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Time-frequency analysis associated with the Hankel-Stockwell transform

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DEDICATION

Here I am reaching the end of my journey in this dream, to start a new another dream awaits me...The moments of my university path have concluded, I experienced a lot of situations and circumstances, days passed where boredom and frustration prevailed, and difficult stations were faced, yet we continued our dream and journey, renewing our determination to achieve our desired goal.

I dedicate my graduation to my family who has always been there for me, to my dear father whose name is inseparable from mine, the source of continuous giving and the spring of endless hope. To my brothers my life's fragrances and mirrors. To everyone who encouraged and supported me in my studies especially my fiance Dr. Aymen.

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Samer Boudjellal

DEDICATION

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I wish to convey my utmost respect and warm appreciation to all my teachers, as well as to everyone who has contributed, near and far. Thank you.

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CONTENTS

RÉSUMÉ

L'objectif de cette mémoire est de définir et d'étudier une nouvelle transformation appelée transformation de Hankel-Stockwell. On établit toute l'analyse harmonique associée à cette transformation, en particulier on démontre une formule de Plancherel, une propriété d'orthogonalité et une formule de reconstruction. Un autre objectif de cette mémoire est de définir les opérateurs de localisation, d'étudier leur bornitude et leur compacité. On montre également que ces opérateurs appartiennent à la classe de Schatten-von Neumann.

Mots clés: Transformation de Hankel, Transformation de Hankel-Stockwell , opérateurs de localisation.

ABSTRACT

Our objective in this thesis is to define and study a new transform called the Hankel-Stockwell transform. We will prove all the harmonic analysis associated to this transform, in particular a Plancherel's formula, an orthogonality property and a reconstruction formula.

Another main purpose of this thesis is to define the localization operators and to study their boundedness and compactness, we will also show that these operators belong to the so-called Schatten-von Neumann class.

Keywords: Hankel transform, Hankel-Stockwell transform, Localization operators.

INTRODUCTION

Many non-stationnary signals as seismic signal, genomic signal, electrocardiograms, and speech are gaining more attentions as they intervene in the real life. So, during the last decades, many methods of determining local spectra have been investigated. In fact, in signal theory, the Fourier transform of a given signal was firstly introduced by Joseph Fourier in 1822, defined for an integrable function f (stable signal) by

$$\hat{f}(\lambda) = \int_{\mathbb{R}} f(x)e^{-i\lambda x}\frac{dx}{\sqrt{2\pi}}; \quad \forall \lambda \in \mathbb{R},$$

represents the set of frequencies that compose the signal with their respective amplitudes that called the spectrum of the signal.

One of the major problems with the Fourier Transform consists of the fact that the frequency representation is global and does not give any temporal localization.

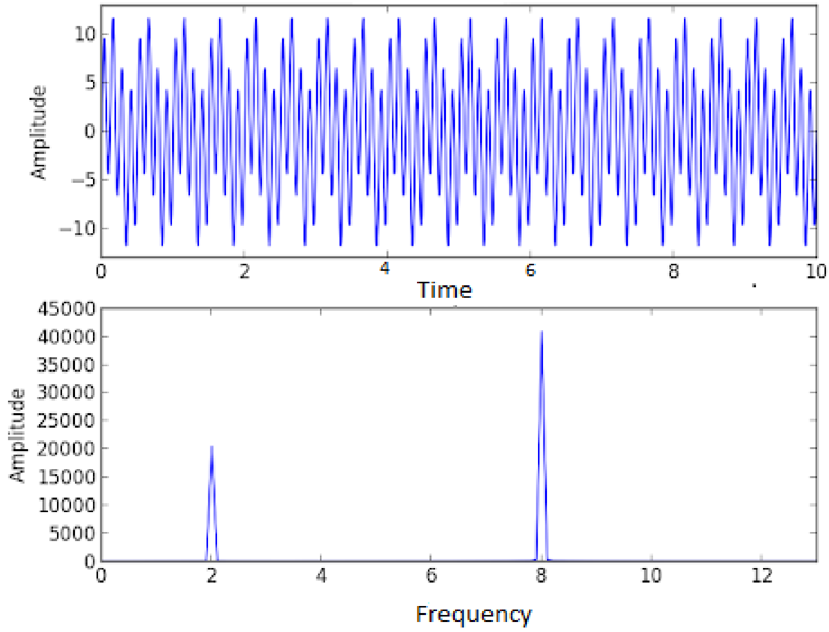


Figure 1: Loss of temporal localization of the Fourier transform

The notion of time-frequency representations was therefore introduced in order to overcome this problem, the basic idea concerning the time-frequency analysis is to introduce into the Fourier analysis, which is a purely spectral analysis, a notion of spatial or temporal locality by replacing the analyzed function f with the product of f by a function ψ suitably chosen having good localization properties, then we apply the Fourier transform to them.

The most famous time-frequency representation was introduced by Denis Gabor [16] called the Short-time Fourier transform (STFT).

Let us consider a non-zero function $\psi \in L^2(\mathbb{R})$ called window. Then, for every $f \in L^2(\mathbb{R})$, the Short-time Fourier transform of f is defined by

$$V_\psi(f)(a,r) = \int_{\mathbb{R}} f(x)\overline{\psi(x-a)}e^{-irx}\frac{dx}{\sqrt{2\pi}}; \quad a,r \in \mathbb{R}.$$

Example of two musical notes played one after the other: time-frequency analysis makes it possible to find both the frequencies (the notes) and the temporal information (the order in which they are played). But quickly, this transform showed many disad-

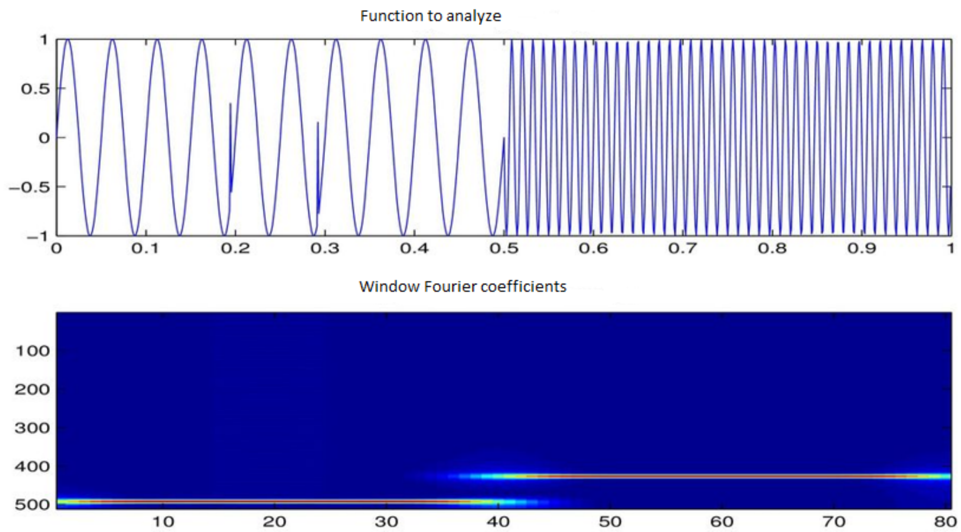


Figure 2: Time frequency localization

vantages like its inability to detect low frequencies and poor time resolution of high frequency events due to the fixed width of the window function this means that the short-time Fourier transform supposes a certain stationary of the signal and it might be unsuitable to non-stationary signals.

In contrast with the STFT, the wavelet transform (WT), introduced by Morlet [20] proposed to use a window of size depending on the analyzed frequency but with a fixed number of oscillations.

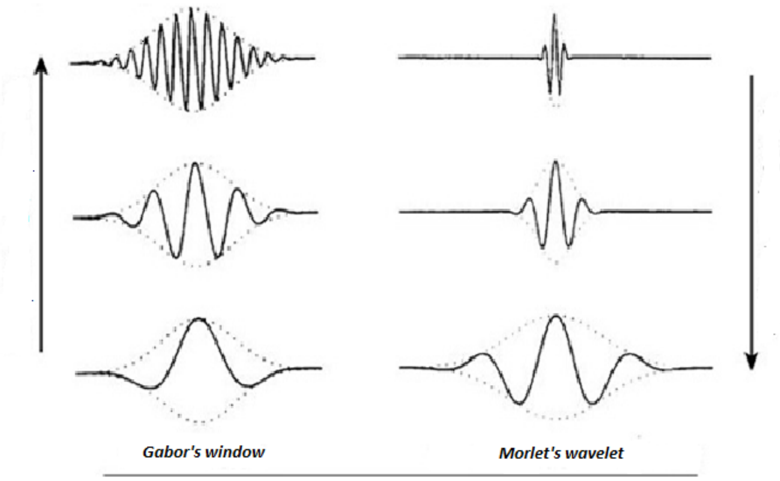
A non-zero function $\psi \in L^2(\mathbb{R})$ is said to be a mother wavelet if

$$\int_0^{+\infty} |\hat{\psi}(a)|^2 \frac{da}{a} < +\infty.$$

The wavelet transform W_ψ with respect to the mother ψ is defined on $L^2(\mathbb{R})$ by

$$W_\psi(f)(a,r) = \frac{1}{\sqrt{a}} \int_{\mathbb{R}} f(x) \overline{\psi\Big(\frac{x-r}{a}\Big)} \frac{dx}{\sqrt{2\pi}}; \quad a,r \in \mathbb{R}_+^* \times \mathbb{R},$$

but this transform produces time-scale plots that are unsuitable for intuitive visual analysis. To circumvent the limitation of the STFT and the WT, a hybrid transform has



been introduced which enjoys the advantages of both STFT and WT called S-transform often known as the Stockwell transform.

The S-transform was introduced firstly by Stockwell, Mansinha and Lowe [31], it is a strong new time-frequency approach that has been described in the scientific literature as a considerable advance over current techniques for localizing spectral information in a range of signal processing settings. it employs a scalable and variable window length and provides many information about spectra, thereby, it does not lose any valuable information and it can reverse back easily.

The S-transform has found many applications in a variety of signal analysis tasks, as geophysics, medical image processing, oceanography and mechanical engineering [1, 6, 9, 15, 26].

Recently, many authors have been interested to extend the classical S-transform to higher dimensional signals as [25, 29]. As the STFT and WT were extended in different settings like the Dunkl [17], the Jacobi [27] and the Hankel settings [2, 18, 22]. In the second chapter of this thesis, The Hankel-Stockwell transform S^{α}_{ψ} is a new transform , where ψ is a window function and $\alpha \geq -\frac{1}{2}$, that extends the S-transform usually defined with the usual Fourier transform to the Hankel settings that is

$$S_{\psi}^{\alpha}(f)(a,r) = \int_0^{+\infty} f(s)\overline{\psi_{a,r}^{\alpha}(s)}\frac{s^{2\alpha+1}}{2^{\alpha}\Gamma(\alpha+1)}ds, \quad (a,r) \in \mathbb{R}_+^* \times \mathbb{R}_+,$$

where $\psi_{a,r}^{\alpha}$ is given by relation (2.11).

The Hankel transform also known as the Fourier-Bessel transform arises as a generalization of the Fourier transform of a radial integrable function in the euclidean space $\mathbb{R}^d$. More precisely, let $f \in L^1(\mathbb{R}^d)$ it is well known that if
$f(x) = F(||x||)$ is radial function on $\mathbb{R}^d$, then $\hat{f}$ is also radial on $\mathbb{R}^d$ and we have

$$\forall \lambda \in \mathbb{R}^d, \quad \hat{f}(\lambda) = \int_0^{+\infty} F(x) j_{\frac{d}{2}-1}(x||\lambda||)\frac{x^{d-1}}{2^{\frac{d}{2}-1}\Gamma(\frac{d}{2})}dx = \mathscr{H}_{\frac{d}{2}-1}(F)(||\lambda||),$$

where $j_{\frac{d}{2}-1}$ is the modified Bessel function of index $\frac{d}{2}-1$ and $\mathscr{H}_{\frac{d}{2}-1}$ is the Hankel transform of index $\frac{d}{2}-1$, defined on $L^1(d\nu_\alpha)$ by

$$\mathscr{H}_\alpha(f)(\lambda) = \int_0^{+\infty} f(r) j_\alpha(\lambda r) d\nu_\alpha(r), \quad \forall \lambda \in \mathbb{R}.$$

The Hankel transform $\mathscr{H}_\alpha$ satisfies the following results:

• **Inversion formula**: Let $f \in L^1(d\nu_\alpha)$ such that $\mathscr{H}_\alpha(f) \in L^1(d\nu_\alpha)$, then we have

$$f(r) = \int_0^{+\infty} \mathscr{H}_\alpha(f)(\lambda) j_\alpha(\lambda r) d\nu_\alpha(\lambda), \quad a.e.$$

• **Plancherel's formula**: The Hankel transform $\mathscr{H}_\alpha$ can be extended to an isometric isomorphism from $L^2(d\nu_\alpha)$ onto itself and we have

$$||\mathscr{H}_\alpha(f)||_{2,\nu_\alpha} = ||f||_{2,\nu_\alpha}.$$

• **Parseval's formula**: For all $f, g \in L^2(d\nu_\alpha)$, we have

$$\int_0^{+\infty} f(r)\overline{g(r)}\, d\nu_\alpha(r) = \int_0^{+\infty} \mathscr{H}_\alpha(f)(\lambda)\overline{\mathscr{H}_\alpha(g)(\lambda)}\, d\nu_\alpha(\lambda).$$

As the harmonic analysis associated to the Hankel transform has shown remarkable development, it is a natural question to ask whether there exists the equivalent of the theory of time-frequency analysis for the Stockwell transform in the Hankel setting. In fact, many results for the Hankel-Stockwell transform have been established in particular, let ψ be an admissible window function in $L^2(d\nu_\alpha)$. Then, we have:

• **Plancherel's formula**: For every f in $L^2(d\nu_\alpha)$, the function $S^\alpha_\psi(f)$ belongs to $L^2(d\mu_\alpha)$ and we have

$$||S^\alpha_\psi(f)||_{2,\mu_\alpha} = \sqrt{C_\psi}||f||_{2,\nu_\alpha}.$$

• **Parseval's formula**: For all f and g in $L^2(d\nu_\alpha)$, we have

$$\int_0^{+\infty}\int_0^{+\infty} S^\alpha_\psi(f)(a,r)\overline{S^\alpha_\psi(g)(a,r)}d\mu_\alpha(a,r) = C_\psi \int_0^{+\infty} f(s)\overline{g(s)}d\nu_\alpha(s).$$

• **Reconstruction formula**: If $|\psi|$ is an admissible window function. Then, for every $f \in L^2(d\nu_\alpha)$, we have

$$f(.) = \frac{1}{C_\psi}\int_0^{+\infty}\int_0^{+\infty} S^\alpha_\psi(f)(a,r)\psi^\alpha_{a,r}(.)d\nu_\alpha(a)d\nu_\alpha(r),$$

weakly in $L^2(d\nu_\alpha)$.

Chapter 3 is devoted to the time-frequency localization operators that were introduced firstly by Daubechies [10, 11, 12]. She pointed out the role of this kind of operators to localize a signal simultaneously in time and frequency.

In the literature, they are also known as anti-Wick operators, wave packets, Toeplitz operators or Gabor multipliers [3, 7, 13]. These operators have many applications to time-frequency analysis, for example, in the theory of differential equations, quantum mechanics, and signal processing [8, 14, 19].

Let $\sigma \in L^p(d\mu_\alpha)$, $p \in [1, +\infty]$. The localization operators for the Hankel-Stockwell transform $L_{\psi_1,\psi_2}(\sigma)$ is defined for all f and $g \in L^2(d\nu_\alpha)$ by

$$\langle L_{\psi_1,\psi_2}(\sigma)(f)|g\rangle_{\nu_\alpha} = \frac{1}{\sqrt{C_{\psi_1}C_{\psi_2}}}\int_0^{+\infty}\int_0^{+\infty}\sigma(a,r)S^\alpha_{\psi_1}(f)(a,r)\overline{S^\alpha_{\psi_2}(g)(a,r)}d\mu_\alpha(a,r),$$

where ψ_1, ψ_2 are two admissible window functions.

In this sense, we recover a filtered version of the signal f, that is why, the localization operators are also called filter operators. It is also common to define $L_{\psi_1,\psi_2}(\sigma)$ by means of scalar product

$$\langle L_{\psi_1,\psi_2}(\sigma)(f)|g\rangle_{\nu_\alpha} = \frac{1}{\sqrt{C_{\psi_1}C_{\psi_2}}}\langle \sigma S^\alpha_{\psi_1}(f)|S^\alpha_{\psi_2}(g)\rangle_{\mu_\alpha}.$$

Firstly, we have shown that the localization operator $L_{\psi_1,\psi_2}(\sigma)$ is bounded.

• Let $\sigma \in L^p(d\mu_\alpha)$, $p \in [1, +\infty]$. For every $f \in L^2(d\nu_\alpha)$, the operator $L_{\psi_1,\psi_2}(\sigma)$ is bounded from $L^2(d\nu_\alpha)$ into itself and we have

$$\|L_{\psi_1,\psi_2}(\sigma)\| \leq \left(\frac{1}{\sqrt{C_{\psi_1}C_{\psi_2}}}\right)^{\frac{1}{p}}\|\sigma\|_{p,\mu_\alpha}.$$

We also studied the compactness of the localization operators and we established the following result:

• Let $\sigma \in L^p(d\mu_\alpha)$; $1 \leq p < +\infty$, then the operator $L_{\psi_1,\psi_2}(\sigma)$ is compact.

We also show that these operators belong to the so-called Schatten-von Neumann class and we give the formula of the trace of the localization operator when σ belongs to

$L^1(d\mu_\alpha)$ by

$$Tr(L_{\psi_1,\psi_2}(\sigma)) = \frac{1}{\sqrt{C_{\psi_1}C_{\psi_2}}}\int_0^{+\infty}\int_0^{+\infty}\sigma(a,r)\langle\psi^\alpha_{2,a,r}|\psi^\alpha_{1,a,r}\rangle_{\nu_\alpha}d\mu_\alpha(a,r).$$

CHAPTER 1

HANKEL TRANSFORM

The Hankel transform also called Fourier-Bessel transform is integral transformation whose kernel is Bessel function. When we are dealing with problems that show circular symmetry, the Hankel transform may be very useful. For example, the Hankel transform is the two-dimensional Fourier transform of a circularly symmetric function. Moreover, the Hankel transform came for the first time by studying the Fourier transform of radial functions and has been generalized later in the general case.
In this chapter, we summarize some harmonic analysis tools related to the Hankel transform that we shall use later (for more details, one can see [21, 23, 28, 30]).

Notations

We denote by

• ν_α is the measure defined on $[0, +\infty[$ by

$$d\nu_\alpha(r) = \frac{r^{2\alpha+1}}{2^\alpha \Gamma(\alpha+1)} dr.$$

• $L^p(d\nu_\alpha)$, $p \in [1, +\infty]$, is the space of measurable functions f on $[0, +\infty[$ such that

$$\|f\|_{p,\nu_\alpha} = \begin{cases} \left(\int_0^{+\infty} | f(r) |^p \, d\nu_\alpha(r) \right)^{\frac{1}{p}} < +\infty, & \text{if } 1 \leq p < +\infty, \\ \underset{r \in [0,+\infty[}{\text{ess sup}} \ | f(r) | < +\infty, & \text{if } p = +\infty. \end{cases}$$

• $\langle . \mid . \rangle_{\nu_\alpha}$ the inner product on $L^2(d\nu_\alpha)$ defined by

$$\langle w | z \rangle_{\nu_\alpha} = \int_0^{+\infty} w(r) \overline{z(r)} d\nu_\alpha(r).$$

• $C_*(\mathbb{R})$ the space of even continuous functions on $\mathbb{R}$.

1.1 Bessel operator

In this section, we define the Bessel operator ℓ_α, the modified Bessel function j_α and we give some related results. We also define the translation operator, the convolution product related to the Bessel operator and we recall some known inequalities which can be useful throughout this manuscript.

Let ℓ_α be the Bessel operator defined on $]0,+\infty[$ by

$$\begin{aligned}\ell_\alpha &= \frac{d^2}{dr^2}+\frac{2\alpha+1}{r}\frac{d}{dr}\\ &= \frac{1}{r^{2\alpha+1}}\frac{d}{dr}\Big(r^{2\alpha+1}\frac{d}{dr}\Big).\end{aligned}$$

Then, for all $\lambda\in\mathbb{C}$, the following problem

$$\begin{cases}\ell_\alpha(u)(r) = -\lambda^2 u(r),\\ u(0)=1,\\ u'(0)=0.\end{cases}$$

admits a unique solution given by the modified Bessel function $j_\alpha(\lambda.)$, where

$$j_\alpha(r) = 2^\alpha\Gamma(\alpha+1)\frac{J_\alpha(r)}{r^\alpha} = \Gamma(\alpha+1)\sum_{k=0}^{+\infty}\frac{(-1)^k}{k!\Gamma(\alpha+k+1)}\Big(\frac{r}{2}\Big)^{2k}, \tag{1.1}$$

and J_α is the Bessel function of the first kind and index α [24, 32].

Proof. Let $\lambda\in\mathbb{C}$. Then, we have

$$\ell_\alpha(j_\alpha(\lambda r)) = j''_\alpha(\lambda r)+\frac{2\alpha+1}{r}j'_\alpha(\lambda r).$$

Since

$$\begin{aligned} j'_\alpha(\lambda r) &= \Big(\Gamma(\alpha+1)\sum_{k=0}^{+\infty}\frac{(-1)^k}{k!\Gamma(\alpha+k+1)}\Big(\frac{\lambda r}{2}\Big)^{2k}\Big)' \\ &= \Gamma(\alpha+1)\sum_{k=1}^{+\infty}\frac{(-1)^k\lambda k}{k!\Gamma(\alpha+k+1)}\Big(\frac{\lambda r}{2}\Big)^{2k-1} \\ &= \lambda\Gamma(\alpha+1)\sum_{k=0}^{+\infty}\frac{(-1)^{k+1}(k+1)}{(k+1)!\Gamma(\alpha+k+2)}\Big(\frac{\lambda r}{2}\Big)^{2k+1} \\ &= -\lambda^2\Big(\frac{r}{2}\Big)\frac{\Gamma(\alpha+2)}{\alpha+1}\sum_{k=0}^{+\infty}\frac{(-1)^k}{k!\Gamma(\alpha+k+2)}\Big(\frac{\lambda r}{2}\Big)^{2k} \\ &= \frac{-\lambda^2 r}{2(\alpha+1)}j_{\alpha+1}(\lambda r), \end{aligned} \tag{1.2}$$

and

$$\begin{aligned} j''_\alpha(\lambda r) &= \Big(\frac{-\lambda^2 r}{2(\alpha+1)}j_{\alpha+1}(\lambda r)\Big)' \\ &= \frac{-\lambda^2}{2(\alpha+1)}j_{\alpha+1}(\lambda r) - \frac{\lambda^2 r}{2(\alpha+1)}j'_{\alpha+1}(\lambda r) \\ &= \frac{-\lambda^2}{2(\alpha+1)}j_{\alpha+1}(\lambda r) + \frac{\lambda^4 r^2}{4(\alpha+1)(\alpha+2)}j_{\alpha+2}(\lambda r). \end{aligned} \tag{1.3}$$

Then, by relations (1.2) and (1.3), we have

$$\begin{aligned} \ell_\alpha(j_\alpha(\lambda r)) &= \frac{-\lambda^2}{2(\alpha+1)}j_{\alpha+1}(\lambda r) + \frac{\lambda^4 r^2}{4(\alpha+1)(\alpha+2)}j_{\alpha+2}(\lambda r) - \frac{\lambda^2(2\alpha+1)}{2(\alpha+1)} - j_{\alpha+1}(\lambda r) \\ &= -\lambda^2\Big(j_{\alpha+1}(\lambda r) - \frac{\lambda^2 r^2}{4(\alpha+1)(\alpha+2)}j_{\alpha+2}(\lambda r)\Big). \end{aligned}$$

Using the fact that (see [32])

$$J_{\alpha+1}(r) + J_{\alpha-1}(r) = \frac{2\alpha}{r}J_\alpha(r),$$

thus, we get

$$j_{\alpha+1}(r) = \frac{r^2}{4(\alpha+1)(\alpha+2)} j_{\alpha+2}(r) + j_\alpha(r).$$

Furthermore, $j_\alpha(0) = 1$ and $j'_\alpha(0) = 0$. The proof is complete. ∎

Proposition 1.1.1 *The function j_α has the following integral representation formula, for all $r \in \mathbb{R}$*

$$j_\alpha(r) = \begin{cases} \dfrac{2\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})} \displaystyle\int_0^1 (1-t^2)^{\alpha-\frac{1}{2}} \cos(tr)dt, & \text{if } \alpha > -\frac{1}{2}, \\ \cos(r), & \text{if } \alpha = -\frac{1}{2}. \end{cases}$$

Proof. 1) For $\alpha = -\frac{1}{2}$, we get

$$\begin{aligned} j_{-\frac{1}{2}}(r) &= \Gamma\Big(\frac{1}{2}\Big) \sum_{k=0}^{+\infty} \frac{(-1)^k}{k!\Gamma(k+\frac{1}{2})} \Big(\frac{r}{2}\Big)^{2k} \\ &= \sqrt{\pi} \sum_{k=0}^{+\infty} \frac{(-1)^k}{\Gamma(k+1)\Gamma(k+\frac{1}{2})} \Big(\frac{r}{2}\Big)^{2k} \\ &= \sqrt{\pi} \sum_{k=0}^{+\infty} \frac{(-1)^k}{2^{2k-1}\Gamma(k)\Gamma(k+\frac{1}{2})} \frac{r^{2k}}{2k}. \end{aligned}$$

Using the fact that

$$2^{2k-1}\Gamma(k)\Gamma(k+\frac{1}{2}) = \sqrt{\pi}\Gamma(2k).$$

Then, we obtain

$$\begin{aligned} j_{-\frac{1}{2}}(r) &= \sum_{k=0}^{+\infty} \frac{(-1)^k}{2k\Gamma(2k)} r^{2k} \\ &= \sum_{k=0}^{+\infty} \frac{(-1)^k}{\Gamma(2k+1)} r^{2k} \\ &= \sum_{k=0}^{+\infty} (-1)^k \frac{r^{2k}}{(2k)!} = \cos(r). \end{aligned}$$

2) For $\alpha > -\frac{1}{2}$, we have

$$\begin{aligned}\int_0^1 (1-t^2)^{\alpha-\frac{1}{2}}\cos(tr)dt &= \int_0^1 (1-t^2)^{\alpha-\frac{1}{2}}\Big(\sum_{k=0}^{+\infty}(-1)^k\frac{(tr)^{2k}}{(2k)!}\Big)dt \\ &= \sum_{k=0}^{+\infty}(-1)^k\frac{r^{2k}}{(2k)!}\int_0^1 (1-t^2)^{\alpha-\frac{1}{2}}t^{2k}dt,\end{aligned}$$

by the change of variable $u = 1-t^2$, we get

$$\begin{aligned}\int_0^1 (1-t^2)^{\alpha-\frac{1}{2}}\cos(tr)dt &= \frac{1}{2}\sum_{k=0}^{+\infty}\frac{(-1)^k}{2k\Gamma(2k)}r^{2k}\int_0^1 u^{\alpha-\frac{1}{2}}(1-u)^{k-\frac{1}{2}}du \\ &= \frac{\sqrt{\pi}}{2}\sum_{k=0}^{+\infty}\frac{(-1)^k}{2k2^{2k-1}\Gamma(k)\Gamma(k+\frac{1}{2})}r^{2k}B\Big(\alpha+\frac{1}{2},k+\frac{1}{2}\Big) \\ &= \frac{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}{2}\sum_{k=0}^{+\infty}\frac{(-1)^k}{k!\Gamma(\alpha+k+1)}\Big(\frac{r}{2}\Big)^{2k} \\ &= \frac{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}{2\Gamma(\alpha+1)}j_\alpha(r).\end{aligned}$$

■

Remark 1.1.1 *The function j_α is bounded, for all $n \in \mathbb{N}$ and $r \in \mathbb{R}$ and we have*

$$|j_\alpha^{(n)}(r)| \le 1. \tag{1.4}$$

We have also the following product formula satisfied by j_α for all $r,s \in \mathbb{R}_+$,

$$j_\alpha(r)j_\alpha(s) = \begin{cases} \dfrac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}\displaystyle\int_0^\pi j_\alpha\Big(\sqrt{r^2+s^2+2rs\cos\theta}\Big)(\sin\theta)^{2\alpha}d\theta, & \text{if } \alpha > -1/2, \\ \\ \dfrac{j_{-1/2}(r+s)+j_{-1/2}(|r-s|)}{2}, & \text{if } \alpha = -1/2. \end{cases} \tag{1.5}$$

1.1.1 Translation operator associated to the Bessel operator

Definition 1.1.1 *We define the Hankel translation operator τ_r^α, $r \in [0, +\infty[$, for all $f \in C_*(\mathbb{R})$ by*

$$\tau_r^\alpha(f)(s) = \begin{cases} \dfrac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})} \displaystyle\int_0^\pi f\Big(\sqrt{r^2+s^2+2rs\cos\theta}\Big)(\sin\theta)^{2\alpha} d\theta, & \text{if } \alpha > -1/2, \\ \\ \dfrac{f(r+s)+f(|r-s|)}{2}, & \text{if } \alpha = -1/2. \end{cases} \tag{1.6}$$

Theorem 1.1.1 *Let $\alpha > -\frac{1}{2}$ and $f \in C_*(\mathbb{R})$. Then, for all $r, s \in]0, +\infty[$, the operator τ_r^α can be also written as*

$$\tau_r^\alpha(f)(s) = \int_0^{+\infty} f(u)\omega_\alpha(u, r, s) d\nu_\alpha(u), \tag{1.7}$$

where ω_α is the Hankel kernel given by

$\omega_\alpha(u, r, s) =$

$$\begin{cases} \dfrac{\Gamma^2(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})2^{\alpha-1}} \dfrac{[u^2-(r-s)^2]^{\alpha-\frac{1}{2}}[(r+s)^2-u^2]^{\alpha-\frac{1}{2}}}{(urs)^{2\alpha}}, & \text{if } |r-s| < u < r+s, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. According to Definition 1.1.1, we have for every $\alpha > -\frac{1}{2}$

$$\tau_r^\alpha(f)(s) = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})} \int_0^\pi f\Big(\sqrt{r^2+s^2+2rs\cos\theta}\Big)(\sin\theta)^{2\alpha} d\theta.$$

We put $u = \sqrt{r^2+s^2+2rs\cos\theta}$, we obtain

$$\tau_r^\alpha(f)(s) = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}\int_{|r-s|}^{r+s} f(u)\frac{[u^2-(r-s)^2]^\alpha[(r+s)^2-u^2]^\alpha}{(2rs)^{2\alpha}}\frac{u}{rs}\frac{[u^2-(r-s)^2]^{-\frac{1}{2}}[(r+s)^2-u^2]^{-\frac{1}{2}}}{(2rs)^{-1}}du$$

$$= \frac{\Gamma(\alpha+1)}{2^{2\alpha-1}\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}\int_{|r-s|}^{r+s} uf(u)\frac{[u^2-(r-s)^2]^{\alpha-\frac{1}{2}}[(r+s)^2-u^2]^{\alpha-\frac{1}{2}}}{(rs)^{2\alpha}}du$$

$$= \frac{\Gamma^2(\alpha+1)}{2^{\alpha-1}\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}\int_{|r-s|}^{r+s} f(u)\frac{[u^2-(r-s)^2]^{\alpha-\frac{1}{2}}[(r+s)^2-u^2]^{\alpha-\frac{1}{2}}}{(urs)^{2\alpha}}\frac{u^{2\alpha+1}}{2^\alpha\Gamma(\alpha+1)}du$$

$$= \int_0^{+\infty} f(u)\omega_\alpha(u,r,s)d\nu_\alpha(u).$$

■

The kernel ω_α is symmetric in the variables u, r, s and we have

$$\int_0^{+\infty}\omega_\alpha(u,r,s)d\nu_\alpha(u) = 1. \tag{1.8}$$

Proposition 1.1.2 *For every $f \in L^1(d\nu_\alpha)$ and for $r \in [0,+\infty[$, the function $\tau_r^\alpha(f)$ belongs to $L^1(d\nu_\alpha)$ and we have*

$$\int_0^{+\infty}\tau_r^\alpha(f)(s)d\nu_\alpha(s) = \int_0^{+\infty} f(u)d\nu_\alpha(u). \tag{1.9}$$

Proof. From relation (1.8) and using Fubini-Tonelli's theorem, we obtain that for $f \in L^1(d\nu_\alpha)$ and for $r \in [0,+\infty[$,

$$\begin{aligned}\int_0^{+\infty}|\tau_r^\alpha(f)(s)|d\nu_\alpha(s) &= \int_0^{+\infty}\Big|\int_0^{+\infty} f(u)\omega_\alpha(u,r,s)d\nu_\alpha(u)\Big|d\nu_\alpha(s)\\ &\leq \int_0^{+\infty}|f(u)|\Big(\int_0^{+\infty}\omega_\alpha(u,r,s)d\nu_\alpha(s)\Big)d\nu_\alpha(u)\\ &= \|f\|_{1,\nu_\alpha} < +\infty.\end{aligned}$$

This shows that $\tau_r^\alpha(f)$ belongs to $L^1(d\nu_\alpha)$ and

$$\begin{aligned}\int_0^{+\infty}\tau_r^\alpha(f)(s)d\nu_\alpha(s) &= \int_0^{+\infty}\int_0^{+\infty} f(u)\omega_\alpha(u,r,s)d\nu_\alpha(u)d\nu_\alpha(s)\\ &= \int_0^{+\infty} f(u)\Big(\int_0^{+\infty}\omega_\alpha(u,r,s)d\nu_\alpha(s)\Big)d\nu_\alpha(u)\\ &= \int_0^{+\infty} f(u)d\nu_\alpha(u).\end{aligned}$$

■

corollary 1.1.1 *For all $r,s\in[0,+\infty[$ and for all $\lambda\in\mathbb{C}$, we have*

$$\tau_r^\alpha(j_\alpha(\lambda.))(s) = j_\alpha(\lambda r)j_\alpha(\lambda s). \tag{1.10}$$

Proof. Let $r,s\in[0,+\infty[$. Then, we get

$$\begin{aligned}\tau_r^\alpha(j_\alpha(\lambda.))(s) &= \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}\int_0^\pi j_\alpha\Big(\lambda\sqrt{r^2+s^2+2rs\cos\theta}\Big)(\sin\theta)^{2\alpha}d\theta\\ &= \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}\int_0^\pi j_\alpha\Big(\sqrt{(\lambda r)^2+(\lambda s)^2+2(\lambda r)(\lambda s)\cos\theta}\Big)(\sin\theta)^{2\alpha}d\theta\\ &= j_\alpha(\lambda r)j_\alpha(\lambda s).\end{aligned}$$

■

Proposition 1.1.3 *For every $f\in L^p(d\nu_\alpha)$, $p\in[1,+\infty]$ and for every $r\in[0,+\infty[$, the function $\tau_r^\alpha(f)$ belongs to $L^p(d\nu_\alpha)$ and we have*

$$\|\tau_r^\alpha(f)\|_{p,\nu_\alpha}\leq\|f\|_{p,\nu_\alpha}. \tag{1.11}$$

Proof. Let $f\in L^p(d\nu_\alpha)$, $p\in[1,+\infty]$

- If $p=+\infty$. Then, for all $r,s\in[0,+\infty[$, we have

$$\tau_r^\alpha(f)(s) = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}\int_0^\pi f\Big(\sqrt{r^2+s^2+2rs\cos\theta}\Big)(\sin\theta)^{2\alpha}d\theta.$$

Then,

$$\begin{aligned}|\tau_r^\alpha(f)(s)| &\leq \|f\|_{\infty,\nu_\alpha}\frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}\int_0^\pi(\sin\theta)^{2\alpha}d\theta \\ &= \|f\|_{\infty,\nu_\alpha}\frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}B\Big(\alpha+\frac{1}{2},\frac{1}{2}\Big) \\ &= \|f\|_{\infty,\nu_\alpha}.\end{aligned}$$

This shows that the function $\tau_r^\alpha(f)$ belongs to $L^\infty(d\nu_\alpha)$ and

$$\|\tau_r^\alpha(f)\|_{\infty,\nu_\alpha} \leq \|f\|_{\infty,\nu_\alpha}.$$

- If $p = 1$. We know that

$$\tau_r^\alpha(f)(s) = \int_0^{+\infty} f(u)\omega_\alpha(u,r,s)d\nu_\alpha(u).$$

According to Fubini-Tonelli's theorem and by relation (1.8), we have

$$\begin{aligned}\|\tau_r^\alpha(f)\|_{1,\nu_\alpha} &\leq \int_0^{+\infty}|f(u)|\Big(\int_0^{+\infty}\omega_\alpha(u,r,s)d\nu_\alpha(s)\Big)d\nu_\alpha(u) \\ &= \int_0^{+\infty}|f(u)|d\nu_\alpha(u) \\ &= \|f\|_{1,\nu_\alpha}.\end{aligned}$$

- If $p \in]1,+\infty[$ and q be the conjugate exponent of p. According to Hölder's inequality and relations (1.7) and (1.8), we obtain

$$\begin{aligned}
|\tau_r^\alpha(f)(s)| &\leq \int_0^{+\infty} |f(u)|\omega_\alpha(u,r,s)^{\frac{1}{p}}\omega_\alpha(u,r,s)^{\frac{1}{q}}d\nu_\alpha(u)\\
&\leq \Big(\int_0^{+\infty} |f(u)|^p\omega_\alpha(u,r,s)d\nu_\alpha(u)\Big)^{\frac{1}{p}}\Big(\int_0^{+\infty} \omega_\alpha(u,r,s)d\nu_\alpha(u)\Big)^{\frac{1}{q}}\\
&= \Big(\int_0^{+\infty} |f(u)|^p\omega_\alpha(u,r,s)d\nu_\alpha(u)\Big)^{\frac{1}{p}}.
\end{aligned}$$

Now, using Fubini-Tonelli's theorem, we have

$$\begin{aligned}
\|\tau_r^\alpha(f)\|_{p,\nu_\alpha}^p &\leq \int_0^{+\infty}\int_0^{+\infty} |f(u)|^p\omega_\alpha(u,r,s)d\nu_\alpha(s)d\nu_\alpha(u)\\
&= \int_0^{+\infty} |f(u)|^p\Big(\int_0^{+\infty} \omega_\alpha(u,r,s)d\nu_\alpha(s)\Big)d\nu_\alpha(u)\\
&= \|f\|_{p,\nu_\alpha}^p.
\end{aligned}$$

■

1.1.2 Convolution product for the Bessel operator

Definition 1.1.2 *The convolution product of $f, g \in L^1(d\nu_\alpha)$ is defined by*

$$\begin{aligned}
f * g(r) &= \int_0^{+\infty} \tau_r^\alpha(f)(s)g(s)d\nu_\alpha(s),\\
&= \int_0^{+\infty} f(s)\tau_r^\alpha(g)(s)d\nu_\alpha(s).
\end{aligned}$$

Theorem 1.1.2 *For all $f, g \in L^1(d\nu_\alpha)$, $f * g \in L^1(d\nu_\alpha)$ and we have*

$$\|f * g\|_{1,\nu_\alpha} \leq \|f\|_{1,\nu_\alpha}\|g\|_{1,\nu_\alpha}.$$

Proof. According to Fubini-Tonnelli's theorem and relation (1.11), we have for every

$f, g \in L^1(d\nu_\alpha)$

$$\begin{aligned}
\int_0^{+\infty} |f * g(r)| d\nu_\alpha(r) &\leq \int_0^{+\infty}\int_0^{+\infty} |\tau_r^\alpha(f)(s)| \ |g(s)| d\nu_\alpha(s) d\nu_\alpha(r) \\
&= \int_0^{+\infty} |g(s)| \Big(\int_0^{+\infty} |\tau_s^\alpha(f)(r)| d\nu_\alpha(r) \Big) d\nu_\alpha(s) \\
&= \int_0^{+\infty} |g(s)| \ \|\tau_s^\alpha f\|_{1,\nu_\alpha} d\nu_\alpha(s) \\
&\leq \|f\|_{1,\nu_\alpha} \|g\|_{1,\nu_\alpha}.
\end{aligned}$$

Then, the function $f * g \in L^1(d\nu_\alpha)$ and

$$\|f * g\|_{1,\nu_\alpha} \leq \|f\|_{1,\nu_\alpha} \|g\|_{1,\nu_\alpha}.$$

■

Theorem 1.1.3 *For all $f \in L^1(d\nu_\alpha)$, $g \in L^p(d\nu_\alpha)$ such that $p \in [1, +\infty[$, $f * g$ in $L^p(d\nu_\alpha)$. Furthermore,*

$$\|f * g\|_{p,\nu_\alpha} \leq \|f\|_{1,\nu_\alpha} \|g\|_{p,\nu_\alpha}.$$

Proof. Let $f \in L^1(d\nu_\alpha)$, $g \in L^p(d\nu_\alpha)$, $p \in [1, +\infty[$ and let q be the conjugate exponent of p. So, from Hölder's inequality, we obtain

$$\begin{aligned}
|f * g(r)| &\leq \int_0^{+\infty} |f(s)| \ |\tau_r^\alpha(g)(s)| d\nu_\alpha(s) \\
&= \int_0^{+\infty} |f(s)|^{\frac{1}{q}} |f(s)|^{\frac{1}{p}} \ |\tau_r^\alpha(g)(s)| d\nu_\alpha(s) \\
&\leq \|f\|_{1,\nu_\alpha}^{\frac{1}{q}} \Big(\int_0^{+\infty} |f(s)| \ |\tau_r^\alpha(g)(s)|^p d\nu_\alpha(s) \Big)^{\frac{1}{p}}.
\end{aligned}$$

Using Fubini-Tonnelli's theorem and relation (1.11), we get

$$\begin{aligned}
\|f*g\|_{p,\nu_\alpha}^p &\le \|f\|_{1,\nu_\alpha}^{\frac{p}{q}} \int_0^{+\infty} |f(s)| \Big(\int_0^{+\infty} |\tau_s^\alpha(g)(r)|^p d\nu_\alpha(r) \Big) d\nu_\alpha(s) \\
&= \|f\|_{1,\nu_\alpha}^{\frac{p}{q}} \int_0^{+\infty} |f(s)| \ \|\tau_s^\alpha g\|_{p,\nu_\alpha}^p d\nu_\alpha(s) \\
&\le \|f\|_{1,\nu_\alpha}^{\frac{p}{q}+1} \|g\|_{p,\nu_\alpha}^p \\
&= \|f\|_{1,\nu_\alpha}^p \|g\|_{p,\nu_\alpha}^p.
\end{aligned}$$

■

Theorem 1.1.4 *For all f in $L^p(d\nu_\alpha)$, g in $L^q(d\nu_\alpha)$ and for all $p, q, r \in [1, +\infty]$ such that $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$, the function $f * g$ belongs to the space $L^r(d\nu_\alpha)$ and we have the followig Young's inequality*

$$\|f*g\|_{r,\nu_\alpha} \le \|f\|_{p,\nu_\alpha} \|g\|_{q,\nu_\alpha}. \tag{1.12}$$

1.2 Hankel transform

Definition 1.2.1 *The Hankel transform $\mathscr{H}_\alpha$ is defined on $L^1(d\nu_\alpha)$ by*

$$\mathscr{H}_\alpha(f)(\lambda) = \int_0^{+\infty} f(r) j_\alpha(\lambda r) d\nu_\alpha(r), \quad \forall \lambda \in \mathbb{R}.$$

where j_α is the modified Bessel function given by (1.1).

The Hankel transform $\mathscr{H}_\alpha$ satisfies the following results:

Theorem 1.2.1 *1. (Inversion formula for the Hankel transform) Let $f \in L^1(d\nu_\alpha)$ such that $\mathscr{H}_\alpha(f) \in L^1(d\nu_\alpha)$, then we have*

$$f(r) = \int_0^{+\infty} \mathscr{H}_\alpha(f)(\lambda) j_\alpha(\lambda r) d\nu_\alpha(\lambda) = \mathscr{H}_\alpha(\mathscr{H}_\alpha(f))(r), \quad a.e. \tag{1.13}$$

2. *(Plancherel's formula) The Hankel transform* $\mathscr{H}_\alpha$ *can be extended to an isometric isomorphism from* $L^2(d\nu_\alpha)$ *onto itself and we have*

$$||\mathscr{H}_\alpha(f)||_{2,\nu_\alpha} = ||f||_{2,\nu_\alpha}. \tag{1.14}$$

3. *(Parseval's formula) For all* $f, g \in L^2(d\nu_\alpha)$*, we have*

$$\int_0^{+\infty} f(r)\overline{g(r)}\, d\nu_\alpha(r) = \int_0^{+\infty} \mathscr{H}_\alpha(f)(\lambda)\overline{\mathscr{H}_\alpha(g)(\lambda)}\, d\nu_\alpha(\lambda).$$

Proposition 1.2.1 1. *For every* $f \in L^2(d\nu_\alpha)$ *and* $r \in [0, +\infty[$*, we have*

$$\mathscr{H}_\alpha(\tau_r^\alpha(f))(\lambda) = j_\alpha(\lambda r)\mathscr{H}_\alpha(f)(\lambda), \quad \forall \lambda \in \mathbb{R}. \tag{1.15}$$

2. *For every* $f \in L^1(d\nu_\alpha)$ *and* $g \in L^2(d\nu_\alpha)$*, the function* $f * g$ *belongs to* $L^2(d\nu_\alpha)$ *and we have*

$$\mathscr{H}_\alpha(f * g) = \mathscr{H}_\alpha(f)\mathscr{H}_\alpha(g). \tag{1.16}$$

3. *Let* $f, g \in L^2(d\nu_\alpha)$*. Then* $f * g \in L^2(d\nu_\alpha)$*, if and only if* $\mathscr{H}_\alpha(f)\mathscr{H}_\alpha(g) \in L^2(d\nu_\alpha)$ *and we have*

$$\mathscr{H}_\alpha(f * g) = \mathscr{H}_\alpha(f)\mathscr{H}_\alpha(g), \tag{1.17}$$

Moreover,

$$\int_0^{+\infty} |f * g(r)|^2 d\nu_\alpha(r) = \int_0^{+\infty} |\mathscr{H}_\alpha(f)(\lambda)|^2 |\mathscr{H}_\alpha(g)(\lambda)|^2 d\nu_\alpha(\lambda).$$

where both integrals are finite or infinite.

Remark 1.2.1 *For every $f, g \in L^2(d\nu_\alpha)$ and $r \in [0, +\infty[$, we have*

$$\tau_r^\alpha(f * g) = \tau_r^\alpha(f) * g = f * \tau_r^\alpha(g). \tag{1.18}$$

CHAPTER 2

HANKEL-STOCKWELL TRANSFORM

The Stockwell transform is a time-frequency spectral localization technique that combines elements of wavelet transform, which analyzes function with respect to position and scale, and Short-Time Fourier Transform which analyzes function concerning position and frequency.

Our investigation in this chapter is to define and study a new transform called the Hankel-Stockwell transform and we establish several basic properties for this transform.

we also prove that $S^{\alpha}_{\psi}(L^2(d\nu_\alpha))$ is a reproducing kernel Hilbert space with kernel function defined by

$$k_\psi\left((a,r);(a',r')\right) = \frac{1}{C_\psi}\langle \psi^{\alpha}_{a,r}|\psi^{\alpha}_{a',r'}\rangle_{\nu_\alpha}, \quad (a,r),(a',r') \in \mathbb{R}^*_+ \times \mathbb{R}_+;$$

where $\psi^{\alpha}_{a,r}$ is the family given by relation (2.11) and C_ψ is the admissible condition for the Hankel-Stockwell transform given by (2.13) .

In the following we denote by

• μ_α the measure defined on $\mathbb{R}^*_+ \times \mathbb{R}_+$ by

$$d\mu_\alpha(a,r) = d\nu_\alpha(a)d\nu_\alpha(r). \tag{2.1}$$

• $L^p(d\mu_\alpha)$, $1 \leq p \leq +\infty$, the Lebesgue space on $\mathbb{R}^*_+ \times \mathbb{R}_+$, with respect to the measure μ_α with the L^p-norm denoted by $\|\ .\ \|_{p,\mu_\alpha}$.

• $\langle .\ |\ .\rangle_{\mu_\alpha}$ the inner product on $L^2(d\mu_\alpha)$ defined by

$$\langle f|g\rangle_{\mu_\alpha} = \int_0^{+\infty}\int_0^{+\infty} f(a,r)\overline{g(a,r)}d\mu_\alpha(a,r).$$

2.1 Dilation operator

For every $a \in \mathbb{R}_+^*$, the dilation operator D_a^α is defined for every measurable function ψ on $\mathbb{R}_+$ by

$$D_a^\alpha(\psi)(r) = a^{\alpha+1}\psi(ar), \quad \forall r \in [0, +\infty[.$$

Then, we have the following properties:

Properties 2.1.1 *1. For every ψ in $L^2(d\nu_\alpha)$,*

$$\|D_a^\alpha(\psi)\|_{2,\nu_\alpha} = \|\psi\|_{2,\nu_\alpha}. \tag{2.2}$$

2. For all ψ, φ in $L^2(d\nu_\alpha)$,

$$\langle D_a^\alpha(\psi)|\varphi\rangle_{\nu_\alpha} = \langle \psi|D_{\frac{1}{a}}^\alpha(\varphi)\rangle_{\nu_\alpha}. \tag{2.3}$$

3. For every ψ in $L^2(d\nu_\alpha)$,

$$|D_a^\alpha(\psi)|^2 = a^{\alpha+1}D_a^\alpha|\psi|^2, \tag{2.4}$$

and

$$\sqrt{D_a^\alpha(|\psi|)} = a^{-\frac{\alpha+1}{2}}D_a^\alpha(\sqrt{|\psi|}). \tag{2.5}$$

4. For every ψ in $L^2(d\nu_\alpha)$,

$$\tau_r^\alpha D_a^\alpha(\psi) = D_a^\alpha \tau_{ar}^\alpha(\psi). \tag{2.6}$$

5. For every ψ in $L^2(d\nu_\alpha)$,

$$\mathscr{H}_\alpha(D_a^\alpha(\psi)) = D_{\frac{1}{a}}^\alpha(\mathscr{H}_\alpha(\psi)). \tag{2.7}$$

Proof.

1. For every ψ in $L^2(d\nu_\alpha)$, we have

$$\begin{aligned} ||D_a^\alpha(\psi)||^2_{2,\nu_\alpha} &= \int_0^{+\infty} |D_a^\alpha(\psi)(r)|^2 d\nu_\alpha(r) \\ &= a^{2\alpha+2}\int_0^{+\infty} |\psi(ar)|^2 d\nu_\alpha(r) \\ &= \int_0^{+\infty} |\psi(s)|^2 d\nu_\alpha(s) \\ &= ||\psi||^2_{2,\nu_\alpha}. \end{aligned}$$

2. For every ψ, φ in $L^2(d\nu_\alpha)$, we get

$$\begin{aligned} \langle D_a^\alpha(\psi)|\varphi\rangle_{\nu_\alpha} &= \int_0^{+\infty} D_a^\alpha(\psi)(r)\overline{\varphi(r)} d\nu_\alpha(r) \\ &= a^{\alpha+1}\int_0^{+\infty} \psi(ar)\overline{\varphi(r)} d\nu_\alpha(r) \\ &= \int_0^{+\infty} \psi(s)\overline{\frac{1}{a^{\alpha+1}}\varphi\Big(\frac{s}{a}\Big)} d\nu_\alpha(s) \\ &= \langle \psi|D^\alpha_{\frac{1}{a}}(\varphi)\rangle_{\nu_\alpha}. \end{aligned}$$

3. For every ψ in $L^2(d\nu_\alpha)$, we obtain

$$\begin{aligned} |D_a^\alpha(\psi)(r)|^2 &= |a^{\alpha+1}\psi(ar)|^2 \\ &= a^{2\alpha+2}|\psi(ar)|^2 \\ &= a^{\alpha+1} D_a^\alpha(|\psi|^2)(r), \end{aligned}$$

and

$$\begin{aligned} \sqrt{D_a^\alpha(|\psi|)(r)} &= \sqrt{a^{\alpha+1}|\psi(ar)|} \\ &= a^{\frac{\alpha+1}{2}}\sqrt{|\psi(ar)|} \\ &= a^{-\frac{\alpha+1}{2}} D_a^\alpha(\sqrt{|\psi|})(r). \end{aligned}$$

4. Let $\psi \in L^2(d\nu_\alpha)$. So, by Definition 1.1.1, we have

$$\begin{aligned}
\tau_r^\alpha(D_a^\alpha(\psi))(s) &= \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}\int_0^\pi D_a^\alpha(\psi)\Big(\sqrt{r^2+s^2+2rs\cos\theta}\Big)(\sin\theta)^{2\alpha}d\theta \\
&= a^{\alpha+1}\frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}\int_0^\pi \psi\Big(\sqrt{(ar)^2+(as)^2+2(ar)(as)\cos\theta}\Big)(\sin\theta)^{2\alpha}d\theta \\
&= a^{\alpha+1}\tau_{ar}^\alpha(\psi)(as) \\
&= D_a^\alpha(\tau_{ar}^\alpha(\psi))(s).
\end{aligned}$$

5. Let $\psi \in L^2(d\nu_\alpha)$, then

$$\begin{aligned}
\mathscr{H}_\alpha(D_a^\alpha(\psi))(\lambda) &= \int_0^{+\infty} D_a^\alpha(\psi)(r)j_\alpha(\lambda r)d\nu_\alpha(r) \\
&= a^{\alpha+1}\int_0^{+\infty}\psi(ar)j_\alpha(\lambda r)d\nu_\alpha(r) \\
&= \frac{1}{a^{\alpha+1}}\int_0^{+\infty}\psi(s)j_\alpha\Big(\frac{\lambda}{a}s\Big)d\nu_\alpha(s) \\
&= \frac{1}{a^{\alpha+1}}\mathscr{H}_\alpha(\psi)\Big(\frac{\lambda}{a}\Big) \\
&= D_{\frac{1}{a}}^\alpha(\mathscr{H}_\alpha(\psi))(\lambda).
\end{aligned}$$

■

2.2 Modulation operator

This section is devoted to define the modulation operator M_a^α and to prove that this operator is an isometry on $L^2(d\nu_\alpha)$.

Definition 2.2.1 *The modulation operator is defined for every function ψ in $L^2(d\nu_\alpha)$ by*

$$M_a^\alpha(\psi) = \mathscr{H}_\alpha\Big(\sqrt{\tau_a^\alpha(|\mathscr{H}_\alpha(\psi)|^2)}\Big), \quad \forall a > 0. \tag{2.8}$$

Proposition 2.2.1 *1. For every $\psi \in L^2(d\nu_\alpha)$, $M_a^\alpha(\psi)$ belongs to $L^2(d\nu_\alpha)$ and we have*

$$\|M_a^\alpha(\psi)\|_{2,\nu_\alpha} = \|\psi\|_{2,\nu_\alpha}. \tag{2.9}$$

2. For every $\psi \in L^2(d\nu_\alpha)$, we get

$$M_a^\alpha D_a^\alpha(\psi) = D_a^\alpha M_1^\alpha(\psi). \tag{2.10}$$

Proof.

1. From relation (1.9) and by using Plancherel's formula for the Hankel transform $\mathscr{H}_\alpha$ (1.14), we have

$$\begin{aligned}\int_0^{+\infty} \Big|\mathscr{H}_\alpha\Big(\sqrt{\tau_a^\alpha(|\mathscr{H}_\alpha(\psi)|^2)}\Big)(\lambda)\Big|^2 d\nu_\alpha(\lambda) &= \int_0^{+\infty} \Big|\sqrt{\tau_a^\alpha(|\mathscr{H}_\alpha(\psi)|^2)(r)}\Big|^2 d\nu_\alpha(r)\\ &= \int_0^{+\infty} \tau_a^\alpha(|\mathscr{H}_\alpha(\psi)|^2)(r) d\nu_\alpha(r)\\ &= \int_0^{+\infty} |\mathscr{H}_\alpha(\psi)(\lambda)|^2 d\nu_\alpha(\lambda)\\ &= \int_0^{+\infty} |\psi(r)|^2 d\nu_\alpha(r)\\ &= \|\psi\|_{2,\nu_\alpha}^2 < +\infty.\end{aligned}$$

   Then, the function $M_a^\alpha(\psi) \in L^2(d\nu_\alpha)$ and we have $\|M_a^\alpha(\psi)\|_{2,\nu_\alpha} = \|\psi\|_{2,\nu_\alpha}$.

2. Let $\psi \in L^2(d\nu_\alpha)$. Then, using relations (2.4), (2.6) and (2.7) we get

$$
\begin{aligned}
M_a^\alpha D_a^\alpha(\psi) &= \mathscr{H}_\alpha\Big(\sqrt{\tau_a^\alpha\Big|\mathscr{H}_\alpha(D_a^\alpha(\psi))\Big|^2}\Big)\\
&= \mathscr{H}_\alpha\Big(\sqrt{\tau_a^\alpha\Big|D^\alpha_{\frac{1}{a}}(\mathscr{H}_\alpha(\psi))\Big|^2}\Big)\\
&= \frac{1}{a^{\frac{\alpha+1}{2}}}\mathscr{H}_\alpha\Big(\sqrt{\tau_a^\alpha D^\alpha_{\frac{1}{a}}(|\mathscr{H}_\alpha(\psi)|^2)}\Big)\\
&= \frac{1}{a^{\frac{\alpha+1}{2}}}\mathscr{H}_\alpha\Big(\sqrt{D^\alpha_{\frac{1}{a}}\tau_1^\alpha(|\mathscr{H}_\alpha(\psi)|^2)}\Big)\\
&= \mathscr{H}_\alpha D^\alpha_{\frac{1}{a}}\Big(\sqrt{\tau_1^\alpha(|\mathscr{H}_\alpha(\psi)|^2)}\Big)\\
&= D_a^\alpha\mathscr{H}_\alpha\Big(\sqrt{\tau_1^\alpha(|\mathscr{H}_\alpha(\psi)|^2)}\Big)\\
&= D_a^\alpha M_1^\alpha(\psi).
\end{aligned}
$$

■

2.3 Hankel-Stockwell transform S_ψ^α

The main aim of this part is to define the Hankel-Stockwell transform S_ψ^α and to prove a Plancherel's formula and a reconstruction formula for this transform. we also prove that the function $S_\psi^\alpha(f)$ belongs to $L^p(d\mu_\alpha)$, $p \in [2, +\infty]$ for every $f \in L^2(d\nu_\alpha)$.

Definition 2.3.1 *For every $\psi \in L^2(d\nu_\alpha)$, the family $\psi_{a,r}^\alpha$, $(a, r) \in \mathbb{R}_+^* \times \mathbb{R}_+$, defined by*

$$\psi_{a,r}^\alpha(s) = \tau_r^\alpha M_a^\alpha D_a^\alpha \psi(s), \quad \forall s \in \mathbb{R}_+. \tag{2.11}$$

By relations (1.11), (2.2) and (2.9), we have

$$||\psi_{a,r}^\alpha||_{2,\nu_\alpha} \leq ||\psi||_{2,\nu_\alpha}. \tag{2.12}$$

Definition 2.3.2 *A nonzero function $\psi \in L^2(d\nu_\alpha)$ is said to be an admissible window function if*

$$0 < C_\psi = c_\alpha \int_0^{+\infty} \tau_1^\alpha\Big(|\mathscr{H}_\alpha(\psi)|^2\Big)(a)\frac{da}{a} < +\infty, \tag{2.13}$$

where

$$c_\alpha = \frac{1}{2^\alpha \Gamma(\alpha+1)}.$$

Definition 2.3.3 *Let ψ be an admissible window function. The continuous Hankel-Stockwell transform S_ψ^α is defined in $L^2(d\nu_\alpha)$ by*

$$S_\psi^\alpha(f)(a,r) = \int_0^{+\infty} f(s)\overline{\psi_{a,r}^\alpha(s)}d\nu_\alpha(s), \quad (a,r) \in \mathbb{R}_+^* \times \mathbb{R}_+.$$

where $\psi_{a,r}^\alpha$ is given by relation (2.11).

The continuous Hankel-Stockwell transform can also be written as

$$\begin{aligned} S_\psi^\alpha(f)(a,r) &= f * M_a^\alpha D_a^\alpha(\overline{\psi})(r) \quad (2.14)\\ &= \langle f|\psi_{a,r}^\alpha\rangle_{\nu_\alpha}. \quad (2.15)\end{aligned}$$

Proposition 2.3.1 *Let ψ be an admissible window function. Then, the continuous Hankel-Stockwell transform S_ψ^α is a bounded linear operator from $L^2(d\nu_\alpha)$ onto $L^\infty(d\mu_\alpha)$ and we have*

$$||S_\psi^\alpha(f)||_{\infty,\mu_\alpha} \leq ||\psi||_{2,\nu_\alpha}||f||_{2,\nu_\alpha}. \tag{2.16}$$

Proof. Let $\psi \in L^2(d\nu_\alpha)$ be an admissible window function. Then, from Cauchy-Schwarz's inequality and relation (2.12), we obtain

$$\begin{aligned} |S^\alpha_\psi(f)(a,r)| &= |\langle f|\psi^\alpha_{a,r}\rangle_{\nu_\alpha}| \\ &\leq \|\psi^\alpha_{a,r}\|_{2,\nu_\alpha}\|f\|_{2,\nu_\alpha} \\ &\leq \|\psi\|_{2,\nu_\alpha}\|f\|_{2,\nu_\alpha}. \end{aligned}$$

Then

$$\|S^\alpha_\psi(f)\|_{\infty,\mu_\alpha} \leq \|\psi\|_{2,\nu_\alpha}\|f\|_{2,\nu_\alpha}.$$

■

The Hankel-Stockwell transform S^α_ψ satisfies the following properties:

Theorem 2.3.1 *(Plancherel's formula) Let ψ be an admissible window function in $L^2(d\nu_\alpha)$, then we have*

$$\|S^\alpha_\psi(f)\|_{2,\mu_\alpha} = \sqrt{C_\psi}\|f\|_{2,\nu_\alpha}. \tag{2.17}$$

Proof. From relations (1.17) and (2.14) and using Fubini-Tonelli's theorem, we get

$$\begin{aligned} &\int_0^{+\infty}\int_0^{+\infty} |S^\alpha_\psi(f)(a,r)|^2 d\mu_\alpha(a,r) \\ &= \int_0^{+\infty}\int_0^{+\infty} |f * M^\alpha_a D^\alpha_a(\psi)(r)|^2 d\nu_\alpha(a)d\nu_\alpha(r) \\ &= \int_0^{+\infty}\int_0^{+\infty} |\mathscr{H}_\alpha(f)(\lambda)|^2 |\mathscr{H}_\alpha(M^\alpha_a D^\alpha_a(\psi))(\lambda)|^2 d\nu_\alpha(a)d\nu_\alpha(\lambda) \\ &= \int_0^{+\infty} |\mathscr{H}_\alpha(f)(\lambda)|^2 \Big(\int_0^{+\infty} |\mathscr{H}_\alpha(M^\alpha_a D^\alpha_a(\psi))(\lambda)|^2 d\nu_\alpha(a)\Big) d\nu_\alpha(\lambda). \end{aligned} \tag{2.18}$$

Now, using relations (2.4), (2.7), (2.8) and (2.10), we get

$$\begin{aligned}\int_0^{+\infty}|\mathscr{H}_\alpha(M_a^\alpha D_a^\alpha(\psi))(\lambda)|^2 d\nu_\alpha(a) &= \int_0^{+\infty}|\mathscr{H}_\alpha(D_a^\alpha M_1^\alpha(\psi))(\lambda)|^2 d\nu_\alpha(a)\\ &= \int_0^{+\infty}|D_{\frac{1}{a}}^\alpha \mathscr{H}_\alpha(M_1^\alpha(\psi))(\lambda)|^2 d\nu_\alpha(a)\\ &= \int_0^{+\infty}\left|D_{\frac{1}{a}}^\alpha\Big(\sqrt{\tau_1^\alpha(|\mathscr{H}_\alpha(\psi)|^2)}\Big)(\lambda)\right|^2 d\nu_\alpha(a)\\ &= \int_0^{+\infty}\frac{1}{a^{\alpha+1}}D_{\frac{1}{a}}^\alpha\Big(\tau_1^\alpha(|\mathscr{H}_\alpha(\psi)|^2)\Big)(\lambda)d\nu_\alpha(a)\\ &= \int_0^{+\infty}\frac{1}{a^{2\alpha+2}}\tau_1^\alpha(|\mathscr{H}_\alpha(\psi)|^2)\Big(\frac{\lambda}{a}\Big)d\nu_\alpha(a)\\ &= \frac{1}{2^\alpha\Gamma(\alpha+1)}\int_0^{+\infty}\tau_1^\alpha(|\mathscr{H}_\alpha(\psi)|^2)(a)\frac{da}{a}.\end{aligned}$$

Then, we get

$$\int_0^{+\infty}|\mathscr{H}_\alpha(M_a^\alpha D_a^\alpha(\psi))(\lambda)|^2 d\nu_\alpha(a) = C_\psi. \tag{2.19}$$

Then, from Plancherel's formula for the Hankel transform $\mathscr{H}_\alpha$ (1.14) and by combining relations (2.18) and (2.19), we obtain

$$\begin{aligned}||S_\psi^\alpha(f)||_{2,\mu_\alpha} &= \sqrt{C_\psi}||\mathscr{H}_\alpha(f)||_{2,\nu_\alpha}\\ &= \sqrt{C_\psi}||f||_{2,\nu_\alpha}.\end{aligned}$$

■

corollary 2.3.1 *(Parseval's formula) Let ψ be an admissible window function in $L^2(d\nu_\alpha)$. Then, for all f and g in $L^2(d\nu_\alpha)$, we have*

$$\int_0^{+\infty}\int_0^{+\infty} S_\psi^\alpha(f)(a,r)\overline{S_\psi^\alpha(g)(a,r)}d\mu_\alpha(a,r) = C_\psi\int_0^{+\infty} f(s)\overline{g(s)}d\nu_\alpha(s).$$

Proof. Using Polarization identity and Plancherel's formula for the Hankel-Stockwell transform (2.17), we have

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty} S_\psi^\alpha(f)(a,r)\overline{S_\psi^\alpha(g)(a,r)}d\mu_\alpha(a,r) = \langle S_\psi^\alpha(f)|S_\psi^\alpha(g)\rangle_{\mu_\alpha}\\
&= \frac{1}{4}\Big(||S_\psi^\alpha(f)+S_\psi^\alpha(g)||_{2,\mu_\alpha}^2 - ||S_\psi^\alpha(f)-S_\psi^\alpha(g)||_{2,\mu_\alpha}^2 + ||S_\psi^\alpha(f)+iS_\psi^\alpha(g)||_{2,\mu_\alpha}^2 - ||S_\psi^\alpha(f)-iS_\psi^\alpha(g)||_{2,\mu_\alpha}^2\Big)\\
&= \frac{1}{4}\Big(||S_\psi^\alpha(f+g)||_{2,\mu_\alpha}^2 - ||S_\psi^\alpha(f-g)||_{2,\mu_\alpha}^2 + ||S_\psi^\alpha(f+ig)||_{2,\mu_\alpha}^2 - ||S_\psi^\alpha(f-ig)||_{2,\mu_\alpha}^2\Big)\\
&= C_\psi\Big(\frac{1}{4}\Big(||f+g||_{2,\nu_\alpha}^2 - ||f-g||_{2,\nu_\alpha}^2 + ||f+ig||_{2,\nu_\alpha}^2 - ||f-ig||_{2,\nu_\alpha}^2\Big)\Big)\\
&= C_\psi\langle f|g\rangle_{\nu_\alpha}\\
&= C_\psi\int_0^{+\infty} f(s)\overline{g(s)}d\nu_\alpha(s).
\end{aligned}$$

■

Theorem 2.3.2 *(Reconstruction formula) Let ψ be an admissible window function in $L^2(d\nu_\alpha)$ such that $|\psi|$ is an admissible window function. Then, for every $f \in L^2(d\nu_\alpha)$, we have*

$$f(.) = \frac{1}{C_\psi}\int_0^{+\infty}\int_0^{+\infty} S_\psi^\alpha(f)(a,r)\psi_{a,r}^\alpha(.)d\mu_\alpha(a,r),$$

weakly in $L^2(d\nu_\alpha)$.

Proof. From Corollary 2.3.1 and Fubini-Tonelli's theorem, we have for all g in $L^2(d\nu_\alpha)$

$$\begin{aligned}
\langle f|g\rangle_{\nu_\alpha} &= \int_0^{+\infty} f(s)\overline{g(s)}d\nu_\alpha(s)\\
&= \frac{1}{C_\psi}\int_0^{+\infty}\int_0^{+\infty} S_\psi^\alpha(f)(a,r)\overline{S_\psi^\alpha(g)(a,r)}d\mu_\alpha(a,r)\\
&= \frac{1}{C_\psi}\int_0^{+\infty}\int_0^{+\infty} S_\psi^\alpha(f)(a,r)\langle\psi_{a,r}^\alpha|g\rangle_{\nu_\alpha}d\mu_\alpha(a,r).
\end{aligned}$$

which gives the result. ■

Remark 2.3.1 *By using the fact that $S^{\alpha}_{\psi}(f)$ belongs to $L^2(d\mu_\alpha)$, for almost $a \in \mathbb{R}^*_+$, the function $r \longmapsto S^{\alpha}_{\psi}(f)(a,r) = f * M^{\alpha}_a D^{\alpha}_a(\overline{\psi})(r)$ belongs to $L^2(d\nu_\alpha)$. Then, by using relations (1.17), (2.7) and (2.10), we obtain*

$$\mathscr{H}_\alpha(S^{\alpha}_{\psi}(f)(a,.))(\lambda) = \frac{1}{a^{\alpha+1}}\mathscr{H}_\alpha(f)(\lambda)\sqrt{\tau^{\alpha}_1\Big(|\mathscr{H}_\alpha(\psi)|^2\Big)\Big(\frac{\lambda}{a}\Big)}. \tag{2.20}$$

Theorem 2.3.3 *Let ψ be an admissible window function in $L^2(d\nu_\alpha)$. For every $f \in L^2(d\nu_\alpha)$, the function $S^{\alpha}_{\psi}(f)$ belongs to $L^p(d\mu_\alpha)$, $p \in [2,+\infty]$ and we have*

$$||S^{\alpha}_{\psi}(f)||_{p,\mu_\alpha} \leq C_{\psi}^{\frac{1}{p}}||\psi||_{2,\nu_\alpha}^{1-\frac{2}{p}}||f||_{2,\nu_\alpha}.$$

Proof. For $p = 2$. The Plancherel's formula for the continuous Hankel-Stockwell transform (2.17) gives

$$||S^{\alpha}_{\psi}(f)||_{2,\mu_\alpha} = C_{\psi}^{\frac{1}{2}}||f||_{2,\nu_\alpha}.$$

For $p = +\infty$ and by relation (2.16), we have

$$||S^{\alpha}_{\psi}(f)||_{\infty,\mu_\alpha} \leq ||\psi||_{2,\nu_\alpha}||f||_{2,\nu_\alpha}.$$

From Riesz-Thorin's interpolation Theorem 3.2.3, we get for every $p \in [2,+\infty]$

$$\begin{aligned} ||S^{\alpha}_{\psi}(f)||_{p,\mu_\alpha} &\leq ||S^{\alpha}_{\psi}(f)||_{\infty,\mu_\alpha}^{1-\frac{2}{p}}||S^{\alpha}_{\psi}(f)||_{2,\mu_\alpha}^{\frac{2}{p}} \\ &\leq C_{\psi}^{\frac{1}{p}}||\psi||_{2,\nu_\alpha}^{1-\frac{2}{p}}||f||_{2,\nu_\alpha}. \end{aligned}$$

■

Proposition 2.3.2 *Let ψ be an admissible window function in $L^2(d\nu_\alpha)$. For every function f in $L^2(d\nu_\alpha)$, we have*

1. *For all* $r_0 \geq 0$,

$$S^\alpha_\psi(\tau^\alpha_{r_0}(f))(a,r) = \tau^\alpha_{r_0}(S^\alpha_\psi(f)(a,.))(r), \quad (a,r) \in \mathbb{R}^*_+ \times \mathbb{R}_+. \tag{2.21}$$

2. *For* $\lambda > 0$, *we have*

$$S^\alpha_\psi(D^\alpha_\lambda(f))(a,r) = \delta_\lambda(S^\alpha_\psi(f))(a,r), \quad (a,r) \in \mathbb{R}^*_+ \times \mathbb{R}_+. \tag{2.22}$$

Proof.

1. From relations (1.18) and (2.14), we have

$$\begin{aligned} S^\alpha_\psi(\tau^\alpha_{r_0}(f))(a,r) &= \tau^\alpha_{r_0}(f) * M^\alpha_a D^\alpha_a(\overline{\psi})(r) \\ &= \tau^\alpha_{r_0}\Big(f * M^\alpha_a D^\alpha_a(\overline{\psi})\Big)(r) \\ &= \tau^\alpha_{r_0}(S^\alpha_\psi(f)(a,.))(r). \end{aligned}$$

2. Using relations (2.3), (2.6), (2.10) and (2.15), we have

$$\begin{aligned} S^\alpha_\psi(D^\alpha_\lambda(f))(a,r) &= \langle D^\alpha_\lambda f | \tau^\alpha_r M^\alpha_a D^\alpha_a \psi\rangle_{\nu_\alpha} \\ &= \langle f | D^\alpha_{\frac{1}{\lambda}} \tau^\alpha_r M^\alpha_a D^\alpha_a \psi\rangle_{\nu_\alpha} \\ &= \langle f | \tau^\alpha_{\lambda r} D^\alpha_{\frac{1}{\lambda}} M^\alpha_a D^\alpha_a \psi\rangle_{\nu_\alpha} \\ &= \langle f | \tau^\alpha_{\lambda r} M^\alpha_{\frac{a}{\lambda}} D^\alpha_{\frac{a}{\lambda}} \psi\rangle_{\nu_\alpha} \\ &= S^\alpha_\psi(f)(\frac{a}{\lambda}, \lambda r) \\ &= \delta_\lambda(S^\alpha_\psi(f))(a,r). \end{aligned}$$

■

2.4 Reproducing kernel Hilbert space $S_\psi^\alpha(L^2(d\nu_\alpha))$

In this section, we prove that $S_\psi^\alpha(L^2(d\nu_\alpha))$ is a reproducing kernel Hilbert space with kernel function defined by (2.23).

Definition 2.4.1 *(Reproducing kernel) Let H be a Hilbert space of functions defined from arbitrary set X into $\mathbb{C}$ issued with the inner product $\langle . \mid . \rangle_H$. Let k be a function defined from $X \times X$ into $\mathbb{C}$, we say that k is a reproducing kernel for H, if*

- *For every $y \in X$, the function $x \mapsto k(x, y) \in H$,*
- *For every $f \in H$ and for every $y \in X$, $f(y) = \langle f | k(., y)\rangle_H$.*

Definition 2.4.2 *(Reproducing kernel Hilbert space) A reproducing kernel Hilbert space is a Hilbert space H with a reproducing kernel whose span is dense in H.*

Proposition 2.4.1 *(Reproducing kernel) Let ψ be an admissible window function in $L^2(d\nu_\alpha)$ and $f \in L^2(d\nu_\alpha)$. Then, $S_\psi^\alpha(L^2(d\nu_\alpha))$ is a reproducing kernel Hilbert space with kernel function*

$$k_\psi\left((a, r); (a', r')\right) = \frac{1}{C_\psi}\langle \psi_{a,r}^\alpha | \psi_{a',r'}^\alpha \rangle_{\nu_\alpha}, \quad (a, r), (a', r') \in \mathbb{R}_+^* \times \mathbb{R}_+. \tag{2.23}$$

Moreover, the kernel k_ψ is pointwise bounded and

$$|k_\psi\left((a, r); (a', r')\right)| \leq \frac{\|\psi\|_{2,\nu_\alpha}^2}{C_\psi}.$$

Proof. For $F \in S_\psi^\alpha(L^2(d\nu_\alpha))$, there exists a function $f \in L^2(d\nu_\alpha)$ such that

$$F(a, r) = S_\psi^\alpha(f)(a, r).$$

Then, from Corollary 2.3.1, we have

$$\begin{aligned} F(a,r) &= \langle f|\psi^{\alpha}_{a,r}\rangle_{\nu_\alpha} \\ &= \frac{1}{C_\psi}\langle S^{\alpha}_{\psi}(f)|S^{\alpha}_{\psi}(\psi^{\alpha}_{a,r})\rangle_{\mu_\alpha} \\ &= \langle S^{\alpha}_{\psi}(f)|k_\psi\left((a,r);(.,.)\right)\rangle_{\mu_\alpha}. \end{aligned}$$

This shows that $k_\psi\left((a,r);(a',r')\right) = \dfrac{1}{C_\psi}S^{\alpha}_{\psi}(\psi^{\alpha}_{a,r})(a',r')$ is a reproducing kernel of the Hilbert space $S^{\alpha}_{\psi}(L^2(d\nu_\alpha))$.

Finally, for all $(a,r),(a',r') \in \mathbb{R}^*_+ \times \mathbb{R}_+$, we have from Cauchy-Schwarz's inequality and relation (2.12) that

$$|k_\psi\left((a,r);(a',r')\right)| = \frac{1}{C_\psi}|\langle\psi^{\alpha}_{a,r}|\psi^{\alpha}_{a',r'}\rangle_{\nu_\alpha}| \le \frac{\|\psi\|^2_{2,\nu_\alpha}}{C_\psi}.$$

■

CHAPTER 3

TIME-FREQUENCY LOCALIZATION OPERATORS FOR THE HANKEL-STOCKWELL TRANSFORM

The localization operators were initiated by Daubechies in [10], she highlighted the role of these operators to localize a signal simultaneously in time and frequency.
This class of operators occurs in various branches of mathematics and have been studied by many authors, we cite for instance Cordero and Gröchening [8], Mari and al.[14], Mari and Nowak [13], Gröchening [19] and Wong [33].
The aim of this chapter is to define and study the localization operators associated with the Hankel-Stockwell transform. We prove that these operators are bounded, from a space of square integrable functions into itself. After that, we define the Schatten-von Neumann class S^p, $p \in [1, +\infty]$ and we show that the localization operators belong to this class. In a particular case, we give also a trace formula.

3.1 Boundedness and compactness of localization operators

Our goal in this section is to define the time-frequency localization operators with two windows for the Hankel-Stockwell transform, we prove that these operators are bounded and compact. For this, we consider two admissible window functions ψ_1 and ψ_2 in $L^2(d\nu_\alpha)$ such that

$$||\psi_1||_{2,\nu_\alpha} = ||\psi_2||_{2,\nu_\alpha} = 1.$$

• We denote by $\mathscr{B}(L^2(d\nu_\alpha))$ the Banach algebra of all bounded linear operators from $L^2(d\nu_\alpha)$ into itself, equipped with the norm

$$||A|| = \sup_{||f|| \leq 1} ||A(f)||_{2,\nu_\alpha}.$$

Lemma 3.1.1 *For every $p \in [1, +\infty]$ and all $f, g \in L^2(d\nu_\alpha)$, the function*

$$(a, r) \longmapsto S^\alpha_{\psi_1}(f)(a, r)\overline{S^\alpha_{\psi_2}(g)(a, r)},$$

belongs to $L^p(d\mu_\alpha)$ and we have

$$\|S^\alpha_{\psi_1}(f)S^\alpha_{\psi_2}(g)\|_{p,\mu_\alpha} \leq \left(\sqrt{C_{\psi_1}C_{\psi_2}}\right)^{\frac{1}{p}} \|f\|_{2,\nu_\alpha}\|g\|_{2,\nu_\alpha}. \tag{3.1}$$

Proof. From Cauchy-Schwarz's inequality and the Plancherel's formula (2.17) for S^α_ψ, for all $f, g \in L^2(d\nu_\alpha)$, we have

$$\begin{aligned}\int_0^{+\infty}\int_0^{+\infty} |S^\alpha_{\psi_1}(f)(a, r)S^\alpha_{\psi_2}(g)(a, r)| d\nu_\alpha(a) d\nu_\alpha(r) &\leq \|S^\alpha_{\psi_1}(f)\|_{2,\mu_\alpha}\|S^\alpha_{\psi_1}(g)\|_{2,\mu_\alpha} \\ &= \sqrt{C_{\psi_1}C_{\psi_2}}\|f\|_{2,\nu_\alpha}\|g\|_{2,\nu_\alpha}.\end{aligned}$$

This means that the function $S^\alpha_{\psi_1}(f)\overline{S^\alpha_{\psi_2}(g)}$ belongs to $L^1(d\mu_\alpha)$ and that

$$\|S^\alpha_{\psi_1}(f)S^\alpha_{\psi_2}(g)\|_{1,\mu_\alpha} \leq \sqrt{C_{\psi_1}C_{\psi_2}}\|f\|_{2,\nu_\alpha}\|g\|_{2,\nu_\alpha}. \tag{3.2}$$

From relation (2.16), and for every $(a, r) \in \mathbb{R}^*_+ \times \mathbb{R}_+$

$$|S^\alpha_{\psi_1}(f)(a, r)S^\alpha_{\psi_2}(g)(a, r)| \leq \|f\|_{2,\nu_\alpha}\|g\|_{2,\nu_\alpha}.$$

Hence, the function $S^\alpha_{\psi_1}(f)\overline{S^\alpha_{\psi_2}(g)}$ belongs to $L^\infty(d\mu_\alpha)$ and we have

$$\|S^\alpha_{\psi_1}(f)S^\alpha_{\psi_2}(g)\|_{\infty,\mu_\alpha} \leq \|f\|_{2,\nu_\alpha}\|g\|_{2,\nu_\alpha}. \tag{3.3}$$

Using relations (3.2) and (3.3), we obtain for every $p \in [1, +\infty]$

$$\begin{aligned} \|S^\alpha_{\psi_1}(f)S^\alpha_{\psi_2}(g)\|_{p,\mu_\alpha} &\leq \|S^\alpha_{\psi_1}(f)S^\alpha_{\psi_2}(g)\|^{1-\frac{1}{p}}_{\infty,\mu_\alpha}\|S^\alpha_{\psi_1}(f)S^\alpha_{\psi_2}(g)\|^{\frac{1}{p}}_{1,\mu_\alpha} \\ &\leq \left(\sqrt{C_{\psi_1}C_{\psi_2}}\right)^{\frac{1}{p}}\|f\|_{2,\nu_\alpha}\|g\|_{2,\nu_\alpha}. \end{aligned}$$

■

Theorem 3.1.1 *(Riesz's representation Theorem)*
Let H be a Hilbert space (real or complex) with the inner product $\langle . \mid . \rangle_H$ and let L be a continuous linear form on H. Then, there exists a unique v in H such that for all $u \in H$, we have

$$L(u) = \langle v \mid u \rangle_H.$$

Proposition 3.1.1 *Let $\sigma \in L^p(d\mu_\alpha)$, $p \in [1, +\infty]$. For every $f \in L^2(d\nu_\alpha)$, there exists a unique function in $L^2(d\nu_\alpha)$ denoted by $L_{\psi_1,\psi_2}(\sigma)(f)$ such that for every $g \in L^2(d\nu_\alpha)$,*

$$\langle L_{\psi_1,\psi_2}(\sigma)(f)|g\rangle_{\nu_\alpha} = \frac{1}{\sqrt{C_{\psi_1}C_{\psi_2}}}\int_0^{+\infty}\int_0^{+\infty}\sigma(a,r)S^\alpha_{\psi_1}(f)(a,r)\overline{S^\alpha_{\psi_2}(g)(a,r)}d\mu_\alpha(a,r).$$

Proof. Let $\sigma \in L^p(d\mu_\alpha)$, $p \in [1, +\infty]$. Let q be the conjugate exponent of p. From Lemma 3.1.1, for all $f, g \in L^2(d\nu_\alpha)$, the function $S^\alpha_{\psi_1}(f)\overline{S^\alpha_{\psi_2}(g)}$ belongs to $L^q(d\mu_\alpha)$. So, by Hölder's inequality, we get

$$\begin{aligned} \frac{1}{\sqrt{C_{\psi_1}C_{\psi_2}}}\int_0^{+\infty}\int_0^{+\infty} |\sigma(a,r)| \quad |S^\alpha_{\psi_1}(f)(a,r)\overline{S^\alpha_{\psi_2}(g)(a,r)}|d\mu_\alpha(a,r) \\ \leq \frac{1}{\sqrt{C_{\psi_1}C_{\psi_2}}}\|\sigma\|_{p,\mu_\alpha}\|S^\alpha_{\psi_1}(f)S^\alpha_{\psi_2}(g)\|_{q,\mu_\alpha} \\ \leq \left(\sqrt{C_{\psi_1}C_{\psi_2}}\right)^{\frac{1}{q}-1}\|f\|_{2,\nu_\alpha}\|g\|_{2,\nu_\alpha}\|\sigma\|_{p,\mu_\alpha} \\ = \left(\frac{1}{\sqrt{C_{\psi_1}C_{\psi_2}}}\right)^{\frac{1}{p}}\|f\|_{2,\nu_\alpha}\|g\|_{2,\nu_\alpha}\|\sigma\|_{p,\mu_\alpha}. \end{aligned}$$

Hence

$$g \longmapsto \frac{1}{\sqrt{C_{\psi_1}C_{\psi_2}}}\int_0^{+\infty}\int_0^{+\infty}\sigma(a,r)S^{\alpha}_{\psi_1}(f)(a,r)\overline{S^{\alpha}_{\psi_2}(g)(a,r)}d\mu_\alpha(a,r),$$

is a continuous anti-linear form on the Hilbert space $L^2(d\nu_\alpha)$. From Riesz's representation Theorem 3.1.1, there exists a unique $L_{\psi_1,\psi_2}(\sigma)(f) \in L^2(d\nu_\alpha)$ such that

$$\langle L_{\psi_1,\psi_2}(\sigma)(f)|g\rangle_{\nu_\alpha} = \frac{1}{\sqrt{C_{\psi_1}C_{\psi_2}}}\int_0^{+\infty}\int_0^{+\infty}\sigma(a,r)S^{\alpha}_{\psi_1}(f)(a,r)\overline{S^{\alpha}_{\psi_2}(g)(a,r)}d\mu_\alpha(a,r).$$

Moreover, for every $\sigma \in L^p(d\mu_\alpha)$, $p \in [1,+\infty]$, the operator

$$L_{\psi_1,\psi_2}(\sigma) : L^2(d\nu_\alpha) \longrightarrow L^2(d\nu_\alpha)$$

is a linear bounded operator and for every $f \in L^2(d\nu_\alpha)$

$$||L_{\psi_1,\psi_2}(\sigma)(f)||_{2,\nu_\alpha} \leq \left(\frac{1}{\sqrt{C_{\psi_1}C_{\psi_2}}}\right)^{\frac{1}{p}}||f||_{2,\nu_\alpha}||\sigma||_{p,\mu_\alpha}. \tag{3.4}$$

■

Definition 3.1.1 *Let $\sigma \in L^p(d\mu_\alpha)$, $p \in [1,+\infty]$. The localization operator for the Hankel-Stockwell transform $L_{\psi_1,\psi_2}(\sigma)$ is defined for all f and $g \in L^2(d\nu_\alpha)$ by*

$$\langle L_{\psi_1,\psi_2}(\sigma)(f)|g\rangle_{\nu_\alpha} = \frac{1}{\sqrt{C_{\psi_1}C_{\psi_2}}}\int_0^{+\infty}\int_0^{+\infty}\sigma(a,r)S^{\alpha}_{\psi_1}(f)(a,r)\overline{S^{\alpha}_{\psi_2}(g)(a,r)}d\mu_\alpha(a,r).$$

From relation (3.4), we deduce that $L_{\psi_1,\psi_2}(\sigma)$ belongs to $\mathscr{B}(L^2(d\nu_\alpha))$ and

$$||L_{\psi_1,\psi_2}(\sigma)|| \leq \left(\frac{1}{\sqrt{C_{\psi_1}C_{\psi_2}}}\right)^{\frac{1}{p}}||\sigma||_{p,\mu_\alpha}. \tag{3.5}$$

Theorem 3.1.2 *Let $\sigma \in L^p(d\mu_\alpha)$; $1 \leq p < +\infty$, then the operator $L_{\psi_1,\psi_2}(\sigma)$ is compact.*

Proof. Let σ in $L^1(d\mu_\alpha)$ and let $(v_k)_k$ be an orthonormal basis of $L^2(d\nu_\alpha)$, then for every $k \in \mathbb{N}$, we get

$$\begin{aligned}
\|L_{\psi_1,\psi_2}(\sigma)(v_k)\|^2_{2,\nu_\alpha} &= \langle L_{\psi_1,\psi_2}(\sigma)(v_k)|L_{\psi_1,\psi_2}(\sigma)(v_k)\rangle_{\nu_\alpha} \\
&= \frac{1}{\sqrt{C_{\psi_1}C_{\psi_2}}}\int_0^{+\infty}\int_0^{+\infty} \sigma(a,r)S^\alpha_{\psi_1}(v_k)(a,r)\overline{S^\alpha_{\psi_2}(L_{\psi_1,\psi_2}(\sigma)(v_k))(a,r)}d\mu_\alpha(a,r) \\
&= \frac{1}{\sqrt{C_{\psi_1}C_{\psi_2}}}\int_0^{+\infty}\int_0^{+\infty} \sigma(a,r)\langle v_k|\psi^\alpha_{1,a,r}\rangle_{\nu_\alpha}\langle \psi^\alpha_{2,a,r}|L_{\psi_1,\psi_2}(\sigma)(v_k)\rangle_{\nu_\alpha}d\mu_\alpha(a,r) \\
&= \frac{1}{\sqrt{C_{\psi_1}C_{\psi_2}}}\int_0^{+\infty}\int_0^{+\infty} \sigma(a,r)\langle v_k|\psi^\alpha_{1,a,r}\rangle_{\nu_\alpha}\langle L^*_{\psi_1,\psi_2}(\sigma)(\psi^\alpha_{2,a,r})|v_k\rangle_{\nu_\alpha}d\mu_\alpha(a,r).
\end{aligned}$$

Thus,

$$\begin{aligned}
\sum_{k=0}^{+\infty}\|L_{\psi_1,\psi_2}(\sigma)(v_k)\|^2_{2,\nu_\alpha} &= \frac{1}{\sqrt{C_{\psi_1}C_{\psi_2}}}\sum_{k=0}^{+\infty}\int_0^{+\infty}\int_0^{+\infty} \sigma(a,r)\langle v_k|\psi^\alpha_{1,a,r}\rangle_{\nu_\alpha}\langle L^*_{\psi_1,\psi_2}(\sigma)(\psi^\alpha_{2,a,r})|v_k\rangle_{\nu_\alpha}d\mu_\alpha(a,r) \\
&= \frac{1}{\sqrt{C_{\psi_1}C_{\psi_2}}}\int_0^{+\infty}\int_0^{+\infty} \sigma(a,r)\langle L^*_{\psi_1,\psi_2}(\sigma)(\psi^\alpha_{2,a,r})|\psi^\alpha_{1,a,r}\rangle_{\nu_\alpha}d\mu_\alpha(a,r) \\
&\leq \frac{1}{\sqrt{C_{\psi_1}C_{\psi_2}}}\int_0^{+\infty}\int_0^{+\infty} |\sigma(a,r)|\ \|L^*_{\psi_1,\psi_2}(\sigma)(\psi^\alpha_{2,a,r})\|_{2,\nu_\alpha}\|\psi^\alpha_{1,a,r}\|_{2,\nu_\alpha}d\mu_\alpha(a,r).
\end{aligned}$$

From relations (2.12) and (3.4), we get

$$\sum_{k=0}^{+\infty}\|L_{\psi_1,\psi_2}(\sigma)(v_k)\|^2_{2,\nu_\alpha} \leq \frac{\|\sigma\|^2_{1,\mu_\alpha}}{C_{\psi_1}C_{\psi_2}}.$$

This shows that the localization operator $L_{\psi_1,\psi_2}(\sigma)$ is in S^2 and that

$$\|L_{\psi_1,\psi_2}(\sigma)\|_{HS} \leq \frac{\|\sigma\|_{1,\mu_\alpha}}{\sqrt{C_{\psi_1}C_{\psi_2}}}.$$

In particular, $L_{\psi_1,\psi_2}(\sigma)$ is a compact operator.

Now, let $\sigma \in L^p(d\mu_\alpha)$, $1 \leq p < +\infty$. Since $L^1(d\mu_\alpha)\bigcap L^p(d\mu_\alpha)$ is dense in $L^p(d\mu_\alpha)$, there exists $(\sigma_k)_k \subset L^1(d\mu_\alpha)$ such that

$$\lim_{k\to+\infty} \|\sigma_k - \sigma\|_{p,\mu_\alpha} = 0.$$

By relation (3.5), we have

$$\|L_{\psi_1,\psi_2}(\sigma_k) - L_{\psi_1,\psi_2}(\sigma)\| \le \left(\frac{1}{\sqrt{C_{\psi_1}C_{\psi_2}}}\right)^{\frac{1}{p}} \|\sigma_k - \sigma\|_{p,\mu_\alpha}.$$

Thus, $\lim\limits_{k\to+\infty} L_{\psi_1,\psi_2}(\sigma_k) = L_{\psi_1,\psi_2}(\sigma)$ in $\mathscr{B}(L^2(d\nu_\alpha))$.
Since the set of compact operators is a closed ideal of $\mathscr{B}(L^2(d\nu_\alpha))$, we deduce that the operator $L_{\psi_1,\psi_2}(\sigma)$ is compact. ■

3.2 Schatten-von Neumann class of localization operators

In this section, we will introduce the Schatten-von Neumann class S^p and we will prove that the localization operator for the Hankel-Stockwell transform $L_{\psi_1,\psi_2}(\sigma)$ is in S^p. Prior to that, we set the following natation:

• $\ell^p(\mathbb{N})$, $1 \le p \le +\infty$, the set of all infinite sequences of real (or complex) numbers $x = (x_j)_{j\in\mathbb{N}}$, such that

$$\|x\|_p = \begin{cases} \left(\sum_{j=0}^{+\infty} |\, x_j \,|^p\right)^{\frac{1}{p}} < +\infty, & \text{if } 1 \le p < +\infty, \\ \sup\limits_{j\in\mathbb{N}} |\, x_j \,| < +\infty, & \text{if } p = +\infty. \end{cases}$$

• The singular values $\left(s_k(A)\right)_{k\in\mathbb{N}}$ of a compact operator A in $\mathscr{B}(L^2(d\nu_\alpha))$ are the eigenvalues of the positive compact self-adjoint operator $|A| = \sqrt{A^*A}$.

• Let A be a compact operator on a separable Hilbert space $\mathscr{H}$. We say that A belongs to the Schatten-von Neumann class S^p, $p \in [1, +\infty[$ if the sequence $\left(s_k(A)\right)_{k\in\mathbb{N}}$ of the singular values of A belongs to $\ell^p(\mathbb{N})$. S^p is equipped with the norm

$$\|A\|_{S^p} = \left(\sum_{k=1}^{+\infty} |s_k|^p\right)^{\frac{1}{p}}.$$

• We denote by S^∞ the $\mathbb{C}^*$-algebra $\mathscr{B}(\mathscr{H})$ of all bounded linear operators on $\mathscr{H}$, S^∞ is equipped with the norm $\|A\|_{S^\infty} = \|A\|, \quad A \in S^\infty$.

• The trace of an operator A in S^1 is defined by

$$Tr(A) = \sum_{k=1}^{+\infty} \langle Av_k | v_k\rangle_{\mathscr{H}}, \tag{3.6}$$

where $(v_k)_k$ is an orthonormal basis of $\mathscr{H}$. $Tr(A)$ is independent of the choice of the orthonormal basis. Moreover, if A is nonnegative, then

$$Tr(A) = \|A\|_{S^1}.$$

S^1 is also known to be the trace class (see[4, 5]).

To prove that the localization operator $L_{\psi_1,\psi_2}(\sigma)$ belongs to the class S^1, we need the following Bessel's inequality.

Theorem 3.2.1 *Let H be a Hilbert space with the inner product $\langle . \mid .\rangle_H$ and let $(e_k)_{k\in\mathbb{N}}$ be an orthonormal family of H. Then, for all x in H, we have the followig Bessel's inequality*

$$\sum_k \left|\langle x \mid e_k\rangle_H\right|^2 \leq \|x\|^2.$$

Theorem 3.2.2 *Let $\sigma \in L^1(d\mu_\alpha)$, then the bounded linear operator $L_{\psi_1,\psi_2}(\sigma)$ belongs to the class*

S^1 and

$$\|L_{\psi_1,\psi_2}(\sigma)\|_{S^1} \leq \frac{1}{\sqrt{C_{\psi_1}C_{\psi_2}}}\|\sigma\|_{1,\mu_\alpha}. \tag{3.7}$$

Proof. Let $\sigma \in L^1(d\mu_\alpha)$ and let $(\upsilon_k)_k$, $(\omega_k)_k$ be two orthonormal basis in $L^2(d\nu_\alpha)$. Then, for every $k \in \mathbb{N}$, we get

$$\begin{aligned}
|\langle L_{\psi_1,\psi_2}(\sigma)\upsilon_k|\omega_k\rangle_{\nu_\alpha}| &\leq \frac{1}{\sqrt{C_{\psi_1}C_{\psi_2}}}\int_0^{+\infty}\int_0^{+\infty}|\sigma(a,r)|\ |\langle \upsilon_k|\psi^\alpha_{1,a,r}\rangle_{\nu_\alpha}|\ |\langle \omega_k|\psi^\alpha_{2,a,r}\rangle_{\nu_\alpha}|d\mu_\alpha(a,r)\\
&\leq \frac{1}{\sqrt{C_{\psi_1}C_{\psi_2}}}\int_0^{+\infty}\int_0^{+\infty}|\sigma(a,r)|\frac{1}{2}\Big(|\langle \upsilon_k|\psi^\alpha_{1,a,r}\rangle_{\nu_\alpha}|^2+|\langle \omega_k|\psi^\alpha_{2,a,r}\rangle_{\nu_\alpha}|^2\Big)d\mu_\alpha(a,r).
\end{aligned}$$

Thus, by applying Bessel's inequality, we get

$$\begin{aligned}
\sum_{k=1}^{+\infty}|\langle L_{\psi_1,\psi_2}(\sigma)\upsilon_k|\omega_k\rangle_{\nu_\alpha}| &\leq \frac{1}{\sqrt{C_{\psi_1}C_{\psi_2}}}\int_0^{+\infty}\int_0^{+\infty}|\sigma(a,r)|\frac{1}{2}\Big(\sum_{k=1}^{+\infty}|\langle \upsilon_k|\psi^\alpha_{1,a,r}\rangle_{\nu_\alpha}|^2+\sum_{k=1}^{+\infty}|\langle \omega_k|\psi^\alpha_{2,a,r}\rangle_{\nu_\alpha}|^2\Big)d\mu_\alpha(a,r)\\
&\leq \frac{1}{\sqrt{C_{\psi_1}C_{\psi_2}}}\int_0^{+\infty}\int_0^{+\infty}|\sigma(a,r)|\frac{1}{2}\Big(\|\psi^\alpha_{1,a,r}\|^2_{2,\nu_\alpha}+\|\psi^\alpha_{2,a,r}\|^2_{2,\nu_\alpha}\Big)d\mu_\alpha(a,r)\\
&\leq \frac{1}{\sqrt{C_{\psi_1}C_{\psi_2}}}\|\sigma\|_{1,\mu_\alpha} < +\infty.
\end{aligned}$$

From [34], the operator $L_{\psi_1,\psi_2}(\sigma)$ belongs to S^1 and

$$\|L_{\psi_1,\psi_2}(\sigma)\|_{S^1} \leq \frac{1}{\sqrt{C_{\psi_1}C_{\psi_2}}}\|\sigma\|_{1,\mu_\alpha}.$$

■

To prove that the localization operator $L_{\psi_1,\psi_2}(\sigma)$ belongs to the class S^p, we need the following Riesz-Thorin's interpolation theorem.

Theorem 3.2.3 *(Riesz-Thorin) Let* $p_0, p_1, q_0, q_1 \in [1, +\infty]$ *and let*

$$A : L^{p_0}(\mathbb{R}^d) + L^{p_1}(\mathbb{R}^d) \longrightarrow L^{q_0}(\mathbb{R}^d) + L^{q_1}(\mathbb{R}^d),$$

be a linear operator. If $A : L^{p_0}(\mathbb{R}^d) \longrightarrow L^{q_0}(\mathbb{R}^d)$ *is bounded of norm* N_0 *and* $A : L^{p_1}(\mathbb{R}^d) \longrightarrow L^{q_1}(\mathbb{R}^d)$ *is bounded of norm* N_1*, then for every* $\theta \in [0, 1]$ *the operator* A *is bounded from* $L^p(\mathbb{R}^d) \longrightarrow L^q(\mathbb{R}^d)$ *of norm* $N \leq N_0^{1-\theta} N_1^{\theta}$*, where*

$$\begin{cases} \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \\ \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}. \end{cases}$$

Theorem 3.2.4 *For every* $\sigma \in L^p(d\mu_\alpha)$*,* $p \in [1, +\infty]$*, the localization operator* $L_{\psi_1,\psi_2}(\sigma) : L^2(d\nu_\alpha) \to L^2(d\nu_\alpha)$ *is in* S^p *and*

$$||L_{\psi_1,\psi_2}(\sigma)||_{S^p} \leq \left(\frac{1}{\sqrt{C_{\psi_1} C_{\psi_2}}} \right)^{\frac{1}{p}} ||\sigma||_{p,\mu_\alpha}. \tag{3.8}$$

Proof. For every $\sigma \in L^1(d\mu_\alpha)$, $L_{\psi_1,\psi_2}(\sigma)$ belongs to S^1 and

$$||L_{\psi_1,\psi_2}(\sigma)||_{S^1} \leq \frac{1}{\sqrt{C_{\psi_1} C_{\psi_2}}} ||\sigma||_{1,\mu_\alpha}.$$

For every $\sigma \in L^\infty(d\mu_\alpha)$, $L_{\psi_1,\psi_2}(\sigma)$ belongs to S^∞ and

$$||L_{\psi_1,\psi_2}(\sigma)||_{S^\infty} \leq ||\sigma||_{\infty,\mu_\alpha}.$$

From Theorem 3.2.3, the localization operator $L_{\psi_1,\psi_2}(\sigma) \in S^p$ and we have

$$||L_{\psi_1,\psi_2}(\sigma)||_{S^p} \quad \leq \quad \left(\frac{1}{\sqrt{C_{\psi_1} C_{\psi_2}}} \right)^{\frac{1}{p}} ||\sigma||_{p,\mu_\alpha}.$$

■

Theorem 3.2.5 *Let σ in $L^1(d\mu_\alpha)$, then*

$$Tr(L_{\psi_1,\psi_2}(\sigma)) = \frac{1}{\sqrt{C_{\psi_1}C_{\psi_2}}}\int_0^{+\infty}\int_0^{+\infty}\sigma(a,r)\langle\psi^\alpha_{2,a,r}|\psi^\alpha_{1,a,r}\rangle_{\nu_\alpha}d\mu_\alpha(a,r).$$

Proof. Let $(v_k)_k$ be an orthonormal basis in $L^2(d\nu_\alpha)$. For every $k\in\mathbb{N}$, we get

$$\begin{aligned}
Tr(L_{\psi_1,\psi_2}(\sigma)(v_k)) &= \sum_{k=1}^{+\infty}\langle L_{\psi_1,\psi_2}(\sigma)v_k|v_k\rangle_{\nu_\alpha}\\
&= \frac{1}{\sqrt{C_{\psi_1}C_{\psi_2}}}\sum_{k=1}^{+\infty}\int_0^{+\infty}\int_0^{+\infty}\sigma(a,r)S^\alpha_{\psi_1}(v_k)(a,r)\overline{S^\alpha_{\psi_2}(v_k)(a,r)}d\mu_\alpha(a,r)\\
&= \frac{1}{\sqrt{C_{\psi_1}C_{\psi_2}}}\sum_{k=1}^{+\infty}\int_0^{+\infty}\int_0^{+\infty}\sigma(a,r)\langle v_k|\psi^\alpha_{1,a,r}\rangle_{\nu_\alpha}\langle\psi^\alpha_{2,a,r}|v_k\rangle_{\nu_\alpha}d\mu_\alpha(a,r).
\end{aligned}$$

Since

$$\frac{1}{\sqrt{C_{\psi_1}C_{\psi_2}}}\sum_{k=1}^{+\infty}\int_0^{+\infty}\int_0^{+\infty}|\sigma(a,r)|\,|\langle v_k|\psi^\alpha_{1,a,r}\rangle_{\nu_\alpha}|\,|\langle\psi^\alpha_{2,a,r}|v_k\rangle_{\nu_\alpha}|d\mu_\alpha(a,r)\leq\frac{1}{\sqrt{C_{\psi_1}C_{\psi_2}}}\|\sigma\|_{1,\mu_\alpha}<+\infty.$$

We conclude that

$$Tr(L_{\psi_1,\psi_2}(\sigma)) = \frac{1}{\sqrt{C_{\psi_1}C_{\psi_2}}}\int_0^{+\infty}\int_0^{+\infty}\sigma(a,r)\langle\psi^\alpha_{2,a,r}|\psi^\alpha_{1,a,r}\rangle_{\nu_\alpha}d\mu_\alpha(a,r).$$

The proof is complete. ■

BIBLIOGRAPHY

[1] S. Assous, A. Humeau, M. Tartas, P. Abraham and J. L'Huillier. *S-transform applied to laser doppler flowmetry reactive hyperemia signals*. IEEE Trans. Biomed. Eng. 53(2006), 1032-1037.

[2] C. Baccar, N. B. Hamadi and H. Herch. *Time-frequency analysis of localization operators associated to the windowed Hankel transform*. Integral Transforms and Special Functions. 27(3)(2016), 245-258.

[3] F. A. Berezin. *Wick and anti-Wick operator symbols*. Mathematics of the USSR-Sbornik. 15(4) (1971), 577-606.

[4] V. Catană. *Two-Wavelet Localization Operators on Homogeneous Spaces and Their Traces*. Integral Equations Operator Theory, 62 (2008), 351-363.

[5] V. Catană. *Schatten-von Neumann norm inequalities for two-wavelet localization operators associated with β-Stockwell transforms*. Applicable Analysis, 91(3) (2012), 503-515.

[6] P. Chu. *A high resolution global sea surface temperature analysis and climatology*. Tropical Oceans Global Atmosphere Notes. 17(1994), 5-7.

[7] A. Córdero and C. Fefferman. *Wave packets and Fourier integral operators*. Communications in Partial Differential Equations. 3 (11) (1978), 979-1005.

[8] E. Córdero and K. Gröcheni. *Time-frequency analysis of localization operators*. Journal of Functional Analysis. 205(1) (2003), 107-131.

[9] P. Dash, B. Panigrahi and G. Panda. *Power quality analysis using S-transform*. IEEE Trans. Power Deliv. 18(2) (1983), 406?411.

[10] I. Daubechies. *Time-frequency localization operators: a geometric phase space approach*. IEEE Trans. Inf. Theory. 34(4) (1988), 605-612.

[11] I. Daubechies. *The wavelet transform, time-frequency localization and signal analysis*. IEEE Trans Inf Theory. 36(5) (1990), 961-1005.

[12] I. Daubechies and T. Paul. *Time-frequency localization operators-a geometric phase space approach: II. The use of dilations*. Inverse Probl. 4(3) (1988), 661-680.

[13] F. De Mari and K. Nowak. *Localization type Berezin-Toeplitz*. The Journal of Geometric Analysis. 12(1) (2002), 9-27.

[14] F. De Mari, H. G. Feichtinger and K. Nowak. *Uniform eigenvalue estimates for time-frequency localization operators*. Journal of the London Mathematical Society. 65(3) (2002), 720-732.

[15] M. Eramian, R. Schincariol, L. Mansinha and R. Stockwell. *Generation of aquifer heterogeneity maps using two-dimensional spectral texture segmentation techniques*. Math. Geol. 31(3) (1999), 327?348.

[16] D. Gabor. *Theory of communication. Part 1: The analysis of information*. Journal of the In- stitution of Electrical Engineers-Part III: Radio and Communication Engineering. 93(26) (1946), 429-441.

[17] S. Ghobber. *Dunkl-Gabor transform and time-frequency concentration*. Czechoslovak. Math. J. 65 (2015), 255-270.

[18] S. Ghobber and S. Omri. *Time-frequency concentration of the windowed Hankel transform*. Integral Transforms Spec Funct. 25 (2014), 481-496.

[19] K. Gröchenig. *Foundations of time-frequency analysis*. Springer Science and Business Media, 2001.

[20] A. Grossmann and J. Morlet. *Decomposition of hardy functions into square integrable wavelets of constant shape*. SIAM journal on mathematical analysis. 15(4) (1984), 723-736. Iberoam. 29 (2013), 115-134.

[21] D. T. Haimo. *Integral equations associated with Hankel convolutions*. Trans. Amer. Math. Soc. 116 (1965), 330-375.

[22] N. B. Hamadi and S. Omri. *Uncertainty principles for the continuous wavelet transform in the Hankel setting*. Appl. Anal. 97 (2018), 513-527.

[23] C. S. Herz. *On the mean inversion of Fourier and Hankel transforms*. Proc. Natl. Acad. Sci. USA. 40 (1954), 996-999.

[24] N. N. Lebedev. *Special functions and their applications*. Dover Publications. New-York, 1972.

[25] Y. Liu and M. Wong. *Inversion formulas for two dimentional Stockwell transforms*. Pseudo differential operators, partiel differential equations and time-frequency analysis. 2007, 323-330.

[26] J. Osler and D. Chapman. *Seismo-acoustic determination of the shear-wave speed of surficial clay and silt sediments on the Scotian shelf*. Canadian Acoustics. 24 (1996), 11-22.

[27] A. Prasad and M. K. Singh. *Pseudo-differential operators associated with the Jacobi differential operator and Fourier-cosine wavelet transform*. Asian-European Journal of Mathematics. 8(1) (2015), 1550010.

[28] A. L. Schwartz. *An inversion theorem for Hankel transforms*. Proc. Amer. Math. Soc. 22 (1969), 713-717.

[29] F. A. Shah and A. Y. Tantary. *Non-isotropic angular Stockwell transform and the associated uncertainty principles*. Applicable Analysis. (2019), 1-25.

[30] K. Stempak. *La théorie de Littlewood-Paley pour la transformation de Fourier-Bessel*. C. R. Acad. Sci. Paris. Sér. I Math. 303 (1986), 15-18.

[31] R. G. Stockwell, L. Mansinha and R. P. Lowe. *Localization of the Complex Spectrum: The S-Transform*. IEEE Transactions on Signal Processing. 44(4) (1996).

[32] G. N. Watson. *A treatise on the theory of Bessel functions*. Cambridge Univ. Press, 1995.

[33] M. W. Wong. *Localization operators*. Seoul National University, Research Institute of Mathematics (1999).

[34] M. W. Wong. *Wavelet transforms and localization operators*. Birkhäuser, 136 (2002).