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ABSTRACT

Our aim in this thesis is to study a first-order linear difference equation with positive fuzzy coefficients.

In the first chapter, we presented some definitions and the main theories related to linear and nonlinear difference equations.

The second chapter focused on fuzzy sets, including some of their important properties and illustrative examples, as well as fuzzy numbers, which are part of them. We discussed their theoretical aspects, fundamental properties, various types, and arithmetic operations on them.

The third chapter was dedicated to studying the existence, uniqueness, boundedness, persistence, and convergence of the positive fuzzy solution.

Key words: fuzzy difference equation, fuzzy numbers, fuzzy sets, boundedness, persistence.

ملخص

هدفنا في هذه المذكرة هو دراسة معادلة فروق خطية من الدرجة الأولى بمعاملات ضبابية موجبة.

في الفصل الأول قدمنا بعض التعاريف و النظريات الرئيسية المتعلقة بمعادلة الفروق الخطية و غير الخطية .

الفصل الثاني تمحور حول المجموعات الضبابية بالإضافة إلى بعض خواصها الهامة و أمثلة توضيحية ,والأعداد الضبابية التي هي جزء منها، حيث تم التطرق لجانبها النظري وخواصها الأساسية والبعض من أنواعها والعمليات الحسابية عليها .

الفصل الثالث خصص لدراسة وجود، وحدانية، محدودية و استمرارية الحل الضبابي الموجب وتقاربه.

الكلمات المفتاحية: معادلة الفروق الضبابية، الأعداد الضبابية، المجموعات الضبابية، المحدودية و الاستمرارية.

RÉSUMÉ

Notre objectif dans ce mémoire est d'étudier une équation aux différences linéaires du premier ordre avec des coefficients flous positifs.

Dans le premier chapitre, nous avons présenté quelques définitions et théories principales relatives aux équations aux différences linéaires et non linéaires.

Le deuxième chapitre est concentré sur les ensembles flous, ainsi que sur certaines de leurs propriétés importantes et des exemples illustratifs, ainsi que sur les nombres flous qui en font partie. Nous avons discuté de leurs aspects théoriques, de leurs propriétés fondamentales, de certains types et des opérations arithmétiques associées.

Le troisième chapitre a été consacré à l'étude de l'existence, de l'unicité, de la bornitude , de la continuité et de la convergence de la solution floue positive.

Mots-clés : équation aux différences floues, nombres flous, ensembles flous, bornitude, continuité.

CONTENTS

INTRODUCTION

Fuzzy sets and fuzzy numbers are fundamental concepts in fuzzy theory, which was developed by Lotfi Zadeh, in 1965 [16]. This theory aims to handle uncertain and imprecise information that is difficult to process using traditional methods.

Additionally, fuzzy sets are a generalization of classical (traditional) sets, allowing an element to belong to the set with degrees of membership ranging between 0 and 1. The concept of a fuzzy number and fuzzy arithmetic operations was introduced by Zadeh, Dubois and Prade [16, 8]. A fuzzy number is represented by a fuzzy set on the real number line and is used in various fields such as artificial intelligence, engineering, and economics.

Difference equations play an important role in mathematics, applied sciences, engineering, and various other fields. Their significance is evident in modeling many biological, physical, and social phenomena, such as the motion of bodies and the spread of diseases. They help in understanding how systems change, analyzing their stability, and predicting the behavior of complex systems.

Difference equations are essential for understanding, describing, and predicting the behavior of continuously changing systems.

A fuzzy difference equation involves sequence differences. Solving a difference equation involves finding a sequence that satisfies the equation. The sequence that satisfies the equation is called a solution of the equation. A fuzzy difference equation is a difference equation where constants and the initial values are fuzzy numbers, and its solutions are sequences of fuzzy numbers. Fuzzy difference equations have been rapidly developed over the years as discrete analogs and numerical solutions of differential equations.

The aim of this research is to verify the existence, uniqueness, and behavior of the global solution. This research consists of three chapters:

In the first chapter, we addresse some basic concepts of linear and nonlinear difference equations and stability.

The work done in the second chapter is divide into two parts. In the first part, we presente some fundamental definitions regarding fuzzy sets and their operations, followed by definitions related to fuzzy numbers, their properties, types, and operations in the second part.

In the final chapter, we solve the first-order fuzzy difference equation, where Deeba and Korvin studied [6] its global behavior, which gives the frequency of genetic patterns. First, we discussed the classical solution of the first-order difference equation with constant coefficients and the initial condition x_0.

$$x_{n+1} = wx_n + q, \quad n = 0, 1, 2, ...$$

Second, we study the existence, boundedness, and persistence of the positive fuzzy solution of the fuzzy difference equation, where x_n is sequences of fuzzy numbers and $w, q, x_0 \in \mathbb{R}_F^+$.

CHAPTER 1

PRELIMINARIES

In this chapter, we present definitions of linear and nonlinear difference equations, stability using the famous method of linearization, periodicity, permanence, as well as some Theorems of convergence.

1.1 Linear difference equations

Definition 1.1.1 [5] *The equation*

$$x_{n+k} = f(x_{n+k-1}, x_{n+k-2}, ..., x_n), \tag{1.1}$$

for a given function f and unknown quantities x_i, $i = 0, 1, \dots$ is called a difference equation of order k.

If f is linear, it is called a linear difference equation and it is in the form

$$a_k x_{n+k} + a_{k-1} x_{n+k-1} + \cdots + a_0 x_n = b, a_0 \neq 0 \tag{1.2}$$

According to whether the coefficients and the right hand side of the equation depend on n or not, it is called an equation with variable or constant coefficients respectively.
When the right hand side $b \neq 0$, then the equation $x_{n+k} = f(x_{n+k-1}, x_{n+k-2}, ..., x_n)$ is non-homogeneous, while for $b = 0$, i.e

$$a_k x_{n+k} + a_{k-1} x_{n+k-1} + \cdots + a_0 x_n = 0. \tag{1.3}$$

This is called a linear homogeneous difference equation.

1.2 Non linear difference equations

Let I be an interval of real numbers and let

$$f : I^{k+1} \rightarrow I,$$

where f is a continuously differentiable function. Consider the difference equation

$$x_{n+1} = f(x_n, x_{n-1}, ..., x_{n-k}), \quad n = 0, 1, 2, ..., \tag{1.4}$$

with the initial conditions $x_{-k}, x_{-k+1}, ..., x_0 \in I$.

Definition 1.2.1 [14] ***(Equilibrium point)*** *A point $\bar{x}$ is said to be an equilibrium point for the equation (1.4) if*

$$\bar{x} = f(\bar{x}, \bar{x}, ..., \bar{x}), \tag{1.5}$$

in other words

$$x_n = \bar{x}, \quad \forall n \geqslant -k. \tag{1.6}$$

Definition 1.2.2 [10] ***(Periodicity)***

A solution $\{x_n\}_{n=-k}^{\infty}$ *of Eq. (1.4) is called periodic with period p (or a period p solution) if there exists an integer* $p \geqslant 1$ *such that*

$$x_{n+p} = x_n, \forall n \geqslant -k. \tag{1.7}$$

We say that the solution is periodic with prime period p if p is the smallest positive integer for which Eq. (1.7) holds. In this case, a p-tuple

$$(x_{n+1}, x_{n+2}, ..., x_{n+p})$$

of any p consecutive values of the solution is called a p-cycle of Eq.(1.4).

Definition 1.2.3 [10] ***(Eventually periodic)***

A solution $\{x_n\}_{n=-k}^{\infty}$ *of Eq.(1.4) is called eventually periodic with period p if there exists an integer* $N \geqslant -k$ *such that* $\{x_n\}_{n=N}^{\infty}$ *is periodic with period p that is,*

$$x_{n+p} = x_n, \forall n \geqslant N.$$

Definition 1.2.4 [9] ***(Permanence)***

The difference equation

$$x_{n+1} = f(x_n, x_{n-1}, ..., x_{n-k}), \quad n = 0, 1, ..., \tag{1.8}$$

is said to be permanent if there exist numbers m and M with $0 < m \leqslant M < \infty$ *such that for any initial conditions* $x_{-k}, x_{-k+1}, ..., x_{-1}, x_0 \in (0, \infty)$ *there exists a positive integer N which depends on the initial conditions such that*

$$m \leqslant x_n \leqslant M, \quad \forall n \geqslant N. \tag{1.9}$$

Definition 1.2.5 [14] ***(Invariant interval)***

An interval $J \subseteq I$ is said to be an invariant interval for equation (1.4) if

$$x_{-k}, x_{-k+1}, ..., x_0 \in J \implies x_n \in J, \quad n > 0. \tag{1.10}$$

1.2.1 Stability of non linear difference equations

Definition 1.2.6 [9] *Let $\overline{x}$ be an equilibrium point of Eq (1.4).*

(i) $\overline{x}$ ***is locally stable*** *if*

$$\forall \epsilon > 0, \exists \delta > 0, \forall x_{-k}, x_{-k+1}, ..., x_{-1}, x_0 \in I : \left|x_{-k} - \overline{x}\right| + \left|x_{-k+1} - \overline{x}\right| + ... + \left|x_0 - \overline{x}\right| < \delta,$$

we have

$$\left|x_n - \overline{x}\right| < \epsilon, \forall n \geqslant -k.$$

(ii) $\overline{x}$ *is* ***locally asymptotically stable*** *if*

. $\overline{x}$ *is locally stable,*

. $\exists \gamma > 0, \forall x_{-k}, x_{-k+1}, ..., x_{-1}, x_0 \in I : \left|x_{-k} - \overline{x}\right| + \left|x_{-k+1} - \overline{x}\right| + ... + \left|x_0 - \overline{x}\right| < \gamma$, *we have*

$$\lim_{n\to\infty} x_n = \overline{x}.$$

(iii) $\overline{x}$ *is* ***global attractor*** *if*

$$\forall x_{-k}, x_{-k+1}, ..., x_{-1}, x_0 \in I, \lim_{n\to\infty} x_n = \overline{x}.$$

(iv) $\overline{x}$ *is* ***globally asymptotically stable*** *if*

. $\overline{x}$ *is locally stable,*

. $\overline{x}$ *is also a global attractor.*

(v) *The equilibrium point $\overline{x}$ of Eq (1.4) is* ***unstable*** *if $\overline{x}$ is not locally stable.*

Definition 1.2.7 [14] *The equation*

$$y_{n+1} = p_0 y_n + p_1 y_{n-1} + \cdots + p_k y_{n-k}, \tag{1.11}$$

is called the linearized equation of Eq. (1.4)*, with*

$$p_i = \frac{\partial f}{\partial u_i}(\overline{x}, \overline{x}, ..., \overline{x}), i = 0, ..., k,$$

and

$$\begin{aligned} f: \quad I^k &\longrightarrow I \\ (u_1, ..., u_k) &\longmapsto f(u_1, ..., u_k), \end{aligned}$$

and

$$p(\lambda) = \lambda^{k+1} - p_0\lambda^k - \cdots - p_k, \tag{1.12}$$

its associated characteristic polynomial.

Theorem 1.2.1 [10]***(Stability by linearization)***
Suppose f is a continuously differentiable function defined on some open neighborhood of $\overline{x}$. Then the following statement are true:
1. *If all the roots of Eq.*(1.12) *have absolute value less than one, then the equilibrium point $\overline{x}$ of equation* (1.4) *is locally asymptotically stable.*
2. *If at least one root of Eq.*(1.12) *has absolute value greater than one, then the equilibrium point $\overline{x}$ of equation* (1.4) *is unstable.*

Theorem 1.2.2 [9]***(The clark Theorem)*** *Assume that $p, q \in \mathbb{R}$ and $k \in \{0, 1, 2, ...\}$. Then*

$$|p| + |q| < 1$$

is a sufficient condition for the asymptotic stability of the difference equation

$$x_{n+1} + px_n + qx_{n-k} = 0, \quad n = 0, 1, ...$$

Remarque 1.2.3 [9] *Theorem* (1.2.2) *can be easily extended to general linear equations of the form*

$$x_{n+k} + p_1x_{n+k-1} + \cdots + p_kx_n = 0, \quad n = 0, 1, ... \tag{1.13}$$

where $p_1, p_2, ..., p_k$ *and* $k \in \{1, 2, ...\}$. *Then Eq* (1.13) *is asymptotically stable provided that*

$$\sum_{i=1}^{k} |p_i| < 1.$$

Theorem 1.2.4 [14]***(Theorem of Rouché)*** *Let* $f(z), g(z)$ *be two holomorphic functions in an open set* Ω *in the complex plane* $\mathbb{C}$, *and let* K *be a compact with boundary contained in* Ω. *If we have*

$$|g(z)| < |f(z)|, \forall z \in \partial K,$$

then the number of zeros of $f(z) + g(z)$ *in* K *is equal to the number of zeros of* $f(z)$ *in* K, *where* ∂K *is the boundary of* K.

1.2.2 Theorems of convergence

Theorem 1.2.5 [10] *Let* $g : [a, b] \times [a, b] \longrightarrow [a, b]$ *be a continuous function, where* a *and* b *are real numbers with* $a < b$, *and consider the difference equation*

$$x_{n+1} = g(x_n, x_{n-1}), \quad n = 0, 1, ... \tag{1.14}$$

Suppose that g *satisfies the following coditions:*

1) $g(x, y)$ *is non-decreasing in* $x \in [a, b]$ *for each fixed* $y \in [a, b]$, *and* $g(x, y)$ *is non-decreasing in* $y \in [a, b]$ *for each fixed* $x \in [a, b]$;

2) *If* (m, M) *is a solution of the system*

$$\begin{cases} m = g(m, m) \\ M = g(M, M), \end{cases}$$

then $\quad m = M$.

Then there exists exactly one equilibrium $\overline{x}$ *of Eq.*(1.14), *and every solution of Eq* (1.14) *converges to* $\overline{x}$.

Theorem 1.2.6 [10] *Let $g : [a, b] \times [a, b] \to [a, b]$ be a continuous function, where a and b are real numbers with $a < b$, and consider the difference equation*

$$x_{n+1} = g(x_n, x_{n-1}), \quad n = 0, 1, ...$$

Suppose that g satisfies the following coditions:

1) *$g(x, y)$ is non-increasing in $x \in [a, b]$ for each fixed $y \in [a, b]$, and $g(x, y)$ is non-decreasing in $y \in [a, b]$ for each fixed $x \in [a, b]$;*

2) *If (m, M) is a solution of the system*

$$\begin{cases} m = g(M, m) \\ M = g(m, M) \end{cases}$$

then $\quad m = M.$

Then there exists exactly one equilibrium $\overline{x}$ of Eq.(1.14), and every solution of Eq (1.14) converges to $\overline{x}$.

Theorem 1.2.7 *[10] Let $g : [a, b] \times [a, b] \to [a, b]$ be a continuous function, where a and b are real numbers with $a < b$, and consider the difference equation*

$$x_{n+1} = g(x_n, x_{n-1}), \quad n = 0, 1, ...$$

Suppose that g satisfies the following coditions:

1) *$g(x, y)$ is non-decreasing in $x \in [a, b]$ for each fixed $y \in [a, b]$, and $g(x, y)$ is non-increasing in $y \in [a, b]$ for each fixed $x \in [a, b]$;*

2) *If (m, M) is a solution of the system*

$$\begin{cases} m = g(m, M) \\ M = g(M, m), \end{cases}$$

then $\quad m = M.$

Then there exists exactly one equilibrium $\overline{x}$ of Eq.(1.14), and every solution of Eq (1.14) converges *to* $\overline{x}$.

Theorem 1.2.8 [10] *Let* $g : [a, b] \times [a, b] \longrightarrow [a, b]$ *be a continuous function, where a and b are real numbers with* $a < b$*, and consider the difference equation*

$$x_{n+1} = g(x_n, x_{n-1}), \quad n = 0, 1, ...$$

Suppose that g satisfies the following coditions:

1) $g(x, y)$ *is non-increasing in* $x \in [a, b]$ *for each fixed* $y \in [a, b]$*, and* $g(x, y)$ *is non-increasing in* $y \in [a, b]$ *for each fixed* $x \in [a, b]$*;*

2) *If* (m, M) *is a solution of the system*

$$\begin{cases} m = g(M, M) \\ M = g(m, m), \end{cases}$$

then $\quad m = M.$

Then there exists exactly one equilibrium $\overline{x}$ *of Eq.*(1.14)*, and every solution of Eq* (1.14) *converges to* $\overline{x}$.

Theorem 1.2.9 [10] *Let* $g : [a, b]^{k+1} \longrightarrow [a, b]$ *be a continuous function, where k is a positive integer, and where [a,b] is an interval of real numbers. Consider the difference equation*

$$x_{n+1} = g(x_n, x_{n-1}, ..., x_{n-k}), \quad n = 0, 1, ... \tag{1.15}$$

Suppose that g satisfies the following coditions:

1. *For each integer i with* $1 \leqslant i \leqslant k+1$*, the function* $g(z_1, z_2, ..., z_{k+1})$ *is weakly monotonic in* z_i *for fixed* $z_1, z_2, ..., z_{i-1}, z_{i+1}, ..., z_{k+1}$.

2. *If* (m, M) *is a solution of the system*

$$\begin{cases} m = g(m_1, m_2, ..., m_{k+1}) \\ M = g(M_1, M_2, ..., M_{k+1}) \end{cases}$$

then $m = M$*, where for* $i = 1, 2, ..., k+1$*, we set*

$$m_i = \begin{cases} m & if \quad g \quad \text{is non} - \text{decreasing in } z_i \\ M & if \quad g \quad \text{is non} - \text{increasing in } z_i. \end{cases}$$

And

$$M_i = \begin{cases} M & if \quad g \quad \text{is non} - \text{decreasing in } z_i \\ m & if \quad g \quad \text{is non} - \text{increasing in } z_i. \end{cases}$$

Then there exists exactly one equilibrium $\overline{x}$ *of Eq.*(1.15)*, and every solution of Eq* (1.15) *converges to* $\overline{x}$*.*

CHAPTER 2

FUZZY SETS AND FUZZY NUMBERS

In this chapter, we present some basic definitions about fuzzy sets and fuzzy numbers. In the first part, we focus on fuzzy sets and operations on them. In the last part, we present some definitions, properties, types and operations on fuzzy numbers.

2.1 Fuzzy sets

2.1.1 Fuzzy logic

Definition 2.1.1 *[7] Fuzzy Logic is an extension of Boolean logic by Lotfi Zadeh in 1965 based on the mathematical theory of fuzzy sets, which is a generalization of the classical set theory. By introducing the notion of degree in the verification of a condition, thus enabling a condition to be in a state other than true or false, fuzzy logic provides a very valuable flexibility for reasoning, which makes it possible to take into account inaccuracies and uncertainties.*
One advantage of fuzzy logic in order to formalize human reasoning is that the rules are set in

natural language.

2.1.2 Classical sets

Definition 2.1.2 *[8] Let X be a classical set of objects, called the universe, whose generic elements are denoted x. Membership in a classical subset A of X is often viewed as a characteristic function ,* μ_A *from X to* $\{0, 1\}$ *such that*

$$\chi_A(x) = \begin{cases} 1 & if\ x \in A \\ 0 & if\ x \notin A \end{cases}$$

Example 2.1.1 *Let X=R be the reference set and A be the set of numbers between 3 and 9, it is characterized by the following characteristic function*

$$\chi_A : R \longrightarrow \{0, 1\}$$

$$\chi_A(x) = \begin{cases} 1 & if\ x \in A \\ 0 & if\ x \notin A \end{cases}$$

$\chi_A(4) = 1, \quad \chi_A(2) = 0.$

2.1.3 Concept of fuzzy sets

Fuzzy sets were introduced by L. Zadeh . The definition of a fuzzy set given by L. Zadeh is as follows: A fuzzy set is a class with a continuum of membership grades. So a fuzzy set A in a referential (universe of discourse) X is characterized by a membership function A which associates with each element $x \in X$ a real number $A(x) \in [0, 1]$, having the interpretation A(x) is the membership grade of x in the fuzzy set A [3].

Definition 2.1.3 *[3] A fuzzy set A (fuzzy subset of X) is defined as a mapping*

$$A : X \longrightarrow [0, 1]$$

where A(x) is the membership degree of x to the fuzzy set A. We denote by F(X) the collection of all fuzzy subsets of X. Fuzzy sets are generalizations of the classical sets represented by their characteristic functions $\chi_A : X \longrightarrow \{0,1\}$. In our case $A(x) = 1$ means full membership of x in A, while $A(x) = 0$ expresses non-membership, but in contrary to the classical case other membership degrees are allowed. We identify a fuzzy set with its membership function. Other notations that can be used are the following $\mu_A(x) = A(x)$. Every classical set is also a fuzzy set. We can define the membership function of a classical set $A \subseteq X$ as its characteristic function

$$\mu_A(x) = \begin{cases} 1 & if\ x \in A \\ 0 & otherwise \end{cases}$$

Definition 2.1.4 *[17] If X is a collection of objects denoted generically by x, then a fuzzy set A in X is a set of ordered pairs:*

$$A = \{(x, \mu_A(x)) | x \in X\}$$

$\mu_A(x)$ is called the membership function or grade of membership .

Example 2.1.2 *A realtor wants to classify the house he offers to his clients. One indicator of comfort of these houses is the number of bedrooms in it. Let X=$\{1, 2, ..., 10\}$ be the set of available types of houses described by x = number of bedrooms in a house. Then the fuzzy set "comfortable type of house for a four-person family" may be described as*

$$A = \{(1, 0.2), (2, 0.5), (3, 0.8), (4, 1), (5, 0.7), (6, 0.3)\}$$

2.1.4 Operations between fuzzy sets

- **Equality**

Two fuzzy sets A and B are equal, whritten as A=B, if and only if $\mu_A(x) = \mu_B(x)$ For all x in X [16].

- **Complement**

The complement of a fuzzy set A is denoted byA^c and is defined by [16].

$$\mu_{A^c} = 1 - \mu_A$$

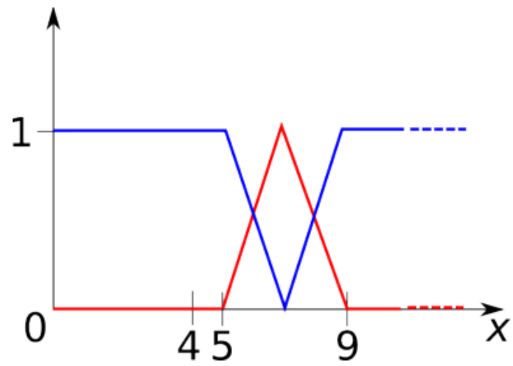

Figure 2.1: *Complement of fuzzy sets.*

- **Containment**

A is contained in B if and only if $\mu_A \leq \mu_B$. In symbols[16]

$$A \subset B \Leftrightarrow \mu_A \leq \mu_B$$

- **Union**

The union of two fuzzy sets A and B with respective membership functions $\mu_A(x)$ and $\mu_B(x)$ is a fuzzy set C, written as $C = A \cup B$, whose membership function is related to those of A and B by [16]

$$\mu_C(x) = \max\left[\mu_A(x), \mu_B(x)\right], x \in X$$

Or, in abbreviated form

$$\mu_C = \mu_A \vee \mu_B$$

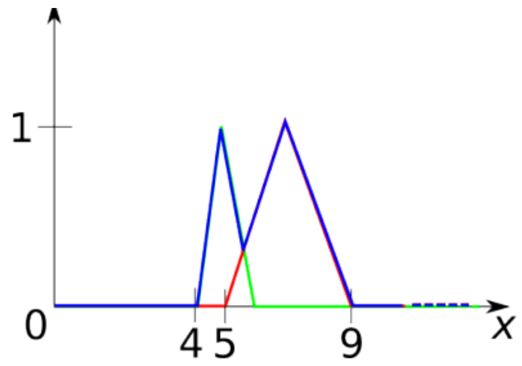

Figure 2.2: *Union of fuzzy sets.*

- **Intersection**

The intersection of two fuzzy sets A and B with respective membership functions $\mu_A(x)$ and $\mu_B(x)$ is a fuzzy set C, written as $C = A \cap B$, whose membership function is related to those of A and B by [16]:

$$\mu_C(x) = \min\left[\mu_A(x), \mu_B(x)\right], x \in X.$$

Or, in abbreviated form:

$$\mu_C = \mu_A \wedge \mu_B.$$

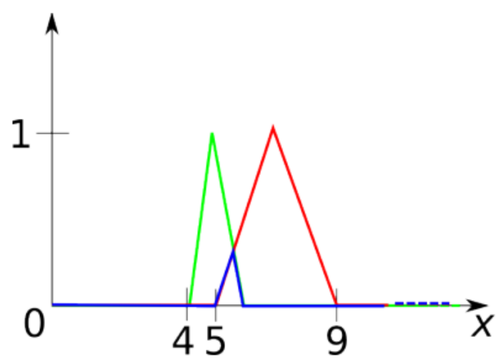

Figure 2.3: *Intersection of fuzzy sets.*

Example 2.1.3 *Let A be the fuzzy set "comfortable type of house for a four-person family" from example 2.1.2 and B be the fuzzy set "large type of house" defined as:*

$B = \{(3, 0.2), (4, 0.4), (5, 0.6), (6, 0.8), (7, 1), (8, 1)\}$.

The intersection $C = A \cap B$ *is then*

$C = \{(3, 0.2), (4, 0.4), (5, 0.6), (6, 0.3)\}$.

The union $D = A \cup B$ *is:*

D= {(1, 0.2), (2, 0.5), (3, 0.8), (4, 1), (5, 0.7), (6, 0.8), (7, 1), (8, 1)}.

The complement B^c*, which might be interpreted as "not large type of house," is :*

$B^c = \{(1, 1), (2, 1), (3, 0.8), (4, 0.6), (5, 0.4), (6, 0.2), (9, 1), (10, 1)\}$.

List the basic properties of complement, union and intersection:

All sets are subsets of the same X [4].

- **Involution**: $\left(A^C\right)^C = A$.
- **Commutativity**: $$A \cup B = B \cup A, \quad A \cap B = B \cap A.$$
- **Associativity**: $$(A \cup B) \cup C = A \cup (B \cup C), \quad (A \cap B) \cap C = A \cap (B \cap C).$$
- **Distributivity**: $$A \cap (B \cup C) = (A \cap B) \cup (A \cap C), \quad A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$$
- **Idempotency**: $$A \cap A = A, \quad A \cup A = A.$$
- **Law of contradiction**: $A \cap A^c = \emptyset$.

- **Law of excluded middle**: $A \cup A^c = X$.

- **De morgan**: $$(A \cup B)^c = A^c \cap B^c,$$ $$(A \cap B)^c = A^c \cup B^c.$$

- **Identity**: $$A \cup \varnothing = A, A \cap \varnothing = \varnothing,$$ $$A \cup X = X, A \cap X = A.$$

- **Absorbtion**: $$A \cup (A \cap B) = A,$$ $$A \cap (A \cup B) = A.$$

2.1.5 Algebraic operations on fuzzy sets

For the time being we return to ordinary fuzzy sets and consider additional operations on them that have been defined in the literature and that will be useful or even necessary for later chapters[17].

- **Product** :

  The algebraic product of two fuzzy sets $C = A \cdot B$ is defined as:

$$C = \{(x, \mu_A(x) \cdot \mu_B(x)), x \in X\}.$$

- **Cartesian Product** :

  Cartesian product applied to multiple fuzzy sets can be defined as follows. Denoting $\mu_{A_1}(x), \mu_{A_2}(x), ..., \mu_{A_n}(x)$ as membership functions of $A_1, A_2, ..., A_n$ for $\forall x_1 \in A_1, x_2 \in A_2, ..., x_n \in A_n$.

  Then, the probability for n-tuple $(x_1, x_2, ..., x_n)$ to be involved in fuzzy set $A_1 \times A_2 \times ... \times A_n$ is:

$$\mu_{A_1 \times A_2 \times ... \times A_n}(x_1, x_2, ..., x_n) = \min[\mu_{A_1}(x_1), \mu_{A_2}(x_2), ..., \mu_{A_n}(x_n)].$$

- **Sum** :

  The algebraic sum (probabilistic sum) $C = A + B$ is defined as :

$$C = \{(x, \mu_{A+B}(x)), x \in X\},$$

  where

$$\mu_{A+B}(x) = \mu_A(x) + \mu_B(x) - \mu_A(x) \cdot \mu_B(x).$$

- **Bounded sum** :

  The bounded sum $C = A \oplus B$ is defined as:

$$C = \{(x, \mu_{A\oplus B}(x)), x \in X\},$$

where

$$\mu_{A\oplus B}(x) = \min\{1, \mu_A(x) + \mu_B(x)\}.$$

- **Bounded difference** :

  The bounded difference $C = A \ominus B$ is defined as:

$$C = \{(x, \mu_{A\ominus B}(x)), x \in X\},$$

where

$$\mu_{A\ominus B}(x) = \max\{0, \mu_A(x) + \mu_B(x) - 1\}.$$

- **Mth power**:

  The mth power of a fuzzy set A is a fuzzy set with the membership function.

$$\mu_{A^m}(x) = [\mu_A(x)]^m, x \in X.$$

Example 2.1.4 *Let A and B two fuzzy sets:*

$A(x) = \{(3, 0.5), (5, 1), (7, 0.6)\}$.

$B(x) = \{(3, 1), (5, 0.6)\}$.

The above definitions are then illustrated by the following results:

$A \times B = \{[(3; 3), 0.5], [(5; 3), 1], [(7; 3), 0.6][(3; 5), 0.5], [(5; 5), 0.6], [(7; 5), 0.6]\}$.

$A^2 = \{(3, 0.25), (5, 1), (7, 0.36)\}$.

$A + B = \{(3, 1), (5, 1), (7, 0.6)\}$.

$A \oplus B = \{(3, 1), (5, 1), (7, 0.6)\}$.

$A \ominus B = \{(3, 0.5), (5, 0.6)\}$.

$A \cdot B = \{(3, 0.5), (5, 0.6)\}$.

2.1.6 Caracteristics of a fuzzy sets

In order to define the characteristics of fuzzy sets, we are redefining and expanding the usual characteristics of classical sets [7].

Definition 2.1.5 *[7] The **height** of A, denoted h(A), corresponds to the upper bound of the codomain of its membership function:*

$$h(A) = sup\{\mu_A(x) | x \in X\}.$$

Definition 2.1.6 *[7] A is said to be **normalised** if and only if h(A) = 1. In practice, it is extremely rare to work on non-normalised fuzzy sets.*

Definition 2.1.7 *[7] The **support** of A is the set of elements of X belonging to at least some A (i.e.The membership degree of x is strictly positive). In other words, the support is the set:*

$$supp(A) = \{x \in X | \mu_A(x) > 0\}.$$

Definition 2.1.8 *[7] The **kernel** of A is the set of elements of X belonging entirely to A. In other words, the kernel* $noy(A) = \{x \in X | \mu_A(x) = 1\}$*. By construction,* $noy(A) \subseteq supp(A)$*.*

Definition 2.1.9 *[17] For a finite fuzzy set A also X finite, the* ***cardinality*** $|A|$ *is defined as:*

$$|A| = \sum_{x \in X} \mu_A(x).$$

$\| A \| = \frac{|A|}{|X|}$ *is called the relative cardinality of A.*
Obviously, the relative cardinality of a fuzzy set depends on the cardinality of the universe. So you have to choose the same universe if you want to compare fuzzy sets by their relative cardinality.

Example 2.1.5 ***1)*** *Let X={A,B,E,F,G,I} finit set and the fuzzy sets A be given by:*
$A = \{(A;0.6);(B;0.7);(E;0.4);(F;0.3);(G;0.8);(I;0.5)\}$.
So $h(A) = 0.8,\ supp(A) = X,\ noy(A) = \emptyset,\ |A| = 3.3$
And $B = \{(A;0);(B;0);(E;1);(F;0.8);(G;0);(I;1)\}$.
So B is a normalised fuzzy set, because $h(B) = 1,\ supp(B) = \{E,F,I\},\ noy(B) = \{E,I\},$
$|B| = 2.8$.
2) *Let X=[0,35] (the set of Ages) such as* $\alpha \in [0,1]$*, and let A be a fuzzy set of X of young ages given by :*

$$\mu_A(x) = \begin{cases} 1 & if\ x \in [20,30] \\ 0 & if\ x \geqslant 35 \quad and \quad x \leqslant 15 \\ \alpha & if\ x \in]15,20[\ and\ x \in]30,35[\end{cases}$$

noy(A)=[20,30], supp(A)=]15,35[, and h(A)=1.

Definition 2.1.10 *[17] A fuzzy set A is* ***convex*** *if:*

$$\mu_A(\lambda x_1 + (1-\lambda)x_2) \geq \min\{\mu_A(x_1), \mu_A(x_2)\}, x_1, x_2 \in X, \lambda \in [0,1].$$

Alternatively*, a fuzzy set is convex if all* $\alpha -$ *level sets are convex.*

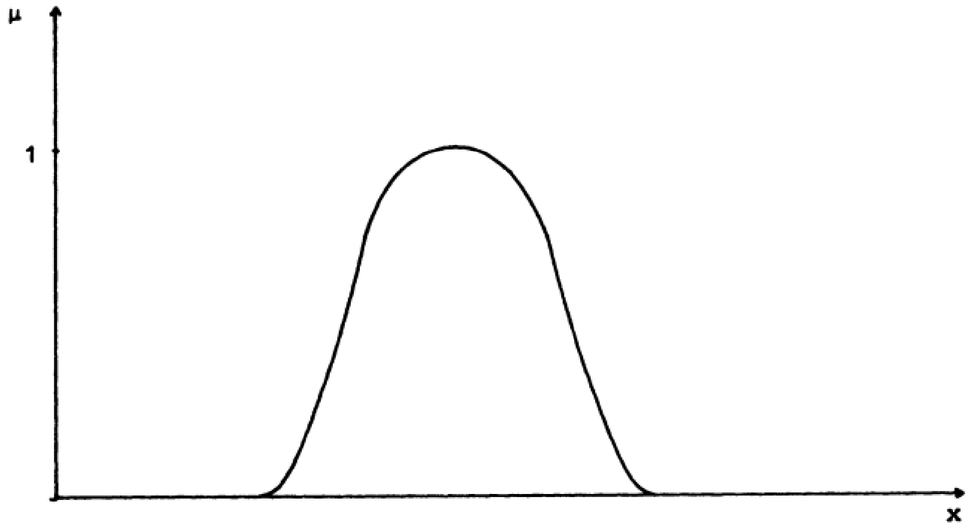

Figure 2.4: *Convex fuzzy set.*

2.1.7 Norms triangular and conorms triangular

We shall investigate the two basic classes of operators: Operators for the intersection and union of fuzzy sets referred to as triangular norms and conorms and the class of averaging operators, which model connectives for fuzzy sets between t-norms and t-conorms. Each class contains parameterized as well as nonparameterized operators [17].

Definition 2.1.11 *[17] t-norms (Triangular Norme) are two-valued functions from* $[0,1] \times [0,1]$ *that satisfy the following conditions:*

1. $T(0,0) = 0; T(\mu_A(x), 1) = T(1, \mu_A(x)) = \mu_A(x), x \in X$.

2. $T(\mu_A(x), \mu_B(x)) \leq T(\mu_C(x), \mu_D(x))$ *if* $\mu_A(x) \leq \mu_C(x)$ *and* $\mu_B(x) \leq \mu_D(x)$ ***(monotonicity)****.*

3. $T(\mu_A(x), \mu_B(x)) = T(\mu_B(x), \mu_A(x))\ \forall \mu_A(x), \mu_B(x) \in [0,1]$ ***(commutativity)****.*

4. $T(\mu_A(x), T(\mu_B(x), \mu_C(x))) = T(T(\mu_A(x), \mu_B(x)), \mu_C(x))\ \forall \mu_A(x), \mu_B(x), \mu_C(x) \in [0,1]$ ***(associativity)****.*

The functions T define a general class of intersection operators for fuzzy sets.

Definition 2.1.12 *[17] t-conorms or s-norms are associative, commutative, and monotonic two-placed functions S that map from* $[0,1] \times [0,1]$ *into* $[0,1]$*. These properties are formulated* *with the following conditions:*

1. $S(1,1) = 1; S(\mu_A(x), 0) = S(0, \mu_A(x)) = \mu_A(x), x \in X$.
2. $S(\mu_A(x), \mu_B(x)) \leq S(\mu_C(x), \mu_D(x))$ *if* $\mu_A(x) \leq \mu_C(x)$ *and a* $\mu_B(x) \leq \mu_D(x)$ ***(monotonicity)*** .
3. $S(\mu_A(x), \mu_B(x)) = S(\mu_B(x), \mu_A(x))\ \forall \mu_A(x), \mu_B(x) \in [0,1]$ ***(commutativity)***.
4. $S(\mu_A(x), S(\mu_B(x), \mu_C(x))) = S(S(\mu_A(x), \mu_B(x)), \mu_C(x))\ \forall \mu_A(x), \mu_B(x), \mu_C(x) \in [0,1]$ ***(associativity)***.

A general class of aggregation operators for the union of fuzzy sets called triangular conorms or t-conorms (sometimes referred to as s-norms).

Remarque 2.1.1 *[17] t-norms and t-conorms are related in a sense of logical duality. Alsina [Alsina 1985] defined a t-conorm as a two-placed function S mapping from* $[0,1] \times [0,1]$ *into* $[0,1]$ *such that the function T, defined as*

$$T(\mu_A(x), \mu_B(x)) = 1 - S(1 - \mu_A(x), 1 - \mu_B(x)).$$

2.1.8 Concept of α-level

A fuzzy subset A of U is "formed" by elements of U with an order (hierarchy) that is given by the membership degrees. An element x of U will be in an "order class" α if its degree of belonging (its membership value) is at least the threshold level $\alpha \in [0,1]$ that defines that class. The classic set of such elements is called an α-level of A, denoted $[A]_\alpha$ [2].

Definition 2.1.13 *[2] (α-level) Let A be a fuzzy subset of U and* $\alpha \in [0,1]$. *The α-level of the subset A is classical set* $[A]_\alpha$ *of U defined by:*

$$[A]_\alpha = \{x \in U : \mu_A(x) \geq \alpha\} \text{ for } 0 < \alpha \leqslant 1,$$

when U is a topological space, the zero α-level of the fuzzy subset A is defined as the smallest closed subset (in the classic sense) in U containing the support set of A. In mathematical terms, $[A]_0$ *is the closure of the support of A and is also denoted by* $\overline{suppA}$. *This consideration becomes* *essential in theoretical situations appearing in this text. Note also that the set* $\{x \in U : \mu_A(x) \geq 0\} = U$ *is not necessarily equal* $[A]_0 = \overline{suppA}$.

Example 2.1.6 *Let* $U = \mathbb{R}$ *be the set of real numbers and let A be a fuzzy subset of* $\mathbb{R}$ *with the following function membership function:*

$$\mu_A(x) = \begin{cases} x-1 & if\ 1 \leqslant x \leqslant 2 \\ 3-x & if\ 2 < x < 3 \\ 0 & if\ x \notin [1,3]. \end{cases}$$

In this case we have: $[A]_\alpha = [\alpha+1, 3-\alpha]$ *for* $0 < \alpha \leqslant 1$ *and* $[A]_0 = \overline{]1,3[} = [1,3]$.

2.2 Fuzzy number

First of all, we'll look into interval, the fundamental concept of fuzzy number, and then operation of fuzzy numbers. In addition, we'all introduce special kind of fuzzy number such as triangular fuzzy number and trapezoidal fuzzy number[6].

2.2.1 Concept of fuzzy number

Definition 2.2.1 *[15]* ***(Interval)*** *When interval is defined on real number* $\mathbb{R}$*, this interval is said to be a subset of* $\mathbb{R}$*. For instance, if interval is denoted as* $A = [a_1, a_3], a_1, a_3 \in \mathbb{R}, a_1 < a_3$*, we may regard this as one kind of sets. Expressing the interval as membership function is shown in the following see Fig(2.5)*

$$\mu_A(x) = \begin{cases} 0, & x < a_1 \\ 1, & a_1 \leq x \leq a_3 \\ 0, & x > a_3 \end{cases}$$

If $a_1 = a_3$*, this interval indicates a point. That is,* $[a_1, a_1] = a_1$

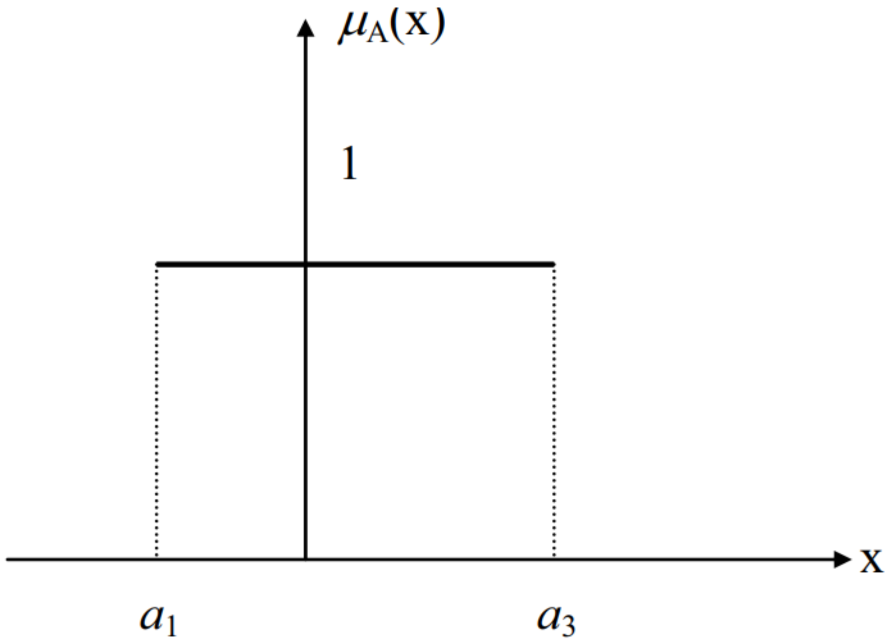

Figure 2.5: *Interval* $A = [a_1, a_3]$.

2.2.2 Some definitions of fuzzy number

Fuzzy number is expressed as a fuzzy set defining a fuzzy interval in the real number $\mathbb{R}$. Since the boundary of this interval is ambiguous, the interval is also a fuzzy set. Generally a fuzzy interval is represented by two end points a_1 and a_3 and a peak point a_2 as $[a_1, a_2, a_3]$ see Fig (2.7). The $\alpha - level$ operation can be also applied to the fuzzy number. If we denote $\alpha - level$ interval for fuzzy number A as A_α, the obtained interval A_α is defined as[15]:

$$A_\alpha = [a_1^{(\alpha)}, a_3^{(\alpha)}].$$

Definition 2.2.2 *[15]* ***(Fuzzy number)*** *It is a fuzzy set the following conditions :*

1. Convex fuzzy set.

2. Normalized fuzzy set.

3. It's membership function is piecewise continuous.

4. It is defined in the real number.

fuzzy number should be normalized and convex. Here the condition of normalization implies that maximum membership value is 1.

$$\exists x \in \mathbb{R}, \mu_A(x) = 1.$$

The convex condition is that the line by α − level is continuous and α − level interval satisfies the following relation.

$$A_\alpha = [a_1^{(\alpha)}, a_3^{(\alpha)}] .$$
$$(\alpha' < \alpha) \Rightarrow (a_1^{(\alpha')} \leq a_1^{(\alpha)}, a_3^{(\alpha')} \geq a_3^{(\alpha)}).$$

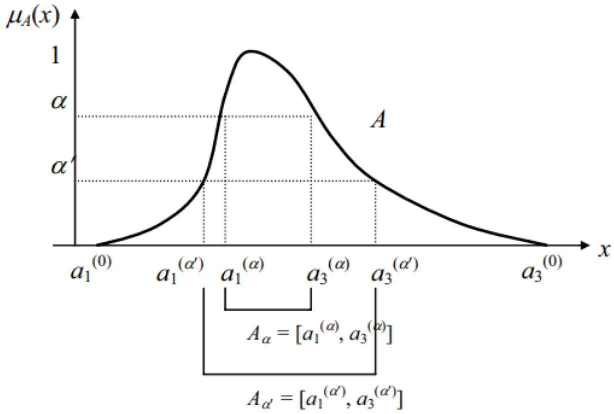

Figure 2.6: *α-level of fuzzy number $(\alpha' < \alpha) \Rightarrow (A_\alpha \subset A_{\alpha'})$.*

The convex condition may also be written as,

$$(\alpha' < \alpha) \Rightarrow (A_\alpha \subset A_{\alpha'}).$$

Definition 2.2.3 *[1] Consider a fuzzy subset of the real line $A : \mathbb{R} \to [0, 1]$. Then we say A is a fuzzy number if it satisfies the following properties:*
(i) A is normal, i.e, $\exists\, x_0 \in \mathbb{R}$ with $\mu_A(x_0) = 1$.
(ii) A is fuzzy convex, i.e, $\mu_A(\lambda x_1 + (1-\lambda)x_2) \geqslant \min\{\mu_A(x_1), \mu_A(x_2)\}, \forall \lambda \in [0, 1], x_1, x_2 \in \mathbb{R}$.
(iii) A is upper semicontinuous on $\mathbb{R}$.
(iv) A is compactly supported i.e, $\overline{\{x \in \mathbb{R}; \mu_A(x) > 0\}}$, is compact.
Let us denote by $\mathbb{R}_F$ the space of all fuzzy numbers. For $0 < \alpha \leqslant 1$ and $A \in \mathbb{R}_F$, we denote α-levels of fuzzy number A by $[A]_\alpha = \{x \in \mathbb{R}; \mu_A(x) \geqslant \alpha\}$ and $[A]_0 = \overline{\{x \in \mathbb{R}; \mu_A(x) > 0\}}$. We call $[A]_0$, the support of fuzzy number A and denote it by supp(A). The fuzzy number A is called positive if supp$(A) \subset]0, \infty[$. We denote by $\mathbb{R}_F^+$, the space of all positive fuzzy numbers.

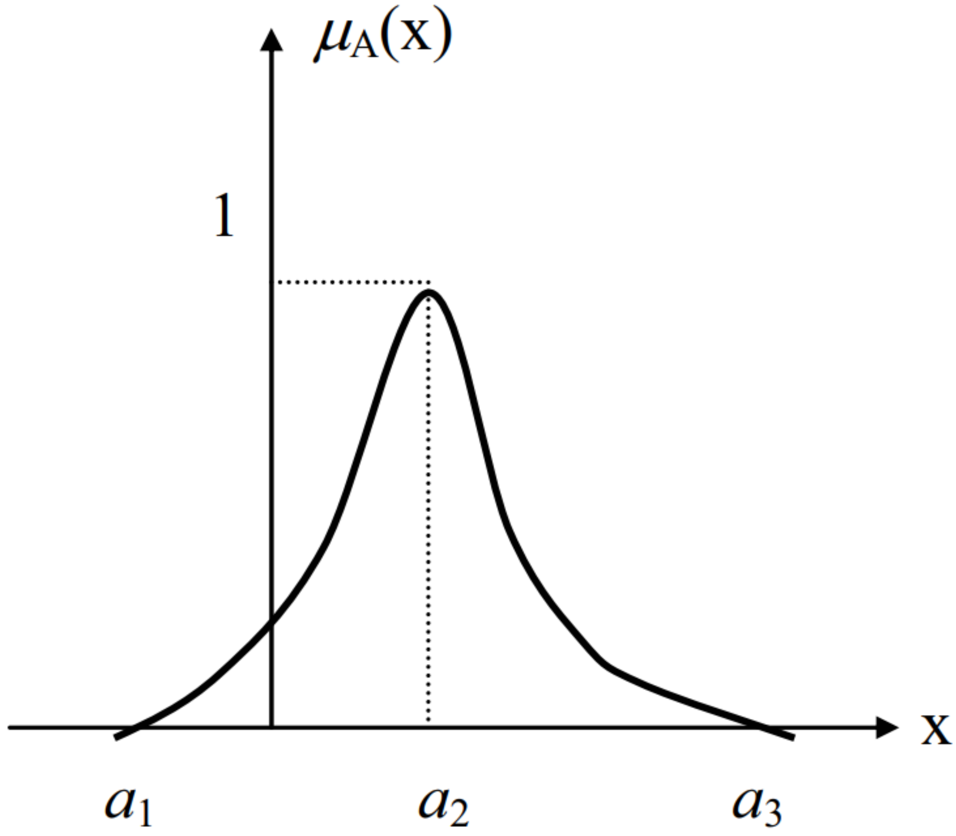

Figure 2.7: *Fuzzy number* $A = [a_1, a_2, a_3]$.

Definition 2.2.4 *[17] A fuzzy number A is called positive (respectivly negative) if its membership function is such that* $\mu_A(x) = 0, \forall x < 0$ (*respectivly* $\forall x > 0$).

Example 2.2.1 *Let X={1,2,3,4,5,6,7,8,9,10}*

μ_A*={numbers close to 5}*

A={(1,0),(2,0.4),(3,0.7),(4,0.9),(5,1),(6,0.9),(7,0.7),(8,0.4),(9,0),(10,0)}

1)A is normalised

$\mu_A = 1 \Rightarrow$ *A is normal fuzzy set.*

2) A is convex($[A]_\alpha$ *convex* $\Rightarrow$ *A is convex).*

Let α *=0.4*

$$[A]_{0.4} = \{2, 3, 4, 5, 6, 7, 8\}.$$

We take two elements $x_1, x_2 \in [A]_\alpha$.

Let λ*=0.5,* $x_1 = 3, x_2 = 4$.

We have $(\lambda x_1 + (1-\lambda)x_2) \in [A]_\alpha, \lambda \in [0,1]$

In compensation, we find:

$$3(0.5) + (1-0.5)4 = 3.5 \in [A]_\alpha$$

$\Rightarrow [A]_\alpha$ *is convex*

$\Rightarrow A$ *is convex*

3)The support of A is bounded

$$[A]_0 = \{2, 3, 4, 5, 6, 7, 8\}$$

We observe that the values of $[A]_0 \in [2, 8]$ *then* $[A]_0$ *is bounded.*

So A is fuzzy number.

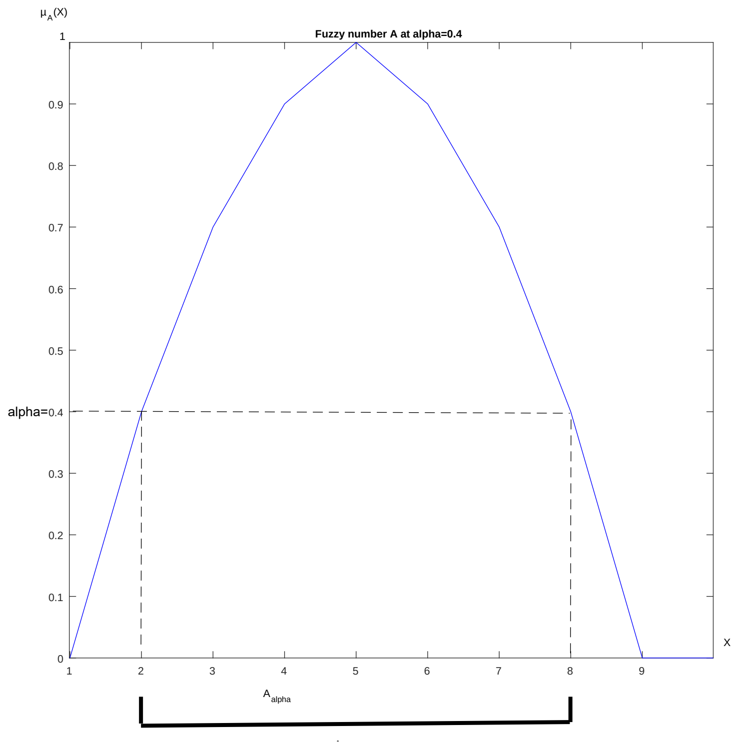

Figure 2.8: *Fuzzy number A at* $\alpha = 0.4$.

2.2.3 Types of fuzzy number

The most common fuzzy numbers are the triangular, trapezoidal and the bell shape numbers [2].

a) Triangulair fuzzy number

Among the various shapes of fuzzy number, triangular fuzzy number(TFN) is the most popular one.

Definition 2.2.5 *[15] Triangulair fuzzy number is a fuzzy number represented with three points as follows :*

$$A = (a_1, a_2, a_3),$$

this representation is interpreted as membership functions see Fig (2.9)

$$\mu_A(x) = \begin{cases} 0, & x < a_1 \\ \frac{x-a_1}{a_2-a_1}, & a_1 \leq x \leq a_2 \\ \frac{a_3-x}{a_3-a_2}, & a_2 \leq x \leq a_3 \\ 0, & x > a_3. \end{cases}$$

Now if you get crisp interval by $\alpha -$ level operation, interval A_α shall be obtained as follows $\forall \alpha \in [0, 1]$.

From

$$\frac{a_1^{(\alpha)} - a_1}{a_2 - a_1} = \alpha, \frac{a_3 - a_3^{(\alpha)}}{a_3 - a_2} = \alpha.$$

We get

$$a_1^{(\alpha)} = (a_2 - a_1)\alpha + a_1,$$

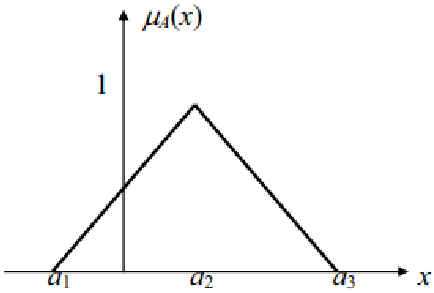

Figure 2.9: *Triangular fuzzy number* $A = (a_1, a_2, a_3)$.

$$a_3^{(\alpha)} = -(a_3 - a_2)\alpha + a_3$$

thus

$$A_\alpha = [a_1^{(\alpha)}, a_3^{(\alpha)}]$$

$$= [(a_2 - a_1)\alpha + a_1, -(a_3 - a_2)\alpha + a_3].$$

Example 2.2.2 *In the case of the triangular fuzzy number A = (-5, -1, 1) see Fig* (2.10)*, the membership function value will be,*

$$\mu_A(x) = \begin{cases} 0, & x < -5 \\ \frac{x+5}{4}, & -5 \leqslant x \leqslant -1 \\ \frac{1-x}{2}, & -1 \leqslant x \leqslant 1 \\ 0, & x > 1. \end{cases}$$

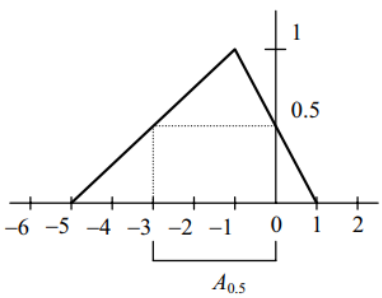

Figure 2.10: $\alpha = 0.5$ *Level of triangular fuzzy number* $A = (-5, -1, 1)$.

α-level interval from this fuzzy number is:

$$\frac{x+5}{4} = \alpha \Rightarrow x = 4\alpha - 5.$$

$$\frac{1-x}{2} = \alpha \Rightarrow x = -2\alpha + 1.$$

$$A_\alpha = [a_1^{(\alpha)}, a_3^{(\alpha)}] = [4\alpha - 5, -2\alpha + 1].$$

If $\alpha = 0.5$, *substituting* 0.5 *for* α, *we get* $A_{0.5}$

$$A_{0.5} = [a_1^{(0.5)}, a_3^{(0.5)}] = [-3, 0].$$

b) Trapezoidal fuzzy number

Another shape of fuzzy number is trapezoidal fuzzy number. This shape is originated from the fact that there are several points whose membership degree is maximum ($\alpha = 1$).

Definition 2.2.6 *[15] We can define trapezoidal fuzzy number A as:*

$$A = (a_1, a_2, a_3, a_4).$$

The membership function of this fuzzy number will be interpreted as follows see Fig (2.11)

$$\mu_A(x) = \begin{cases} 0, & x < a_1 \\ \frac{x-a_1}{a_2-a_1}, & a_1 \le x \le a_2 \\ 1, & a_2 \le x \le a_3 \\ \frac{a_4-x}{a_4-a_3}, & a_3 \le x \le a_4 \\ 0, & x > a_4. \end{cases}$$

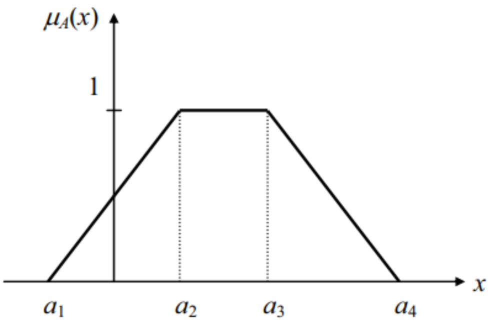

Figure 2.11: *Trapezoidal fuzzy number* $A = (a_1, a_2, a_3, a_4)$.

α-cut interval for this shape is written below. $\forall \alpha \in [0, 1]$

$$A_\alpha = [(a_2 - a_1)\alpha + a_1, -(a_4 - a_3)\alpha + a_4].$$

When $a_2 = a_3$*, the trapezoidal fuzzy number coincides with triangular one.*

Example 2.2.3 *The fuzzy set of the teenagers can be represented by the trapezoidal fuzzy number with the membership function:*

$$\mu_A(x) = \begin{cases} \frac{x-11}{3} & if\ 11 \leqslant x < 14 \\ 1 & if\ 14 \leqslant x \leqslant 17 \\ \frac{20-x}{3} & if\ 17 < x \leqslant 20 \\ 0 & otherwise. \end{cases}$$

And it is illustrated in Fig (2.12). *The α-levels for this example* $[3\alpha + 11, -3\alpha + 20]$, *with* $\alpha \in [0, 1]$.

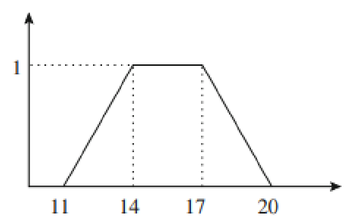

Figure 2.12: *Trapezoidal fuzzy number .*

c) Bell shape fuzzy number

Definition 2.2.7 *[2] A fuzzy number has the bell shape if the membership function is smooth and symmetric in relation to a given real number. The following membership function has those properties for fixed u, a and δ see Fig (2.13)*

$$\mu_A(x) = \begin{cases} e\left(-\left(\frac{x-u}{a}\right)^2\right) & if\ u-\delta \leqslant x \leqslant u+\delta \\ 0 & otherwise. \end{cases}$$

The α-levels of fuzzy numbers in bell shape are the intervals:

$$[a_1^{(\alpha)}, a_3^{(\alpha)}] = \begin{cases} \left[u - \sqrt{\ln\left(\frac{1}{\alpha^{a^2}}\right)}, u + \sqrt{\ln\left(\frac{1}{\alpha^{a^2}}\right)}\right] & if\ \alpha \geqslant \bar{\alpha} = e - \left(\frac{\delta}{a}\right)^2 \\ [u-\delta, u+\delta] & if\ \alpha < \bar{\alpha} = e - \left(\frac{\delta}{a}\right)^2. \end{cases}$$

We next present the arithmetic operations for fuzzy numbers, that is, the operations that allow us "to compute" with fuzzy sets

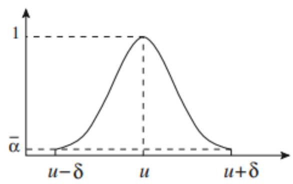

Figure 2.13: *Fuzzy number in the bell shape .*

2.2.4 Arithmetic operations with fuzzy numbers

The arithmetic operations involving fuzzy numbers are closely linked to the interval arithmetic operations. Let us list some of those operations for closed intervals on the real line $\mathbb{R}$ [2].

1) Interval arithmetic operations

Let λ be a real number and, A and B two closed intervals on the real line given by:

$$A = [a_1, a_2] \text{ and } B = [b_1, b_2].$$

Definition 2.2.8 *[2] **(Interval operations)** The arithmetic operations between intervals can be defined as:*

***(a) The Sum** between A and B is the interval:*

$$A + B = [a_1 + b_1, a_2 + b_2].$$

***(b) The difference** between A and B is the interval:*

$$A - B = [a_1 - b_2, a_2 - b_1].$$

***(c) The multiplication** of A by a scalar λ is the interval:*

$$\lambda A = \begin{cases} [\lambda a_1, \lambda a_2] & if\ \lambda \geq 0 \\ [\lambda a_2, \lambda a_1] & if\ \lambda < 0. \end{cases}$$

***(d) The multiplication** of A by B is the interval:*

$$A \cdot B = [\min P, \max P],$$

where $P = \{a_1 b_1, a_1 b_2, a_2 b_1, a_2 b_2\}$.

***(e) The quotient** of A by B, if $0 \notin B$, is the interval:*

$$A/B = [a_1, a_2] \cdot [\tfrac{1}{b_2}, \tfrac{1}{b_1}].$$

***(f)] The Inverse interval** of A is:*

$$[a_1, a_3]^{-1} = [\min(\tfrac{1}{a_1}, \tfrac{1}{a_3},), \max(\tfrac{1}{a_1}, \tfrac{1}{a_3})].$$

Example 2.2.4 *There are two intervals A and B, A=[3,5], B=[-2,7]. Then following operations might be set.*
$A + B = [3 - 2, 5 + 7] = [1, 12]$.
$A - B = [3 - 7, 5 - (-2)] = [-4, 7]$.
$A \cdot B = [\min P, \max P] = [-10, 35]$ *where* $P = \{-6, 21, -10, 35\}$.
$A/B = [3, 5].[\frac{1}{7}, \frac{-1}{2}] = [\frac{-5}{2}, \frac{5}{7}]$ *Where* $P = \{\frac{3}{7}, \frac{-3}{2}, \frac{5}{7}, \frac{-5}{2}\}$.
$B^{-1} = [-2, 7]^{-1} = [\min(\frac{-1}{2}, \frac{1}{7}), \max(\frac{-1}{2}, \frac{1}{7})] = [\frac{-1}{2}, \frac{1}{7}]$.

2) Interval α-level operation

Theorem 2.2.1 *The α-levels of the fuzzy set $A \otimes B$ are given by:*

$$[A \otimes B]_\alpha = [A]_\alpha \otimes [B]_\alpha.$$

For all $\alpha \in [0, 1]$, where $\otimes$ is any arithmetic operations $\{+, -, \times, \div\}$ [2].

Proposition 2.2.1 *[2] Let A and B be fuzzy numbers with α-levels respectively given by $[A]_\alpha = [a_1^\alpha, a_2^\alpha]$ and $[B]_\alpha = [b_1^\alpha, b_2^\alpha]$. Then the following properties hold:*
(a) *The sum of A and B is the fuzzy number $A + B$ whose α-levels are:*

$$[A + B]_\alpha = [A]_\alpha + [B]_\alpha = [a_1^\alpha + b_1^\alpha, a_2^\alpha + b_2^\alpha].$$

(b) *The difference of A and B is the fuzzy number $A - B$ whose α-levels are:*

$$[A - B]_\alpha = [A]_\alpha - [B]_\alpha = [a_1^\alpha - b_2^\alpha, a_2^\alpha - b_1^\alpha].$$

(c) *The multiplication of A by a scalar λ is the fuzzy number λA whose α-levels are:*

$$[\lambda A]_\alpha = \lambda [A]_\alpha = \begin{cases} [\lambda a_1^\alpha, \lambda a_2^\alpha] & if\ \lambda \geqslant 0 \\ [\lambda a_2^\alpha, \lambda a_1^\alpha] & if\ \lambda < 0. \end{cases}$$

(d) *The multiplication of A by B is the fuzzy number A· B whose α-levels are:*

$$[A \cdot B]_\alpha = [A]_\alpha [B]_\alpha = [\min p^\alpha, \max p^\alpha],$$

where $p^\alpha = [a_1^\alpha b_1^\alpha, a_1^\alpha b_2^\alpha, a_2^\alpha b_1^\alpha, a_2^\alpha b_2^\alpha]$.
(e) *The division of A by B, if 0 ∉ supp B, is the fuzzy number whose α-levels are:*

$$\left[\frac{A}{B}\right]_\alpha = \frac{[A]_\alpha}{[B]_\alpha} = \left[a_1^\alpha, a_2^\alpha\right]\left[\frac{1}{b_2^\alpha}, \frac{1}{b_1^\alpha}\right].$$

Theorem 2.2.2 *[2]* ***(Extension principle for real intervals)*** *Let A and B be two closed intervals of* $\mathbb{R}$ *and* ⊗ *one of the arithmetic operations between real numbers. Then:*

$$\mu_{A \otimes B}(z) = \sup_{\{(x,y): x \otimes y = z\}} \min[\mu_A(x), \mu_B(y)].$$

It is simple to verify that:

$$\min(\mu_A(x), \mu_B(y)) = \begin{cases} 1 & if\ x \in A \text{ and } y \in B \\ 0 & if\ x \notin A \text{ and } y \notin B. \end{cases}$$

Thus, for the sum case (⊗ = +)*, we have*

$$\sup_{\{(x,y): x+y=z\}} \min[\mu_A(x), \mu_B(y)] = \begin{cases} 1 & if\ x \in A + B \\ 0 & if\ x \notin A + B. \end{cases}$$

The other cases can be obtained analogously.

Definition 2.2.9 *[17] A binary operation* $*$ *in* $\mathbb{R}$ *is called increasing (decreasing) if for* $x_1 > y_1$ *and* $x_2 > y_2$. $x_1 * x_2 > y_1 * y_2$ $(x_1 * x_2 < y_1 * y_2)$.

Example 2.2.5 *Let the following functions:*
$f(x, y) = x + y$ *is an increasing operation.*
$f(x, y) = -(x + y)$ *is an decreasing operation.*

2.2.5 Extended operations with fuzzy numbers

If the normal algebraic operations $+, -, \cdot, \div$ are extended to operations on fuzzy numbers, they shall be denoted by $\oplus, \ominus, \odot, \oslash$ [17].

Definition 2.2.10 *[2] Let A and B be two fuzzy numbers and λ a real number.*
(a) *The sum of the fuzzy numbers A and B is the fuzzy number $A \oplus B$, whose membership function is:*

$$\mu_{A\oplus B}(z) = \sup_{\{(x,y):x+y=z\}} \min[\mu_A(x), \mu_B(y)].$$

(b) *The difference $A \ominus B$ is the fuzzy number whose membership function is given by:*

$$\mu_{A\ominus B}(z) = \sup_{\{(x,y):x-y=z\}} \min[\mu_A(x), \mu_B(y)].$$

(c) *The multiplication of A by B is the fuzzy number $A \odot B$, whose membership function is given by:*

$$\mu_{A\odot B}(z) = \sup_{\{(x,y):x\cdot y=z\}} \min[\mu_A(x), \mu_B(y)].$$

(d) *The quotient is the fuzzy number $A \oslash B$ whose membership function is:*

$$\mu_{A\oslash B}(z) = \sup_{\{(x,y):x/y=z\}} \min[\mu_A(x), \mu_B(y)].$$

Properties

The following points are properties of the extended operations with fuzzy numbers[17]:
1. $\oplus, \ominus, \odot, \oslash$ are commutative.
2. $\oplus, \ominus, \odot, \oslash$ are associative.
3. $\ominus(A \oplus B) = (\ominus A) \oplus (\ominus B)$.
4. $0 \in \mathbb{R} \subseteq \mathbb{R}_F$ is the neutral element for $\oplus$, that is,$A \oplus 0, \forall A \in \mathbb{R}_F$.
5. For $\oplus$ there does not exist an invers element, that is, $\forall A \in \mathbb{R}_F \backslash \mathbb{R} : A \oplus (\ominus A) \neq 0 \in \mathbb{R}$.
6. $(\ominus A) \odot B = \ominus(A \odot B)$.

7. $A \odot 1 = A \in \mathbb{R} \subseteq \mathbb{R}_F$ is the neutral element for $\odot$, that is, $A \odot 1 = A, \forall A \in \mathbb{R}_F$.
8. For $\odot$ there does not exist an inverse element, that is, $\forall A \in \mathbb{R}_F \backslash \mathbb{R} : A \odot A^{-1} \neq 1$.

CHAPTER 3

FUZZY DIFFERENCE EQUATIONS

In this chapter, first we provide the solution of first-order difference equation
$x_{n+1} = wx_n + q$, with w, q are constant coefficients in real numbers.
Second, we study the existence, boundedness and persistence to positive fuzzy solution, also the existence of equilibrium point of fuzzy difference equation

$$x_{n+1} = wx_n + q, \quad n = 0, 1, 2, ... \tag{3.1}$$

where x_n is a sequence of positive fuzzy numbers,the parameters w, q are positive fuzzy numbers and the initial condition x_0 is arbitrary positive fuzzy number.

3.1 Linear difference equations

Definition 3.1.1 [5] *Given constant w and q, a difference equation of the form*

$$x_{n+1} = wx_n + q. \tag{3.2}$$

$n = 0, 1, 2, ...$ *is called a first-order linear difference equation. A procedure analogous to the method we used to solve* $x_{n+1} = wx_n$ *will enable to solve this equation as well.*
Namely,

$$\begin{aligned}
x_n &= wx_{n-1} + q \\
&= w(wx_{n-2} + q) + q \\
&= w^2x_{n-2} + q(w + 1) \\
&= w^2(wx_{n-3} + q) + (w + 1) \\
&= w^3x_{n-3} + q(w^2 + w + 1) \\
&\vdots \\
&= w^nx_0 + q(w^{n-1} + w^{n-2} + \cdots + w^2 + w + 1).
\end{aligned}$$

Note that $w = 1$*, this gives*

$$x_n = x_0 + nq. \tag{3.3}$$

$n = 0, 1, 2, \ldots$*as the solution of the difference equation* $x_{n+1} = x_n + q$
for $w \neq 1$ *known that*

$$w^{n-1} + w^{n-2} + \cdots + w^2 + w + 1 = \frac{1 - w^n}{1 - w},$$

hence

$$x_n = w^nx_0 + q\left(\frac{1 - w^n}{1 - w}\right). \tag{3.4}$$

$n = 0, 1, 2, \ldots$ *is the solution of the first-order linear difference equation* $x_{n+1} = wx_n + q$ *when* $w \neq 1$.

3.2 Fuzzy difference equations

Lemma 3.2.1 [1] *Let f be a continuous function from* $\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+$ *into* $\mathbb{R}^+$ *and A, B, C be fuzzy numbers, then*

$$[f(A,B,C)]_\alpha = f([A]_\alpha,[B]_\alpha,[C]_\alpha), \quad \alpha \in [0,1].$$

Theorem 3.2.1 [1]*(Stacking Theorem) If* $A \in \mathbb{R}_F$ *is a fuzzy number and* A_α, $\alpha \in [0,1]$ *are its* α*-cuts, then*

(i) A_α *is a closed intervale* $A_\alpha = [A_{l,\alpha}, A_{r,\alpha}]$,*for any* $\alpha \in [0,1]$,

(ii) *If* $0 \leqslant \alpha_1 \leqslant \alpha_2 \leqslant 1$, *then* $A_{\alpha_2} \subseteq A_{\alpha_1}$,

(iii) *For any sequence* α_n *which converges from below to* $\alpha \in (0,1]$, *we have*

$$\bigcap_{n=1}^{\infty} A_{\alpha_n} = A_\alpha,$$

(iv) *For any sequence* α_n *which converges from above to 0, we have*

$$\overline{\bigcup_{n=1}^{\infty} A_{\alpha_h}} = A_0.$$

Theorem 3.2.2 [1] *Let us consider the functions*

$$A_{l,\alpha}, A_{r,\alpha} : [0,1] \to \mathbb{R},$$

satisfy the following conditions

(i) $A_{l,\alpha} \in \mathbb{R}$ *is bounded, non-decreasing, left-continuous function on]0,1] and it is right-continuous at 0.*

(ii) $A_{r,\alpha} \in \mathbb{R}$ *is bounded, non-increasing, left-continuous function on]0,1] and it is right-continuous at 0.*

(iii) $A_{l,1} \leq A_{r,1}$.

Then there is a fuzzy number $A \in \mathbb{R}_F$ that has $A_{l,\alpha}, A_{r,\alpha}$ as endpoints of its α-cuts, A_α. Conversely let $A \in \mathbb{R}_\mathbb{F}$ with endpoints $A_{l,\alpha}, A_{r,\alpha}$, then conditions (i)-(iii) are satisfied.

Definition 3.2.1 [1] *Let A, B be fuzzy numbers with $[A]_\alpha = [A_{l,\alpha}, A_{r,\alpha}]$, $[B]_\alpha = [B_{l,\alpha}, B_{r,\alpha}]$, $\alpha \in [0,1]$. Then the metric on the fuzzy numbers space is defined as follow*

$$D(A, B) = sup \quad max\{|A_{l,\alpha} - B_{l,\alpha}|, |A_{r,\alpha} - B_{r,\alpha}|\}$$

where sup is taken for all $\alpha \in [0, 1]$.

Definition 3.2.2 *[11] We say that x_n is a positive solution of (3.1) if x_n is a sequence of positive fuzzy numbers, which satisfies (3.1).*
We say that a sequence of positive fuzzy numbers x_n is persistent (resp. is bounded) if there exists a positive number M (resp., N) such that

$$suppx_n \subset [M, +\infty[, (resp. suppx_n \subset [0, N]), \quad n=1,2, ...$$

In addition, we say that x_n is bounded and persists if there exist numbers $M, N \in [0, +\infty[$ such that

$$suppx_n \subset [M, N], \quad n = 1,2,...$$

Remark 3.2.1 *Let $u, v \in \mathbb{R}_F$, if $U \div V = W \in \mathbb{R}_F$ exists, then there are two cases:*
***case (i)**: if $U_{l,\alpha}V_{r,\alpha} \leqslant U_{r,\alpha}V_{l,\alpha}, \forall\alpha \in [0, 1]$, then $W_{l,\alpha} = \frac{U_{l,\alpha}}{V_{l,\alpha}}, W_{r,\alpha} = \frac{U_{r,\alpha}}{V_{r,\alpha}}$,*
***case (ii)**: if $U_{l,\alpha}V_{r,\alpha} \geqslant U_{r,\alpha}V_{l,\alpha}, \forall\alpha \in [0, 1]$, then $W_{l,\alpha} = \frac{U_{r,\alpha}}{V_{r,\alpha}}, W_{r,\alpha} = \frac{U_{l,\alpha}}{V_{l,\alpha}}$.*

3.3 Existence and uniqueness of positive fuzzy solution

Theorem 3.3.1 [11] *For any positive fuzzy numbers x_0, fuzzy difference equation (3.1), there exists a unique positive solution x_n whose initial value is x_0.*

Proof.

***Firstly*: existence of the solution**

For all positive fuzzy numbers x_0, where w, q $\in \mathbb{R}_F^+$, suppose there exists a fuzzy number sequence that satisfies equation (3.1) whose initial value is x_0. Consider their α-cuts, $\alpha \in [0, 1]$,

$$\begin{cases} [x_n]_\alpha = [L_{n,\alpha}, R_{n,\alpha}] \\ [w]_\alpha = [w_{l,\alpha}, w_{r,\alpha}] \\ [q]_\alpha = [q_{l,\alpha}, q_{r,\alpha}] \end{cases} \tag{3.5}$$

following (3.1), (3.5) , and Lemma (3.2.1), we have:

$$\begin{aligned} [x_{n+1}]_\alpha &= [L_{n+1,\alpha}, R_{n+1,\alpha}] \\ &= [wx_n + q]_\alpha \\ &= [wx_n]_\alpha + [q]_\alpha \\ &= [w]_\alpha [x_n]_\alpha + [q]_\alpha \\ &= [w_{l,\alpha}, w_{r,\alpha}][L_{n,\alpha}, R_{n,\alpha}] + [q_{l,\alpha}, q_{r,\alpha}] \\ &= [w_{l,\alpha} L_{n,\alpha} + q_{l,\alpha}, w_{r,\alpha} R_{n,\alpha} + q_{r,\alpha}]. \end{aligned}$$

So we obtain the related equation system

$$\begin{cases} L_{n+1,\alpha} = w_{l,\alpha} L_{n,\alpha} + q_{l,\alpha} \\ R_{n+1,\alpha} = w_{r,\alpha} R_{n,\alpha} + q_{r,\alpha}. \end{cases} \tag{3.6}$$

For any given initial values $(L_{i,\alpha}, R_{i,\alpha}), i = 0, \alpha \in [0, 1]$, system (3.6), there exists a unique positive solution $(L_{n,\alpha}, R_{n,\alpha}), \alpha \in [0, 1]$.

Now we demonstrate that $[L_{n,\alpha}, R_{n,\alpha}], \alpha \in [0, 1]$ determines the solution x_n of (3.1) whose initial value is x_0, where $(L_{n,\alpha}, R_{n,\alpha})$ is the positive solution of system (3.6) with initial value $(L_{i,\alpha}, R_{i,\alpha}), i = 0$, such that

$$[x_n]_\alpha = [L_{n,\alpha}, R_{n,\alpha}], \alpha \in [0, 1]. \tag{3.7}$$

By Theorem (3.2.1) and w, q, x_0 are positive fuzzy numbers, for any $\alpha_1, \alpha_2 \in [0, 1]$, $\alpha_1 \leqslant \alpha_2$, we have:

$$\begin{cases} 0 < w_{l,\alpha_1} \leqslant w_{l,\alpha_2} \leqslant w_{r,\alpha_2} \leqslant w_{r,\alpha_1}, \\ 0 < q_{l,\alpha_1} \leqslant q_{l,\alpha_2} \leqslant q_{r,\alpha_2} \leqslant q_{r,\alpha_1}, \\ 0 < L_{0,\alpha_1} \leqslant L_{0,\alpha_2} \leqslant R_{0,\alpha_2} \leqslant R_{0,\alpha_1} \end{cases} \tag{3.8}$$

By induction and (3.6), (3.8), we will show

$$L_{n,\alpha_1} \leqslant L_{n,\alpha_2} \leqslant R_{n,\alpha_2} \leqslant R_{n,\alpha_1}. \tag{3.9}$$

We prove that (3.9) is true.

1) For n=0 (3.9) is true by (3.8).

2) Suppose (3.9) is true for n and prove that (3.9) is true for n+1,

$$\begin{aligned} L_{n+1,\alpha_1} &= w_{l,\alpha_1} L_{n,\alpha_1} + q_{l,\alpha_1} \\ &\leqslant w_{l,\alpha_2} L_{n,\alpha_2} + q_{l,\alpha_2} = L_{n+1,\alpha_2} \\ &\leqslant w_{r,\alpha_2} R_{n,\alpha_2} + q_{r,\alpha_2} = R_{n+1,\alpha_2} \\ &\leqslant w_{r,\alpha_1} R_{n,\alpha_1} + q_{r,\alpha_1} = R_{n+1,\alpha_1}. \end{aligned}$$

Hence

$$L_{n+1,\alpha_1} \leqslant L_{n+1,\alpha_2} \leqslant R_{n+1,\alpha_2} \leqslant R_{n+1,\alpha_1} \tag{3.10}$$

Therefore (3.10) is true.

So by induction (3.9) is true.

Following (3.6) and we put n=0, we have:

$$\begin{cases} L_{1,\alpha} = w_{l,\alpha} L_{0,\alpha} + q_{l,\alpha} \\ R_{1,\alpha} = w_{r,\alpha} R_{0,\alpha} + q_{r,\alpha}. \end{cases} \tag{3.11}$$

• By theoreme (3.2.2): $x_0 \in \mathbb{R}_F^+$ **exist**, then $w_{l,\alpha}, w_{r,\alpha}, q_{l,\alpha}, q_{r,\alpha}, L_{0,\alpha}, R_{0,\alpha}$ are left continous, and by (3.11) $L_{1,\alpha}, R_{1,\alpha}$ are left continous.

• Also working inductively, we can prove that $L_{n,\alpha}, R_{n,\alpha}$ are left continous.

• We have L_α is non decreasing and R_α is non increasing by (3.10).

• By systeme (3.11) and (3.10) we have $L_1 \leqslant R_1$.

Secondly: positivity of the solution

To prove that the solution is a positive fuzzy number, we must prove $\operatorname{supp} x_n = \overline{\bigcup_{\alpha\in[0,1]} [L_{n,\alpha}, R_{n,\alpha}]}$ (support of fuzzy number x_n) is compact. It is sufficient to prove that $\bigcup_{\alpha\in(0,1]} [L_{n,\alpha}, R_{n,\alpha}]$ is bounded.

Since w, q, x_0 are positive fuzzy numbers, there exist constants $M_w > 0, N_w > 0, M_q > 0, N_q > 0, M_0 > 0, N_0 > 0$ such that for all $\alpha \in [0, 1]$

$$\begin{cases} [w_{l,\alpha}, w_{r,\alpha}] \subset \overline{\bigcup_{\alpha\in[0,1]} [w_{l,\alpha}, w_{r,\alpha}]} \subset [M_w, N_w]. \\ [q_{l,\alpha}, q_{r,\alpha}] \subset \overline{\bigcup_{\alpha\in[0,1]} [q_{l,\alpha}, q_{r,\alpha}]} \subset [M_q, N_q]. \\ [L_{0,\alpha}, R_{0,\alpha}] \subset \overline{\bigcup_{\alpha\in[0,1]} [L_{0,\alpha}, R_{0,\alpha}]} \subset [M_0, N_0]. \end{cases} \tag{3.12}$$

Hence from (3.11) and (3.12) we can easily get

$$[L_{1,\alpha}, R_{1,\alpha}] \subset [M_w M_0 + M_q, N_w N_0 + N_q], \quad \alpha \in (0, 1]. \tag{3.13}$$

From which it is obvious that

$$\bigcup_{\alpha\in(0,1]} [L_{1,\alpha}, R_{1,\alpha}] \subset [M_w M_0 + M_q, N_w N_0 + N_q], \quad \alpha \in (0, 1]. \tag{3.14}$$

There for (3.14) implies that $\overline{\bigcup_{\alpha\in(0,1]} [L_{1,\alpha}, R_{1,\alpha}]}$ is compact and $\overline{\bigcup_{\alpha\in(0,1]} [L_{1,\alpha}, R_{1,\alpha}]} \subset (0, \infty)$. Deducing inductively, one can get that $\overline{\bigcup_{\alpha\in(0,1]} [L_{n,\alpha}, R_{n,\alpha}]}$ is compact, moreover,for $n = 1, 2, ...,$

$$\overline{\bigcup_{\alpha\in(0,1]} [L_{n,\alpha}, R_{n,\alpha}]} \subset (0, \infty)$$

Therefore (3.9), (3.14) and that $L_{n,\alpha}, R_{n,\alpha}$ are left continuous we have that $[L_{n,\alpha}, R_{n,\alpha}]$ determines a sequence of positive fuzzy numbers (x_n) such that (3.7) holds.

Thirdly: uniqueness of the solution

We prove the uniqueness of the solution. Suppose that there exists another solution $\tilde{x}_n$ of (3.1) with initial data $x_0 \in \mathbb{R}_F^+$.

For $[x_n]_\alpha$ solution of (3.1):

$$[x_{n+1}]_\alpha = [w_{l,\alpha} L_{n,\alpha} + q_{l,\alpha}, w_{r,\alpha} R_{n,\alpha} + q_{r,\alpha}]$$
$$[x_0]_\alpha = [c_{l,\alpha}, c_{r,\alpha}]$$

For $[\tilde{x}_n]_\alpha$ solution of (3.1):

$$[\tilde{x}_{n+1}]_\alpha = [w_{l,\alpha} \tilde{L}_{n,\alpha} + q_{l,\alpha}, w_{r,\alpha} \tilde{R}_{n,\alpha} + q_{r,\alpha}]$$
$$[\tilde{x}_0]_\alpha = [c_{l,\alpha}, c_{r,\alpha}]$$

We get $[\tilde{x}_n]_\alpha = [x_n]_\alpha$ for any $\alpha \in [0, 1]$, so $\tilde{x}_n = x_n$ which is contradictory .

So the positive solution of fuzzy difference equation (3.1) is unique. ■

3.4 The boundedness and persistence of positive fuzzy solution

Lemma 3.4.1 *[12] Consider the difference equations:*

$$y_{n+1} = w_1 y_n + q_1, \quad z_{n+1} = w_2 z_n + q_2, \quad n = 1, 2, ..., \quad y_0, z_0 \in \mathbb{R}^+. \tag{3.15}$$

Suppose that there exist positive numbers P, Q, P', Q', G such that $Q' < 1, y_0 \leqslant z_0 \leqslant G$, $P \leqslant q_1 \leqslant q_2 \leqslant Q$ and $P' \leqslant w_1 \leqslant w_2 \leqslant Q'$, then there exists positive number T such that the following statements are true.

$$P \leqslant y_n \leqslant T, \quad P \leqslant z_n \leqslant T.$$

Proof. 1) Let y_n be a positive solution of (3.15). Then we have

$$\begin{aligned}
y_n &= w_1 y_{n-1} + q_1 \leqslant Q' y_{n-1} + Q = Q'(w_1 y_{n-2} + q_1) + Q \\
&\leqslant Q'(Q' y_{n-2} + Q) + Q = Q'^2 y_{n-2} + Q'Q + Q \\
&\leqslant Q'^2 (Q' y_{n-3} + Q) + Q'Q + Q = Q'^3 y_{n-3} + Q'^2 Q + Q'Q + Q \\
&\leqslant Q'^4 y_{n-4} + Q'^3 Q + Q'^2 Q + Q'Q + Q \\
&\ \ \vdots \\
&\leqslant Q'^n y_0 + Q(Q'^{n-1} + Q'^{n-2} + ... + Q' + 1) = Q'^n y_0 + Q\frac{1 - Q'^n}{1 - Q'}.
\end{aligned}$$

We are going to the limit, we have

$$\lim_{n\to+\infty} y_n \leqslant \lim_{n\to+\infty} Q'^n y_0 + Q\tfrac{1-Q'^n}{1-Q'} \leqslant G + \tfrac{Q}{1-Q'}.$$

So we have $P \leqslant y_n \leqslant T$, where $T = G + \frac{Q}{1-Q'}$.
2) Let z_n be a positive solution of (3.15). Then we have

$$\begin{aligned}
z_n &= w_2 z_{n-1} + q_2 \leqslant Q'_{n-1} + Q = Q'(w_2 y_{n-2} + q_2) + Q \\
&\leqslant Q'(Q' z_{n-2} + Q) + Q = Q'^2 z_{n-2} + Q'Q + Q \\
&\leqslant Q'^2 (Q' z_{n-3} + Q) + Q'Q + Q = Q'^3 z_{n-3} + Q'^2 Q + Q'Q + Q \\
&\leqslant Q'^4 z_{n-4} + Q'^3 Q + Q'^2 Q + Q'Q + Q \\
&\ \ \vdots \\
&\leqslant Q'^n z_0 + Q(Q'^{n-1} + Q'^{n-2} + ... + Q' + 1) = Q'^n z_0 + Q\frac{1 - Q'^n}{1 - Q'}.
\end{aligned}$$

We are going to the limit, we have

$$\lim_{n\to+\infty} z_n \leqslant \lim_{n\to+\infty} Q'^n z_0 + Q\tfrac{1-Q'^n}{1-Q'} \leqslant G + \tfrac{Q}{1-Q'}.$$

So we have $P \leqslant z_n \leqslant T$, where $T = G + \frac{Q}{1-Q'}$. ∎

Theorem 3.4.1 [12] *Let $q, w, x_0 \in \mathbb{R}_F^+$ and $w_{r,0} < 1$. Then every positive solution of Eq.* (3.1) *is bounded and persists.*

Proof:Let x_n be a positive solution of Eq. (3.1) Since $q, w, x_0 \in \mathbb{R}_f^+$ and $w_{r,0} < 1$, then there exist positive numbers P, Q, P', Q', R, for each $\alpha \in [0,1]$ such that $Q' < 1$,
$L_{0,\alpha} \leqslant R_{0,\alpha} \leqslant R, P \leqslant q_{l,\alpha} \leqslant q_{r,\alpha} \leqslant Q$ and $P' \leqslant w_{l,\alpha} \leqslant w_{r,\alpha} \leqslant Q'$.
By lemma (3.4.1) we obtain:
$P < L_{n,\alpha} \leqslant R_{n,\alpha} < T, \forall \alpha \in [0,1]$. Therefore $[L_{n,\alpha}, R_{n,\alpha}] \subset [P, T], \forall \alpha \in [0,1]$ and so

$$supp(x_n) \subset [P, T].$$

Then the positive solution is bounded and persists.

3.5 The existence of positive equilibrium point

Definition 3.5.1 *[12] We say that fuzzy numbers x is a equilibrium for (3.1), if $x = wx + q$.*

Proposition 3.5.1 *If $w < 1$, then every positive solution x_n of Eq. (3.1) converges to the positive equilibrium x as $n \to \infty$.*

Proof: The solution of Eq. (3.1) is given by:

$$x_n = w^n x_0 + q\left(\frac{1-w^n}{1-w}\right).$$

If $w < 1, w^n = 0$ when $n \to \infty$.
So

$$\lim_{n\to+\infty} x_n = \lim_{n\to+\infty} w^n x_0 + q\left(\frac{1-w^n}{1-w}\right) = \frac{q}{1-w}$$

Theorem 3.5.1 *[13] Consider (3.1) wkere $w, x_0, q \in \mathbb{R}_F^+$ such that $w_{r,\alpha}, R_{0,\alpha} < 1$. then the following statments are true:*
i) (3.1) has unique equilibrium point.
ii) Every positive solution x_n of (3.1) converges to the unique equilibrium x with respect to D as $n \to \infty$.

Proof: i) By definition (3.5.1), if x is a equilibrium point of (3.1), then

$$\begin{cases} L_\alpha = w_{l,\alpha} L_\alpha + q_{l,\alpha} ...(1) \\ R_\alpha = w_{r,\alpha} R_\alpha + q_{r,\alpha} ...(2). \end{cases} \tag{3.16}$$

According to (1) we have

$$\begin{aligned} L_\alpha - w_{l,\alpha} L_\alpha &= q_{l,\alpha} \\ L_\alpha (1 - w_{l,\alpha}) &= q_{l,\alpha} \\ L_\alpha &= \tfrac{q_{l,\alpha}}{1 - w_{l,\alpha}}. \end{aligned}$$

Based on (2) we have

$$\begin{aligned} R_\alpha - w_{r,\alpha} R_\alpha &= q_{r,\alpha} \\ R_\alpha (1 - w_{r,\alpha}) &= q_{r,\alpha} \\ R_\alpha &= \tfrac{q_{r,\alpha}}{1 - w_{r,\alpha}}. \end{aligned}$$

The system (3.16) has one solution: $[x]_\alpha = [L_\alpha, R_\alpha] = \left[\frac{q_{l,\alpha}}{1-w_{l,\alpha}}, \frac{q_{r,\alpha}}{1-w_{r,\alpha}}\right]$.
so $[x]_\alpha$ is an equilibrium point of (3.1). For uniqueness, suppose there exist another equilibrium point $\tilde{x} \in \mathbb{R}_F^+$ for (3.1)

$$[\tilde{x}]_\alpha = [\tilde{L}_\alpha, \tilde{R}_\alpha]$$

$$\begin{cases} \tilde{L}_\alpha = w_{l,\alpha} \tilde{L}_\alpha + q_{l,\alpha} ...(3) \\ \tilde{R}_\alpha = w_{r,\alpha} \tilde{R}_\alpha + q_{r,\alpha} ...(4). \end{cases} \tag{3.17}$$

According to (3) we have

$$\begin{aligned} \tilde{L}_\alpha - w_{l,\alpha} \tilde{L}_\alpha &= q_{l,\alpha} \\ \tilde{L}_\alpha (1 - w_{l,\alpha}) &= q_{l,\alpha} \\ \tilde{L}_\alpha &= \tfrac{q_{l,\alpha}}{1 - w_{l,\alpha}}. \end{aligned}$$

Based on (4) we have

$$\tilde{R}_\alpha - w_{r,\alpha}\tilde{R}_\alpha = q_{r,\alpha}$$
$$\tilde{R}_\alpha(1 - w_{r,\alpha}) = q_{r,\alpha}$$
$$\tilde{R}_\alpha = \frac{q_{r,\alpha}}{1-w_{r,\alpha}}.$$

So $[\tilde{x}]_\alpha = [\tilde{L}_\alpha, \tilde{R}_\alpha] = \left[\frac{q_{l,\alpha}}{1-w_{l,\alpha}}, \frac{q_{r,\alpha}}{1-w_{r,\alpha}}\right], \forall\alpha \in]0,1]$.
From this we have $L_\alpha = \tilde{L}_\alpha = \frac{q_{l,\alpha}}{1-w_{l,\alpha}}$ and $R_\alpha = \tilde{R}_\alpha = \frac{q_{r,\alpha}}{1-w_{r,\alpha}}, \forall\alpha \in]0,1]$.
So the eqilibrium point of (3.1) is unique.
ii) From (3.6), proposition (3.5.1) and since $w_{r,\alpha}, R_{0,\alpha} < 1$, we have $\lim_{n\to+\infty} L_{n,\alpha} = L_\alpha = \frac{q_{l,\alpha}}{1-w_{l,\alpha}}$ and
$\lim_{n\to+\infty} R_{n,\alpha} = R_\alpha = \frac{q_{r,\alpha}}{1-w_{r,\alpha}}$. So we have

$$\lim_{n\to+\infty} D(x_n, x) = \lim_{n\to+\infty} sup \max\left\{\left|L_{n,\alpha} - L_\alpha\right|, \left|R_{n,\alpha} - R_\alpha\right|\right\} = 0.$$

So x_n converge to the unique equilibrium point.

Example 3.5.1 *Consider the following fuzzy difference equation*

$$x_{n+1} = wx_n + q, \quad n = 0, 1, 2, ...$$

Take $w, q \in \mathbb{R}_F^+$ and initial value $x_0 \in \mathbb{R}_F^+$ are triangular fuzzy numbers with membership functions as

$$w(x) = \begin{cases} 5x - \frac{1}{2} & , \quad \frac{1}{10} \leqslant x \leqslant \frac{3}{10} \\ -5x + \frac{5}{2} & , \quad \frac{3}{10} \leqslant x \leqslant \frac{1}{2}. \end{cases} \quad , \quad q(x) = \begin{cases} x - 6 & , \quad 6 \leqslant x \leqslant 7 \\ -x + 8 & , \quad 7 \leqslant x \leqslant 8. \end{cases}$$

$$x_0(x) = \begin{cases} x - 2 & , \quad 2 \leqslant x \leqslant 3 \\ -x + 4 & , \quad 3 \leqslant x \leqslant 4. \end{cases}$$

Then we have,

$$[w]_\alpha = \begin{cases} 5x - \frac{1}{2} = \alpha \implies x = \frac{\alpha}{5} + \frac{1}{10} \\ -5x + \frac{5}{2} = \alpha \implies x = \frac{-\alpha}{5} + \frac{1}{2}. \end{cases} \quad , \quad [q]_\alpha = \begin{cases} x - 6 = \alpha \implies x = \alpha + 6 \\ -x + 8 = \alpha \implies x = -\alpha + 6. \end{cases}$$

$$So \quad [w]_\alpha = \left[\frac{\alpha}{5} + \frac{1}{10}, \frac{-\alpha}{5} + \frac{1}{2}\right] \quad , \quad [q]_\alpha = [\alpha + 6, -\alpha + 6] \quad , \quad \alpha \in]0,1].$$

$$[x_0]_\alpha = \begin{cases} x - 2 = \alpha \implies x = \alpha + 2 \\ -x + 4 = \alpha \implies x = -\alpha + 4. \end{cases}$$

$$So \quad [x_0]_\alpha = [\alpha + 2, -\alpha + 4] \quad , \quad \alpha \in]0,1].$$

Therefore, it follows that

$$\begin{cases} \overline{\cup_{\alpha\in(0,1]}[w]_\alpha} = \left[\frac{1}{10}, \frac{1}{2}\right] \\ \overline{\cup_{\alpha\in(0,1]}[q]_\alpha} = [6, 8] \\ \overline{\cup_{\alpha\in(0,1]}[x_0]_\alpha} = [2, 4]\,. \end{cases} \tag{3.18}$$

Since $w_{l,\alpha}, w_{r,\alpha} < 1$, *by theorem* (3.3.1) *there exist a unique positive solution to* (3.1).
By theorem (3.4.1) *every positive solution of Eq* (3.1) *is bounded and persists. Also, by theorem* (3.5.1) *there is unique positive equilibrium.*

$$[x]_\alpha = [L_\alpha, R_\alpha] = \left[\frac{q_{l,\alpha}}{1-w_{l,\alpha}}, \frac{q_{r,\alpha}}{1-w_{r,\alpha}}\right].$$

For $\alpha = 0$

$$[x]_0 = [L_0, R_0] = \left[\frac{q_{l,0}}{1-w_{l,0}}, \frac{q_{r,0}}{1-w_{r,0}}\right] = [6.66, 12].$$

For $\alpha = 0.5$

$$[x]_{0.5} = [L_{0.5}, R_{0.5}] = \left[\frac{q_{l,0.5}}{1-w_{l,0.5}}, \frac{q_{r,0.5}}{1-w_{r,0.5}}\right] = [8.13, 9.16].$$

Moreover, every positive solution x_n *of* (3.1) *converges to the unique positive equilibrium* x.

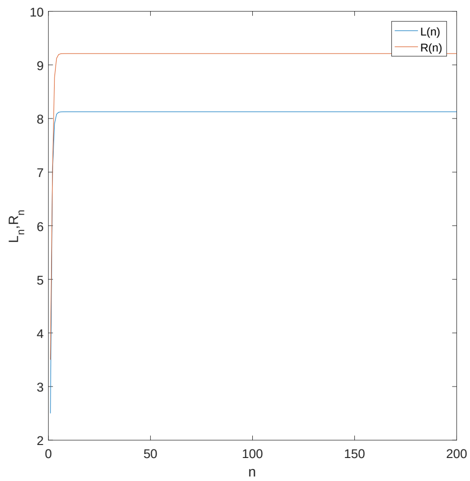

Figure 3.1: *The solution of Eq*(3.1) *at* $\alpha = 0.5$.

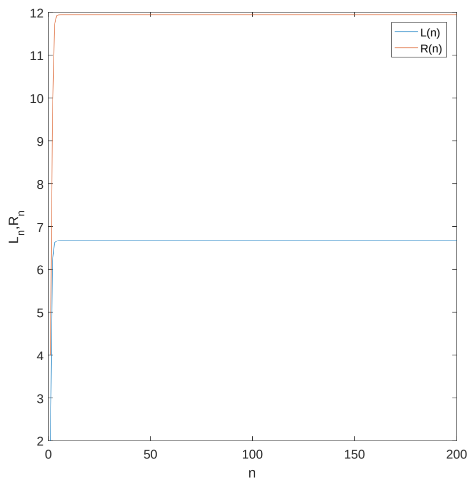

Figure 3.2: *The solution of Eq*(3.1) *at* $\alpha = 0$.

CONCLUSION

This dessertation explores the theoretical and practical aspects of first-order linear difference equations with positive fuzzy coefficients. Through the three chapters, we systematically developed and analyzed the mathematical foundation and behavior of fuzzy difference equations, providing valuable insights into their characteristics and applications.

Key Findings:

- Existence and Uniqueness: The research confirmed that positive fuzzy solutions exist and are unique under certain conditions. This finding is pivotal as it ensures that the equations are solvable within the defined fuzzy framework.
- Boundedness and Persistence: The boundedness and persistence of solutions indicate the stability and resilience of fuzzy solutions, making them reliable for modeling and predicting the behavior of systems involving uncertainty.
- Convergence: The analysis of convergence demonstrated that fuzzy solutions tend to stabilize over time, reflecting the consistency and reliability of these solutions in dynamic systems.

In conclusion, this dissertation contributes to the broader understanding of fuzzy mathematics and its application to difference equations. It opens new avenues for research and application, providing a valuable tool for dealing with uncertainty in mathematical modeling and analysis. The work underscores the potential of fuzzy systems to offer more flexible and realistic models for a variety of complex, real-world problems.

BIBLIOGRAPHY

[1] Z. Alijani, F. Tchier, On the fuzzy difference equation of higher order, J. Comput. Complex. Appl , 3(1)(2017), 44-49.

[2] L. C. Barros, R. C. Bassanezi, W. A. Lodwick, A first course in logic, Fuzzy Dynamical Systems, and Biomathematics, Theory and Applications, Springer-Verlag Berlin Heidelberg ,(2017).

[3] B. Bede, Mathematics of fuzzy sets and fuzzy logic, Springer-Verlag Berlin Heidelberg, (2013).

[4] J. J. Buckley, E. Eslami, An introduction to fuzzy logic and Fuzzy sets, Springer-Verlag Berlin Heidelberg GmbH, (2002).

[5] M. Darus, A. F. Wahab, Review on fuzzy difference equation, Malaysian Journal of Fundamental and Applied Sciences, 8(4)(2012), 176-180.

[6] E. Deeba, A. Korvin and E. L. Koh, A fuzzy difference equation with an application, Journal of Difference Equations and Applications, 2(4)(1996), 365-374.

[7] F. Dernoncourt, Introduction to fuzzy logic, Massachusetts institute of technology, 21(2013), 50-56.

[8] D. Dubois, H. Parde, Fuzzy sets and systems : Theory and applications, Academic Press, New York, 144(1980).

[9] E. M. Elsayed, Asymptoti behavio of certain difference equations, Doctoral thesis, Mansoura University(2006).

[10] E. A. Grove, G. Ladas, Periodicities in nonlinear difference equations, Chapman & Hall/CRC, (2005).

[11] C. Han, L. Li, G. Su, and T. Sun, Dynamical behaviors of a k-order fuzzy difference equation, Open Mathematics, 20(2022), 391-403.

[12] A. KHastan, New solutions for first order linear fuzzy difference equations, Journal of Computational and Applied Mathematics, 312(2017), 156-166.

[13] A. KHastan, Fuzzy logistic difference equation, Iranian Journal of Fuzzy Systems, 15(7)(2018), 55-66.

[14] A. Khelifa, Stabilité globale des solutions de quelques modèles d'équations et systèmes d'équations aux différences, Thèse de Doctorat, Université de Jijel, (2022).

[15] H. L. Kwang, First course on fuzzy theory and applications, Springer Berlin Heidelberg New YorK, (2005).

[16] L. Zadeh, Fuzzy sets, Information and control, 8(1965), 338-353.

[17] H. J. Zimmermann, Fuzzy sets theory and its applications, 4 ed Springer Science+Business Media, (2001).