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On generating functions of some bivariate polynomials and different numbers

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Thanks

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♡

Résumée

Dans ce travail, nous avons introduit ces concepts liés aux relations de récurrence des suites numériques et aux fonctions symétriques. Nous avons ensuite étudié les propriétés des fonctions symétriques concernant les nombres et les polynomes spéciaux. Plus précisément, nous nous sommes concentrés sur les (p, q)-nombres et nous avons calculé de nouvelles fonctions génératrices pour les produits de ces nombres avec les k- nombres de Fibonacci, k-nombres de Pell, k-nombres balancing, et les polynomes de Fibonacci et de Lucas bivariés complexes.

Mots-clées : ***Fonctions symétriques, fonctions génératrices,*** (p, q)***-nombres Fibonacci et*** (p, q)***-nombres Lucas.***

Abstract

In this work we introduced some notions related to recurrence relations of number sequences and symmetric functions, then we studie properties of symmetric functions of special numbers and polynomials. In particular we were interested on (p, q)-numbers and we have calculated new generating functions of products of them with k-Fibonacci numbers, k-Pell numbers, k-balacing numbers, complex bivariate Fibonacci polynomials and complex bivariate Lucas polynomials.

Key-words: ***Symmetric functions, generating functions,*** (p, q)***-Fibonacci numbers and*** (p, q)***-Lucas numbers.***

ملخص

في هذا العمل قدمنا بعض المفاهيم المتعلقة بالعلاقات التراجعية لتتابع الأرقام و الدوال المتناظرة. ثم قمنا بدراسة خصائص الدوال المتناظرة المتعلقة بالأرقام و كثيرات الحدود المتعامدة الخاصة. بشكل خاص ركزنا على الأرقام (p, q) وحسبنا دوال توليد جديدة لجداءات هذه الأرقام مع أعداد فيبوناتشي k و أعداد التوازن k و متعامدات فيبوناتشي و لوكاس المركبة.

كلمات مفتاحية: الدوال المتناظرة، الدوال المولدة، أرقام (p, q) فيبوناتشي و (p, q) لوكاس.

Dedication

TO

The source of tenderness,

TO

The one my soul longs to see,

IF

A sword were to pierce my heart, it would not have taken from me what your eyes have taken.

MY

Dear mother, my heart has deserted me to you, to dwell between your sides and draw strength and determination from the warmth of your love and tenderness.

TO

My beloved mother, to the soul I yearn to see, who has always been a beacon for me in my paths,

TO

My dear support, my beloved father, who has never hesitated to support and encourage me continuously,

I

Dedicate to you both this wonderful work with pride and honor.

TO

My father and mother, thank you for all the sacrifices and support you have given me. This success is the result of your efforts and support throughout the years."

Hadjer Afrid

Dedication

I

Dedicate my work to my dear family

TO

My mother and father, who spared no effort in supporting and encouraging me. Thanks to your guidance and love, I have become who I am today.

TO

My husband and life partner, Omar, you have been my steadfast support forever. Thank you for your patience and invaluable continuous support.

AND

To the joy of my existence, my children AbdElssamad and Miral, you are the heartbeat of my heart and the source of my happiness. My days and dreams blossom because of you."

Roumaissa Merzouki ✍

Contents

INTRODUCTION

RESEARCH and studies in the field of generating functions and special numbers patterns continue, with the introduction of new approaches and tools that contribute to deepening our understanding of these numerical patterns and thier various applications. Recent explorations by scientists using generating functions for complex and polynomial numbers offer insightful views on how these numbers interact in multi-dimensional environments and under changing conditions.

Theoretical methods have diversified and evolved to include the use of complex algebraic and analytical technicians to derive new properties of these numbers, enhancing our ability to solve mathematical problem more effectively. On the computational front, technological and programming developments have improved our ability to analyse complex numerical patterns at increasing speeds, opening up new avenues for exploration.

The importance of these studies extends beyond mathematics to fields such as computer science, where special numbers like Fibonacci and Lucas numbers are used in developing encryption algorithms and cybersecurity, as well as in physics and engineering for prodective modelling and system analysis.

Recent trends show an increase in joint research projects across different disciplines enriching and amplifying the impact of research. This collaboration leads to the production innovative solutions that transcend the traditional boundaries of any specific scientific discipline.

The future of generating functions and special numbers is expected to see new developments that merge theoretical and practical methods, along with expanding the use of modern technologies such as artificial intelligence to understand and analyse these patterns more effectively. Continuing expand research and analysis in this field not only enhances our understanding of mathematics but also contributes to advancing technology and science in many field.

In the first chapter we define formal series, recurrence relation, orthogonal polynomials, and ordinary generating functions.

In the second chapter we introduce to define symmetric functions and their types with some properties.

In the last chapter we introduce to calculate new generating functions of numbers sequences with (p, q)-numbers and bivariate complex polynomials.

CHAPTER 1

PRELIMINARY CONCEPTS

In this chapter, we introduce all the definitions and basic concepts used throughout this work. Firstly, we define formal series, the recurrence relations followed by a recapitulation of orthogonal polynomials. Finally, we introduce ordinary generating functions.

1.1 Formal series

Let $\mathbb{K}$ be a commutative field ($\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$)

Definition 1.1.1. *[3] The elements of the set* $\mathbb{K}[[t]] = \left\{ \sum_{n=0}^{\infty} a_n t^n, a_n \in \mathbb{N} \right\}$ *are called the ring of formal series (with one indeterminate) with coefficients in* $\mathbb{K}$*,* t^n *is called the monomial of degree* n*, and* a_n *is its coefficient, for* $a_n \in \mathbb{N}$

- $\mathbb{C}[[t]]$ *is denoted as the set of formal series with coefficients in* $\mathbb{C}$.

- $\mathbb{R}[[t]]$ *is denoted as the set of formal series with coefficients in* $\mathbb{R}$.

Remark 1.1.1. $\mathbb{K}[[t]]$*, the set of polynomials with coefficients in* $\mathbb{K}$*, is a subset of* $\mathbb{K}[[t]]$.

1.1.1 Operations on formal series

-Addition [3] Let $\alpha(t)=\sum\limits_{n\in\mathbb{N}} a_n t^n$ and $\beta(t)=\sum\limits_{n\in\mathbb{N}} b_n t^n$ be two formal series, then :

$$(\alpha+\beta)(t)=\sum_{n\in\mathbb{N}}(a_n+b_n)t^n.$$

Example : Let $\alpha(t)=\sum\limits_{n\in\mathbb{N}}\left(\frac{1-2^n}{2^n}\right)^n t^n$, $\beta(t)=\sum\limits_{n\in\mathbb{N}} t^n$, then :

$$\begin{aligned}(\alpha+\beta)(t)&=\sum_{n\in\mathbb{N}}\left(1+\frac{1-2^n}{2^n}\right)t^n\\&=\sum_{n\in\mathbb{N}}\frac{1}{2^n}t^n.\end{aligned}$$

.

-Convolution product [3] Let $\alpha(t)=\sum\limits_{n\in\mathbb{N}} a_n t^n$ and $\beta(t)=\sum\limits_{n\in\mathbb{N}} b_n t^n$ two formal series, then :

$$(\alpha*\beta)(t)=\sum_{n\in\mathbb{N}}\left(\sum_{k=0}^{n}a_k b_{n-k}\right)t^n.$$

Example: Let $\alpha(t)=\sum\limits_{n\in\mathbb{N}} t^n=\frac{1}{1-t}$, $\beta(t)=\sum\limits_{n\in\mathbb{N}} nt^n=\frac{t}{(1-t)^2}$, then:

$$(\alpha*\beta)(t)=\sum_{n\in\mathbb{N}}\frac{n(n+1)}{2}t^n.$$

-Scalar multiplication [3] $\beta(t)=\sum\limits_{n=0}^{\infty} kt^n$ is the product of $\alpha(t)=\sum\limits_{n\in\mathbb{N}} t^n$ by the scalar k.

Example : $\beta(t)=\sum\limits_{n=0}^{\infty} 3t^n$ is the product of $\alpha(t)=\sum\limits_{n\in\mathbb{N}} t^n$ by 3.

-Derivation [3] $\beta(t)=\sum\limits_{n=0}^{\infty}(n+1)a_{n+1}t^n$ is the result of derivation of $\alpha(t)=\sum\limits_{n=0}^{\infty} a_n t^n$ with respect to t.

Example : $\frac{2}{(1-2t)^2}$ is the result of derivation of $\frac{1}{(1-2t)}$ with respect to t.

-Integration [3] $\beta(t)=\sum\limits_{n=0}^{\infty}\frac{a_n}{n+1}t^{n+1}$ is the result of integrating the series

$\alpha(t) = \sum_{n=0}^{\infty} a_n t^n.$

Example: $\beta(t) = \sum_{n=0}^{\infty} \frac{(3n+1)}{n+1} t^{n+1}$ is the result of integrating $\alpha(t) = \sum_{n=0}^{\infty} (3n+1)^n t^n$.

-Division [3] Let $\alpha(t) = \sum_{n=0}^{\infty} a_n t^n$ and $\beta(t) = \sum_{n=0}^{\infty} b_n t^n$ two formal series such that $\beta(t) \neq 0$, $\beta(t)$ devise $\alpha(t)$ if and only if there exists a formal series $\omega(t)$ such that:

$$\alpha(t) = \beta(t)\,\omega(t).$$

Proposition 1.1.1. *[3] Every formal series $\alpha(t) = \sum_{n=0}^{\infty} a_n t^n$ has an additive inverse given by:*

$$-\alpha(t) = \sum_{n=0}^{\infty} (-a_n)\, t^n.$$

Proposition 1.1.2. *[3] If $\alpha(t) \neq 0$ and $\beta(t) \neq 0$ are two formal series, then $\alpha(t)\beta(t) \neq 0$ as well.*

Proof. Let $\alpha(t) = \sum_{n=0}^{\infty} a_n t^n \neq 0 \Leftrightarrow \exists n_1 \in \mathbb{N}$ (the smallest integer); such that: $a_{n_1} \neq 0$ and let $\beta(t) = \sum_{n=0}^{\infty} b_n t^n \neq 0 \Leftrightarrow \exists n_2 \in \mathbb{N}$ (the smallest integer); such that : $b_{n_2} \neq 0$ then :
$\alpha(t)\beta(t) = \sum_{n\in\mathbb{N}} \left(\sum_{k=0}^{n} a_k b_{n-k}\right) t^n = (a_{n_1} b_{n_2})\, t^{n_1+n_2} + \sum_{n>n_1+n_2\in\mathbb{N}} \left(\sum_{k=0}^{n} a_k b_{n-k}\right) t^n$. Since :
$a_{n_1} b_{n_2} \neq 0 \Rightarrow \alpha(t)\beta(t) \neq 0.$ □

1.1.2 Invertible series

Definition 1.1.2. *[3] A series $\alpha(t)$ in $\mathbb{K}[[t]]$ is called invertible if there exists a series $\beta(t) \in \mathbb{K}[[t]]$ satisfaying $\alpha(t)\beta(t) = \beta(t)\alpha(t) = 1$. In this case, $\beta(t)$ is called the inverse of $\alpha(t)$.*

Example 1.1.1. *The serie $\sum_{n=0}^{+\infty} (-1)^n t^n \in \mathbb{K}[t]$ is the inverse of $1-t$ indeed:*

$$\begin{aligned}\left(\sum_{n=0}^{+\infty} (-1)^n t^n\right)(1-t) &= \sum_{n=0}^{+\infty} (-1)^n t^n - \sum_{n=0}^{+\infty} (-1)^n t^{n+1} \\ &= t^0 + \sum_{n=1}^{+\infty} (-1)^n t^n - \sum_{n=1}^{+\infty} (-1)^n t^n \\ &= 1.\end{aligned}$$

Theorem 1.1.1. *[3]* $\alpha(t) = \sum_{n=0}^{\infty} a_n t^n \in \mathbb{K}[[t]]$ *is invertible under multiplication if and only if* $a_0 \neq 0$.

Proof. $\alpha(t) = \sum_{n=0}^{\infty} a_n t^n$ is invertible in $\mathbb{K}[[t]]$ if and only if there exists $\alpha^{-1}(t) = (c_n)_{n\in\mathbb{N}}$ in $\mathbb{K}[[t]]$ such that $\alpha(t)\alpha^{-1}(t) = -1$, that is: if and only if $a_0 c_0 = 1$ and
$\forall n \in \mathbb{N}^*, \sum_{j=0}^{n} a_j c_{n-j} = 0.$

- If α is invertible, since $a_0 c_0 = 1$, $a_0 \neq 0$
- Conversely, if $a_0 \neq 0$, the triangular system of equation

$$\begin{cases} a_0 c_0 & = 1 \\ a_1 c0 + a_0 c_1 & = 0 \\ a_2 c_0 + a_1 c_1 + a_0 c_2 & = 0 \\ \vdots & \\ a_n c_0 + a_{n-1} c_1 + \cdots + a_0 c_n & = 0 \end{cases}$$

has a unique solution. □

Proposition 1.1.3. *[3] If* $\alpha(t)$ *is an invertible series, its inverse is unique.*

Proof. Let $\alpha(t)$ is an invertible series, $\beta(t)$ and $\omega(t)$ be two inverses of $\alpha(t)$, then:

$$\alpha(t)\beta(t) = \alpha(t)\omega(t) = 1.$$

thus:

$$\omega(t)\alpha(t)\beta(t) = \omega(t)1,$$

which implies

$$\beta(t) = \omega(t).$$

□

1.2 Recurrence Relations

1.2.1 Homogeneous Linear Recurrence Relations

Definition 1.2.1. *[31] A recurrence relation is called a homogeneous linear recurrence relation of order k with constant coefficients if it is of the form :*

$$a_n - c_1 a_{n-1} - c_2 a_{n-2} - \cdots - c_k a_{n-k} = 0. \tag{1.1}$$

where $c_1, c_2, \cdots, c_k$ are real numbers and $c_k \neq 0$.

Remark 1.2.1. *[31] If $a_i = 0 \forall i$, $\exists [n-k, n]$, a solution of equation (1.1) is called a trivial solution.*

Remark 1.2.2. *[31] Let $a_n = t^n$ be a solution of equation (1.1), with $a_n \neq 0$, then we obtain:*

$$t^n - c_1 t^{n-1} - \cdots - c_k t^{n-k} = 0, \Longleftrightarrow t^k - c_1 t^{k-1} - \cdots - c_k = 0. \tag{1.2}$$

This latter equation is the characteristic equation of the recurrence relation (1.1).

Definition 1.2.2. *[31] Consider the homogeneous linear recurrence relation of order k with constant coefficients.*

$$a_n - c_1 a_{n-1} - c_2 a_{n-2} - \cdots - c_k a_{n-k} = 0. \tag{1.3}$$

The corresponding characteristic polynomial is :

$$P(t) = t^k - c_1 t^{k-1} - \cdots - c_k. \tag{1.4}$$

Theorem 1.2.1. *[3] Let $c_1, c_2, \cdots, c_k$ be real numbers such that $c_k \neq 0$. Suppose the characteristic equation :*

$$t^k - c_1 t^{k-1} - \cdots - c_k = 0.$$

has k distinct roots $t_1, t_2, \cdots, t_k$. Then, a sequence a_n is a solution of the recurrence relation :

$$a_n = \alpha_1 t_1^n + \alpha_2 t_2^n + \cdots + \alpha_k t_k^n, \forall n = 0, 1, 2, \cdots, \tag{1.5}$$

where $\alpha_1, \alpha_2, ..., \alpha_k$ are real constants.

Example 1.2.1. *Consider the Fibonacci sequence , it's recurrence relation is given by :*

$$\begin{cases} F_n = F_{n-1} + F_{n-2}, n \geq 2 \\ \\ F_0 = 0, F_1 = 1. \end{cases}$$

The characteristic equation is: $t^2 - t - 1 = 0$ *which has simple roots* $t_1 = \dfrac{1+\sqrt{5}}{2}$ *and* $t_2 = \dfrac{1-\sqrt{5}}{2}$. *The general solution is therfore of the form* $F_n = \lambda_1 t_1^n + \lambda_2 t_2^n$.
The values of λ_1 *and* λ_2 *are provided by the initial conditions:* $\lambda_1 = \dfrac{1}{\sqrt{5}}$ *and* $\lambda_2 = \dfrac{-1}{\sqrt{5}}$. *Then :*

$$F_n = \left(\frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right) \right).$$

1.2.2 Linear Recurrence Relations of Order 2

Generalized Fibonacci sequence $(G_n)_{n\in\mathbb{N}}$ is defined by the following recurrence relation:

$$\begin{cases} G_n = pG_{n-1} + qG_{n-2}, n \geq 2 \\ \\ G_0 = \alpha,\ G_1 = \beta \end{cases} \tag{1.6}$$

with $p, q \in \mathbb{R}_+$ and $\alpha, \beta \in \mathbb{C}$.

Lemma 1.2.1. *[3] Consider* $t^2 - pt - q = 0$*, the characteristic equation associated with* (1.6). *Then:*

1. If the characteristic equation has two distinct real solutions t_1 *and* t_2*, the general solution of* (1.6) *is given by:*

$$G_n = \frac{\lambda_1 t_1^n - \lambda_2 t_2^n}{t_1 - t_2},$$

with $\lambda_1 = \beta - \alpha t_2$ *and* $\lambda_2 = \beta - \alpha t_1$.

2. If the characteristic equation has a double solution t *in* $\mathbb{R}$ *the general solution of* (1.6) *is given by :*

$$G_n = (c_1 + c_2 n)\, t^n,$$

with $c_1 = \alpha$ *and* $c_2 = \dfrac{\beta - \alpha t}{t}$.

Proof. The characteristic equation associated with relation (1.6) is :

$$t^2 - pt - q = 0.$$

1. If $t_1 \neq t_2$, are the roots of this equation then:

$$t_1 = \frac{p + \sqrt{p^2 + 4q}}{2}, t_2 = \frac{p - \sqrt{p^2 + 4q}}{2}.$$

The general solution is:

$$G_n = c_1 t_1^n + c_2 t_2^n.$$

The constants c_1 and c_2 are determined by the initial conditions as follows:

$$\begin{cases} G_0 = c_1 + c_2 = \alpha, \\ G_1 = c_1 t_1 + c_2 t_2 = \beta. \end{cases}$$

By solving this system of two equations and two indeterminate, we obtain :

$$\begin{cases} c_1 = \dfrac{\beta - \alpha t_2}{t_1 - t_2}, \\ c_2 = \dfrac{\alpha t_1 - \beta}{t_1 - t_2}. \end{cases}$$

The final solution is:

$$G_n = \frac{\lambda_1 t_1^n + \lambda_2 t_2^n}{t_1 - t_2},$$

with $\lambda_1 = \beta - \alpha t_2$ and $\lambda_2 = \beta - \alpha t_1$.

2. If the characteristic equation of relation (1.6) has a double root t:

$$t = \frac{1}{2}p.$$

Therefore, the general solution is:

$$G_n = (c_1 + c_2 n)\, t^n.$$

The constants c_1 and c_2 are determined by the initial conditins as follows:

$$\begin{cases} G_0 = c_1 = \alpha, \\ \\ G_1 = (c_1 t_1 + c_2)\, t = \beta. \end{cases}$$

By solving this system of two equations and two unknowns, we obtain:

$$\begin{cases} c_1 = \alpha, \\ c_2 = \dfrac{\beta - \alpha t}{t}. \end{cases}$$

The final solution is:

$$G_n = (c_1 + c_2 n)\, t^n,$$

with $c_1 = \alpha$ and $c_2 = \dfrac{\beta - \alpha t}{t}$. □

Definition 1.2.3. *[11] The k-Fibonacci numbers are defined by the following reccurence relation:*

$$\begin{cases} F_{k,n} = kF_{k,n-1} + F_{k,n-2}, \forall n \geq 2 \\ \\ F_{k,0} = 1, F_{k,1} = 1. \end{cases} \tag{1.7}$$

The first terms of the k-Fibonacci numbers are given by:

$$F_{k,0} = 1,$$

$$F_{k,1} = 1,$$

$$F_{k,2} = k + 1,$$

$$F_{k,3} = k^2 + k + 1,$$

$$F_{k,4} = k^3 + k^2 + 2k + 1,$$

$$F_{k,5} = k^4 + k^3 + 3k^2 + 2k + 1,$$

$$F_{k,6} = k^5 + 4k^3 + 3k^2 + 3k + 1.$$

Its Binet's formula is given by:

$$F_{k,n} = \frac{1}{\sqrt{k^2+4}}\left(\left(\frac{k+\sqrt{k^2+4}}{2}\right)^n - \left(\frac{k-\sqrt{k^2+4}}{2}\right)^n\right). \tag{1.8}$$

Definition 1.2.4. *[11] The Fibonacci numbers are defined by the following recurrence relation:*

$$\begin{cases} F_n = F_{n-1} + F_{n-2}, \forall n \geq 2 \\ \\ F_0 = 1, F_1. \end{cases} \tag{1.9}$$

The first terms of the Fibonacci numbers are given by:

$$\{F_0 = 1, F_1 = 1, F_2 = 2, F_3 = 3, F_4 = 5, F_5 = 8, F_6 = 12\}.$$

Its Binet's formula is given by:

$$F_n = \left(\frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)\right). \tag{1.10}$$

Definition 1.2.5. *[20] The k-Mersenne numbers are defined by the following recurrence relation:*

$$\begin{cases} M_{k,n} = 3kM_{k,n-1} - 2M_{k,n-2}, \forall n \geq 2 \\ \\ M_{k,0} = 0, M_{k,1} = 1. \end{cases} \tag{1.11}$$

The first terms of the k-Mersenne numbers are given by:

$$M_{k,0} = 0,$$

$$M_{k,1} = 1,$$

$$M_{k,2} = 3k,$$

$$M_{k,3} = 9k^2 - 2,$$

$$M_{k,4} = 27k^3 - 12k,$$

$$M_{k,5} = 81k^4 - 54k^2 - 4,$$

$$M_{k,6} = 243k^5 - 216k^3 + 12k.$$

Its Binet's formula is given by:

$$M_{k,n} = \frac{t_1^n - t_2^n}{t_1 - t_2}, \tag{1.12}$$

with:

$$t_1 = \frac{3k + \sqrt{9k^2 - 8}}{2}.$$

and

$$r_2 = \frac{3k - \sqrt{9k^2 - 8}}{2}.$$

Definition 1.2.6. *[20] The Mersenne numbers are defined by the following recurrence relation:*

$$\begin{cases} M_n = 3M_{n-1} - 2M_{n-2}, \forall n \geq 2 \\ M_0 = 2, M_1 = 1. \end{cases} \tag{1.13}$$

The first terms of the Mersenne numbers are given by:

$$\{M_0 = 0, M_1 = 1, M_2 = 3, M_3 = 7, M_4 = 15, M_5 = 23, M_6 = 39\}.$$

Its Binet's formula is given by:

$$M_n = \frac{t_1^n - t_2^n}{t_1 - t_2} = 2^n - 1. \tag{1.14}$$

with: $t_1 = 2$ *and* $t_2 = 1$.

Definition 1.2.7. *[20] The k-Mersenne-Lucas numbers are defined by the following recurrence relation:*

$$\begin{cases} m_{k,n} = 3km_{k,n} - 2m_{k,n-2}, \forall n \geq 1 \\ m_{k,0} = 2, m_{k,1} = 3k. \end{cases} \tag{1.15}$$

The first terms of the k-Mersenne numbers are given by:

$$m_{k,0} = 2,$$

$$m_{k,1} = 3k,$$

$$m_{k,2} = 9k^2 - 4,$$

$$m_{k,3} = 27k^3 - 18k$$

$$m_{k,4} = 81k^4 - 72k^2 + 8.$$

The sequence $(m_{k,n})_{n\in\mathbb{N}}$*, has a second-order recurrence relation, the characteristic equation given by:*

$$t^2 - 3kt + 2 = 0.$$

Its Binet's formula is given by:

$$m_{n,k} = t_1^n + t_2^n, \tag{1.16}$$

where $t_1 > t_2$ *with:*

$$t_1 = \frac{3k + \sqrt{9k^2 - 8}}{2}.$$

and

$$t_2 = \frac{3k - \sqrt{9k^2 - 8}}{2}.$$

• *For* $k = 1$ *we have :*

$$t_1 = 2, t_2 = 1.$$

Definition 1.2.8. *[20] The Mersenne-Lucas numbers are defined by the following recurrence relation:*

$$\begin{cases} m_n = 3m_{n-1} - 2m_{n-2}, \forall n \geq 1 \\ \\ m_0 = 2, m_1 = 3. \end{cases} \tag{1.17}$$

The first terms of the Mersenne-Lucas numbers are given by:

$$\{m_0 = 2, m_1 = 3, m_2 = 5, m_3 = 9, m_4 = 18.\}.$$

The sequence $(m_{k,n})_{n\in\mathbb{N}}$*, with a second-order recurrence relation, has the characteristic equation given by:*

$$t^2 - 3kt + 2 = 0.$$

Its Binet's formula is given by:

$$m_n = t_1^n + t_2^n, \tag{1.18}$$

where $t - 1 > t_2$ *with :*

$$t_1 = 2.$$

and

$$t_2 = 1.$$

Proposition 1.2.1. *[20] The n^{th} terms of the negative indices of the k-Mersenne-Lucas numbers are given by:*

$$m_{k,-n} = \frac{1}{2^n} m_{k,n}.$$

Proposition 1.2.2. *The negative indices of the Mersenne-Lucas numbers are given by:*

$$m_{-n} = \frac{1}{2^n} m_n.$$

1.2.3 Linear Recurrence Relations of ordre 3

The generalized Fibonacci sequence of ordre three $(W_n)_{n\in\mathbb{N}}$ is defined by the following recurrence relation:

$$\begin{cases} W_n = aW_{n-1} + bW_{n-2} + cW_{n-3}, n \geq 3 \\ \\ W_0 = \alpha, W_1 = \beta, W_2 = \gamma. \end{cases} \tag{1.19}$$

where $a, b, c \in \mathbb{R}$ and $\alpha, \beta \in \mathbb{R}_+$.

Form the recurrence relation (1.19), we obtainthe characteristic equation
$t^3 - \alpha t^2 - bt - c = 0$, the solutions t_1, t_2, t_3 for the characteristic equation are given by:

$$\begin{cases} t_1 = \dfrac{a}{3} + A + B, \\ \\ t_2 = \dfrac{a}{3} + \omega A + \omega B, \\ \\ t_3 = \dfrac{a}{3} + \omega^2 A + \omega B. \end{cases}$$

with:

$$\begin{cases} A = \left(\dfrac{a^3}{27} + \dfrac{ab}{6} + \dfrac{c}{2} + \sqrt{\Delta} \right)^{\frac{1}{3}}, \\ \\ B = \left(\dfrac{a^3}{27} + \dfrac{ab}{6} + \dfrac{c}{2} - \sqrt{\Delta} \right)^{\frac{1}{3}}, \\ \\ \Delta = \dfrac{a^3 c}{27} - \dfrac{a^2 b^2}{108} + \dfrac{abc}{6} - \dfrac{b^3}{27} + \dfrac{c^2}{4}. \end{cases}$$

and $\omega = \dfrac{-1+i\sqrt{3}}{2}$.
Its Binet's formula is given by:

$$W\left(n\right) = \frac{R}{\left(t_1-t_2\right)\left(t_1-t_2\right)}t_1^n + \frac{S}{\left(t_1-t_2\right)\left(t_1-t_3\right)}t_2^n - \frac{T}{\left(t_1-t_3\right)\left(t_1-t_3\right)}t_3^n.$$

with:

$$\begin{cases} R = \gamma - \left(t_2+t_3\right)\beta + t_2t_3\alpha, \\ S = \gamma - \left(t_1+t_3\right)\beta + t_1t_3\alpha, \\ T = \gamma - \left(t_1+t_2\right)\beta + t_1t_2\alpha. \end{cases}$$

An example associated with the recurrence relation (1.19)

1. For $a = 0$, $b = c = 1$ and $\alpha = \beta = \gamma = 1$, we obtain the Padovan sequence $\left(P_n\right)_{n\in\mathbb{N}}$:

$$\begin{cases} P_n^{(3)} = P_{n-2}^{(3)} + P_{n-3}^{(3)}, n \geq 3, \\ P_0^{(3)} = P_1^{(3)} = P_2^{(3)} = 1, \\ \{P_n^{(3)}\} = \{1,1,1,2,2,3,4,5,7,9,12,16,21,28,37,...\}. \end{cases} \quad (1.20)$$

Its Binet's formula is given by:

$$P_n(3) = \frac{\left(t_2-1\right)\left(t_3-1\right)}{\left(t_1-t_2\right)\left(t_1-t_3\right)}t_1^n + \frac{\left(t_1-1\right)\left(t_3-1\right)}{\left(t_2-t_1\right)\left(t_2-t_3\right)}t_2^n - \frac{\left(t_1-1\right)\left(t_2-1\right)}{\left(t_3-t_1\right)\left(t_3-t_2\right)}t_3^n.$$

with:

$$\begin{cases} t_1 = \sqrt[3]{\dfrac{9+\sqrt{69}}{18}} + \sqrt[3]{\dfrac{9-\sqrt{69}}{18}}, \\ t_2 = \omega\sqrt[3]{\dfrac{9+\sqrt{69}}{18}} + \overline{\omega}\sqrt[3]{\dfrac{9-\sqrt{69}}{18}}, \\ t_3 = \overline{\omega}\sqrt[3]{\dfrac{9+\sqrt{69}}{18}} + \omega\sqrt[3]{\dfrac{9-\sqrt{69}}{18}}. \end{cases}$$

with $W = \dfrac{-1+i\sqrt{3}}{2}$.

2. For $a=1, c=1, b=0$, and $\alpha=0, \beta=\gamma=1$, we obtain the Nayarana sequence $(N_n)_{n\in\mathbb{N}}$:

$$\begin{cases} N_n^{(3)} = N_{n-1}^{(3)} + N_{n-3}^{(3)}, n \geq 3, \\ N_0^{(3)} = N_1^{(3)} = N_2^{(3)} = 1, \\ \{N_n^{(3)}\} = \{0,1,1,1,2,2,3,4,6,9,13,16,19,28,41,...\}. \end{cases} \tag{1.21}$$

Its Binet's formula is given by:

$$N_n^{(3)} = \frac{1}{(t_1-t_2)(t_1-t_3)}t_1^{n+1} + \frac{1}{(t_2-t_1)(t_2-t_3)}t_2^{n+1} + \frac{1}{(t_3-t_1)(t_3-t_2)}t_3^{n+1}.$$

with:

$$\begin{cases} t_1 = \frac{1}{3}\left(1 + \sqrt[3]{\frac{2}{29+3\sqrt{93}}} + \sqrt[3]{\frac{29+3\sqrt{93}}{2}}\right), \\ t_2 = \frac{1}{3}\left(1 - \omega\sqrt[3]{\frac{2}{29+3\sqrt{93}}} + \omega^2\sqrt[3]{\frac{29+3\sqrt{93}}{2}}\right), \\ t_3 = \frac{1}{3}\left(1 - \omega^2\sqrt[3]{\frac{2}{29+3\sqrt{93}}} - \omega\sqrt[3]{\frac{29+3\sqrt{93}}{2}}\right). \end{cases}$$

with : $\omega = \frac{1+i\sqrt{3}}{2}$.

3. For $a=b=1, c=2$ and $\alpha=0, \beta=\gamma=1$, we obtain the Jacobsthal sequence of the third order $(J_n)_{n\in\mathbb{N}}$:

$$\begin{cases} J^{(3)} = J_{n+2}^{(3)} + J_{n+1}^{(3)} + 2J_n^{(3)}, n \geq 0, \\ J_0^{(3)} = J_1^{(3)} = J_2^{(3)} = 1, \\ \{J_n^{(3)}\} = \{0,1,1,2,3,5,9,18,37,146,293...\}. \end{cases} \tag{1.22}$$

Its Binet's formula is given by :

$$J_n(3) = \frac{2}{7}2^n + \frac{3+2i\sqrt{3}}{21}\omega_1^n - \frac{3-2i\sqrt{3}}{21}\omega_2^n.$$

with: $\omega_1 = \frac{-1+i\sqrt{3}}{2}$ and $\omega_2 = \overline{\omega_1}$.

4. For $a=0, b=1, c=2$ and $\alpha=3, \beta=0, \gamma=2$, we obtain the Jacobsthal-Padovan

sequence $(JP_n)_{n\in\mathbb{N}}$:

$$\begin{cases} JP_n = JP_{n-2} + 2JP_{n-3}, n \geq 3, \\ JP_0 = JP_1 = JP_2 = 1, \\ \{JP_n\} = \{1, 1, 1, 3, 3, 5, 9, 11, 19, 29, 41...\}. \end{cases} \tag{1.23}$$

5. For $a = 0, b = c = 1$ and $\alpha = 3, \beta = 0, \gamma = 2$, we obtain the Perin sequence $(R_n)_{n\in\mathbb{N}}$:

$$\begin{cases} R_n = R_{n-2} + R_{n-3}, n \geq 3, \\ R_0 = 3, R_1 = 0, R_2 = 2, \\ \{R_n\} = \{3, 0, 2, 3, 2, 5, 5, 7, 10, 12, 17...\}. \end{cases} \tag{1.24}$$

6. For $a = 0, b = 2, c = 1$ and $\alpha = 3, \beta = 0, \gamma = 2$, we obtain the Pell-Perin sequence $(PR_n)_{n\in\mathbb{N}}$:

$$\begin{cases} PR_n = 2PR_{n-2} + PR_{n-3}, n \geq 3, \\ PR_0 = 3, PR_1 = 0, PR_2 = 2, \\ \{PR_n\} = \{3, 0, 2, 3, 4, 8, 11, 20, 30, 51, 80...\}. \end{cases} \tag{1.25}$$

7. For $a = 0, b = c = 1$ and $\alpha = \beta = 0, \gamma = 1$,

we obtain the Padovan-Perin sequence $(S_n)_{n\in\mathbb{N}}$:

$$\begin{cases} S_n = S_{n-2} + S_{n-3}, n \geq 3, \\ S_0 = 3, S_1 = 0, S_2 = 2, \\ \{S_n\} = \{0, 0, 1, 0, 1, 1, 1, 2, 2, 3, 4...\}. \end{cases} \tag{1.26}$$

1.3 Orthogonal Polynomials

Theorem 1.3.1. *[10] Every second-order recurrence relation of polynomials sequences $(P_n(x))_{n\in\mathbb{N}}$ is that of orthogonal polynomials.*

Theorem 1.3.2. *[10] Let $(P_n(t))_{n\in\mathbb{N}}$ be a sequence of normalized polynomials:*

$$P_n(t) = t^n + a_{n-1}t^{n-1} + a_{n-2}t^{n-2} + a_{n-3}t^{n-3} + \cdots$$

Then, $(P_n(t))_{n\in\mathbb{N}}$ *is a sequence of normalized orthogonal polynomials if and only if there exist two sequences of complex numbers* $(\alpha_n)_{n\in\mathbb{N}}$ *and* $(\beta_n)_{n\in\mathbb{N}}$ *satisfaying the following recurrence relation:*

$$\begin{cases} P_{n+1}(t) = (t-\alpha_n) P_n(t) - \beta_n P_{n-1}(t), \forall n \geq 0 \\ \\ P_1(t) = 3, P_0(t) = 1. \end{cases}$$

Theorem 1.3.3. ***[10]*** *Let* $(P_n(t))_{n\in\mathbb{N}}$ *be a sequence of orthogonal polynomials. The following statements are equivalent:*

1. $P_n(-t) = (-1)^n P_n(t)$.
2. $P_{n+1}(t) = tP_n(t) - \beta_n P_{n-1}(t), P_{-1}(t) = 0, P_0(t) = 1, \forall n \geq 0.$

Definition 1.3.1. ***[10]*** *The Chebyshev polynomials of the first kind, denoted* $T_n(x)$ *are polynomials in* x *of degree* n *defined by the relation:*

$$T_n(x) = \cos(n\theta), \text{ with } x = \cos(\theta).$$

For all x *in* $[-1,1]$, $\theta \in [0,\pi]$.

The first few terms of Chebyshev polynomials of the first kind $T_n(x)$ *are given by:*

$$T_0(x) = 1,$$

$$T_1(x) = x,$$

$$T_2(x) = 2x^2 - 1,$$

$$T_3(x) = 4x^3 - 3x,$$

$$T_4(x) = 8x^4 + 8x^2 + 1,$$

$$T_5(x) = 16x^5 - 20x^3 + 5x,$$

$$T_6(x) = 32x^6 - 48x^4 + 18x^2 - 1,$$

$$T_7(x) = 64x^7 - 112x^5 + 56x^3 - 7x.$$

Definition 1.3.2. ***[24]*** *Chebyshev polynomials of the second kind, denoted* $T_n(x)$ *satisfies the* *second-order recurrence relation:*

$$\begin{cases} T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \forall n \geq 1. \\ \\ T_0(x) = 1, T_1(x) = x. \end{cases} \tag{1.27}$$

Definition 1.3.3. *Chebyshev polynomials of the second kind, denoted $U_n(x)$ are polynomials in x of degree n defined by the relation :*

$$U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta}, \text{ with } x = \cos\theta.$$

For all x in $[-1, 1]$, $\theta \in [0, \pi]$.
The first few terms of Chebyshev polynomials of the second kind $U_n(x)$ are given by:

$$U_0(x) = 1,$$

$$U_1(x) = 2x,$$

$$U_2(x) = 4x^2 - 1,$$

$$U_3(x) = 8x^3 - 4x,$$

$$U_4(x) = 16x^4 - 12x^2 + 1,$$

$$U_5(x) = 32x^5 - 32x^3 + 6x,$$

$$U_6(x) = 64x^6 - 80x^4 + 24x^3 - 1,$$

$$U_7(x) = 128x^7 - 192x^5 + 80x^3 - 8x.$$

Definition 1.3.4. *[24] Chebyshev polynomials of the second kind, denoted $V_n(x)$ satisfies the second-order recurrence relation :*

$$\begin{cases} U_n(x) = 2xU_{n-1} - U_{n-2}, \forall n \geq 2. \\ \\ U_0(x) = 1, U_1(x) = 2x. \end{cases} \tag{1.28}$$

Definition 1.3.5. *[29] Chebyshev polynomials of the third kind, denoted* $V_n(x)$ *are polynomials in* x *of degree* n *defined by the relation :*

$$V_n(x) = \frac{\cos\left(n+\frac{1}{2}\right)\theta}{\cos\left(\frac{1}{2}\right)\theta}, \text{ with } x = \cos\theta.$$

For all x *in* $[-1,1]$, $\theta \in [0,\pi]$.

The first few terms of Chebyshev polynomials of the third kind $U_n(x)$ *are given by [19]:*

$$V_0(x) = 1,$$

$$V_1(x) = 2x - 1,$$

$$V_2(x) = 4x^2 - 2x - 1,$$

$$V_3(x) = 8x^3 - 4x^2 - 4x + 1,$$

$$V_4(x) = 16x^4 - 8x^3 - 12x^2 + 4x + 1,$$

$$V_5(x) = 32x^5 - 16x^4 + 32x^3 + 12x^2 + 6x - 1,$$

$$V_6(x) = 64x^6 - 32x^5 - 80x^4 - 32x^3 + 24x^2 - 6x - 1,$$

$$V_7(x) = 128x^7 - 64x^6 - 192x^5 + 80x^4 + 80x^3 - 24x^2 - 8x + 1.$$

Definition 1.3.6. *Chebyshev polynomials of the third kind, denoted* $V_n(x)$ *satisfies the second-order recurrence relation :*

$$\begin{cases} V_n(x) = 2xV_{n-1}(x) - V_{n-2}(x), \forall n \geq 2. \\ V_0(x) = 1, V_1(x) = 2x - 1. \end{cases} \tag{1.29}$$

Definition 1.3.7. *Chebyshev polynomials of the fourth kind, denoted* $W_n(x)$ *are polynomials in* x *of degree* n *defined by the relation :*

$$W_n(x) = \frac{\sin\left(n+\frac{1}{2}\right)\theta}{\sin\left(\frac{1}{2}\right)\theta}, \text{ with } x = \cos\theta.$$

For all x in $[-1,1]$, $\theta \in [0,\pi]$.

The first few terms of Chebyshev polynomials of the fourth kind $W_n(x)$ are given by:

$$W_0(x) = 1,$$

$$W_1(x) = 2x + 1,$$

$$W_2(x) = 4x^2 + 2x - 1,$$

$$W_3(x) = 8x^3 + 4x^2 - 4x - 1,$$

$$W_4(x) = 16x^4 + 8x^3 - 12x^2 - 4x - 1,$$

$$W_5(x) = 32x^5 + 16x^4 - 32x^3 - 12x^2 + 6x + 1,$$

$$W_6(x) = 64x^6 + 32x^5 - 80x^4 - 32x^3 + 24x^2 + 6x - 1,$$

$$W_7(x) = 128x^7 + 64x^6 - 192x^5 - 80x^4 + 80x^3 + 24x^2 - 8x - 1.$$

Definition 1.3.8. *[24] Chebyshev polynomials of the fourth kind, denoted $V_n(x)$ satisfies the second-order recurrence relation :*

$$\begin{cases} W_n(x) = 2xW_{n-1} - W_{n-2}, \forall n \geq 2. \\ W_0(x) = 1, W_1(x) = 2x + 1. \end{cases} \tag{1.30}$$

Definition 1.3.9. *Vieta Fibonacci polynomials denote $(v_n(x))_{n\in\mathbb{N}}$ are defined by the recurrence relation :*

$$\begin{cases} v_n(x) = xv_{n-1}(x) - v_{n-2}(x), \forall n \geq 2. \\ v_0(x) = 0, v_1(x) = 1. \end{cases} \tag{1.31}$$

Definition 1.3.10. *Vieta Lucas polynomials denote $(u_n(x))_{n\in\mathbb{N}}$ are defined by the recurrence relation :*

$$\begin{cases} u_n(x) = xu_{n-1}(x) - u_{n-2}(x), \forall n \geq 2. \\ u_0(x) = 2, u_1(x) = x. \end{cases} \tag{1.32}$$

Definition 1.3.11. *Vieta Pell polynomials denote $(t_n(x))_{n\in\mathbb{N}}$ are defined by the recurrence* *relation :*

$$\begin{cases} t_n(x) = xt_{n-1}(x) - t_{n-2}(x), \forall n \geq 2. \\ \\ t_0(x) = 0, t_1(x) = 1. \end{cases} \tag{1.33}$$

Definition 1.3.12. *Vieta Pell Lucas polynomials denote* $(s_n(x))_{n\in\mathbb{N}}$ *are defined by the recurrence relation :*

$$\begin{cases} s_n(x) = xs_{n-1}(x) - s_{n-2}(x), \forall n \geq 2. \\ \\ s_0(x) = 2, s_1(x) = 2x. \end{cases} \tag{1.34}$$

1.4 Ordinary Generating Functions

Definition 1.4.1. *[17] The Ordinary Generating Function* (OGF) *of the sequence :*

$$(a_n)_{n\in\mathbb{N}} = (a_0, a_1, a_2, ...),$$

is defined by :

$$G(t) = \sum_{n=0}^{\infty} a_n t^n. \tag{1.35}$$

Example 1.4.1. - *The OGF of* $(1, 1, 1, ...)$ *is :*

$$G(t) = \sum_{n=0}^{\infty} t^n = 1 + t + t^2 + t^3 + \cdots = \frac{1}{1-t}.$$

- *The OGF of the sequence* (2^n) *is :*

$$\begin{aligned} G(t) &= \sum_{n=0}^{\infty} (2^n) t^n. \\ &= 1 + 2t + 2^2t^2 + \cdots + 2t^n. \end{aligned}$$

Theorem 1.4.1. *[17] Let* $A(t)$ *be the OGF of* $(a_n)_{n\in\mathbb{N}}$*, and* $B(t)$ *be the OGF of* $(b_n)_{n\in\mathbb{N}}$ *then:*

1. $A(t) + B(t)$ *is the OGF of* $(a_n + b_n)_{n\in\mathbb{N}}$ *.*
2. $tA(t)$ *is the OGF of* $(0, a_0, a_1, a_2, ..., a_{n-1}, ...)$ *.*
3. $A'(t)$ *is the OGF of* $(a_1, 2a_2, 3a_3, ..., (n+1)a_{n+1}, ...)$ *.*

4. $A(t)B(t)$ *is the OGF of* $(a_0, a_0b_1 + a_1b_0, a_0b_2 + a_1b_1 + a_2b_0, ...)$ *.*

5. $(1-t)A(t)$ *is the OGF of* $(a_0, a_1 - a_0, a_2 - a_1, ..., a_n - a_{n-1}, ...)$ *.*

Proof. 1. $A(t) + B(t) = \sum_{n=0}^{\infty} a_n t^n + \sum_{n=0}^{\infty} b_n t^n = \sum_{n=0}^{\infty} (a_n + b_n) t^n.$

2. $tA(t) = t\sum_{n=0}^{\infty} a_n t^n = \sum_{n=0}^{\infty} a_n t^{n+1} = 0 + \sum_{n=1}^{\infty} (a_{n-1}) t^n.$

3. $A'(t) = \left(\sum_{n=0}^{\infty} a_n t^n\right)' = \sum_{n=0}^{\infty} n a_n t^{n-1} = \sum_{n=0}^{\infty} (n+1)(a_{n+1}) t^n.$

4.

$$A(t)B(t) = \left(a_0 + a_1 t + a_2 t^2 + \cdots\right)\left(b_0 + b_1 + b_2 t^2 + \cdots\right).$$

$$= a_0b_0 + (a_0b_1 + a_1b_0)t + (a_0b_2 + a_1b_1 + a_2b_0)t^2 + \cdots$$

$$= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a_k b^{n-k}\right) t^n.$$

5. If $B(t)$ is the OGF of $(1, -1, 0, 0, ...)$, then $(1-t)A(t) = A(t)(1-t) = A(t)B(t)$, is the OGF of:

$$\Bigg(a_0, \underbrace{a_0(-1) + a_1(1)}_{= a_1 - a_0}, ..., \underbrace{a_0b_n + a_1b_{n-1}}_{= 0} + \cdots + \underbrace{a_{n-1}(-1) + a_n(1)}_{= a_n - a_{n-1}}, ...\Bigg).$$

□

Theorem 1.4.2. *[3] Let the sequence* $(G_n)_{n\in\mathbb{N}}$ *be defined by the recurrence relation:*

$$\begin{cases} G_n(t) = pG_{n-1} + qG_{n-2}, n \geq 2 \\ \\ G_0 = \alpha, G_1 = \beta. \end{cases} \tag{1.36}$$

where $p, q \in \mathbb{R}_+^*$ *and* $\alpha, \beta \in \mathbb{C}$*.*

Then the associated generating function for $(G_n)_{n\in\mathbb{N}}$ *is given by :*

$$G(t) = \frac{\alpha + (\beta - p\alpha)t}{1 - pt - qt^2}. \tag{1.37}$$

Proof. We have:

$$G(t) = \sum_{n=0}^{\infty} G_n t^n$$

$$= G_0 + G_1 t + \sum_{n=2}^{\infty} G_n t^n$$

$$= \alpha + \beta t + \sum_{n=2}^{\infty} \left(pG_{n-1} + qG_n - 2\right) t^n$$

$$= \alpha + \beta t + pt \sum_{n=2}^{\infty} G_{n-1} t^{n-1} + qt^2 \sum_{n=2}^{\infty} G_{n-2} t^{n-2}$$

$$= \alpha + \beta t + pt \sum_{n=1}^{\infty} G_n t^n + qt^2 \sum_{n=0}^{\infty} G_n t^n$$

$$= \alpha + \beta t + pt \left(\sum_{n=0}^{\infty} G_n t^n - \alpha \right) + qt^2 \sum_{n=0}^{\infty} G_n t^n$$

$$= \alpha + (\beta - \alpha p)\, t + ptG(t) + qt^2 G(t)\,.$$

so :

$$G(t) \left(1 - pt - qt^2\right) = \alpha + (\beta - p\alpha)\, t.$$

This gives us :

$$G(t) = \frac{\alpha + (\beta - p\alpha)\, t}{1 - pt - qt^2}.$$

From the previous theorem we obtain the following generating function :

1. For $\alpha = k$, $\beta = q = 1$, $p = k$, we get the generating function of k-Fibonacci numbers:

$$G(t) = \frac{1}{1 - t - t^2}.$$

2. For $\alpha = 0$, $\beta = 1$, $p = 3k$, $q = -2$, we get the generating function of k-Mersenne numbers :

$$G(t) = \frac{1}{1 - 3kt - 2t^2}.$$

3. For $\alpha = 2$, $\beta = p = 3k$, $q = -2$, we get the generating function of k-Mersenne-Lucas numbers :

$$G(t) = \frac{2 - 3kt}{1 - 3kt + 2t^2}.$$

• For $k = 1$ in the above relation we obtain the following generating functions:

- The generating function of Fibonacci numbers is given by:

$$G(t) = \frac{1}{1-t-t^2}.$$

- The generating function of Mersenne numbers is given by:

$$G(t) = \frac{1}{1-3t+2t^2}.$$

- The generating function of Mersenne-Lucas numbers is given by:

$$G(t) = \frac{2-3t}{1-3t+2t^2}.$$

□

Theorem 1.4.3. *[3] Let the sequence $(W_n)_{n\in\mathbb{N}}$ be defined by the recurrence relation:*

$$\begin{cases} W_n(t) = aW_{n-1} + bWG_{n-2} + cW_{n-3}, n \geq 3 \\ \\ W_0 = \alpha, W_2 = \beta, W_3 = \gamma. \end{cases} \tag{1.38}$$

where $a, b, c \in \mathbb{R}$ and $\alpha, \beta, \gamma \in \mathbb{C}$. Then the generating function associated with $(W_n)_{n\in\mathbb{N}}$ is given us:

$$G(t) = \frac{\alpha + (\beta - \alpha a)t + (\gamma - \beta a - b\alpha)t^2}{1 - at - bt^2 - ct^3}. \tag{1.39}$$

Proof.

$$\begin{aligned} G(t) &= \sum_{n=0}^{\infty} W_n t^n \\ &= W_0 + W_1 t + W_2 t^2 + \sum_{n=3}^{\infty}(aW_{n-1} + bW_{n-2} + cW_{n-3})t^n \\ &= \alpha + \beta t + \gamma t^2 + at\sum_{n=3}^{\infty} W_{n-1}t^{n-1} + bt^2\sum_{n=3}^{\infty} W_{n-2}t^{n-2} + ct^3\sum_{n=3}^{\infty} W_{n-3}t^{n-3} \\ &= \alpha + \beta t + \gamma t^2 + at\sum_{n=2}^{\infty} W_n t^n + bt^2\sum_{n=1}^{\infty} W_n t^n + ct^3\sum_{n=0}^{\infty} W_n t^n \\ &= \alpha + \beta t + \gamma t^2 + at\left(\sum_{n=0}^{\infty} W_n t^n - \alpha - \beta t\right) + bt^2\left(\sum_{n=0}^{\infty} W_n t^n - \alpha\right) + ct^3\left(\sum_{n=0}^{\infty} W_n t^n\right) \end{aligned}$$

$$= \alpha + \beta t + \gamma t^2 + at\sum_{n=0}^{\infty} W_n t^n - \alpha a t - a\beta t^2 + bt^2 \sum_{n=0}^{\infty} W_n t^n - \alpha b t^2 + ct^3 \sum_{n=0}^{\infty} W_n t^n$$

$$= \alpha + \beta t + \gamma t^2 + atG(t) - \alpha a t - a\beta t^2 + bt^2 G(t) - \alpha b t^2 + ct^3 G(t).$$

Therefore:

$$G(t)\left(1 - \alpha t - bt^2 - ct^3\right) = \alpha + (\beta - \alpha a)t + (\gamma - \beta a - b\alpha)t^2.$$

Hence:

$$G(t) = \frac{\alpha + (\beta - \alpha a)\, t + (\gamma - \beta a - b\alpha)\, t^2}{1 - at - bt^2 - ct^3}.$$

Now, let's proceed with the translation:

1. For $\alpha = \beta = \gamma = 1$, $a = 0$, $b = c = 1$, we obtain the generating function for the sequence $(P_n)_{n\in\mathbb{N}}$:

$$G(t) = \frac{1+t}{1 - t^2 - t^3}.$$

2. For $\alpha = 0$, $\beta = \gamma = 1$, $b = c = 1$, $b = 0$, we obtain the generating function for the sequence $(N_n)_{n\in\mathbb{N}}$:

$$G(t) = \frac{t}{1 - t - t^3}.$$

3. For $\alpha = 0$, $\beta = \gamma = 1$, $a = b = 1$, $c = 2$, we obtain the generating function for the sequence $(J_n)_{n\in\mathbb{N}}$:

$$G(t) = \frac{t}{1 - t - t^2 - 2t^3}.$$

4. For $\alpha = \beta = \gamma = 1$, $a = 0$, $b = 1$, $c = 2$, we obtain the generating function for the sequence $(JP_n)_{n\in\mathbb{N}}$:

$$G(t) = \frac{1+t}{1 - t^2 - 2t^3}.$$

5. For $\alpha = 3$, $\beta = 0$, $\gamma = 4$, $a = 0$, $b = c = 1$, we obtain the generating function for the sequence $(R_n)_{n\in\mathbb{N}}$:

$$G(t) = \frac{3 - t^2}{1 - t^2 - t^3}.$$

6. For $\alpha = 3$, $\beta = 0$, $\gamma = 10$, $a = 0$, $b = 2$, $c = 1$, we obtain the generating function for the sequence $(PR_n)_{n\in\mathbb{N}}$:

$$G(t) = \frac{3 - 4t^2}{1 - 2t^2 - t^3}.$$

7. For $\alpha=\beta=0$, $\gamma=1$, $a=0$, $b=c=1$, we obtain the generating function for the sequence $(S_n)_{n\in\mathbb{N}}$:

$$G(t)=\frac{t^2}{1-2t^2-t^3}.$$

□

Theorem 1.4.4. *[3] Let the sequence $(P_n(x))_{n\in\mathbb{N}}$ defined by the recurrence relation :*

$$\begin{cases} P_n(x)=pxP_{n-1}(x)+qP_{n-2}(x), n\geq 2 \\ \\ P_0=\alpha, P_1=\beta x+\gamma. \end{cases} \tag{1.40}$$

where $p,q\in\mathbb{R}$ and $\alpha,\beta,\gamma\in\mathbb{C}$. Then the generating function associated with $(P_n(x))_{n\in\mathbb{N}}$ is given by:

$$G(t)=\frac{\alpha+((\beta-\alpha p)x+\gamma)}{1-pxt-qt^2}. \tag{1.41}$$

Proof. We have:

$$\begin{aligned} G(t) &= \sum_{n=0}^{\infty} GP_n(x)t^n \\ &= P_0(x)+P_1(x)t+\sum_{n=2}^{\infty}P_n(x)t^n \\ &= \alpha+(\beta x+\gamma)t+px\sum_{n=2}^{\infty}P_{n-1}(x)t^n+q\sum_{n=2}^{\infty}P_{n-2}(x)t^n \\ &= \alpha+(\beta x+\gamma)t+pxt\sum_{n=2}^{\infty}P_{n-1}(x)t^{n-1}+qt^2\sum_{n=0}^{\infty}P_n(x)t^n \\ &= \alpha+(\beta x+\gamma)t+pxt\left(\sum_{n=0}^{\infty}P_n(x)t^n-\alpha\right)+qt^2\sum_{n=0}^{\infty}P_n(x)t^n \\ &= \alpha+(\beta x+\gamma)t+pxt\sum_{n=0}^{\infty}P_n(x)t^n-px\alpha t+qt^2\sum_{n=0}^{\infty}P_n(x)t^n \\ &= \alpha+((\beta-\alpha p)x+\gamma)t+pxtG(t)+qt^2G(t), \end{aligned}$$

so

$$G(t)\left(1-pxt-qt^2\right)=\alpha+((\beta-\alpha p)x+\gamma)t.$$

Therefore:

$$G(t)=\frac{\alpha+((\beta-\alpha p)x+\gamma)}{1-pxt-qt^2}.$$

According to the previous theorem, we deduce the following generating functions :

1. For $\alpha = \beta = 0, \gamma = p = 1, q = -1$, we obtain the generating functions associated with $(v_n(x))_{n\in\mathbb{N}}$:

$$G(t) = \frac{t}{1 - xt + t^2}.$$

2. For $\alpha = 2, \beta = 1, \gamma = 0, p = 1, q = -1$, we obtain the generating functions associated with $(u_n(x))_{n\in\mathbb{N}}$:

$$G(t) = \frac{2 - xt}{1 - xt + t^2}.$$

3. For $\alpha = \beta = 0, \gamma = 1, p = 2, q = -1$, we obtain the generating functions associated with $(t_n(x))_{n\in\mathbb{N}}$:

$$G(t) = \frac{t}{1 - 2xt + t^2}.$$

4. For $\alpha = 2, \beta = \gamma = 0, p = 1, q = -1$, we obtain the generating functions associated with $(s_n(x))_{n\in\mathbb{N}}$:

$$G(t) = \frac{2 - 2xt}{1 - xt + t^2}.$$

5. For $\alpha = 1, \beta = 1, \gamma = 0, p = 2, q = -1$, the generating functions of the Chebyshev polynomials of the first kind $T_n(x)$ is :

$$G(t) = \frac{1 - xt}{1 - 2xt + t^2}.$$

6. $\alpha = 1, \beta = 2, \gamma = 0, p = 2, q = -1$, the generating functions of the Chebyshev polynomials of the second kind $U_n(x)$ is :

$$G(t) = \frac{1}{1 - 2xt + t^2}.$$

7. $\alpha = 1, \beta = 2, \gamma = -1, p = 2, q = -1$, the generating functions of the Chebyshev polynomials of the third kind $V_n(x)$ is :

$$G(t) = \frac{1 - t}{1 - 2xt + t^2}.$$

8. $\alpha = 1, \beta = 2, \gamma = 1, p = 2, q = -1$, the generating functions of the Chebyshev polynomi-

als of the fourth kind $W_n(x)$ is:

$$G(t) = \frac{1+t}{1-2xt+t^2}.$$

□

CHAPTER 2

SYMMETRIC FUNCTIONS

In this chapter we define the elementary and the complete symmetric functions and we recall some properties about these functions.

2.1 second order equations

Consider the second order equation: $P(t) = t^2 - t - 1$:

$$M = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix},$$

this matrix is called: "companion matrix "of the polynomial $P(t) = t^2 - t - 1$.

we are looking for the eigenvectors : $\begin{pmatrix} u_{n+1} \\ u_n \end{pmatrix} = M \begin{pmatrix} u_n \\ u_{n-1} \end{pmatrix}$ and by the initial values: $u_0 = 0,\ u_1 = 1$ We obtain :

$$\begin{pmatrix} u_{n+1} \\ u_n \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u_n \\ u_{n-1} \end{pmatrix},$$

so :

$$u_{n+1} = u_n + u_{n-1}. \tag{2.1}$$

and :

$$\begin{pmatrix} u_{n+1} \\ u_n \end{pmatrix} = M^n \begin{pmatrix} u_1 \\ u_0 \end{pmatrix} = M^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

we look for the eigenvalues by daigonalization of the matrix M:

$$\begin{vmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - \lambda - 1.$$

we have: $P_M(\lambda) = 0$.
so: the eigenvalues are: $\dfrac{1+\sqrt{5}}{2}$ and $\dfrac{1-\sqrt{5}}{2}$.
this matrix is called: "companion matrix" of the polynomial $t^2 = t + 1$.
on the other hand, let's define the sequence by the recurrence relation:

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \lambda_i \begin{pmatrix} x \\ y \end{pmatrix} \text{ (i=1 or i=2)}$$

so:

$$\begin{cases} x + y = \dfrac{1+\sqrt{5}}{2} x \\ x = \dfrac{1+\sqrt{5}}{2} y. \end{cases} \text{ and } \begin{cases} x + y = \dfrac{1-\sqrt{5}}{2} x \\ x = \dfrac{1-\sqrt{5}}{2} y. \end{cases}$$

these two equations are equivalent to: $x = \lambda_i$
so the eigenvectors of M are proportional to:

$$\vec{U_1} = \begin{pmatrix} \dfrac{1+\sqrt{5}}{2} \\ 1 \end{pmatrix} \text{ and } \vec{U_2} = \begin{pmatrix} \dfrac{1-\sqrt{5}}{2} \\ 1 \end{pmatrix}.$$

note that:

$$M \vec{v_1} = \frac{1+\sqrt{5}}{2} \vec{v_1} = \begin{pmatrix} \dfrac{3+\sqrt{5}}{2} \\ \dfrac{1+\sqrt{5}}{2} \end{pmatrix} \text{ and } M^n \vec{v_1} = \left(\frac{1+\sqrt{5}}{2}\right)^n \vec{v_1} = \begin{pmatrix} \left(\dfrac{1+\sqrt{5}}{2}\right)^{n+1} \\ \left(\dfrac{1+\sqrt{5}}{2}\right)^{n} \end{pmatrix}.$$

$$M\ \vec{v_2}=\frac{1-\sqrt{5}}{2}\ \vec{v_2}=\begin{pmatrix}\frac{3-\sqrt{5}}{2}\\ \frac{1+\sqrt{5}}{2}\end{pmatrix} \text{ and } M^n\ \vec{v_2}=\left(\frac{1-\sqrt{5}}{2}\right)^n\ \vec{v_2}=\begin{pmatrix}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\\ \left(\frac{1-\sqrt{5}}{2}\right)^{n}\end{pmatrix}.$$

to pass from the eigenvectors to the canonical basis, we use the matrix

$$\begin{pmatrix}\frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2}\\ 1 & 1\end{pmatrix}.$$

and to go from the canonical basis to the eigenvector basis, we will use the inverse matrix which is :

$$\frac{1}{\sqrt{5}}\begin{pmatrix}1 & -\frac{1-\sqrt{5}}{2}\\ -1 & \frac{1+\sqrt{5}}{2}\end{pmatrix}.$$

we will assume, for the moment : $\lambda_1 \neq \lambda_2$, and even, more precisely:
$|\lambda_1| \geq |\lambda_2|$

$$M=\frac{1}{\sqrt{5}}\begin{pmatrix}\frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2}\\ 1 & 1\end{pmatrix}\begin{pmatrix}\frac{1+\sqrt{5}}{2} & 0\\ 0 & \frac{1-\sqrt{5}}{2}\end{pmatrix}\begin{pmatrix}1 & -\frac{1-\sqrt{5}}{2}\\ -1 & \frac{1+\sqrt{5}}{2}\end{pmatrix}.$$

and

$$M^n=\frac{1}{\sqrt{5}}\begin{pmatrix}\frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2}\\ 1 & 1\end{pmatrix}\begin{pmatrix}\left(\frac{1+\sqrt{5}}{2}\right)^n & 0\\ 0 & \left(\frac{1-\sqrt{5}}{2}\right)^n\end{pmatrix}\begin{pmatrix}1 & -\frac{1-\sqrt{5}}{2}\\ -1 & \frac{1+\sqrt{5}}{2}\end{pmatrix}.$$

$$M^n=\begin{pmatrix}\frac{\left(\frac{1+\sqrt{5}}{2}\right)^{n+1}-\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}}{\sqrt{5}} & \frac{-\left(\frac{1+\sqrt{5}}{2}\right)^{n}-\left(\frac{1-\sqrt{5}}{2}\right)^{n}}{\sqrt{5}}\\ \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{n}-\left(\frac{1-\sqrt{5}}{2}\right)^{n}}{\sqrt{5}} & \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{n-1}-\left(\frac{1-\sqrt{5}}{2}\right)^{n-1}}{\sqrt{5}}\end{pmatrix}.$$

2.2 Symmetric functions

Definition 2.2.1. *[33] a function $f(x_1; x_2; ...; x_n)$, in n variables is symmetric if for all permutations of the set of indices $(1; 2; ...; n)$ the following equality is verified:*

$$f(x_1; x_2; ...; x_n) = f\left(x_{s(1)}; x_{s(2)}; ...; x_{s(n)}\right).$$

Example 2.2.1. • *The function* $f(x_1; x_2) = x_1 + x_2$ *is symmetric, because*
$f(x_2; x_1) = x_2 + x_1 = f(x_1; x_2)$
• *The function* $h(x_1; x_2) = x_1x_2 + x_1^2$ *is not symmetric , because*
$h(x_2; x_1) = x_2x_1 + x_2^2 \neq h(x_1; x_2)$;

when the functions are real or complex values, symmetric functions form a subalgebra of the algebra of functions of n variables, that is:

Proposition 2.2.1.

- *The product of two symmetric functions is still a symmetric function.*
- *Any symmetric rational function (on a commutative field) is the quotient of two symmetric polynomials*
- *The sum of two symmetric functions is still a symmetric function.*

2.2.1 Elementary symmetric functions

Definition 2.2.2. *[3] We call* k^{th} *elementary symmetric functions* $e_k(\lambda_1, \lambda_2, \lambda_3, ..., \lambda_n)$ *the function defined by:*

$$e_k^{(n)} = e_k(\lambda_1, \lambda_2, \lambda_3, ..., \lambda_n) = \sum_{i_1+i_2,.......+i_n=k} \lambda_1^{i_1}, \lambda_2^{i_2}, \lambda_3^{i_3}, ..., \lambda_n^{i_n}. \tag{2.2}$$

$k \leq n$, *with*$i_1, i_2, i_3, ..., i_n = 0$ *or* 1

Example 2.2.2. *For an equation of degree* 2 ($n = 2$, *roots* : λ_1 *and* λ_2)

$$\begin{cases} e_0^{(2)} = 1 \\ e_1^{(2)} = \lambda_1 + \lambda_2 \\ e_2^{(2)} = \lambda_1\lambda_2. \end{cases}$$

Example 2.2.3. *For an equation of degree 2 ($n = 2$, roots : λ_1, λ_2 and λ_3)*

$$\begin{cases} e_0^{(3)} = 1 \\ e_1^{(3)} = \lambda_1 + \lambda_2 + \lambda_3 \\ e_2^{(3)} = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 \\ e_3^{(3)} = \lambda_1\lambda_2\lambda_3. \end{cases}$$

Proposition 2.2.2. *[1] Let $e_k^{(n)}$ is an elementary symmetric function, then:*

1. $e_k^{(n+1)} = \lambda_{(n+1)} e_{k-1}^{(n)} + e_k^{(n)}$.
2. $e_k^{(n+1)} = \lambda_n e_{k-1}^{(n-1)} + \lambda_{n-1} e_{k-1}^{(n-2)} + \cdots + \lambda_{n-i} e_{k-1}^{(n-i-1)} + \cdots + \lambda_k e_{k-1}^{(k-1)}$.

Proposition 2.2.3. *[18] We can also define the k^{th} elementary functions as:*

$$E(t) = \sum_{k \geq 0} e_k^{(n)} t^k = \prod_{i=1}^{n} (1 + \lambda_i t). \tag{2.3}$$

with $e_k(\lambda_1, \lambda_2, \lambda_3, ..., \lambda_n)$, for $(k \geq 0)$

Proof. We have :

$$e_k^{(n)} = e_k(\lambda_1, \lambda_2, \lambda_3, ..., \lambda_n) = \sum_{i_1+i_2,.......+i_n=k} \lambda_1^{i_1}, \lambda_2^{i_2}, \lambda_3^{i_3}, ..., \lambda_n^{i_n}.$$

with $e_k^{(n)} = 0$ if $(k \geq n)$

Let's see that

$$\sum_{k \geq 0} e_k^{(n)} t^k = \prod_{i=1}^{n} (1 + \lambda_i t)$$

For $n = 2$, we have:

$$\prod_{i=1}^{1} (1 + \lambda_i t) = (1 + \lambda_1 t)(1 + \lambda_2 t)$$

$$= 1 + (\lambda_1 + \lambda_2) t + \lambda_1\lambda_2 t^2$$

$$= e_0 + e_1 t + e_2 t^2$$

$$= \sum_{k=0}^{2} e_k t^k.$$

suppose the property is true for n: $\sum_{k\geq 0}^{n} e_k t^k = \prod_{i=1}^{n} (1+\lambda_i t)$.

and let's show that the property is true for $n+1$: $\sum_{k\geq 0}^{n+1} e_k t^k = \prod_{i=1}^{n} (1+\lambda_i t)$.

$$\prod_{i=1}^{n+1} (1+\lambda_i t) = \prod_{i=1}^{n} (1+\lambda_i t)(1+\lambda_{n+1} t)$$

$$= \left(\sum_{k=0}^{n} e_k t^k\right)(1+\lambda_{n+1} t)$$

$$= \sum_{k=0}^{n} e_k t^k + \lambda_{n+1} \sum_{k=0}^{n} e_k t^{k+1}$$

$$= \sum_{k=0}^{n} e_k t^k + \lambda_{n+1} \sum_{k=1}^{n} e_{k-1} t^k$$

$$= \sum_{k=0}^{n} e_k t^k + \lambda_{n+1} \sum_{k=0}^{n} e_{k-1} t^k$$

$$= \sum_{k\geq 0} \left(e_k^{(n)} + \lambda_{n+1} e_{k-1}^{(n)}\right) t^k$$

$$= \sum_{k\geq 0} e_k^{(n+1)} t^k$$

$$= \sum_{k\geq 0}^{n+1} e_k t^k.$$

□

2.2.2 Complete symmetric functions

Definition 2.2.3. *[3] We also define the complete symmetric functions* $h_k(\lambda_1, \lambda_2, \lambda_3, ..., \lambda_n)$ *of the roots in the following way:*

$$h_k^{(n)} = h_k(\lambda_1, \lambda_2, \lambda_3, ..., \lambda_n) = \sum_{i_1+i_2,.......+i_n=k} \lambda_1^{i_1} \lambda_2^{i_2} \lambda_3^{i_3} .... \lambda_n^{i_n}. \quad (2.4)$$

with: $i_1, i_2, i_3, ..., i_n \geq 0$

Example 2.2.4. *For an equation of degree* 2 $(n = 2)$*, with* λ_1 *and* λ_2 *are root*

$$\begin{cases} h_0^{(2)} & = 1 \\ h_1^{(2)} & = \lambda_1 + \lambda_2 \\ h_2^{(2)} & = \lambda_1^2 + \lambda_1\lambda_2 + \lambda_2^2 \\ h_2^{(2)} & = \lambda_1^3 + \lambda_1^2\lambda_2 + \lambda_1\lambda_2^2 + \lambda_2^3. \\ \vdots \end{cases}$$

Example 2.2.5. *For an equation of degree* 3 $(n = 3)$*, for roots:* λ_1*,* λ_2 *and* λ_3*.*

$$\begin{cases} h_0^{(3)} & = 1 \\ h_1^{(3)} & = \lambda_1 + \lambda_2 + \lambda_3 \\ h_2^{(3)} & = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 \\ h_3^{(3)} & = \lambda_1^3 + \lambda_2^3 + \lambda_3^3 + \lambda_1^2\lambda_2 + \lambda_1^2\lambda_3 + \lambda_2^2\lambda_1 + \lambda_2^2\lambda_3 + \lambda_3^2\lambda_1 + \lambda_3^2\lambda_2 + \lambda_1\lambda_2\lambda_3. \\ \vdots \end{cases}$$

Proposition 2.2.4. *[1] Let* $h_k^{(n)}$ *is a complete symmetric function, then:*

1. $h_k^{(n+1)} = \lambda_{n+1} h_{k-1}^{(n+1)} + h_k^{(n)}$.
2. $h_k^{(n+1)} = \lambda_{n+1}^{(k)} + \lambda_{n+1}^{(k-1)} h_1^{(n)} + \lambda_{n+1}^{(k-2)} h_2^{(n)} + \cdots + \lambda_{n+1} h_{k-1}^{(n)} + h_k^{(n)}$.

Proposition 2.2.5. *[18] We define the* k^{th} *complete symmetric functions as the coefficients of the formal series expansion:*

$$H(t) = \sum_{k \geq 0} h_k^{(n)} t^k = \frac{1}{\prod_{i=1}^{n} (1 + \lambda_i t)}. \tag{2.5}$$

Proof. $\sum_{k \geq 0} h_k^{(n)} = h_k(\lambda_1, \lambda_2, \lambda_3 ...., \lambda_n) = \sum_{i_1 + i_2, ... + i_n} \lambda_1^{i_1} \lambda_2^{i_2} \lambda_3^{i_3} .... \lambda_n^{i_n}$

For $n = 2$, we have:

$$\begin{aligned}
\sum_{k\geq 0} h_k^{(2)} =& h_0^{(2)} + h_1^{(2)} + h_2^{(2)} + \cdots \\
=& 1 + (\lambda_1 + \lambda_2)\, t + \left(\lambda_1^2 + \lambda_1\lambda_2 + \lambda_2^2\right) t^2 + \cdots \\
=& \left(1 + \lambda_1 t + \lambda_1^2 t^2 + \cdots\right) + \left(1 + \lambda_2 t + \lambda_2^2 t^2 + \cdots\right) \\
=& \left(1 + \lambda_1 t + \lambda_1^2 t^2 + \cdots\right) + \left(1 + \lambda_2 t + \lambda_2^2 t^2 + \cdots\right) \\
=& \left(\sum_{k\geq 0} (\lambda_1 t)^k\right) \left(\sum_{k\geq 0} (\lambda_2 t)^k\right) \\
=& \frac{1}{(1 - \lambda_1 t)\,(1 - \lambda_2 t)} \\
=& \frac{1}{\prod_{i=1}^{2} (1 - \lambda_i t)}.
\end{aligned}$$

suppose the property is true for n, then:

$$\sum_{k\geq 0} h_k^{(n)} t^k = \frac{1}{\prod_{i=1}^{n} (1 - \lambda_i t)}.$$

and let's show that the property is true for $n + 1$:

$$\sum_{k\geq 0} h_k^{(n+1)} t^k = \frac{1}{\prod_{i=1}^{n+1} (1 - \lambda_i t)}.$$

we have : $h_k^{(n+1)} = \lambda_{n+1} h_{k-1}^{(n+1)} + h_k^{(n)}$.

$$\begin{aligned}
\sum_{k\geq 0} h_k^{(n+1)} t^k =& \sum_{k\geq 0} \left(\lambda_{n+1} h_{k-1}^{(n+1)} + h_k^{(n)}\right) t^k \\
=& \lambda_{n+1} \sum_{k\geq 0} h_{k-1}^{(n+1)} t^k + \sum_{k\geq 0} h_k^{(n)} t^k \\
=& \lambda_{n+1} t \sum_{k\geq 0} h_k^{(n+1)} t^k + \sum_{k\geq 0} h_k^{(n)} t^k
\end{aligned}$$

$$\sum_{k\geq 0} h_k^{(n+1)} t^k - \lambda_{n+1} t \sum_{k\geq 0} h_k^{(n+1)} t^k = \sum_{k\geq 0} h_k^{(n)} t^k$$

$$\sum_{k\geq 0} h_k^{(n+1)} t^k \left(1 - \lambda_{n+1} t\right) = \frac{1}{\prod_{i=1}^{n} \left(1 - \lambda_i t\right)}$$

$$\sum_{k\geq 0} h_k^{(n+1)} t^k = \frac{\left(1 - \lambda_i t\right)^{-1}}{\prod_{i=1}^{n} \left(1 - \lambda_i t\right)}$$

$$= \frac{1}{\prod_{i=1}^{n+1} \left(1 - \lambda_i t\right)}.$$

□

2.2.3 exponential symmetric functions

Definition 2.2.4. *[37] Let k be a positive integer, we call a k^{th} exponential symmetric function, the series defined by:*

$$p_k^{(n)} = \sum_{i\geq 1} \lambda_i^k. \tag{2.6}$$

Example 2.2.6. *The exponential symmetric function $P_2{}^{(3)}$ on a 3-letter alphabet is:*

$$P_2^{(3)} \left(\lambda_1, \lambda_2, \lambda_3\right) = \lambda_1^2 + \lambda_1^2 + \lambda_3^2.$$

Proposition 2.2.6. *[37] The k^{th} symmetric functions of exponential can also be defined as the coefficients of the series:*

$$P_k\left(t\right) = \sum_{k\geq 1} P_k t^{(k-1)}$$

$$= \frac{\partial}{\partial t} \log H\left(t\right)$$

$$= \sum_{i\geq 1} \frac{\lambda_i}{1 - \lambda_i t}.$$

Proof.

$$\log H\left(t\right) = \log \prod_{i=1}^{n} \frac{1}{1 - \lambda_i t}$$

$$= \sum_{i=1}^{n} \log \frac{1}{1-\lambda_i t}$$

$$\frac{\partial}{\partial t} \log H(t) = \sum_{i \geq 1} \frac{\lambda_i}{(1-\lambda_i t)}$$

$$= \sum_{i \geq 0} \lambda_i \sum_{i \geq 1} (\lambda_i t)^k$$

$$= \sum_{i \geq 1} \sum_{k \geq 1} \lambda_i^k t^{k-1}$$

$$= \sum_{i \geq 1} P_k(\lambda) t^{k-1}.$$

□

2.2.4 Relations between symmetric functions

Proposition 2.2.7. *[3] Let $E(t)$, $H(t)$ and $P(t)$ be three symmetric functions, then:*

1. $H(t) E(-t) = 1.$
2. $P(t) = \dfrac{H'(t)}{H(t)}.$
3. $\prod_{i \geq 1} \dfrac{1}{(1-\lambda_i t)} = \exp \sum_{n \geq 1} P_n(\lambda) \dfrac{t^n}{n}.$

Proof. 1. We have:

$$E(t) = \sum_{k \geq 0} e_k t^k = \prod_{i=1}^{n} (1+\lambda_i t).$$

$$E(-t) = \sum_{k \geq 0} e_k (-t)^k = \prod_{i=1}^{n} (1-\lambda_i t).$$

$$H(t) = \sum_{k \geq 0} h_k t^k = \prod_{i=1}^{n} (1+\lambda_i t).$$

so:

$$E(-t) H(t) = \left(\prod_{i \geq 1} (1+\lambda_i t)^{-1} \right) \left(\prod_{i \geq 1} (1+\lambda_i t) \right) = 1.$$

2. we have:

$$\begin{aligned}
\frac{H'(t)}{H(t)} &= \frac{\partial}{\partial t} \log H(t) \\
&= \frac{\partial}{\partial t} \log \prod_{i\geq 1} \left(\frac{1}{1-\lambda i t} \right) \\
&= \sum_{i\geq 1} \frac{\partial}{\partial t} \log \frac{1}{(1-\lambda_i t)} \\
&= \sum_{i\geq 1} \frac{\lambda_i}{(1-\lambda_i t)} \\
&= \sum_{i\geq 1} \frac{\lambda_i}{(1-\lambda_i t)} \\
&= \sum_{i\geq 1} \lambda_i \sum_{j\geq 0} (\lambda_i t)^j \\
&= \sum_{i\geq 1} \sum_{j\geq 0} \lambda_i^{j+1} t^j .
\end{aligned}$$

we set: $j = n-1$, then:

$$\begin{aligned}
\frac{H'(t)}{H(t)} &= \sum_{i\geq 1} \sum_{n\geq 1} \lambda_i^n t^{n-1} \\
&= \sum_{i\geq 1} \sum_{n\geq 1} \frac{\lambda_i^n t^n}{n} \\
&= \sum_{n\geq 1} P_n(\lambda) \frac{t^n}{n} \\
\prod_{i\geq 1} \frac{1}{(1-\lambda_i t)} &= \exp \sum_{n\geq 1} P_n(\lambda) \frac{t^n}{n} .
\end{aligned}$$

□

2.2.5 Some properties of symmetric functions

Definition 2.2.5. *[3] Any set of finite characters is called an alphabet.*

Definition 2.2.6. *[3] Consider the alphabet $E_2 = \{e_1, e_2\}$, and we define the associated symmetric function S_n by:*

$$S_n(E_2) = S_n(e_1 + e_2) = \frac{e_1^{n+1} - e_2^{n+1}}{e_1 - e_2}. \tag{2.7}$$

with :

$$\begin{aligned} S_0(E_2) &= h_0 = 1 \\ S_1(E_2) &= h_1 = e_1 + e_2 \\ S_2(E_2) &= h_2 = e_1^2 + e_1 e_2 + e_2^2. \\ &= \vdots \end{aligned}$$

Definition 2.2.7. *[2] Let A and B be two alphabets, we denote $S_j(A - B)$ for the coefficients of the following rational series:*

$$\frac{\prod_{b \in B}(1 - bt)}{\prod_{a \in A}(1 - at)} = \sum_{j=0}^{\infty} S_j(A - B)\, t^j. \tag{2.8}$$

with $S_j(A - B) = 0$ for $n \leq 0$.

Proposition 2.2.8. *[3] If A has the cardinality 1 ($A = x$), then:*

$$\frac{\prod_{b \in B}(1 - bt)}{(1 - xt)} = 1 + \cdots + t^{j-1} s_j(x - B) + t^j \frac{S_j(x - B)}{1 - xt}.$$

Proof. We have:

$$\frac{\prod_{b \in B}(1 - bt)}{(1 - xt)} = \sum_{j=0}^{\infty} S_j(x - B)\, t^j$$

so:

$$\begin{aligned} \sum_{j=0}^{\infty} S_j(x - B)\, t^j =& 1 + \cdots + S_{j-1}(x - B)\, t^{j-1} + S_j(x - B)\, t^j + S_{j+1}(x - B)\, t^{j+1} + \cdots \\ =& 1 + \cdots + t^{j-1} S_{j-1}(x - B) + t^j \left(S_j(x - B) + S_{j+1}(x - B)t + \cdots\right) \\ =& 1 + \cdots + t^{j-1} S_{j-1}(x - B) + t^j \left(S_j(x - B) + xtSj(x - B) + \cdots\right) \end{aligned}$$

$$=1+\cdots+t^{j-1}S_{j-1}\left(x-B\right)+t^{j}S_{j}\left(x-B\right)\left(1+xt+x^{2}t^{2}+\cdots\right)$$

$$=1+\cdots+t^{j-1}S_{j}\left(x-B\right)+t^{j}\frac{S_{j}\left(x-B\right)}{1-xt}$$

so:

$$\frac{\prod\limits_{b\in B}(1-bt)}{(1-xt)}=1+\cdots+t^{j-1}S_{j}\left(x-B\right)+t^{j}\frac{S_{j-1}\left(x-B\right)}{1-xt}.$$

□

Proposition 2.2.9. *[1] Consider successively the case $A=\phi$ or $B=\phi$, then we obtain the factorization:*

$$\sum_{j=0}^{\infty}S_{j}\left(A-B\right)t^{j}=\sum_{j=0}^{\infty}S_{j}\left(A\right)t^{j}\sum_{j=0}^{\infty}S_{j}\left(-B\right)t^{j}.$$

If $B=\phi$ we obtain:

$$\sum_{j=0}^{\infty}S_{j}(A)t^{j}=\frac{1}{\prod\limits_{a\in A}(1-at)}.$$

If $A=\phi$ we obtain:

$$\sum_{j=0}^{\infty}S_{j}\left(-B\right)t^{j}=\frac{1}{\prod\limits_{b\in B}(1-bt)}.$$

so:

$$\sum_{j=0}^{\infty}S_{j}\left(A\right)t^{j}\sum_{j=0}^{\infty}S_{j}\left(-B\right)t^{j}=\frac{\prod\limits_{b\in B}(1-bt)}{\prod\limits_{a\in A}(1-at)}.$$

that is to say:

$$\sum_{j=0}^{\infty}S_{j}\left(A-B\right)t^{j}=\sum_{j\geq 0}\left(\sum_{k=0}^{n}S_{j-k}\left(A\right)S_{k}\left(-B\right)\right)t^{j}.$$

Proposition 2.2.10. *[3] Let $A=x$, we have:*

$$S_{n}\left(x-B\right)=x^{n}S_{0}\left(-B\right)+x^{n-1}S_{1}\left(-B\right)+\cdots+S_{0}\left(-B\right).$$

Proof. We have according to the formula

$$S_{n}\left(x-B\right)=\sum_{k=0}^{n}S_{n-k}S_{k}\left(-B\right)$$

$$=\sum_{k=0}^{n}x^{n-k}S_{k}\left(-B\right)$$

$$=S_{n}\left(x-B\right)=x^{n}S_{0}\left(-B\right)+x^{n-1}S_{1}\left(-B\right)+\cdots+S_{0}\left(-B\right).$$

□

2.2.6 Symmetric functions and generating functions

Definition 2.2.8. *[3] Let f be a function on R^n, the divided difference $\partial_{a_i,a_{i+1}}$ is defined by:*

$$\partial_{a_i,a_{i+1}}(f) = \frac{f(a_1, ..., a_i, a_{i+1}, ..., a_n) - f(a_1, ..., a_{i-1}, a_{i+1}, a_i, ..., a_n)}{a_i - a_{i+1}}. \quad (2.9)$$

.

Definition 2.2.9. *[3] We define the operator $\delta_{a_1a_2}$ by:*

$$\delta_{a_1a_2} f(a_1) = \frac{a_1^k f(a_1) - a_2^k f(a_2)}{a_1 - a_2}, \forall k \in \mathbb{N}. \quad (2.10)$$

Remark 2.2.1. *[3] If $f(a_1) = a_1$, in the formula (2.10), we obtain:*

$$\delta_{a_1a_2} f(a_1) = S_k(a_1 + a_2).$$

Proposition 2.2.11. *Let $A = \{a_1, a_2\}$, we define the operator $\delta_{a_1a_2}^{-k}$, by:*

$$\delta_{a_1a_2} f(a_1) = S_k(a_1 + a_2) f(a_1) + a_2^k \partial_{a_1a_2} f(a_1), \forall k \in \mathbb{N}.$$

Proof. We have :

$$\delta_{a_1a_2}^{-k} f(a_1) = \frac{a_1^k f(a_1) - a_2^k f(a_2)}{a_1 - a_2}$$

. so:

$$\begin{aligned}
\delta_{a_1a_2} f(a_1) =& \frac{a_1^k f(a_1) - a_2^k f(a_1) + a_2^k f(a_1) - a_2^k f(a_2)}{a_1 - a_2} \\
=& \frac{a_1^k - a_2^k}{a_1 - a_2} f(a_1) + a_2^k \frac{f(a_1) - f(a_1)}{a_1 - a_2} \\
=& S_k(a_1 + a_2) f(a_1) + a_2^k \partial_{a_1a_2} f(a_1).
\end{aligned}$$

□

Definition 2.2.10. *[3] We define the operator $\delta_{a_1a_2}$ by:*

$$\delta_{a_1a_2}^{-k} f(a_1) = \frac{a_2^k f(a_1) - a_1^k f(a_2)}{(a_1a_2)^k a_1 - a_2}, \forall k \in \mathbb{N}. \tag{2.11}$$

Proposition 2.2.12. *Let $A = \{a_1, a_2\}$, we define the operator $\delta_{a_1a_2}^{-k}$, by:*

$$\delta_{a_1a_2}^{-k} f(a_1) = -\frac{s_{k-1}(a_1+a_2)}{(a_1a_2)} f(a_1) + \frac{a_k}{(a_1a_2)} \partial_{a_1a_2} f(a_1), \forall k \in \mathbb{N}.$$

Proof.

$$\begin{aligned}
\delta_{a_1a_2}^{-k} f(a_1) =& \frac{a_1^k f(a_1) - a_2^k f(a_2)}{a_1 - a_2} \\
=& \frac{a_2^k f(a_1) - a_1^k f(a_1) + a_1^k f(a_1) - a_2)^k f(a_2)}{a_1 - a_2} \\
=& -\frac{f(a_1(a_1-a_2))}{(a_1a_2)^k (a_1-a_2)} + \frac{a_1^k (f(a_1) - f(a_2))}{(a_1-a_1)(a_1a_2)^k} \\
=& -\frac{S_{k-1}(a_1+a_2)}{(a_1a_2)} f(a_1) + \frac{a_k}{(a_1a_2)} \partial_{a_1a_2} f(a_1).
\end{aligned}$$

□

2.2.7 Symmetric functions of certain numbers and polynomials

Theorem 2.2.1. *Given an alphabet $A = \{a_1, a_2\}$, so:*

$$\sum_{n=0}^{\infty} S_n(a_1+a_2) t^n = \frac{1}{\prod_{a\in A}(1-at)}. \tag{2.12}$$

Proof. Let $\sum_{n=0}^{\infty} a_1^n t^n$ and $\frac{1}{(1-at)}$ two series such that :

$$\left(\sum_{n=0}^{\infty} a_1^n t^n\right)\left(\frac{1}{1-at}\right) = 1.$$

Let $f(a_1) = \sum\limits_{n=0}^{\infty} a_1^n t^n$ then the first member of formula (2.12) is given by:

$$\begin{aligned}\delta_{a_1a_2} f(a_1) =& \frac{a_1 \sum\limits_{n=0}^{\infty} a_1^n t^n - a_2 \sum\limits_{n=0}^{\infty} a_2^n t^n}{a_1 - a_2} \\ =& \sum_{n=0}^{\infty} \frac{a_1^{n+1} - a_2^{n+1}}{a_1^{n+1} - a_2^{n+1}} t^n \\ =& \sum_{n=0}^{\infty} S_n(a_1 + a_2) t^n.\end{aligned}$$

Let $f(a_1) = \dfrac{1}{1 - a_1 t}$ then the second member of formula (2.12) is given by:

$$\begin{aligned}\delta_{a_1a_2} f(a_1) =& \frac{1}{a_1 - a_2}\left(a_1 \frac{1}{1 - a_1 t} - a_2 \frac{1}{1 - a_2 t}\right) \\ =& \frac{1}{a_1 - a_2}\left(\frac{a_1(1 - a_2 t) - a_2(1 - a_1 t)}{(1 - a_1 t)(1 - a_2 t)}\right) \\ =& \frac{1}{(1 - a_1 t)(1 - a_2 t)}.\end{aligned}$$

□

Lemma 2.2.1. *Given an alphabet* $A = \{a_1, a_2\}$ *,*

then:

$$\sum_{n=0}^{\infty} S_{n-1}(a_1 + a_2) t^n = \frac{t}{\prod\limits_{a \in A}(1 - at)}. \tag{2.13}$$

Theorem 2.2.2. *Given an alphabet* $A = \{a_1, a_2\}$ *,*

then:

$$\sum_{n=0}^{\infty} S_{n+1}(a_1 + [-a_2]) t^n = \frac{a_1 - a_2 + a_1 a_2 t}{1 - (a_1 - a_2) t - a_1 a_2 t^2}. \tag{2.14}$$

Proof. The action of the operator $\delta^2_{a_1 a_{[-2]}}$ on the series

$$f(a_1 t) = \sum_{n=0}^{\infty} a_1^n t^n$$

gives us the left side of equality (2.13)

$$\delta^2_{a_1[-a_2]} f(a_1) = \frac{a_1^2 \sum\limits_{n=0}^{\infty} a_1^n t^n - [-a_2]^2 \sum\limits_{n=0}^{\infty} a_2^n t^n}{a_1 - [-a_2]}$$

$$= \sum_{n=0}^{\infty} \frac{a_1^{n+2} - [-a_2]^{n+2}}{a_1 - [-a_2]} t^n$$

$$= \sum_{n=0}^{\infty} S_{n+1}(a_1 + [-a_2]) t^n.$$

Let $f(a_1 t) = \dfrac{1}{1 - a_1 t}$ then the second member of the equality (2.13) is given by:

$$\delta^2_{a_1[a_2]} f(a_1 t) = \delta^2_{a_1[a_2]^2} \left(\frac{1}{1 - a_1 t}\right)$$

$$= \frac{a_1^2 \dfrac{1}{1 - a_1 t} - [-a_2]^2 \dfrac{1}{1 - a_2 t}}{a_1 - [-a_2]}$$

$$= \frac{a_1^2 (1 + a_2 t) - a_2^2 (1 - a_1 t)}{(a_1 + a_2)(1 - a_1 t)(1 + a_1 t)}$$

$$= \frac{a_1^2 - a_2^2 + a_1 a_2 t + a_2^2 a_1 t}{(a_1 + a_2)(1 - (a_1 - a_2) t - a_1 a_2 t^2)}$$

$$= \frac{a_1 - a_2 + a_1 a_2 t}{1 - (a_1 - a_2) t + a_1 a_2 t^2}.$$

□

- By replacing (a_2) by $(-a_2)$, and let $a_1 - a_2 = k$ and $a_1 a_2 = 1$ in the formula (2.12) we obtain:

$$\sum_{n=0}^{\infty} S_n(a_1 + [-a_2]) t^n = \frac{1}{1 - kt - t^2} = \sum_{n=0}^{\infty} F_{k,n} t^n. \tag{2.15}$$

which represents the generating function of k-Fibonacci numbers, then we deduce the following proposition.

Proposition 2.2.13. *[33] Let $(F_{k,n})_{n\in\mathbb{N}}$ be the sequence of k-Fibonacci, then:*
$\forall\ n \in N : F_{k,n} = S_n(a_2 + [-a_2])$, with:

$$a_1 = \frac{k + \sqrt{k^2 + 4}}{2} \text{ and } a_2 = \frac{k - \sqrt{k^2 + 4}}{2}.$$

- *By replacing* (a_2) *by* $(-a_2)$, *and let* $a_1 - a_2 = k$ *and* $a_1a_2 = 1$. *in the formula* (2.12) *and* (2.13) *we obtain:*

$$\sum_{n=0}^{\infty} S_n\left(a_1 + [-a_2]\right) = \frac{1}{1 - kt - t^2}. \tag{2.16}$$

$$\sum_{n=0}^{\infty} S_{n-1}\left(a_1 + [-a_2]\right) = \frac{t}{1 - kt - t^2}. \tag{2.17}$$

- *By multiplying* (2.16) *by* 2 *and* (2.17) *by* $(-k)$ *and adding the results we get*

$$\sum_{n=0}^{\infty} \left(2S_n\left(a_1 + [-a_2]\right) - kS_{n-1}\left(a_1 + [-a_2]\right)\right) t^n = \frac{2 - kt}{1 - kt - t^2} = \sum_{n=0}^{\infty} L_{k,n} t^2. \tag{2.18}$$

Proposition 2.2.14. *[33] Let* $\left(L_{k,n}\right)_{n\in\mathbb{N}}$ *be the sequence of k-Pell-Lucas, then:*
$\forall\ n \in \mathbb{N}, Q_{k,n} = \left(2S_n\left(a_1 + [-a_2]\right) - kS_{n-1}\left(a_1 + [-a_2]\right)\right)$,

with:

$$a_1 = \frac{k + \sqrt{k^2 + 4}}{2} \text{ and } a_2 = \frac{k - \sqrt{k^2 + 4}}{2}.$$

Remark 2.2.2.
1. *Let* $k = 1$ *in formula* (2.15) *we obtain the generating function of Fibonacci numbers.*
2. *Let* $k = 1$ *in formula* (2.18) *we obtain the generating function of Lucas numbers.*

- *By replacing* (a_2) *by* $(-a_2)$, *and let* $a_1 - a_2 = 2$ *and* $a_1a_2 = k$ *in the formula* (2.13) *we obtain:*

$$\sum_{n=0}^{\infty} S_{n-1}\left(a_1 + [-a_2]\right) t^n = \frac{1}{1 - 2t - kt^2} = \sum_{n=0}^{\infty} P_n t^n. \tag{2.19}$$

Proposition 2.2.15. *[33] Let* $\left(P_{k,n}\right)_{n\in\mathbb{N}}$ *be the sequence of k-Pell, then:*
$\forall n \in \mathbb{N} : P_{n,k} = S_{n-1}\left(a_1 + [-a_2]\right)$
with:

$$a_1 = 1 + \sqrt{k+1} \text{ and } a_2 = 1 - \sqrt{k+1}.$$

- *By replacing* (a_2) *by* $(-a_2)$, *and let* $a_1 - a_2 = 2$ *and* $a_1a_2 = k$ *in the formula* (2.12) *we obtain:*

$$\sum_{n=0}^{\infty} S_n\left(a_2 + [-a_2]\right) t^n = \frac{1}{1 - 2t - kt^2}. \tag{2.20}$$

$$\sum_{n=0}^{\infty} S_{n-1}\left(a_2 + [-a_2]\right) t^n = \frac{t}{1 - 2t - kt^2}. \tag{2.21}$$

• By multiplying (2.19) by 2 and (2.20) by (-2) and adding the results we gets

$$\sum_{n=0}^{\infty}\left(2S_n\left(a_1+[-a_2]\right)-2S_{n-1}\left(a_1+[-a_2]\right)\right)t^n=\frac{2-2t}{1-2t-kt^2}=\sum_{n=0}^{\infty}Q_{k,n}t^n. \tag{2.22}$$

Proposition 2.2.16. *[33] Let $\left(Q_{k,n}\right)_{n\in\mathbb{N}}$ be the sequence of k-Pell-Lucas, then:*
$\forall n\in\mathbb{N}N, Q_{n,k}=2S_n\left(a_1+[-a_2]\right)-2S_{n-1}\left(a_1+[-a_2]\right).$ *with:*

$$a_1=1+\sqrt{k+1} \text{ and } a_2=1-\sqrt{k+1}.$$

Remark 2.2.3.
1. *Let $k=1$ in formula (2.19) we obtain the generating function of Pell numbers.*
2. *Let $k=1$ in formula (2.22) we obtain the generating Pell-Lucas numbers.*

• *By replacing (a_1) by $(2a_1)$ and (a_2) by $(-2a_2)$, and let $a_1-a_2=x$ and $4a_1a_2=-1$ in the formula (2.13) we obtain:*

$$\sum_{n=0}^{\infty}S_n\left(2a_1+[-2a_2]\right)t^n=\frac{1}{1-2xt-t^2}=\sum_{n=0}^{\infty}U_n\left(x\right)t^n. \tag{2.23}$$

Proposition 2.2.17. $\forall n\in\mathbb{N}$: *We have*

$$U_n\left(x\right)=S_n\left(2a_1+[-2a_2]\right).$$

where $U_n(x)$ is the Chebyshev polynomials of the second kind.
• *By replacing (a_1) by $(2a_1)$ and a_2 by $(-2a_2)$, and let $a_1-a_2=x$ and $4a_1a_2=-1$ in the formula (2.12) and (2.13) we obtain:*

$$S_n\left(2a_2+[-2a_2]\right)=\frac{1}{1+2xt-t^2}. \tag{2.24}$$

$$S_{n-1}\left(2a_2+[-2a_2]\right)=\frac{t}{1+2xt-t^2}. \tag{2.25}$$

• By multiplying (2.25) by $(-x)$ and adding the result with (2.24) we obtain: $\sum_{n=0}^{\infty}\left(S_n\left(2a_1+[-2a_2]\right)-xS_{n-1}\left(2a_1+[-2a_2]\right)\right)t^n=\dfrac{1-xt}{1-2t-t^2}=\sum_{n=0}^{\infty}T_{k,n}\left(x\right)t^2.$

Proposition 2.2.18. $\forall n\in\mathbb{N}$ *We have:*

$$T_n\left(x\right)=S_n\left(2a_1+[-2a_2]\right)-xS_{n-1}\left(2a_2+[-2a_2]\right).$$

where $T_n(x)$ are the Chebyshev polynomials of the first kind.

2.3 Symmetric functions of products of certain numbers and polynomials

Theorem 2.3.1. *Given two alphabets $A = \{a_1, a_2\}$ and $B = \{b_1, b_2\}$*

$$\sum_{n=0}^{\infty} S_n(a_1 + a_2) S_{n+k-1}(b_1 + b_2) t^n = \frac{b_1^k - b_2^k - (a_1 + a_2)\left(b_1^k b_2 - b_2^k b_1^k\right) t - a_1 a_2 \left(b_2^k b_1^2 - b_1^k b_2^2\right) t^2}{(b_1 - b_2) \prod_{a \in A} (1 - ab_1 t) \prod_{a \in A} (1 - ab_2 t)} \tag{2.26}$$

$\forall n \in \mathbb{N}$.

Proof. The action of the operator $\delta_{b_1 b_2}^k$ on the series

$f(b_1 t) = \sum_{n=0}^{\infty} s_n(a_1 + a_2) b_1^n t^n$ we gives the left side of equality (2.27) then:

$$\begin{aligned}
\delta_{b_1 b_2}^k f(b_1 t) =& \delta_{b_1 b_2}^k \left(S_n(a_1 + a_2) b_1^n t^n\right) \\
=& \frac{\sum_{n=0}^{\infty} S_n(a_1 + a_2) b_1^n b_1^k t^n - \sum_{n=0}^{\infty} S_n(a_1 + a_2) b_2^n b_2^k t^n}{b_1 - b_2} \\
=& \sum_{n=0}^{\infty} S_n(a_1 + a_2) \frac{b_1^{n+k} - b_2^{k+n}}{(b_1 - b_2)} t^n \\
=& \sum_{n=0}^{\infty} S_n(a_1 + a_2) S_{n+k-1}(b_1 + b_2) t^n.
\end{aligned}$$

□

Let $f(a_1 t) = \frac{1}{\prod_{a \in A} (1 - ab_1 t)}$ then the second member of the equality (2.27) is written:

$$\begin{aligned}
\delta_{b_1 b_2}^k f(b_1 t) =& \delta_{b_1 b_2}^k \left(\frac{1}{\prod_{a \in A} (1 - ab_1 t)}\right) \\
=& \frac{1}{b_1 - b_2} \left(\frac{b_1^k}{\prod_{a \in A} (1 - ab_1 t)} - \frac{b_2^k}{\prod_{a \in A} (1 - ab_2 t)}\right)
\end{aligned}$$

$$=\frac{b_1^k \prod\limits_{a\in A}(1-ab_2t)-b_2^k\prod\limits_{a\in A}(1-ab_1t)}{\prod\limits_{a\in A}(1-ab_1t)\prod\limits_{a\in A}(1-ab_2t)\,(b_1-b_2)}$$

$$=\frac{b_1^k-b_2^k-(a_1+a_2)\left(b_1^kb_2-b_2^kb_1\right)t-a_1a_2+\left(b_2^kb_1-b_1^kb_2\right)t^2}{\prod\limits_{a\in A}(1-ab_1t)\prod\limits_{a\in A}(1-ab_2t)\,(b_1-b_2)}.$$

Let $k=0,1$ in theorem (2.3.1) we obtain the following lemmas

Lemma 2.3.1. *Given two alphabet $A=\{a_1,a_2\}$ and $B=\{b_1,b_2\}$, then:*

$$\sum_{n=0}^{\infty} S_n(a_1+a_2)\,S_{n-1}(b_1+b_2)\,t^n=\frac{(a_1+a_2)\,t+a_1a_2\,(b_1+b_2)\,t^2}{\sum\limits_{n=0}^{\infty}S_n(-A)\,b_1^nt^n\sum\limits_{n=0}^{\infty}S_n(-A)\,b_2^nt^n}. \tag{2.27}$$

Lemma 2.3.2. *Let $k=0,1$ in theorem* (2.3.1)*; we obtain the following lemmas*

$$\sum_{n=0}^{\infty} S_n(a_1+a_2)\,S_{n-1}(b_1+b_2)\,t^n=\frac{1-a_1a_2b_1b_2t^2}{\sum\limits_{n=0}^{\infty}S_n(-A)\,b_1^nt^n\sum\limits_{n=0}^{\infty}S_n(-A)\,b_2^nt^n}. \tag{2.28}$$

Lemma 2.3.3. *Let $k=0,1$ in theorem* (2.3.1) *we obtain the following lemmas:*

$$\sum_{n=0}^{\infty} S_n(a_1+a_2)\,S_{n-1}(b_1+b_2)\,t^n=\frac{t-a_1a_2b_1b_2t^3}{\sum\limits_{n=0}^{\infty}S_n(-A)\,b_1^nt^n\sum\limits_{n=0}^{\infty}S_n(-A)\,b_2^nt^n}. \tag{2.29}$$

- By replacing a_2 by $(-a_2)$, b_2 by $(-b_2)$ and let $a_1-a_2=b_1-b_2=1$ and $a_1a_2=b_1b_2=1$

$$\sum_{n=0}^{\infty} S_n(a_1+[-a_2])\,S_{n-1}(b_1+[-b_2])\,t^n=\frac{1-t^2}{1-t-4t^2-t^3+t^4}.$$

Proposition 2.3.1. $\forall n\in\mathbb{N}$*; The generating function of the product of the numbers of Fibonacci is given by:*

$$\sum_{n=0}^{\infty}F_nt^n=\frac{1-t^2}{1-t-4t^2-t^3+t^4}.$$

Theorem 2.3.2. $\forall n\in\mathbb{N}$*; The generating function of the product of the Fibonacci numbers and the Lucas numbers is given by:*

$$\sum_{n=0}^{\infty}F_nL_nt^n=\frac{2-t-2t^2}{1-t-4t^2-t^3+t^4}.$$

Proof. We have:

$$L_n = 2S_n\left(b_1 + [-b_2]\right) - S_{n-1}\left(b_1 + [-b_2]\right).$$

so:

$$\sum_{n=0}^{\infty} F_n L_n t^n = \sum_{n=0}^{\infty} S_n\left(a_1 + [-a_2]\right)\left[2S_n\left(b_1 + [-b_2]\right) - S_{n-1}\left(b_1 + [-b_2]\right)\right]t^n$$

$$= \sum_{n=0}^{\infty} 2S_n\left(a_1 + [-a_2]\right) S_n\left(b_1 + [-b_2]\right) t^n - \sum_{n=0}^{\infty} S_n\left(a_1 + [-a_2]\right)$$

$$\left(S_{n-1}\left(b_1 + [-b_2]\right)\right) t^n$$

$$= \frac{2\left(1 - t^2\right)}{1 - t - 4t^2 - t^3 + t^4} - \frac{t + t^2}{1 - t - 4t^2 - t^3 + t^4}$$

$$= \frac{2 - t - 3t^2}{1 - t - 4t^2 - t^3 + t^4}.$$

□

- By replacing a_2 with $(-a_2)$ and b_2 with $(-b_2)$; and setting

$a_1 - a_2 = 1; b_1 - b_2 = 2$ and $b_1 b_2 = a_1 a_2 = 1$ in (2.27), we obtain:

$$\sum_{n=0}^{\infty} S_n\left(a_1 + [-a_2]\right) S_{n-1}\left(b_1 + [-b_2]\right) t^n = \frac{t + 2t^2}{1 - 2t - 7t^2 - 2t^3 + t^4}.$$

Proposition 2.3.2. $\forall n \in \mathbb{N}$; *The generating function of the product of Fibonacci numbers and Pell numbers is given by:*

$$\sum_{n=0}^{\infty} F_n P_n t^n = \frac{t + 2t^2}{1 - 2t - 7t^2 - 2t^3 + t^4}.$$

Theorem 2.3.3. $\forall n \in \mathbb{N}$; *The generating function of the product of Pell numbers and Pell-Lucas numbers is given by:*

$$\sum_{n=0}^{\infty} Q_n P_n t^n = \frac{2t}{1 - 6t + t^2}.$$

Proof. We have:

$$Q_n = 2S_n\left(b_1 + [-b_2]\right) - 2S_{n-1}\left(b_1 + [-b_2]\right)$$

so:

$$\sum_{n=0}^{\infty} Q_n P_n t^n = \sum_{n=0}^{\infty} S_{n-1}\left(a_1+[-a_2]\right) 2S_n\left(b_1+[-b_2]\right) - 2S_{n-1}\left(b_1+[-b_2]\right)]t^n$$

$$= \sum_{n=0}^{\infty} 2S_{n-1}\left(a_1+[-a_2]\right) S_n\left(b_1+[-b_2]\right) t^n - 2\sum_{n=0}^{\infty} S_{n-1}\left(a_1+[-a_2]\right)$$

$$S_{n-1}\left(b_1+[-b_2]\right) t^n$$

$$= \frac{2\left(2+2t^2\right)}{1-4t-10t^2-4t^3+t^4} - \frac{2\left(t-t^3\right)}{1-4t-10t^2-4t^3+t^4}$$

$$= \frac{2t}{1-6t+t^2}.$$

• By replacing a_2 with $(-a_2)$ and b_1 with $(-2b_2)$; and setting $a_1 - a_2 = 1$; $b_1 - b_2 = x$ and $4b_1b_2 = -1, a_1a_2 = 1$ in (2.27), we obtain:

$$\sum_{n=0}^{\infty} S_n\left(a_1+[-a_2]\right) S_n\left(2b_1+[-2b_2]\right) t^n = \frac{1+t^2}{1+2xt+(4x^2-3)+2xt^3+t^4}$$

$$= \sum_{n=0}^{\infty} F_n U_n(x) t^n.$$

□

Proposition 2.3.3. $\forall n \in \mathbb{N}$; *The generating function of the product of Fibonacci numbers and Chebyshev polynomials of the second kind is given by:*

$$\sum_{n=0}^{\infty} F_n U_n(x) t^n = \frac{1+t^2}{1+2xt+(4x^2-3)+2xt^3+t^4}.$$

with

$$F_n U_n(x) = S_n\left(a_1+[-a_2]\right) S_n\left(2b_1+[-2b_2]\right).$$

$$a_1 = \frac{1+\sqrt{5}}{2}, a_1 = \frac{1-\sqrt{5}}{2}, b_1 = x+\sqrt{x^2-1}, b_2 = x-\sqrt{x^2-1}.$$

Proof. We have:

$$T_n = S_n\left(2a_2+[-2a_2]\right) - xS_{n-1}\left(2a_2+[-2a_2]\right).$$

so:

$$\sum_{n=0}^{\infty} F_n T_n t^n = \sum_{n=0}^{\infty} S_n(a_1 + [-a_2]) \left(S_n\left(2a_2 + [-2a_2]\right)\right) - xS_{n-1}\left(2a_2 + [-2a_2]\right)$$

$$= \sum_{n=0}^{\infty} S_n(a_1 + [-a_2]) S_n(2b_1 + [-2b_2]) t_n - x \sum_{n=0}^{\infty} S_n(a_1 + [-a_2]) S_{n-1}$$

$$(2b_1 + [-2b_2]) t^n$$

$$= \frac{1 + t^2}{1 + 2xt + (4x^2 - 3) + 2xt^3 + t^4} - x \frac{t + 2xt^2}{1 + 2xt + (4x^2 - 3) + 2xt^3 + t^4}$$

$$= \frac{1 - xt + \left(1 - 2x^2\right) t^2}{1 + 2xt + (4x^2 - 3) + 2xt^3 + t^4}.$$

$\square$

Theorem 2.3.4. $\forall n \in \mathbb{N}$; *The generating function of the products of the Chebyshev polynomials of the first and second kind is given by:*

$$\sum_{n=0}^{\infty} U_n(a_1 + a_2) T_n(b_1 - b_2) t^n = \frac{1 + t + (2x + 1)\, t^2}{1 + 2xt + (4x^2 - 3) + 2xt^3 + t^4}.$$

with:

$U_n(a_1 - a_2) T_n(b_1 - b_2) t^n = S_n\left(2a_1 + [-2a_2]\right) S_n\left(2b_1 + [-2b_2]\right) - (b_1 - b_2)\, S_n\left(2b_1 + [-2b_2]\right).$

Proof.

$$\sum_{n=0}^{\infty} U_n(a_1 - a_2) T_n(b_1 - b_2) t^n = \sum_{n=0}^{\infty} S_n\left(2a_1 + [-2a_2]\right) S_n\left(2b_1 + [-2b_2 b_2]\right)$$

$$-(b_1 - b_2) S_{n-1}(2b_1 + [-2b_2]) t^n$$

$$= \sum_{n=0}^{\infty} S_n\left(2a_1 + [-2_2]\right) S_n\left(2b_1 + [-2b_2]\right) t^n$$

$$-(b_1 - b_2) \sum_{n=0}^{\infty} S_{n-1}\left(2a_1 + [-2_2]\right) S_n\left(2b_1 + [-2b_2]\right) t^n$$

$$=\sum_{n=0}^{\infty} S_n\left(2a_1+[-2a_2]\right)S_n\left(2b_1+[-2b_2]\right)t^n$$

$$-\left(b_1-b_2\right)t^n\left(\sum_{n=0}^{\infty} S_n\left(2a_1+[-2a_2]\right)\left[\left(2b_1t\right)^n-\left(-2b_2t\right)^2\right]\right)$$

$$=\frac{1+t^2}{1-4\left(a_1-a_2\right)\left(a_1-a_2\right)t+\left(4\left(a_1-a_2\right)^2+4\right)\left(b_1-b_2\right)^2-20t^2-4\left(a_1-a_2\right)\left(b_1-b_2t^3+t^4\right)}$$

$$-\frac{b_1-b_2}{2\left(b_1+b_2\right)}\left(\frac{1}{1-2\left(a_1-a_2\right)b_1t+\left(2b_1t\right)^2}-\frac{1}{1+2\left(a_1-a_2\right)b_1t-\left(2b_1t\right)^2}\right)$$

$$=\frac{1-t^2}{1-4\left(a_1-a_2\right)\left(a_1-a_2\right)t+\left(4\left(a_1-a_2\right)^2+4\right)\left(b_1-b_2\right)^2-20t^2-4\left(a_1-a_2\right)\left(b_1-b_2t^3+t^4\right)}$$

$$-\frac{\left(b_1-b_2\right)\left[2\left(a_1-a_2\right)t-\left(b_1-b_2\right)t^2\right]}{1-4\left(a_1-a_2\right)\left(a_1-a_2\right)t+\left(4\left(a_1-a_2\right)^2+4\right)\left(b_1-b_2\right)^2-20t^2-4\left(a_1-a_2\right)\left(b_1-b_2t^3+t^4\right)}$$

$$=\frac{1-2\left(a_1-a_2\right)\left(a_1-a_2\right)t+\left(2\left(b_1-b_2\right)^2-1\right)t^2}{1-4\left(a_1-a_2\right)\left(a_1-a_2\right)t+\left(4\left(a_1-a_2\right)^2+4\right)\left(b_1-b_2\right)^2-20t^2-4\left(a_1-a_2\right)\left(b_1-b_2t^3+t^4\right)}.$$

□

Theorem 2.3.5. *[6] Given two alphabets $A=a_1,a_2,a_3$ and $B=b_1,b_2$, then:*

$$\sum_{n=0}^{\infty} S_n\left(A\right)S_{n+k}\left(B\right)t^n=\frac{\sum_{n=0}^{\infty} S_n\left(-A\right)\delta_{b_1b_2}^{k+1}\left(b_1^n\right)t^n}{\sum_{n=0}^{\infty} S_n\left(-A\right)\delta_{b_1b_2}^{k+1}\left(b_2^n\right)t^n}, k\in\mathbb{N}. \tag{2.30}$$

Proof. The action of the operator $\delta_{b_1b_2}^{k}$ on the series $f\left(b_1t\right)=\sum_{n=0}^{\infty} S_n\left(A\right)b_1^{n+1}t^n$ gives the left-hand side of equation (2.30) then:

$$\delta_{b_1b_2^k}f\left(b_1t\right)=\frac{\sum_{n=0}^{\infty} S_n\left(A\right)b_1^{n+1}t^n-\sum_{n=0}^{\infty} S_n\left(A\right)b_2^{n+1}t^n}{a_1-a_2}$$

$$=\sum_{n=0}^{\infty} S_n\left(A\right)\frac{b_1^{n+k+1}-b_2^{n+k+1}}{b_1-b_2}$$

$$=\sum_{n=0}^{\infty} S_n\left(A\right)S_{n+k}\left(B\right)t^n.$$

, Let $\sum_{n=0}^{\infty} S_n(A)\, b_1^{n+1}t^n = \frac{b_1}{\prod_{a\in A}(1-ab_1t)}$ be the second member of the equality (2.30), then it can be written as:

$$\begin{aligned}\delta_{b_1b_2^k} f(b_1t) =& \delta_{b_1b_2}^k \frac{b_1}{\prod_{a\in A}(1-ab_1t)}\\ =& \frac{1}{b_1-b_2}\left(\frac{b_1^{k+1}}{\prod_{a\in A}(1-ab_1t)} - \frac{b_1^{k+1}}{\prod_{a\in A}(1-ab_2t)}\right)\\ =& \frac{b_1^{k+1}\prod_{a\in A}(1-b_2t) - b_1^{k+1}\prod_{a\in A}(1-b_2t)}{\left(b_1-b_2\prod_{a\in A}(1-ab_1t)\prod_{a\in A}(1-ab_2t)\right)}.\end{aligned}$$

and according to identity:

$$\sum_{n=0}^{\infty} S_n(A)\, b_1^{n+1}t^n = \prod_{a\in A}(1-ab_1t)\,.$$

the result is:

$$\begin{aligned}\delta_{b_1b_2}^k f(b_1t) =& \delta_{b_1b_2}^k \prod_{a\in A}(1-ab_1t)\\ =& \frac{b_1^{k+1}\prod_{a\in A}(1-b_2t) - b_1^{k+1}\prod_{a\in A}(1-b_2t)}{(b_1-b_2)\left(\sum_{n=0}^{\infty} S_n(-A)\, b_2^n\right)\left(\sum_{n=0}^{\infty} S_n(-A)\, b_1^n\right)}\\ =& \frac{\sum_{n=0}^{\infty} S_n \frac{b_1^{k+1}b_1 - b_2^{k+1}b_2}{b_1-b_2} t^n}{\left(\sum_{n=0}^{\infty} S_n(-A)\, b_2^n\right)\left(\sum_{n=0}^{\infty} S_n(-A)\, b_1^n\right)}\\ =& \frac{\sum_{n=0}^{\infty} S_n(A)\, \delta_{b_1b_2}^k b_2^n t^n}{\left(\sum_{n=0}^{\infty} S_n(-A)\, b_2^n\right)\left(\sum_{n=0}^{\infty} S_n(-A)\, b_1^n\right)}.\end{aligned}$$

□

Theorem 2.3.6. *[4] Let $A = (a_1, a_2)$, $B = (b_1, b_2)$, and $C = (c_1, c_2)$ be three alphabets, then:*

$$\sum_{n=0}^{\infty} S_n(A)\, S_{n+k-1}(B)\, S_{n+k-1}(C)$$

$$= \frac{b_1^k b_2^k}{c_1 - c_2} \times \frac{\left(\begin{array}{l} \left(\sum_{n=0}^{\infty} S_n(-A)\, b_2^n c_1^n t^n\right)\left(\sum_{n=0}^{\infty} S_n(-A)\, b_1^n c_1^n t^n\right)\left(\sum_{n=0}^{\infty} S_n(-A)\, S_{n-k-1} B c_2^{n+1} t^n\right) \\ -\left(\sum_{n=0}^{\infty} S_n(-A)\, b_2^n c_2^n t^n\right)\left(\sum_{n=0}^{\infty} S_n(-A)\, b_1^n c_2^n t^n\right)\left(\sum_{n=0}^{\infty} S_n(-A)\, S_{n-k-1} B c_1^{n+1} t^n\right) \end{array}\right)}{\prod_{a\in A}(1 - ab_1c_1t)\prod_{a\in A}(1 - ab_2c_1t)\prod_{a\in A}(1 - ab_2c_1t)\prod_{a\in A}(1 - ab_1c_2t)\prod_{a\in A}(1 - ab_1c_1t)} \tag{2.31}$$

, $\forall k \in \mathbb{N}$.

Proof. The action of the operator $\delta^k_{c_1c_2}\delta^k_{b_1b_2}$ on the series $f(b_1^n c_1^n t) = \sum_{n=0}^{\infty} S_n(A)\, b_1^{n+1} t^n$ gives the left-hand side of equation (2.31) then:

$$\begin{aligned}
\delta_{b_1b_2^k}\delta_{c_1c_2^k} f(b_1t) =& \delta_{b_1b_2^k}\delta_{c_1c_2^k}\left(\sum_{n=0}^{\infty} S_n(A)\, b_1^{n+1} t^n\right) \\
=& \frac{\sum_{n=0}^{\infty} S_n(A)\, b_1^{n+k} c_1^n t^n - \sum_{n=0}^{\infty} S_n(A) b_2^{n+k} c_1^n t^n}{b_1 - b_2} \\
=& \delta_{c_1c_2^k}\left(\sum_{n=0}^{\infty} S_n(A)\, \frac{b_1^{n+k} - b_2^{n+k}}{b_1 - b_2}\right) \\
=& \delta_{c_1c_2^k}\left(\sum_{n=0}^{\infty} S_n(A)\, S_{n+k=1}(B)\, c_1^n t^n\right) \\
=& \frac{c_1^k \sum_{n=0}^{\infty} S_n(A)\, S_{n+k=1}(B)\, c_1^n t^n - c_2^k \sum_{n=0}^{\infty} S_n(A)\, S_{n+k=1}(B)\, c_2^n t^n}{c_1 - c_2} \\
=& \sum_{n=0}^{\infty} S_n(A)\, S_{n+k=1}(B)\, \frac{c_1^{n+k} - c_2^{n+k}}{c_1 - c_2} t^n \\
=& \sum_{n=0}^{\infty} S_n(A)\, S_{n+k-1}(b)\, S_{n+k-1}(c)\,.
\end{aligned}$$

Let $f(b_1c_1t) = \dfrac{1}{\prod_{a\in A}(1 - ab_1c_1t)}$ be the second member of the equality (2.31), then it can be written as:

$$\delta^k_{b_1b_2}\delta^k_{c_1c_2}f\left(b_1c_1t\right)=\delta^k_{b_1b_2}\delta^k_{c_1c_2}\left(\frac{b_1}{\prod\limits_{a\in A}\left(1-ab_1c_1t\right)}\right)$$

$$=\delta^k_{c_1c_2}\left(\frac{1}{b_1-b_2}\left(\frac{b_1^k}{\prod\limits_{a\in A}\left(1-ab_1c_1t\right)}-\frac{b_1^k}{\prod\limits_{a\in A}\left(1-ab_2c_1t\right)}\right)\right)$$

$$=\delta^k_{c_1c_2}\left(\frac{b_1^k\prod\limits_{a\in A}\left(1-b_2c_1t\right)-b_1^k\prod\limits_{a\in A}\left(1-b_2c_1t\right)}{\left(b_1-b_2\right)\prod\limits_{a\in A}\left(1-ab_1c_1t\right)\prod\limits_{a\in A}\left(1-ab_2c_1t\right)}\right).$$

and according to identity $\sum\limits_{n=0}^{\infty}S_n\left(A\right)b_1^nc_1^nt^n=\prod\limits_{a\in A}\left(1-ab_1c_1t\right)$ the result is:

$$\delta^k_{b_1b_2}\delta^k_{c_1c2}f\left(b_1c_1t\right)=\delta^k_{c_1c_2}\left(\frac{{b_1}^k\prod\limits_{a\in A}\left(1-b_2c_1t\right)-b_1^k\prod\limits_{a\in A}\left(1-b_2c_1t\right)}{\left(b_1-b_2\right)\left(\sum\limits_{n=0}^{\infty}S_n\left(-A\right)b_2^n\right)\left(\sum\limits_{n=0}^{\infty}S_n\left(-A\right)b_1^n\right)}\right)$$

$$=\delta^k_{b_1b_2}\left(\frac{-{b_1}^k{b_2}^k\sum\limits_{n=0}^{\infty}S_n(A)\frac{b_1^{n-k}-b_2^{n-k}}{b_1-b_2}c_1^nt^n}{\prod\limits_{a\in A}\left(1-b_1c_1t\right)\prod\limits_{a\in A}\left(1-b_2c_1t\right)}\right)$$

$$=\delta^k_{b_1b_2}\left(\frac{-{b_1}^k{b_2}^k\sum\limits_{n=0}^{\infty}S_n(A)S_{n-k-1}(B)c_1^nt^n}{\prod\limits_{a\in A}\left(1-b_1c_1t\right)\prod\limits_{a\in A}\left(1-b_2c_1t\right)}\right)$$

$$=\frac{1}{c_1-c_2}\left(\frac{-c_1^k{b_1}^k{b_2}^k\sum\limits_{n=0}^{\infty}S_n\left(A\right)S_{n-k-1}\left(B\right)c_1^nt^n}{\prod\limits_{a\in A}\left(1-b_1c_1t\right)\prod\limits_{a\in A}\left(1-b_2c_1t\right)}+\frac{-c_2^k{b_1}^k{b_2}^k\sum\limits_{n=0}^{\infty}S_n\left(A\right)S_{n-k-1}\left(B\right)c_2^nt^n}{\prod\limits_{a\in A}\left(1-b_1c_1t\right)\prod\limits_{a\in A}\left(1-b_2c_1t\right)}\right)$$

$$=\frac{{b_1}^k{b_2}^k}{c_1-c_2}\frac{\left(\begin{array}{c}\left(\sum\limits_{n=0}^{\infty}S_n\left(A\right)b_2^nc_1^nt^n\right)\left(\sum\limits_{n=0}^{\infty}S_n\left(A\right)b_1^nc_1^nt^n\right)\left(\sum\limits_{n=0}^{\infty}S_n\left(A\right)S_{n-k-1}\left(B\right)c_2^{n+k}t^n\right)\\-\left(\sum\limits_{n=0}^{\infty}S_n\left(A\right)b_2^nc_2^nt^n\right)\left(\sum\limits_{n=0}^{\infty}S_n\left(A\right)b_1^nc_2^nt^n\right)\left(\sum\limits_{n=0}^{\infty}S_n\left(A\right)S_{n-k-1}\left(B\right)c_1^{n+k}t^n\right)\end{array}\right)}{\prod\limits_{a\in A}\left(1-ab_1c_1t\right)\prod\limits_{a\in A}\left(1-ab_2c_1t\right)\prod\limits_{a\in A}\left(1-b_1c_2t\right)\prod\limits_{a\in A}\left(1-b_2c_2t\right)}.$$

□

CHAPTER 3

NEW GENERATING FUNCTIONS OF PRODUCTS OF (p, q)-NUMBERS WITH SOME SPECIAL NUMBERS AND POLYNOMIALS.

IN this chapter we introduce new generating functions of triple products of squares of k-Fibonacci numbers with (p, q)-Fibonacci and (p, q)-Lucas numbers , then squares of k-Pell numbers with (p, q)-Lucas and (p, q) Fibonacci numbers , then (p, q)-Fibonacci numbers with bivariate complex Fibonacci polynomials , (p, q)-Lucas numbers with bivariete complex Lucas polynomials, (p, q)-Fiboncci with bivariate complex Lucas plynomiales. Finally we calculate the new generating functions of products of squares of k-balancing numbers with (p, q)-Fibonacci numbers and (p, q)-Lucas numbers

The reccurence relation of k-Fibonacci numbers $\{F_{k,n}\}_{n\in\mathbb{N}}$ is defined by:

$$F_{k,n} = kF_{k,n-1} + F_{k,n-2} \text{ for } k \geq 1 \text{ and } n \leq 2.$$

with initial conditions $F_{k,0} = 1$ and $F_{k,1} = k$.

The bivariate complex fibonacci polynomials $\{F_n(x, y)\}_{n=0}^{\infty}$ were initially defined in [**7**] using the following recrusive formula:

$$F_{n+1}(x, y) = ixF_n(x, y) + yF_{n-1}(x, y) \text{ for } n \geq 1.$$

with initial conditions $F_0(x, y) = 0$ and $F_1(x, y) = 1$.

their Binet's formula and explicit formula are respectively expressed as:

$$F_n(x, y) = \frac{\alpha^n(x, y) - \beta^n(x, y)}{\alpha(x, y) - \beta(x, y)}.$$

and

$$F_n(x, y) = \sum_{j=0}^{\frac{n-1}{2}} \begin{pmatrix} n-j-1 \\ 1 \end{pmatrix} (ix)^{n-2j-1} y^j.$$

where $\alpha(x, y)$, $\beta(x, y)$ represent the roots of the characteristic equation

$$t^2 - ixt - y = 0.$$

similarly, the bivariate complex Lucas polynomials $\{L_n(x, y)\}_{n=0}^{\infty}$ were defined in [**7**] by:

$$L_{n+1}(x, y) = ixL_n(x, y) + yL_{n-1}(x, y) \text{ for } n \geq 1.$$

with initial conditions $L_0(x, y) = 2$ and $L_1(x, y) = ix$.

their Binet's and explicit formulas are respectively presented by:

$$L_n(x, y) = \alpha^n(x, y) - \beta^n(x, y).$$

and

$$L_n(x, y) = \sum_{j=0}^{\frac{n}{2}} \begin{pmatrix} n-j \\ j \end{pmatrix} (ix)^{n-2j} y^j.$$

For any positive integer k, the k-Pell sequence denoted as $(P_{k,n})_{n\in\mathbb{N}}$. follows:

$$P_{k,n+1} = 2P_{k,n} + kP_{k,n-1}, \text{ for } n \geq 1,$$

with initial conditions ,

$$P_{k,0} = 0, P_{k,1} = 1.$$

The characteristic equation linked to the k-Pell numbers recurrence relation, expressed as:

$$r^2 - 2r - k = 0.$$

yields roots $r_1 = 1 + \sqrt{1+k}$ and $r_2 = 1 - \sqrt{1+k}$. Notably, since the value of $\sqrt{1+k} > 1$, exceeds 1, it follows that $r_2 < 0$ is it between $r1$ and 0.
Furthermore, the relationships $r_1 + r_2 = 2, r_1 - r_2 = 2\sqrt{1+k}$ and $r_1 r_2 = -k$, emerge from the equation's solutions. Specifically, when $k = 1$, r_1 becomes the silver ratio, denoted by $r_1 = 1 + \sqrt{2}$, which holds significance in the context of the Pell numbers sequence.
Its Binet's formula is given by:

$$P_{k,n} = \frac{r_1^n - r_2^n}{r_1 - r_2}.$$

where r_1, r_2 are the roots of the characteristic equation $r^2 - 2r - k = 0$ and $r_1 > r_2$.
Balancing numbers, were introduced by R.P. Finkelstein satisfy a specific Diophantine equation, k-balancing numbers being a generalized form proposed by Ray in [**16**]. Their recursive definition and explicit formula are respectively outlined as follows:

$$B_{k,n+1} = 6kB_{k,n} - B_{k,n-1}, k \geqslant 1.$$

The explicit formula of k-balancing numbers is given by:

$$\alpha_1^{n+2} = 6k\alpha_1^{n+1} - \alpha_1^n \text{ and } \alpha_2^{n+2} = 6k\alpha_2^{n+1} - \alpha_2^n.$$

α_1 and α_2 represent the roots of the equation: $\alpha_1^2 = 6k\alpha_1 - 1$ and $\alpha_2^2 = 6k\alpha_2 - 1$, note that $3k + \sqrt{9k^2 - 1}$ and $3k - \sqrt{9k^2 - 1}$ represent the roots of the equation $\alpha^2 = 6k\alpha - 1$
Its Binet's formula of n^{th} k-balancing numbers is given by:

$$B_{k,n} = \frac{\alpha_1^n - \alpha_2^n}{\alpha_1 - \alpha_2}.$$

Definition 3.0.1. *[34] The (p, q)-Fibonacci numbers $\{F_{p,q,n}\}_{n\in\mathbb{N}}$ are defined by the following* *recurrence relation:*

$$F_{p,q,n} = pF_{p,q,n-1} + qF_{p,q,n-2} \text{ for } n \geq 2, \text{ with } F_{p,q,0} = 0, F_{p,q,1} = 1.$$

Its Binet's formula for is given by:

$$F_{p,q,n} = \frac{x_1^n - x_2^n}{x_1 - x_2}.$$

Definition 3.0.2. *[34] The (p, q)-Lucas numbers $\{L_{p,q,n}\}_{n\in\mathbb{N}}$ are defined by the following recurrence relation:*

$$L_{p,q,n} = pL_{p,q,n-1} + qL_{p,q,n-2} \text{ for } n \geq 2, \text{ with } L_{p,q,0} = 2, L_{p,q,1} = p.$$

Its Binet's formula is given by:

$$L_{p,q,n} = x_1^n + x_2.$$

3.1 Principle Theorem

The following theorem is the bases of all the next calculations and results it was proven in [**10**]:

Theorem 3.1.1. *Let A, B and C be three alphabets, respectively, $\{a_1, a_2, a_3\}$, $\{b_1, b_2\}$ and $\{c_1, c_2\}$ then we have*

$$\sum_{n=0}^{\infty} S_n(A) S_{n+k-1}(B) S_{n+k-1}(C) z^n = \frac{b_1^k b_2^k \begin{pmatrix} \left(\sum_{n=0}^{\infty} S_n(-A) b_2^n c_1^n z^n\right)\left(\sum_{n=0}^{\infty} S_n(-A) b_1^n c_1^n z^n\right) \sum_{n=0}^{\infty} S_n(-A) S_{n-k-1}(B) C_2^{n+k} z^n \\ - \left(\sum_{n=0}^{\infty} S_n(-A) b_2^n c_2^n z^n\right)\left(\sum_{n=0}^{\infty} S_n(-A) b_1^n c_2^n z^n\right) \sum_{n=0}^{\infty} S_n(-A) S_{n-k-1}(B) C_1^{n+k} z^n \end{pmatrix}}{(c_1 - c_2) \prod_{a\in A}(1 - ab_1c_1z) \prod_{a\in A}(1 - ab_2c_1z) \prod_{a\in A}(1 - ab_1c_2z) \prod_{a\in A}(1 - ab_2c_2z)}, \quad (3.1)$$

for all $k \in \mathbb{N}$.

If $k = 0, 1, 2$ and $a_3 = 0$ in the theorem (3.1.1), we deduce the following lemmas.

Lemma 3.1.1. *[4] Let A, B and C be three alphabets, respectively, $\{a_1,a_2\}$, $\{b_1,b_2\}$ and $\{c_1,c_2\}$ then we have:*

$$\sum_{n=0}^{\infty} S_n(A)\,S_{n-1}(B)\,S_{n-1}(C)\,z^n = \frac{N}{D}, \quad n \in \mathbb{N}, \tag{3.2}$$

with:

$$\begin{aligned} N = {} & (a_1+a_2)\,z - a_1a_2\,(b_1+b_2)\,(c_1+c_2)\,z^2 + b_1b_2c_1c_2\,(a_1+a_2)\left(2a_1a_2 - (a_1+a_2)^2\right) z^3 \\ & + a_1a_2b_1b_2c_1c_2\,(b_1+b_2)\,(c_1+c_2)\,(a_1+a_2)^2\,z^4 - b_1b_2c_1c_2a_1^2a_2^2\,(a_1+a_2) \\ & \left(b_1b_2\,(c_1+c_2)^2 + c_1c_2\,(b_1+b_2)^2 - c_1c_2b_1b_2\right) z^5 + a_1^3a_2^3b_1^2b_2^2c_1^2c_2^2\,(b_1+b_2)\,(c_1+c_2)\,z^6. \end{aligned}$$

$$\begin{aligned} D = {} & 1 - (a_1+a_2)\,(b_1+b_2)\,(c_1+c_2)\,z + (b_1b_2\,(a_1+a_2)^2\,(c_1+c_2)^2 + \left((b_1+b_2)^2 - 2b_1b_2\right) \\ & ((a_1+a_2)^2\,c_1c_2 - 2a_1a_2c_1c_2 + a_1a_2\,(c_1+c_2)^2))z^2 - (a_1+a_2)\,(b_1+b_2)\,(c_1+c_2) \\ & \left(b_1b_2c_1c_2\,(a_1+a_2)^2 + b_1b_2a_1a_2\,(c_1+c_2)^2 + a_1a_2c_1c_2\,(b_1+b_2)^2 - 5a_1a_2c_1c_2b_1b_2\right) z^3 + \\ & (a_1^2a_2^2c_1^2c_2^2\,(b_1+b_2)^4 + c_1^2c_2^2b_1^2b_2^2\,(a_1+a_2)^4 + a_1^2a_2^2b_1^2b_2^2\,(c_1+c_2)^4 - a_1a_2b_1b_2c_1c_2 \\ & (4b_1b_2c_1c_2\,(a_1+a_2)^2 + 4a_1a_2c_1c_2\,(b_1+b_2)^2 + 4a_1a_2b_1b_2\,(c_1+c_2)^2 - (a_1+a_2)^2 \\ & (b_1+b_2)^2\,(c_1+c_2)^2) + 6a_1^2a_2^2b_1^2b_2^2c_1^2c_2^2)z^4 - a_1a_2b_1b_2c_1c_2\,(a_1+a_2)\,(b_1+b_2)\,(c_1+c_2) \\ & \left(a_1a_2c_1c_2\,(b_1+b_2)^2 + b_1b_2c_1c_2\,(a_1+a_2)^2 + b_1b_2a_1a_2\,(c_1+c_2)^2 - 5a_1a_2b_1b_2c_1c_2\right) z^5 + \\ & (a_1^2a_2^2b_1^3b_2^3c_1^2c_2^2\,(a_1+a_2)^2\,(c_1+c_2)^2 + a_1^2a_2^2b_1^2b_2^2c_1^2c_2^2\left((b_1+b_2)^2 - 2b_1b_2\right) \\ & \left((a_1+a_2)^2\,c_1c_2 - 2a_1a_2c_1c_2 + a_1a_2\,(c_1+c_2)^2\right) z^6 - a_1^3a_2^3b_1^3b_2^3c_1^3c_2^3\,(a_1+a_2)\,(b_1+b_2) \\ & (c_1+c_2)\,z^7 + a_1^4a_2^4b_1^4b_2^4c_1^4c_2^4z^8. \end{aligned}$$

Lemma 3.1.2. *[4] Let A, B and C be three alphabets, respectively, $\{a_1,a_2\}$, $\{b_1,b_2\}$ and $\{c_1,c_2\}$ then we have*

$$\sum_{n=0}^{\infty} S_n(A)\,S_{n+1}(B)\,S_{n+1}(C)\,z^n = \frac{N_1}{D}, \quad n \in \mathbb{N}, \tag{3.3}$$

with:

$$\begin{aligned} N_1 = {} & (a_1+a_2)\,z - a_1a_2\,(c_1+c_2)\,(b_1+b_2)\,z^2 + c_1c_2b_1b_2\,(a_1+a_2)\left(-(a_1+a_2)^2 + 2a_1a_2\right) z^3 \\ & + a_1a_2b_1b_2c_1c_2\,(a_1+a_2)^2\,(b_1+b_2)\,(c_1+c_2)\,z^4 - a_1^2a_2^2b_1b_2c_1c_2\,(a_1+a_2) \\ & \left(b_1b_2\,(c_1+c_2)^2 + c_1c_2\,(b_1+b_2)^2 - b_1b_2c_1c_2\right) z^5 + a_1^3a_2^3b_1^2b_2^2c_1^2c_2^2\,(c_1+c_2)\,(b_1+b_2)\,z^6. \end{aligned}$$

From the previous lemma we deduce the following relationship

$$\sum_{n=0}^{\infty} S_{n-1}(A) S_n(B) S_n(C) z^n = \frac{N_2}{D}, \quad n \in \mathbb{N}, \tag{3.4}$$

with:

$$N_2 = (c_1 + c_2)(b_1 + b_2) z - (a_1 + a_2)\Big(c_1c_2(b_1 + b_2)^2 + b_1b_2(c_1 + c_2)^2 - c_1c_2b_1b_2\Big) z^2$$
$$+ c_1c_2b_1b_2(a_1 + a_2)^2(b_1 + b_1)(c_1 + c_1) z^3 - c_1^2c_2^2b_1^2b_2^2(a_1 + a_2)\Big((a_1 + a_2)^2 - 2a_1a_2\Big) z^4$$
$$- a_1^2a_2^2b_1^2b_2^2c_1^2c_2^2(b_1 + b_2)(c_1 + c_2) z^5 + a_1^2a_2^2b_1^3b_2^3c_1^3c_2^3(a_1 + a_2)z^6.$$

Lemma 3.1.3. *[4] Let A, B and C be three alphabets, respectively, $\{a_1, a_2\}$, $\{b_1, b_2\}$ and $\{c_1, c_2\}$ then we have*

$$\sum_{n=0}^{\infty} S_n(A) S_n(B) S_n(C) z^n = \frac{N_3}{D}, \quad n \in \mathbb{N}, \tag{3.5}$$

with:

$$N_3 = 1 - \Big(a_1a_2c_1c_2(b_1 + b_2)^2 + a_1a_2b_1b_2(c_1 + c_2)^2 + b_1b_2c_1c_2(a_1 + a_2)^2 - 3a_1a_2c_1c_2b_1b_2\Big) z^2$$
$$+ 2a_1a_2b_1b_2c_1c_2(a_1 + a_2)(b_1 + b_2)(c_1 + c_2) z^3$$
$$- (b_1b_2a_1^2a_2^2c_1^2c_2^2(b_1 + b_2)^2 + c_1c_2a1^2a_2^2b_1^2b_2^2(c_1 + c_2)^2 + a_1a_2b_1^2b_2^2c_1^2c_2^2(a_1 + a_2)^2$$
$$- 3a_1^2a_2^2b1^2b_2^2c_1^2c_2^2)z^4 + a_1^3a_2^3b1^3b_2^3c_1^3c_2^3z^6.$$

From the previous lemma we deduce the following relationship

$$\sum_{n=0}^{\infty} S_{n-1}(A) S_{n-1}(B) S_{n-1}(C) z^n = \frac{N_4}{D}, \quad n \in \mathbb{N}, \tag{3.6}$$

with

$$N_4 = z - \Big(a_1a_2c_1c_2(b_1 + b_2)^2 + a_1a_2b_1b_2(c_1 + c_2)^2 + b_1b_2c_1c_2(a_1 + a_2)^2 - 3a_1a_2c_1c_2b_1b_2\Big) z^3$$
$$+ 2a_1a_2b_1b_2c_1c_2(a_1 + a_2)(b_1 + b_2)(c_1 + c_2) z^4 - (b_1b_2a_1^2a_2^2c_1^2c_2^2(b_1 + b_2)^2$$
$$+ c_1c_2a_1^2a_2^2b_1^2b_2^2(c_1 + c_2)^2 + a_1a_2b_1^2b_2^2c_1^2c_2^2(a_1 + a_2)^2 - 3a_1^2a_2^2b_1^2b_2^2c_1^2c_2^2)z^5 + a_1^3a_2^3b_1^3b_2^3c_1^3c_2^3z^7.$$

3.2 Main results

In this section, we use the aforementioned theorem with the objective of deriving novel generating functions for the products involving established numbers and polynomials.

First step

In this section we will introduce new generating function of triple product of squares of k-Fibonacci numbers with (p,q) -Fibonacci numbers and (p,q) -Lucas numbers then we need to do the following substitutions in (3.4) and (3.5)

We replace a_2 by $[-a_2]$, b_2 by $[-b_2]$ and c_2 by $[-c_2]$ and we put:

$$\begin{cases} a_1 - a_2 = p, \\ b_1 - b_2 = k, \\ c_1 - c_2 = k \end{cases} \quad \text{and} \quad \begin{cases} a_1 a_2 = q, \\ b_1 b_2 = 1, \\ c_1 c_2 = 1 \end{cases}$$

we obtain:

$$\sum_{n=0}^{\infty} S_{n-1}(a_1,[-a_2])\, S_n(b_1,[-b_2])\, S_n(c_1,[-c_2])\, z^n = \frac{P_1}{D_1}.$$

$$\sum_{n=0}^{\infty} S_n(a_1,[-a_2])\, S_n(b_1,[-b_2])\, S_n(c_1,[-c_2])\, z^n = \frac{P_2}{D_1}.$$

with:

$$D_1 = 1 - pk^2 z - \left(q\left(k + 4k^2 + 4\right) + 2p^2\left(k^2+1\right)\right) z^2 - \left(k^2pq\left(2k^2+5\right) + k^2p^3\right) z^3$$

$$+ \left(\left(-k^2+4\right)p^2q + 2k^2q^2\left(k^2+4\right) + p^4 + 6q^2\right) z^4 + \left(k^2pq\left(2k^2q + p^2 + 5q\right)\right) z^5$$

$$- \left(k^2q^3\left(k^2+4\right) + 2q^2p^2\left(k^2+1\right) + 4q^3\left(k^2+1\right)\right) z^6 + k^2pq^3z^7 + q^4z^8.$$

$$P_1 = k^2z - p\left(-2k^2-1\right) z^2 + (pk)^2 z^3 - \left(p^3 - 2qp\right) z^4 - q^2k^2z^5 + q^2pz^6.$$

$$P_2 = 1 - \left(2qk^2 + p^2 + 3q\right) z^2 - 2pqk^2z^3 - \left(-2p^2k^2 - qp^2 - 3q^2\right) z^4 - q^3z^6.$$

Theorem 3.2.1. *For $n \in \mathbb{N}$, the new generating function of product of (p,q)-Fibonacci numbers with squares of k-Fibonacci numbers is given by:*

$$\sum_{n=0}^{\infty} P_{p,q} F_{k,n}^2 z^n = \frac{P_1}{D_1}.$$

where:

$P_1 = k^2 z - p\left(-2k^2 - 1\right) z^2 + (pk)^2 z^3 - \left(p^3 - 2qp\right) z^4 - q^2 k^2 z^5 + q^2 p z^6.$

For $k = 1$ we have:

$$D_1' = 1 - pz - \left(4p^2 + 9q\right) z^2 - \left(p^3 + 7pq\right) z^3 + \left(16q^2 + 3p^2 q + p^4\right) z^4 + \left(p^3 q + 7pq^2\right) z^5$$
$$- \left(4q^2 p^2 + 9q^3 p\right) z^6 + pq^3 z^7 + q^4 z^8.$$

$P_1' = z + 3pz^2 + p^2 z^3 - \left(p^3 - 2qp\right) z^4 - q^2 z^5 + q^2 p z^6.$

$P_2' = 1 - \left(p^2 + 5q\right) z^2 - 2pqz^3 - \left(-2p^2 - qp^2 - 3q^2\right) z^4 - q^3 z^6.$

corollary 3.2.1. *For $n \in \mathbb{N}$, the new generating functions of product of (p,q)-Fibonacci numbers with squares of Fibonacci numbers is given by:*

$$\sum_{n=0}^{\infty} P_{p,q} F_n^2 z^n = \frac{P_1'}{D_1'}.$$

Theorem 3.2.2. *For $n \in \mathbb{N}$, the new generating function of the product of (p,q)- Lucas numbers with squares of k-Fibonacci numbers is given by:*

$$\sum_{n=0}^{\infty} L_{p,q} F_{k,n}^2 z^n = \frac{P_3}{D_1}.$$

where:

$$P_3 = 2 - pk^2 z - \left(2q(2k^2 + 3) + p^2(-2k^2 + 5)\right) z^2 - \left(k^2 p(p^2 + 4q)\right) z^3$$
$$+ \left(2q^2\left(2k^2 + 3\right) + p^2(p^2 + 4q)\right) z^4 + pq^2 k^2 z^5 - \left(2q^3 + q^2 p^2\right) z^6.$$

Proof. The symmetric function of (p,q)-Lucas numbers is given by $2S_n(a_1, [-a_2]) - pS_{n-1}(a_1, [-a_2])$ **[34]** and we know that k-Fibonacci numbers are given by the symmetric function: $S_n(b_1, [-b_2])$ **[34]** then we have:

$$\sum_{n=0}^{\infty} P_{p,q} F_{k,n}^2 z^n = \sum_{n=0}^{\infty} (2S_n(A) - pS_{n-1}(A)(S_n(B))(s_n(C))$$
$$= \sum_{n=0}^{\infty} 2\left(S_n(A) S_n(B) s_n(C)\right) - p\left(S_{n-1}(A) S_n(B) s_n(C)\right)$$
$$= \frac{P_3}{D_1}.$$

where:

$$P_3 = 2 - pk^2z - \left(2q\left(2k^2+3\right)+p^2\left(-2k^2+5\right)\right)z^2 - \left(k^2p\left(p^2+4q\right)\right)z^3$$
$$+\left(2q^2\left(2k^2+3\right)+p^2\left(p^2+4q\right)\right)z^4 + pq^2k^2z^5 - \left(2q^3+q^2p^2\right)z^6.$$

□

For $k = 1$ we have:

$$P_3' = 2 - pz - \left(10q+5p^2\right)z^2 - \left(4pq+p^3\right)z^3 + \left(10q^2+p^4+4p^2q\right)z^4 + pq^2z^5$$

$$-\left(p^2q^2+2q^3\right)z^6.$$

$$D_1' = 1 - pz - \left(4p^2+9q\right)z^2 - \left(p^3+7pq\right)z^3 + \left(16q^2+3p^2q+p^4\right)z^4 + \left(p^3q+7pq^2\right)z^5$$
$$-\left(4q^2p^2+9q^3p\right)z^6 + pq^3z^7 + q^4z^8.$$

corollary 3.2.2. *for $n \in \mathbb{N}$, the generating function of product of (p, q)-Lucas numbers with squares of Fibonacci numbers is given by:*

$$\sum_{n=0}^{\infty} L_{p,q}F_n^2z^n = \frac{P_3'}{D_1'}.$$

Second step

As in the previous step we will now give te new generating function of the prodauct of squares of k-Pell numbers with (p, q)-Fibonacci numbers and (p, q)-Lucas numbers that's why we will effectuate the following replacements in (3.6) and (3.2):

$$\begin{cases} a_2 \longrightarrow [-a_2] \\ b_2 \longrightarrow [-b_2] \\ c_2 \longrightarrow [-c_2] \end{cases} \text{and} \begin{cases} a_1 - a_2 = p \\ b_1 - b_2 = 2 \\ c_1 - c_2 = 2 \end{cases} \text{and} \begin{cases} a_1a_2 = q \\ b_1b_2 = k \\ c_1c_2 = k \end{cases}$$

here we obtain:

$$\sum_{n=0}^{\infty} S_{n-1}\left(a_1, [-a_2]\right)S_{n-1}\left(b_1, [-b_2]\right)S_{n-1}\left(c_1, [-c_2]\right)z^n = \frac{Q_1}{D_2}.$$

$$\sum_{n=0}^{\infty} S_n\left(a_1, [-a_2]\right)S_{n-1}\left(b_1, [-b_2]\right)S_{n-1}\left(c_1, [-c_2]\right)z^n = \frac{Q_3}{D_2}.$$

with:

$$D_2 = 1 - 4pz - \left(2kp^2\left(k+4\right) + 4kq\left(k+4\right) + 16q\right)z^2 - \left(4k^2p^3 + 4kpq\left(5k+8\right)\right)z^3$$

$$+ \left(4k^2p^2q\left(k^2-4\right) + k^3q^2\left(6k+32\right) + k^2\left(k^2p^4 + 32q^2\right)\right)z^4$$

$$+ \left(4k^3pq\left(kp^2 + 5kq + q\right)\right)z^5 - \left(2k^5p^2q^2\left(k+4\right) + 4k^4q^3\left(k^2 + 4q + 4\right)\right)z^6$$

$$+ 4k^6pq^3z^7 + k^8q^4z^8.$$

$$Q_1 = z - \left(k^2\left(p^2 + 3q\right) + 8kq\right)z^3 - 8k^2qpz^4 + \left(k^3q\left(kp^2 + 3kq + 8q\right)\right)z^5 - k^6q^3z^7.$$

$$Q_2 = pz + 4qz^2 - \left(k^2p\left(2q + p^2\right)\right)z^3 - 4k^2qp^2z^4 + \left(k^3pq^2\left(k+8\right)\right)z^5 - 4k^4q^3z^6.$$

Theorem 3.2.3. *For $n \in \mathbb{N}$ the new generating function of Product of (p,q)-Fibonacci with squares of k-Pell numbers is given by:*

$$\sum_{n=0}^{\infty} Q_{p,q}P_{k,n}^2 z^n = \frac{Q_1}{D_2}.$$

For $k = 1$ we have:

$$D_2' = 1 - 4pz - \left(10p^2 + 36q\right)z^2 - \left(52pq + 4p^3\right)z^3 + \left(p^4 - 12p^2q + 70q^2\right)z^4$$

$$+ \left(52q^2p + 4qp^3\right)z^5 - \left(10q^2p^2 + 36q^3\right)z^6 + 4q^3pz^7 + q^4z^8.$$

$$Q_1' = z - \left(p^2 + 11q\right)z^3 - 8pqz^4 + \left(q\left(p^2 + 11q\right)\right)z^5 - q^3z^7.$$

$$Q_2' = pz + 4qz^2 - \left(p\left(p^2 + 2q\right)\right)z^3 - 4p^2qz^4 + 9pq^2z^5 - 4q^3z^6.$$

corollary 3.2.3. *For $n \in \mathbb{N}$, the new generating functions of product of (p,q)-Fibonacci with squares of k-Pell numbers is given by:*

$$\sum_{n=0}^{\infty} Q_{p,q}P_n^2 z^n = \frac{Q_1'}{D_2'}.$$

Theorem 3.2.4. *For $n \in \mathbb{N}$, the new generating function of Prodect of (p,q)-Lucas with squares of k-Pell numbers is given by:*

$$\sum_{n=0}^{\infty} P_{p,q}P_{k,n}^2 z^n = \frac{Q_3}{D_2}.$$

where:

$$Q_3 = pz + 8qz^2 + \left(-k^2p^3 - k^2pq + 8kpq\right)z^3 + \left(-k^4p^3q - k^4pq^2 + 8k^3pq^2\right)z^5 - 8k^4q^3z^6$$
$$+ k^6pq^3z^7.$$

Proof. As we mentiond in the first step we have the symmetric function of (p, q)- Lucas numbers is: $2S_n(a_1, [-a_2]) - pS_{n-1}(a_1, [-a_2])$ and the k-Pell one is given by: $S_{n-1}(b_1, [-b_2])$ then:

$$\sum_{n=0}^{\infty} P_{p,q}P_{k,n}^2 z^n = \sum_{n=0}^{\infty}(2S_n(A) - pS_{n-1}(A)(S_{n-1}(B))(s_{n-1}(C))$$
$$= \sum_{n=0}^{\infty} 2(S_n(A)S_{n-1}(B)s_{n-1}(C)) - p(S_{n-1}(A)S_{n-1}(B)s_{n-1}(C))$$
$$= \frac{Q_3}{D_2}.$$

such that:

$$Q_3 = pz + 8qz^2 + \left(-k^2p^3 - k^2pq + 8kpq\right)z^3 + \left(-k^4p^3q - k^4pq^2 + 8k^3pq^2\right)z^5 - 8k^4q^3z^6$$
$$+ k^6pq^3z^7.$$

□

For $k = 1$ we have:

$$Q_3' = pz + 8qz^2 + \left(7pq - p^3\right)z^3 + \left(\left(7q - p^2\right)\right)z^5 - 8q^3z^6 + q^3pz^7.$$

$$D_2' = 1 - 4pz - \left(10p^2 + 36q\right)z^2 - \left(52pq + 4p^3\right)z^3 + \left(p^4 - 12p^2q + 70q^2\right)z^4 + \left(52q^2p + 4qp^3\right)z^5$$
$$- \left(10q^2p^2 + 36q^3\right)z^6 + 4q^3pz^7 + q^4z^8.$$

corollary 3.2.4. *For $n \in \mathbb{N}$, the new generating function of product of (p, q)-Lucas with squares of k-Pell numbers is given by:*

$$\sum_{n=0}^{\infty} Q_{p,q}P_n^2 z^n = \frac{Q_3'}{D_2'}.$$

Third step

We will do the same procedure as the first and second steps by involving new generating functions of product of (p, q)- Fibonacci numbers and (p, q)- Lucas numbers with squares of bivariate complex Fibonacci and Lucas polynomials and also we need the following replacements in (3.6), (3.5), (3.2), (3.4)

$$\begin{cases} a_2 \longrightarrow [-a_2] \\ b_2 \longrightarrow [-b_2] \\ c_2 \longrightarrow [-c_2] \end{cases} \text{and} \begin{cases} a_1 - a_2 = p \\ b_1 - b_2 = ix \\ c_1 - c_2 = is \end{cases} \text{and} \begin{cases} a_1 a_2 = q \\ b_1 b_2 = y \\ c_1 c_2 = t \end{cases}$$

then we have:

$$\sum_{n=0}^{\infty} S_{n-1}(a_1,[-a_2])\,S_{n-1}(b_1,[-b_2])\,S_{n-1}(c_1,[-c_2])\,z^n = \frac{B_1}{D_3}.$$

$$\sum_{n=0}^{\infty} S_n(A)\,S_n(B)\,s_n(C)\,z^n = \frac{B_2}{D_3}.$$

$$\sum_{n=0}^{\infty} S_n(A)\,S_{n-1}(B)\,s_{n-1}(C)\,z^n = \frac{B_3}{D_3}.$$

$$\sum_{n=0}^{\infty} S_{n-1}(A)\,S_n(B)\,s_n(C)z^n = \frac{B_4}{D_3}.$$

with:

$$B_1 = z - \left(qy\left(s^2-3t\right) + t\left(qx^2-p^2y\right)\right)z^3 + 2pqstxyz^4$$
$$+\left(qty\left(-qs^2y - qtx^2 + p^2ty + 3qty\right)\right)z^5 - q^3t^3y^3z^7.$$

$$B_2 = 1 - \left(qy\left(-s^2+3t\right) + t\left(-qx^2+p^2y\right)\right)z^2 + 2pqstxyz^3$$
$$-\left(qty\left(qs^2y + qtx^2 - p^2ty - 3qty\right)\right)z^4 - q^3t^3y^3z^6.$$

$$B_3 = pz - qsxz^2 - \left(pty\left(p^2+2q\right)\right)z^3 + p^2qstxyz^4 + \left(pq^2ty\left(-s^2y - tx^2 + ty\right)\right)z^5 + q^3st^2xy^2z^6.$$

$$B_4 = -sxz - \left(p\left(s^2y + tx^2 - ty\right)\right)z^2 - p^2stxyz^3 - \left(pt^2y^2\left(p^2+2q\right)\right)z^4 + q^2st^2xy^2z^5$$
$$+ pq^2t^3y^3z^6.$$

$$D_3 = 1 + psxz + \left(qs^2\left(-x^2+2y\right) + 2qt\left(x^2-2y\right) + p^2\left(s^2y+tx^2-2ty\right)\right)z^2$$
$$+ \left(-pqs^3xy + pstx\left(-qx^2+p^2y+5qy\right)\right)z^3$$
$$+ \left(p^2qty\left(-s^2x^2+4ty\right) + q^2y^2s^2\left(s^2-4t\right) + q^2t^2x^2\left(-4y+x^2\right) + t^2y^2\left(p^4+6q^2\right)\right)z^4$$
$$+ \left(pqstxy\left(qs^2y+qtx^2-p^2y-5qt\right)\right)z^5$$
$$+ \left(q^3t^2x^2y^2\left(-s^2+2t\right) + p^2q^2t^2y^2\left(s^2y+tx^2-2ty\right) + 2q^3t^2y^3\left(s^2-2t\right)\right)z^6$$
$$- pq^3st^3xy^3z^7 + q^4t^4y^4z^8.$$

Theorem 3.2.5. *For $n \in \mathbb{N}$, the new generating function of the product of (p,q)-Fibonacci with bivariate complex Fibonacci is given by:*

$$\sum_{n=0}^{\infty} P_{p,q}B^2Fz^n = \frac{B_5}{D_3}.$$

where:

$$B_5 = pz - 2qsxz^2 - \left(py\left(qs^2+p^2t\right) + pqt\left(x^2+y\right)\right)z^3 - \left(pqty\left(qs^2y+qtx+p^2ty+qty\right)\right)z^5$$
$$+ 2q^3st^2xy^2z^6 + pq^3t^3y^3z^7.$$

Proof. The symmetric function of produc of (p,q)-Fibonacci is given by: $2S_n(a_1,[a_2]) - pS_{n-1}(a_1,[-a_2])$ and we know that the symmetric function of bivariate complex Fibonacci is given by: $S_{n-1}(b_1,[-b_2])$

then:

$$\sum_{n=0}^{\infty} P_{p,q}B^2Fz^n = \sum_{n=0}^{\infty}(2S_n(A) - pS_{n-1}(A)(S_{n-1}(B))(s_{n-1}(C))$$
$$= \sum_{n=0}^{\infty} 2\left(S_n(A)S_{n-1}(B)s_{n-1}(C)\right) - p\left(S_{n-1}(A)S_{n-1}(B)s_{n-1}(C)\right)$$
$$= \frac{B_5}{D_3}.$$

where:

$$B_5 = pz - 2qsxz^2 - \left(py\left(qs^2+p^2t\right) + pqt\left(x^2+y\right)\right)z^3 - \left(pqty\left(qs^2y+qtx+p^2ty+qty\right)\right)z^5$$
$$+ 2q^3st^2xy^2z^6 + pq^3t^3y^3z^7.$$

□

Theorem 3.2.6. *For $n \in \mathbb{N}$, the new generating function of product of (p,q)-Fibonacci with* *bivaraite complex Lucas is given by:*

$$\sum_{n=0}^{\infty} P_{p,q}FBL^2(x,y)\,z^n = \frac{B_6}{D_3}.$$

where:

$$B_6 = z + \left(4p\left(s^2y - tx^2 + ty\right)\right)z^2 + \left(p^2txy(-4s+x) - qx^2y\left(s^2-3t\right) - qtx^4\right)z^3$$
$$- \left(2pty\left(qsx^3 + 2p^2ty + 4qty\right)\right)z^4$$
$$- \left(-q^2stxy^2(sx+4t) + qt^2x^2y\left(-x^2q + p^2y + 3qy\right)\right)z^5 + 4pq^2t^3y^3z^6 + q^3t^3x^2y^3z^7.$$

Proof. The symmetric function of product of (p, q)-Fibonacci is given by: $S_{n-1}(a_1, [-a_2])$ and we know that the symmetric function of bivariate Lucas is given by:
$2S_n(b_1, [-b_2]) - ixS_{n-1}(b_1, [-b_2])$
then:

$$\sum_{n=0}^{\infty} P_{p,q}FBL^2(x,y)\,z^n = \sum_{n=0}^{\infty} S_{n-1}(A)\left(2S_n(B) - ixS_{n-1}(B)\right)\left(2S_n(C) - ixS_{n-1}(C)\right)$$

$$= \sum_{n=0}^{\infty} 4S_{n-1}(A)\,S_n(B)\,S_n(C) - x^2S_{n-1}(A)\,S_{n-1}(B)\,S_{n-1}(C)$$

$$= \frac{B_6}{D_3}.$$

where:

$$B_6 = z + \left(4p\left(s^2y - tx^2 + ty\right)\right)z^2 + \left(p^2txy(-4s+x) - qx^2y\left(s^2-3t\right) - qtx^4\right)z^3$$
$$- \left(2pty\left(qsx^3 + 2p^2ty + 4qty\right)\right)z^4$$
$$- \left(-q^2stxy^2(sx+4t) + qt^2x^2y\left(-x^2q + p^2y + 3qy\right)\right)z^5 + 4pq^2t^3y^3z^6 + q^3t^3x^2y^3z^7.$$

□

Theorem 3.2.7. *For $n \in \mathbb{N}$, the new generating function of the product (p, q)-Lucas with squares of bivariate complex Lucas is given by:*

$$\sum_{n=0}^{\infty} P_{p,q}BL^2(x,y)\,z^n = \frac{B_7}{D_3}.$$

where:

$$B_7 = 8-(px(-4s+x))z-\left(-4s^2y(p+2q)-4tx^2\left(p^2+2q\right)+12ty\left(p^2+2q\right)-2qsx^3\right)$$
$$z^2+\left(p^3txy(4s+x)+pqsxy(xs+16t)+pqtx^2\left(x^2+y\right)\right)z^3$$
$$+\left(8q^2ty\left(-s^2y-x^2t+3ty\right)+4p^2t^2y^2\left(p^2+4q\right)\right)z^4$$
$$+\left(pq^2stxy^2(sx-4t)+pqt^2x^2y\left(qx^2+p^2y+qy\right)\right)z^5$$
$$-\left(2q^2t^2y^2\left(qsx^3+2p^2ty+4qty\right)\right)z^6-pq^3t^3x^2y^3z^7.$$

Proof. We have the symmetric function of (p,q)-Lucas numbers is:
$(2S_n(a_1,[-a_2])-pS_{n-1}(a_1,[-a_2]))$ and the symmetric function of bivariate complex Lucas polynomial is given by: $(2S_n(b_1,[-b_2])-ixS_{n-1}(b_1,[-b_2]))$ then:

$$\sum_{n=0}^{\infty}P_{p,q}*BL^2(x,y)z^n=\sum_{n=0}^{\infty}(2S_n(A)-pS_{n-1}(A))(2S_n(B)-ixS_{n-1}(B))$$
$$(2S_n(C)-ixS_{n-1}(C))$$
$$=\sum_{n=0}^{\infty}8S_n(A)S_n(B)S_n(C)-2x^2S_n(A)S_{n-1}(B)S_{n-1}(C)$$
$$-4pS_{n-1}(A)S_n(B)S_n(C)+px^2S_{n-1}(A)S_{n-1}(B)S_{n-1}(C)$$
$$=\frac{B_7}{D_3}.$$

where:

$$B_7 = 8-(px(-4s+x))z-\left(-4s^2y(p+2q)-4tx^2\left(p^2+2q\right)+12ty\left(p^2+2q\right)-2qsx^3\right)$$
$$z^2+\left(p^3txy(4s+x)+pqsxy(xs+16t)+pqtx^2\left(x^2+y\right)\right)z^3$$
$$+\left(8q^2ty\left(-s^2y-x^2t+3ty\right)+4p^2t^2y^2\left(p^2+4q\right)\right)z^4 \qquad \square$$
$$+\left(pq^2stxy^2(sx-4t)+pqt^2x^2y\left(qx^2+p^2y+qy\right)\right)z^5$$
$$-\left(2q^2t^2y^2\left(qsx^3+2p^2ty+4qty\right)\right)z^6-pq^3t^3x^2y^3z^7.$$

Final step

Finally we calculate new generating functions of the product of squares k-balancing numbers with (p,q)-Fibonacci numbers and (p,q)-Lucas numbers, that's why we need the following replacements in (3.6) and (3.2):

$$\begin{cases} a_2 \longrightarrow [-a_2] \\ b_2 \longrightarrow [-b_2] \\ c_2 \longrightarrow [-c_2] \end{cases} \text{ and } \begin{cases} a_1 - a_2 = p \\ a_1 a_2 = q \end{cases} \text{ and } \begin{cases} b_1 - b_2 = 6k \\ b_1 b_2 = -1 \end{cases} \text{ and } \begin{cases} c_1 - c_2 = 6k \\ c_1 c_2 = -1 \end{cases}$$

then we obtain:

$$\sum_{n=0}^{\infty} S_{n-1}\left(a_1, [-a_2]\right) S_{n-1}\left(b_1, [-b_2]\right) S_{n-1}\left(c_1, [-c_2]\right) = \frac{H_1}{D_4}.$$

$$\sum_{n=0}^{\infty} S_{n}\left(a_1, [-a_2]\right) S_{n-1}\left(b_1, [-b_2]\right) S_{n-1}\left(c_1, [-c_2]\right) = \frac{H_2}{D_4}.$$

with:

$$\begin{aligned} D_4 = {} & q^4z^8 + 36k^2pq^3z^7 + \left(-1296k^4q^3 + 72k^2p^2q^2 + 144k^2q^3 - 2p^2q^2 - 4q^3\right) z^6 \\ & + \left(-2592k^4pq^2 + 36k^2p^3q + 180k^2pq^2\right) z^5 \\ & + \left(-1296k^4p^2q + 2592k^4q^2 - 288k^2q^2 + p^4 + 4p^2q + 6q^2\right) z^4 \\ & + \left(2592k^4pq - 36k^2p^3 - 180k^2pq\right) z^3 + \left(-1296k^4q + 72k^2p^2 + 144k^2q - 2p^2 - 4q\right) z^2 \\ & - 36k^2pz + 1. \end{aligned}$$

$$H_1 = -q^3z^7 + \left(-72k^2q^2 + p^2q + 3q^2\right) z^5 - 72k^2pqz^4 + \left(72k^2q - p^2 - 3q\right) z^3 + z.$$

$$H_2 = -36k^2q^3z^6 + \left(-72k^2pq^2 + pq^2\right) z^5 - 36k^2p^2qz^4 + \left(-p^3 - 2pq\right) z^3 + 36k^2qz^2 + pz.$$

Theorem 3.2.8. *For $n \in \mathbb{N}$, the new generating function of the product of (p,q)-Fibonacci numbers with squares of k-balancing numbers is given by:*

$$\sum_{n=0}^{\infty} P_{p,q} B_{(k,n)}^2 = \frac{H_1}{D_4}.$$

where:

$$H_1 = -q^3z^7 + \left(-72k^2q^2 + p^2q + 3q^2\right) z^5 - 72k^2pqz^4 + \left(72k^2q - p^2 - 3q\right) z^3 + z.$$

Theorem 3.2.9. *For $n \in \mathbb{N}$, the new generating function of the product of (p,q)-Lucas numbers* *with squares of k-balancing numbers is given by:*

$$\sum_{n=0}^{\infty} L_{p,q} B_{(k,n)}^2 = \frac{H_3}{D_4}.$$

where:
$$H_3 = pq^3z^7 - 72k^2q^3z^6 + \left(-72k^2pq^2 - p^3q - pq^2\right)z^5 + \left(-72k^2pq - p^3 - pq\right)z^3 + 72k^2qz^2 + pz.$$

Proof. The semmytric function of (p, q)-Lucas numbers is given by: $2S_n(a_1, [-a_2]) - pS_{n-1}(a_1, [-a2])$ **[34]** then we have:

$$\begin{aligned}\sum_{n=0}^{\infty} L_{p,q} B_{(k,n)}^2 &= \sum_{n=0}^{\infty} (2S_n(A) - pS_{n-1}(A))\, S_{n-1}(B)\, S_{n-1}(C) \\ &= \sum_{n=0}^{\infty} 2S_n(A)\, S_{n-1}(B)\, S_{n-1}(C) - pS_{n-1}(A)\, S_{n-1}(B)\, S_{n-1}(C) \\ &= \frac{H_3}{D_4}.\end{aligned}$$

where:
$$H_3 = pq^3z^7 - 72k^2q^3z^6 + \left(-72k^2pq^2 - p^3q - pq^2\right)z^5 + \left(-72k^2pq - p^3 - pq\right)z^3 + 72k^2qz^2 + pz.$$ □

For $k = 1$ we have:

$$D_4' = q^4z^8 + 36pq^3z^7 + \left(70p^2q^2 - 1156q^3\right)z^6 + \left(36p^3q - 2412pq^2\right)z^5 + \left(p^4 - 1292p^2q + 2310q^2\right)z^4 + \left(-36p^3 + 2412pq\right)z^3 + \left(70p^2 - 1156q\right)z^2 - 36pz + 1.$$

$$H_3' = pq^3z^7 - 72q^3z^6 + \left(-p^3q - 73pq^2\right)z^5 + \left(-p^3 - 73pq\right)z^3 + 72qz^2 + pz$$

$$H_1' = -q^3z^7 + \left(p^2q - 69q^2\right)z^5 - 72pqz^4 + \left(-p^2 + 69q\right)z^3 + z.$$

$$H_2' = -36q^3z^6 - 71pq^2z^5 - 36p^2qz^4 + \left(-p^3 - 2pq\right)z^3 + 36qz^2 + pz.$$

corollary 3.2.5. *For $n \in \mathbb{N}$, the new generating function of (p, q)-Lucas with squares of balancing numbers is given by:*

$$\sum_{n=0}^{\infty} L_{p,q} B_n^2 = \frac{H_3'}{D_4'}.$$

CONCLUSION

This study presents findings that demonstrate the use of generating functions and the multiplication of numbers (p, q) can lead to effective solutions for a variety of mathematical problems, including their applications in calculus and number theory.

Understanding symmetric functions and their role in mathematics, physics, and computer science is fundamental to developing complex mathematical models and practical applications in these fields.

The findings of this study highlight the importance of establishing a strong and appropriate mathematical foundation for diverse scientific and technological applications, thereby contributing to progress and development in various domains.

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