الجمهوريـة الجزائـريـة الديمقراطيـة الشعبيـة
People's Democratic Republic of Algeria
وزارة التعليـم العالـي والبحـث العلمـي
Ministry of Higher Education and Scientific Research

Nº Réf :……………

Abdelhafid Boussouf
University Center – Mila

Institute of Mathematics and Computer Science

Department of Mathematics

Submitted for the Degree of Master
In: Mathematics
Specialty: Applied Mathematics

Stability analysis and chaos control of a discrete Rosenzeig-MacArthur prey predator model

Presented by: Teqwa Bouguettoucha

Nedjla Boukhalfa

Defended on …/06/2024 in front of the Jury

Rabeh Bououden	MCA	Abdelhafid Boussouf U. C. of Mila,	Chairman
Mohammed-Salah Abdelouahab	Prof	Abdelhafid Boussouf U. C. of Mila,	Supervisor
Widad Laouira	MCA	Abdelhafid Boussouf U. C. of Mila,	Examiner

Academic Year: 2023/2024

ACKNOWLEDGMENTS

Praise be to ALLAH the All-Hearing, the All-Knowing, the Possessor of Glory and Great Favor. May blessings and peace be upon the Chosen One, the Noble Guide, and upon all his family and companions. We thank ALLAH, the Most High, the Almighty, who has illuminated for us the path of science and knowledge and helped us complete this work.

A sincere greeting and special thanks to **Professor Mohammed Salah Abdelouahab** for his determination, wisdom, patience, constant care, continuous work with us, and his valuable advice and guidance.

We also extend our thanks and gratitude to the discussion committee, **Dr. Bououden Rabeh** and **Dr. Laouira Widad**, for reading, reviewing, and discussing this dissertation.

Additionally, we thank the management of the Mathematics Institute, including professors and employees. We thank all the people who helped us, both near and far, in this work.

الشكر أولا لله عز و جل المعين العظيم على توفيقه لي في كل خطوة في هذا العمل، فلك الحمد كما ينبغي لجلال وجهك و لعظيم سلطانك.

قال **رسول الله** (صلى الله عليه وسلم): "مَنْ صَنَعَ إِلَيْكُمْ مَعْرُوفًا فَكَافِئُوهُ، فَإِنْ لَمْ تَجِدُوا مَا تُكَافِئُونَهُ فَادْعُوا لَهُ حَتَّى تَرَوْا أَنَّكُمْ قَدْ كَافَأْتُمُوهُ". (رواه أبو داوود) أخيرًا، أتقدم بجزيل شكري إلى كل من مدوا لي يد العون والمساعدة في إخراج هذه الدراسة على أكمل وجه.

تتسابق عبارات الحب ويتزاحم الكلام الرقيق لقول أسمى عبارات الشكر و العرفان لمن كان له أثر في حياتي كل بإسمه و مقامه.

ثانيا أتقدم بالشكر لعائلتي، أثمن جهودكم الكبيرة في جميع أوقات الدراسة التي مرت علي.

أبي ، بكل الحب أتقدم إليك بالشكر على وجودك بقربي، فالحياة كانت مظلمة لولا وجودك، ومهما شكرتك لن أوفيك حقك.

أمي، لا يسعني الشكر فأنت من قدمت لي الحياة ،لا أعرف كيف أشكرك يا أمي، فكلمة شكرًا هي لا شيء مقارنة بسخائك وعطائك، إني أود حقًا أن أرد لكِ الجميل، وليتني أعرف كيف السبيل.

أختي، لك جزيل الشكر يا اختي الغالية على ما قدمتيه لي ومشاعرك النبيلة الغير مستغربة و هذا من نبلك و أصلك.

و أيضا عائلتي الكبيرة، إن قلت شكراً فشكري لن يوفيكم، حقاً سعيتم فكان السعي مشكوراً، إن جف حبري عن التعبير يكتبكم قلب به صفاء الحب تعبيراً.

و أخيرا و ليس آخرا **نجلاء** أرسل الطيور برسائل شكري وعرفاني لك، تتطير بالفضاء اللامع بين ملايين النجوم، لتوصل لك كلمات شكري التي تفوق النجوم عدداً. شكرا على عملك الدؤوب طيلة إنجاز هذا العمل و تشجيعك المتواصل عند التعثر، و أعتذر عن أي تقصير بدر مني.

أشكر ريان جامع على عطاءها و مساعدتها بكل ما تستطيع تقديمه و لم تبخل.

و أخص بالذكر أيضا **صديقاتي**:مريم، منار، وفاء، زهرة، رميساء، غادة، شيماء و نجاة. مهما أوتيت من بلاغة وأفنيت بحر النطق في النظم لما كنت بعد القول إلّا مقصراً، ومعترفاً بالعجز عن واجب الشكر، شكرا على مساندتكم و دعمكم الدائم.

تقوى

الحمد لله الذي أعانني على إنجاز هذا العمل

أما بعد أهدي ثمرة جهدي المتواضع إلى من وهبوني الحياة

و الأمل و النشأة على شغف الإطلاع و المعرفة، **أبي العزيز** و **أمي الغالية** ...

إلى من وهبني الله نعمة وجودهما أخي **صهيب** و أختي **سندس**...

إلى حبيبة قلبي **أبرار** رحمة الله عليها ...

إلى من كنت محظوظة بصداقتها **رونق**...

إلى من كاتفتني و نحن نشق الطريق معا نحو النجاح في مسيرتنا العلمية ...

إلى رفيقة دربي **تقوى** ...

إلى **عائلتي** الكبيرة و كل **صديقاتي** و **أساتذتي**...

نجلاء

ABSTRACT

In this work, we explored some fundamental concepts for studying a discrete dynamical system (equilibrium points, bifurcations, chaos, chaotic attractor, Lyapunov exponents, and some methods for controlling chaos).

As an application, we studied a system consisting of two recursive equations (difference equations), each representing a population known as the Rosenzweig-MacArthur system. We explored the stability, bifurcation, and chaos of the system. We also examined the existence of bifurcations at equilibrium solutions (Flip and Neimark) when parameters traverse certain curves. Notably, the fold bifurcation does not exist in this system. We studied these bifurcations using the center manifold theorem and bifurcation theory. Chaos was analyzed through two methods: Feedback and OGY. The theoretical results were confirmed numerically.

Keywords: Rosenzweig-MacArthur system, stability, bifurcation (Flip and Neimark-Sacker), center manifold, Lyapunov exponent, control (Feedback and OGY).

RÉSUMÉ

Dans ce travail, nous avons exploré certains concepts fondamentaux pour l'étude d'un système dynamique discret (points d'équilibre, bifurcations, chaos, attracteur chaotique, exposants de Lyapunov, et quelques méthodes pour contrôler le chaos).
À titre d'application, nous avons étudié un système composé de deux équations récursives (équations aux différences), chacune représentant une population connue sous le nom de système de Rosenzweig-MacArthur. Nous y avons exploré la stabilité, les bifurcations et le chaos du système. Nous avons également étudié l'existence de bifurcations aux solutions d'équilibre (Flip, Neimark) lorsque les paramètres traversent certaines courbes, et nous avons constaté qu'il n'existe pas de bifurcation en pli. Nous avons étudié ces bifurcations à l'aide du théorème de la variété centrale et de la théorie des bifurcations. Nous avons aussi étudié le chaos en utilisant deux méthodes (Feedback et OGY). Les résultats théoriques ont été confirmés numériquement.
Mots-clés: système de Rosenzweig-MacArthur, stabilité, bifurcation (Flip et Neimark-Sacker), variété centrale, exposant de Lyapunov, contrôle (Feedback et OGY).

ملخص

في هذا العمل، إستعرضنا بعض المفاهيم الأساسية لدراسة نظام ديناميكي متقطع (نقاط التوازن، التشعبات، الفوضى، الجاذب الفوضوي، مؤشرات ليابونوف، وبعض طرق التحكم في الفوضى). كجزء من التطبيق، درسنا نظاما يتكون من معادلتين تراجعيتين (معادلتين فروق)، كل منهما يمثل مجموعة تعرف بنظام روزنزاويغ-ماكارثر. حيث قمنا بدراسة الاستقرار، التشعب والفوضى في النظام. كما درسنا وجود التشعبات عند حلول التوازن (فليب ، نيمارك) عندما تمر المعلمات عبر منحنيات معينة، ولا يوجد تشعب الطي. ودرسنا هذه التشعبات باستخدام نظرية متعددات الحدود المركزية ونظرية التشعب. كما درسنا الفوضى باستخدام طريقتين (التغذية الرجعية و أوجي) تم تأكيد النتائج النظرية عدديا.

<u>الكلمات المفتاحية:</u> نظام روزنزوايغ-ماك آرثر،الاستقرار،التشعب(فليب و نيمارك-ساكر)،المتعدد المركزي،أسس ليابونوف،التحكم(التغذية الرجعية و أوجي).

CONTENTS

LIST OF FIGURES

LIST OF TABLES

INTRODUCTION

Dynamical system is the mathematical formalization of the general scientific concept of a deterministic process. They describe the future and past states of many physical, chemical, biological, ecological, economical [23].

Differential equations were invented by Isaac Newton in the late 17th century, and their various properties were discovered over time. Subsequent generations attempted to develop them further, but faced some difficulties [41].

In the 19th century; Henri Poincare introduced a new perspective focusing on the qualitative aspects of the question.One of the questions he posed was whether the solar system is stable forever, or if some planets will eventually become unstable, highlighting the sensitivity to initial conditions [41].

In 1975; Li and York published a paper titled "Period three implies chaos," demonstrating the existence of periodic orbits of period 3 for a dynamical system defined by a continuous function $F : I \mapsto I$. This implies the existence of countless non-periodic and unstable orbits [30, 25].

The term "chaos" appeared for the first time in this research to indicate the complexity studied by Li and York [30]. And Devaney has defined fundamental definitions for chaos in "The field of dynamical systems and especially the study of chaotic systems has been hailed as one of the important breakthroughs in science in this centry " [24].

A chaotic system is a simple and complex in it bihavior system sensitive to initial conditions, exhibiting properties of recurrence and high complexity. Small disturbances can lead to non-repetition or biased imbalance, making long-term predictability impossible [24]. Since then, chaos theory has become important, especially in control systems, where Li and York's contributions to chaos control have been significant. Many techniques have been developed to control chaos, most of which are variations of the OGY (Ott-Grebogi-Yorke) and PRC (Pyragas) approaches. The notion of population is based on the existence of an intermediate structure between the individual and the species. The definitions of population proposed by geneticists and population biologists have followed quite parallel paths since the beginning of the century [6].

The first definitions of population date back to Wright (1931).They are defined it a group of individuals sharing the same gene pool.Population geneticists have also introduced the concept of local populations, neighborhoods, or communities, which are entities relatively isolated from other entities in terms of reproduction and the dispersion of individuals [6]. Local populations are interpreted as demographic units in which the main population processes take place, such as reproduction, competition, and predation [6].

Population dynamics lies at the heart of the interface between dynamical systems and biology [3].

Population dynamics is a crucial research area for understanding the evolution and interaction of species within an ecosystem. Since the work of Thomas Robert Malthus, several models have been developed to describe variations in the size and structure of populations. Among these models, the most well-known are the Malthusian model, the Verhulst model, and the Lotka-Volterra model.

Our dissertation is composed of an introduction and four chapters organized as follows:

- In the first chapter: we present some essential concepts on discrete dynamical systems.
- In the second chapter: we present the important notions of stability and bifurcation with their types.
- In the third chapter: we focus on the notions of chaos.
- In the fourth chapter: we discuss the two different types of chaos control in dynamical

systems, which are OGY and Feedback.

- Finally, we conclude the work with a chapter dedicated to applying the basic notions (fixed point, stability, bifurcation, Lyapunov exponent, OGY, Feedback, Center-Manifold).

CHAPTER 1

GENERALITIES ABOUT DISCRETE DYNAMICAL SYSTEM

Dynamics is a process that evolves over time, which can be either deterministic or stochastic.Dynamical systems are generally described by differential or difference equations. They are used to model a wide range of natural phenomena in different fields such as: (physics, chemical, electromechanical, biological, economic, etc) [26].
In this chapter, we discuss the definitions and types of dynamic systems, especially the discrete dynamic system and its theories (flow, phase portrait, fixed point, periodic point).

1.1 Discrete dynamical systems

Definition 1.1.1 *[23]*
A dynamical system is defined as a triplet $(T, \mathbb{X}, \phi)$*, where* T *is a time* $x \in \mathbb{X}$*,* $\mathbb{X}$ *a state space,* $\phi : T \times \mathbb{X} \to \mathbb{X}$ *a family of evolution operators parametrized by* $t \in T$*;* $x \in \mathbb{X}$ *and satisfying the properties* $\phi(0, x) = Id$ *and* $\phi(\phi(t, x), s) = \phi(t + s, x)$ $\forall t, s > 0$ *we distinguish two types of dynamic systems.*

- *Continous-time systems if* $T = \mathbb{R}^+$ *or* $\mathbb{R}$ *.*

- *Discrete-time systems if* $T = \mathbb{N}$ *or* $\mathbb{Z}$ *.*

1.1.1 Types of dynamical systems

a) **Continuous-time dynamical systems:**

Definition 1.1.2 *[26]*
In continuous time a dynamical system is represented by differential equations :

$$\dot{x} = f(x, t, p), \tag{1.1}$$

where $x \in \mathbb{R}^n$ *,* $p \in \mathbb{R}^m$*,* $t \in \mathbb{R}$*.*
$f : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$ *: an application representing the dynamics of the system .*
If we associate an initial state with this dynamic:

$$x_0 = x(t_0).$$

For each chosen pair (t_0, x_0)*, we can identify a unique solution :*

$$\phi(., t_0, x_0) : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n.$$

Such as :

$$\begin{aligned} \phi_f(t_0, t_0, x_0) &= x_0; \\ \dot{\phi}_f(t, t_0, x_0) &= f(\phi_f(t, t_0, x_0), t). \end{aligned}$$

This solution provides the successive states occupied by the system at each time t. When the function x is continuous within a certain interval $\mathbb{I} \subset \mathbb{R}$ of the variable x there is existence and uniqueness of the solution for any initial condition $x \in \mathbb{I}$ more precisely we have the following theorem.

Theorem 1.1.1 *[12]*

We consider a differential equation:

$$\frac{dx}{dt} = f(t, x, \lambda),$$

and we suppose that the second member of the equation is given by a function f which is Lipschitzian with ratio λ with respect to x uniformly with respect to a parameter λ and with respect to $t \in [-a, +a]$. There is only one solution maximum $\phi = \phi(t, t_0, x_0)$ such that $\phi(t_0, t_0, x_0) = x_0$.

Example 1.1.1 *The Lorenz map represented by:*

$$\begin{cases} \dot{x} &= \sigma(y - x), \\ \dot{y} &= -xz + rx - y, \\ \dot{z} &= xy - bz. \end{cases}$$

b) **Discrete-time dynamical systems:**

Definition 1.1.3 *[30]*
In discrete time a dynamical system is represented by an application (iterative function) in the form :

$$x_{k+1} = f(x_k, c, k), \tag{1.2}$$

$x_k \in \mathbb{R}^n$, $c \in \mathbb{R}^m$ and $k = 1, 2, 3 \cdots$
Where $f : \mathbb{R}^n \times \mathbb{Z}^+ \to \mathbb{R}^n$ indicates the dynamics of the system in discrete time .
We can also identify (x_0, k_0) a unique solution :

$$\phi_f(., x_0, k_0) : \mathbb{R}^+ \times \mathbb{R}^n;$$

Such as:

$$\phi_f(k_0, x_0, k_0) = x_0,$$
$$\phi_f(k+1, x_0, k_0) = f(\phi_f(k, x_0, k_0), c), k.$$

Example 1.1.2 *The Henon map:*

$$\begin{cases} x_{k+1} &= y_k + 1 - ax_k^2. \\ y_{k+1} &= bx_k. \end{cases}$$

b.1) **Linear discrete systems:**

Definition 1.1.4 *[11]*
A linear discrete dynamic system is represented by equation of the form :

$$\begin{cases} x_{n+1} &= Ax_n, \\ x(0) &= x_0, \end{cases}$$

given by: $x = (x_0, x_1, x_2 \cdots, x_n) = (x_0, Ax_0, A^2x_0, \cdots, A^n x_0)$.
Where A *is square matrix of dimention* $n \times n$.

b.2) **Non linear discrete systems:**

Definition 1.1.5 *[11]*
A non linear discrete dynamic system is represented by an equation of the form:

$$\begin{cases} x_{n+1} &= f(x_n, t), \\ x(0) &= x_0 \in \mathbb{R}^n. \end{cases}$$

Where $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ *is a differentiable function.*

c) **Autonomous or non autonomous systems:**
A dynamical system is a non autonomous [26] if the variable t appears explicitly in an expression:

$$\begin{cases} x_{n+1} &= f(x_n), \\ x(0) &= x_0. \end{cases}$$

and autonomous if the variable t not appear explicitly in the expression f :

$$\begin{cases} x_{n+1} &= f(x_n, t), \\ x(0) &= x_0. \end{cases}$$

• **Phase Space:**
A dynamical system is defined by a set of state variables, each of which completely describes the system's state at any particular time. The system's dynamic behavior is

connected to how these state variables change over time. This concept is represented in phase space, where each point represents a state and the path associated with that point illustrates a trajectory [39].

•Transition from continuous time to discrete time:

Euler's method:

A simple way to obtain an equation in discrete time is to perform the Euler approximation of a continuous time equation.

Let $x(t)$ be a real variable depending on time t. A differential equation of first autonomous order is written in the following general form:

$$\frac{dx}{dt} = f(x),$$

where the function f depends on the variable x.

Let x(t) be the solution at time t and $x(t + \Delta t)$ be the solution at time $t + \Delta t$. Derivative $\frac{dx}{dt}$ can be approximated by the following relation:

$$\frac{dx}{dt} \approx \frac{x(t + \Delta t) - x(t)}{\Delta t}.$$

This Euler approximation is all the more valid as the time interval Δt is small. From the two previous equations we obtain a discrete time equation:

$$x(t + \Delta t) = x(t) + f(x)\Delta t.$$

The previous equation allows, from an initial condition x_0, to calculate the solution at consecutive time intervals, Δt , $2\Delta t$, $3\Delta t$, ..., $n\Delta t$, and so on.

By choosing $\Delta t = 1$ as the unit of time, it is possible to rewrite this equation in discrete time as follows:

$$x(t + 1) = g(x(t)),$$

where the function $g(x(t)) = x(t) + f(x(t))$ [4].

1.1.2 Flow, trajectory (orbit), phase portrait

a) ***Flow:***

Definition 1.1.6 *[12]*
The correspondence $\phi_t : x_0 \mapsto x(t)$ which associates with each initial state value x_0 the value of the maximal solution x(t) at time t, that corresponds to this initial data is called the flow at time t of the vector field. The flow of the vector field is the map which associates with (t, x) the maximal solution x(t) at time t that corresponds to the initial data x:

$$(t, x) \mapsto \phi(t, x) = \phi_t = x(t).$$

The flow is said to be complete when this correspondence is defined for any value of $t \in]-\infty, +\infty[$.

Properties
$\phi_t(x_0)$ has the following properties :

i) $\phi_t(x_0)$ **is of class** c^r.

ii) $\phi_0(x_0) = x_0$.

iii) $\phi_{t+s}(x_0) = \phi_t(\phi_s(x_0)) \qquad \forall t, s > 0.$

For the demonstration of these properties in [41]

Figure 1.1: Flow representation.

Figure 1.2: Illustration of stationarity.

b) ***Trajectory:***

Definition 1.1.7 *[5]*
An orbit of the system (1.2) starting at x_0 is an ordered subset of the state space X, $x \in X$. the following:

$$O(x_0) = \{x(0) = x_0, x(1) = f(x(0)), ..., x(n+1) = f(x(n))\}.$$

c) ***Phase portrait:***

Phase portrait are frequently used in dynamical system to represent the dynamics of a map graphically. A phase portrait consists of a diagram exhibiting possible changing positions of a map function and the arrows indicate the change of positions under iterations of the map [26].

Example 1.1.3 *Consider a simple one dimensional map:*

$$f : [0, 2\pi] \to [0, 2\pi].$$

Defined by $f(\theta) = \theta + 0.3\sin(3\theta)$. The phase portrait of this map is displayer in figure (1.3). The figure shows that the six points satisfy the relationship $f(\theta) = \theta$. The arrows indicate that the flow moves toward the three points $\frac{\pi}{3}$,π,$\frac{5\pi}{3}$ and the flow moves away from the other three points 0,$\frac{2\pi}{3}$,$\frac{4\pi}{3}$, the points have special interest.

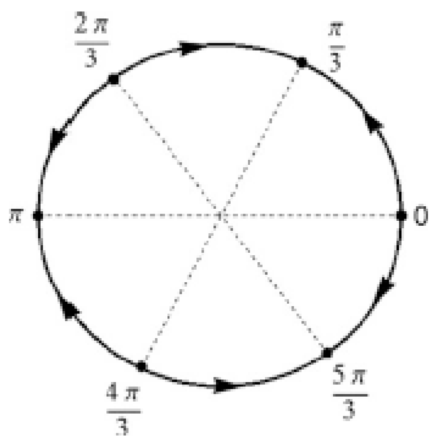


Figure 1.3: Phase portrait of $f(\theta) = \theta + 0.3\sin(3\theta)$.

1.1.3 Fixed points, periodic points

Consider a discrete-time equation of the following general form:

$$x_{n+1} = f(x_n).$$

1) Fixed point

Definition 1.1.8 *[11]*
The vector $x^* \in \mathbb{R}^n$ is called an equilibrium point of the system if $x^* = f(x^*)$.

Example 1.1.4 *: Consider the following discrete system:*

$$\begin{cases} x_1(k+1) = \alpha x_1(k) + x_2^2(k), \\ x_2(k+1) = x_1(k) + \beta x_2(k). \end{cases}$$

An equilibrium point of this system is a vector $x^* = (x_1, x_2)$ that solves the system:

$$\begin{cases} x_1 = \alpha x_1 + x_2^2, \\ x_2 = x_1 + \beta x_2. \end{cases}$$

Which gives two equilibrium points: $x^* = (0,0)$ and $x^* = ((1-\alpha)(1-\beta)^2, (1-\alpha)(1-\beta))$.

2) Periodic Point:

x is a periodic point of the system (1.2) if there exists $k \geq 1$, such that $f^k(x) = x$, the period of a periodic point x is the smallest integer $k \geq 1$ such that $f^k(x) = x$ [33].

CHAPTER 2

STABILITY AND BIFURCATION

2.1 Stability

2.1.1 Stability of fixed points

Definition 2.1.1 *[9]*
Let $f : I \to I$ be a map and x^* be a fixed point of f, where I is an interval in the set of real numbers $\mathbb{R}$. Then

(1) x^* is said to be stable if for every $\epsilon > 0$, there exists $\delta > 0$ and $n > 0$ such that $||x_0 - x^*|| \leq \delta$ implies $||x_k - x^*|| \leq \epsilon$ for all $k > n$. Otherwise, it is said to be unstable.

(2) x^* is said to be asymptotically stable (a.s) if it is stable and $\lim_{k\to\infty} ||x_k - x^*|| = 0$.

(3) x^* is said to be globally asymptotically stable (g.a.s) if it is asymptotically stable (a.s) and for every x_0^*, $\lim_{k\to\infty} ||x_k - x^*|| = 0$.

Henceforth, unless otherwise stated, "stable" (asymptotically stable).

- The exisstence of a fixed point can be assured by:

Theorem 2.1.1 *(Brouwer's Fixed Point Theorem) [13]*
Suppose $F : B \to \mathbb{R}^n$ is continuous, B is a compact convex subset of $\mathbb{R}^n$, and $F(B) \subseteq B$. Then there exists $x^* \in B$ such that $F(x^*) = x^*$.

A proof of this theorem can be found in Heuser [1994].

- The uniqueness of a fixed point can be assured by:

Theorem 2.1.2 *(Contraction Mapping Theorem) [13]*
Suppose $F : B \to B$ where B is a closed subset of a Banach space X and F is a contraction on B, i.e. there exists $\mu < 1$ such that

$$\|F(x) - F(y)\| \leqslant \mu\|x - y\| \quad forall \quad x, y \in B.$$

Then there exists a unique fixed point x^* and each trajectory starting in B converges exponentially fast to it.

Linear dynamical systems

Let's consider the autonomous systems, which are time-invariant and take the form:

$$x_{n+1} = Ax_n. \tag{2.1}$$

The next theorem summarizes the key stability results for such linear autonomous systems.
We make the assumption that the system (2.1) is at its equilibrium state, meaning that: $x^* = Ax^*$. Therefore:

$$x^* = (0, 0, \cdots, 0) \quad if \quad \det(I - A) \neq 0.$$

So there exists a unique fixed point ($x^* = 0$) In the next theorem we summarize the main stability results for the linear autonomous systems.

Theorem 2.1.3 *(m-dimensional linear systems) [9]*
The following statements hold for equation (2.1) :

(i) The zero solution of (2.1) is stable if and only if $\rho(A) \leq 1$ and the eigenvalues of unit

modulus are semisimple (An eigenvalue is said to be semisimple if the corresponding Jordan block is diagonal).

(i) *The zero solution of* (2.1) *is asymptotically stable if and only if* $\rho(A) < 1$.

Where:
$\rho(A) = max\,\{|\lambda| : \lambda$ is an eigenvalue of A $\}$ is the spectral radius of A.
Proof. The proof can be found in reference [9]. ∎

Theorem 2.1.4 *(2-dimentional linear systems) [10]*
The following statements hold for Equation (2.1) *:*

(a) *If* $\rho(A) < 1$*, then the origin is asymptotically stable.*

(b) *If* $\rho(A) > 1$*, then the origin is unstable.*

(c) *If* $\rho(A) = 1$*, then the origin is unstable if the Jordan form is of the form* $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$*, and stable otherwise.*

Proof. The proof can be found in reference [10]. ∎

- **Trace-determinant method:**
  Let's examine the two-dimensional linear dynamic system $X_{n+1} = AX_n$. The system's qualitative characteristics can be categorized according to the values of tr(A) and det(A).
  The eigenvalues of A are derived by solving the characteristic equation.

$$P(\lambda) = \lambda^2 - tr(A)\lambda + det(A).$$
$$P(\lambda) = 0.$$

Proposition 2.1.1
Consider the two-dimensional linear dynamical system $X_{n+1} = AX_n$*, and let* λ_1 *and* λ_2 *be the eigenvalues of A.*

1. *if* $(tr(A))^2 > 4det(A)$ *then* $\lambda_{1,2}$ *are real* $(\lambda_1 > \lambda_2)$*, moreover*

(a) *The origin is a Saddle* ($|\lambda_1| > 1$ *and* $|\lambda_2| < 1$) *or* ($|\lambda_1| < 1$ *and* $|\lambda_2| > 1$) *if and only if:*

$$\begin{cases} P(1) < 0 \quad \textit{and} \quad P(-1) > 0, \\ \qquad\qquad \textit{or} \\ P(1) > 0 \quad \textit{and} \quad P(-1) < 0, \end{cases}$$

i.e. , if and only if:

$$\begin{cases} -tr(A) - 1 < det(A) < tr(A) - 1. \\ \qquad\qquad \textit{or} \\ tr(A) - 1 < det(A) < -tr(A) - 1. \end{cases}$$

(b) *The origin is a sink* ($|\lambda_{1,2}| < 1$) *if and only if:*

$$P(1) > 0 \textit{ and } P(-1) > 0,$$

i.e., if and only if:

$$det(A) > tr(A) - 1 \textit{ and } det(A) > -tr(A) - 1.$$

(c) *The origin is a Source* ($|\lambda_{1,2}| > 1$) *if and only if:*

$$P(1) < 0 \textit{ and } P(-1) < 0,$$

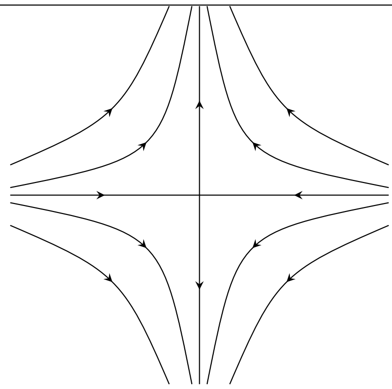
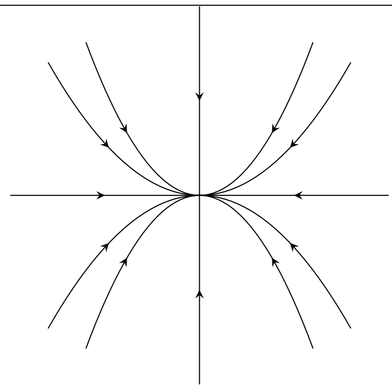
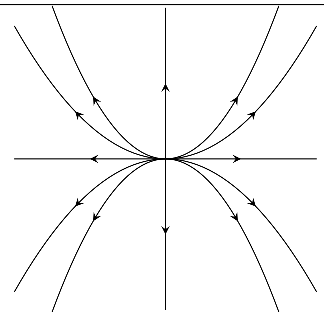
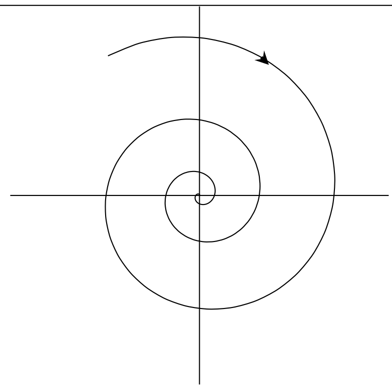
i.e., if and only if:

$$det(A) < tr(A) - 1 \textit{ and } det(A) < -tr(A) - 1.$$

2. *if* $(tr(A))^2 < 4det(A)$ *then* $\lambda_{1,2}$ *are complex, moreover*

   (a) *The origin is a spiral sink if and only if* $det(A) < 1$.

   (b) *The origin is a spiral source if and only if* $det(A) > 1$.

Type	Eigenvalue	Phase Portrait
Saddle	$\lambda_1, \lambda_2 \in \mathbb{R}, \lvert\lambda_1\rvert > 1$ and $\lvert\lambda_2\rvert < 1$ or $\lvert\lambda_1\rvert < 1$ and $\lvert\lambda_2\rvert > 1$	
Sink	$\lambda_1, \lambda_2 \in \mathbb{R}, 0 < \lvert\lambda_{1,2}\rvert < 1$	
Source	$\lambda_1, \lambda_2 \in \mathbb{R}, \lvert\lambda_{1,2}\rvert > 1$	
Stable focus	$\lambda_1, \lambda_2 \in \mathbb{C}, \lambda_{1,2} = \alpha \pm \iota\beta , \lvert\lambda_{1,2}\rvert < 1 , \beta \neq 0$	

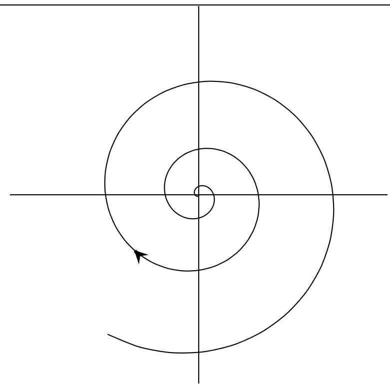
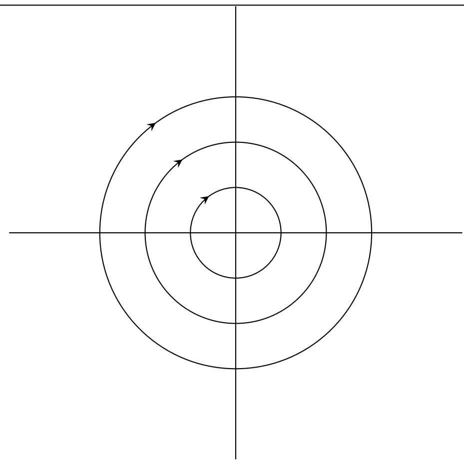
Instable focus	$\lambda_1, \lambda_2 \in \mathbb{C}$, $\lambda_{1,2} = \alpha \pm \iota\beta$, $\lvert\lambda_{1,2}\rvert > 1$, $\beta \neq 0$	
center	$\lambda_1, \lambda_2 \in \mathbb{C}$, $\lambda_{1,2} = \alpha \pm \iota\beta$, $\lvert\lambda_{1,2}\rvert = 1$, $\beta \neq 0$	

Table 2.1: The types of fixed points for 2-dimentional linear systems and their phase portrait.

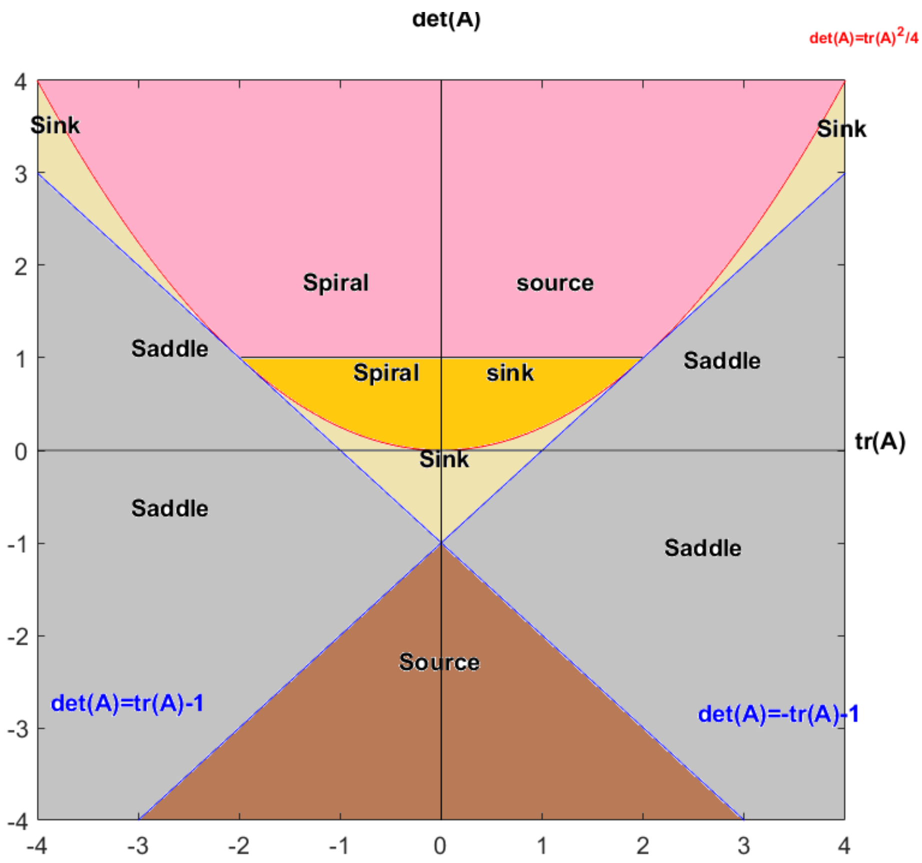


Figure 2.1: Stability by trace-determinent plane.

Non linear dynamical systems

• Indirect method (Linearization):

Definition 2.1.2

The first-order Taylor expansion of $f(x_k) = x_{k+1}$ *around* x^* *is :*

$$\begin{aligned} x_{k+1} &= f(x^*) + Df(x^*)(x_k - x^*) + O(x_k - x^*)^2, \\ &= x^* + A(x_k - x^*) + O(x_k - x^*)^2. \end{aligned}$$

Puting $X_k = x_k - x^*$.
We get :

$$X_{k+1} = AX_k.$$

Where:

$$A = Df(x^*) = \begin{pmatrix} \frac{\partial f_1(x^*)}{\partial x_1} & \frac{\partial f_1(x^*)}{\partial x_2} & \cdots & \frac{\partial f_1(x^*)}{\partial x_m} \\ \frac{\partial f_2(x^*)}{\partial x_1} & \frac{\partial f_2(x^*)}{\partial x_2} & \cdots & \frac{\partial f_2(x^*)}{\partial x_m} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n(x^*)}{\partial x_1} & \frac{\partial f_n(x^*)}{\partial x_2} & \cdots & \frac{\partial f_n(x^*)}{\partial x_m} \end{pmatrix}.$$

Since $\|x - x^*\| \to 0$ *in the vicinity of* x^**, by neglecting the second-order terms, the system* (1.2) *is linearized as :*

$$x_{k+1} = Ax_k.$$

The mapping $X \to AX$ *is called the linearization of* f *in the vicinity of* x^**. We say that the system* (1.2) *is approximated in the vicinity of the equilibrium point* x^**.*

Let $f : \mathbb{R}^n \to \mathbb{R}^n$, to determine the attractivity of an equilibrium point, we need to calculate the eigenvalues of the Jacobian matrix:

$$J(x^*) = Df(x^*) \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_m} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_m} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_m} \end{pmatrix}.$$

Theorem 2.1.5 *The fixed point* x^* *of* (1.2) *is:*

1. *Stable if all the eigenvalues of* $J(x^*) = Df(x^*)$ *are inside the unit circle (their modules are less than 1).*

2. *Unstable if any of these eigenvalues of* $J(x^*) = Df(x^*)$ *has module greater than 1 (outside the unit circle).*

Remark 2.1.6 *[18]*
Sometimes, these points are also called stationary points or equilibrium points. Let x_f *be a point of the equation* (1.2)*, and* λ_i*,* $1 \le i \le n$*, be the eigenvalues of the Jacobian matrix* $Df(x)$ *associated with* x*.*
1. x_f *is a hyperbolic point if* $|\lambda_i| \neq 1 \forall i \in [1, n]$.
2. x_f *is an elliptic point if* $|\lambda_i| = 1 \forall i \in [1, n]$.

• Direct method (Lyapunov function):

The stability of the system (1.2) can be studied using a well-chosen function, called the Lyapunov function. This is an incredible method, called direct, which is useful for non-linear systems, with the advantage of being applicable in non-standard situations [11].

Let's consider the system :

$$\begin{cases} x_{k+1} &= f(x_k), \\ x(0) &= x_0, x_0 \in \mathbb{R}^n. \end{cases} \tag{2.2}$$

If x^* is an equilibrium point of this system, then we have the following definition.

Definition 2.1.3

A function V defined on a region $\Omega \subset \mathbb{R}^n$ of the state space of the discrete system (2.2) containing x^* is a Lyapunov function if it satisfies the following conditions:

1. *V is continuous on Ω.*
2. *V has a unique minimum at the point x^* on Ω.*
3. *The function $\Delta V(x) = V(f(x)) - V(x) \leqslant 0$ on Ω.*

Remark 2.1.7 *[11]*

1) *Condition 3 of the definition is equivalent to saying that along the trajectory of the system contained in Ω, the function V is decreasing. Indeed:*

- *If at time k, the state of the system is x, then at time k+1, the state of the system is f(x). The values of Lyapunov function at these points are V(x) and V(f(x)), so the variation is $\Delta V(x) = V(f(x)) - V(x)$.*
- *If V is a Lyapunov function on Ω, then $\Delta V(x) \leqslant 0$ for all $x \in \Omega$.*

2) *The geometric interpretation allows us to conclude that if a Lyapunov function exists, the equilibrium point must be stable.*

3) *Condition 2 of the definition can be replaced by $V(x^*) = 0$ and $V(x) > 0$ on Ω. Indeed, it suffices to consider the function W defined by $W(x) = V(x) - V(x^*)$.*

Theorem 2.1.8
If there exists a Lyapunov function $V(x)$ associated with the system (2.2) in a ball $B(x^*, R_0)$, then the equilibrium point x^* is stable. If, furthermore, $\Delta V(x) < 0$ at every point (except x^*), then x^* is asymptotically stable.

Theorem 2.1.9 *(**Global asymptotic Lyapunov stability**)*
Let $x = 0$ be an equilibrium point of the autonomous system:

$$x_{k+1} = f(x_k),$$

where $f : D \to \mathbb{R}^n$ is locally Lipschitz in $D \subset \mathbb{R}^n$ and $0 \in D$.
Let $V : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous function such that:

$$V(0) = 0, \text{ and } V(x) > 0, \forall x \in D - \{0\},$$
$$\|x\| \to \infty \Rightarrow V(x) \to \infty,$$
$$V(f(x)) - V(x) \leq 0, \forall x \in D,$$

then x=0 is asymptotically globally stable.

•Jury's criterion
The stability test of the jury, is directly applied to the characteristic polynomial of the system:

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0.$$

Where $a_0, a_1, ..., a_{n-1}, a_n$ are real coefficients and $a_n > 0$ Let:

$$b_k = \begin{pmatrix} a_0 & a_{n-k} \\ a_n & a_k \end{pmatrix}, c_k = \begin{pmatrix} b_0 & n_{n-1-k} \\ b_{n-1} & b_k \end{pmatrix}, d_k = \begin{pmatrix} c_0 & c_{n-2-k} \\ c_{n-2} & c_k \end{pmatrix}, \cdots$$

The necessary and sufficient conditions for the polynomial $P(z)$ to have no roots outside or on the unit circle are as follows [11]:

$$P(1) > 0,$$

$$(-1)^n P(-1) > 0,$$

$$|a_0| < |a_n|,$$

$$|b_0| > |b_{n-1}|,$$

$$|c_0| > |c_{n-2}|,$$

$$|d_0| > |d_{n-3}|.$$

$$\vdots$$

Remark 2.1.10
Verify the three conditions that are easy to compute: $P(1) > 0$, $(-1)^n P(-1) > 0$ *and* $|a_0| < |a_n|$. *Stop if any of these conditions are not met.*

Remark 2.1.11
If $a_n < 0$*:*

- *First, construct another polynomial* $Q(z) = -P(z)$.
- *Then, treat the new polynomial* $Q(z)$.

Example 2.1.1 *Consider the characteristic polynomial:*

$$P(\lambda) = \lambda^4 + \frac{\lambda^3}{2} - \frac{\lambda^2}{4} - \frac{\lambda}{8}.$$

We have: $P(1) = \frac{9}{8}$, $P(-1) = \frac{3}{8}$, $a_4 = 0$, $|b_4| = 1$, $b_1 = \frac{1}{8}$, $|c_4| = \frac{7}{8}$, $|c_2| = \frac{17}{64}$.
Therefore, the corresponding discrete system is asymptotically stable.

2.1.2 Stability of periodic points

Theorem 2.1.12 *[26]*
Let $O(x^*) = x^*, f(x^*), ..., f^{k-1}(x^*)$ *be the orbit of the k-periodic point* x^**, where f is a continuously differentiable function at* x^**. Then the following statements hold true:*

- x^* *is asymptotically stable if*

$$|f'(x^*) f'(f(x^*)) ... f'(f^{k-1}(x^*))| < 1.$$

- *x^* is unstable if*

$$|f'(x^*)f'(f(x^*))...f'(f^{k-1}(x^*))| > 1.$$

2.1.3 Attractors

Definition 2.1.4
An attractor is a closed subset of the phase space that draws all other trajectories towards it [31, 1].

There are two types of attractors: regular attractors and strange or chaotic attractors.

a) ***Regular attractors:*** which characterize the evolution of non-chaotic systems, they can be classified into three types:

i) ***Fixed points:*** points that all trajectories of nearby points are attracted to them. Thus, they represent a constant stationary solution satisfying $f(x) = x$.

ii) ***A periodic attractor:*** are periodic orbits (orbits of trajectories that cycle around a finite point set) which are attractive. Thus, it represents a periodic solution of the system.

iii) ***Invariant Curve:*** in discrete systems, invariant curves are similar to the torus found in continuous flows. The dynamics on a closed invariant curve can be simplified to those of a map of the unit circle onto itself, known as a "circle map". The behavior on these closed curves can be complex, particularly when parameters change, causing these curves to lose their regularity and potentially become invariant sets.

b) ***Strange attractor:*** the strange attractor is another intriguing type of attractor introduced by Ruelle and Takens. Its characteristics include:

- In the phase space, the attractor occupies zero volume.
- The dimension d of the attractor is fractal (non-integer) with $0 < d < n$, where n is the dimension of the phase space.
- Sensitivity to initial conditions: two trajectories of the attractor that are initially close to each other always diverge over time.

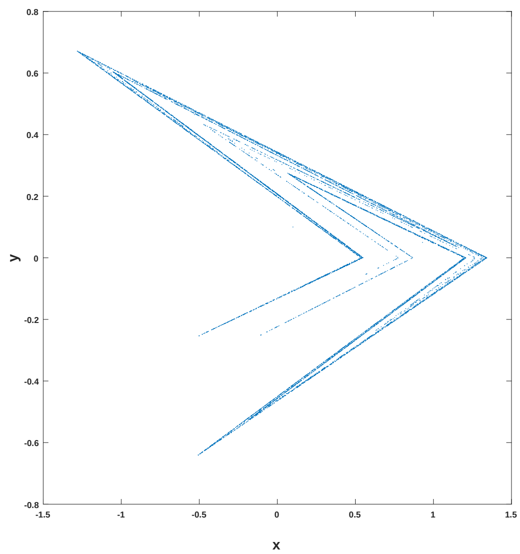


Figure 2.2: Lozi attractor obtained for a = 1.7 and b = 0.5.

2.1.4 Basin of attraction

Given an attractor A, we call the basin of attraction of A the set of all initial conditions x_0, such that $d(x_n, A) \longrightarrow 0$ as $n \longrightarrow \infty$.
Different basins of attraction are separated by basin boundaries. The geometry of these boundaries is frequently as complex as the geometry of the attractors themselves.

2.1.5 Stable, unstable, and center eigenspaces

The stable and unstable eigenspaces provide an essential reference point to the local characterization of a nonlinear dynamical system in the proximity of a steady-state equilibrium [15].

Definition 2.1.5 *(Stable, unstable, and center eigenspaces)*
Let $f(x) : \mathbb{R}^n \to \mathbb{R}^n$ be a continuously differentiable single-value function, and let $Df(x^*)$ be the Jacobian matrix of $f(x)$ evaluated at a steady-state equilibrium, x^*, i.e.

$$Df(x^*) = \begin{bmatrix} \frac{\partial f^1(x^*)}{\partial x_{1t}} & \frac{\partial f^1(x^*)}{\partial x_{2t}} & \cdots & \frac{\partial f^1(x^*)}{\partial x_{nt}} \\ \frac{\partial f^2(x^*)}{\partial x_{1t}} & \frac{\partial f^2(x^*)}{\partial x_{2t}} & \cdots & \frac{\partial f^2(x^*)}{\partial x_{nt}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f^n(x^*)}{\partial x_{1t}} & \frac{\partial f^n(x^*)}{\partial x_{2t}} & \cdots & \frac{\partial f^n(x^*)}{\partial x_{nt}} \end{bmatrix}.$$

- *The stable eigenspace $E^s(x^*)$ of the steady-state equilibrium x^* is:*

  $E^s(x^*)$=*span* {*eigenvectors of* $Df(x^*)$ *whose eigenvalues have modulus* < 1} .

- *The unstable eigenspace $E^u(x^*)$ of the steady-state equilibrium x^* is:*

  $E^u(x^*)$=*span* {*eigenvectors of* $Df(x^*)$ *whose eigenvalues have modulus* > 1}.

- *The center eigenspace $E^c(x^*)$ of the steady-state equilibrium x^* is:*

  $E^c(x^*)$=*span* {*eigenvectors of* $Df(x^*)$ *whose eigenvalues have modulus* = 1}.

2.1.6 Stable and unstable manifolds

The stable and unstable manifolds provide the nonlinear counterparts for the stable and unstable eigenspaces.

Definition 2.1.6 *(Local stable and unstable manifolds)*
Consider the nonlinear dynamical system:

$$x(t+1) = f(x_t).$$

- *A local stable manifold, $W^s_{loc}(x^*)$, a local stable manifold, x^*:*

$$\begin{aligned} W^s_{loc}(x^*) &= \{x \in U | \lim_{n\to\infty} f^{\{n\}}(x)\}, \\ &= \{x^* \quad and \quad f^{\{n\}}(x) \in U \quad \forall n \in \mathbb{N}\}. \end{aligned}$$

- *A local unstable manifold, $W^u_{loc}(x^*)$, A local unstable manifold,$\bar{x}$,*

$$\begin{aligned} W^u_{loc}(x^*) &= \{x \in U | \lim_{n\to\infty} \phi^{\{-n\}}(x)\}, \\ &= \{x^* \quad and \quad \phi^{\{n\}}(x^*) \in U \quad \forall n \in \mathbb{N}\}. \end{aligned}$$

where $U \equiv B_\varepsilon(x^*)$ *for some* $\varepsilon > 0$, *and* $f^{\{n\}}(x)$ *is the* n^{th} *iteration over* x *under the map* f *[15].*
$B_\varepsilon(x^*) \equiv \left\{x \in \mathbb{R}^n : |x_i - x_i^*| < \epsilon \forall i = 1, 2, 3, \dots n\right\}$

Definition 2.1.7 *(Globally stable and unstable manifolds)*
Consider the nonlinear dynamical system:

$$x(t+1) = \phi(x_t), \tag{2.3}$$

and let x^* *be the steady-state equilibrium of the system (2.3), [12].*

- *The global stable manifold,* $W^s(x^*)$, *of a steady-state equilibrium,* x^*, *is:*

$$W^s(x^*) = \bigcup_{n \in N} \{\phi^{-\{n\}}(W^s_{loc}(x^*))\}.$$

- *The global unstable manifold,* $W^u(x^*)$, *of a steady-state equilibrium,* x^*, *is:*

$$W^u(x^*) = \bigcup_{n \in N} \{\phi^{\{n\}}(W^u_{loc}(x^*))\} .$$

2.1.7 Cobweb diagram

One of the most effective graphical methods for iterating to determine the stability of fixed points of the system (1.2) is the cobweb diagram. On the x-y plane, we plot the curve $y = f(x)$ and the diagonal line $y = x$. Starting from an initial point x_0, we move vertically to the graph of f at the point $(x_0, f(x_0))$. Then, we move horizontally to intersect the line $y = x$ at $(f(x_0), f(x_0))$, which determines $f(x_0)$ on the x-axis.
To find $f^2(x_0)$, we repeat this process by moving vertically to the graph of f at $(f(x_0), f^2(x_0))$ and then horizontally to $(f^2(x_0), f^2(x_0))$.
This iterative process allows us to evaluate all points in the orbit of x_0, denoted by $\{x_0, f(x_0), f^2(x_0), \dots, f^n(x_0), \dots\}$ [10].

Example 2.1.2 *we use the cobweb diagram to find the fixed points for the quadratic map:*

$$f_c(x) = x^2 + c,$$

on the interval $[-2, 2]$ *, where* $c \in [-2, 0]$. *Then determine the stability of all fixed points.*

► *To find the fixed point of* $f_c(x)$, *we solve the equation* $x^2 + c = x$ *or* $x^2 - x + c = 0$.
This yields the two fixed points $x_1^* = \frac{1}{2} - \frac{1}{2}\sqrt{1-4c}$ *and* $x_2^* = \frac{1}{2} + \frac{1}{2}\sqrt{1-4c}$.
Since we have not developed enough machinery to treat the general case for arbitrary c, let us examine few values of c. We begin with $c = -0.5$ *and an initial point* $x_0 = 1.1$ *. It is clear from that the fixed point* $x_1^* = \frac{1}{2} - \frac{\sqrt{3}}{2} \approx -0.366$ *is asymptotically stable, whereas the second fixed point* $x_2^* = \frac{1}{2} + \frac{\sqrt{3}}{2} \approx 1.366$ *is unstable.*

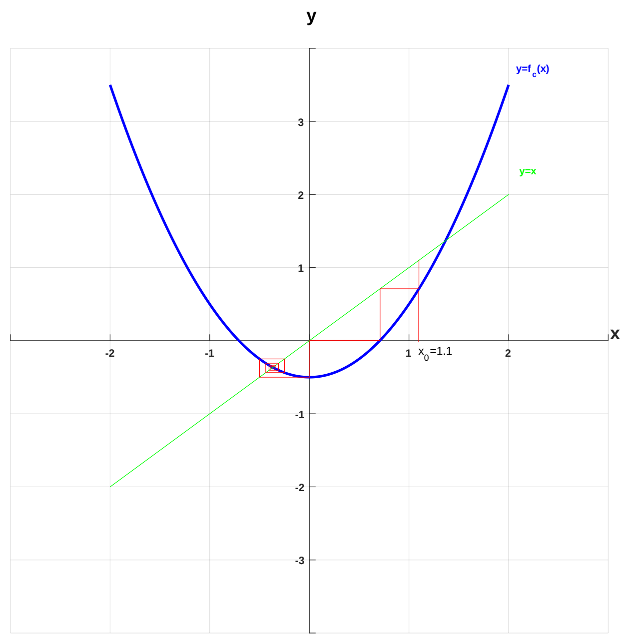


Figure 2.3: Cobweb diagram of $f_c(x) = x^2 + c$ with $c = -0.5$.

2.2 Bifurcation

2.2.1 Bifurcation theory

A discrete dynamical systems undergo a bifurcation when its bihavior changes as a parameter change [14]. In the following section, we explore the center manifold theoremthe and different types of bifurcations in discrete dynamical systems. We will discuss their conditions with examples, as well as the Center Manifold Theorem .

Center manifold

We have seen in the second section of **chapter2** the general concepts of stability of two dimensional maps via linearization, when the fixed point is hyperbolic in other words the eigenvalues of the Jacobian matrix are off the unit circle. But it does not give us an exact stability in the nonhyperbolic case. So we are obligated to use the **center manifold theory**, and we will discuss it later.

Center manifold is a set M_c in a lower dimensional space, where the dynamics of the original system can be obtained by studying the dynamics on M_c [10].

Let consider the system:

$$F : \begin{cases} x \longrightarrow Ax + f(x, y), \\ y \longrightarrow By + g(x, y). \end{cases} \tag{2.4}$$

We studied in **Chapter2** the stability of hyperbolic fixed points of F. We have

$$J = D_x F(c^*, x^*), \tag{2.5}$$

where J in Equation (2.5) has the form $J = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$.

Moreover, all of the eigenvalues of A lie on the unit circle and all of the eigenvalues of B are off the unit circle. Furthermore,

$$\begin{aligned} f(0,0) &= 0, & g(0,0) &= 0, \\ Df(0,0) &= 0, & Dg(0,0) &= 0. \end{aligned}$$

Theorem 2.2.1 *[10]*

There is a C^r-center manifold for system (2.4) that can be represented locally as:

$$M_c = \{\, (x, y) \in \mathbb{R}^n \times \mathbb{R}^s : y = h(x), |x| < \delta, h(0) = 0, \\ Dh(0) = 0, \textit{for a sufficiently small } \delta \,\}.$$

Furthermore, the dynamics restricted to M_c are given locally by the map

$$x \longrightarrow Ax + f(x, h(x)), \quad x \in \mathbb{R}^n. \tag{2.6}$$

This theorem asserts the existence of a center manifold, i.e., a curve $y = h(x)$ on which the dynamics of System (2.4) is given by Equation (2.6). The next result states that the dynamics *on the center manifold M_c determines completely the dynamics of System (2.4).*

Center manifolds depending on parameters

Suppose that system (2.4) depends on a vector of parameters, $\mu \in \mathbb{R}^m$ Then, System (2.4) takes the form:

$$\begin{cases} x(n+1) = Ax(n) + f(\mu, x(n), y(n)), \\ y(n+1) = By(n) + g(\mu, x(n), y(n)). \end{cases} \tag{2.7}$$

The center manifold M_c now takes the form:

$$M_c = \{ (\mu, x, y) : y = h(x, \mu)|x| < \delta_1, |\mu| < \delta_2, h(0,0) = 0, Dh(0,0) = 0 \}.$$

After substituting y we get the latter equations lead to the functional equation :

$$F(h(x, \mu)) = h[Ax + f(\mu, h(x, \mu), x), \mu] - Bh(x, \mu) - g(\mu, h(x, \mu), x). \tag{2.8}$$

Definition 2.2.1 *[33]*

Consider the following nonlinear dynamical system:

$$x_{k+1} = F_\mu(x_k). \tag{2.9}$$

Where $x_k \in \mathbb{R}^n$, $\mu \in \mathbb{R}^m$, $k \in \mathbb{N}$, and $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, F is continious function.

Definition 2.2.2

A bifurcation is a quantitative or qualitative change in the solution of a dynamical system when the parameters on which it depends are modified, and more precisely, the disappearance or change in stability or the appearance of new solutions.

Theorem 2.2.2 *[14] (The bifurcation criterion)*

Let f_μ be a family of functions depending smoothly on the parameter μ. Suppose that $f_{\mu_0}(x_0) = x_0$ and $\left.\frac{\partial f}{\partial \mu}\right|_{\mu_0}(x_0) \neq 1$. Then there are intervals I about x_0 and J about μ_0 and a smooth function $p : J \to I$ such that $p(\mu_0) = x_0$ and $f_\mu(p(\mu)) = p(\mu)$. Moreover, f_μ has no other fixed points in I.

Bifurcation diagram

It tracks the points of the system's steady state according to the bifurcation parameter. These graphs are called bifurcation diagrams.

In general, we choose the initial state x_0 such that the horizontal axis represents the values of μ and the vertical axis represents the higher iterations F_μ^n. Then we track its limit value as a function of a single parameter.

In the discrete case, we plot the successive values of the state and the variable. It summarizes the information about the state space, and the variation with respect to the parameter μ can visualize the transition from a steady state to chaos.

Types of bifurcation

Consider the one-parameter family of maps:

$$F(u, \mu) : \mathbb{R} \times \mathbb{R}^2 \longrightarrow \mathbb{R}.$$

where $u = (x, y) \in \mathbb{R}^2$, $\mu \in \mathbb{R}$ and $F \in C^r$, $r \geqslant 5$. If (u^*, μ^*) is a fixed point, we can change the variables so that the fixed point is at $(0, 0)$.
Let $J = D_\mu F(0, 0)$. Using the center manifold theorem, we can derive a one-dimensional map $f_\mu(x)$ defined on the center manifold M_c. By Theorem (2.2.1), we can infer the following statements.

Where λ the eigenvalues of J lies on the unit circle,meaning $|\lambda| = 1$. There are three distinct cases in which the fixed point $(0, 0)$ is nonhyperbolic.

I. J has an eigenvalue equal to 1. Then we havethree kinds of bifurcations:
Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a $\mathbb{C}^r$ function with $r \geq 2$ [10]:

(a) Saddle-node bifurcation (fold bifurcation:)
This type of bifurcation is characterized by the sudden loss or acquisition of multiple stable or unstable equilibrium solutions when the value of a parameter crosses a critical value. It satisfies the following conditions:

1. $\frac{\partial^2 f(x,\mu)}{\partial x^2}|_{(0,0)} \neq 0.$

2. $\frac{\partial f(x,\mu)}{\partial c}|_{(0,0)} \neq 0.$

Theorem 2.2.3 ***(Topological normal form for the fold bifurcation)*** *[23]*
Any generic scalar one-parameter system

$$x \mapsto f(x, \mu),$$

having at $\mu = 0$ the fixed point $x_0 = 0$ with $\lambda = f_x(0, 0) = 1$, is locally topologically equivalent near the origin to one of the following normal forms:

$$x \mapsto \mu + x \pm x^2.$$

▶ Consider the map:

$$x \mapsto f(x, \mu) = x + \mu \pm x^2, x \in \mathbb{R}^1, \mu \in \mathbb{R}^1. \tag{2.10}$$

We can solve for the fixed points directly as follows

$$f(x, \mu) - x = \mu \pm x^2 = 0.$$

We are interested in the nature of the fixed points for (2.10) near $(x, \mu) = (0, 0)$ the map possesses a unique curve of fixed points in the $x - \mu$ plane passing through the bifurcation point which locally lies on one side of $\mu = 0$. Then we must check the conditions (1) and (2) we have:

$$\frac{\partial f}{\partial x}(0, 0) = 1.$$

$$\frac{\partial f^2}{\partial x^2}(0, 0) \neq 0.$$

$$\frac{\partial f}{\partial \mu}(0, 0) = 1 \neq 0.$$

Thus, (2.10) can be viewed as a normal form for the saddle-node bifurcation of maps. Notice that, with the exception of the condition $\dfrac{\partial f}{\partial x}(0, 0) = 1$, the conditions for a one-parameter family of one-dimensional maps to undergo a saddle-node bifurcation in terms of derivatives of the map at the bifurcation point are exactly the same as those for vector fields.

In Figure 2.4, we show a curve of fixed points and refer to the bifurcation occurring at $(x,\mu) = (0,0)$ as a saddle-node bifurcation [44].

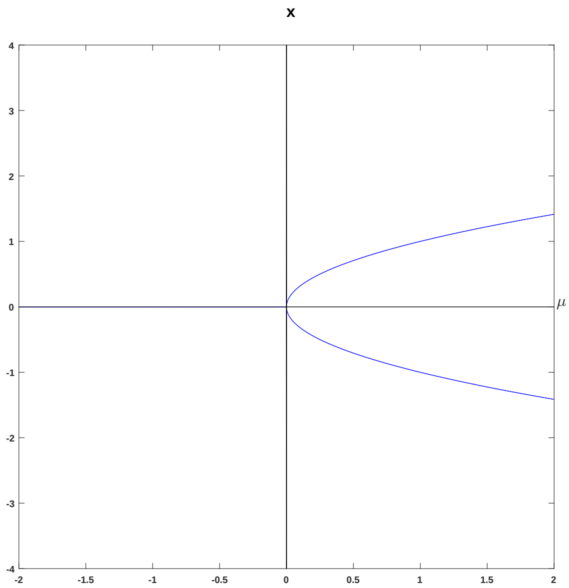


Figure 2.4: Fold bifurcation of $f(x,\mu) = x + \mu + x^2$.

(b) Pitchfork bifurcation:

In general, a Pitchfork bifurcation occurs near the bifurcation point (x_0, μ_0). The model has two fixed-point curves in the (x_n, μ) plane that pass through the bifurcation point, with one of them being on both sides of the line $\mu = \mu_0$. It satisfies the following conditions:

1. $\frac{\partial f}{\partial \mu}(0,0) = 0$, $\frac{\partial^2 f}{\partial \mu^2}(0,0) = 0$.

2. $\frac{\partial f}{\partial x}(0,0) = 1$, $\frac{\partial^2 f}{\partial x^2}(0,0) = 0$.

Theorem 2.2.4 *[44]*

Consider the application $x_{k+1} = f(x_k, \mu)$, such that $f(-x,\mu) = -f(x,\mu)$ for all μ near $\mu = 0$. If this application has a non-hyperbolic fixed point at $x^* = 0$, $\mu = 0$ and $\frac{\partial f}{\partial x}(0,0) = 1$, and if

$$\frac{\partial^2 f}{\partial x \partial \mu}(0,0) \neq 0, \frac{\partial^3 f}{\partial x^3}(0,0) \neq 0,$$

then in the neighborhood of $(0,0)$, this application is locally equivalent to one of the *following normal forms:*

$$x_{k+1} = x_k + x_k(\pm\mu \pm \beta x_k^2).$$

▶ Consider the map:

$$x \mapsto f(x,\mu) = x + \mu x \pm x^3, x \in \mathbb{R}^1, \mu \in \mathbb{R}^1, \tag{2.11}$$

we can solve for the fixed points directly as follows:

$$f(x,\mu) - x = \mu x \pm x^3.$$

We are interested in the nature of the fixed points for (2.11) near $(x,\mu) = (0,0)$, in the $x - \mu$ plane the map has two curves of fixed points passing through the bifurcation point; one curve exists on both sides of $\mu = 0$ and the other lies locally to one side of $\mu = 0$. Now we seek general conditions for a one-parameter family of $C^r(r \geq 3)$ one-dimensional maps to undergo a pitchfork bifurcation:

- $\dfrac{\partial f}{\partial \mu}(0,0) = 0, \dfrac{\partial^2 f}{\partial x^2}(0,0) = 0.$
- $\dfrac{\partial f}{\partial x \partial \mu}(0,0) \neq 0, \dfrac{\partial^3 f}{\partial x^3}(0,0) \neq 0.$

$$l = \frac{-\dfrac{\partial^3 f}{\partial x^3}(0,0)}{\dfrac{\partial f}{\partial x \partial \mu}(0,0)}.$$

Moreover, the sign of l tells us on which side of $\mu = 0$ that one of the curves of fixed points lies. Thus, we can view (2.11) as a normal form for the pitchfork bifurcation.

We end our discussion of the pitchfork bifurcation by graphically showing the bifurcation for $x \mapsto x + \mu x \pm x^3$ in Figure 2.5 [44].

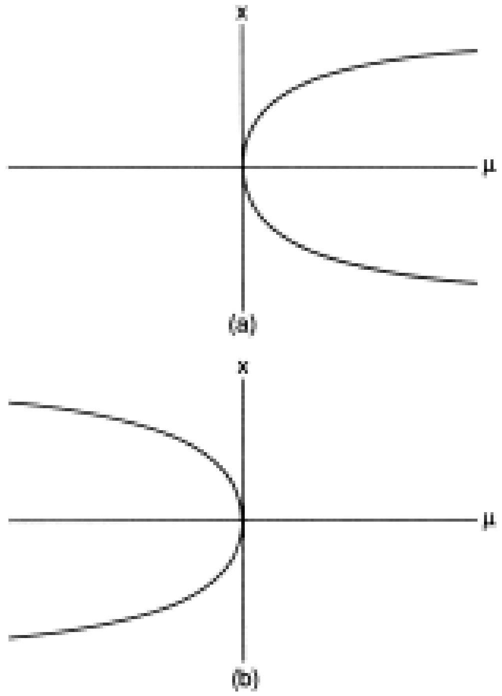


Figure 2.5: Pitchfork bifurcation of $f(x,\mu) = x + \mu x \pm x^3$ a) $-\dfrac{\partial^3 f}{\partial x^3}(0,0)/\dfrac{\partial f}{\partial x \partial \mu}(0,0) > 0$ b) $-\dfrac{\partial^3 f}{\partial x^3}(0,0)/\dfrac{\partial f}{\partial x \partial \mu}(0,0) < 0$

(c) Transcritical bifurcation:

This type of bifurcation is characterized by an exchange of stability between two equilibrium solutions. Initially, the system has a stable equilibrium solution and an unstable equilibrium solution. When a parameter varies and reaches a critical value, the stable equilibrium solution becomes unstable, while the unstable equilibrium becomes stable It satisfies the following conditions:

1. $\frac{\partial f}{\partial c}(0,0) = 0.$

2. $\frac{\partial f}{\partial x}(0,0) = 1.$

3. $\frac{\partial^2 f}{\partial x^2}(0,0) \neq 0.$

Theorem 2.2.5 *[44]*

Let $x_{k+1} = f(x_k,\mu) = x_k g(x_k,\mu)$ be a point x_e that is non-hyperbolic at $x^* = 0$, $\mu = 0$

$$g(0,0) = 1.$$

$\frac{\partial g}{\partial x}(0,0) \neq 0$; *and* $\frac{\partial g}{\partial x}(0,0) \neq 0$.

Then, in the neighborhood of $(0,0)$, *the application is locally equivalent to one of the following normal forms:*

$$x_{k+1} = x_k \pm \mu x_k \pm x_k^2.$$

▶ Consider the map:

$$x \mapsto x + \mu x \pm x^2, x \in \mathbb{R}^1, \mu \in \mathbb{R}^1. \tag{2.12}$$

We can solve for the fixed points directly as follows:

$$f(x,\mu) - x = \mu x \pm x^2$$

Hence, there are two curves of fixed points passing through the bifurcation point:

$$x = 0,$$

and

$$\mu = \pm x^2.$$

We are interested in the nature of the fixed points for (2.11) near $(x,\mu) = (0,0)$, in the $x - \mu$ plane the map has two curves of fixed points passing through the origin and existing on both sides of $\mu = 0$. Then we must check the conditions $(1,2,3)$ we have:

- $\dfrac{\partial f}{\partial \mu}(0,0) = (0,0)$.
- $\dfrac{\partial^2 f}{\partial x \partial \mu}(0,0) \neq 0$.
- $\dfrac{\partial^2 f}{\partial x^2}(0,0) \neq 0$.

$$k = \frac{-\dfrac{\partial^2 f}{\partial x^2}}{\dfrac{\partial^2 f}{\partial x \partial \mu}}. \tag{2.13}$$

Moreover, the sign of (2.13) gives us the slope of the curve of fixed points that is not $x = 0$.

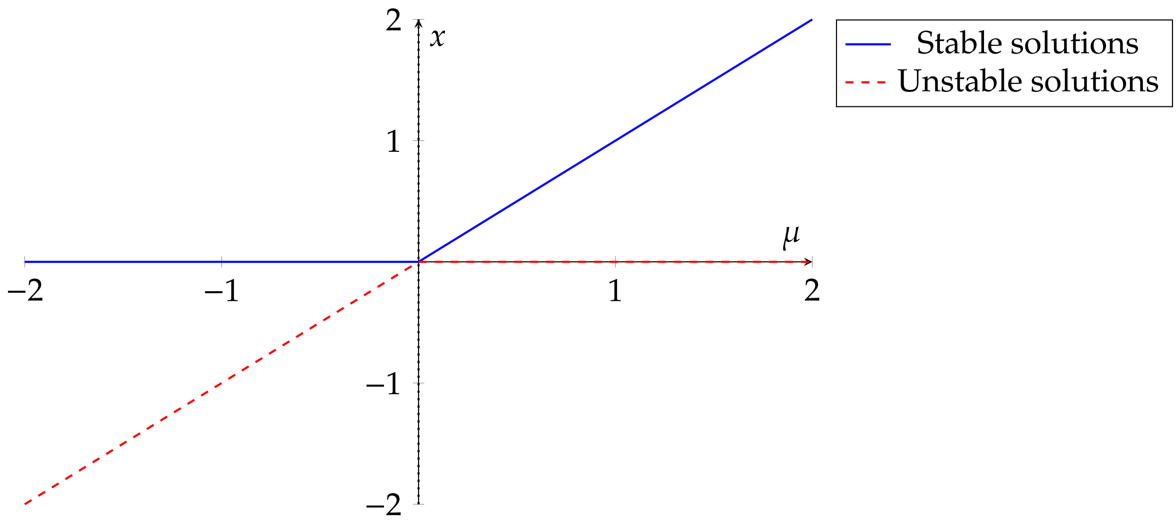


Figure 2.6: Transcritical bifurcation diagram for $f(x, \mu) = x + \mu x \pm x^2$ when $k > 0$.

II. If J has an eigenvalue equal to -1, then we have :

- **Period-doubling (flip):**

  This bifurcation occurs when a stable cycle of order k has a multiplier that passes through the value $\lambda = 1$. This cycle then becomes unstable and gives rise to a cycle of order 2k. It satisfies the following conditions:

  1. $\frac{\partial f}{\partial x}(0,0) = -1 \Rightarrow \frac{\partial}{\partial x}[F(x,\mu) - x]_{(0,0)} \neq 0.$

  2. The derivate of $\frac{\partial f}{\partial x}$ with respect to λ at the point $(0,0)$ is nonzero, that is:
  $$\alpha = \left[\frac{\partial^2 f}{\partial \mu \partial x} + \frac{1}{2}\frac{\partial f}{\partial \mu}\frac{\partial^2 f}{\partial x^2}{}_{(0,0)}\right] \neq 0.$$

  3. $\beta = \left[\frac{1}{3!}\frac{\partial^3 f}{\partial x^3} + \frac{1}{2!}\left(\frac{\partial f}{\partial x}\right)^2\right]_{(0,0)} \neq 0.$

  Then:

  (a) There occurs a differentiable curve $x(\mu)$ of fixed points passing through the point (x_0, μ_0); the stability of the fixed points changes at (x_0, μ_0): the curvature changes from stable to unstable as μ increases past the value μ_0 if $\alpha < 0$ while the converse occurs if $\alpha > 0$.

  (b) Moreover, there occurs a period-doubling bifurcation at the point (x_0, μ_0), there is a differentiable curve $\mu a = l(x)$ passing through the point (x_0, μ_0) such that all points on the curve except the point (x_0, μ_0) are hyperbolic period-2 points, $f^2_{\mu=l(x)}(x) = x$; the curve $\lambda = l(x)$ is tangential to $\mu = \mu_0$ at (x_0, μ_0); $l'(x) = 0$ and $l''(x_0) = -2\beta/\alpha \neq 0$. Finally, the period 2 orbits are attracting if $\beta > 0$ and are repelling if $\beta < 0$.

Theorem 2.2.6 *[10] (Topological normal form for the flip bifurcation)*
Any generic, scalar, one-parameter system

$$x \mapsto f(x, \mu),$$

having at $\mu = 0$ the fixed point $x_0 = 0$ with $\mu = f_x(0, 0) = -1$, is locally topologically equivalent near the origin to one of the following normal forms:

$$x \mapsto -(1 + \mu)x \pm x^3.$$

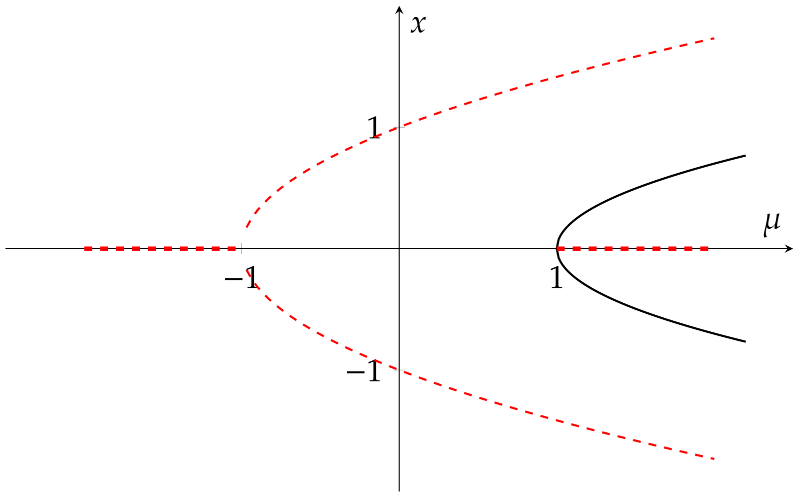


Figure 2.7: Flip bifurcation diagram for μ of $f(x, \mu) = -(1 + \mu)x \pm x^3$.

Example 2.2.1 *Consider the following simple population model [Ricker 1954]:*

$$x_{k+1} = \alpha x_k e^{-x_k}.$$

where x_k is population, k : the year,α :growt rate. The above recurrence relation corresponds to the discrete-time dynamical system

$$x \mapsto \alpha x e^{-x} \equiv f(x, a).$$

System has a trivial fixed point $x_0 = 0$ for all values of the parameter α. At $\alpha_0 = 1$, however, a nontrivial positive fixed point appears:

$$x_1 \alpha = \ln(\alpha).$$

The multiplier of this point is given by the expression:

$$\mu(\alpha) = 1 - \ln(\alpha).$$

Thus, x_1 is stable for $1 < \alpha < \alpha_1$ and unstable for $\alpha > \alpha_1$, where $\alpha_1 = e^2$.At the critical parameter value $\alpha = \alpha_1$, the fixed point has multiplier $\mu(\alpha_1) = -1$. Therefore, a flip bifurcation takes place. To apply Theorem (2.2.6), one needs to verify the corresponding nondegeneracy conditions in which all the derivatives must be computed at the fixed point $x_1(\alpha_1) = 2$ and at the critical parameter value al. One can check that:

$$c(0) = \frac{1}{6} > 0, f_{x\alpha} = -\frac{1}{e^2} \neq 0.$$

The Neimark-Sacker bifurcation is the birth of a closed invariant curve from a fixed point in discrete dynamical systems, occurring when the fixed point changes stability through a pair of complex eigenvalues with modulus equal to 1. This bifurcation occurs only in discrete dynamical systems with dimension greater than or equal to 2 ($m <= 2$) and is analogous to the Hopf bifurcation in continuous dynamical systems (ODEs), where the complex conjugate eigenvalues are written in the form of Euler's complex numbers.

$\lambda_{1,2}(\mu) = \rho(\mu) \exp(\pm i\theta(\mu))$

If these eigenvalues cross the unit circle for $\mu = \mu_0$ such that $0 < \theta(\mu_0) < \pi$, a closed invariant curve appears, which is an attractor for the system's orbits. This phenomenon is called the Neimark-Sacker bifurcation. Note that the condition on the angle $\theta(\mu_0)$ at the critical parameter value μ_0 implies that the eigenvalues must be strictly complex. In this case, defining the complex number

$$z_n = x_n + iy_n.$$

it can be shown that as long as $\rho_0(\mu_0) \neq 0$ and $\exp(ik\theta(\mu_0)) \neq 0$ for $k = 1, 2, 3, 4$, the system is locally equivalent to:

$$z_{n+1} = (1 + \epsilon) \exp(i\theta(\epsilon))z_n + c(\epsilon)z_n|z_n|^2 + O(|z_n|^4).$$

Where ϵ is a new parameter. The situation where $\exp(ik\theta(\mu_0)) = 0$ for all $k \in \{1, 2, 3, 4\}$ is known as a strong resonance and is associated with the first four roots of 1 on the complex unit circle.

As in the case of the Andronov-Hopf bifurcation, in the supercritical Neimark-

Sacker bifurcation, a fixed point loses stability and a closed orbit appears with an increasing radius. There is also a subcritical case, but it will not be treated here.

To illustrate the Neimark-Sacker bifurcation, consider the following example:

- **Neimark-Sacker bifurcation** :
  Let $F(x,\mu) \equiv F_\mu(x)$: $\mathbb{R}^2 \to \mathbb{R}$ be a $\mathbb{C}^r$ function in variables x, μ with $r \geq 3$. If $F(x,\mu)$ satisfies the following conditions:

  Theorem 2.2.7 *[29] Consider the family of C^r maps ($r \geq 5$), $f_\mu : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$ such that the following conditions hold:*

  (1) $f_\mu(0) = 0$, i.e., the origin is a fixed point of f_μ.

  (2) $Df_\mu(0)$ has two complex conjugate eigenvalues $\lambda_{1,2}(\mu) = r(\mu)e^{\pm i\theta(\mu)}$, where $r(0) = 1$, $r'(0) \neq 0$, $\theta(0) = \theta_0$.

  (3) $e^{ik\theta_0} \neq 1$ for $k \in \{1,2,3,4\}$ (absence of strong resonances condition).

$$\psi = -\Re\left(\frac{(1-2\bar{\lambda})\bar{\lambda}^2}{1-\lambda}\rho_{11}\rho_{20}\right) - \frac{1}{2}\|\rho_{11}\|^2 - \|\rho_{02}\|^2 + \Re(\bar{\lambda}\rho_{21}), \tag{2.14}$$

  ψ is called the first Lyapunov coefficient.
  Where:

$$\begin{cases} \rho_{02} &= \frac{1}{8}\left(\frac{\partial^2 P}{\partial x_t{}^2} - \frac{\partial^2 P}{\partial y_t{}^2} + 2\frac{\partial^2 Q}{\partial x_t \partial y_t} + \iota\left(\frac{\partial^2 Q}{\partial x_t^2} - \frac{\partial^2 Q}{\partial y_t^2} + 2\frac{\partial^2 P}{\partial x_t \partial y_t}\right)\right)|_{(0,0)} \ . \\ \rho_{11} &= \frac{1}{4}\left(\frac{\partial^2 P}{\partial x_t{}^2} + \frac{\partial^2 P}{\partial y_t{}^2} + \iota\left(\frac{\partial^2 Q}{\partial x_t^2} + \frac{\partial^2 Q}{\partial y_t^2}\right)\right)|_{(0,0)} \ . \\ \rho_{20} &= \frac{1}{8}\left(\frac{\partial^2 P}{\partial x_t{}^2} - \frac{\partial^2 P}{\partial y_t{}^2} + 2\frac{\partial^2 Q}{\partial x_t \partial y_t} + \iota\left(\frac{\partial^2 Q}{\partial x_t^2} - \frac{\partial^2 Q}{\partial y_t^2} - 2\frac{\partial^2 P}{\partial x_t \partial y_t}\right)\right)|_{(0,0)} \ . \\ \rho_{21} &= \frac{1}{16}\left(\frac{\partial^3 P}{\partial x_t^3} + \frac{\partial^3 P}{\partial y_t^3} + \frac{\partial^3 Q}{\partial x_t^2 \partial y_t} + \frac{\partial^3 Q}{\partial y_t^3} + \iota\left(\frac{\partial^3 Q}{\partial x_t^3} + \frac{\partial^3 Q}{\partial x_t \partial y_t^2} - \frac{\partial^3 P}{\partial x_t^2 \partial y_t} - \frac{\partial^3 P}{\partial y_t^3}\right)\right)|_{(0,0)} \ . \end{cases}$$

  Then, for sufficiently small μ and F_μ, there exists a unique invariant closed curve enclosing that bifurcates from the origin as a passes through 0. If ψ<0, we have a supercritical Neimark-Sacker bifurcation. If ψ>0, we have a subcritical Neimark-Sacker bifurcation.

  Theorem 2.2.8 *Consider the following application:*

$$\begin{cases} x_{k+1} &= f(x_k, y_k, \mu), \\ y_{k+1} &= g(x_k, y_k, \mu). \end{cases}$$

At a non-hyperbolic point $(x, y, \mu) = (0, 0, 0)$*, the eigenvalues of the Jacobian matrix are:*

$$\lambda = |\lambda(c)|e^{i\theta(c)};\ |\lambda(0)| = 1.$$

Then, if:

$$\frac{\partial|\lambda|}{\partial c}(0) \neq 0\ ;\ \lambda = e^{ik\theta(0)}, |\lambda(0)| \neq 1\ \ (k = 1, 2, 3, 4)\ ;\ d(0) \neq 0,$$

there is a change to polar coordinates that transforms the application into the following form:

$$\begin{cases} \gamma_{k+1} &= |\lambda|\gamma_k + d\gamma_k^3, \\ \theta_{k+1} &= \theta_k + \phi(c) + b(c)\gamma_k^2. \end{cases}$$

There are two types distinguished, node-col bifurcations that give rise to fixed points and period-doubling bifurcations that produce periodic orbits.

Remark 2.2.9 *[18] :*
Note that the pli bifurcation and flip bifurcation take place in systems of dimension $n \geq 1$*, but for the Neimark-Sacker bifurcation it requires* $n \geq 2$*.*

Example 2.2.2 *Consider the discrete dynamical system generated by the transformation F defined as follows:*

$$F : \begin{cases} X_{n+1} &= \mu X_n(1 - Y_n), \quad \mu > 0 \\ Y_{n+1} &= X_n. \end{cases}$$

This system has two fixed points:

$$\begin{cases} (X_1, Y_1) &= (0, 0). \\ (X_2, Y_2) &= (1 - \frac{1}{\mu}, 1 - \frac{1}{\mu}). \end{cases}$$

The Jacobian matrix evaluated at the fixed point (X_2, Y_2) *is:*

$$DF(X_2, Y_2) = \begin{pmatrix} 1 & 1 - \mu \\ 1 & 0 \end{pmatrix}.$$

Its characteristic equation is:

$$\lambda^2 - \lambda + \mu - 1 = 0.$$

From which we deduce the eigenvalues:

$$\lambda_{1,2} = \frac{1}{2} \pm \sqrt{\frac{5}{4} - \mu}.$$

If $\mu > \frac{5}{4}$, *the eigenvalues are complex with* $|\lambda_{1,2}|^2 = \mu - 1$. *For* $\mu = 2$, *the fixed point* (X_2, Y_2) *loses its stability. The eigenvalues are then* $\lambda_{1,2} = e^{\pm i\frac{\pi}{3}}$ *and the system undergoes a Neimark bifurcation.*

Let $T = tr(J)$, $D = det(J)$. Then the following trace-determinant diagram figure illustrate the main bifurcation phenomena.

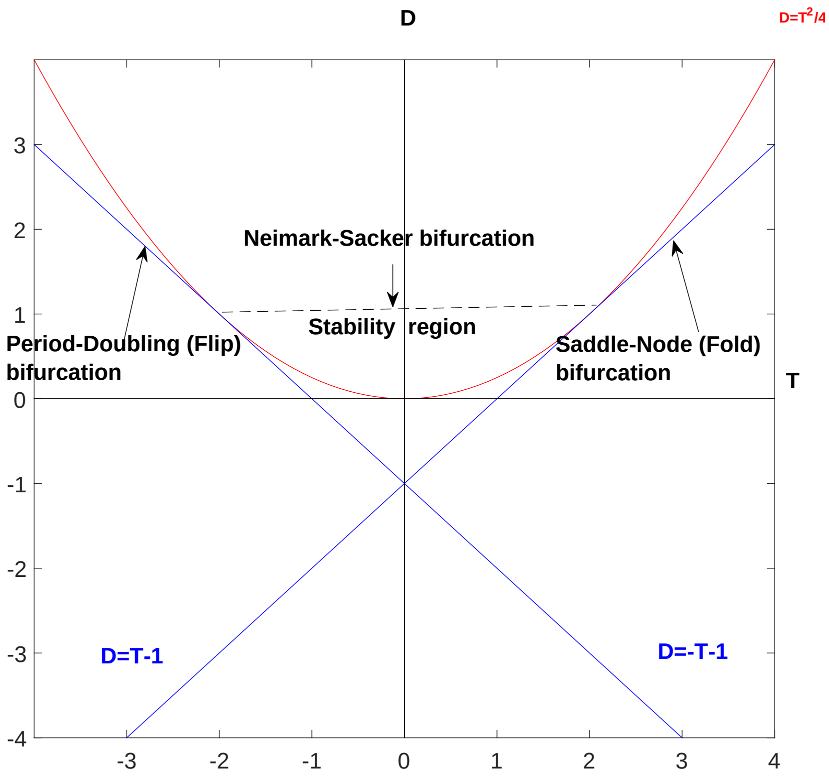


Figure 2.8: The occurrence of the three main types of bifurcation.

CHAPTER 3

NOTIONS OF CHAOS

Chaos is a deterministically unpredictable phenomenon. In the evolution of chaotic orbit there are trajectories which do not settle down to fixed points or periodic orbits or quasi-periodic orbits as time tends to infinity. Even a deterministic system has no random or noisy inputs; an irregular behavior may appear due to presence of nonlinearity, dimensionality, or nondifferentiability of the system. Although the time evolution obeys strict deterministic laws, the system seems to behave according to its own free will. The mathematical definition of chaos introduces two notions , the topological transitive property implying the mixing and the metrical property measuring the distance. Chaotic orbit may be expressed by fractals. Before defining chaos under the mathematical framework we discuss some preliminary concepts and definitions of topological and metric spaces which are essential for chaos theory.

3.1 Definitions of chaos

There exist numerous mathematical definitions of chaos in literature; however, as of now, there is no universally accepted mathematical definition of chaos. Prior to

presenting a definition of chaos by Devaney [7], it is essential to establish some basic definitions.
Let $(J \subset \mathbb{R}, d)$ designate a compact metric space (d is a distance), and let f be the function:

$$f : J \to J \ , \ x_{k+1} = f(x_k) \ , \ x_0 \in \ J.$$

Definition 3.1.1 *[26]*
(Dense set) In a topological space (X, τ), a subset A of X is said to be a dense set (or an everywhere dense set) if $\bar{A} = X$. In other words, A is said to be dense subset of X if for any $x \in X$, any neighborhood of x contains at least one point of A.

Definition 3.1.2 *[7]*
f is said to be topologically transitive if, for any pair of open sets $U, V \subset J$, there exists $k > 0$ such that $f^k(U) \cap V \neq \emptyset$.

Definition 3.1.3 *[7]*
f has sensitive dependence on initial conditions on J if there exists $\delta > 0$ such that, for any $x \in J$ and any neighborhood N of x, there exists $y \in J$ and $n \geqslant 0$ such that $|f^n(x) - f^n(y)| > \delta$.

Example 3.1.1 *The logistic map $\mu x(1 - x)$ with $\mu > 2 + \sqrt{5}$ possesses sensitive dependence on initial conditions on Λ.*
To see this, choose δ less than the width of A_0, where A_0 is the gap between I_0 and I_1 in which all points immediately escape from I. Let $x, y \in \Lambda$. If $x \neq y$, then $S(x) \neq S(y)$, so the itineraries of x and y must differ in at least one spot, say the $n^t h$. But this means that $f_\mu^n(x)$ and $f_\mu^n(y)$ lie on opposite sides of A_0, so that

$$|f_\mu^n(x) - f_\mu^n(y)| > \delta.$$

Definition 3.1.4 *(Devaney) [7]*
Let V be a set. $f : V \to V$is said to be chaotic on V if f has the following three properties:

1. *periodic points are dense in V.*
2. *f is topologically transitive.*
3. *f has sensitive dependence on initial conditions.*

3.2 Characteristics of chaos

- **Sensitivity to initial conditions**: sensitivity to initial conditions was first observed by Poincare in the late 19th century and later rediscovered by Lorenz in 1963 during his meteorological research. This discovery sparked significant interest in mathematics, leading to numerous important works. This sensitivity is responsible for the unpredictable long-term outcomes of chaotic systems, where even small changes in initial conditions can lead to vastly different results. The degree of sensitivity to initial conditions is a measure of the system's chaotic behavior [33].

- **Non-linearity**: if the system is linear, it cannot be chaotic.

- **Determinism**: a chaotic system has fundamental deterministic rules (rather than probabilistic ones).

- **Unpredictability**: due to sensitivity to initial conditions, which can only be known to a finite degree of precision.

- **Irregularity**: hidden order comprising an infinite number of unstable periodic patterns (or motions). This hidden order forms the infrastructure of chaotic systems, more simply put as "order in disorder."

3.3 Lyapunov exponents

The Lyapunov exponent is an important quantitative measure in chaos theory, used to measure the potential difference between orbits arising from neighboring initial conditions and to quantify the sensitivity of a chaotic system to its initial conditions. It is also used to study the stability (or instability) of equilibrium points in nonlinear systems.
Let the following discrete nonlinear dynamical system be given:

$$x_{k+1} = f(x_k),$$

with $x_k \in \mathbb{R}$, we assume that the trajectory emanating from an initial state $x(0)$ reaches an attractor. x_k is thus bounded inside the attractor [33].
We choose two very close initial conditions, denoted x_0 and $x'(0)$, and observe how the trajectories emanating from them behave. Assuming that the two trajectories x_k and $x'(k)$ diverge exponentially, after k steps we have:

$$|x'(k) - x(k)| = |x'(0) - x(0)|e^{\lambda k},$$

λ indicate the divergence rate per iteration of the two trajectories, whose expression is as follows:

$$\lambda = \frac{1}{k} \ln \left| \frac{x'(k) - x(k)}{x'(0) - x(0)} \right|.$$

For $x(0)$ and $x'(0)$ close, if the absolute value of the difference $\varepsilon = |x'(0) - x(0)|$ tends to converge to zero, we obtain:

$$\lambda = \lim_{k\to\infty} \frac{1}{k} \lim_{\varepsilon\to 0} \ln \left| \frac{x'(k) - x(k)}{x'(0) - x(0)} \right|.$$

This gives:

$$\begin{aligned}
\lambda &= \lim_{k\to\infty} \lim_{\varepsilon\to 0} \frac{1}{k} \ln \left| \frac{x'(k) - x(k)}{x'(k-1) - x(k-1)} \times \frac{x'(k-1) - x(k-1)}{x'(k-2) - x(k-2)} \times \cdots \times \frac{x'(1) - x(1)}{x'(0) - x(0)} \right|. \\
&= \lim_{k\to\infty} \lim_{\varepsilon\to 0} \frac{1}{k} \sum_{i=0}^{k-1} \ln \left| \frac{x'(i-1) - x(i+1)}{x'(i) - x(i)} \right|. \\
&= \lim_{k\to\infty} \lim_{\varepsilon\to 0} \frac{1}{k} \sum_{i=0}^{k-1} \ln \left| \frac{f(x'(i)) - f(x(i))}{x'(i) - x(i)} \right|.
\end{aligned}$$

Finally, we have:

$$\lambda = \lim_{k\to\infty} \lim_{\varepsilon\to 0} \frac{1}{k} \sum_{i=0}^{k-1} \ln \left| \frac{df(x(i))}{dx(i)} \right|,$$

λ the Lyapunov exponent, measures the average rate of divergence of two distinct trajectories starting from two very close initial conditions.
In the case of a system of dimension $n > 1$, there exist n Lyapunov exponents $Lj(j = 1, 2, \cdots, n)$, each of them measures the divergence rate along one of the axes of the phase space. To calculate the Lyapunov exponent, we start from an initial point $x(0) \in \mathbb{R}$, to characterize the infinitesimal behavior around the point $x(k)$, we use the first derivative

matrix Df(x(i)).

$$Df(x(i)) = \begin{pmatrix} \frac{\partial f_1(x(i))}{\partial x_1(i)} & \cdots & \frac{\partial f_1(x(i))}{\partial x_n(i)} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n(x(1))}{\partial x_1(i)} & \cdots & \frac{\partial f_n(x(i))}{\partial x_n(i)} \end{pmatrix}.$$

Let's denote: $J_k = Df(x(k-1)) \cdots Df(x(0))$, with: $J_0 = Df(x(0))$.
The Lyapunov exponent is calculated by the following expression:

$$\lambda_i = \lim_{k\to\infty} \frac{1}{k} \ln|\lambda_i(J_k \cdots J_1)| \ , \ i = 1, 2, ..., n.$$

By analyzing the Lyapunov exponents of a system, we can conclude about the behavior of the system as follows:
- If all Lyapunov exponents are negative, there exist asymptotically stable fixed points or periodic points.
- If one or more Lyapunov exponents are zero, and the others are negative, the attractor is quasi-periodic.
- If at least one of the Lyapunov exponents is positive, and the others can be negative or zero, the attractor is chaotic.

Example 3.3.1 *Lyapunov exponent for the tent map:*
For the general tent map

$$T(x) = \begin{cases} 2rx, & 0 \le x \le \frac{1}{2}. \\ 2r(1-x), & \frac{1}{2} \le x \le 1. \end{cases}$$

we calculate $|T'(x)| = 2r, \forall x \in [0,1]$, *except at* $x = \frac{1}{2}$, *the point of non differentiability. Here the parameter* r *lies in the interval* $0 \leqslant r \leqslant 1$.
Thus the Lyapunov exponent of the tent map is given by

$$\lambda = \lim_{N\to\infty} \frac{1}{N} \sum_{i=0}^{N-1} \ln|T'(x_i)| = \lim_{N\to\infty} \frac{1}{N} \sum_{i=0}^{N-1} \ln 2r = \lim_{N\to\infty} \frac{1}{N}.N \ln 2r$$
$$= \ln 2r.$$

Since $\lambda > 0$ *for* $2r > 1$, *that is, for* $r > \frac{1}{2}$, *the tent map is chaotic for* $r > \frac{1}{2}$.
It is nonchaotic for $r \le \frac{1}{2}$. *The transition from nonchaotic to chaotic behavior occurs at*

$r = r_c = \frac{1}{2}$.

3.4 Fractal dimension

Our intuitive idea of dimension assigns an integer to common geometric objects. For example, a point has dimension 0, a line segment has dimension 1, a full square (interior and boundary) has dimension 2, a full cube (interior and boundary) has dimension 3, etc. This intuitive idea is not sophisticated enough for complex geometric objects, including strange attractors. More elaborate definitions of dimension have been proposed [30], we will present some of them here :

1) **Hausdorff dimension**

Definition 3.4.1 *(The Hausdorff outer measure of order $s \in \mathbb{R}^+$ in a separable metric space X) [38]*
Let $G \subseteq X$. Let's denote by $C(G, \delta)$ the set of δ-coverings of G, i.e., the set of countable families of open sets (C_i) with diameters less than or equal to δ such that $G \subseteq \bigcup_i C_i$. Let

$$H_{s,\delta}(G) = \inf_{(C_i)\in C(G,\delta)} \sum_i |C_i|^s.$$

The Hausdorff outer measure is the measure defined by:

$$H_s(G) = \lim_{\delta\to 0} (H_{s,\delta}(G)).$$

The Hausdorff dimension of $G \subseteq X$ is defined as follows:

$$\dim_H(G) = \inf\{s : H_s(G) = 0\} = \sup\{s : H_s(G) = \infty\}.$$

2) **Correlation dimension**

Let's consider $O(x_1)$ as a trajectory of a dynamical system, where the initial condition is denoted by x_1. The correlation dimension of the set $O(x_1)$ is calculated as follows:
Given a positive real number r, we form the "correlation integral

$$C(r) = \lim_{n\to\infty} \frac{1}{n^2 - n} \sum_{i\neq j}^{n} H(r - \|x_i - x_j\|),$$

where:

$$H(x) = \begin{cases} 0 & x < 0, \\ 1 & x \geqslant 0, \end{cases}$$

is the unit-step function.

The summation counts how many pairs of vectors are closer than r when l<i, j<n, and $i \neq j$, while $n^2 - n$ is the total number of pairs with $i \neq j$. Hence, the ration between the two represents the fraction of pairs that are closer than r, and $C(r)$ measures the density of pairs of distinct vectors x_i and x_j that are closer than r [30].

Definition 3.4.2 *[30]*
The correlation dimension D_C of $O(x_1)$ is defined as:

$$D_C = \lim_{r \to 0} \frac{\ln C(r)}{\ln r}.$$

Example 3.4.1 *Consider the dynamical system governed by the function*

$$F(x) = \begin{cases} x^2 & if -3 \leqslant x \leqslant 1. \\ 4\sqrt{x} - 3 & if 1 < x \leqslant 9. \end{cases}$$

in the interval $[-3, 9]$. Suppose that we start from the point $x_0 = -2$. We have

$$x_1 = 4, x_2 = 5, x_3 = 4\sqrt{5} - 3, \cdots .$$

And the sequence $\{x_n\}$ is increasing and converges to 9. For every $r > 0$ there is an index $n(r)$ such that all elements of the sequence with index larger than $n(r)$ are closer than r. This implies that $C(r) = 1$ and $D_C = 0$.

3) **Capacity dimension:**

The concept of the capacity dimension works like this: imagine you have a bounded subset A of $\mathbb{R}^n$. Take a positive number ϵ and pick hypercubes (line segments in $\mathbb{R}$, squares in $\mathbb{R}^2$, cubes in $\mathbb{R}^3$, and so on) with side lengths of ϵ. Define $N(\epsilon)$ as the smallest number of these hypercubes needed to cover A.

$$D_C = \lim_{\epsilon \to 0} \frac{N(\epsilon)}{-\ln(\epsilon)}.$$

D_C is called the capacity dimension of the set A [5].

4) **Kaplan and Yorke dimension (Lyapunov)** :
Let $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$, the n Lyapunov exponents of an attractor of a dynamical system, and let j be the large natural number such that: $\lambda_1 + \lambda_2 + \cdots + \lambda_j \geqslant 0$. Then the Karlan and Yorke (Lyapunov) dimension is given by [5]:

$$D_{KY} = j + \frac{\sum_{i=1}^{j} \lambda_i}{|\lambda_{j+1}|}.$$

3.5 Scenarios of transition to chaos

The study of the transition to chaos involves examining the series of bifurcations in a system's dynamics as its parameters change, particularly focusing on dissipative phenomena and the shift towards dissipative chaos. While the chaotic behavior itself is not the primary focus, understanding its relationship to the transition is crucial, and this section explores three scenarios of transitioning from regular dynamics to chaotic dynamics when a parameter is varied.
Three common paths have been identified for dissipative systems, each linked to a specific type of bifurcation: the period-doubling route associated with the flip bifurcation or period doubling, the intermittency route related to the fold or saddle node bifurcation, and the Ruelle-Takens route linked to the Neimark-Sacker bifurcation [18].

- **Period-doubling cascade**
  This passage discusses the process by which a dynamical system transitions from an equilibrium state (a stable fixed point) to chaos through a series of period-doubling bifurcations. As the bifurcation increases, the stable fixed point is replaced by a stable 2-cycle at $\mu = \mu_0$ parameter μ (the fixed point still exists for $\mu > \mu_0$ but becomes unstable). Subsequently, at $\mu = \mu_1$, this 2-cycle loses its stability and is replaced by a stable 4-cycle. This pattern continues: the periodic orbit of period 2^{j-1} present for $\mu < \mu_j$ loses its stability at $\mu = \mu_j$ and is replaced by a stable periodic orbit of period 2^j. The sequence $(\mu_j)_{j\geqslant 0}$ of bifurcation values converges to a limit μ_C. At $\mu = \mu_C$, the system enters a chaotic regime. The

bifurcation values $(\mu_j)_{j\geqslant 0}$ are unique to the system, as is their limit value μ_C. However, the accumulation of these values at μ_C follows a geometric progression:

$$\lim_{j\to+\infty} \frac{\mu_{j+1}-\mu_j}{\mu_{j+2}-\mu_{j+1}} = \delta.$$

where δ is a universal number [5] : $\delta = 4.66920...$. This means that the exponent δ is identical in all systems where such a sequence of period-doubling leading to chaos is observed: a qualitative similarity between asymptotic behaviors implies a quantitative identity .

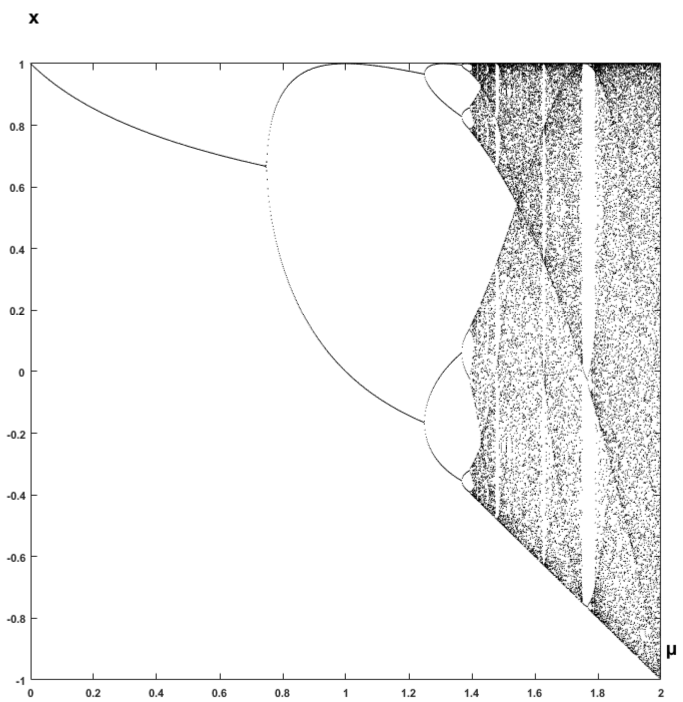


Figure 3.1: Bifurcation diagram of the logistic map $f(x) = 1 - \mu x^2$. The parameter μ is put in abscissa and the attractor along the other axis. This plot clearly displays an accumulation of period doublings leading to chaos in $\mu_c = 1.4011550...$

- **By intermittency**

  The route named intermittency describes the persistence of regular and predictable phases in a globally chaotic dynamics. The key idea is that after the disappearance of a stable fixed point x^*_μ through a saddle-node bifurcation in

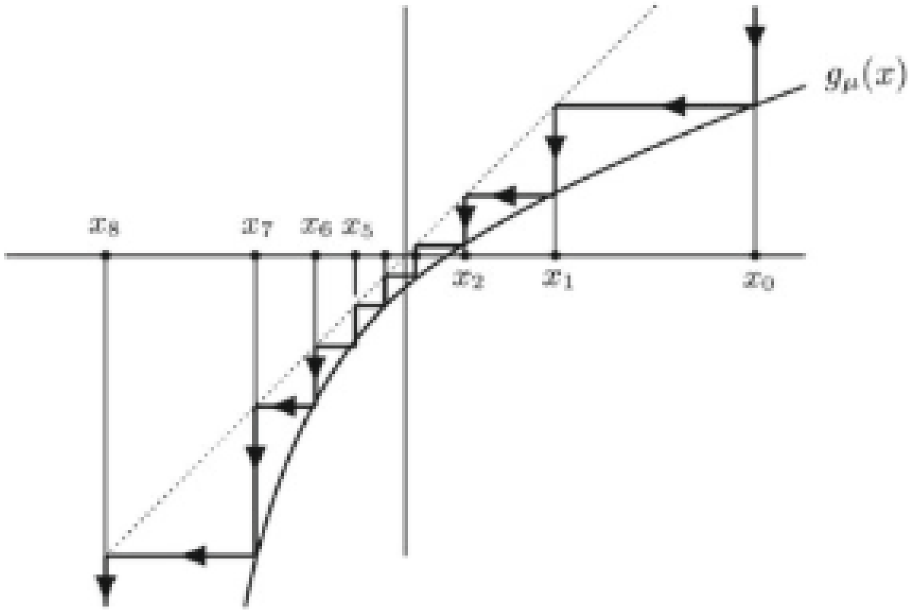


Figure 3.2: Temporal intermittency. This sketch explains the slow and regular regime displayed by the discrete evolution $x_{n+1} = g_\mu(x_n)$ if μ is slightly larger than the value μ_0 associated with the bifurcation $g'_{\mu_0}(0) = +1$.

$\mu = 0$, the dynamics remains slow in the neighborhood of x_0^*, as if it were experiencing the presence of a ghost fixed point. The typical example (actually, the normal form) is the discrete evolution:

$$x_{(n+1)} = g_\mu(x_n) = -\mu + x_n - Ax_n^2.$$

In $\mu = 0$, the fixed points $\pm\sqrt{-\mu/A}$ observed for $\mu < 0$ (respectively stable and unstable) merge in $x_0^* = 0$ and for $\mu > 0$, there is no longer fixed points. However, $g_m u(x) \approx x$ in the neighborhood of 0, so that the trajectory loiters a long time in this region and a regular and slow regime is observed, that roughly follows the evolution law $x_{(n+1)} = x_n - \mu$ as long as $Ax_n << 1$ and $A^2 x_n << 1$.

- **Ruelle and Takens' scenario**
  The exact statement of this route is quite technical because it requires introducing a topology on the space of vector fields in order to define the proximity of two continuous dynamical systems. An approximate formulation is as follows: a continuous dynamical system undergoing three successive Hopf bifurcations

generally has a strange attractor. Each Hopf bifurcation corresponds to the appearance of an unstable mode. The statement above can be reformulated as follows: the loss of stability of 3 modes with frequencies whose pairwise ratios are irrational leads to chaos. This result has profoundly modified scientists' understanding of chaos: the prevailing view before, due to Landau, required the loss of stability of an infinity of modes for the evolution to become apparently erratic and unpredictable. Consequently, it was believed that chaos would only occur in systems with an infinity of degrees of freedom. Landau's scenario was found to be much too restrictive: the nonlinear coupling of three modes with pairwise irrational frequency ratios is sufficient to generate a strange attractor [5].

CHAPTER 4

CONTROL OF CHAOTIC SYSTEMS

A chaotic attractor contains an infinite number of unstable periodic orbits as it evolves over time. The system will visit a small neighborhood of each point on these periodic orbits (which are unstable) within the attractor. This suggests that we can describe chaotic dynamics as a sequence of irregular jumps from one periodic orbit to another, leading to the concept of chaos control.

Chaos control is the stabilization of one of these unstable periodic orbits through small perturbations in the system, which makes chaotic motion more stable and predictable. The disturbance should be small compared to the overall size of the system's attractor to avoid large alterations in the system's natural dynamics. Several techniques have been devised for chaos control, and we will focus on two types: **The Ott, Grebogie and Yorke (OGY) method** and the **Feedback** method.
We consider the discrete model:

$$\begin{cases} x_{k+1} &= F_d(x_k, u_k), \\ y_k &= h(k). \end{cases} \tag{4.1}$$

Where $x_k \in \mathbb{R}^n$, $u_k \in \mathbb{R}^m$ and $y_k \in \mathbb{R}^l$ are the state vector, input vector, and output vector at the k-th iteration, respectively. The control of a chaotic system aims to stabilize an unstable periodic orbit. Let $\hat{x}_k$ be the periodic solution of period T of the system (4.1) , with the initial condition $\hat{x}_0$. If the solution x_k is unstable, stability can be understood in a certain sense, for example:

$$\lim_{k\to\infty}(x_k - \hat{x}_k) = 0, \tag{4.2}$$

or

$$\lim_{k\to\infty}(y_k - \hat{y}_k) = 0, \tag{4.3}$$

for any solution x_k of equation (4.1) with $x_0 \in \mathbb{R}^n$ as the initial condition, and Ω as a set of any initial conditions. Additionally, $\hat{y}_k$ is the desired output function. The problem is to find a control function for both forms: non-feedback control:

$$u_k = u(k, x_0),$$

or in feedback form:

$$u_k = u(x_k),$$

which verifies equation (4.2) or equation (4.3) [45].

4.1 Chaos control methods

4.1.1 The OGY method

As previously mentioned, OGY is one of the control methods used in dynamic systems. This method relies on a fundamental concept: that within a chaotic attractor, there are numerous unstable periodic orbits. By perturbing a specific parameter, it becomes possible to access and stabilize one or more of these orbits. How is this achieved ? [45]. In the case of a continuous-time dynamical system, it must first be converted into a discrete-time system using Poincare maps or other methods [45].
n-dimensional system can be written in the form :

$$x_{i+1} = f(x_i, p). \tag{4.4}$$

Where $x_i \in \mathbb{R}^n$: n dimensional state at i.
p : the accessible parameter .
$\bar{p}$: nominal value.
Since the perturbation of p is assumed to be small, the value of p is restricted

$$|p - \bar{p}| < \delta.$$

In discrete-time maps with $p = \bar{p}$, the fixed point satisfies $x^*_{i+1} = x^*_i$. Generally, $x^*_{i+T} = x^*_i$. system (4.4) can be linearized around x^*:

$$x_{i+1} - x^* \approx A(x_i - x^*) + B(p - \bar{p}),$$

where A is the Jacobian matrix and B represents the influence of the (input) p (for $j, k = 1, 2, \cdots, n$) [45]:

$$A = \frac{\partial f}{\partial x}(x^*, \bar{p}) = D_x f(x^*, \bar{p}), \tag{4.5}$$

$$B = \frac{\partial f}{\partial p}(x^*, \bar{p}) = D_p f(x^*, \bar{p}). \tag{4.6}$$

Let x^* be a given hyperbolic fixed point. The linearization of the system (4.4) around this fixed point is given by:

$$\Delta x(i + 1) = A\Delta x(i) + B(p - \bar{p}),$$

where:

$$\Delta x(i + 1) = x(i + 1) - x_F(p),$$

and

$$A = \lambda_u e_u f_u + \lambda_s e_s f_s.$$

If small changes are made to the parameter p, then the coordinate of the fixed point is also shifted to a nearby point $x_F(\bar{p})$. Around this point, we can write the following approximations:

$$\begin{aligned} x_F(\bar{p}) &= x_F(p) + (\bar{p} - p_0)(\frac{\partial x_F}{\partial p})_{p=p_0} \\ &= x_F(p) + \Delta p(i)B. \end{aligned}$$

where $\Delta p(i) = (\bar{p} - p_0)$ and $B = (\frac{\partial x_F}{\partial p})_{p=p_0}$. The expression for $\Delta x(i+1)$ can then be rewritten as:

$$\Delta x(i+1) = \Delta p(i)B + A[\Delta x(i) - \Delta p(i)B].$$

If we want the imposed variation to correspond to the unstable fixed point, meaning that the system trajectory follows the stable direction and that:

$$f_u \Delta x(i+1) = 0,$$

then:

$$\Delta p(i) = \frac{\lambda_u}{\lambda_u - 1} \frac{f_u}{f_u B} \Delta x(i) = K \Delta x(i).$$

This parametric variation is activated only when $x(i)$ is located within an interval $|\Delta x(i)| < \Delta p_{max}$ [25].

Example 4.1.1 ***Control of the Henon map using the OGY method***

The Henon map is described by the following equations:

$$\begin{cases} x_{n+1} = 1 - a x_n^2 + y_n, \\ y_{n+1} = b x_n, \end{cases} \tag{4.7}$$

where a and b represent the control parameters.

- ***Stability and chaos:***

  To determine the fixed points, we set $x_{n+1} = x_n$ and $y_{n+1} = y_n$, yielding:

  $$\begin{cases} x &= 1 - a x^2 + y, \\ y &= b x. \end{cases}$$

  This means:

  $$x_f, y_f = -\frac{(1-b)}{2} \pm \sqrt{\frac{(1-b)^2}{4} + a}. \tag{4.8}$$

  Setting: $c = \frac{1-b}{2}$.
  We obtain: $x_f, y_f = -c \pm \sqrt{c^2 + a}$.

- ***Application of the control algorithm:***

  The control algorithm is applied to the system with chaotic parameter values $a = 1.4$ and

$b = 0.3$.

Control using the OGY method consists of the following operations:

a- Identification of the fixed point to be stabilized:

Substituting a and b in equation (4.8) yields:

$x_{f1}, y_{f1} = 0.8839$.

$x_{f2}, y_{f2} = -1.5839$.

In our case, we choose the point $x_{f1} = 0.8839$.

b- Calculation of the matrices A and B:

We have $A = D_x F(x^*, \bar{p})$ *and* $B = \dfrac{\partial F}{\partial p}(x^*, \bar{p})$.

$$A = \begin{bmatrix} -2x_{f1} & b \\ 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

So:

$$A = \begin{bmatrix} -1.7678 & 0.3 \\ 1 & 0 \end{bmatrix}.$$

c- Calculation of the eigenvalues λ_u and λ_s:

λ_u *and* λ_s *are defined by*

$$\lambda_{s,u} = -x_{f1} \pm \sqrt{x_{f1}^2 + b}.$$

Thus:

$$\lambda_s = 0.1559 \quad \text{and} \quad \lambda_u = -1,9237.$$

d- Calculation of the eigenvectors $\{v_s, v_u\}$ and the covariance vectors $\{f_s, f_u\}$:

The eigenvectors are calculated using the following equation:

$$[\lambda I - A]\, e = 0.$$

The eigenvector are chosen in the form:

$$e = \begin{bmatrix} \lambda \\ 1 \end{bmatrix} \text{With } e_s = \begin{bmatrix} \lambda_s \\ 1 \end{bmatrix} \text{and } e_u = \begin{bmatrix} \lambda_u \\ 1 \end{bmatrix}.$$

So:

$$e_s = \begin{bmatrix} 0.1559 \\ 1 \end{bmatrix}, \; e_u = \begin{bmatrix} -1.9237 \\ 1 \end{bmatrix}.$$

Knowing that :$f_s e_s = f_u e_u = 1$ *and* $f_s e_u = f_u e_s = 0$. *Which give :*

$$f_s = \begin{bmatrix} \frac{1}{\lambda_s - \lambda_u} & \frac{\lambda_u}{\lambda_s - \lambda_u} \end{bmatrix} \text{ and } f_u = \begin{bmatrix} \frac{1}{\lambda_u - \lambda_s} & \frac{\lambda_s}{\lambda_u - \lambda_s} \end{bmatrix}.$$

$$f_s = \begin{bmatrix} 0.4808 & 0.9250 \end{bmatrix} \text{ and } f_u = \begin{bmatrix} -0.4787 & 0.0746 \end{bmatrix}.$$

e- *Calculation of* k:

The parameter k is represented by:

$$k = \frac{\lambda_u f_u}{f_u B} = \frac{\lambda_u \begin{bmatrix} \frac{1}{\lambda_u - \lambda_s} & \frac{\lambda_s}{\lambda_s - \lambda_u} \end{bmatrix}}{\begin{bmatrix} \frac{1}{\lambda_u - \lambda_s} & \frac{\lambda_s}{\lambda_s - \lambda_u} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}} = \begin{bmatrix} \lambda_u & -\lambda_u \lambda_s \end{bmatrix},$$

$$k = \begin{bmatrix} -1.9237 & 0.3011 \end{bmatrix}.$$

We choose $\delta = 0.01$.

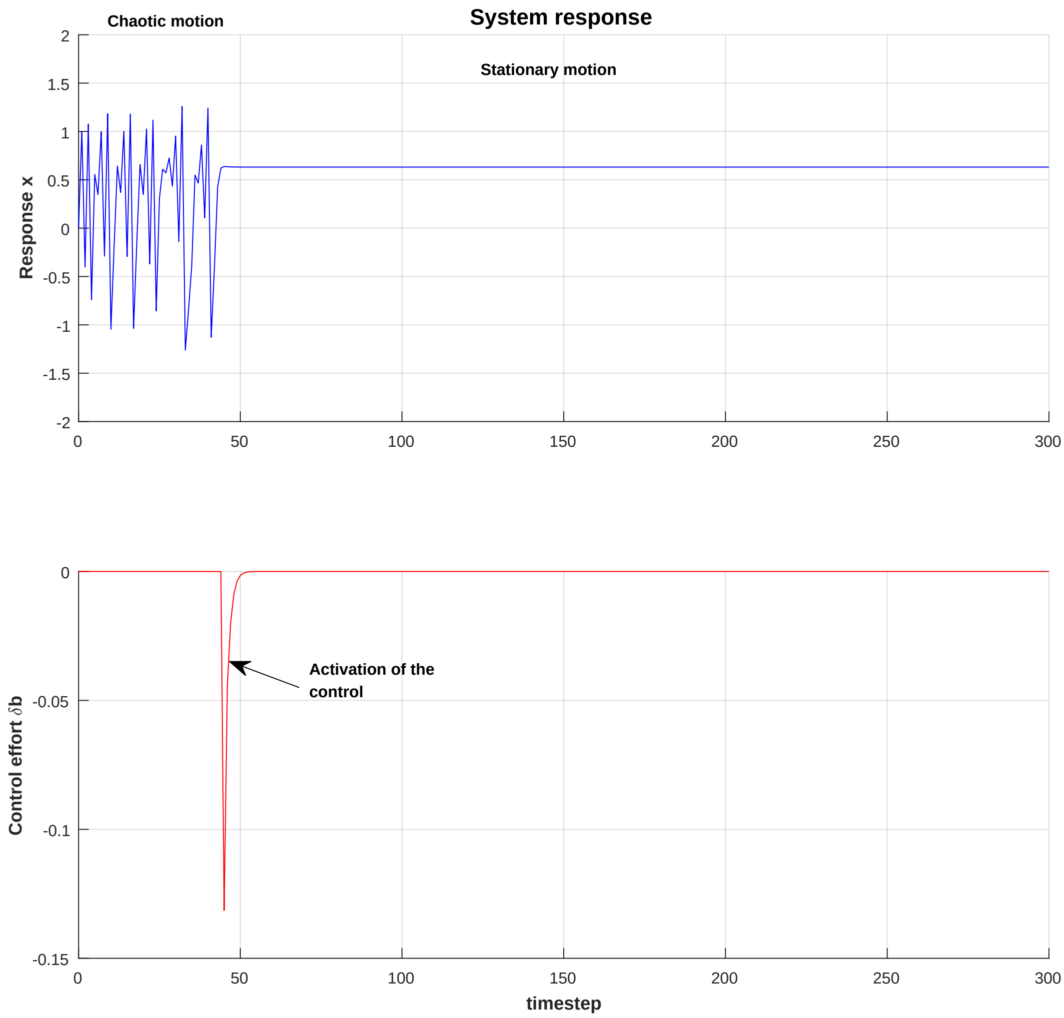


Figure 4.1: Control of the Henon system by the OGY method

4.1.2 The closed-loop control method (Feedback)

This method consists of perturbing the systems state variables to reach the target orbit. It has the advantage of guaranteeing robust stability and a strong noise rejection capability, let:

$$x_{n+1} = f(x_n, u_n). \tag{4.9}$$

where $f : \mathbb{R}^k \times \mathbb{R}^m \to \mathbb{R}^k$. The objective is to find a feedback control

$$u_n = h(x_n). \tag{4.10}$$

in such a way that the equilibrium point $x^* = 0$ of the closed-loop system

$$x_{n+1} = f(x_n, h(x_n)), \tag{4.11}$$

is asymptotically stable (locally). We make the following assumptions:

- $f(0, 0) = 0$,
- f is continuously differentiable, $A = \frac{\partial f}{\partial x}(0, 0)$ is a $k \times k$ matrix, $B = \frac{\partial f}{\partial u}(0, 0)$ is a $k \times m$ matrix.

Under the above conditions, we have the following surprising result.

Theorem 4.1.1
If the pair $\{A, B\}$ is controllable, then the nonlinear system (4.9) is stabilizable. Moreover, if K is the gain matrix for the pair $\{A, B\}$, then the control $u_n = -Kx_n$ may be used to stabilize system (4.9).

Proof.
Since the pair $\{A, B\}$ is controllable, there exists a feedback control $u(n) = -Kx(n)$ that stabilizes the linear part of the system, namely:

$$y_{n+1} = Ay(n) + Bu(n).$$

We are going to use the same control on the nonlinear system. So let $g : \mathbb{R}^k \to \mathbb{R}^k$ be a function defined by $g(x) = f(x, -Kx)$. Then system equation (4.9) becomes:

$$x_{n+1} = g(x_n). \tag{4.12}$$

With:

$$\frac{\partial g}{\partial x}|_{x=0} = A - Bk.$$

Since by assumption the zero solution of the linearized system:

$$y_{n+1} = (A - Bk)y_n.$$

is asymptotically stable, it follows by Theorem (Lyapunov stability theorem) that the zero solution of system (4.12) is also locally asymptotically stable. This completes the proof of the theorem. ∎

CHAPTER 5

NON LINEAR DYNAMICS AND CHAOS CONTROL OF A DISCRETE ROSENZEIG-MACARTHUR PREY-PREDATOR MODEL

The Rosenzweig-MacArthur model has been powerful in describing and predicting various phenomena in ecological systems of predator-prey interactions (e.g. [32]). In the model, the prey is a biotic or abiotic factor that promotes growth of its predator, while the predator utilizes the prey and reduces its growth. Thus the prey is beneficial to its predator without any harmful effect. In natural environment, the prey does not always have only positive effects on its predator. Hence the Rosenzweig-MacArthur model needs to be extended to characterize the interactions in which the prey has both positive and negative effects on its predator. Indeed, this type of predator-prey interactions has been displayed in real situations for years.

In plant-animal systems, the plant (prey) may have non-trophic, negative effect on its predator. As shown by Stamp [36], some plants carry specific chemicals that are toxic to herbivores. When the plants are at low density, the herbivores may have strategies to avoid ingestion of the toxins and the mortality rate is small. However, when the plants are at high density, both of the ingestion and mortality rates increase. Thus the plants have a non-trophic, harmful effect on the herbivores, while they are prey of the herbivores [42]. The Rosenzeig-MacArthur model is a system of two differantial equations used in population dynamics to modelise the predator-prey relationship [27].

In this chapter we will apply the hydra effect to the Rosenzweig system, in which the effect element is of the Holling type II, shows us two types of bifurcation: Flip and Neimark-Sacker, and we use the center manifold in this study. We will also draw region of stability, bifurcation diagram,lyapunov exponent and phase portrait moreover we will apply the feedback and OGY controls [42, 21, 22].

5.1 Model formulation

To develop the continuous-time Rosenzweig-Macarthur predator-prey model, one should follow this general framework [24]:

$$\begin{cases} \dot{x} &= rx(1-\frac{x}{k}) - h(x)y, \\ \dot{y} &= y(dh(x)-e). \end{cases} \tag{5.1}$$

where $rx(1-\frac{x}{k})$ is the logistic growth function, parameters k, r, d, e are all positive and h(x) denote the per predator kill rate with the following mathematical form (type II functional response) [24]:

$$h(x) = \frac{sx}{1+s\tau x}, \tag{5.2}$$

where $\frac{1}{\tau}$ is the maximum kill rate.
With substituting (5.2) in (5.1) we find:

$$\begin{cases} \dot{x} &= rx(1-\frac{x}{k}) - \frac{sxy}{1+s\tau x}, \\ \dot{y} &= y(d\frac{sx}{1+s\tau x}-e). \end{cases} \tag{5.3}$$

We put :

$$\begin{cases} u &= \frac{x}{X} \quad so \quad \dot{u} = \frac{\dot{x}}{X}, \\ v &= \frac{y}{Y} \quad so \quad \dot{v} = \frac{\dot{y}}{Y}. \end{cases}$$

The model (5.3) becomes:

$$\begin{cases} \dot{u} &= ru(1-\frac{uX}{k}) - \frac{suYv}{1+s\tau uX}, \\ \dot{v} &= v(\frac{suX}{1+s\tau uX}-e). \end{cases} \tag{5.4}$$

By denoting $c = \dfrac{d}{\tau}$, $b = \dfrac{1}{\tau}$, $H = \dfrac{1}{s\tau X}$, $Y = X$, $e = m$ and $K = \dfrac{k}{X}$, then model (5.4) becomes:

$$\begin{cases} \dot{u} &= ru(1-\frac{u}{K}) - \frac{buv}{u+H}, \\ \dot{v} &= v(-m+\frac{cu}{u+H}). \end{cases} \tag{5.5}$$

Finally, in the original variables, the model (5.5) takes the following from [24, 40]:

$$\begin{cases} \dot{x} &= rx(1-\frac{x}{K}) - \frac{bxy}{x+H}, \\ \dot{y} &= y(-m+\frac{cx}{x+H}). \end{cases} \tag{5.6}$$

Using Euler's method, setting $\Delta t = h$ we find:

$$\begin{cases} x_{t+1} &= x_t + hf_1(x_t), \\ y_{t+1} &= y_t + hf_2(y_t), \end{cases} \quad \textit{where} \quad \begin{cases} f_1(x_t) &= rx_t(1 - \frac{x_t}{K}) - \frac{bx_t y_t}{x_t+H}, \\ f_2(y_t) &= y_t(-m + \frac{cx_t}{x_t+H}). \end{cases}$$

The discrete version of model (5.6) is :

$$\begin{cases} x_{t+1} &= x_t(1 + hr) - \frac{hrx_t^2}{K} - \frac{hbx_t y_t}{x_t+H}, \\ y_{t+1} &= y_t + hy_t(-m + \frac{cx_t}{x_t+H}). \end{cases} \tag{5.7}$$

5.2 Dynamical analysis

We explore local dynamical analysis of discrete model (5.7) in the present section. For study local dynamics, first we study the existence of equilibrium solutions of discrete model (5.21) in $\mathbb{R}^2 = \{(x, y) : x, y \geq 0\}$.

5.2.1 Existence of equilibrium solutions:

Lemma 5.2.1

(i) Forall b, c, m, r, H, h, K discrete model (5.7) has trivial and semitrivial equilibrium solutions $P_1 = (0, 0)$ and $P_2 = (K, 0)$, respectively.

(ii) If $c > \max\left\{m, \frac{m(H+K)}{K}\right\}$ then discrete model (5.7) has positive equilibrium solution $P_3 = \left(\frac{Hm}{c-m}, \frac{cHr(cK-Hm-Km)}{bK(c-m)^2}\right)$.

Proof.

The study state satisfy: $\begin{cases} x_{t+1} &= x \\ y_{t+1} &= y \end{cases}$ thus: $\begin{cases} x &= x(1 + hr) - \frac{hr}{K}x^2 - \frac{hbxy}{x+H} \\ y &= y(1 - hm) + \frac{hcxy}{x+H} \end{cases}$ then:

$$\begin{cases} hrx - \frac{hr}{K}x^2 - \frac{hbxy}{x+H} = 0, \\ \quad -hmy + \frac{hcxy}{x+H} = 0, \end{cases}$$

it follows:

$$\left(hr - \frac{hr}{K}x - \frac{hby}{x+H}\right)x = 0. \tag{5.8}$$

$$\left(-hm + \frac{hcx}{x+H}\right)y = 0. \tag{5.9}$$

After the calculations we find:

$$x_1 = 0, y_1 = 0.$$

$$x_2 = K, y_2 = 0.$$

$$x_3 = \frac{mH}{c-m}, y_3 = \frac{cHr(K(c-m) - Hm)}{bK(c-m)^2}.$$

Now the Jacobian matrix of the discrete model (5.7) at an equilibrium solution $P = (x, y)$ under the map (5.7) is:

$$J\mid_P = \begin{pmatrix} 1 + hr - \frac{2hrx}{K} & -\frac{bhx}{x+H} \\ \frac{cHhy}{(x+H)^2} & 1 - hm + \frac{chx}{x+H} \end{pmatrix}. \tag{5.10}$$

■

5.2.2 Stability of fixed points

Stability of P_1:

Theorem 5.2.1

The fixed point P_1 of model (5.7) will never be a sink and it will be :

(i) a source if $m > \frac{2}{h}$;

(ii) a saddle if $0 < m < \frac{2}{h}$;

(iii) non-hyperbolic if $m = \frac{2}{h}$;

Proof.

Around P_1, (5.10) becomes:

$$J|_{P_1} = \begin{pmatrix} 1 + hr & 0 \\ 0 & 1 - hr \end{pmatrix}.$$

With eigenvalues:

$$\lambda_1 = 1 + hr, \quad \lambda_2 = 1 - hr. \tag{5.11}$$

$| \lambda_1 |=| 1 + hr |= 1 + hr > 1.$

and

$| \lambda_2 |=| 1 - hm |< 1,$

$-1 < 1 - hm < 1,$

$-2 < -hm < 0,$

$\frac{2}{h} > m > 0.$

■ By stability theory, P_1 of discrete model (5.7) is a sink if $| \lambda_{1,2} |< 1$ but $| \lambda_1 |> 1$, and so for all allowed parametric values b, c, K, H, m, r, h, P_1 is never sink. In similar way, one can obtain that P_1 is a source if $m > \frac{2}{h}$, sadlle if $0 < m < \frac{2}{h}$ and non-hyperbolic if $m = \dfrac{2}{h}$.

Stability of P_2:

Theorem 5.2.2

P_2 of model (5.7) is:

(i) a sink if $\frac{-2H+hmH}{2-hm+ch} < K < \frac{hmH}{cH-hm}$ and $0 < r < \frac{2}{h}$;

(ii) a source if $K < \frac{-2H+hmH}{2-hm+ch}$ and $r > \frac{2}{h}$;

(iii) a saddle if $K < \frac{-2H+hmH}{2-hm+ch}$ and $0 < r < \frac{2}{h}$;

(iv) non-hyperbolic if $K = \frac{-2H+hmH}{2-hm+ch}$ or $r = \frac{2}{h}$.

Proof.

Around P_2, (5.10) becomes:

$$J\,|_{P_3} = \begin{pmatrix} 1 - hr & \frac{-bhK}{K+H} \\ 0 & 1 - hm + \frac{chK}{K+H} \end{pmatrix}.$$

$$\lambda_1 = 1 - hr, \lambda_2 = 1 - hm + \frac{chK}{K+H}.$$

- $|\lambda_1| = |1 - hr| < 1$
  $-1 < 1 - hr < 1$
  $-2 < -hr < 0$
  $\frac{2}{h} > r > 0$

- $|\lambda_2| = |1 - hm + \frac{chK}{K+H}| < 1$
  $-1 < 1 - hm + \frac{chK}{K+H} < 1$
  $-2 < -hm + \frac{chK}{K+H} < 0$
  $-2 < -hm + \frac{chK}{K+H}$
  $-2 < \frac{-hmK-hmH+chK}{K+H}$
  $-2K - 2H < -hmK - hmH + chK$
  $-2H < -hmK - hmH + chK + 2K$
  $-2H + hmH < (-hm + ch + 2)K$
  $\frac{-2H+hmH}{2-hm+ch} < K$ ....(1)
  $-hm + \frac{chK}{K+H} < 0$
  $-hm + \frac{chK}{K+H} < 0$
  $\frac{-hmK-hmH+chK}{K+H} < 0$
  $-hmK - hmH + chK < 0$
  $(-hm + ch)K < hmH$
  $K < \frac{hmH}{ch-hm}$ ....(2)

from (1) and (2) $\frac{-2H+hmH}{2-hm+ch} < K < \frac{hmH}{ch-hm}$ ■
By linear stability theory, if $|\lambda_1| = |1 - hr| < 1$ and $|\lambda_2| = |1 - hm + \frac{chK}{H+K}| < 1$, i.e, $0 < r < \frac{2}{h}$ and $\frac{-2H+hmH}{2-hm+ch} < K < \frac{hmH}{cH-hm}$ then P_2 of discrete model (5.7) is a sink. Similarly, its easy to prove that P_2 of discrete model (5.7) is a source if $K < \frac{-2H+hmH}{2-hm+ch}$ and $r > \frac{2}{h}$ saddle $K < \frac{-2H+hmH}{2-hm+ch}$ or $0 < r < \frac{2}{h}$ and non-hyperbolic if $K = \frac{-2H+hmH}{2-hm+ch}$ or $r = \frac{2}{h}$.

Stability of P_3:

Theorem 5.2.3

If $\Delta < 0$ then P_3 is:

(i) a stable focus if $0 < K < \frac{H(hcm+c+m-hm^2)}{(c-m)(1+ch-hm)}$ *with* $c > \max\left\{m, \frac{hm-1}{h}, \frac{hm^2-m}{hm+1}\right\}$;

(ii) an unstable focus if $K > \frac{H(hcm+c+m-hm^2)}{(c-m)(1+ch-hm)}$*;*

(iii) non-hyperbolic if $K = \frac{H(hcm+c+m-hm^2)}{(c-m)(1+ch-hm)}$*;*

Proof.

Further at P_3 , (5.10) becomes:

$$J|_{P_3} = \begin{pmatrix} 1 - \frac{hmr(c(H-K)+m(H+K))}{cK(c-m)} & -\frac{bhm}{c} \\ \frac{hr(K(c-m)-Hm)}{bK} & 1 \end{pmatrix}. \tag{5.12}$$

With characteristic equation is:

$$\lambda^2 - T\lambda + D = 0. \tag{5.13}$$

Where:

$$\begin{cases} T &= 2 - \frac{hmr(c(H-K)+m(H+K))}{cK(c-m)}, \\ D &= 1 - \frac{hmr(c(H-K)+m(H+K))}{cK(c-m)} + \frac{h^2mr(cK-m(H+K))}{cK}. \end{cases}$$

Finally, the roots of (5.13) are:

$$\lambda_{1,2} = \frac{T \pm \sqrt{\Delta}}{2},$$

where:

$$\begin{cases} \Delta &= T^2 - 4D, \\ &= \left(2 - \frac{hmr(c(H-K)+m(H+K))}{cK(c-m)}\right)^2 - 4\left(1 - \frac{hmr(c(H-K)+m(H+K))}{cK(c-m)} + \frac{h^2mr(cK-m(H+K))}{cK}\right). \end{cases}$$

If $\Delta < 0$ then the characteristics roots of $J|_{P_3}$ at P_3 are:

$$\lambda_{1,2} = 2 - \frac{hmr(c(H-K)+m(H+K))}{cK(c-m)} \pm \frac{\iota}{2}\sqrt{\left(4\left(\frac{hmr(c(H-K)+m(H+K))}{cK(c-m)} + \frac{h^2mr(cK-m(H+K))}{cK}\right) - \left(\frac{hmr(c(H-K)+m(H+K))}{cK(c-m)}\right)^2\right)}.$$

Which give that if:

$|\lambda_{1,2}| = \sqrt{D} = \sqrt{1 - \frac{hmr(c(H-K)+m(H+K))}{cK(c-m)} + \frac{h^2mr(cK-m(H+K))}{cK}} < 1.$

then:

$$\frac{-hmrcH + hmrcK - hm^2rH - hm^2rK + h^2mrc^2K - 2h^2m^2rcH - 2h^2m^2rcK + h^2m^3rH - h^2m^3rK}{cK(c-m)} < 0$$

$-hmrcH+hmrcK-hm^2rH-hm^2rK+h^2mrc^2K-2h^2m^2rcH-2h^2m^2rcK+h^2m^3rH-h^2m^3rK<0$
$hmrcK-hm^2rK+h^2mrc^2K-2h^2m^2rcK-h^2m^3rK<hmrcH+hm^2rH+h^2m^2rcH-h^2m^3rH$

$$K<\frac{hmrcH+hm^2rH+h^2m^2rcH-h^2m^3rH}{hmrc-hm^2r+h^2mrc^2-2h^2m^2rc-h^2m^3r}$$
$$K<\frac{hmrH(c+m+hmc-hm^2)}{hmr(c-m+hc^2-2hmc-hm^2)}$$

So we find:
$0<K<\frac{H(c+m+hmc-hm^2)}{(c-m)(1+ch-hm)}$
• So P_3 is stable focus if $0<K<\frac{H(c+m+hmc-hm^2)}{(c-m)(1+ch-hm)}$, unstable focus if $K>\frac{H(c+m+hmc-hm^2)}{(c-m)(1+ch-hm)}$ and non-hyperbolic if $K=\frac{H(c+m+hmc-hm^2)}{(c-m)(1+ch-hm)}$ ■

Theorem 5.2.4
If $\Delta>0$ then P_3 is :

(i) A stable node if $\frac{mH}{c-m}<K<\frac{hHmr(-2c-2m-hmc+hm^2)}{(h^2m^2r-4c-h^2mrc-2hmr)(c-m)}$ with $c<min\left\{\frac{h^2m^2r-2hmr}{4+h^2mr},\frac{hm^2-2m}{mh+2}\right\}$ and $c>m$.

(ii) An unstable node if $\frac{hHmr(-2c-2m-hmc+hm^2)}{(h^2m^2r-4c-h^2mrc-2hmr)(c-m)}<K<\frac{mH}{c-m}$.

(iii) Non-hyperbolic if $K=\frac{mH}{c-m}$ or $K=\frac{hHmr(-2c-2m-hmc+hm^2)}{(h^2m^2r-4c-h^2mrc-2hmr)(c-m)}$.

Proof.
$\Delta>0$ i.e $|tr(J)|-1<det(J)$

<u>**case a**</u> $det(J)<tr(J)-1$
$1-\frac{hmr(c(H-K)+m(H+K))}{cK(c-m)}+\frac{h^2mr(cK-m(H+K))}{cK}<2-\frac{hmr(c(H-K)+m(H+K))}{cK(c-m)}-1$
$\frac{h^2mr(cK-m(H+K))}{cK}<0$
$h^2mrcK-h^2m^2rH-h^2m^2rK<0$
$h^2m^2rH<K(h^2mrc-h^2m^2r)$
$\frac{h^2m^2rH}{h^2mrc-h^2m^2r}<K$
$\frac{h^2mr(mH)}{h^2mr(c-m)}<K$
$\frac{mH}{c-m}<K$

case b $-tr(J) - 1 < det(J)$

$-2 + \frac{hmr(c(H-K)+m(H+K))}{cK(c-m)} - 1 < 1 - \frac{hmr(c(H-K)+m(H+K))}{cK(c-m)} + \frac{h^2mr(cK-m(H+K))}{cK}$

$\frac{-2hmrcH+2hmrcK-2hm^2rH-2hm^2rK+h^2mrc^2K-h^2m^2rcH-h^2m^2rcK-h^2m^2rcK+h^2m^3rH+h^2m^3rK}{cK(c-m)} > -4$

$$\frac{-2hmrcH - 2hm^2rH - h^2m^2rcH + h^2m^3rH}{-4c^2 + 4cm - 2hmrc + 2hm^2r - h^2mrc^2 + 2h^2m^2rc - h^2m^3r} > K$$

$$\frac{hHmr(-2c - 2m - hmc + hm^2)}{(h^2m^2r - 4c - h^2mrc - 2hmr)(c - m)} > K$$

■ So, P_3 is a stable node if $|\lambda_{1,2}| < 1$, i.e, $\frac{mH}{c-m} < K < \frac{hHmr(-2c-2m-hmc+hm^2)}{(h^2m^2r-4c-h^2mrc-2hmr)(c-m)}$. Moreover, simple manipulation also shows that P_3 is unstable node if $\frac{hHmr(-2c-2m-hmc+hm^2)}{(h^2m^2r-4c-h^2mrc-2hmr)(c-m)} < K < \frac{mH}{c-m}$ and non-hyperbolic if $K = \frac{hHmr(-2c-2m-hmc+hm^2)}{(h^2m^2r-4c-h^2mrc-2hmr)(c-m)}$ or $K = \frac{mH}{c-m}$.

5.2.3 Bifurcation analysis

Bifurcation at P_1:

The model (5.7) does undergo flip bifurcation at P_1:
if non hyperbolic condition $m = \frac{2}{h}$ holds then from (5.11) one has $\lambda_2|_{m=\frac{2}{h}} = -1$ but $\lambda_1|_{m=\frac{2}{h}} = 1 + hr \neq \pm 1$. This implies that at P_1 the under study model (5.7) may undergoes a flip bifurcation when (b, c, m, r, H, h, K) passes through the region:

$$F|_{P_1} = \left\{(b, c, K, m, r, H, h) \in \mathbb{R}^*_+, m = \frac{2}{h}\right\}.$$

Proof.

$$g(x, y) = y + hy\left(-m + \frac{cx}{x + H}\right).$$

We check the flip conditions at the first fixed point P_1, $m = \frac{2}{h}$:

- $\frac{\partial g}{\partial y} = 1 + h(-m + \frac{cx}{x+H})$, $\frac{\partial g}{\partial y}|_{(\frac{2}{h},0)} = -1$.
- $\alpha = \frac{\partial^2 g}{\partial m \partial y} + \frac{1}{2}\frac{\partial g}{\partial m}\frac{\partial^2 g}{\partial y^2}|_{(\frac{2}{h},0)} = -h \neq 0$.
- $\beta = \frac{1}{3!}\frac{\partial^3 g}{\partial y^3} + \frac{1}{2!}\left(\frac{\partial g}{\partial y}\right)^2|_{(\frac{2}{h},0)} = \frac{1}{2} \neq 0$.

So we have a flip bifurcation in $m = \frac{2}{h}$ if $(b, c, K, m, r, H, h) \in F|_{P_1}$. ■

Bifurcation at P_2:

- If $r = \frac{2}{h}$ and $hm \neq \frac{chK}{H+K}$, we get $\lambda_1|_{r=\frac{2}{h}} = -1$ but $\lambda_2|_{r=\frac{2}{h}} = 1 - hm + \frac{chK}{H+K} \neq \pm 1$ this implies that if (b, c, K, m, r, H, h) goes through the curve :

$$F_1|_{P_2} = \left\{ (b, c, K, m, r, H, h), r = \frac{2}{h}, hm \neq \frac{chK}{H+K} \right\}.$$

then at P_2 the model (5.7) may undergoes a flip bifurcation.

Theorem 5.2.5

If $(b, c, K, m, r, H, h) \in F_1|_{P_2}$ then at P_2 the under study model (5.7) undergoes a flip bifurcation.

Proof.

$$f(x, y) = x + hrx - \frac{hrx^2}{K} - \frac{hbxy}{x+H}.$$

We check the flip conditions at the second fixed point P_2, $r = \frac{2}{h}$:

- $\frac{\partial f}{\partial x} = 1 + hr - 2\frac{hr}{K}x - \frac{hbHy}{(x+H)^2}$, $\frac{\partial f}{\partial x} |_{(\frac{2}{h}, K)} = -1.$
- $\alpha = \frac{\partial^2 f}{\partial r \partial x} + \frac{1}{2}\frac{\partial f}{\partial r}\frac{\partial^2 f}{\partial x^2} |_{(\frac{2}{h}, K)} = -h \neq 0.$
- $\beta = \frac{1}{3!}\frac{\partial^3 f}{\partial x^3} + \frac{1}{2!}\left(\frac{\partial f}{\partial x}\right)^2 |_{(\frac{2}{h}, K)} = \frac{1}{2} \neq 0.$

So we have a flip bifurcation in $r = \frac{2}{h}$ if $(b, c, K, m, r, H, h) \in F_1|_{P_2}$. ■

- If $K = \frac{-2H+hmH}{2-hm+ch}$ so we have $\lambda_1|_{K=\frac{-2H+hmH}{2-hm+ch}} = 1 - hr \neq \pm 1$ but $\lambda_2|_{K=\frac{-2H+hmH}{2-hm+ch}} = -1$,this implies that if (b, c, m, r, H, h, K) goes through the curve:

$$F_2 |_{P_2} = \left\{ (b, c, m, r, H, h, K), K = \frac{-2H + hmH}{2 - hm + ch} \right\}.$$

then at P_2 the model (5.7) may under goes a flip bifurcation .

Theorem 5.2.6

If $(b, c, K, m, r, H, h) \in F_2 |_{P_2}$ then at P_2 no flip bifurcation occurs.

Proof.
$$\frac{\partial f}{\partial x} = 1 + hr - 2\frac{hr}{K}x - \frac{hbHy}{(x+H)^2}, \quad \frac{\partial f}{\partial x}|_{(k=\frac{-2H+hmH}{2-hm+ch},K)} = -1. \tag{5.14}$$
$$\alpha \neq 0. \tag{5.15}$$

and
$$\beta = 0. \tag{5.16}$$

■ The obtained condition (5.16) violates the non-degenerate condition for its existence, and hence at P_2 no flip bifurcation occurs if $(b, c, K, m, r, H, h) \in F|_{P_2}$.

Bifurcation at P_3:

► If $K = \frac{H(hcm+c+m-hm^2)}{(c-m)(1+ch-hm)}$ (non-hyperbolic condition) $|\lambda_{1,2}|_{K=\frac{H(hcm+c+m-hm^2)}{(c-m)(1+ch-hm)}} = 1$. This implies that if (b, c, K, m, r, H, h) passes through the following indicated curve then at P_3 model (5.7) may undergoes a Neimark-Sacker bifurcation:
$$N|_{P_3} = \left\{(b, c, m, r, H, h, K), K = \frac{H(hcm + c + m - hm^2)}{(c-m)(1+ch-hm)}\right\}.$$

Theorem 5.2.7
If $(b, c, m, r, H, h, K) \in N\ |_{P_3}$ then at P_3 the model (5.7) undergoes a Neimark-Sacker bifurcation by considering K as a bifurcation parameter.

Proof.
If K varies in a small neighborhood of K^* that is $K = K^* + \epsilon$ where $\epsilon << 1$ then the discrete model (5.7) can be written as:
$$\begin{cases} x_{t+1} &= x_t(1+hr) - \frac{hrx_t^2}{K^*+\epsilon} - \frac{hbx_ty_t}{x_t+H}, \\ y_{t+1} &= y_t + hy_t(-m + \frac{cx_t}{x_t+H}). \end{cases} \tag{5.17}$$

Now the pair of complex characteristics roots of $J\ |_{P_3}$ of the discrete model (5.17) is:
$$\lambda_{1,2} = \frac{T(\epsilon) \pm \iota\sqrt{4D(\epsilon) - T^2(\epsilon))}}{2}. \tag{5.18}$$

Where:

$$\begin{cases} T(\epsilon) & = 2 - \frac{hmr(c(H-(K^*+\epsilon))+m(H+(K^*+\epsilon)))}{cK(c-m)}, \\ D(\epsilon) & = 1 - \frac{hmr(c(H-(K^*+\epsilon))+m(H+(K^*+\epsilon)))}{c(K^*+\epsilon)(c-m)} + \frac{h^2mr(c(K^*+\epsilon)-m(H+(K^*+\epsilon)))}{c(K^*+\epsilon)}. \end{cases} \tag{5.19}$$

From (5.18) and (5.19), we get:

$$\left.\frac{d|\lambda_{1,2}|}{d\epsilon}\right|_{\epsilon=0} = \frac{hmr(c-m)(1+ch-hm)^2}{2cH(c+m+chm-hm^2)}. \tag{5.20}$$

$$\begin{cases} u_t & = x_t - x^*, \\ v_t & = v_t - y^*. \end{cases} \tag{5.21}$$

with $x^* = \frac{mH}{c-m}$ $y^* = \frac{cHr(cK-Hm-Km)}{bK(c-m)^2}$:

$$\begin{cases} u_{t+1} & = x_{t+1} - x^* = (1+hr)(u_t + x^*) - \frac{hr}{K^*+\epsilon}(u_t+x^*)^2 - \frac{hb(u_t+x^*)(v_t+y^*)}{u_t+x^*+H} - x^*, \\ v_{t+1} & = y_{t+1} - y^* = (1-hm)(v_t+y^*) + \frac{hc(u_t+x^*)(v_t+y^*)}{u_t+x^*+H} - y^*. \end{cases}$$

$$\begin{cases} u_{t+1} & = u_t + hru_t + hrx^* - \frac{hr}{K^*+\epsilon}u_t^2 - \frac{hr}{K^*+\epsilon}x^{*2} - \frac{2hrx^*}{K^*+\epsilon}u_t - \frac{hb}{u_t+x^*+H}[u_tv_t + y^*u_t + x^*v_t + x^*y^*], \\ v_{t+1} & = v_t - hmv_t - hmy^* + \frac{hc}{u_t+x^*+H}[u_tv_t + y^*u_t + x^*v_t + x^*y^*]. \end{cases}$$

Let $L(u,v) = \begin{pmatrix} L_1(u,v) \\ L_2(u,v) \end{pmatrix}$.

Where:

$$L_1(u,v) = \frac{hb}{u_t + x^* + H}[u_tv_t + y^*u_t + x^*v_t + x^*y^*],$$
$$L_2(u,v) = \frac{hc}{u_t + x^* + H}[u_tv_t + y^*u_t + x^*v_t + x^*y^*].$$

We limited to the Taylor development with two variables u and v :

$L(u,v) = L(0,0) + \frac{\partial L}{\partial u}(0,0)u + \frac{\partial L}{\partial v}(0,0)v + \frac{1}{2}\left(\frac{\partial^2 L}{\partial u^2}(0,0)u^2 + 2\frac{\partial^2 L}{\partial u \partial v}(0,0)uv + \frac{\partial^2 L}{\partial v^2}(0,0)v^2\right).$

We have:

$$\begin{cases} L_1(0,0) & = \frac{hbx^*y^*}{x^*+H}, \\ \frac{\partial L_1}{\partial u}(u,v) & = \frac{hbHv_t+hbHy^*}{(u_t+x^*+H)^2}, \quad \frac{\partial L_1}{\partial u}(0,0) = \frac{hbHy^*}{(x^*+H)^2}, \\ \frac{\partial L_1}{\partial v}(u,v) & = \frac{hbu_t}{u_t+x^*+H} + \frac{hbx^*}{u_t+x^*+H}, \quad \frac{\partial L_1}{\partial v}(0,0) = \frac{hbx^*}{x^*+H}, \\ \frac{\partial^2 L_1}{\partial u^2}(u,v) & = \frac{-2hbHy^*}{(u_t+x^*+H)^3}, \quad \frac{\partial^2 L_1}{\partial u^2}(0,0) = \frac{-2hbHy^*}{(x^*+H)^3}, \\ \frac{\partial L_1}{\partial u \partial v}(u,v) & = \frac{hbH}{(u_t+x^*+H)^2}, \quad \frac{\partial L_1}{\partial u \partial v}(0,0) = \frac{hbH}{(x^*+H)^2}, \\ \frac{\partial^2 L_1}{\partial v^2}(u,v) & = 0. \end{cases}$$

Then:

$$\begin{cases} u_{t+1} & \approx u_t + hru_t + hrx^* - \frac{hr}{K^*+\epsilon}u_t^2 - \frac{hr}{K^*+\epsilon}x^{*2} - \frac{2hrx^*}{K^*}u_t - \frac{hb}{x^*+H}[x^*y^* + x^*v_t] - \frac{hb}{(x^*+H)^2}[Hy^*u_t + Hu_tv_t] \\ & + \frac{hbHy^*}{(x^*+H)^3}u_t^2, \\ v_{t+1} & \approx v_t - hmv_t - hmy^* + \frac{hcx^*y^*}{x^*+H} + \frac{hcHy^*}{(x^*+H)^2}u_t + \frac{hcx^*}{x^*+H}v_t - \frac{hcHy^*}{(x^*+H)^3}u_t^2 + \frac{hcH}{(x^*+H)^2}u_tv_t. \end{cases} \tag{5.22}$$

That we can be written as:

$$\begin{cases} u_{t+1} & = \beta_{11}u_t + \beta_{12}v_t + \beta_{13}u_t^2 + \beta_{14}u_tv_t, \\ v_{t+1} & = \beta_{21}u_t + \beta_{22}v_t + \beta_{23}u_t^2 + \beta_{24}u_tv_t. \end{cases}$$

With:

$$\begin{cases} \beta_{11} & = 1 + hr - \frac{2hrx^*}{K} - \frac{bHhy^*}{(x+H)^2}, \\ \beta_{12} & = -\frac{bhx^*}{x^*+H}, \\ \beta_{13} & = -\frac{hr}{K} + \frac{bhHy^*}{(x^*+H)^3}, \\ \beta_{14} & = -\frac{bHh}{(x^*+H)^2}, \\ \beta_{21} & = \frac{cHhy^*}{(x^*+H)^2}, \\ \beta_{22} & = 1 - hm + \frac{chx^*}{x^*+H}, \\ \beta_{23} & = -\frac{cHhy^*}{(x^*+H)^3}, \\ \beta_{24} & = \frac{cHh}{(x^*+H)^2}. \end{cases} \tag{5.23}$$

Using the transformation:

$$\begin{cases} u_t & = \beta_{12}x_t, \\ v_t & = (\eta - \beta_{11})x_t - \zeta y_t, \end{cases}$$

which can be written in the vectorial form :

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = T \begin{pmatrix} x_t \\ y_t \end{pmatrix}.$$

Where: $T = \begin{pmatrix} \beta_{12} & 0 \\ \eta - \beta_{11} & -\zeta \end{pmatrix}$.

$T^{-1} = \frac{1}{det(T)}(Co(T))^t = \frac{1}{-\beta_{12}\zeta}\begin{pmatrix} -\zeta & -\eta + \beta_{11} \\ 0 & \beta_{12} \end{pmatrix}^t = -\frac{1}{\beta_{12}\zeta}\begin{pmatrix} -\zeta & 0 \\ -\eta + \beta_{11} & \beta_{12} \end{pmatrix} = \begin{pmatrix} \frac{1}{\beta_{12}} & 0 \\ \frac{\eta-\beta_{11}}{\beta_{12}\zeta} & -\frac{1}{\zeta} \end{pmatrix}$.

So:

$$\begin{pmatrix} x_t \\ y_t \end{pmatrix} = T^{-1} \begin{pmatrix} u_t \\ v_t \end{pmatrix}.$$

Then:

$$\begin{cases} x_{t+1} & = \frac{1}{\beta_{12}} u_{t+1}, \\ y_{t+1} & = \frac{\eta-\beta_{11}}{\beta_{12}\zeta} u_{t+1} - \frac{1}{\zeta} v_{t+1}. \end{cases}$$

$$\begin{aligned}
x_{t+1} &= \frac{1}{\beta_{12}} (\beta_{11} u_t + \beta_{12} v_t + \beta_{13} u_t^2 + \beta_{14} u_t v_t), \\
x_{t+1} &= \frac{1}{\beta_{12}} [\beta_{11}\beta_{12} x_t + \beta_{12}((\eta - \beta_{11}) x_t - \zeta y_t) + \beta_{13}\beta_{12}^2 x_t^2 + \beta_{14}(\beta_{12} x_t)((\eta - \beta_{11}) x_t - \zeta y_t)] \\
&= \frac{1}{\beta_{12}} [\beta_{11}\beta_{12} x_t + \beta_{12}((\eta - \beta_{11}) x_t - \zeta y_t) + \beta_{13}\beta_{12}^2 x_t^2 + \beta_{14}(\beta_{12} x_t)((\eta - \beta_{11}) x_t - \zeta y_t)] \\
&= \beta_{11} x_t + (\eta - \beta_{11} x_t) - \zeta y_t + \beta_{13}\beta_{12} x_t^2 + \beta_{14} x_t((\eta - \beta_{11}) x_t - \zeta y_t), \\
x_{t+1} &= \eta x_t - \zeta y_t + \beta_{13}\beta_{12} x_t^2 + \beta_{14}(\eta - \beta_{11}) x_t^2 - \beta_{14}\zeta x_t y_t. \\
y_{t+1} &= \frac{\eta - \beta_{11}}{\beta_{12}\zeta} u_{t+1} - \frac{1}{\zeta} v_{t+1}, \\
y_{t+1} &= \frac{\eta - \beta_{11}}{\beta_{12}\zeta} (\beta_{11} u_t + \beta_{12} v_t + \beta_{13} u_t^2 + \beta_{14} u_t v_t) - \frac{1}{\zeta} (\beta_{21} u_t + \beta_{22} v_t + \beta_{23} u_t^2 + \beta_{24} u_t v_t), \\
y_{t+1} &= \frac{\eta - \beta_{11}}{\beta_{12}\zeta} (\beta_{12}\eta x_t - \beta_{12}\zeta y_t + \beta_{13}\beta_{12}^2 x_t^2 + \beta_{14}\beta_{12}\eta x_t^2 - \beta_{14}\beta_{11}\beta_{12} x_t^2 - \beta_{14}\beta_{12}\zeta x_t y_t) \\
&- \frac{1}{\zeta} (\beta_{21}\beta_{12} x_t + \beta_{22}\eta x_t - \beta_{22}\beta_{11} x_t - \beta_{22}\zeta y_t + \beta_{23}\beta_{12} x_t^2 + \beta_{24}\beta_{12}\eta x_t^2 - \beta_{21}\beta_{11}\beta_{12} x_t^2 - \beta_{24}\beta_{12}\zeta x_t y_t), \\
y_{t+1} &= \frac{(\eta - \beta_{11})\eta}{\zeta} x_t - (\eta - \beta_{11}) y_t + \frac{(\eta - \beta_{11})\eta}{\zeta} \beta_{13}\beta_{12} x_t^2 + \frac{(\eta - \beta_{11})\eta}{\zeta} \beta_{14}\eta x_t^2 - \frac{(\eta - \beta_{11})\eta}{\zeta} \beta_{14}\beta_{11} x_t^2 \\
&- (\eta - \beta_{11})\beta_{14} x_t y_t - \frac{\beta_{21}\beta_{12}}{\zeta} x_t - \frac{\beta_{22}}{\zeta} \eta x_t + \frac{\beta_{22}\beta_{11}}{\zeta} x_t + \beta_{22} y_t - \frac{\beta_{23}\beta_{12}^2}{\zeta} x_t^2 + \frac{\beta_{24}\beta_{12}}{\zeta} \eta x_t^2 \\
&+ \frac{\beta_{24}\beta_{11}\beta_{12}}{\zeta} x_t^2 + \beta_{24}\beta_{12} x_t y_t \\
y_{t+1} &= \frac{1}{\zeta} (\eta^2 - \beta_{11}\eta - \beta_{21}\beta_{12} - \beta_{22}\eta + \beta_{22}\beta_{11}) x_t + (-\eta + \beta_{11} + \beta_{22}) y_t \\
&+ \left[\frac{(\eta - \beta_{11})}{\zeta} \beta_{13}\beta_{12} + \frac{(\eta - \beta_{11})}{\zeta} \beta_{14}\eta - \frac{(\eta - \beta_{11})}{\zeta} \beta_{14}\beta_{11} - \frac{\beta_{23}\beta_{12}}{\zeta} + \frac{\beta_{24}\beta_{12}}{\zeta} \eta + \frac{\beta_{24}\beta_{11}\beta_{12}}{\eta} \right] x_t^2 \\
&+ (-(\eta - \beta_{11})\beta_{14} + \beta_{24}\beta_{12}) x_t y_t.
\end{aligned}$$

Furthermore, (5.22) becomes the following form:

$$\begin{cases} x_{t+1} &= \eta x_t - \zeta y_t + P(x_t, y_t), \\ y_{t+1} &= \zeta x_t + \eta y_t + Q(x_t, y_t). \end{cases} \tag{5.24}$$

With:

$$\begin{cases} \eta &= 1 - \frac{hmr(c(H-K)+m(H+K))}{2cK(c-m)}, \\ \zeta &= \frac{\sqrt{4\left(1-\frac{hmr(c(H-K)+m(H+K))}{cK(c-m)}+\frac{h^2mr(K(c-m)-Hm)}{cK}\right)-\left(2-\frac{hmr(c(H-K)+m(H+K))}{cK(c-m)}\right)^2}}{2}. \end{cases}$$

where:

$$\begin{cases} P(x_t, y_t) &= \sigma_{11}x_t^2 + \sigma_{12}x_t y_t, \\ Q(x_t, y_t) &= \sigma_{21}x_t^2 + \sigma_{22}x_t y_t, \end{cases} \tag{5.25}$$

$$\begin{cases} \sigma_{11} &= \beta_{12}\beta_{13} + \eta\beta_{14} - \beta_{11}\beta_{14}, \\ \sigma_{12} &= -\beta_{14}\zeta, \\ \sigma_{21} &= \frac{1}{\zeta}\left[\beta_{12}\beta_{13}(\eta-\beta_{11}) + \beta_{14}(\eta-\beta_{11})^2 - \beta_{12}\beta_{24}(\eta-\beta_{11}) - \beta_{23}\beta_{12}^2\right], \\ \sigma_{22} &= \beta_{12}\beta_{24} - \beta_{14}(\eta-\beta_{11}). \end{cases} \tag{5.26}$$

Now following quantity required to be non-zero in order to answer that (5.24) undergoes a Neimark-Sacker bifurcation:

$$\psi = -\Re\left(\frac{(1-2\bar{\lambda})\bar{\lambda}^2}{1-\lambda}\rho_{11}\rho_{20}\right) - \frac{1}{2}\|\rho_{11}\|^2 - \|\rho_{02}\|^2 + \Re(\bar{\lambda}\rho_{21}). \tag{5.27}$$

Where:

$$\begin{cases} \rho_{02} &= \frac{1}{8}\left(\frac{\partial^2 P}{\partial {x_t}^2} - \frac{\partial^2 P}{\partial {y_t}^2} + 2\frac{\partial^2 Q}{\partial x_t \partial y_t} + \iota\left(\frac{\partial^2 Q}{\partial x_t^2} - \frac{\partial^2 Q}{\partial y_t^2} + 2\frac{\partial^2 P}{\partial x_t \partial y_t}\right)\right)|_{(0,0)}, \\ \rho_{11} &= \frac{1}{4}\left(\frac{\partial^2 P}{\partial {x_t}^2} + \frac{\partial^2 P}{\partial {y_t}^2} + \iota\left(\frac{\partial^2 Q}{\partial x_t^2} + \frac{\partial^2 Q}{\partial y_t^2}\right)\right)|_{(0,0)}, \\ \rho_{20} &= \frac{1}{8}\left(\frac{\partial^2 P}{\partial {x_t}^2} - \frac{\partial^2 P}{\partial {y_t}^2} + 2\frac{\partial^2 Q}{\partial x_t \partial y_t} + \iota\left(\frac{\partial^2 Q}{\partial x_t^2} - \frac{\partial^2 Q}{\partial y_t^2} - 2\frac{\partial^2 P}{\partial x_t \partial y_t}\right)\right)|_{(0,0)}, \\ \rho_{21} &= \frac{1}{16}\left(\frac{\partial^3 P}{\partial x_t^3} + \frac{\partial^3 P}{\partial y_t^3} + \frac{\partial^3 Q}{\partial x_t^2 \partial y_t} + \frac{\partial^3 Q}{\partial y_t^3} + \iota\left(\frac{\partial^3 Q}{\partial x_t^3} + \frac{\partial^3 Q}{\partial x_t \partial y_t^2} - \frac{\partial^3 P}{\partial x_t^2 \partial y_t} - \frac{\partial^3 P}{\partial y_t^3}\right)\right)|_{(0,0)}. \end{cases}$$

From (5.25), we get :

$$\begin{array}{llllll}
\frac{\partial^2 P}{\partial x_t^2}\mid_{(0,0)} & = 2\sigma_{11}, & \frac{\partial^2 P}{\partial x_t \partial y_t}\mid_{(0,0)} & = \sigma_{12}, & \frac{\partial^2 P}{\partial y_t^2}\mid_{(0,0)} & = 0,\\
\frac{\partial^3 P}{\partial x_t^2 \partial y_t}\mid_{(0,0)} & = 0, & \frac{\partial^3 P}{\partial x_t \partial y_t^2}\mid_{(0,0)} & = 0, & \frac{\partial^3 P}{\partial y_t^3}\mid_{(0,0)} & = 0,\\
\frac{\partial^2 Q}{\partial x_t \partial y_t}\mid_{(0,0)} & = \sigma_{22}, & \frac{\partial^2 Q}{\partial y_t^2}\mid_{(0,0)} & = 0, & \frac{\partial^3 Q}{\partial x_t^3}\mid_{(0,0)} & = 0,\\
\frac{\partial^3 Q}{\partial x_t \partial y_t^2}\mid_{(0,0)} & = 0, & \frac{\partial^3 Q}{\partial y_t^3}\mid_{(0,0)} & = 0, & \frac{\partial^3 Q}{\partial x_t^2 \partial y_t}\mid_{(0,0)} & = 0.\\
\frac{\partial^3 P}{\partial x_t^3}\mid_{(0,0)} & = 0, & \frac{\partial^2 Q}{\partial x_t^2}\mid_{(0,0)} & = 2\sigma_{21}, & &
\end{array}$$

After manipulation we gets:

$$\begin{cases}
\rho_{02} & = \frac{1}{4}(\sigma_{11} + \sigma_{22} + \iota(\sigma_{21} + \sigma_{12})),\\
\rho_{11} & = \frac{1}{2}(\sigma_{11} + \iota\sigma_{21}),\\
\rho_{20} & = \frac{1}{4}(\sigma_{11} + \sigma_{22} + \iota(\sigma_{21} - \sigma_{12})),\\
\rho_{21} & = 0.
\end{cases} \tag{5.28}$$

Using (5.28) in (5.27) if one obtains $\psi \neq 0$ as $(b, c, m, r, H, h, K) \in N|_{P_3}$ then at P_3 the discrete model (5.7) undergoes Neimark-Sacker bifurcation. Additionally, for its existence, it is also necessary that $\lambda_{1,2}^m \neq 1, m = 1, ..., 4$ if $\epsilon = 0$ that corresponds to $T(0) \neq -2, 0, 1, 2$. But if $K = \frac{H(hcm+c+m-hm^2)}{(c-m)(1+ch-hm)}$ holds then from (5.19), we get $D(0) = 1$, and so $D(0) \neq -2, 2$. Therefore, we only require that $D(0) \neq 0, 1$, i.e.,

$$r \neq -\frac{2(c + m + chm - hm^2)}{h^2 m(m - c)}, -\frac{(c + m + chm - hm^2)}{h^2 m(m - c)}.$$

Moreover, at P_3 the existence of Neimark-Sacker bifurcation also classify to supercritical (resp.subcritical) Neimark-Sacker bifurcation if $\psi < 0$ (resp. $\psi > 0$). ∎

► If $K = \frac{mH}{c-m}$ (non-hyperbolic condition), we have: $\lambda_1 \mid_{K=\frac{mH}{c-m}} = 1$ but $\lambda_2 \mid_{K=\frac{mH}{c-m}} = 1 - hr \neq \pm 1$. This implies that if (b, c, m, r, H, h, K) passes through the following curve then at P_3 there may exists fold bifurcation:

$$F_3 \mid_{P_3} = \left\{(b, c, m, r, H, h, K), K = \frac{mH}{c - m}\right\}.$$

Theorem 5.2.8

If $(b, c, K, m, r, H, h) \in F_{|P_3}$ then at the P_3 discrete model (5.7) cannot undergoes the fold bifurcation.

Proof.

If K varies in a small neighborhood of K^* then (5.7) will be written in the form (5.17).

Let: $L_3 = \frac{hr}{K^*+\epsilon}$.

We limited the Taylor development of L_3 with one variable :

$L(\epsilon) = L(0) + L'(0)\epsilon$.

$$\begin{cases} L(0) &= \frac{hr}{K^*}, \\ L'(\epsilon) &= -\frac{hr}{(K^*+\epsilon)^2}, \end{cases} \quad , \qquad L'(0) = -\frac{hr}{K^{*2}}.$$

So : $L_3(\epsilon) = \frac{hr}{K^*} - \frac{hr}{K^{*2}}\epsilon$.

Then:

$$\begin{cases} u_{t+1} &= (1+hr)(u_t+x^*) - (\frac{hr}{K^*} - \frac{hr}{K^{*2}})(u_t+x^*)^2 - \frac{hb(u_t+x^*)(v_t+y^*)}{(u_t+x^*)+H} - x^*, \\ v_{t+1} &= (1-hm)(v_t+y^*) + \frac{hc(u_t+x^*)(v_t+y^*)}{(u_t+x^*)+H} - y^*. \end{cases}$$

Moreover, by (5.21), the discrete model (5.7) takes the form:

$$\begin{cases} u_{t+1} &= \beta_{11}u_t + \beta_{12}v + \beta_{13}u_t^2 + \beta_{14}u_tv_t + \delta_{01}u_t\epsilon + \delta_{02}u_t^2\epsilon, \\ v_{t+1} &= \beta_{21}u_t + \beta_{22}v_t + \beta_{23}u_t^2 + \beta_{24}u_tv_t. \end{cases} \tag{5.29}$$

With:

$$\begin{cases} \beta_{11} &= 1 + hr - \frac{2hrx^*}{K} - \frac{bHhy^*}{(x^*+H)^2}, \\ \beta_{12} &= -\frac{bhx^*}{x^*+H}, \\ \beta_{13} &= -\frac{hr}{K} + \frac{bhHy^*}{(x^*+H)^3}, \\ \beta_{14} &= -\frac{bhH}{(x^*+H)^2}, \\ \delta_{01} &= \frac{2hr}{K^{*2}}x^*, \\ \delta_{02} &= \frac{hr}{K^{*2}}, \\ \beta_{21} &= \frac{chHy^*}{(x^*+H)^2}, \\ \beta_{22} &= 1 - hm + \frac{chx^*}{x^*+H}, \\ \beta_{23} &= -\frac{chHy^*}{(x^*+H)^3}, \\ \beta_{24} &= \frac{chH}{(x^*+H)^2}, \end{cases} \tag{5.30}$$

Using the transformation:

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = T \begin{pmatrix} x_t \\ y_t \end{pmatrix} \quad , \quad T = \begin{pmatrix} 1 & 1 \\ -\frac{rc}{bm} & 0 \end{pmatrix}.$$

So:

$$\begin{pmatrix} x_t \\ y_t \end{pmatrix} = T^{-1} \begin{pmatrix} u_t \\ v_t \end{pmatrix} \quad , \quad T^{-1} = \begin{pmatrix} 0 & \frac{rc}{bm} \\ -1 & \frac{bm}{rc} \end{pmatrix}.$$

$$\begin{cases} x_{t+1} &= -\frac{bm}{rc} v_{t+1}, \\ y_{t+1} &= u_{t+1} + \frac{bm}{rc} v_{t+1}, \end{cases}$$

$$\begin{cases} x_{t+1} &= -\frac{bm}{rc}(\beta_{21}u_t + \beta_{21}v_t + \beta_{23}u_t^2 + \beta_{24}u_t v_t), \\ y_{t+1} &= \beta_{11}u_t + \beta_{12}v_t + \beta_{13}u_t^2 + \beta_{14}u_t v_t + \delta_{01}u_t\epsilon + \delta_{02}u_t^2\epsilon + \frac{bm}{rc}(\beta_{21}u_t + \beta_{22}v_t + \beta_{23}u_t^2 + \beta_{24}u_t v_t). \end{cases}$$

$$\begin{cases} x_{t+1} &= -\frac{bm}{rc}(\beta_{21}(x_t+y_t) + \beta_{22}(-\frac{rc}{bm}x_t) + \beta_{23}(x_t^2+y_t^2+2x_t y_t) + \beta_{24}(x_t+y_t)(-\frac{rc}{bm}x_t)), \\ y_{t+1} &= \beta_{11}(x_t+y_t) + \beta_{12}(-\frac{rc}{bm}x_t) + \beta_{13}(x_t^2+y_t^2+2x_t y_t) + \beta_{14}(x_t+y_t)(-\frac{rc}{bm}x_t) + \delta_{01}(x_t+y_t)\epsilon \\ & +\delta_{02}(x_t^2+y_t^2+2x_t y_t)\epsilon + \frac{bm}{rc}(\beta_{21}(x_t+y_t) + \beta_{22}(-\frac{rc}{bm}x_t) + \beta_{23}(x_t^2+y_t^2+2x_t y_t) \\ & +\beta_{24}(x_t+y_t)(-\frac{rc}{bm}x_t)). \end{cases}$$

System (5.29) becomes:

$$\begin{pmatrix} x_{t+1} \\ y_{t+1} \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} x_t \\ y_t \end{pmatrix} + \begin{pmatrix} P(x_t, y_t, \epsilon) \\ Q(x_t, y_t, \epsilon) \end{pmatrix} \quad / \quad A = 1, B = \lambda_2, \tag{5.31}$$

where:

$$\begin{cases} P(x_t, y_t, \epsilon) &= -\frac{bm}{rc}\beta_{23}(x_t^2+y_t^2+2x_t y_t) - \frac{bm}{rc}\beta_{24}(x_t+y_t)(-\frac{rc}{bm}x_t), \\ Q(x_t, y_t, \epsilon) &= (\beta_{13} + \frac{bm}{rc}\beta_{23})(x_t^2+y_t^2+2x_t y_t) + (\beta_{14} + \frac{bm}{rc}\beta_{24})(x_t+y_t)(-\frac{rc}{bm}x_t) + \delta_{01}(x_t+y_t)\epsilon \\ & +\delta_{02}(x_t^2+y_t^2+2x_t y_t)\epsilon. \end{cases}$$

It is recall here at $\epsilon = 0$, we can check the stability of $O(0,0)$ by the center manifold theory, according to the theorem of implicit functions there exists $F^C O$.
Where $F^C O = y$:
then:

$$M_c = \{(x, y, \epsilon) \in \mathbb{R}^2 : y = F^C O(x, \epsilon), |x| < \delta_1, |\epsilon| < \delta_2, F^C O(0,0) = DF^C O(0,0) = 0\}. \tag{5.32}$$

The function $F^C O$ must satisfy equation (5.32), we suppose that:

$$F^C O(x_t,\epsilon) = b_0\epsilon + b_1 x_t\epsilon + b_2\epsilon^2 + b_3 x_t^2 + O((|x_t| + |y_t|)^3).$$

By (5.32):

$$\begin{aligned}
F(F^C O(x_t,\epsilon)) &= F^C O[Ax_t + P(x, F^C O(x_t,\epsilon),\epsilon)] + BF^C O(x_t,\epsilon) - Q(x_t, F^C O(x_t,\epsilon),\epsilon),\\
F(F^C O(x_t,\epsilon)) &= F^C O\left[x_t - \frac{bm}{rc}\beta_{23}(x_t^2 + F^C O(x,\epsilon)^2 + 2x_t F^C O(x,\epsilon)) - \frac{bm}{rc}\beta_{24}(x_t + F^C O(x,\epsilon))(-\frac{rc}{bm}x_t)\right]\\
&- \lambda_2 F^C O(x_t,\epsilon) - (\beta_{13} + \frac{bm}{rc}\beta_{23})(x_t^2 + F^C O(x_t,\epsilon)^2 + 2x_t F^C O(x,\epsilon)) + (\beta_{14} + \frac{bm}{rc}\beta_{24})\\
&(x_t + F^C O(x,\epsilon))(-\frac{rc}{bm}x_t) + \delta_{01}(x_t + F^C O(x,\epsilon))\epsilon + \delta_{02}(x_t^2 + y_t^2 + 2x_t y_t)\epsilon,\\
F(F^C O(x_t,\epsilon)) &= b_0\epsilon + b_1\epsilon(x_t - \frac{bm}{rc}\beta_{23}x_t^2 - \frac{bm}{rc}\beta_{23}(b_0\epsilon + b_1 x_t\epsilon + b_2\epsilon^2 + b_3 x_t^2)^2 - 2\frac{bm}{rc}\beta_{23}x_t(b_0\epsilon + b_1 x_t\epsilon\\
&+ b_2\epsilon^2 + b_3 x_t^2)) + b_2\epsilon^2 + b_3(x_t - \frac{bm}{rc}\beta_{23}x^2 - \frac{bm}{rc}\beta_{23}(b_0\epsilon + b_1 x_t\epsilon + b_2\epsilon^2 + b_3 x_t^2)^2\\
&- 2\frac{bm}{rc}\beta_{23}x_t(b_0\epsilon + b_1 x_t\epsilon + b_2\epsilon^2 + b_3 x_t^2))^2 - \lambda_2(b_0\epsilon + b_1 x_t\epsilon + b_2\epsilon^2 + b_3 x_t^2) - (\beta_{13} + \frac{bm}{rc}\beta_{23})\\
&+ (b_0\epsilon + b_1 x_t\epsilon + b_2\epsilon^2 + b_3 x_t^2)^2 + 2x_t(b_0\epsilon + b_1 x_t\epsilon + b_2\epsilon^2 + b_3 x_t^2)) + (\beta_{14} + \frac{bm}{rc}\beta_{24})\\
&(x_t + (b_0\epsilon + b_1 x_t\epsilon + b_2\epsilon^2 + b_3 x_t^2))(-\frac{rc}{bm}x_t) + \delta_{01}(x_t + (b_0\epsilon + b_1 x_t\epsilon + b_2\epsilon^2 + b_3 x_t^2))\epsilon\\
&+ \delta_{02}(x_t^2 + y_t^2 + 2x_t(b_0\epsilon + b_1 x_t\epsilon + b_2\epsilon^2 + b_3 x_t^2))\epsilon,\\
F(F^C O(x_t,\epsilon)) &= b_0\epsilon + b_1 x_t\epsilon - 2\frac{bm}{rc}\beta_{23}b_0 x_t\epsilon + b_2\epsilon^2 + b_3 x_t^2 - \lambda_2 b_0\epsilon - \lambda_2 b_1 x_t\epsilon - \lambda_2 b_2\epsilon^2 - \lambda_2 b_3 x_t^2 - \beta_{13}x_t^2\\
&- \beta_{13}b_0^2\epsilon^2 - 2\beta_{13}b_0\epsilon x_t - \frac{bm}{rc}\beta_{23}x_t^2 - \frac{bm}{rc}\beta_{23}b_0^2\epsilon^2 - 2\frac{bm}{rc}\beta_{23}b_0 x_t\epsilon + \beta_{14}\frac{rc}{bm}x_t^2 + \beta_{14}\frac{rc}{bm}b_0 x_t\epsilon\\
&+ \beta_{24}x_t^2 + \beta_{24}b_0 x_t\epsilon - \delta_{01}x_t\epsilon - \delta_{01}b_0\epsilon^2,\\
F(F^C O(x_t,\epsilon)) &= (b_0 - \lambda_2 b_0)\epsilon + (b_1 - 2\frac{bm}{rc}\beta_{23}b_0 - \lambda_2 b_1 - 2\beta_{13}b_0 - 2\frac{bm}{rc}\beta_{23}b_0 + \beta_{14}\frac{rc}{bm}b_0 + \beta_{24}b_0 - \delta_{01})x\\
&+ (b_2 - \lambda_2 b_2 - \beta_{13}b_0^2 - \frac{bm}{rc}\beta_{23}b_0^2 - \delta_{01}b_0)\epsilon^2 + (b_3 - \lambda_2 b_3 - \beta_{13} - \frac{bm}{rc}\beta_{23} + \beta_{14}\frac{rc}{bm} + \beta_{24})x^2
\end{aligned}$$

So:

$$b_0 - \lambda_2 b_0 = 0 \Rightarrow b_0 = 0, b_2 = 0,$$

$$b_1 - \lambda_2 b_1 - \delta_{01} = 0 \Rightarrow b_1 = \frac{\delta_{01}}{1-\lambda_2},$$

$$b_3 - \lambda_2 b_3 - \beta_{13} - \frac{bm}{rc}\beta_{23} + \frac{rc}{bm}\beta_{14} + \beta_{24} = 0 \Rightarrow b_3 = \frac{\beta_{13} - \frac{rc}{bm}\beta_{14} + \frac{bm}{rc}\beta_{23} - \beta_{24}}{1-\lambda_2}.$$

Now, we write (5.31) restrict to F^cO as:

$$g_1(x_t,\epsilon) = x_t + d_1x_t^2 + d_2x_t\epsilon + d_3\epsilon^2 + d_4x_t^2\epsilon + d_5x_t\epsilon^2 + O((|x_t| + |\epsilon|)^3). \tag{5.33}$$

$$\begin{aligned}
x_{t+1} &= x_t + (-\frac{bm}{rc}\beta_{23}u_t^2 - \frac{bm}{rc}\beta_{24}u_tv_t),\\
&= x_t - \frac{bm}{rc}\beta_{23}x_t^2 - \frac{bm}{rc}\beta_{23}(b_1x_t\epsilon + b_3x_t^2)^2 - 2\frac{bm}{rc}\beta_{23}x_t(b_1x_t\epsilon + b_3x_t^2) + \beta_{24}x_t^2 + \beta_{24}x(b_1x_t\epsilon + b_3x_t^2),\\
&= x_t - \frac{bm}{rc}\beta_{23}x_t^2 - \frac{bm}{rc}\beta_{23}b_1^2x_t^2\epsilon^2 - \frac{bm}{rc}\beta_{23}b_3^2x_t^4 - 2\frac{bm}{rc}\beta_{23}b_1b_3x_t^2\epsilon + \beta_{24}x_t^2 + \beta_{24}b_1x_t^2\epsilon + \beta_{24}b_3x_t^3,\\
&= x_t + \left(-\frac{bm}{rc}\beta_{23} + \beta_{24}\right)x_t^2 + \beta_{24}b_3x_t^3 - \frac{bm}{rc}\beta_{23}b_1^2x_t^2\epsilon^2 + \left(\beta_{24}b_1 - 2\frac{bm}{rc}\beta_{23}b_1b_3\right)x_t^2\epsilon - \frac{bm}{rc}\beta_{23}b_t^2x_t^4.
\end{aligned}$$

Where:

$$d_1 = -\frac{bm}{rc}\beta_{23} + \beta_{24},$$

$$d_2 = d_3 = d_5 = 0,$$

$$d_4 = \beta_{24}b_1 - 2\frac{bm}{rc}\beta_{23}b_1b_3.$$

We obtain:

$$g_1(x_t,\epsilon) = x_t + d_1x_t^2 + d_4x_t^2\epsilon.$$

After that we discuss the conditions on fold bifurcation:

– $\frac{\partial g_1}{\partial \epsilon} \mid_{(O)} = 0.$

The first condition is not satisfaied; so if $(b,c,m,r,H,h,K) \in F_3 \mid P_3$, the fold bifurcation does not exist. ■

- If $K = \frac{hHmr(-2c-2m-hmc+hm^2)}{(h^2m^2r-4c-h^2mrc-2hmr)(c-m)}$ $\lambda_1 \mid_{K=\frac{hHmr(-2c-2m-hmc+hm^2)}{(h^2m^2r-4c-h^2mrc-2hmr)(c-m)}} = -1$,

$\lambda_2 \mid_{K=\frac{hHmr(-2c-2m-hmc+hm^2)}{(h^2m^2r-4c-h^2mrc-2hmr)(c-m)}} = \frac{c(-2+hm(-3+hr))-m(2+hm(-3+hr))}{m(-2+hm)-c(2+hm)} \neq \pm 1$ This implies that if (b,c,m,r,H,h,K) passes through the following curve then system(5.7) may undergoes a flip bifurcation at P_3:

$$F_4 \mid_{P_3} = \left\{(b,c,m,r,H,h,K), K = \frac{hHmr(-2c-2m-hmc+hm^2)}{(h^2m^2r-4c-h^2mrc-2hmr)(c-m)}\right\}.$$

Theorem 5.2.9

If $(b,c,m,r,H,h,K) \in F_4 \mid_{P_3}$ then at P_3 discrete model (5.7) undergoes a flip bifurcation.

Proof.

By (5.21), the system (5.29) takes the form:

$$\begin{pmatrix} x_{t+1} \\ y_{t+1} \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} x_t \\ y_t \end{pmatrix} + \begin{pmatrix} F(x_t, y_t, \epsilon) \\ G(x_t, y_t, \epsilon) \end{pmatrix}. \tag{5.34}$$

Where:

$$\begin{aligned}
F(u_t, v_t, \epsilon) &= \frac{-2m(-2+hm)+2c(2+hm)}{c(4+hm(4-hr))+m(4+hm(-4+hr))}(\beta_{13}u_t^2+\beta_{14}u_tv_t+\delta_{01}u_t\epsilon+\delta_{02}u_t^2\epsilon) \\
&\quad - \frac{bhm(m(2-hm)+c(2+hm))}{-4c(c+m+chm-hm^2)+ch^{(}c-m)mr}(\beta_{23}u_t^2+\beta_{24}u_tv_t), \\
G(u_t, v_t, \epsilon) &= \frac{h(c-m)m(-2+hr)}{c(-4+hm(-4+hr))-m(4+hm(-4+hr))}(\beta_{13}u_t^2+\beta_{14}u_tv_t+\delta_{01}u_t\epsilon+\delta_{02}u_t^2\epsilon) \\
&\quad + \frac{bhm(m(2-hm)+c(2+hm))}{-4c(c+m+chm-hm^2)+ch^{(}c-m)mr}(\beta_{23}u_t^2+\beta_{24}u_tv_t), \\
u_t &= x_t + y_t, \\
v_t &= \frac{c(2c-2m-chr+hmr)}{b(2c+2m+chm-hm^2)}x_t - \frac{2c}{bhm}y_t, \\
u_t^2 &= x_t^2 + y_t^2 + 2x_ty_t, \\
u_tv_t &= \frac{c(2c-2m-chr+hmr)}{b(2c+2m+chm-hm^2)}(x_t^2+x_ty_t) - \frac{2c}{bhm}(y_tx_t+y_t^2), \\
u_t\epsilon &= x_t\epsilon + y_t\epsilon, \quad , \quad u_t^2\epsilon = x_t^2\epsilon + y_t^2\epsilon + 2x_ty_t.
\end{aligned}$$

By:

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ A & B \end{pmatrix} \begin{pmatrix} x_t \\ y_t \end{pmatrix}, \quad A = \frac{c(2c-2m-chr+hmr)}{b(2c+2m+chm-hm^2)}, \quad B = -\frac{2c}{bhm}.$$

Now from (5.34), and by (5.32) the center manifold is:

$$M^cF_{00}(0,0) = \left\{(x_t, y_t) : y_t = C_0\epsilon + C_1x_t^2 + C_2x_t\epsilon + C_3\epsilon^3 + O\left((|x_t|+|\epsilon|)^3\right)\right\},$$

$$
\begin{aligned}
M^c(h(x,\epsilon)) &= h[Ax + F(x,h(x,\epsilon),\epsilon),\epsilon] - Bh(x,\epsilon) - G(x,h(x,\epsilon),\epsilon),\\
M^c(h(x,\epsilon)) &= h[-x + F(x,h(x,\epsilon),\epsilon),\epsilon] - \lambda_2 h(x,\epsilon) - G(x,h(x,\epsilon),\epsilon),\\
M^c(h(x,\epsilon)) &= h[-x + A(\beta_{13}x_t^2 + \beta_{14}Bx_t^2 + \beta_{13}y_t^2 + 2\beta_{13}x_ty_t + \beta_{14}Bx_ty_t - \frac{2c}{bhm}\beta_{14}x_ty_t - \frac{2c}{bhm}\beta_{14}y_t^2\\
&\quad + \delta_{01}x_t\epsilon + \delta_{01}y_t\epsilon + \delta_{02}x_t^2\epsilon + \delta_{02}y_t^2\epsilon + 2\delta_{02}x_ty_t\epsilon) - C(\beta_{23}(x_t^2 + y_t^2 + 2x_ty_t) + \beta_{24}B(x_t^2 + x_ty_t)\\
&\quad - \frac{2c}{bhm}\beta_{24}(x_ty_t + y_t^2))] - \lambda_2(C_0\epsilon + C_1x_t^2 + C_2x_t\epsilon + C_3\epsilon^3) - D(\beta_{13}(x_t^2 + y_t^2 + 2x_ty_t))\\
&\quad + \beta_{14}B(x_t^2 + x_ty_t) - \frac{2c}{bhm}\beta_{14}(x_ty_t + y_t^2) + \delta_{01}x_t\epsilon + \delta_{01}y_t\epsilon + \delta_{02}\epsilon(x_t^2 + y_t^2 + 2x_ty_t),\\
M^c(h(\epsilon,x)) &= (C_0 - \lambda_2C_0)\epsilon + (AC_1\beta_{13} + AC_1\beta_{14}B - CC_1\beta_{23} - CC_1\beta_{24} - \lambda_2C_1 - D\beta_{13} - D\beta_{14}B)x_t^2\\
&\quad + (2AC_1\beta_{14}C_0 + AC_1\beta_{14}C_0 - \frac{2CAC_1C_0}{bhm} + AC_1\delta_{01} - \lambda_2C_2C_0 - 2D\beta_{14}C_0 - D\beta_{14}BC_0\\
&\quad + \frac{2c}{bhm}D\beta_{14}C_0 - D\delta_{01} - 2CC_1\beta_{23}C_0 - CC_1C_0\beta_{24}B + \frac{2CC_1C_0\beta_{24}}{bhm} + CC_1C_2)x_t\epsilon\\
&\quad + (C_3 - \lambda_2C_3)\epsilon^3.
\end{aligned}
$$

Suppose that:

$$
C = \frac{bhm(m(2-hm) + c(2+hm))}{-4c(c+m+chm-hm^2) + ch^2(c-m)mr},
$$

$$
D = \frac{h(c-m)m(-2+hr)}{c(-4+hm(-4+hr)) - m(4+hm(-4+hr))}.
$$

Where:

$$
C_0 = C_3 = 0 \quad /C_1 = \frac{1}{1-\lambda_2}(D(\beta_{13} + A\beta_{14}) + C(\beta_{23} + A\beta_{24})) \quad /C_2 = \frac{1}{1-\lambda_2}(D\delta_{01}). \tag{5.35}
$$

Now, we write (5.34) as:

$$
g_1(x_t) = -x_t + h_1x_t^2 + h_2x_t\epsilon + h_3x_t^2\epsilon + h_4x_t\epsilon^2 + h_5x_t^3 + O\left((|x_t| + |\epsilon|)^4\right). \tag{5.36}
$$

$$
\begin{aligned}
x_{t+1} &= -x_t + F(u_t, v_t, \epsilon),\\
x_{t+1} &= -x_t + \frac{-2m(-2+hm)+2c(2+hm)}{c(4+hm(4-hr))+m(4+hm(-4+hr))}(\beta_{13}u_t^2 + \beta_{14}u_tv_t + \delta_{01}u_t\epsilon + \delta_{02}u_t^2\epsilon)\\
&\quad - \frac{bhm(m(2-hm)+c(2+hm))}{-4c(c+m+chm-hm^2)+hmrc(c-m)}(\beta_{23}u_t^2 + \beta_{24}u_tv_t),\\
x_{t+1} &= -x_t - \frac{-2m(-2+hm)+2c(2+hm)}{c(4+hm(4-hr))+m(4+hm(-4+hr))}(\beta_{13}(x_t^2 + y_t^2 + 2x_ty_t) + \beta_{14}\\
&\quad (\frac{c(2c-2m-chr+hmr)}{b(2c+2m+chm-hm^2)}(x_t^2 + x_ty_t) - \frac{2c}{bhm}(x_ty_t + y_t^2)) + \delta_{01}(x_t + y_t) + \delta_{02}(x_t^2 + y_t^2 + 2x_ty_t)\epsilon)\\
&\quad - \frac{bhm(m(2-hm)+c(2+hm))}{-4c(c+m+chm-hm^2)+ch^(c-m)mr}(\beta_{23}(x_t^2 + y_t^2 + 2x_ty_t) + \beta_{24}(\frac{c(2c-2m-chr+hmr)}{b(2c+2m+chm-hm^2)}\\
&\quad (x^2 + x_ty_t) - \frac{2c}{bhm}(x_ty_t + y_t^2)),\\
x_{t+1} &= -x_t - \frac{-2m(-2+hm)+2c(2+hm)}{c(4+hm(4-hr))+m(4+hm(-4+hr))}(\beta_{13}(x_t^2 + h^2(x,\epsilon) + 2x_th(x,\epsilon) + \beta_{14}\\
&\quad (\frac{c(2c-2m-chr+hmr)}{b(2c+2m+chm-hm^2)}(x_t^2 + x_th(x,\epsilon)) - \frac{2c}{bhm}(x_th(x,\epsilon) + h^2(x,\epsilon))) + \delta_{01}(x_t + h(x,\epsilon))\\
&\quad + \delta_{02}(x_t^2 + h^2(x,\epsilon) + 2x_th(x,\epsilon))\epsilon) - \frac{bhm(m(2-hm)+c(2+hm))}{-4c(c+m+chm-hm^2)+ch^(c-m)mr}\\
&\quad (\beta_{23}(x_t^2 + h^2(x,\epsilon) + 2x_th(x,\epsilon))) + \beta_{24}(\frac{c(2c-2m-chr+hmr)}{b(2c+2m+chm-hm^2)}(x^2 + x_th(x,\epsilon))\\
&\quad - \frac{2c}{bhm}(x_th(x,\epsilon) + h^2(x,\epsilon))),\\
x_{t+1} &= -x_t - \frac{-2m(-2+hm)+2c(2+hm)}{c(4+hm(4-hr))+m(4+hm(-4+hr))}(\beta_{13}(x_t^2 + (C_0\epsilon + C_1x_t^2 + C_2x_t\epsilon + C_3\epsilon^3)^2\\
&\quad + 2x_t(C_0\epsilon + C_1x_t^2 + C_2x_t\epsilon + C_3\epsilon^3) + \beta_{14}(\frac{c(2c-2m-chr+hmr)}{b(2c+2m+chm-hm^2)}(x_t^2 + x_t(C_0\epsilon + C_1x_t^2 + C_2x_t\epsilon\\
&\quad + C_3\epsilon^3)) - \frac{2c}{bhm}(x_t(C_0\epsilon + C_1x_t^2 + C_2x_t\epsilon + C_3\epsilon^3) + (C_0\epsilon + C_1x_t^2 + C_2x_t\epsilon + C_3\epsilon^3)^2)) + \delta_{01}(x_t\\
&\quad + (C_0\epsilon + C_1x_t^2 + C_2x_t\epsilon + C_3\epsilon^3)) + \delta_{02}(x_t^2 + (C_0\epsilon + C_1x_t^2 + C_2x_t\epsilon + C_3\epsilon^3)^2 + 2x_t(C_0\epsilon + C_1x_t^2
\end{aligned}
$$

$$
\begin{aligned}
& + C_2x_t\epsilon + C_3\epsilon^3))\epsilon) - \frac{bhm(m(2-hm)+c(2+hm))}{-4c(c+m+chm-hm^2)+ch^{(}c-m)mr}(\beta_{23}(x_t^2 + (C_0\epsilon + C_1x_t^2 + C_2x_t\epsilon \\
& + C_3\epsilon^3)^2 + 2x_t(C_0\epsilon + C_1x_t^2 + C_2x_t\epsilon + C_3\epsilon^3)) + \beta_{24}(\frac{c(2c-2m-chr+hmr)}{b(2c+2m+chm-hm^2)}(x^2 + x_t(C_0\epsilon + C_1x_t^2 \\
& + C_2x_t\epsilon + C_3\epsilon^3)) - \frac{2c}{bhm}(x_t(C_0\epsilon + C_1x_t^2 + C_2x_t\epsilon + C_3\epsilon^3) + (C_0\epsilon + C_1x_t^2 + C_2x_t\epsilon + C_3\epsilon^3)^2)), \\
x_{t+1} = & -x_t - \frac{-2m(-2+hm)+2c(2+hm)}{c(4+hm(4-hr))+m(4+hm(-4+hr))}(\beta_{13}((x_t^2 + C_1x_t^2 + 2x_t(C_1x_t^2 + C_2x_t\epsilon) \\
& + \beta_{14}(\frac{c(2c-2m-chr+hmr)}{b(2c+2m+chm-hm^2)}(x_t^2 + x_t(C_1x_t^2 + C_2x_t\epsilon)) - \frac{2c}{bhm}(x_t(c_1x_t^2 + C_2x_t\epsilon) \\
& + C_1^2x_t^2)) + \delta_{01}(x_t\epsilon + C_1x_t^2 + C_2x_t\epsilon)\epsilon + \delta_{02}(x_t^2 + c^21x_t^2 + 2x_t(C_1x_t^2 + C_2x_t\epsilon)\epsilon)))) \\
& - \frac{bhm(m(2-hm)+c(2+hm))}{-4c(c+m+chm-hm^2)+ch^{(}c-m)mr}(\beta_{23}(x_t^2 + C_1^2x_t^2 + 2x_t(C_1x_t^2 + C_2x_t\epsilon)) \\
& + \beta_{24}(\frac{c(2c-2m-chr+hmr)}{b(2c+2m+chm-hm^2)}(x_t^2 + x_t(C_1x_t^2 + C_2x_t\epsilon))) - \frac{2c}{bhm}(x_t(C_1x_t^2 + C_2x_t\epsilon) + C_1^2x_t^2)).
\end{aligned}
$$

So we find :

$$
\left\{
\begin{array}{lll}
h_1 & = & \left\{ \begin{array}{c} \frac{-2m(-2+hm)+2c(2+hm)}{c(4+hm(4-hr))+m(4+hm(-4+hr))} \times (\beta_{13} + \frac{c(2c-2m-chr+hmr)}{b(2c+2m+chm-hm^2)}\beta_{14}) \\ -\frac{bhm(m(2-hm)+c(2+hm))}{-4c(c+m+chm-hm^2)+ch^2(c-m)mr} \times (\beta_{23} + \frac{c(2c-2m-chr+hmr)}{b(2c+2m+chm-hm^2)}\beta_{24}) \end{array} \right\}, \\
h_2 & = & \frac{-2m(-2m+hm)+2c(2+hm)}{c(4+hm(4-hr))+m(4+hm(-4+hr))}\delta_{01}, \\
h_3 & = & \left\{ \begin{array}{c} \frac{-2m(-2m+hm)+2c(2+hm)}{c(4+hm(4-hr))+m(4+hm(-4+hr))} \times (2\beta_{13}C_2 + (\frac{c(2c-2m-chr+hmr)}{b(2c+2m+chm-hm^2)} - \frac{2c}{bhm})\beta_{14}C_2 + \delta_{01}C_1 + \delta_{02}C_1) \\ -\frac{bhm(m(2-hm)+c(2+hm))}{-4c(c+m+chm-hm^2)+ch^2(c-m)mr} \times (2\beta_{23}C_2 + (\frac{c(2c-2m-chr+hmr)}{b(2c+2m+chm-hm^2)} - \frac{2c}{bhm})\beta_{24}C_2) \end{array} \right\}, \\
h_4 & = & \frac{-2m(-2m+hm)+2c(2+hm)}{c(4+hm(4-hr))+m(4+hm(-4+hr))}\delta_{01}C_2, \\
h_5 & = & \left\{ \begin{array}{c} \frac{-2m(-2m+hm)+2c(2+hm)}{c(4+hm(4-hr))+m(4+hm(-4+hr))} \times (2\beta_{13}C_1 + (\frac{c(2c-2m-chr+hmr)}{b(2c+2m+chm-hm^2)} - \frac{2c}{bhm})\beta_{14}C_1) \\ -\frac{bhm(m(2-hm)+c(2+hm))}{-4c(c+m+chm-hm^2)+ch^2(c-m)mr}(2\beta_{23}C_1 + (\frac{c(2c-2m-chr+hmr)}{b(2c+2m+chm-hm^2)} - \frac{2c}{bhm})\beta_{24}C_1) \end{array} \right\}.
\end{array}
\right.
$$

We calculate:

$$
\Gamma_1 = \left(\frac{\partial^2 g_1}{\partial x_t \partial \epsilon} + \frac{1}{2}\frac{\partial g_1}{\partial \epsilon}\frac{\partial^2 g_1}{\partial x_t^2} \right)\Bigg|_{(0,0)}. \tag{5.37}
$$

$$
\Gamma_2 = \left(\frac{1}{6}\frac{\partial^3 g_1}{\partial x_t^3} + \left(\frac{1}{2}\frac{\partial^2 g_1}{\partial x_t^2} \right)^2 \right)\Bigg|_{(0,0)}. \tag{5.38}
$$

$\frac{\partial g_1}{\partial \epsilon} = h_2x_t + h_3x_t^2 + 2h_4x_t\epsilon.$
$\frac{\partial^2 g_1}{\partial x_t \partial \epsilon} = h_2 + 2h_3x_t + 2h_4\epsilon.$
$\frac{\partial g_1}{\partial x_t} = -1 + 2h_1x_t + h_2\epsilon + 2h_3x_t\epsilon + h_4\epsilon^2 + 3h_3x_t^2.$
$\frac{\partial^2 g_1}{\partial x_t^2} = 2h_1 + 2h_3\epsilon + 6h_5x_t.$
$\frac{\partial^3 g_1}{\partial x_t^3} = 6h_5.$

In view of (5.36), (5.37) and (5.38) one gets:

$$\Gamma_1 = h_2 \neq 0, \tag{5.39}$$

and

$$\Gamma_2 = h_5 + h_1^2. \tag{5.40}$$

Finally, from (5.40) if $\Gamma_1 \neq 0$ as $(b, c, H, m, r, h, K) \in F_4 \mid_{P_3}$ then at P_3 the model (5.7) undergoes a flip bifurcation, moreover, if $\Gamma_2 > 0$ (respectively, $\Gamma_2 < 0$) then the period-2 points bifurcating from P_3 are stable (respectively, unstable). ■

Chaos control

We will explore chaos control first by OGY control method than by Feedback method.

- **OGY method**

The first method is OGY that we first verify controllability by:
Lets consider the system:

$$X_{n+1} = F(X_n, p) \simeq AX_n + B. \tag{5.41}$$

The system (5.41) is controllable if the the controllability matrix:

$$P = [B, AB],$$

has full rank 2.
Where $A = \frac{\partial F}{\partial x}\Big|_{x=x_F(\bar{p})}$ and $B = \frac{\partial F}{\partial p}\Big|_{p=\bar{p}}$, $X_F = [x_F, y_F]$.
So for the system (5.7) we have:

$$A = \frac{\partial F}{\partial X}\Big|_{(x_F,y_F)} = \begin{pmatrix} 1 + hr - \frac{2hrx_F}{K} & -\frac{bhx_F}{x_F+H} \\ \frac{cHhy_F}{(x_F+H)^2} & 1 - hm + \frac{chx_F}{x_F+H} \end{pmatrix}, B_K = \frac{\partial F}{\partial K}\Big|_{(x_F,y_F)} = \begin{bmatrix} \frac{hrx_F^2}{K^2} \\ 0 \end{bmatrix}.$$

The K controllability matrix is:

$$P_K = [B_K, AB_K] = \begin{bmatrix} \frac{hrx_F^2}{K^*} & \frac{hrx_F^2}{(K^*)^2} - \frac{h^2r^2mx_F^2(c(H-K)+m(H+K))}{cK^3(c-m)} \\ 0 & \frac{h^2r^2(K(c-m)-Hm)}{bK^3}. \end{bmatrix}.$$

Then we conclude that the Rosenzeig MacArthur system can stabilize around equilibruim point by OGY method.

- **Feedback method**

For the second method we adding control force $U_t = -k_1(x_t - x) - k_2(y_t - y)$
Let the system:

$$X_{t+1} = F(X_n, p) \simeq AX_t + U_t \tag{5.42}$$

The system (5.42) is controllable if the controllability matrix:

$$P = [B, AB]$$

Where $A = \frac{\partial F}{\partial x}\Big|_{x=x_F}$ and $B = K = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}$ and $x_F = (x, y)$
The discrete model (5.7) becomes:

$$\begin{cases} x_{t+1} & = (1 + hr)x_t - \frac{hr}{K}x_t^2 - \frac{hb}{x_t+H}x_t y_t - k_1(x_t - x) - k_2(y_t - y), \\ y_{t+1} & = (1 - hm)y_t + \frac{hc}{x_t+H}x_t y_t. \end{cases} \tag{5.43}$$

where $x = \frac{mH}{c-m}$, $y = \frac{cHr(cK-Hm-Km)}{bK(c-m)^2}$, the Jackobien of (5.43) is:

$$J^c\mid_{P_3} = \begin{pmatrix} \ell_{11} - k_1 & \ell_{12} - k_2 \\ \ell_{21} & \ell_{22} \end{pmatrix}. \tag{5.44}$$

$$\begin{matrix} \ell_{11} & = 1 - \frac{hmr(c(H-K)+m(H+K))}{cK(c-m)}, \\ \ell_{12} & = -\frac{bhm}{c}, \end{matrix} \quad , \quad \begin{matrix} \ell_{21} & = \frac{hr(K(c-m)-Hm)}{bK}, \\ \ell_{22} & = 1. \end{matrix}$$

$$\lambda_1 + \lambda_2 = \ell_{11} + \ell_{22} - k_1, \tag{5.45}$$

$$\lambda_1\lambda_2 = \ell_{22}(\ell_{11} - k_1) - \ell_{21}(\ell_{12} - k_2). \tag{5.46}$$

Lemma 5.2.2
If the characteristics roots of $J^c(P_3)$ satisfying $|\lambda_{1,2}| < 1$ then (5.43) is asymptotically stable.

Proof.
If $\lambda_1 = \pm 1$

- If $\lambda_1\lambda_2 = 1$, from (5.46) : $\ell_{22}(\ell_{11} - k_1) - \ell_{21}(\ell_{12} - k_2) = 1$

$-\frac{hmr(c(H-K)+m(H+K))}{cK(c-m)} - k_1 - \frac{hr(K(c-m)-Hm)}{bK}\left(-\frac{bhm}{c} - k_2\right) = 0$

$\frac{-hmrb(c(H-K)+m(H+K))-k_1cbK(c-m)+hr(K(c-m)-Hm)(bhm(c-m))+k_2hrc(K(c-m)-Hm)(c-m)}{bKc(c-m)} = 0$

$-hmrb(c(H-K)+m(H+K)) - k_1cbK(c-m) + hr(K(c-m)-Hm)(bhm(c-m)) + k_2hrc(K(c-m)-Hm)(c-m) = 0$

We get :

$$L_1 : \begin{cases} -hmrb(c(H-K)+m(H+K)) - k_1cbK(c-m) \\ +hr(K(c-m)-Hm)(bhm(c-m)) + k_2hrc(K(c-m)-Hm)(c-m) = 0. \end{cases} \tag{5.47}$$

- If $\lambda_1 = 1$, by substitute λ_1 in (5.45) and (5.46), they becomes:

$$\lambda_2 = \ell_{11} + \ell_{22} - k_1 - 1, \tag{5.48}$$

$$\lambda_2 = \ell_{22}(\ell_{11} - k_1) - \ell_{21}(\ell_{12} - k_2). \tag{5.49}$$

$\ell_{11} + \ell_{22} - k_1 - 1 = \ell_{22}(\ell_{11} - k_1) - \ell_{21}(\ell_{12} - k_2)$

$\ell_{11} + \ell_{22} - k_1 - 1 - \ell_{22}(\ell_{11} - k_1) + \ell_{21}(\ell_{12} - k_2) = 0$

$\frac{hr(K(c-m)-Hm)}{bK}(-\frac{bhm}{c} - k_2) = 0$

$\frac{h^2bmr(K(c-m)-Hm)-hrbck_2(K(c-m)-Hm)}{bKc} = 0$

$h^2bmr(K(c-m)-Hm) - hrk_2bKc(K(c-m)-Hm) = 0$

$h^2bmr - hrkck_2 = 0$

$hbm - kck_2 = 0$

We get:

$$L_2 : \left\{ hbm - kck_2 = 0. \right. \tag{5.50}$$

- If $\lambda_1 = -1$ by substitute it in (5.45) and (5.46) :

$$\lambda_2 = \ell_{11} + \ell_{22} - k_1 + 1, \tag{5.51}$$

$$\lambda_2 = -\ell_{22}(\ell_{11} - k_1) + \ell_{21}(\ell_{12} - k_2). \tag{5.52}$$

$\ell_{11} + \ell_{22} - k_1 + 1 = -\ell_{22}(\ell_{11} - k_1) + \ell_{21}(\ell_{12} - k_2)$

$\ell_{11} + \ell_{22} - k_1 + 1 = -\ell_{11} + k_1 + \ell_{21}(\ell_{12} - k_2)$

$\ell_{11} + \ell_{22} - k_1 + 1 + \ell_{11} - k_1 - \ell_{21}(\ell_{12} - k_2) = 0$

$2\ell_{11} + 2 - 2k_1 - \ell_{21}(\ell_{12} - k_2) = 0$

$2(1-\frac{hmr(c(H-K)+m(H+K))}{cK(c-m)})+2-2k_1+-\frac{hr(K(c-m))}{bK}(-\frac{bhm}{c}-k_2)=0$
$4-\frac{2hmr(c(H-K)+m(H+K))}{cK(c-m)}-2K_1+\frac{hr(K(c-m)-Hm)}{bK}(\frac{bhm}{c}-k_2)=0$
We get:

$$L_3:\left\{\ 4-\frac{2hmr(c(H-K)+m(H+K))}{cK(c-m)}-2K_1+\frac{hr(K(c-m)-Hm)}{bK}(\frac{bhm}{c}-k_2)=0.\right. \tag{5.53}$$

Thus from (5.47), (5.50) and (5.53) lines L_1, L_2 and L_3 gives the conditions for the eigenvalues satisfying $|\lambda_{1,2}|<1$. Moreover, triangular region bounded by L_1, L_2 and L_3 contains stable eigenvalues. ■

5.3 Numerical simulations

In this section, we will give some numerical simulations for the system (5.7) to support our theoretical results.
We presente this in two cases with different values of parametres and intial conditions.

5.3.1 Case(1): Numerical simulation for the set of parametre: $b=0.55$, $c=2.05$, $H=0.8$, $m=0.15$, $r=0.7$, $h=1$

If

$$b=0.55, c=2.05, H=0.8, m=0.15, r=0.7, h=1, \tag{5.54}$$

and $K=[0.1,1.2]$. With the initial condition $(x_0,y_0)=(0.1,0.1)$ then at $K=0.36$ discrete model (5.7) undergoes a Neimark-Sacker bifurcation. Two 2D bifurcation diagrams are drawn in Figure 5.1. Additionally, corresponding to Figure 5.1. 2D Lyapunov exponent is also plotted in Figure 5.2 and the positive Lyapunov exponent indicative chaotic behavior.Further, at $(b,c,H,m,r,h,K)=(0.55,2.05,0.8,0.15,0.7,1.0,0.36)$ model (5.7) has interior equilibrium solution $P_3=(0.0632,0.9058)$ and, from (5.12), one gets the Jackobien matrix:

$$J^c\mid_{P_3}=\begin{pmatrix}0.9194 & -0.0402\\ 1.9938 & 1\end{pmatrix},$$

with the characteristic equation:

$$\lambda^2 - 1.9194\lambda + 0.9996 = 0. \tag{5.55}$$

The roots of (5.55) are $\lambda_{1,2} \approx 0.9597 \pm 0.2804\iota$ with $|\lambda_{1,2}| \approx 1$ which implies the fact that parametric condition:
$(b, c, H, m, r, h, K) = (0.55, 2.05, 0.8, 0.15, 0.7, 1.0, 0.36) \in N\,|_{P_3}$.
Furthermore, for the set of parameter values (5.54), and varying the value of the bifurcation parameter $K < 0.36$ then one can obtain that the respective interior equilibrium solution is a stable focus. Similarly, if one varies $K = 0.2965, 0.3165, 0.3355, 0.3445, 0.3575, 0.359 < 0.36$ then Figure 5.4 also show that the corresponding equilibrium solution $P_3 = (0.0632, 0.8646)$,
$(0.0632, 0.8793)(0.0632, 0.8918), (0.0632, 0.8972), (0.0632, 0.9045), (0.0632, 0.9053)$ is also stable focus. On the other hand, if one varies the bifurcation parameter $K > 0.36$ then we can conclude that respective interior equilibrium solution is unstable focus, and meanwhile supercritical Neimark-Sacker bifurcation occurs.
For instance, for the set of parameter (5.54), then from (5.20) the nondegenerate conditions, i.e, $\frac{d|\lambda_{1,2}|}{d\epsilon}|_{\epsilon=0}= 0.206 \neq 0$ holds, and if $K = 0.3599$ then from (5.23) and (5.2.3), one gets:

$$\begin{cases} \beta_{11} &= 0.9194, \\ \beta_{12} &= -0.0402, \\ \beta_{13} &= -1.3253, \\ \beta_{14} &= -0.5906, \\ \beta_{21} &= 1.9938, \\ \beta_{22} &= 1, \\ \beta_{23} &= -2.3099, \\ \beta_{24} &= 2.2012. \end{cases} \tag{5.56}$$

$$\begin{cases} \eta &= 0.9597, \\ \zeta &= 0.9378. \end{cases} \tag{5.57}$$

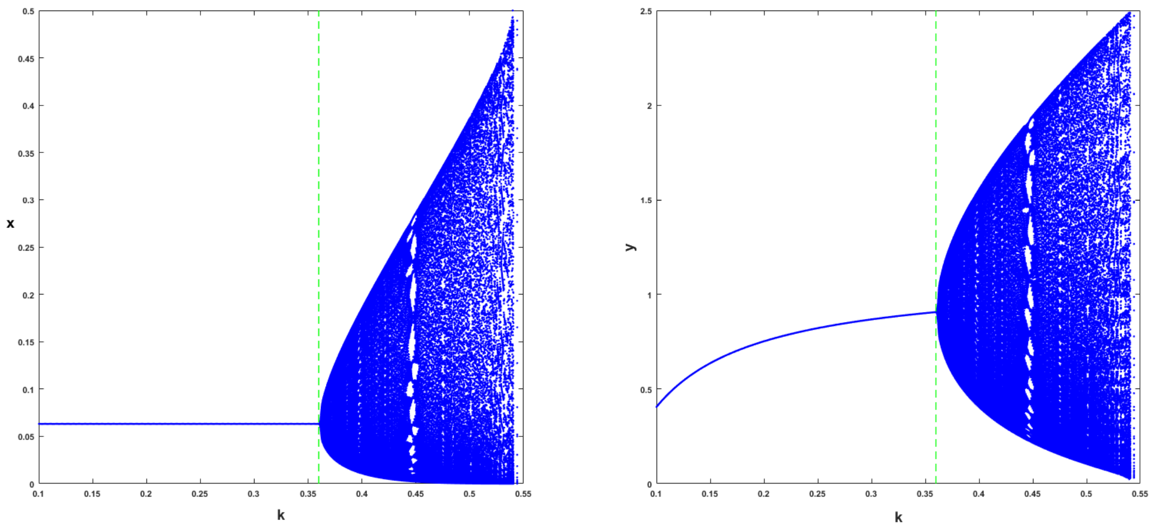
Utilizing (5.56) and (5.57) into (5.26) one gets:

$$\begin{cases} \sigma_{11} &= 0.0295, \\ \sigma_{12} &= 0.5538, \\ \sigma_{21} &= 0.0091, \\ \sigma_{22} &= -0.0648. \end{cases} \tag{5.58}$$

Now in view of (5.58) and (5.28), one obtains

$$\begin{cases} \rho_{02} &= -0.0088 + 0.1407\iota, \\ \rho_{11} &= 0.0147 + 0.0045\iota, \\ \rho_{20} &= -0.0088 - 0.1361\iota, \\ \rho_{21} &= 0. \end{cases} \tag{5.59}$$

Finally, using (5.59) along with $\bar{\lambda} = 0.9597 \pm 0.2804\iota$ into (5.27) one gets:
$\psi = -0.1944 < 0$ which give the fact closed invariant curve must exists, and hence discrete model undergoes supercritical Neimark-Sacker bifurcation at indicated interior fixed point $P_3 = (0.0632, 0.9058)$ (see Figure (5.5a)).
In similar manner, we can also obtain that if one varies $K = 0.361, 0.366, 0.40$ then stable invariant curves also appears which are depicted in Figure (5.5c).



(a) (b)

Figure 5.1: Neimark-Sacker bifurcation of discrete system (5.7) (a) for x_t and (b) for y_t for the set of (5.54) at $P_3 = (0.0632, 0.9058)$.

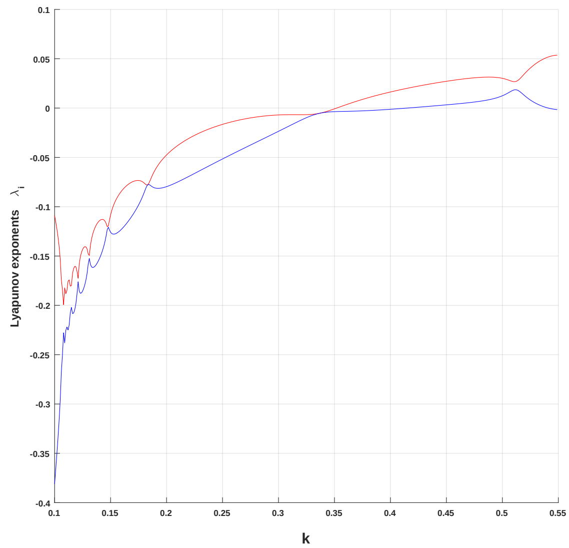


Figure 5.2: Lyapunov exponent of the system (5.7) varsus k for the set of parameter values (5.54) at $P_3 = (0.0632, 0.9058)$.

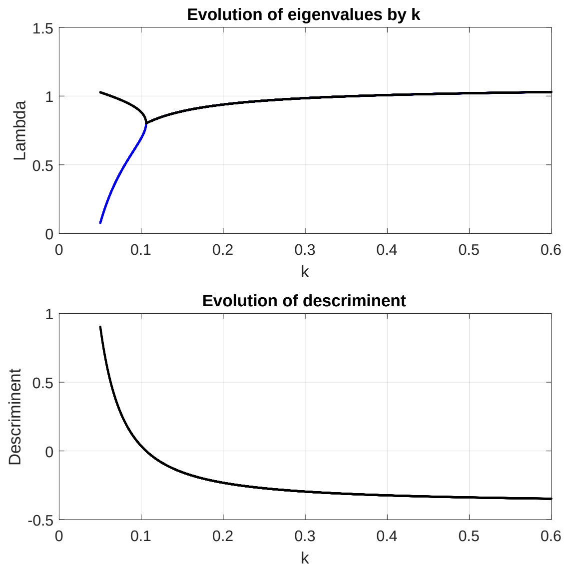


Figure 5.3: Evolution of eigenvalues and descriminenet by k for the set of parameter values (5.54) at $P_3 = (0.0632, 0.9058)$

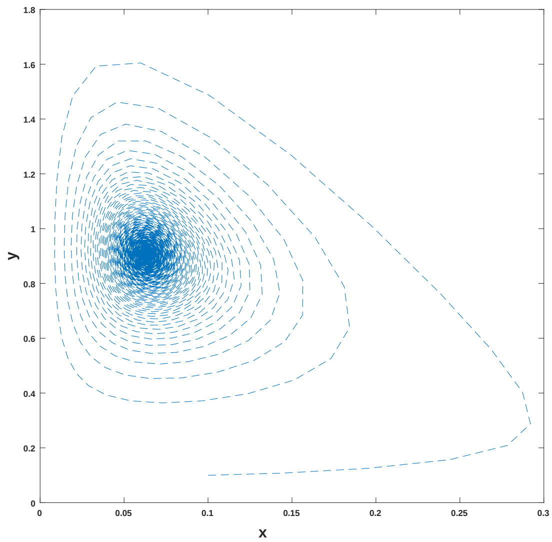
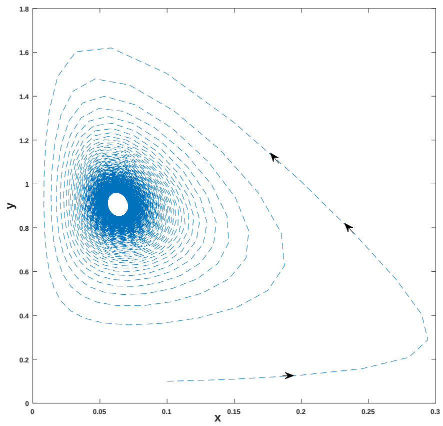
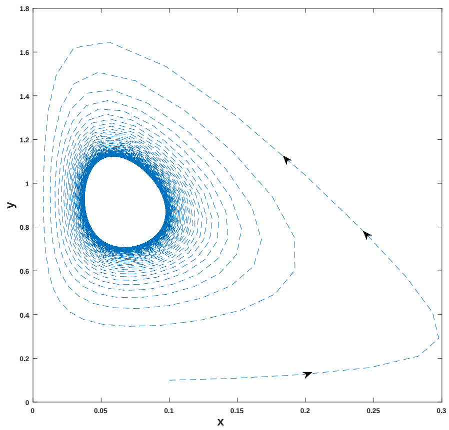


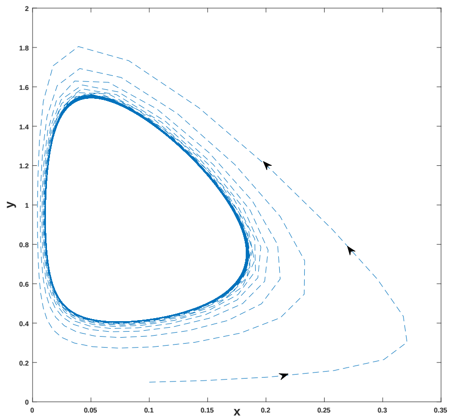
Figure 5.4: Phase portrait of the discrete model (5.7) for the set of parameter values (5.54) in $P_3 = (0.0632, 0.9058)$, *k=0.359* and the initial condition: (0.1760, 0.5259)



(a) *k=0.361*



(b) *k=0.366*



(c) *k=0.40*

Figure 5.5: Invariant closed curves of the discrete model (5.7) for the set of parameter values (5.54) in $P_3 = (0.0632, 0.9058)$ and the initial condition: (0.1760,0.5259)

5.3.2 Case(2): Numerical simulation for the set of parametre: $b = 0.9$, $c = 3.5$, $H = 0.7$, $m = 0.7$, $r = 2.8$, $h = 1$

Flip bifurcation

If

$$b = 0.9, c = 3.5, H = 0.7, m = 0.7, r = 2.8, h = 1 \tag{5.60}$$

$K \in [0.01, 0.55]$. With $(x_0, y_0) = (0.1760, 0.5259)$ then at $K = 0.2168$ discrete model undergoes the flip bifurcation. The Lyapunov exponent with flip bifurcation diagrams are drawn in figure (5.6a).

Further, at $(b, c, H, m, r, h, K) = (0.9, 3.5, 0.7, 0.7, 2.8, 1, 0.2168)$ the discrete model (5.7) has an equilibrium solution $P_3 = (0.1750, 0.5249)$ and the Jackobian matrix is :

$$J^c \mid_{P_3} = \begin{pmatrix} -1.1522 & -0.1800 \\ 1.6795 & 1 \end{pmatrix}.$$

with the eigenvalues $\lambda_1 = -1$ and $\lambda_2 = 0.8489 \neq \pm 1$ and hence based on these simulations one can obtain that $(b, c, H, m, r, h, K) = (0.9, 3.5, 0.7, 0.7, 2.8, 1, 0.2168) \in F_3 \mid_{P_3}$. From (5.30), (5.35), (5.2.3) one gets:

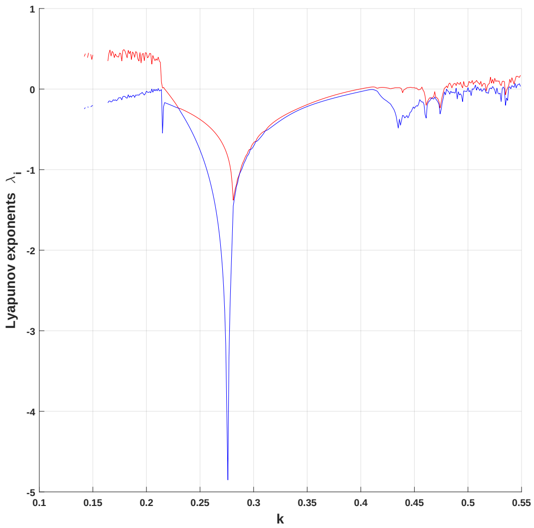
$$\begin{cases} \beta_{11} &= -1.1522, \\ \beta_{12} &= -0.1800, \\ \beta_{13} &= -12.4215, \\ \beta_{14} &= -0.8229, \\ \delta_{01} &= 20.8501, \\ \delta_{02} &= 59.5716, \\ \beta_{21} &= 1.6795, \\ \beta_{22} &= 1, \\ \beta_{23} &= -1.9195, \\ \beta_{24} &= 3.2. \end{cases} \tag{5.61}$$

$$\begin{cases} C_0 &= C_3 = 0, \\ C_1 &= 0.8055, \\ C_2 &= -11.2991. \end{cases} \tag{5.62}$$

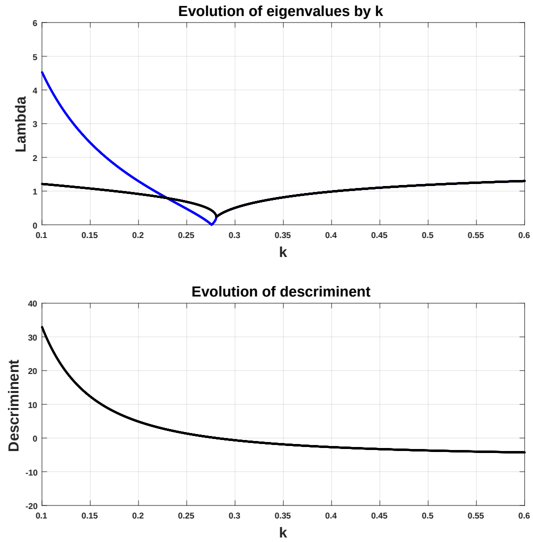
And:

$$\begin{cases} h_1 &= -15.3803, \\ h_2 &= 23.8379, \\ h_3 &= -163.5835, \\ h_4 &= -269.3474, \\ h_5 &= -144.6623. \end{cases} \tag{5.63}$$

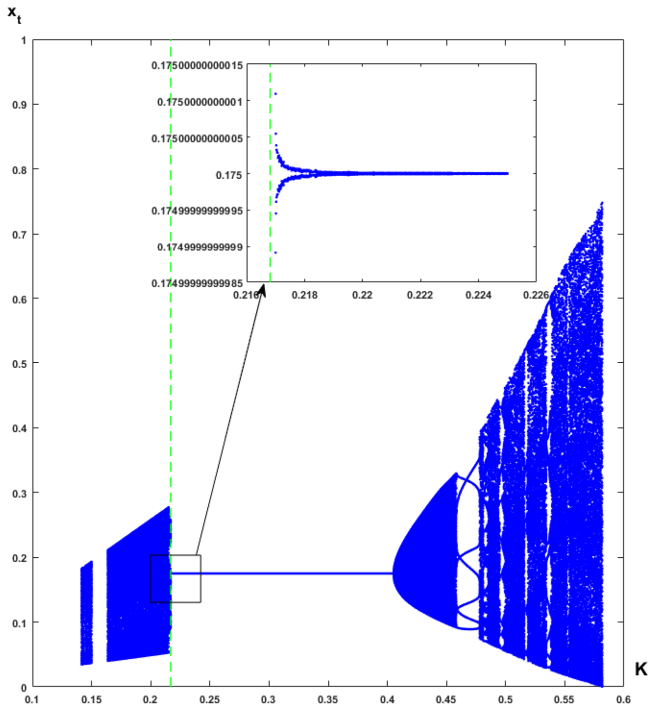
Substituting (5.63) in (5.39) and (5.40) one gets: $\Gamma_1 = 23.8379 \neq 0$, $\Gamma_2 = 91.8928 > 0$. Since $\Gamma_2 = 91.8928 > 0$ and so it can be concluded that stable period-2 points bifurcate from $P_3 = (0.175, 0.5249)$.



(a) Lyapunov exponents of Flip bifurcation at $P_3 = (0.175, 0.5249)$.



(b) Evolution of eigenvalues and descriminenet by K at $P_3 = (0.175, 0.5249)$



(c) Flip bifurcation diagram for x_t at $P_3 = (0.175, 0.5249)$

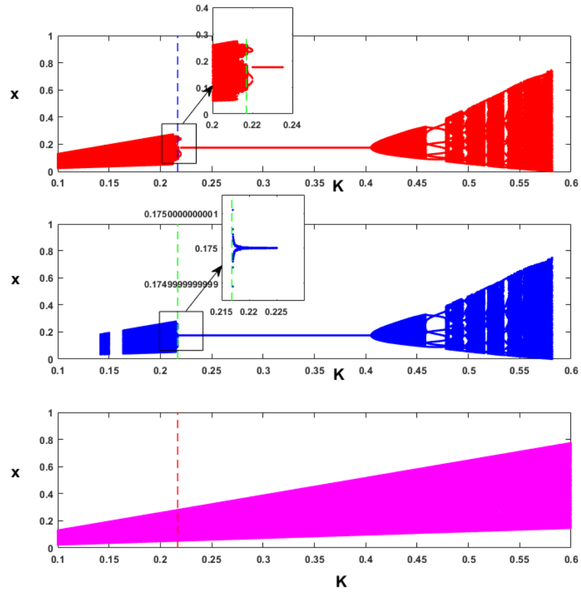
Figure 5.6: a)Lyapunov exponents of Flip bifurcation b)Evolution of eigenvalues and descriminenet by K. c)Flip bifurcation diagram for x_t at $P_3 = (0.175, 0.5249)$.

- In this section, we explain the chaotic coexistence in the Rosenzweig-MacArthur system. Table 5.3.2 illustrates the cases of changes in attractors and Lyapunov exponents with the evolution of K.

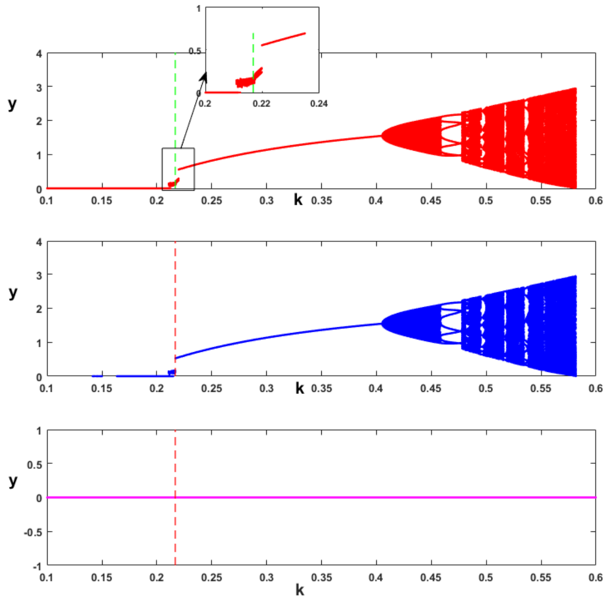
Attractors	K	$\lambda_{1,2}$
Chaotic	[0.164, 0.2169]	$\lambda_1 > 0$ and $\lambda_2 < 0$
Quasi-periodic	0.217	$\lambda_1 = 0$ and $\lambda_2 < 0$
Periodic	[0.217, 0.399]	$\lambda_1 < 0$ and $\lambda_2 < 0$
chaotic	[0.3992, 0.437]	$\lambda_1 > 0$ and $\lambda_2 < 0$
Quasi-periodic	0.438	$\lambda_1 = 0$ and $\lambda_2 < 0$
periodic	[0.438, 0.442]	$\lambda_1 < 0$ and $\lambda_2 < 0$
Quasi-periodic	0.457	$\lambda_1 = 0$ and $\lambda_2 < 0$
Periodic	[0.457, 0.477]	$\lambda_1 < 0$ and $\lambda_2 < 0$
Chaotic	[0.477, 0.538]	$\lambda_1 > 0$ and $\lambda_2 < 0$
Hyper-chaotic	[0.538, 0.5816]	$\lambda_1 > 0$ and $\lambda_2 > 0$

Table 5.1: The Lyapunov exponents and the type of attractors of the system with the first set of parameters (5.60) at $P_3 = (0.175, 0.5249)$

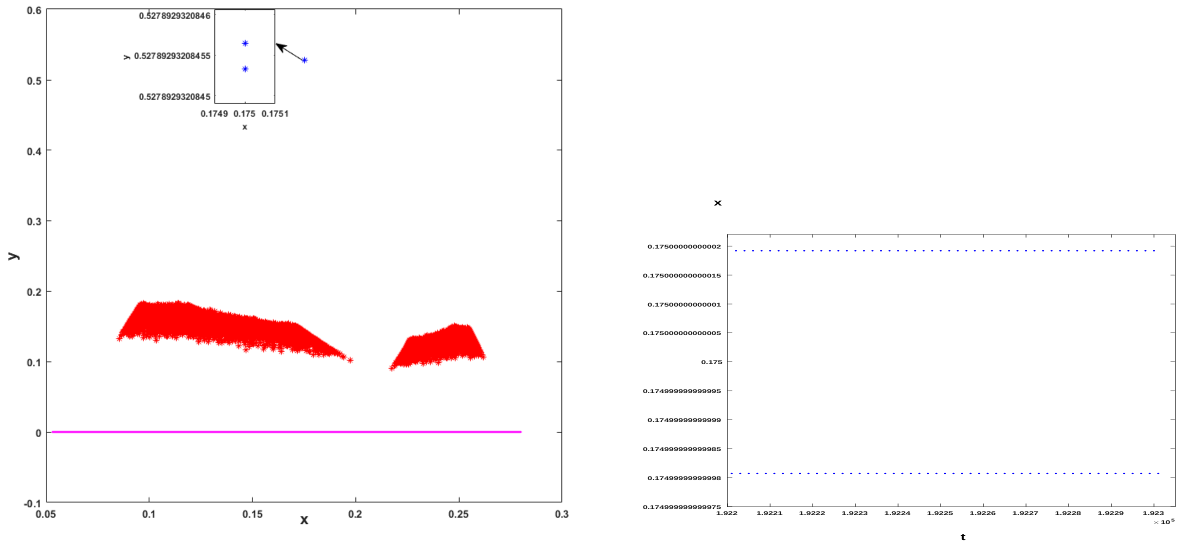
  - **Coexistence of attractor:** In the diagram, we observe the coexistence of different attractors for several values of k, such as k=0.2171 and k=0.47, with different initial conditions.
    * If k=0.2171 with three initial conditions for $(x_1, y_1) = (0.1749, 0.5278)$ we have a chaotic attractor, for $(x_2, y_2) = (0.2505, 0.1405)$ we have(periodic points), for $(x_3, y_3) = (0.1931, 0)$ (invariant curve), where we observe two periodic points, corresponding to two Lyapunov exponent values, both of which are negative (periodic attractor), (see figure 5.8*a*,5.8*b*)
    * If k=0.47 with three initial conditions for $(x_1, y_1) = (0.1831, 0.9633)$ we have a periodic points , for $(x_2, y_2) = (0.1750, 1.4346)$ we have (periodic points), for $(x_3, y_3) = (0.5886, 0)$ (invariant curve), where observe seven periodic points,corresponding to two Lyapunov exponent values, both of which are negative (periodic attractor), (see figure 5.9*a*,5.9*b*)



(a) Bifurcation diagram for x_t with three conditions: $(x_1, y_1) = (0.1749, 0.5278)$, $(x_2, y_2) = (0.2505, 0.1405)$, $(x_3, y_3) = (0.1931, 0)$ at $P_3 = (0.175, 0.5249)$.

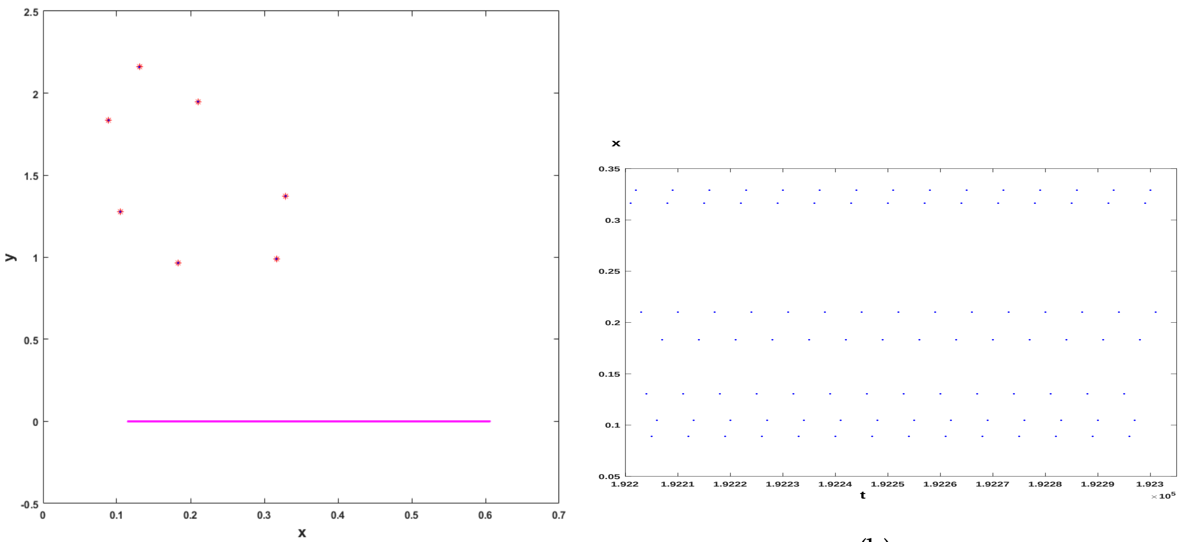


(b) Bifurcation diagram for y_t with three conditions: $(x_1, y_1) = (0.1831, 0.9633)$, $(x_2, y_2) = (0.1750, 1.4346)$, $(x_3, y_3) = (0.5886, 0)$ at $P_3 = (0.175, 0.5249)$.



(a) (b)

Figure 5.8: a) Coexistence of a chaotic attractor and period-2 points for the set of parameter (5.60) at $P_3 = (0.175, 0.5249)$ with $k = 0.2171$ and the initial condition $(0.2505, 0.1405)$ for the chaotic attractor and $(0.1749, 0.5278)$ for the period-2 points. b) Time evolution of x for the periodic points



(b)

(a)

Figure 5.9: a) Coexistence of a chaotic attractor and period-2 points for the set of parameter (5.60) at $P_3 = (0.175, 0.5249)$ with $k = 0.47$ and the initial condition $(0.1750, 1.4346)$ for the chaotic attractor and $(0.1831, 0.9633)$ for the period-2 points. b) Time evolution of x for the periodic points

Control methods

- **Feedback control** From (5.47), (5.50) and (5.53), one gets:
  The first strigh line equation:

$$L_1 : 0.5954K_1 - K_2 + 1.2814 = 0. \tag{5.64}$$

The second strigh line equation:

$$L_2 : K_2 + -0.00002545 = 0. \tag{5.65}$$

The third strigh line equation:

$$L_3 : 1.1908K_1 - K_2 + 0.0012 = 0. \tag{5.66}$$

Hence, lines (5.64), (5.65) and (5.66) determine triangular region that gives $|\lambda_{1,2}| < 1$ (See Figure 5.11). The Figures (5.10a,5.10b,5.10c,5.10d) are without control. Finally t vs x_t and y_t for (5.43) with $k1 = 0.02$, $k2 = 0.24$ have been plotted that implies that unstable trajectories are stabilized (See Figure 5.12a;5.12a;5.12b;5.12c;5.12d)

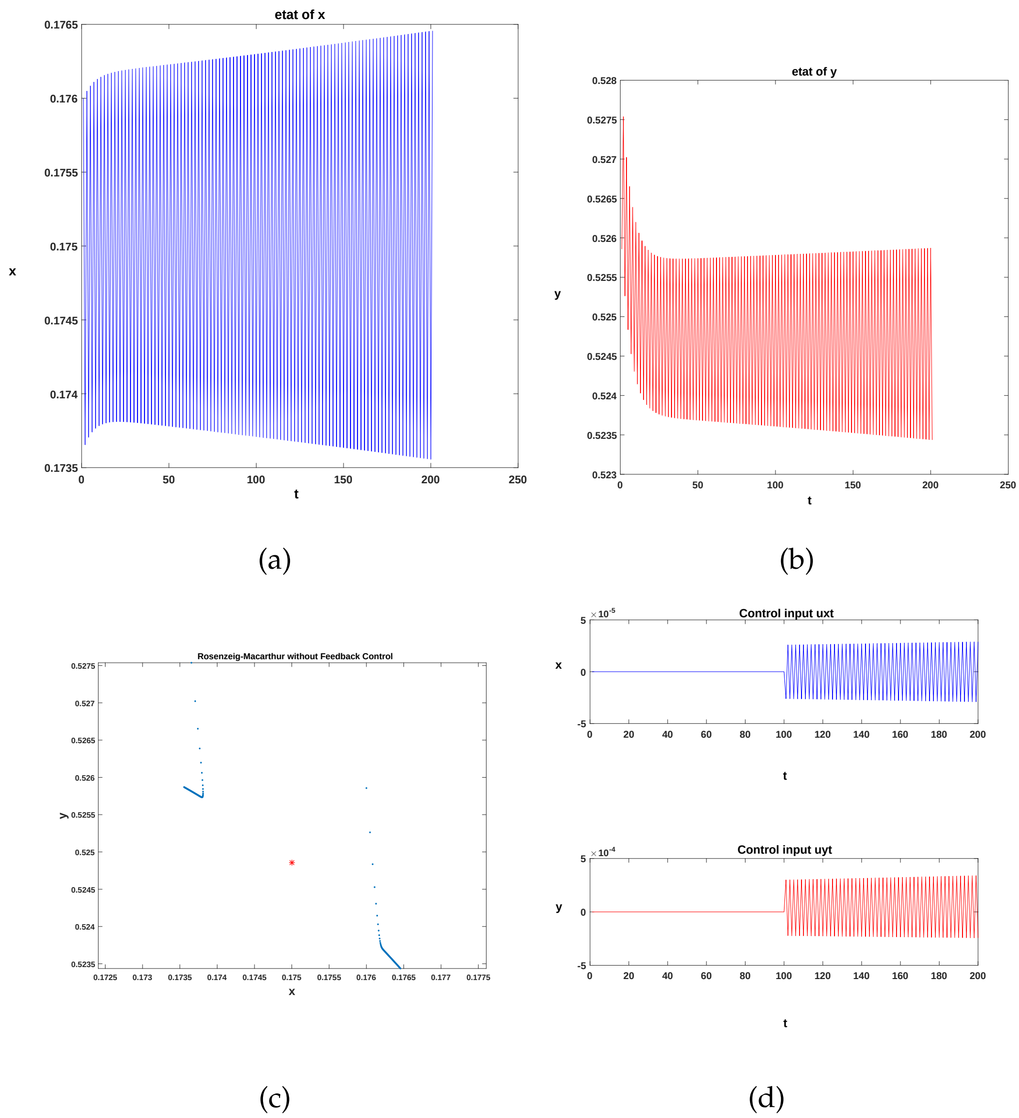


Figure 5.10: (a)Time evolution of x_t,(b)time evolution of y_t,(c)time evolution of fixed point. (d)Plot of x_t and y_t for the non controlled system (5.43) for the set of parameter values (5.60) in $P_3 = (0.175, 0.5249)$ and initial codition $(0.176, 0.5259)$ with $K1 = 0.02$, $K2 = 0.24$ by control Feedback

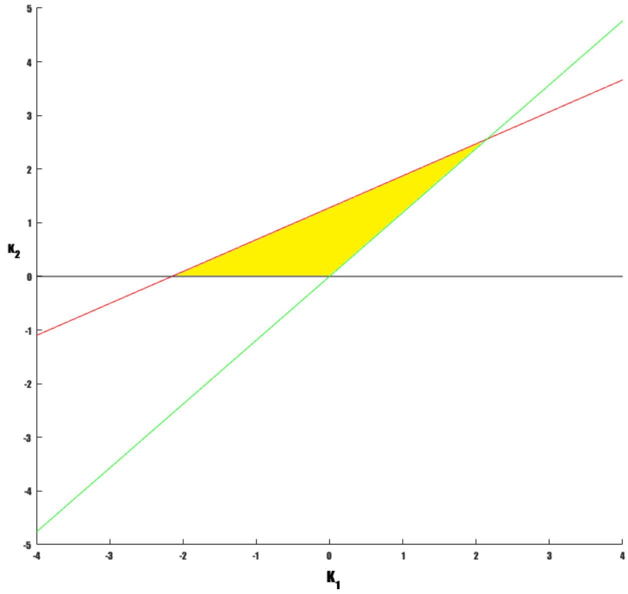
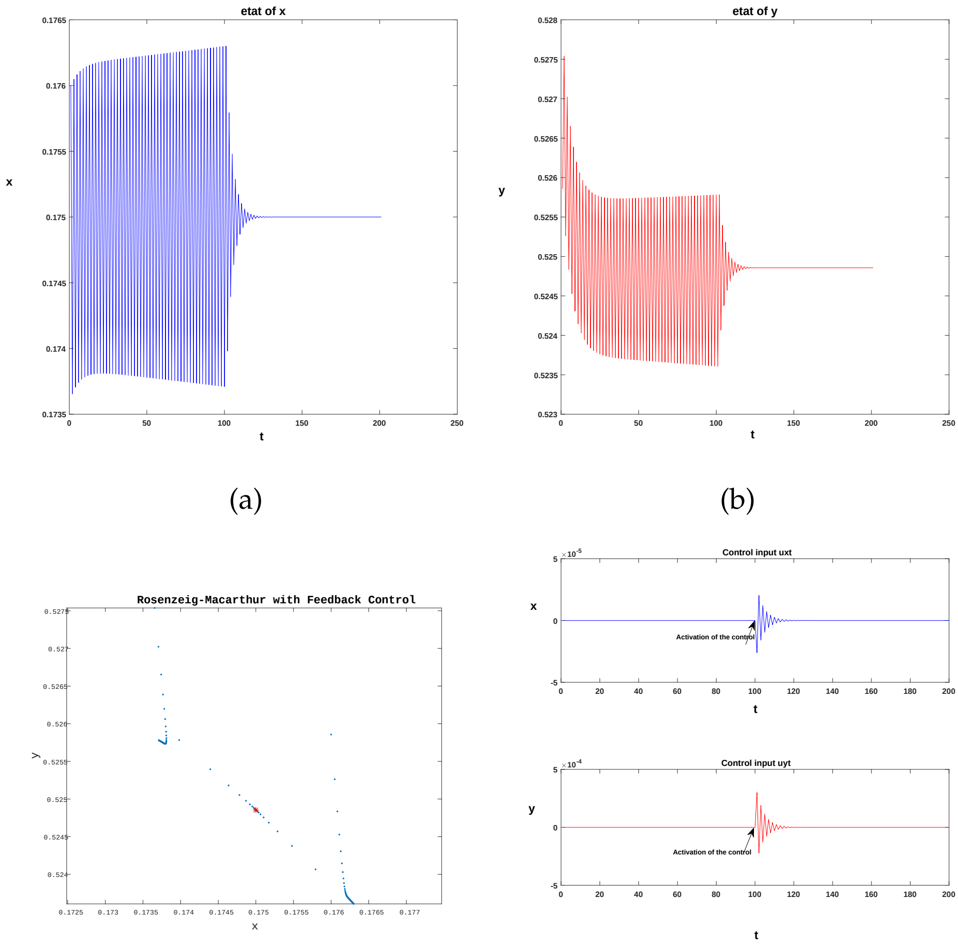


Figure 5.11: Stability region ($|\lambda_{1,2}| < 1$) for the set of parameter values (5.60) at $P_3 = (0.175, 0.5249)$.



(a) (b) (c) (d)

Figure 5.12: (a)Time evolution of x_t,(b)time evolution of y_t,(c)time evolution of fixed point. (d)Plot of x_t and y_t for the controlled system (5.43) for the set of parameter values (5.60) and initial codition $(0.176, 0.5259)$ with $K1 = 0.02$, $K2 = 0.24$ by control feedback

- **OGY method**

  Let us consider K as a control parameter of the system (5.7) for the set of parameter values (5.60) and the Jacobian matrix at the fixed point P_3 is:

$$A = \frac{\partial f}{\partial x}(0.1750, 0.5249) = \begin{bmatrix} -1.1522 & -0.1800 \\ 1.6795 & 1 \end{bmatrix},$$

$$\text{with} \quad B = \frac{\partial f}{\partial r}(0.1750, 0.5249) = \begin{bmatrix} 1,8243 \\ 0 \end{bmatrix}.$$

  The matrix A has two eigenvalues : $\lambda_u = -1.0011$ and $\lambda_s = 0.8489$, indicating that

P_3 is saddle point (hyperbolic)

Whith two right eigenvectors

$$v_u = \begin{pmatrix} -0.7660 \\ 0.6429 \end{pmatrix} \quad \text{and} \quad v_s = \begin{pmatrix} 0.0896 \\ -0.9960 \end{pmatrix},$$

and two left eigenvectors:

$$f_u = \begin{pmatrix} -0.9960 \\ -0.0896 \end{pmatrix} \quad \text{and} \quad f_s = \begin{pmatrix} -0.6429 \\ -0.7660 \end{pmatrix},$$

and $K^T = (-0.5487, -0.0494)$.

Figure 5.13 illustrates the response of the controlled Rosenzeig-Macarthur model with the applied control effort. The control is activated when the system state approaches the unstable equilibrium P_3 at $t = 100$ and the parameter K is adjusted by a small perturbation of order 10^{-4} during the short time period $t \in [100, 101]$. Subsequently, the control is quickly established at $t = 101$, stabilizing the Rosenzeig-Macarthur model to its P_3 equilibrium.

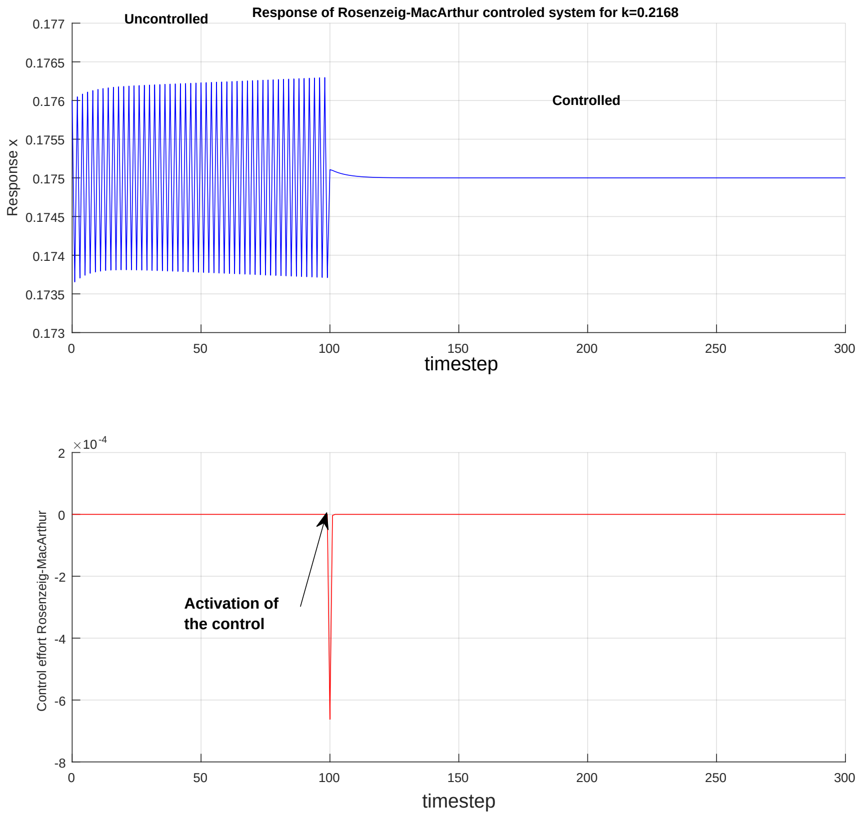


Figure 5.13: Response of the controlled system (5.7) using the OGY control with the set of parameter values (5.60) in $P_3 = (0.175, 0.5249)$, $k = 0.2168$, and initial condition $(0.1760, 0.5259)$.

CONCLUSION

The goal of this work is to study the Rosenzweig-MacArthur model for prey and predator interactions, finding that it has three fixed points whose stability depends on the parameter k. We showed that the model undergoes two types of bifurcations (Neimark and Flip) and that their calculations are very complicated. In addition, the bifurcations indicate that the system is sensitive to initial conditions. Furthermore, we discovered that chaotic behavior can be eliminated by applying two types of control (Feedback and OGY). Finally, we confirmed the results numerically.

It is also possible to propose the development of this system into one where the interaction part is of the Holling Type III with $k = 2$ for both the prey and the predator.

BIBLIOGRAPHY

[1] Abraham. R, Mira. C, Gardini. L, Chaos in discrete dynamical systems, Springer Science-Business Media, New York: Telos, 1997.

[2] Amira. R, Abdelouahab. M. S, Touafek. N, Mesmouli. M. B, Zaidi. H. N, Hassan. T. S, Nonlinear dynamic in a remanufacturing duopoly game: spectral entropy analysis and chaos control, Aims Mathematics, **9(3)**, 7711-7727, (2024).

[3] Arlot. S, Etude d'un modele de dynamique des populations, hal-00679905, 2012.

[4] Auger. P, Lett. C, Poggiale. J. C, Modélisation mathématique en écologie-2e éd.: Cours et exercices corrigés, Dunod , Paris, 2015.

[5] Bououden. R, Systemes dynamiques chaotiques : Application à l'optimisation, doctoral thesis, Université freres mentouri -Constantine 1-, 2019.

[6] Debouzie. D, La notion de population en dynamique et génétique des populations, Natures Sciences Sociétés, **7(4)**, 19-26, (1999).

[7] Devaney. R. L, An introduction to chaotic dynamical dystems, CRC Press, third edition, Boca Raton, 2022.

[8] Dietrich. S, Eugene. S. H, From Newton to Mandelbrot: a primer in theoretical physics, Berlin, 1990.

[9] Elaydi. S, An Introduction to difference equations, 3 edition, Sprigner, San Antonio, 2005.

[10] Elaydi. S, Discrete chaos:With Applications in Science and Engineering, Second Edition,CRC Press, San Antonio, 2007.

[11] El jai .A, Zerrike .A, Stabilité des systèmes dynamiques, Presses universitaires de Perpignan 2014.

[12] Françoise. J. P, Oscillations en biologie : Analyse qualitative et modèles,**46**, Springer-Verlag Berlin, Heidelberg, 2005.

[13] Haschke. R, Bifurcations in discrete-time neural networks controlling complex network behaviour with inputs, 2003.

[14] Hirsch. M. W, Smale. S, Devaney. R. L, Differential equations, dynamical systems, and an introduction to chaos, Elsevier, Waltham, 2013.

[15] Galor. O, Discrete dynamical systems, Springer Science-Business Media, first edition, Berlin, 2007.

[16] Gasri .A, Chaos et synchronisation (généralisé) dans les systèmes dynamiques, doctoral thesis,Université freres mentouri -Constantine 1-, 2018.

[17] Grognard. F, Feedback stabilization of predator-prey systems for impulsive biological control. IFAC Proceedings Volumes, **47(3)**, 5264-5269, (2014).

[18] Jouini .L, Quantification et synchronisation du chaos dans les systèmes dynamiques discrets, doctoral thesis, Université Laarbi Tebessi- Tébessa, 2021.

[19] Hadeler. K. P, Gerstmann.I, The discrete Rosenzweig model, Mathematical Bio sciences, **98(1)**, 49-72, (1990).

[20] Khan. A. Q, Ma. J, Xiao. D, Global dynamics and bifurcation analysis of a host-parasitoid model with strong Allee effect. Journal of Biological Dynamics, **11(1)**, 121-146, (2017).

[21] Khan. A. Q, Maqbool. A, Alharbi. T. D, Bifurcations and chaos control in a discrete Rosenzweig-Macarthur prey-predator model. Chaos: An Interdisciplinary Journal of Nonlinear Science, **34(3)**,(2024).

[22] Kumar. S, Kharbanda. H, Chaotic behavior of predator-prey model with group defense and non-linear harvesting in prey. Chaos, Solitons Fractals, **119**, 19-28, (2019).

[23] Kuznetsov, Y. A, Kuznetsov. I. A, Kuznetsov. Y, Elements of applied bifurcation theory, **112**, 2 ed Springer, NewYork, 1998.

[24] Kh.Lakhdari.Analyse et contrôle de systèmes dynamiques chaotiques,memoire de master, (2020).

[25] Laouira. W, Contrôle des systèmes dynamiques chaotiques, thèse de doctorat, Université freres mentouri -Constantine 1-, 2018.

[26] Layek, G. C. An introduction to dynamical systems and chaos, **449**, Springer, New Delhi, 2015.

[27] Lobry, C. Sari, T. Migrations in the Rosenzweig-MacArthur model and the" atto-fox" problem. Revue Africaine de Recherche en Informatique et Mathématiques Appliquées, **20**, 95-125, (2015).

[28] Lynch. S, Dynamical systems with applications using mathematica, Boston, Birkhuser, 2007.

[29] Mareno. A and English. Lars Q, Flip and Neimark-Sacker bifurcations in a coupled logistic map system, Discrete Dynamics in Nature and Society, **2020(1)**, 1-14 (2020).

[30] Martelli. M, Introduction to discrete dynamical systems and chaos, John Wiley and Sons, Canada, (1999).

[31] Mehazzem. A, Etude des bifurcations et chaos dans quelques systèmes spatio-temporels discrets, doctoral thesis, Université Larbi Tébessi -Tébessa, 2022.

[32] Murdoch. W, W, Briggs. C, J, Nisbet. R, M, Consumer-Resource Dynamics (MPB-36), Princeton University Press, (2003).

[33] Ouanas. A, Sur La synchronisation des systèmes chaotiques discrets , doctoral thesis, Université Frères Mentouri à Constantine, 2015.

[34] Liu. P, Elaydi. S. N, Discrete competitive and cooperative models of Lotka-Volterra type, Journal of Computational Analysis and Applications, **3**, 53-73, (2001).

[35] Turchin. P, Complex population dynamics: a theoretical/empirical synthesis (MPB-35), Princeton university press, New Jersey, 2013.

[36] Polis. G. A, Myers. C. A, Holt. R. D, The ecology and evolution of intraguild predation: potential competitors that eat each other, Annual review of ecology and systematics, **20(1)**, 297-330, (1989).

[37] Ghosh. B, Multistability, chaos and mean population density in a discrete-time predator-prey system, Chaos, Solitons & Fractals, **162**, 112497,(2022).

[38] Roueff. F,Dimension de Hausdorff du graphe d'une fonction continue: nouvelles majorations déterministes et minorations presque sûres, Comptes Rendus de l'Académie des Sciences-Series I-Mathematics, **332(10)**, 875-880, (2001).

[39] Senouci. A, Systèmes dynamiques chaotiques et synchronisation, Thèse de doctorat, Université Mohamed Khider, Biskra. , (2020).

[40] Smith. H. L, The Rosenzweig-Macarthur predator-prey model. School of Mathematical and Statistical Sciences, (2008).

[41] Strogatz. S. H, Nonlinear dynamics and chaos: with applications to physics, biology, chemistry, and engineering, CRC press, Reading Mass 1994.

[42] Wang. Y, Wu, H. Wang, A predator-prey model characterizing negative effect of prey on its predator, Applied Mathematics and Computation, **219(19)**, 9992-9999,(2013).

[43] Weide. V, Varriale. M.C., Hilker, F. M, Hydra effect and paradox of enrichment in discrete-time predator-prey models, Mathematical biosciences, **310**, 120-127, (2019)

[44] Wiggins.S, Introduction to Applied Nonlinear Dynamical Systems and Chaos, second edition, Springer-Verlag, New York, 2000.

[45] Witvoet.G, Control of chaotic dynamical systems using OGY , 2005.

[46] Yue. Y, Xie. J, Lyapunov exponents and coexistence of attractors in vibro-impact systems with symmetric two-sided rigid constraints, Physics Letters A ,**373(23-24)**, 2041-2046, (2009).

[47] Zoghlami.N, Stabilité et stabilisation en temps fini des systèmes dynamiques interconnectés et problème de consensus en temps fini, thèse Doctorat de l'ueve et Enit, Université d'Evryval d'Essonne et Lim ,Ecole polytechnique de Tunisie,2014.