.3.3. *Let f be a function defined in the neighborhood of x_0 except at x_0 ($x_0 \notin D_f$), and $\lim_{x\to x_0} f(x) = l$. Then the function which is defined by*

$$\tilde{f} = \begin{cases} f(x) & : \ x \neq x_0, \\ l & : \ x = x_0. \end{cases}$$

is continuous at x_0. $\tilde{f}$ is the continuous extension of f at x_0.

Example 4.3.2. *Show that:*

$$f(x) = \frac{x^2 + x - 6}{x^2 - 4},\ x \neq 2.$$

has a continuous extension to $x = 2$, and find that extension.

Solution:

$\lim_{x\to 2} f(x) = \lim\limits_{x\to 2} \dfrac{x^2 + x - 6}{x^2 - 4} = \lim\limits_{x\to 2} \dfrac{(x-2)(x+3)}{(x-2)(x+2)} = \dfrac{5}{4}$, *exists. So f has a continuous extension at $x = 2$ defined by*

$$\tilde{f} = \begin{cases} \dfrac{x^2 + x - 6}{x^2 - 4} & : \ x \neq 2, \\[2ex] \dfrac{5}{4} & : \ x = 2. \end{cases}$$

4.3.5 *Discontinuous functions*

When f is not continuous at x_0, we say f is discontinuous at x_0, or that it has a discontinuity at x_0.

We say that the function f is not continuous in the following cases:

1. If f is not defined at x_0, then f is discontinuous at x_0.

2. If f defined in the neighborhood of x_0, then f is discontinuous at x_0 if

$$\exists\, \varepsilon > 0,\ \forall \delta > 0,\ \exists\, x \in I:\ |x - x_0| < \delta,\ and\ |f(x) - f(x_0)| \geq \varepsilon.$$

3. If $\lim_{x \overset{>}{\longrightarrow} x_0} f(x) \neq \lim_{x \overset{<}{\longrightarrow} x_0} f(x)$, then f is discontinuous at x_0, and x_0 is a discontinuous point of the first kind.

4. If one of the two limits $\lim_{x \overset{>}{\longrightarrow} x_0} f(x)$, $\lim_{x \overset{<}{\longrightarrow} x_0} f(x)$ or both limits does not exist or are not finite, then f is discontinuous at x_0, and x_0 is a discontinuous point of the second kind.

5. If $\lim_{x \overset{<}{\longrightarrow} x_0} f(x) = \lim_{x \overset{>}{\longrightarrow} x_0} f(x) \neq f(x_0)$, then f is discontinuous at x_0.

4.3.6 *Uniform continuity*

Definition 4.3.4. *Let $f : I \longrightarrow \mathbb{R}$. Then f is uniformly continuous on I if:*

$$\forall \varepsilon > 0,\ \exists\, \delta > 0,\ \forall x',\ x'' \in I:\ |x' - x''| < \delta \Longrightarrow |f(x') - f(x'')| < \varepsilon.$$

Remark 4.3.1.

1. *Uniform continuity is a property of the interval form, whereas continuity can be defined at a point.*

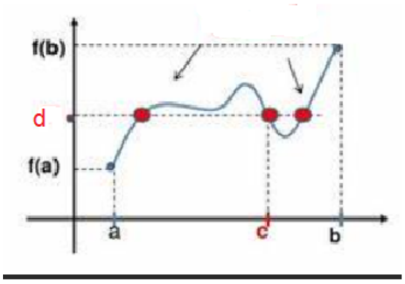
2. *The number δ does not depend on ε whereas for continuity δ depends on ε and x_0.*

3. *Let $f : I \longrightarrow \mathbb{R}$ be a function. If f is uniformly continuous, then f is continuous.*

Example 4.3.3. *$f(x) = x$ and $g(x) = \sin x$ are uniformly continuous on $\mathbb{R}$ (we find $\delta = \varepsilon$).*

4.3.7 *The intermediate value theorem*

Theorem 4.3.3. *Suppose that $f : [a, b] \longrightarrow \mathbb{R}$ is a continuous function on a closed bounded interval. Then for every d strictly between $f(a)$ and $f(b)$ there is a point $a < c < b$ such that $f(c) = d$.*



Corollary 4.3.1. *Suppose that $f : [a, b] \longrightarrow \mathbb{R}$ is a continuous function on a closed bounded interval. If $f(a).f(b) < 0$, then there is a point $a < c < b$ such that $f(c) = 0$.*

Corollary 4.3.2. *Let $f : D \longrightarrow \mathbb{R}$ is a continuous function and $I \subseteq D$ is an interval, then $f(I)$ is an interval.*

Theorem 4.3.4. *Let $I = [a, b]$ be a closed interval, and $f : [a, b] \longrightarrow \mathbb{R}$ be a continuous function. Then f is uniformly continuous.*

Theorem 4.3.5. *Any continuous function on a closed interval $[a, b]$ is bounded on $[a, b]$, i.e:* $\sup_{[a,b]} |f(x)| < +\infty$.

Remark 4.3.2.
1. *The image by a continuous function of a closed interval of $\mathbb{R}$ is a closed interval.*
2. *If I is not closed then the interval $f(I)$ is not necessarily of the nature of I. For example: $f(x) = x^2$, then $f(]-1, 1[) = [0, 1[$.*

4.3.8 *Fixed point theorem*

Definition 4.3.5. *Let $f : I \longrightarrow I$ and let $\dot{x} \in I$, we say that $\dot{x} \in I$ is a fixed point of f if: $f(\dot{x}) = \dot{x}$.*

Theorem 4.3.6. *Let $f : [a,b] \longrightarrow [a,b]$ be a continuous function, then f admits at least one fixed point in $[a,b]$ i.e: $\exists \dot{x} \in [a,b]$ such that $f(\dot{x}) = \dot{x}$.*

Exercise 4.3.1. *Let f be a continuous function on $[a,b]$ and $x_1,\ x_2,\ \cdots, x_n \in [a,b]$. Prove that there exists $c \in [a,b]$ with*

$$f(c) = \frac{f(x_1) + f(x_2) + \cdots + f(x_n)}{n}.$$

Solution:

Let $\alpha = \min\{f(x) :\ x \in [a,\ b]\}$, and $\beta = \max\{f(x) :\ x \in [a,\ b]\}$. Then

$$\frac{f(x_1) + f(x_2) + \cdots + f(x_n)}{n} \leq \frac{n\beta}{n} = \beta.$$

Similarly,

$$\frac{f(x_1) + f(x_2) + \cdots + f(x_n)}{n} \geq \alpha.$$

Then the conclusion follows from the Intermediate Value Theorem.

Exercise 4.3.2. *Consider k distinct points $x_1,\ x_2, \cdots, x_k \in \mathbb{R},\ k \geq 1$. Find a function defined on $\mathbb{R}$ that is continuous at each $x_i,\ i = 1, \cdots, k$ and discontinuous at all other points.*

Solution: Consider

$$f(x) = \begin{cases} (x-a_1)(x-a_2)\cdots(x-a_k), & if\ x \in \mathbb{Q}, \\ 0, & if\ x \in \mathbb{Q}^c. \end{cases}$$

5

Differentiable Functions

5.1 The Derivative

5.1.1 *Definition and basic properties*

Definition 5.1.1. *Let I be an interval, and $c \in I$, let $f : I \longrightarrow \mathbb{R}$ be a function defined in the neighborhood of c. If the limit*

$$l = \lim_{x \longrightarrow c} \frac{f(x) - f(c)}{x - c},$$

exists in $\mathbb{R}$, then we say that f is differentiable at c. When this limit exists, it is denoted by $f'(c)$ and called the derivative of f at c.
If f is differentiable at all $c \in I$, then we simply say that f is differentiable. The derivative is sometimes written as $\frac{df}{dx}$ or $\frac{d}{dx}(f(x))$. The expression $\frac{f(x) - f(c)}{x - c}$ is called the difference quotient.

The graphical interpretation of the derivative is depicted in Figure 5.1. The left-hand plot gives the line through $(c, f(c))$ and $(x, f(x))$ with slope $\frac{f(x) - f(c)}{x - c}$, that is, the so-called secant line. When we take the limit as x goes to c, we get the right-hand plot, where we see that the derivative of the function at the point c is the slope of the line tangent to the graph of f at the point $(c, f(c))$.

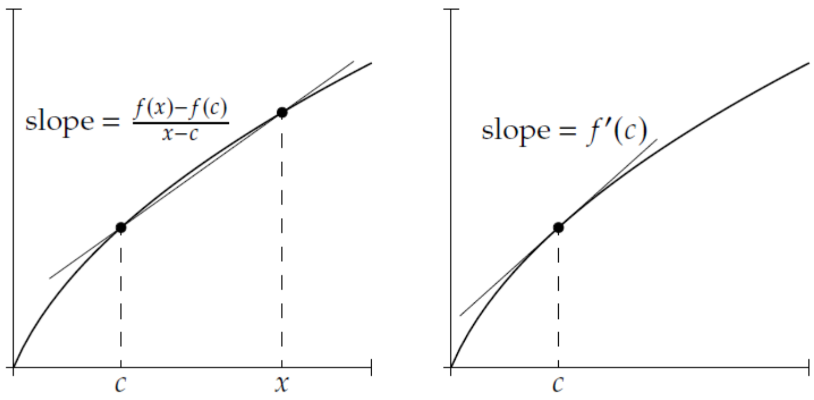


Figure 5.1: Graphical interpretation of the derivative

Example 5.1.1. *Let $f(x) = x^2$ defined on the whole real line, and let $c \in \mathbb{R}$ be arbitrary. We find that if $x \neq c$,*

$$\frac{x^2 - c^2}{x - c} = \frac{(x+c)(x-c)}{x-c} = x + c.$$

Therefore,

$$f'(c) = \lim_{x \longrightarrow c} \frac{x^2 - c^2}{x - c} = \lim_{x \longrightarrow c} (x + c) = 2c.$$

Example 5.1.2. *The function $f(x) = \sqrt{x}$ is differentiable for $x > 0$. To see this fact, fix $c > 0$, and suppose $x \neq c$ and $x > 0$. Compute*

$$\frac{\sqrt{x} - \sqrt{c}}{x - c} = \frac{\sqrt{x} - \sqrt{c}}{(\sqrt{x} - \sqrt{c})(\sqrt{x} + \sqrt{c})} = \frac{1}{\sqrt{x} + \sqrt{c}}.$$

Therefore,

$$f'(c) = \lim_{x \longrightarrow c} \frac{\sqrt{x} - \sqrt{c}}{x - c} = \lim_{x \longrightarrow c} \frac{1}{\sqrt{x} + \sqrt{c}} = \frac{1}{2\sqrt{c}}.$$

Remark 5.1.1. *If we put $x - c = h$, the quantity $\frac{f(x) - f(c)}{x - c}$ becomes $\frac{f(c+h) - f(c)}{h}$. So we can define the notion of differentiability of f at c in the following way:*

$$f \text{ is differentiable at } c \Leftrightarrow \lim_{h\longrightarrow 0}\frac{f(c+h)-f(c)}{h} \text{ exists in } \mathbb{R}.$$

Proposition 5.1.1. *Let $f : I \longrightarrow \mathbb{R}$ be differentiable at $c \in I$, then it is continuous at c.*

Proof 16. *We know the limits*

$$\lim_{x\longrightarrow c}\frac{f(x)-f(c)}{x-c} = f'(c), \qquad \text{and} \qquad \lim_{x\longrightarrow c}(x-c) = 0.$$

exists. Furthermore,

$$f(x)-f(c) = \left(\frac{f(x)-f(c)}{x-c}\right)(x-c),$$

Therefore, the limit of $f(x)-f(c)$ exists and

$$\lim_{x\longrightarrow c}(f(x)-f(c)) = \left(\lim_{x\longrightarrow c}\frac{f(x)-f(c)}{x-c}\right)\left(\lim_{x\longrightarrow c}(x-c)\right) = f'(c).0 = 0.$$

Hence $\lim_{x\longrightarrow c} f(x) = f(c)$, and f is continuous at c.

Proposition 5.1.2. *If f is differentiable over I, then f is continuous over I.*

Proposition 5.1.3. *Let I be an interval, let $f : I \longrightarrow \mathbb{R}$ and $g : I \longrightarrow \mathbb{R}$ be a differentiable functions at $c \in I$, and let $\alpha \in \mathbb{R}$, then:*

1. ***The linearity:***
   - *Define $h : I \longrightarrow \mathbb{R}$ by $h(x) = \alpha.f(x)$. Then h is differentiable at c and $h'(c) = \alpha.f'(c)$.*
   - *Define $h : I \longrightarrow \mathbb{R}$ by $h(x) = f(x) + g(x)$. Then h is differentiable at c and $h'(c) = f'(c) + g'(c)$.*
2. ***Product rule:***
   If $h : I \longrightarrow \mathbb{R}$ is defined by $h(x) = g(x)f(x)$, then h is differentiable at c and

$$h'(c) = f(c)g'(c) + f'(c)g(c).$$

Proof 17. *We have:*

$$\begin{aligned}
\lim_{h\to 0}\frac{(f.g)(c+h)-(f.g)(c)}{h} &= \lim_{h\to 0}\frac{f(c+h).g(c+h)-f(c).g(c)}{h}\\
&= \lim_{h\to 0}\left[\frac{f(c+h)[g(c+h)-g(c)]}{h}+\frac{[f(c+h)-f(c)]g(c)}{h}\right]\\
&= \lim_{h\to 0}f(c+h)\lim_{h\to 0}\frac{g(c+h)-g(c)}{h}+\lim_{h\to 0}\frac{f(c+h)-f(c)}{h}\lim_{h\to 0}g(c)\\
&= f'(c)g(c)+f(c)g'(c).
\end{aligned}$$

3. ***Quotient rule:***
If $g(x)\neq 0$ for all $x\in I$, and if $h: I\longrightarrow\mathbb{R}$ is defined by $h(x)=\dfrac{f(x)}{g(x)}$, then h is differentiable at c and

$$h'(c)=\frac{f'(c)g(c)-f(c)g'(c)}{(g(c))^2}.$$

5.1.2 *Chain rule*

Proposition 5.1.4. *Let I, and J be an intervals, let $g: I\longrightarrow J$ be a differentiable at $c\in I$, and $f: J\longrightarrow\mathbb{R}$ be differentiable at $g(c)$. If $h: I\longrightarrow\mathbb{R}$ is defined by*

$$h(x)=(f\circ g)(x)=f(g(x)),$$

then h is differentiable at c and

$$h'(c)=f'(g(c))g'(c).$$

5.1.3 *Inverse function*

Proposition 5.1.5. *Let $I\subset\mathbb{R}$ be an interval, and let f be an injective and continuous function on I. If f is differentiable at a point c with $f'(c)\neq 0$, then the inverse function: $f^{-1}: f(I)\longrightarrow\mathbb{R}$ is differentiable at $f(c)$ and*

$$(f^{-1}(f(c)))'=\frac{1}{f'(c)}.$$

5.2 Left and Right Derivatives

Definition 5.2.1. *Let $f : [a, b] \longrightarrow \mathbb{R}$ be a function, we say that f is right-differentiable at $a \le c < b$ with right derivative $f'(c^+)$ if*

$$\lim_{x \overset{>}{\longrightarrow} c} \frac{f(x) - f(c)}{x - c} = f'(c^+),$$

exists, and f is left-differentiable at $a < c \le b$ with left derivative $f'(c^-)$ if

$$\lim_{x \overset{<}{\longrightarrow} c} \frac{f(x) - f(c)}{x - c} = f'(c^-) \text{ exists.}$$

A function is differentiable at $a < c < b$ if and only if the left and right derivatives exist at c and are equal.

Remark 5.2.1. *If $f'(c^+)$ and $f'(c^-)$ exist but $f'(c^+) \neq f'(c^-)$ then f is not differentiable at c and point $(c, f(c))$ is an angular point.*

Example 5.2.1. *The absolute value function $f(x) = |x|$ is left and right differentiable at 0 with left and right derivatives*

$$f'(0^+) = 1 \quad \text{and} \quad f'(0^-) = -1.$$

These are not equal, and f is not differentiable at 0.

5.3 Successive Derivatives and Leibnitz's Rule

5.3.1 *Successive derivatives*

Let f be a function differentiable on I, then f' is called the first order derivative of f, if f' is differentiable on I, then its derivative is called the second order derivative of f and is denoted by f'' or $f^{(2)}$. Recursively, we define the derivative of order n of f as follows: $f^{(n)}(x) = (f^{(n-1)}(x))'$.

Example 5.3.1. *1). Let $f(x) = \sin(x)$. Calculate $f^{(n)}(x)$. We have:*

$$\begin{aligned}
f^{(0)}(x) &= \sin(x),\\
f'(x) &= f^{(1)}(x) = \cos(x) = \sin(x+\frac{\pi}{2}),\\
f^{(2)}(x) &= -\sin(x) = \sin(x+\pi),\\
f^{(3)}(x) &= -\cos(x) = \sin(x+\frac{3\pi}{2}),\\
f^{(4)}(x) &= \sin(x) = \sin(x+2\pi),\\
&\vdots\\
f^{(n)}(x) &= \sin(x+\frac{n\pi}{2}).
\end{aligned}$$

2). $f(x) = \ln x$. Calculate $f^{(n)}(x)$. We have:

$$f^{(0)}(x) = \ln x, \qquad f'(x) = \frac{1}{x},$$

$$f^{(2)}(x) = \frac{-1}{x^2}, \qquad f^{(3)}(x) = \frac{2}{x^3},$$

$$f^{(4)}(x) = \frac{-2\times 3}{x^4}, \qquad f^{(5)}(x) = \frac{2\times 3\times 4}{x^5} = \frac{4!}{x^5},$$

$$\vdots$$

$$f^{(n)}(x) = (-1)^{n+1}\frac{(n-1)!}{x^n},\ n\in\mathbb{N}^*.$$

Definition 5.3.1. *(Class Functions: C^n)*
Let n be a non-zero natural number. A function f defined on I is said to be of class C^n or n times continuously differentiable if it is n times differentiable and $f^{(n)}$ is continuous on I, and we note $f \in C^n(I)$.

Remark 5.3.1. *A function f is said to be Ťof class C^0Ť if it is continuous on I.*

Definition 5.3.2. *(Class Functions: C^∞)*
A function f is said to be of class C^∞ on I if it is in the class C^n. $\forall n \in \mathbb{N}$. For example $f(x) = e^x$.

5.3.2 *Leibnitz formula*

Theorem 5.3.1. *Let f and g be two functions n times differentiable on I, then $f \times g$ is $n-$times differentiable on I, and we have:*

$$(f\times g)^{(n)} = \sum_{k=0}^{n} C_n^k\, f^{(n-k)}\, g^{(k)},\ C_n^k = \frac{n!}{k!(n-k)!}.$$

Example 5.3.2. *For $n = 2$, we have:*

$$\begin{aligned}(f\times g)^{(2)} &= C_2^0 f'' g + C_2^1 f' g' + C_2^2 f g'' \\ &= f'' g + 2f' g' + f g''.\end{aligned}$$

For $n = 6$, we have:

$$\begin{aligned}(f\times g)^{(6)} &= C_6^0 f^{(6)} g + C_6^1 f^{(5)} g' + C_6^2 f^{(4)} g'' + C_6^3 f^{(3)} g^{(3)} + C_6^4 f'' g^{(4)} + C_6^5 f' g^{(5)} + C_6^6 f g^{(6)} \\ &= f^{(6)} g + 6f^{(5)} g' + 15f^{(4)} g'' + 20f^{(3)} g^{(3)} + 15f'' g^{(4)} + 6f' g^{(5)} + f g^{(6)}.\end{aligned}$$

If $h(x) = \left(x^3 + 5x + 1\right) e^x = f(x)g(x)$, then:

$$\begin{array}{ll} f'(x) = 3x^2 + 5, & g'(x) = e^x, \\ f''(x) = 6x, & g''(x) = e^x, \\ f^{(3)}(x) = 6, & g^{(3)}(x) = e^x, \\ f^{(4)}(x) = 0, & g^{(4)}(x) = e^x, \\ f^{(n)}(x) = 0,\ \forall n \geq 4, & g^{(n)}(x) = e^x. \end{array}$$

So:

$$\begin{aligned}h^{(n)}(x) &= C_n^0 f g^{(n)} + C_n^1 f' g^{(n-1)} + C_n^2 f'' g^{(n-2)} + C_n^3 f^{(3)} g^{(n-3)} + C_n^4 f^{(4)} g^{(n-4)} + \cdots \\ &= (x^3 + 5x + 1)e^x + n(3x^2 + 5)e^x + \frac{n(n-1)}{2}(6x)e^x + \frac{n(n-1)(n-2)}{6} 6e^x.\end{aligned}$$

5.4 The Mean Value Theorem

5.4.1 *Extreme values*

Definition 5.4.1. *A critical point of a function $f(x)$, is a value c in the domain of f where f is not differentiable or its derivative is* 0 *(i.e. $f'(c) = 0$).*

Definition 5.4.2. *A function f is said to have a local maximum (local minimum) at c if f is defined on an open interval I containing c and $f(x) \leq f(c)$ ($f(x) \geq f(c)$) for all $x \in I$. In either case, f is said to have a local extremum at c.*

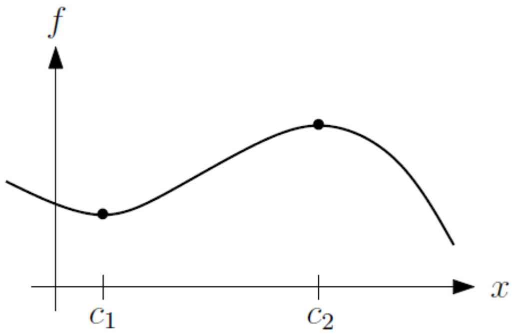


Figure 5.2: Local extrema of f

5.4.2 *Local extremum theorem*

Theorem 5.4.1. *If f has a local extremum at c and if f is differentiable at c, then $f'(c) = 0$.*

Proof. Suppose that f has a local maximum at c. Let I be an open interval containing c such that $f(x) \leq f(c)$ for all $x \in I$. Then:

$$\frac{f(x) - f(c)}{x - c} = \begin{cases} \geq 0, & if\ x \in I \text{ and } x < c, \\ \leq 0, & if\ x \in I \text{ and } x > c. \end{cases}$$

It follows that the left-hand derivative of f at c is ≥ 0 and the right-hand derivative is ≤ 0, hence $f'(c) = 0$. The proof for the local minimum case is similar. □

5.4.3 *Rolle's theorem*

Theorem 5.4.2. *Let f be a continuous function on $[a, b]$ and differentiable on $]a, b[$. If $f(a) = f(b)$, then there exists a point $c \in]a, b[$ such that $f'(c) = 0$.*

Proof. By the extreme value theorem there exist $x_m, x_M \in [a, b]$ such that $f(x_m) \leq f(x) \leq f(x_M)$ for all $x \in [a, b]$. If $f(x_m) = f(x_M)$, then f is a constant function and the assertion of the theorem holds trivially. If $f(x_m) \neq f(x_M)$, then either $x_m \in]a, b[$ or $x_M \in]a, b[$, and the conclusion follows from the local extremum theorem. □

5.4.4 *Mean value theorem*

Theorem 5.4.3. *If f is continuous on $[a, b]$ and differentiable on $]a, b[$, then there exists $c \in]a, b[$ such that:*

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

Proof. The function $g : [a, b] \longrightarrow \mathbb{R}$ defined by:

$$g(x) = f(x) - f(a) - \left[\frac{f(b) - f(a)}{b - a}\right](x - a),$$

is continuous on $[a, b]$ and differentiable on $]a, b[$ with

$$g'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}.$$

Moreover, $g(a) = g(b) = 0$. Rolle's theorem implies that there exists $a < c < b$ such that $g'(c) = 0$, which proves the result. □

5.4.5 *Mean value inequality*

Let f be a continuous function on $[a, b]$, and differentiable on $]a, b[$. If there exists a constant M such that: $\forall\, x \in]a, b[$: $|f'(x)| \leq M$, then

$$\forall\, x,\ y \in [a, b] :\ |f(x) - f(y)| \leq M\,|x - y|.$$

According to the Mean value theorem on $[x, y]$, $\exists\, c \in]x, y[$: $f'(c) = \dfrac{f(x) - f(y)}{x - y}$. Then

$$|f'(c)| \leq M \Longrightarrow \left|\frac{f(x) - f(y)}{x - y}\right| \leq M \Longrightarrow M\,|x - y|\,.$$

5.5 Variation of a Functions

Let f be a continuous function on $[a,b]$, and differentiable on $]a,b[$ then:

1. $\forall\ x \in]a,b[$: $f'(x) > 0 \Longleftrightarrow f$ is strictly increasing on $[a,b]$.

2. $\forall\ x \in]a,b[$: $f'(x) < 0 \Longleftrightarrow f$ is strictly decreasing on $[a,b]$.

3. $\forall\ x \in]a,b[$: $f'(x) = 0 \Longleftrightarrow f$ is a constant.

5.6 L'Hôpital's Rule

Let f and g be two continuous functions on I (I is a neighborhood of c), differentiable on $I - \{c\}$, and satisfying the following conditions:

- $\lim\limits_{x \longrightarrow c} f(x) = \lim\limits_{x \longrightarrow c} g(x) = 0$ or $\pm\infty$.

- $g'(x) \neq 0,\ \forall x \in I - \{c\}$.

then:

$$\text{if } \lim_{x \longrightarrow c} \frac{f'(x)}{g'(x)} = l \Longrightarrow \lim_{x \longrightarrow c} \frac{f(x)}{g(x)} = l.$$

Example 5.6.1. *Using L'Hopital's rule:*

1. $\lim\limits_{x \longrightarrow 0} \dfrac{3x - \sin x}{x} = \lim\limits_{x \longrightarrow 0} \dfrac{3 - \cos x}{1} = 2.$

2. $\lim\limits_{x \longrightarrow 0} \dfrac{\sqrt{1+x} - 1}{x} = \lim\limits_{x \longrightarrow 0} \dfrac{\frac{1}{2\sqrt{1+x}}}{1} = \dfrac{1}{2}.$

Remark 5.6.1. *The converse is generally false. For example:* $f(x) = x^2 \cos(\frac{1}{x})$*, and* $g(x) = x$*, so we have* $\lim\limits_{x \longrightarrow 0} \dfrac{f(x)}{g(x)} = \lim\limits_{x \longrightarrow 0} x \cos(\frac{1}{x}) = 0$ *while* $\lim\limits_{x \longrightarrow 0} \dfrac{f'(x)}{g'(X)} = \lim\limits_{x \longrightarrow 0} (2x \cos(\frac{1}{x}) + \sin(\frac{1}{x}))$ *does not exists because (*$\lim\limits_{x \longrightarrow 0} \sin(\frac{1}{x})$ *does not exists).*

5.7 Convex Functions

Definition 5.7.1. *A function f is said to be convex on an interval I if*

$$f(tx+(1-t)y) \leq tf(x)+(1-t)f(y),\ \forall\ t \in [0.1],\ x,\ y \in I.$$

f is concave if $-f$ is convex.

Example 5.7.1.

1. *The function $x \longrightarrow |x|$ is convex on $\mathbb{R}$ because $|tx+(1+t)y| \leq t|x|+(1-t)|y|$.*
2. *The affine functions $f : x \longrightarrow \alpha x+\beta$ are both convex and concave on $\mathbb{R}$, because they indeed satisfy $f(xt+(1-t)y) = tf(x)+(1-t)f(y)$. Conversely, if a function is both convex and concave then it is affine.*

Theorem 5.7.1. *If $f :]a,b[\longrightarrow \mathbb{R}$ has an increasing derivative, then f is convex. In particular, f is convex if $f'' \geq 0$.*

Example 5.7.2. *Consider the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ given by $f(x) = \sqrt{x^2+1}$. We have $f'(x) = \dfrac{x}{\sqrt{x^2+1}}$, and $f''(x) = \dfrac{1}{(x^2+1)^{\frac{3}{2}}}$. Since $f''(x) \geq 0$ for all x, it follows from the corollary that f is convex.*

Remark 5.7.1. *If $f : I \longrightarrow \mathbb{R}$ is convex then:*

- *f differentiable on the left and right (therefore continues) and $f'_l \leq f'_r$.*
- *The functions $f'_l,\ f'_r$ are increasing.*
- *f is continuous at every interior point of I.*
- *Let $f : I \longrightarrow \mathbb{R}$ a differentiable function. Then f is convex $\Longleftrightarrow$ f' is increasing on I.*
- *A concave function on I is continuous at all points interior to I.*
- *If f is differentiable and concave $\Longleftrightarrow$ f is decreasing.*

6

Elementary Functions

In our calculus course, we are going to deal mostly with **elementary functions**. They are

- Power functions $\left(x^2,\ \sqrt{x},\ x^{\frac{1}{3}}, \cdots\right)$,
- Exponential functions $(2^x,\ e^x,\ \pi^x, \cdots)$,
- Logarithmic functions $(\ln x,\ \log_2 x, \cdots)$,
- Trigonometric functions $(\sin x,\ \cos x,\ \tan x, \cdots)$,
- Inverse trigonometric functions $(\arcsin x,\ \arccos x,\ \arctan x, \cdots)$,
- Hyperbolic functions $(chx,\ shx,\ thx, \cdots)$,

and their sums, differences, products, quotients, and compositions. For example

$$f(x) = \frac{\arcsin \sqrt{x^2 - 3}}{\ln(x^4 + 3) - \tan e^{\cos x}} \text{ is an elementary function.}$$

6.1 Power functions

6.1.1 *Review of exponents*

We start at the beginning. For a number a and a positive integer n,

$$a^n = \underbrace{a.a.a.\cdots.a}_{n\ times}.$$

6.1.2 *Basic laws of exponents*

$$a^1 = a, \qquad (ab)^n = a^n b^n, \qquad \left(\frac{a}{b}\right)^n = \frac{a^n}{b^n},$$
$$a^m a^n = a^{m+n}, \qquad \frac{a^m}{b^n} = a^{m-n}, \qquad (a^m)^n = a^{mn}.$$

6.1.3 *Definition of power functions*

Definition 6.1.1. *Let $a \in \mathbb{R}$, we name power function of exponent a, the function defined by*

$$\forall x \in]0, +\infty[, \; x^a = e^{a\ln(x)}.$$

For example, $y = x$, $y = x^4$, $y = x^{\frac{2}{3}}$ are power functions.

In a power function $f(x) = x^a$, the base x is a variable, and the exponent a is a constant.

The appearance of the graph of a power function depends on the constant a .

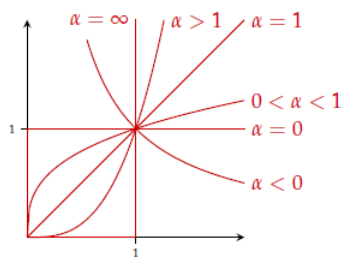


Figure 6.1: Power function with real exponents.

Definition 6.1.2. ***(Power functions** $y = x^n$)*

If n is an integer greater than 1*, then the overall shape of the graph of* $y = x^n$ *is determined by the parity of n (whether n is even or odd).*

- *If n is even, then the graph has a shape similar to the parabola* $y = x^2$*.*
- *If n is odd, then the graph has a shape similar to the cubic parabola* $y = x^3$*.*

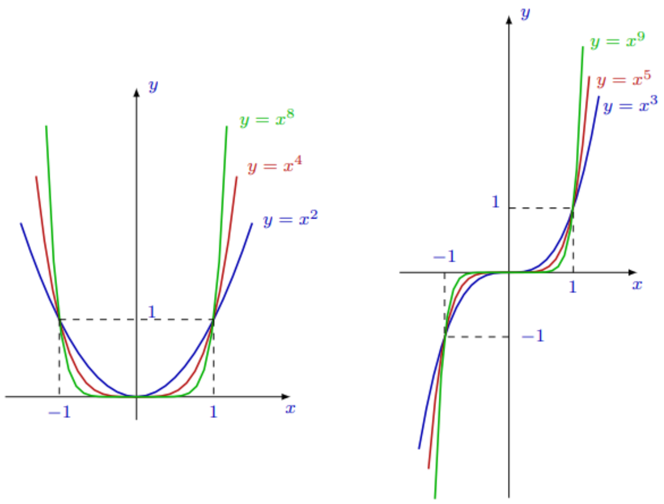


Figure 6.2: Power function with integer exponents.

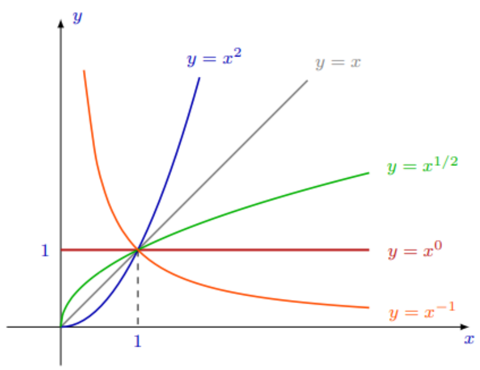


Figure 6.3: The graphs of $y = x^n$ for some rational n and $x > 0$.

Proposition 6.1.1. *1. For* $a \in \mathbb{R}^*$*, the power function with exponent a is a continuous function* *on* $]0, +\infty[$*, and strictly monotonic (strictly increasing if* $a > 0$ *and strictly decreasing if* $a < 0$*).*

2. *It is differentiable on* $]0, +\infty[$ *with derivative:* $(x^a)' = ax^{a-1}, \ \forall x \in]0, +\infty[$.

3. *We have:*

$$\lim_{x\to+\infty} x^a = \begin{cases} 0 & : a < 0 \\ 1 & : a = 0 \\ +\infty & : a > 0 \end{cases} \quad \text{and} \quad \lim_{x\to 0^+} x^a = \begin{cases} +\infty & : a < 0 \\ 1 & : a = 0 \\ 0 & : a > 0 \end{cases}$$

6.2 Logarithm and Exponential Functions

6.2.1 *Logarithm*

Definition 6.2.1. *The function that satisfies the following two conditions is called the neperian logarithm function and is denoted by* ln

- $\forall x \in \mathbb{R}_+^*, \ \ln'(x) = \dfrac{1}{x}$.
- $\ln(1) = 0$.

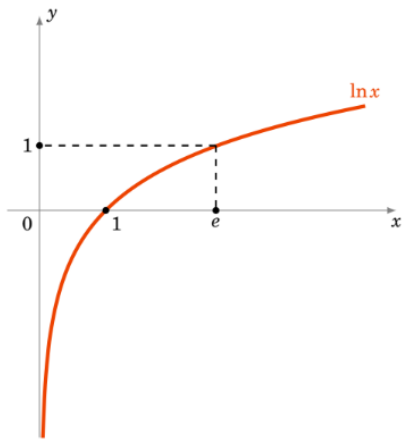


Figure 6.4: Logarithm function

Remark 6.2.1. *(Properties of derivatives)*

1. *According to the previous definition, the function* $\ln(x)$ *is differentiable on* $\mathbb{R}_+^*$ *and* $\forall x \in \mathbb{R}_+^*$ $(\ln(x))' = \frac{1}{x}$.
2. *The function ln*$(|x|)$ *is differentiable on* $\mathbb{R}^*$ *and* $\forall x \in \mathbb{R}^*$ $(\ln|x|)' = \frac{1}{x}$.
3. *Let g be a function differentiable and non-zero on I then the function* $\ln(|g(x)|)$ *is differentiable on I and its derivative:* $\ln(|g(x)|)' = \frac{g'(x)}{g(x)}$.

Proposition 6.2.1. *(Algebraic properties of the function* $\ln(x)$*)*
The logarithm function satisfies the following properties: (for all $a,\ b > 0$ *):*

1. $\ln(a \times b) = \ln a + \ln b$,
2. $\ln(\frac{a}{b}) = \ln a - \ln b$,
3. $\ln(\frac{1}{a}) = -\ln a$,
4. $\ln(a^n) = n \ln a$, *for all* $n \in \mathbb{N}$.

Proposition 6.2.2. *(Limits and classical inequalities)*

1. $\lim_{x\to+\infty} \ln(x) = +\infty$, *and* $\lim_{x\to 0^+} \ln(x) = -\infty$.
2. $\lim_{x\to+\infty} \frac{\ln(x)}{x} = 0$.
3. $\lim_{x\to+\infty} \frac{\ln(x)}{x^p} = 0,\ p \in \mathbb{R}_+^*$.
4. $\lim_{x\to 0^+} \frac{\ln(x+1)}{x} = 1$.
5. $\lim_{x\to 0^+} x \ln(x) = 0$.
6. $\forall x \in]-1, +\infty[,\ \ln(x+1) \leq x$.

Remark 6.2.2. *Let* $a \in]0,1[\cup]1,+\infty[$, *we call the logarithm function with base a and denote* $\log_a$, *the function defined by:*

$$\log_a = \frac{\ln x}{\ln a}, \ \forall x > 0.$$

- *We have:* $\ln(x) = \log_e(x)$ *i.e., the neperian logarithm function is the logarithm function with base e.*
- $\log_a(a) = 1$.

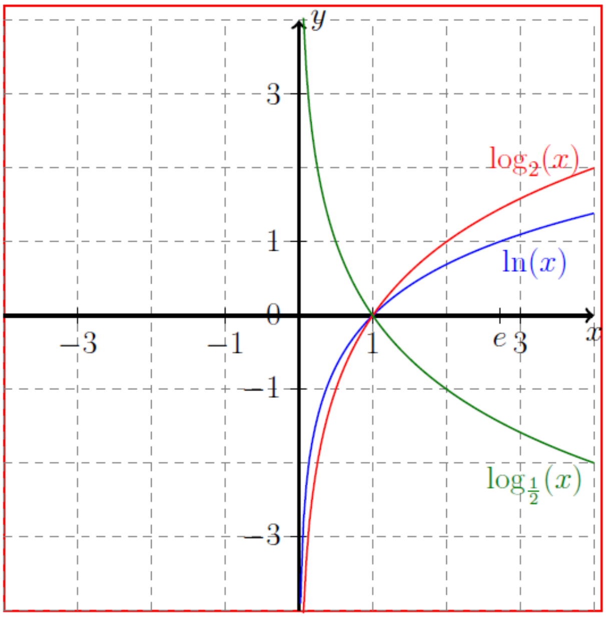


Figure 6.5: Graphical representation of the logarithmic functions and logarithms with base a for $a = \frac{1}{2}$, $a = 2$

6.2.2 *Exponential*

Definition 6.2.2. *The inverse function of the function* $\ln(x)$ *is called the exponential function and is denoted by:* $\exp(x)$ *or* e^x, *and satisfies the following properties:*

1. $\forall x > 0, \ x = e^{\ln(x)}$.
2. $\forall \ y \in \mathbb{R}, \ y = \ln(e^y)$.

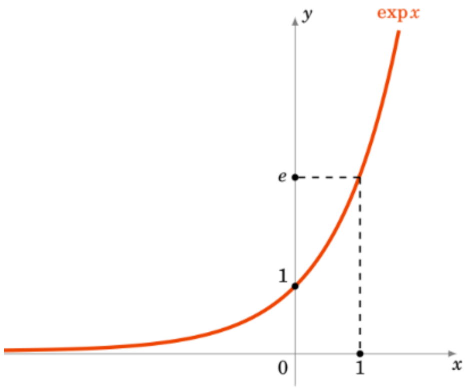


Figure 6.6: Exponential function

Proposition 6.2.3. *1. The function e^x is continuous and strictly increasing on $\mathbb{R}$.*

2. The function e^x is differentiable on $\mathbb{R}$ and we have: $\forall x \in \mathbb{R},\ (e^x)' = e^x$.

3. If u is differentiable on I then: the function $e^{u(x)}$ is differentiable on I and its derivative defined by: $\forall x \in I,\ (e^{u(x)})' = u'(x).e^{u(x)}$.

Proposition 6.2.4. *(Algebraic properties of the function e^x):*

1. $e^{x+y} = e^x \times e^y,\ \forall\ x,\ y \in \mathbb{R}.$
2. $e^{-x} = \frac{1}{x},\ \forall\ x \in \mathbb{R}.$
3. $e^{x-y} = \frac{x}{y},\ \forall\ x,\ y \in \mathbb{R}.$
4. $e^{nx} = (e^x)^n,$

Proposition 6.2.5. *(Limits and inequalities):*

1. $\lim_{x\to-\infty} e^x = 0.$
2. $\lim_{x\to+\infty} e^x = +\infty.$
3. $\lim_{x\to+\infty} xe^{-x} = 0,\ \lim_{x\to+\infty} \frac{e^x}{x^a} = +\infty,\ \lim_{x\to+\infty} \frac{x^a}{e^x} = 0,\ a \in \mathbb{R}.$
4. $\lim_{x\to 0} \frac{e^x - 1}{x} = 1.$

5. $\forall x \in \mathbb{R},\ e^x \geq 1 + x.$

Remark 6.2.3. *Let $a \in]0,1[\cup]1,+\infty[$. The inverse function of the function $log_a(x)$ is called the exponential function with base a and is denoted a^x:*

- $\forall x \in \mathbb{R},\ a^x = e^{x\ln(a)}.$
- $\forall x \in \mathbb{R},\ log_a(a^x) = log_a(e^{x\ln(a)}) = \dfrac{\ln(e^{x\ln(a)})}{\ln(a)} = x.$

6.3 Trigonometric Functions

6.3.1 *Sine function*

Definition 6.3.1. *The sine function $y = \sin x$ is defined as follows*

$$\begin{aligned}\sin :\ \mathbb{R} &\longrightarrow [-1,1]\\ x &\longrightarrow \sin x.\end{aligned}$$

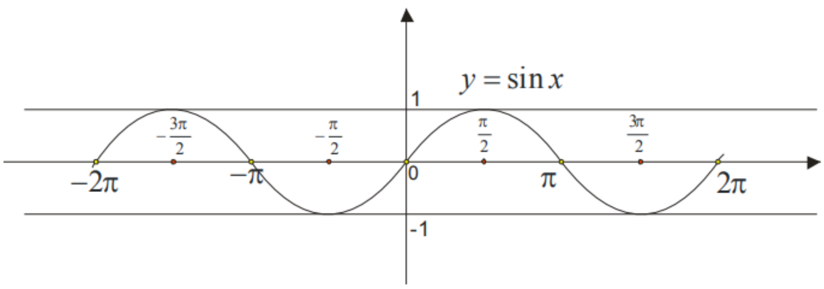


Figure 6.7: Sine function

6.3.2 *Cosine function*

Definition 6.3.2. *The cosine function $y = \cos x$ is defined as follows*

$$\begin{aligned}\cos :\ \mathbb{R} &\longrightarrow [-1,1]\\ x &\longrightarrow \cos x.\end{aligned}$$

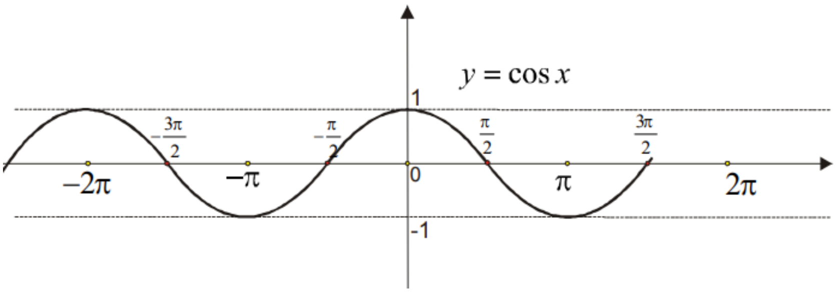


Figure 6.8: Cosine function

Properties: For all $x, \in \mathbb{R}$, we have

- $|\cos(x)| \leq 1,$ *and* $|\sin(x)| \leq 1.$
- $\sin^2 x + \cos^2 x = 1.$
- $\cos(x)$ and $\sin(x)$ are $2\pi-$periodic, and

$$\begin{cases} \cos(x + 2\pi) = \cos(x) \\ \sin(x + 2\pi) = \sin(x) \end{cases}$$

- The function $\cos(x)$ is even and the function $\sin(x)$ is odd.
- The functions $\cos(x)$ and $\sin(x)$ belong to $C^{+\infty}(\mathbb{R})$ and we have:

$$\forall x \in \mathbb{R}, \begin{cases} (\cos(x))' = -\sin(x) \\ \text{and} \\ (\sin(x))' = \cos(x) \end{cases}$$

$$\forall x \in \mathbb{R}, \ \forall n \in \mathbb{N}, \begin{cases} \cos(x)^{(n)} = \cos(x + \frac{n\pi}{2}) \\ \text{and} \\ \sin(x)^{(n)} = \sin(x + \frac{n\pi}{2}) \end{cases}.$$

Properties: For all $(x, y) \in \mathbb{R}^2$, we have the following formulas:

- $\cos(x + y) = \cos(x)\cos(y) - \sin(x)\sin(y).$
- $\cos(x - y) = \cos(x)\cos(y) + \sin(x)\sin(y).$

- $\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$.
- $\sin(x-y) = \sin(x)\cos(y) - \cos(x)\sin(y)$.
- $\cos(2x) = \cos^2(x) - \sin^2(x) = 2\cos^2(x) - 1 = 1 - 2\sin^2(x)$.
- $\sin(2x) = 2\sin(x)\cos(x)$.
- $\sin(x) + \sin(y) = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$.
- $\sin(x) - \sin(y) = 2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$.
- $\cos(x) + \cos(y) = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$.
- $\cos(x) - \cos(y) = -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$.

6.3.3 *Tangent function*

Definition 6.3.3. *The tangent function is one of the main trigonometric functions and defined by:*

$$\begin{aligned} \tan:\ \mathbb{R}|\left\{\frac{\pi}{2}+k\pi\right\} &\longrightarrow \mathbb{R} \\ x &\longrightarrow \tan x = \frac{\sin x}{\cos x} \end{aligned}, \ k \in \mathbb{Z}$$

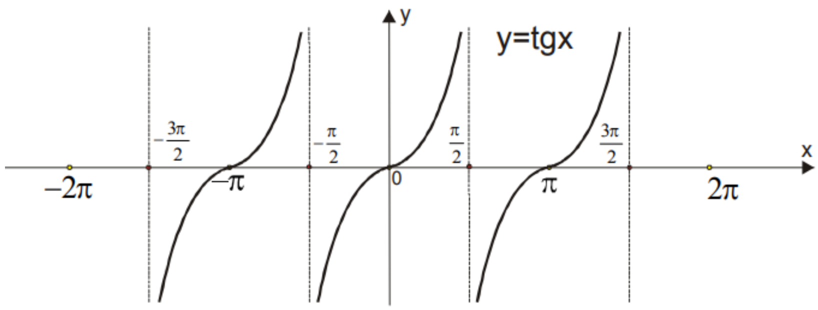


Figure 6.9: Tangent function

Proposition 6.3.1. *The function* $\tan(x)$ *checks the following properties:*

- *The function* $\tan(x)$ *is differentiable on* $\mathbb{R}|\left\{\frac{\pi}{2}+k\pi\right\},\ k\in\mathbb{Z}$ *and we have:*

$$(\tan(x))' = \frac{1}{\cos^2(x)} = 1+\tan^2(x).$$

- *The function* $\tan(x)$ *is* $\pi-$*periodic i.e:* $\tan(x+\pi)=\tan(x)$.

- *For any* $x,\ y\in\mathbb{R}|\left\{\frac{\pi}{2}+k\pi\right\},\ k\in\mathbb{Z}$ *we have:*

$$\begin{cases} \tan(x+y) = \dfrac{\tan(x)+\tan(y)}{1-\tan(x)\tan(y)} \\ \textit{and} \\ \tan(x-y) = \dfrac{\tan(x)-\tan(y)}{1+\tan(x)\tan(y)} \end{cases}$$

- $x\in\mathbb{R}|\left\{\frac{\pi}{2}+k\pi\right\},\ k\in\mathbb{Z}$, *we have* $\tan(2x)=\dfrac{2\tan(x)}{1-\tan^2(x)}$.

Proposition 6.3.2. *(Some usual limits)*

1. $\lim\limits_{x\to 0}\dfrac{\sin(x)}{x}=1$.
2. $\lim\limits_{x\to 0}\dfrac{1-\cos(x)}{x^2}=\dfrac{1}{2}$.
3. $\lim\limits_{x\to 0}\dfrac{\cos(x)-1}{x}=0$.
4. $\lim\limits_{x\to -\frac{\pi}{2}}\tan(x)=-\infty$.
5. $\lim\limits_{x\to \frac{\pi}{2}}\tan(x)=+\infty$.
6. $\lim\limits_{x\to 0}\dfrac{\tan(x)}{x}=1$.

6.3.4 *Cotangent function*

Definition 6.3.4. *The cotangent function* $y=\cot x$ *is defined by:*

$$\begin{aligned} \cot:\ \ \mathbb{R}|\{k\pi\} &\longrightarrow \mathbb{R} \\ x &\longrightarrow \cot x = \frac{\cos x}{\sin x} \end{aligned},\ k\in\mathbb{Z}$$

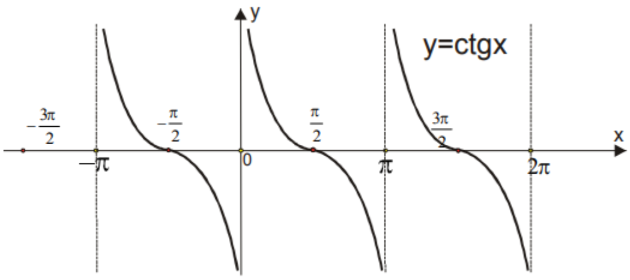


Figure 6.10: Cotangent function

6.4 Inverse Trigonometric Functions

6.4.1 *The function arc-sinus*

According to the variation table below, we have: the function $\sin(x)$ is continuous and strictly increasing on $[-\frac{\pi}{2}, \frac{\pi}{2}]$, then the function $\sin(x)$ represents a bijection from $[-\frac{\pi}{2}, \frac{\pi}{2}]$ to $[-1, 1]$.

x	$-\frac{\pi}{2}$	0	$+\frac{\pi}{2}$
$\sin(x)' = \cos(x)$		$+$	
$\sin(x)$	-1 ↗	0 ↗	1

Definition 6.4.1. *The inverse function of the restriction of* $\sin(x)$ *on* $[-\frac{\pi}{2}, \frac{\pi}{2}]$ *is called the arcsine function and is denoted by* $\arcsin(x)$ *or* $\sin^{-1}(x)$*:*

$$\begin{aligned} \arcsin : [-1, 1] &\longrightarrow [-\frac{\pi}{2}, \frac{\pi}{2}] \\ x &\longrightarrow \arcsin(x) \end{aligned}$$

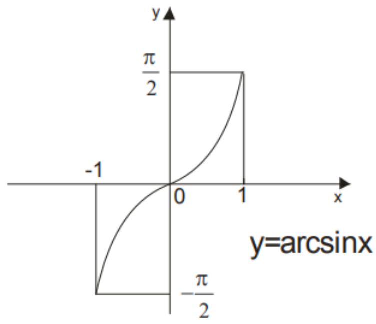


Figure 6.11: Arcsine function

Proposition 6.4.1. *The function* $\arcsin(x)$ *has the following properties:*

1. *The function* $\arcsin(x)$ *is continuous and strictly increasing on* $[-1, 1]$.
2. $\arcsin(\sin x) = x, \quad x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.
3. $\sin(\arcsin(x)) = x, \quad x \in [-1, 1]$.
4. *The function* $\arcsin(x)$ *is odd.*
5. *Thehe* arcsin *function is indefinitely differentiable on* $]-1, 1[$, *and*

$$\arcsin'(x) = \frac{1}{\sqrt{1-x^2}}.$$

More general

$$\arcsin'(f(x)) = \frac{f'(x)}{\sqrt{1-f(x)^2}}.$$

Remark 6.4.1. *some usual values for the function* $\arcsin(x)$*:*

$$\arcsin(-1) = -\frac{\pi}{2} \qquad \arcsin(0) = 0 \qquad \arcsin(1) = \frac{\pi}{2}$$
$$\arcsin\left(-\frac{1}{2}\right) = -\frac{\pi}{6} \qquad \arcsin\left(\frac{1}{2}\right) = \frac{\pi}{6} \qquad \arcsin\left(-\frac{\sqrt{2}}{2}\right) = -\frac{\pi}{4}$$
$$\arcsin\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4} \qquad \arcsin\left(-\frac{\sqrt{3}}{2}\right) = -\frac{\pi}{3} \qquad \arcsin\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$$

6.4.2 *The Arccosine Function*

In the variation table below, we have, the function $\cos(x)$ is continuous and strictly decreasing on $[0,\pi]$, so the function $\cos(x)$ makes a bijection from $[0,\pi]$ into $[-1,1]$.

x	0 $\qquad\qquad$ π
$(\cos(x))' = -\sin(x)$	$-$
$\cos(x)$	$1 \searrow -1$

Definition 6.4.2. *The inverse function of the restriction of* $\cos(x)$ *on* $[0,\pi]$ *is called the arccosine function and is denoted by* $\arccos(x)$ *or* $\cos^{-1}(x)$*:*

$$\begin{aligned} \arccos : [-1,1] &\longrightarrow [0,\pi] \\ x &\longrightarrow \arccos(x) \end{aligned}$$

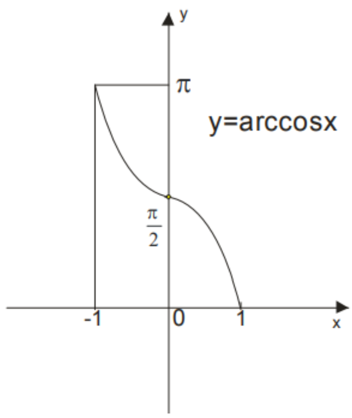


Figure 6.12: Arccosine function

Proposition 6.4.2. *The function* $\arccos(x)$ *has the following properties:*

1. *The function* $\arccos(x)$ *is continuous and strictly decreasing on* $[-1,1]$.

2. $\arccos(\cos x) = x, \quad x \in [0, \pi]$.

3. $\cos(\arccos(x)) = x, \quad x \in [-1, 1]$.

4. *The function* $\arccos(x)$ *is neither even nor odd.*

5. *The* arccos *function is indefinitely differentiable on* $]-1, 1[$, *and*

$$\arccos'(x) = -\frac{1}{\sqrt{1-x^2}}.$$

More general

$$\arccos'(f(x)) = -\frac{f'(x)}{\sqrt{1-f(x)^2}}.$$

Remark 6.4.2. *some usual values for the function* $\arccos(x)$*:*

$$\begin{array}{ccc}
\arccos(-1) = \pi & \arccos(0) = \frac{\pi}{2} & \arccos(1) = 0 \\
\arccos\left(-\frac{1}{2}\right) = -\frac{2\pi}{3} & \arccos\left(\frac{1}{2}\right) = \frac{\pi}{3} & \arccos\left(-\frac{\sqrt{2}}{2}\right) = \frac{3\pi}{4} \\
\arccos\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4} & \arccos\left(-\frac{\sqrt{3}}{2}\right) = \frac{5\pi}{6} & \arccos\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}
\end{array}$$

6.4.3 *The Arctangent function*

The function $\tan(x) = \frac{sin(x)}{cos(x)}$ is defined on $D = \mathbb{R} | \{\frac{\pi}{2} + k\pi, \ k \in \mathbb{Z}\}$. It is continuous and differentiable on its domain of definition and for all $x \in D$ we have:

$$(\tan(x))' = \frac{1}{\cos^2(x)} = 1 + \tan^2(x)$$

Consider the restriction of the function $\tan(x)$ on the interval $]-\frac{\pi}{2}, \frac{\pi}{2}[$, from the table of variation below we have: the function $\tan(x)$ is continuous and strictly increasing on $]-\frac{\pi}{2}, \frac{\pi}{2}[$, then the function $\tan(x)$ makes a bijection from $]-\frac{\pi}{2}, \frac{\pi}{2}[$ into $\mathbb{R}$.

x	$-\frac{\pi}{2}$		$\frac{\pi}{2}$
$(\tan(x))' = \dfrac{1}{\cos^2}$	||	$+$	||
$\tan(x)$	$-\infty$	$\nearrow$	$+\infty$

Definition 6.4.3. *We call the arctangent function* $\arctan(x)$ *or* $\tan^{-1}(x)$ *the inverse of the tangent function on* $]-\frac{\pi}{2}, \frac{\pi}{2}[$ *defined by:*

$$\begin{aligned} \arctan :]-\infty, +\infty[&\longrightarrow]-\frac{\pi}{2}, \frac{\pi}{2}[\\ x &\longrightarrow \arctan(x) \end{aligned}$$

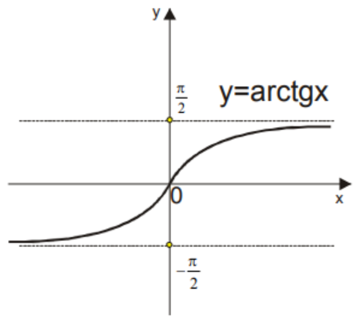


Figure 6.13: Arctan function

Proposition 6.4.3. *The function* $\arctan(x)$ *has the following properties:*

1. *The function* $\arctan(x)$ *is continuous and strictly increasing on* $\mathbb{R}$, *with values in* $]-\frac{\pi}{2}, \frac{\pi}{2}[$.
2. $\arctan(\tan x) = x, \quad x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.
3. $\tan(\arctan(x)) = x, \quad x \in \mathbb{R}$.
4. *The function* $\arctan(x)$ *is odd.*

5. *The function* arctan $\in C^\infty(\mathbb{R})$, *and we have*

$$\arctan'(x) = \frac{1}{1+x^2}.$$

More general

$$\arctan'(f(x)) = \frac{f'(x)}{1+f^2(x)}.$$

Remark 6.4.3. *The table below shows some usual values for the function* arctan(x).

$\tan(0) = 0$	$\arctan(0) = 0$
$\tan(\frac{\pi}{6}) = \frac{1}{\sqrt{3}}$	$\arctan(\frac{1}{\sqrt{3}}) = \frac{\pi}{6}$
$\tan(\frac{\pi}{4}) = 1$	$\arctan(1) = \frac{\sqrt{2}}{2}$
$\tan(\frac{\pi}{3}) = \sqrt{3}$	$\arctan(\sqrt{3}) = \frac{\pi}{3}$

6.4.4 *The Arccotangent function*

$k^{-1} : \mathbb{R} \longrightarrow [0, \pi]$

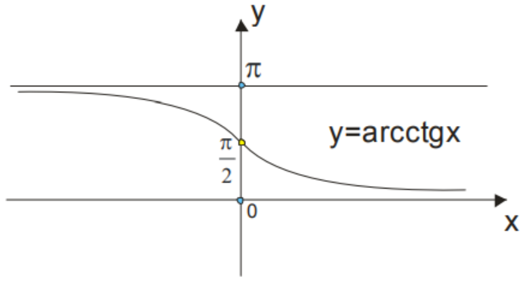


Figure 6.14: Arcctan function

Valid:

- $\operatorname{arcctan}(\cot x) = x, \quad x \in [0, \pi]$.
- $\cot(\operatorname{arcctan}(x)) = x, \quad x \in \mathbb{R}$.

The function *arcctan* $\in C^{\infty}(\mathbb{R})$, and we have

$$arcctan'(x) = -\frac{1}{1+x^2}.$$

More general

$$arcctan'(f(x)) = -\frac{f'(x)}{1+f^2(x)}.$$

We have

$$arcctan\,(0) = \frac{\pi}{2}, \ \lim_{x\to-\infty} arcctan\,(x) = \pi, \ \lim_{x\to+\infty} arcctan\,(x) = 0.$$

It can easily be shown that:

$$\begin{aligned} \arctan x + arcctan\ x &= \frac{\pi}{2}, \ \forall x \in \mathbb{R}. \\ \arctan x + \arctan\ \frac{1}{x} &= \frac{\pi}{2}, \ \forall x > 0. \\ \arctan x + \arctan\ \frac{1}{x} &= -\frac{\pi}{2}, \ \forall x < 0. \end{aligned}$$

6.5 Hyperbolic Functions

6.5.1 *Hyperbolic cosine*

Definition 6.5.1. *We call the hyperbolic cosine function and denoted (ch or cosh), the even part of the exponential function defined by:*

$$ch\,(x) = \frac{e^x + e^{-x}}{2}$$

6.5.2 *Hyperbolic sine*

Definition 6.5.2. *The hyperbolic sine function, denoted by (sh or sinh), is the odd part of the exponential function defined by:*

$$sh\ (x) = \frac{e^x - e^{-x}}{2}$$

6.5.3 *Hyperbolic tangent*

Definition 6.5.3. *The hyperbolic tangent function, denoted by (th or tanh), is the quotient of the hyperbolic sine function with the hyperbolic cosine function and defined by:*

$$th\ (x) = \frac{sh\ (x)}{ch\ (x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

6.5.4 *Hyperbolic cotangent*

Definition 6.5.4. *The hyperbolic tangent function, denoted by (cth or ctanh), is the quotient of the hyperbolic cosine function with the hyperbolic sine function and defined by:*

$$cth\ (x) = \frac{ch\ (x)}{sh\ (x)} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

Graphs of these functions are obtained from graphics: $y = e^x$ and $y = e^{-x}$, $\left(y = \frac{1}{2}e^x,\ and\ y = \frac{1}{2}e^{-x}\right)$.

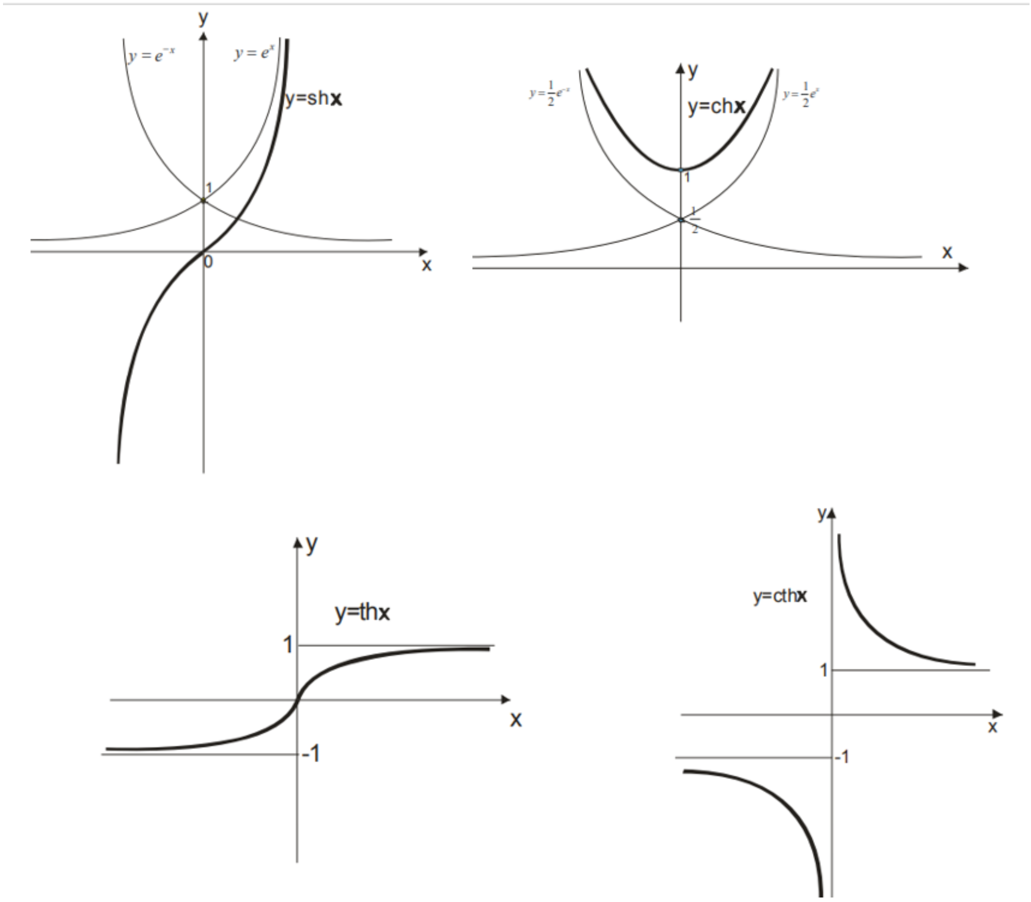


Figure 6.15: Hyperbolic functions

Proposition 6.5.1.

- *The function ch* (x) *is a function defined on* $\mathbb{R}$*, continuous and even.*
- *The function sh* (x) *is a function defined on* $\mathbb{R}$*, continuous and odd.*
- *The function th* (x) *is a function defined on* $\mathbb{R}$*, continuous and odd.*
- *The function cth* (x) *is a function defined on* $\mathbb{R}$*, continuous and odd.*
- *The functions ch* (x), $sh(x)$, $th(x)$ *and cth* (x) *are differentiable on* $\mathbb{R}$ *and their derivatives are defined by:*

$$\forall\, x \in \mathbb{R};\ \begin{cases} (ch\ (x))' = sh\ (x) \\ (sh\ (x))' = ch\ (x) \\ (th\ (x))' = \dfrac{1}{ch^2\ (x)} = 1 - th^2(x) \\ (cth\ (x))' = -\dfrac{1}{sh^2\ (x)} \end{cases}$$

Remark 6.5.1. *The hyperbolic functions have the following properties:*

1. $ch\ (0) = 1,\ sh\ (0) = 0,\ th(0) = 0.$
2. $\lim\limits_{x\to-\infty} ch\ (x) = +\infty,\ \lim\limits_{x\to-\infty} sh\ (x) = -\infty,\ \lim\limits_{x\to-\infty} th\ (x) = -1,\ \lim\limits_{x\to-\infty} cth\ (x) = -1.$
3. $\lim\limits_{x\to+\infty} ch\ (x) = +\infty,\ \lim\limits_{x\to+\infty} sh\ (x) = +\infty,\ \lim\limits_{x\to+\infty} th\ (x) = 1,\ \lim\limits_{x\to+\infty} cth\ (x) = 1.$

Proposition 6.5.2. *For every real x, we have:*

- $ch\ (x) + sh\ (x) = e^x$,
- $ch\ (x) - sh\ (x) = e^{-x}$,
- $ch^2(x) - sh^2(x) = 1$,
- $sh\ (2x) = 2.sh\ (x).ch\ (x)$,
- $ch\ (2x) = ch^2(x) + sh^2(x)$.

Proposition 6.5.3. *(Addition formulas):*
For all $(x, y) \in \mathbb{R}^2$, we have the following formulas:

- $ch(x + y) = ch\ (x).ch\ (y) + sh\ (x).sh\ (y)$,
- $ch(x - y) = ch\ (x).ch\ (y) - sh\ (x).sh\ (y)$,
- $sh(x + y) = sh\ (x).ch\ (y) + ch\ (x).sh\ (y)$,
- $sh(x - y) = sh\ (x).ch\ (y) - ch\ (x).sh\ (y)$,
- $th(x + y) = \dfrac{th\ (x) + th\ (y)}{1 + th\ (x).th\ (y)}$,

- $th(x-y) = \dfrac{th\,(x) - th\,(y)}{1 - th\,(x).th\,(y)}$,

Proposition 6.5.4. *(Some usual limits of hyperbolic functions):*

1. $\lim\limits_{x\to+\infty} \dfrac{ch\,(x)}{e^x} = \dfrac{1}{2}$,
2. $\lim\limits_{x\to+\infty} \dfrac{sh\,(x)}{e^x} = \dfrac{1}{2}$,
3. $\lim\limits_{x\to 0} \dfrac{sh\,(x)}{x} = 1$,
4. $\lim\limits_{x\to 0} \dfrac{ch\,(x) - 1}{x^2} = \dfrac{1}{2}$.

Exercise 6.5.1. *Show that for all real numbers x and y:*

$$e^{\frac{x+y}{2}} \leq \frac{e^x+e^y}{2}.$$

Solution:

Let $x,\ y \in \mathbb{R}$, we have:

$$\begin{aligned}\left(e^{\frac{x}{2}} - e^{\frac{y}{2}}\right)^2 \geq 0 \quad &\Rightarrow \quad e^x + e^y - 2\cdot e^{\frac{x+y}{2}} \geq 0\\ &\Rightarrow \quad 2\cdot e^{\frac{x+y}{2}} \leq e^x + e^y\\ &\Rightarrow \quad e^{\frac{x+y}{2}} \leq \frac{e^x+e^y}{2}.\end{aligned}$$

Exercise 6.5.2. *According to the values of x, find the limits of x^n when $n \to +\infty$.*

Solution:

Let $x \in \mathbb{R}$, then if:

1. $x \leq -1 \Rightarrow x^n$ diverges.
2. $-1 < x < 1 \Rightarrow x^n \to 0$.
3. $x = 1 \Rightarrow x^n \to 1$.
4. $x > 1 \Rightarrow x^n$ diverges.

Exercise 6.5.3. 1. *Compute: $ch\,(\frac{1}{2}\ln(3))$, and $sh\,(\frac{1}{2}\ln(3))$.*

2. *Show that: $ch\,(a+b) = ch(a)ch(b) + sh(a)sh(b)$.*
3. *Deduce the solutions of the equation: $2ch(x) + sh(x) = \sqrt{3}ch(5x)$.*

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