الجمهوريـة الجزائـريـة الديمقراطيـة الشعبيـة
People's Democratic Republic of Algeria
وزارة التعليـم العالـي والبحـث العلمـي
Ministry of Higher Education and Scientific Research

Nº Réf :……………

Abdelhafid Boussouf
University Center– Mila

Institute of Mathematics and Computer Sciences

Department of mathematics

Submitted for the degree of Master
In: Mathematics

Specialty: Fundamental Mathematics

Numerical solution of first integro-differential equations using the reproducing kernel Hilbert space method

Presented by: -Yekhlef Loubna
- Daoudi Maroua

In front of the jury

Hafi rassou zineb	MCB	Abdelhafid Boussouf	U. C. of Mila	president
Chellouf Yassamine	MCB	Abdelhafid Boussouf	U. C. of Mila	Supervisor
Rouibah khaoula	MCB	Abdelhafid Boussouf	U. C. of Mila	Examiner

Academic year: 2023/2024

Acknowledgement

Reaching the end of this thesis, we would like to thank Almighty Allah, who gave us the courage, patience, and strength to complete this modest work. We thank all the people who, near or far, helped us realize it, especially our supervisor, **Dr."Chellouf yassamine"**, who made every effort to ensure we did a good job. who has all the credit for supporting us, advising us, and being gentle in treating us. We thank the jury members **Dr."Hafi rassou zineb"** and **Dr."Rouibah khaoula"**. We also thank the college administration, **"Institute of Science and Technology"**. Finally, congratulations to us on an edifice of which we are leaders. What distinguishes us is the fruit of our giving, so we have the right to celebrate.

Dedication

To the man of my life, my eternal example, the one who has always sacrificed to see me succeed, for his confidence, to you my dearest father **"Zidane"**, thank you for always being there to make me happy.

To the light of my days, the source of my efforts, the flame of my heart, for her love, to you my dearest mother **"Saida"**, thank you for doing the impossible for me.

To my adorable sisters: **"Sidra"**, **"Roukia"**, and her husband **"Nassim"**; to my dear brothers: **"Ayoub"** and his wife **"Aya"**, **"Mouad"** and his wife **"Amira"**, for giving flavor and meaning to our family life, and for the love they reserve for me. I wish them a life full of happiness and success.

And I haven't forgotten you, my little angel **"Rital"**, because you are the air I breathe. I am very happy that you are in my life.

To all my friends who are always there to help me, thank you for your understanding, tenderness, and kindness.

To my colleague who shared this work with me **"Daoudi Maroua"**. To my wonderful and beautiful supervisor: **Dr."Chellouf yassamine"**.

Yekhlef Loubna

Dedication

This dedication can only be dedicated to **Mrs.”Rahima Nasseri”** and **Mr.”Yassin Daoudi”** ... Yes, they are my mother and father. .....
I also dedicate it to those whose names are absent in my dedication, but they were found in my heart and throughout my academic career From the day I chose my major until the last paper
I wrote in my diary Without forgetting my professor **Dr.”Chellouf yassamine”**, where I did not expect to have this much respect and kindness,
I thank you for everything. Of course, and to my colleague **”Yekhlef Loubna”**, with whom I started my story, thank you, my dear.

Daoudi Maroua

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$\mathscr{H}$: Hilbert Space.

$L^p[a,b]$: Lebesgue Spaces.

$\mathcal{A}^m[a,b]$: Set of functions with absolutely continuous derivative of order m-1.

$\mathscr{W}_2^m[a,b]$: Sobolev Space.

$\mathrm{k}(x,y)$: Reproducing kernel function.

$\{\psi_i\}_{i=1}^{\infty}$: Orthogonal functions system.

$\{\overline{\psi}_i\}_{i=1}^{\infty}$: Orthonormal functions system.

B_{ik} : Orthogonalization coefficients.

$\mathcal{L}$: Bounded linear operator.

Abbreviations

$Abs.C$: Absolutely Continuous.

$BC's$: Boundary Conditions.

$BVPs$: Boundary Value Problems.

$IDEs$: Integro Differential Equations.

$PDEs$: Partial Differential Equations.

$RKHS$: Reproducing Kernel Hilbert Space.

Abstract

This thesis discusses the idea of reproducing kernel Hilbert spaces connected to positive definite kernels and provides examples of how to apply it to a certain class of first integro differential equations. We build a new reproducing kernel space, examine the Sobolev space which is one of the most practical reproducing kernel Hilbert spaces, and provide an expression for reproducing kernel functions. In meantime, we built the whole orthonormal basis in the space $\mathscr{W}_2^m[a,b]$ using a replicating kernel function and its conjugate operator.

This thesis uses the replicating kernel Hilbert space approach to study the solutions of a general class of first-order integro-differential equations. In the space $\mathscr{W}_2^m[a,b]$, the analytical and approximate solutions are shown as series. It is demonstrated that the $n-$term approximation and all of its derivatives converge uniformly to the analytical solution and all of its derivatives, respectively. To demonstrate the correctness, dependability, and computing efficiency of the suggested technique for first-order integro-differential equations, a number of numerical examples are given.

KeyWords: Integro-Differential Equation, Positive Definite Functions, Sobolev Spaces, Reproducing Kernel Theory, Reproducing Kernel Method.

ملخص

تناقش هذه الأطروحة فكرة إعادة إنتاج فضاءات نواة هلبرت المرتبطة بنواة محددة موجبة وتقدم أمثلة لكيفية تطبيقها على فئة معينة من المعادلات التفاضلية التكاملية الأولى. نحن نبني مساحة نواة جديدة لإعادة الإنتاج، ونفحص مساحة سوبوليف التي تعد واحدة من أكثر مساحات هيلبرت للنواة عملية. في غضون ذلك، قمنا ببناء الأساس المتعامد بالكامل في الفضاء $W_2^m[a,b]$ باستخدام النواة المتماثلة والمشغل المرافق لها.

تستخدم هذه الأطروحة منهج فضاء هيلبرت التكراري لدراسة حلول فئة عامة من المعادلات التفاضلية التكاملية من الدرجة الأولى. في الفضاء $W_2^m[a,b]$ تظهر الحلول التحليلية والتقريبية على شكل سلاسل. لقد ثبت أن تقريب الحد n وجميع مشتقاته يتقارب بشكل موحد مع الحل التحليلي وجميع مشتقاته على التوالي.

لإثبات صحة واعتمادية وكفاءة الحوسبة للتقنية المقترحة للمعادلات التفاضلية التكاملية من الدرجة الأولى، تم تقديم عدد من الأمثلة العددية.

الكلمات المفتاحية: المعادلة التكاملية التفاضلية، الدوال المحددة الموجبة، فضاءات سوبوليف، إعادة إنتاج نظرية النواة، إعادة إنتاج طريقة النواة.

Résumé

Cette thèse discute de l'idée de reproduire des espaces de Hilbert à noyau connectés à des noyaux définis positifs et fournit des exemples de la façon de l'appliquer à une certaine classe des équations intégro-différentielles du premier ordre. Nous construisons un nouvel espace de noyau de reproduction, examinons l'espace de Sobolev qui est l'un des espaces de Hilbert de noyau de reproduction les plus pratiques, et fournissons une expression pour reproduire les fonctions du noyau. En même temps, nous avons construit toute la base orthonormée dans l'espace W_2^m [a,b] en utilisant une fonction noyau de réplication et son opérateur conjugué.

Cette thèse utilise l'approche spatiale de Hilbert à noyau de réproduction pour étudier les solutions d'une classe générale d'équations intégro-différentielles du premier ordre. Dans l'espace W_2^m [a,b], les solutions analytiques et approximatives sont présentées sous forme de séries. Il est démontré que l'approximation n-terme et toutes ses dérivées convergent uniformément vers la solution analytique et toutes ses dérivées, respectivement.

Pour démontrer l'exactitude, la fiabilité et l'efficacité informatique de la technique suggérée pour les équations intégro-différentielles du premier ordre, un certain nombre d'exemples numériques sont donnés.

Mots clés: équation intégro-différentielle, fonctions définies positives, espaces de Sobolev, théorie du noyau de reproduction, méthode du noyau de reproduction.

Introduction

In the early 20th century, S. Zaremba used the reproducing kernel for the first time when he worked on boundary value problems (BVPs) for harmonic and biharmonic functions. He was the first to introduce and express the reproducing property of the kernel corresponding to a class of functions in a particular case in 1907. But he did not develop any theory and did not give any particular name to the kernels he introduced.

J. Mercer studied the functions that satisfy the reproducing property in Hilbert's theory of integral equations in 1909. He devised a theory known as "positive-definite kernels" based on the kernels that were taken into consideration at the time, which were continuous kernels of positive-definite integral operators. Additionally, he demonstrated that among all continuous kernels of integral equations, these positive definite kernels have good properties.

Three Berlin mathematicians, G. Szego (1921), S. Bergman (1922), and S. Bochner (1922), presented the concept of reproducing kernels in their dissertations in the 1920s. Specifically, for the class of harmonic and analytic functions, S. Bergman presented reproducing kernels in one or more variables, which he dubbed "kernel functions."

Nonetheless, positive definite kernels were first proposed by E. H. Moore (1935) in the general analysis under the term "positive Hermitian matrices," with the intention of using them in a manner that was similar to the generalization of integral equations. In 1943, N. Aronszajn developed the theory of reproducing kernels which contains the Bergman kernel functions.

Bergman and Schiffer (1947) developed the original Zaremba proposal to apply the kernels to the solution of BVPs. These studies demonstrated the effectiveness of the kernels as a tool for solving BVPs of elliptic-type partial differential equations (PDEs). Moreover, S. Bergman and M. Schiffer (1948) achieved very nice results by using kernels to conformal mapping of multiply-connected domains.

N. Aronszajn organized the general theory of reproducing kernels in 1950. The groundwork for reproducing kernel theory was laid when he compiled the findings of other studies in the same field and utilized the same term "reproducing kernel functions," for each of these functions. The basic result by Aronszajn is the existence and uniqueness of a reproducing kernel Hilbert space (RKHS) corresponding to any self-adjoint nonnegative-definite kernel.

In 1986, M.G. Cui showed that space $\mathscr{W}_2^1[a,b]$ is a Hilbert space with reproducing kernel function which is expressed by finite terms. Hence, the implementation of reproducing kernel theory started in many fields. In 1988, the general theory of reproducing kernel Hilbert space and its different applications was given by S. Saitoh. By redefining the inner product of the space $\mathscr{W}_2^m[a,b]$ and based on the reproducing kernel created by M.G. Cui 1998, the reproducing kernel functions of $\mathscr{W}_2^m[a,b]$ space can be represented by piecewise polynomials and the higher order of derivatives without changing any other conditions.

The RKHS algorithm provides the possibility to pick any point in the integration interval, this algorithm has been successfully applied to various fields of numerical analysis, computational mathematics, probability, and statistics (17), and machine learning (34). Therefore, a wide range of research works have been directed to its applications in various stochastic categories (36), and defined problems involving operator equations (12), partial differential equations (1), integrative equations ((18), (35)), and differential integration equations ((11), (7), (2), (8)). In addition, many studies have focused in recent years on the use of the RKHS method as a framework for seeking approximate numerical solutions to different problems ((20), (43)). Moreover, the numerical solutions of the different groups of BVP can be found in ((6), (19),(21), (40)).

The thesis is organized as follows: In Chapter One, we give a preliminaries, basic concepts, definitions and theorems relevant to our study. Afterward, we present the reproducing kernel function by re-defining the inner product of a reproducing kernel space in order to obtain the analytical approximate solution for a general form of IDEs. Furthermore, the analysis of the RKHS method is described and an effective algorithm is introduced.

In Chapter Two, the RKHS method is applied to approximate the solution of a general form of first-order IDEs. It is a relatively new analytical technique. The analytical solution $u(x)$ and approximate solution $u_n(x)$ are represented in the form of series in the space $\mathscr{W}_2^2[a,b]$. Meanwhile, we give an iterative method to solve a nonlinear first-order IDEs. Various numerical examples are presented to illustrate the computational efficiency and the accuracy of the proposed method.

This thesis ends in Chapter Three with some concluding remarks and future recommendations.

Chapter 1

Basic Fundamentals

1.1 Functional Analysis

Functional analysis is a branch of mathematics that focuses on the study of infinite-dimensional vector spaces and the mappings between these spaces that respect the algebraic and topological structures defined on them. These structures include inner products, norms, and topology, which are essential for understanding the properties of functions and operators defined in these spaces. Functional analysis also extends the concepts of calculus to infinite-dimensional spaces, providing a framework for solving differential and integral equations in these contexts.
Most of the definitions and properties of this section are taken from (37).

1.1.1 Normed Spaces

Definition 1.1.1. *Let X be a vector space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. A norm on X is a map $\|.\| : X \longrightarrow [0, \infty)$ that satisfies the following three properties.*

1. *(Positive definite) For all $x \in X$, $\|x\| \geq 0$, if $x \in X$, then $\|x\| = 0$ iff $x = 0$,*

2. *For all $\lambda \in \mathbb{R}$ (or $\mathbb{C}$) and for all $x \in X$, $\|\lambda x\| = |\lambda| \, \|x\|$,*

3. *(Triangle inequality) For all* $x, y \in X$, $\|x+y\| \leq \|x\| + \|y\|$.

Definition 1.1.2. *A vector space with norm defined on it is called* ***normed space****.*

Example 1.1.1.
1. $\mathbb{R}$ *is a vector space over* $\mathbb{R}$*, and if we define* $\|.\| : \mathbb{R} \longrightarrow [0, \infty)$ *by* $\|x\| = |x|$, $x \in \mathbb{R}$*, then it becomes a normed space.*

2. $\mathbb{R}^n$ *is a vector space over* $\mathbb{R}$*, and let* $\|x\|_2 = \left(\sum_{i=1}^n |x_i|^2\right)^{\frac{1}{2}}, x = [x_1, \ldots, x_n] \in \mathbb{R}^n$*, then* $\mathbb{R}^n$ *is a normed space.*

We have a notion of "distance" between vectors in a normed space, and we can say whether two vectors are close or far away. Thus, in a normed space, we can discuss Cauchy sequences and convergent sequences.

Definition 1.1.3. *Let* $(X, \|\cdot\|)$ *be a normed space. A sequence* $(x_n)_{n\in\mathbb{N}}$ *in* X *is said to be converge to* $a \in X$ *if*

$$\forall \epsilon > 0, \exists N \in \mathbb{N} : \forall n \geq N : \|x_n - a\| < \varepsilon. \tag{1.1}$$

Note that (1.1) *says that the real sequence* $(\|x_n - a\|)_{n\in\mathbb{N}}$ *converges to* 0*, i.e*

$$\lim_{n\to\infty} \|x_n - a\| = 0.$$

In this case, we writes $\lim_{n\to\infty} x_n = a$ *or* $x_n \to a$ *as* $n \to \infty$*.*

Definition 1.1.4. *A sequence* $(x_n)_{n\in\mathbb{N}}$ *in* X *is called a Cauchy sequence if*

$$\forall \epsilon > 0, \exists N \in \mathbb{N} : \forall n, m \geq N : \|x_n - x_m\| < \epsilon. \tag{1.2}$$

Remark 1.1.1. *Every convergent sequence is a Cauchy sequence, since* *for* $\epsilon > 0, \exists N_1$ *s.t* $\forall n \geq N_1 : |x_n - x| < \frac{\epsilon}{2} < \epsilon$, *and for* $m > n \geq N_1$ *we have also:* $|x_m - x| < \frac{\epsilon}{2} < \epsilon$.

Let $N \geq N_1$, *then for all* $n, m \geq N$ *we have:*

$$\|x_n - x_m\| \leq \|x_m - x\| + \|x - x_n\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Definition 1.1.5. *A normed space* X *is called complete (or Banach space), if every Cauchy sequence* $(x_n)_{n\in\mathbb{N}}$ *is converges in* X.

1.1.2 Inner Product Spaces

Let X be a vector space.

Definition 1.1.6. *A function* $\langle . \rangle : X \times X \longrightarrow \mathbb{K}$ *is called an inner product if*

1. $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$ *for all* $u, v, w \in X$,
2. $\langle \alpha u, v \rangle = \alpha \langle u, v \rangle$ *for* $u, v \in X$ *and* $\alpha \in \mathbb{K}$,
3. $\langle u, v \rangle = \overline{\langle v, u \rangle}$ *for* $u, v \in X$,
4. $\langle u, u \rangle \geq 0$ *for all* $u \in X$ *and* $\langle u, u \rangle = 0 \Leftrightarrow u = 0$.

Every inner product naturally induces a norm of the form

$$\|x\| = \sqrt{\langle x, x \rangle}.$$

Definition 1.1.7. *A Hilbert space* $\mathscr{H}$ *is a complete inner product space.*

Definition 1.1.8. *Let* X *and* Y *be a vector spaces over* $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ *or* $\mathbb{C}$) *and let* $D(T)$ *be a subspace of* X. *We say that* $T : D(T) \subset X \longrightarrow Y$ *is a linear operator if*

$$T(x+y) = T(x)+T(y)$$

$$T(\alpha\, x) = \alpha\, T(x)$$

When $D(T)=X$, so we write $T: X \longrightarrow Y$.

Definition 1.1.9. *Let X and Y be two normed spaces. The linear operator $T: D(T)\subset X \longrightarrow Y$ is bounded, if there exist a real number $C>0$ such that*

$$\|Tx\|_Y \leq C\,\|x\|_X\,,\ \forall x\in D(T). \tag{1.3}$$

Definition 1.1.10. *Let $T: D(T)\subset X \longrightarrow Y$ be any operator, not necessarily linear, where $D(T)\subset X$ and X, Y are normed spaces. The operator T is continuous at an $x_0\in D(T)$ if for every $\varepsilon>0$, $\exists\delta>0$ such that $\|T_x - T_{x_0}\|<\varepsilon, \forall x\in D(T)$ satisfying $\|x-x_0\|<\delta$.*

T is continuous if T is continuous at every $x\in D(T)$.

Definition 1.1.11. *We say that a linear operator f is a linear functional if it is defined from X to $\mathbb{K}$, $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$, where X is a vector space and $\mathbb{K}$ is a scalar field.*

Note that, a bounded linear functional f is a bounded linear oparator i.e

$$\exists c\in\mathbb{R} \text{ such that } \forall x\in D(T),\ |f(x)|\leq c\,\|x\|\,.$$

Theorem 1.1.1. *(41) (**Riesz's Theorem**) If f is a bounded linear functional on a Hilbert space $\mathscr{H}$, then there exists some $y\in\mathscr{H}$ such that for every $x\in\mathscr{H}$ we have $f(x)=\langle x,y\rangle$. Where y is uniquely determined by f and has norm $\|f\|=\|y\|$.*

Definition 1.1.12. *Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be two Hilbert spaces, and T a bounded linear operator from $\mathscr{H}_1$ to $\mathscr{H}_2$. The bounded opertor $T^*: \mathscr{H}_2 \longrightarrow \mathscr{H}_1$ such that $\langle Tx,y\rangle =$*

$\langle x, T^*y \rangle$, $\forall x \in \mathscr{H}_1$ *and* $y \in \mathscr{H}_2$ *is called the adjoint of* T*. If* $T = T^*$*, then* T *is self-adjoint.*

Definition 1.1.13. *Let* $\Omega \subseteq \mathbb{R}^n$*, thus* $L^p(\Omega)$*,* $1 \leq p < \infty$ *represents the linear space of* p^{th} *order of integrable functions* u *on* Ω*, and* $L^\infty(\Omega)$ *as the linear space of essentially bounded functions.*

The spaces $L^p(\Omega)$, $1 \leq p < \infty$, and $L^\infty(\Omega)$ are Banach spaces with respect to the norms $\|u\|_{L^p} = \left(\int_\Omega |u(x)|^p \, dx\right)^{\frac{1}{p}} < \infty$, and $\|u\|_{L^\infty} = \underset{x\in\Omega}{esssup}\, |u(x)|$, respectively. If $p = 2$, then the space $L^2(\Omega) = \left\{u : \left(\int_\Omega |u|^2 (x) dx\right)^{\frac{1}{2}} < \infty\right\}$ is a Hilbert space with respect to the inner product $\langle u, v\rangle_{L^2} = \int_\Omega u(x)v(x)dx$.

Definition 1.1.14. *A function* $u : [a, b] \longrightarrow \mathbb{R}$ *is called absolutely continuous (Abs.C) if* $\forall \varepsilon > 0$*,* $\exists\ \delta$ *such that for any finite set* $(x_1, y_1), (x_2, y_2), \cdots, (x_k, y_k) (\subset [a, b])$ *satisfies* $\sum_{i=1}^k |y_i - x_i| < \delta$ *then*

$$\sum_{i=1}^{k} |u(y_i) - u(x_i)| < \varepsilon.$$

1.2 Integral Equations

An equation is called an integral equation when a function that has to be determined and is unknown appears under one or more integral sing. Naturally, such an equation can have additional terms. The usual format for integral equations is as follows:

$$h(x)g(x) = f(x) + \lambda \int_{\alpha(x)}^{\beta(x)} k(x, t, g(t))dt \tag{1.4}$$

where:

$h(x)$, $f(x)$, and $k(x, t, g(t))$ three functions are given, $k(x, t, g(x))$ is called the kernel,

$\alpha(x)$ and $\beta(x)$ are the limits of integration, $g(x)$ it the unknown function appears under the integral sign, and λ a no null constant parameter in to $\mathbb{R}$ or $\mathbb{C}$.

1.2.1 Classification of Integral Equations

Integral equations can be expressed in several ways. Fundamentally, the types rely on the integration limits and kernel of the equation. In this part, we will talk about different kinds of integral equations.

Fredholm integral equations

There are several scientific uses for Fredholm integral equations. Furthermore, the construction of Fredholm integral equations from boundary value problems was demonstrated. The greatest known contributions to integral equations and spectral theory are those made by Erik Ivar Fredholm (1866-1927). The theory of integral equations was developed by Swedish mathematician Fredholm, and operator theory was largely developed as a result of his 1903 work published in Acta Mathematica. In all Fredholm integral equations the limits of integration are finite and the upper limit of integration b is fixed.

1. First Kind Fredholm Integral Equation:

$$h(x) = 0$$

$$f(x) + \lambda \int_a^b k(x,t)g(t)dt = 0 \tag{1.5}$$

2. Second Kind Fredholm Integral Equation:

$$h(x) = 1$$

$$g(x) = f(x) + \lambda \int_a^b k(x,t)g(t)dt \tag{1.6}$$

Volterra integral equations

Numerous scientific applications, including population dynamics, epidemic spread, and semi-conductor devices, contain Volterra integral equations. It was also demonstrated that initial value problems may be used to create Volterra integral equations. Although Volterra began working on integral equations in 1884, he didn't start studying them seriously until 1896. The term "integral equation" was first used in 1888 by du Bois-Reymond. But Lalesco was the one who initially came up with the term Volterra integral equation in 1908.

In all Volterra Equations, the upper limit of integration b is variable, $b = x$.

1. First Kind Volterra Integral Equation:

$$h(x) = 0$$

$$f(x) + \lambda \int_a^x k(x,t)g(t)dt = 0 \tag{1.7}$$

2. Second Kind Volterra Integral Equation:

$$h(x) = 1$$

$$g(x) = f(x) + \lambda \int_a^x k(x,t)g(t)dt \tag{1.8}$$

Singular Integral Equations

When one or both limits of integration become infinite or when the kernel becomes infinite at one or more points within the range of integration, the integral equation is called singular.

For example, the integral equations

$$g(x) = f(x) + \lambda \int_{-\infty}^{\infty} \exp^{-|x-t|} g(t)dt \tag{1.9}$$

and

$$f(x) = \int_0^x \frac{1}{(x-t)^\alpha} g(t), 0 < \alpha < 1 \tag{1.10}$$

are singular equations.

1.3 Reproducing Kernel Hilbert Spaces

A linear and continuous map ϕ from a Hilbert space $\mathscr{H}$ to the real numbers is called a functional, that is, ϕ is an element of the dual space $\mathscr{H}^*$. We denote the point evaluation functional by δ_x, where $x \in X$.

A Dirac functional at an element $x \in X$ is a functional $\delta_x \in \mathscr{H}$ such that $\delta_x(f) = f(x)$. Note that δ_x is bounded if $\exists \mathcal{M} > 0$ *such that* $\|\delta_x f\|_{\mathbb{R}} \leq \mathcal{M} \|f\|_{\mathscr{H}}, \ \forall f \in \mathscr{H}$. When we convert this theorem into Dirac evaluation functionals, we obtain that for every δ_x, there is a unique vector k_x in $\mathscr{H}$ such that $\delta_x(f) = f(x) = \langle f, \mathrm{k}_x \rangle_{\mathscr{H}}$.

Definition 1.3.1. *Let $\mathscr{H}$ be a Hilbert space of function $f : X \longrightarrow \mathbb{K}$ on a set X. A function $\mathrm{k} : X \times X \longrightarrow \mathbb{C}$ is a reproducting kernel of $\mathscr{H}$ if the following properties* *are satisfied*

1. $\mathrm{k}(.,x) \in \mathscr{H}$, *For every* $x \in X$,

2. $\langle f, \mathrm{k}(.,x)\rangle_{\mathscr{H}} = f(x)$, *For every* $f \in \mathscr{H}$ *and* $x \in X$.

The second condition infers that the function f *evaluated at* x *is produced by the inner product of* f *with* k. *Also, we can write the first condition as follows: for all* $x \in X$, $\mathrm{k}_x(y) = \mathrm{k}(x,y) \in \mathscr{H}$, $\forall y \in X$. *Thus, using the reproducing property to the function* k_x *at* y, *we get:*

$$\mathrm{k}_x(y) = \langle \mathrm{k}_x, \mathrm{k}_y \rangle, \ \forall x, y \in X.$$

So, $\forall x \in X$, *we obtain* $\|\mathrm{k}_x\|^2 = \langle \mathrm{k}_x, \mathrm{k}_x \rangle = \mathrm{k}(x,x)$.

If a Hilbert function space $\mathscr{H}$ *has a reproducing kernel* k, *then* $\mathscr{H}$ *is called a reproducing kernel Hilbert space (RKHS). We denote the RKHS by* $\mathscr{H}_{\mathrm{k}}$ *and* $\langle .,. \rangle_{\mathscr{H}_{\mathrm{k}}}$, $\|.\|_{\mathscr{H}_{\mathrm{k}}}$ *are representd the norm and the inner product, respectively.*

Theorem 1.3.1. *If a Hilbert space* $\mathscr{H}$ *of functions defined on a set* X *has a reproducing kernel, then the reproducing kernel* $\mathrm{k}(x,y)$ *is uniquely determined by the Hilbert space* $\mathscr{H}$.

Theorem 1.3.2. *Let* $\mathscr{H}$ *be a Hilbert functions space on* X, *then there is a reproducing kernel* k *of* $\mathscr{H}$ *if and only if for every* $x \in X$, *the Dirac functional* $\delta_x : f \longrightarrow f(x)$ *is a bounded linear functional on* $\mathscr{H}$.

Definition 1.3.2. *Let* $\mathrm{k} : X \times X \longrightarrow \mathbb{C}$ *be a complex-valued fucntion on a set* X, *then*

1. k *is Hermitian if for every finite set of points* $y_1, ..., y_n \subseteq X$, *and any complex numbers* $c_1, ..., c_2$, *we have*

$$\sum_{i,j=1}^{\infty} \overline{c_i} c_j \ \mathrm{k}(y_i, y_j) \ \in \mathbb{R}$$

2. k *is positive definite if*

$$\sum_{i,j=1}^{\infty} \overline{c_i} c_j \ \mathrm{k}(y_i, y_j) \ \geq 0$$

Theorem 1.3.3. *(10) The reproducing kernel* $\mathrm{k}(x, y)$ *of a reproducing kernel Hilbert space* $\mathscr{H}$ *is a positive definite kernel.*

Proof : We have

$$\begin{aligned} 0 &\leq \left\|\textstyle\sum_{i=1}^{n} c_i \mathrm{k}_{x_i}\right\|^2 = \left\langle \textstyle\sum_{i=1}^{n} c_i \mathrm{k}_{x_i} \ , \ \textstyle\sum_{i=1}^{n} c_i \mathrm{k}_{x_i} \right\rangle, \\ &= \textstyle\sum_{i=1}^{n} \sum_{j=1}^{n} c_i \overline{c_j} \left\langle \mathrm{k}_{x_i} \ , \ \mathrm{k}_{x_j} \right\rangle, \\ &= \textstyle\sum_{i=1}^{n} \sum_{j=1}^{n} c_i \overline{c_j} \mathrm{k}(x_i, x_j). \end{aligned}$$

Hence, $\sum_{i=1}^{n} \sum_{j=1}^{n} c_i \overline{c_j} \mathrm{k}(x_i, x_j) \geq 0$. □

Remark 1.3.1. *Let* $\mathscr{H}$ *be a RKHS, and* $\mathrm{k}(x, y)$ *its kernel on* X. *Then for every* $x, y \in X$, *we have*

1. $|\mathrm{k}(x, y)|^2 \leq \mathrm{k}(x, x)\mathrm{k}(y, y)$,

2. *for* $x^* \in X$, *then the following are equivalent*

   - $\mathrm{k}(x^*, x^*) = 0$,

   - $\mathrm{k}(x^*, y) = 0, \ \forall y \in X$,

   - $f(x^*) = 0, \ \forall f \in \mathscr{H}$.

We can clarify (1) by applying the Schwarz inequality in $\mathscr{H}$, *so we oget*

$$|\mathrm{k}(x,y)|^2 = |\langle \mathrm{k}_x, \mathrm{k}_y\rangle|^2 \le \|\mathrm{k}_x\|^2 \|\mathrm{k}_y\|^2 = \langle \mathrm{k}_x, \mathrm{k}_x\rangle\langle \mathrm{k}_y \mathrm{k}_y\rangle = \mathrm{k}(x,x)\mathrm{k}(y,y).$$

For (2) it follows by (1) that

$$|\mathrm{k}(x^*,y)|^2 \le \mathrm{k}(x^*,x^*)\mathrm{k}(y,y) = 0.$$

Therefore, $\mathrm{k}(x^*,x^*) = 0$ *is equivalent to* $\mathrm{k}(x^*,y) = 0, \forall y \in X$. *Moreover, by the reproducing property* $\mathrm{k}(x^*,y) = 0$, $\forall y \in X$ *if and only if* $f(x^*) = 0$ *for every* $f \in \mathscr{H}$.

Theorem 1.3.4. *Any sequence of functions* $(g_n)_{n\ge1}$ *that strongly converge to a function* g *in* $\mathscr{H}_{\mathrm{k}}(x)$, *it also converges in the point-wise sense, which means* $\lim_{n\to\infty} g_n(x) = g(x)$, $\forall x \in X$. *Furthermore, this convergence is uniform on every subset of* X *on which* $x \longrightarrow \mathrm{k}(x,x)$ *is bounded.*

Proof : For $x \in X$, using the Schwwarz inequality and the reproducing property, we have:

$$\begin{aligned} |g(x) - g_n(x)| &= |\langle g\ ,\ \mathrm{k}_x\rangle - \langle g_n\ ,\ \mathrm{k}_x\rangle|\,, \\ &= |\langle g - g_n\ ,\ \mathrm{k}_x\rangle|\,, \\ &\le \|g - g_n\|\,\|\mathrm{k}_x\|\,, \\ &= \|g - g_n\|\,\mathrm{k}(x,x)^{\frac{1}{2}}. \end{aligned}$$

Therefore, $\lim_{n\to\infty} g_n(x) = g(x),\ \forall x \in X$.

Moreover it is clear from the above inequality that this convergence is uniform on every subset of X on which $x \longrightarrow \mathrm{k}(x,x)$ is bounded □

Definition 1.3.3. *Consider the non-negative integer m, and let $u \in L^2[a,b]$, then the function space $\mathscr{W}_2^m[a,b]$ is defined as follows*

$$\mathscr{W}_2^m[a,b] = \{u | u^{(i)} \text{ is Abs.C},\ i = 1, \ldots\ldots, m-1,\ \text{and } u^{(m)} \in L^2[a,b]\}.$$

The inner product and the norm are defined respectively in the function space $\mathscr{W}_2^m[a,b]$ as follows:

$$\langle u, v \rangle_{\mathscr{W}_2^m[a,b]} := \sum_{i=0}^{m-1} u^{(i)}(a) v^{(i)}(a) + \int_a^b u^{(m)}(x) v^{(m)}(x) dx, \tag{1.11}$$

and

$$\|u\|_{\mathscr{W}_2^m[a,b]} = (\langle u, u \rangle_{\mathscr{W}_2^m[a,b]})^{\frac{1}{2}}, \tag{1.12}$$

for all functions $u(x), v(x)$ in $\mathscr{W}_2^m[a,b]$.

Theorem 1.3.5. *The function space $\mathscr{W}_2^m[a,b]$ is a Hilbert space.*

Theorem 1.3.6. *The function space $\mathscr{W}_m^2[a,b]$ is a reproducing kernel space. That is, $\forall x \in [a,b]$, $\forall u(y) \in \mathscr{W}_2^m[a,b]$, $\exists\ \mathrm{k}_x(y) \in \mathscr{W}_2^m[a,b]$, $y \in [a,b]$ such that $\langle u(y), \mathrm{k}_x(y) \rangle = u(x)$, and $\mathrm{k}_x(y)$ is called the reproducing kernel function of the space $\mathscr{W}_2^m[a,b]$.*

1.4 Reproducing Kernel Function

Different ways to express the reproducing kernel functions in the space $\mathscr{W}_2^m[a,b]$ are discussed in this section. The expressions are displayed as $2m-1$ degree piecewise polynomials. We will also talk about some related results and crucial points about these kernel functions. Examples of such kernel functions in the space $\mathscr{W}_2^1[a,b]$ are given towards the end of this section.

In the space $\mathscr{W}_2^m[a,b]$, let's now determine the reproducing kernel function $\mathrm{k}_x(y)$ expression form. Assume that the reproducing kernel function of the space $\mathscr{W}_2^m[a,b]$ is $\mathrm{k}_x(y)$. Thus, by using the equations (1.11) and (1.12) for each fixed $x \in [a,b]$ and each $u(y) \in \mathscr{W}_2^m[a,b]$, we obtain $\langle u(y), \mathrm{k}_x(y)\rangle = u(x).y \in [a,b]$. We obtain

$$\langle u(y), \mathrm{k}_x(y)\rangle_{\mathscr{W}_2^m[a,b]} = \sum_{i=0}^{m-1} u^{(i)}(a)\mathrm{k}_x^{(i)}(a) + \int_a^b u^{(m)}(y)\mathrm{k}_x^{(m)}(y)dy, \tag{1.13}$$

using the integration by part for the right-hand of equation (1.13) we obtain

$$\int_a^b u^{(m)}(y)\mathrm{k}_x^{(m)}(y)dy = \sum_{i=0}^{m-1}(-1)^i u^{(m-i-1)}(y)\mathrm{k}_x^{(m+i)}(y)|_{y=a}^b + \int_a^b (-1)^m u(y)\mathrm{k}_x^{(2m)}(y)dy.$$

Assume that $j = m - i - 1$, then the

rst term from the right side of the above formula can be rewritten as follows

$$\sum_{i=0}^{m-1}(-1)^i u^{(m-i-1)}(y)\mathrm{k}_x^{(m+i)}(y)|_{y=a}^b = \sum_{j=0}^{m-1}(-1)^{m-j-1}u^{(j)}(y)\mathrm{k}_x^{(2m-j-1)}(y)|_{y=a}^b.$$

After a certain simplification, equation (1.13) becomes

$$\begin{aligned}\langle u(y), \mathrm{k}_x(y)\rangle_{\mathscr{W}_2^m[a,b]} &= \sum_{i=0}^{m-1} u^{(i)}(a)(\mathrm{k}_x^{(i)}(a) - (-1)^{m-i-1}\mathrm{k}_x^{(2m-i-1)}(a)) \\ &+ \sum_{i=0}^{m-1}(-1)^{m-i-1}u^{(i)}(b)\mathrm{k}_x^{(2m-i-1)}(b) + \int_a^b (-1)^m u(y)\mathrm{k}_x^{(2m)}(y)dy.\end{aligned}$$

Since $\mathrm{k}_x(y), u(y) \in \mathscr{W}_2^m[a,b]$, it suggests that

$$\mathrm{k}_x^{(i)}(a) - (-1)^{m-i-1}\mathrm{k}_x^{(2m-i-1)}(a) = 0,\ \mathrm{k}_x^{(2m-i-1)}(b) = 0,\ i = 0, \cdots, m-1.$$

Then $\langle u(y), \mathrm{k}_x(y)\rangle_{\mathscr{W}_2^m[a,b]} = \int_a^b u(y)((-1)^m \mathrm{k}_x^{(2m)}(y))dy$.

Now, let δ the dirac-delta function, for all $x \in [a,b]$, if $(-1)^{(m)}\mathrm{k}_x^{(2m)}(y) = \delta(x-y)$, then

$$\langle u(y), \mathrm{k}_x(y)\rangle_{\mathscr{W}_2^m[a,b]} = \int_a^b u(y)\delta(x-y)dy = u(x),$$

Obviously, $\mathrm{k}_x(y)$ is the reproducing kernel of the space $\mathscr{W}_2^m[a,b]$, then $\mathrm{k}_x(y)$ is solution of the following generalized differential equations

$$\begin{cases} (-1)^m \mathrm{k}_x^{(2m)}(y) & = & \delta(x-y), \\ \mathrm{k}_x^{(i)}(a) - (-1)^{m-i-1}\mathrm{k}_x^{(2m-i-1)}(a) & = & 0,\ i = 0, \cdots, m-1\ , \\ \mathrm{k}_x^{(2m-i-1)}(b) & = & 0,\ i = 0, \cdots, m-1. \end{cases} \tag{1.14}$$

When $x \neq y$

$$(-1)^m \mathrm{k}_x^{(2m)}(y) = 0, \tag{1.15}$$

with the boundary conditions

$$\mathrm{k}_x^{(i)}(a) - (-1)^{m-i-1}\mathrm{k}_x^{(2m-i-1)}(a) = 0,\ \mathrm{k}_x^{(2m-i-1)}(b) = 0,\ i = 1, \cdots, m-1. \tag{1.16}$$

For the equations (1.15), $\lambda^{2m} = 0$ is the characteristic equation, and $\lambda = 0$ is their characteristic values with $2m$ multiple roots, so the general solution of equation (1.15) is as follows:

$$\mathrm{k}_x(y) = \begin{cases} \sum_{i=0}^{2m-1} \mathcal{P}_i(x) y^i, & y \leq x, \\ \sum_{i=0}^{2m-1} \mathcal{Q}_i y^i, & y > x. \end{cases} \tag{1.17}$$

Additionally, since $(-1)^m \mathrm{k}_x^{2m}(y) = \delta(x-y)$, we have

$$\mathrm{k}_x^{(i)}(x+0) = \mathrm{k}_x^{(i)}(x-0),\ i = 0, \cdots, 2m-2, \tag{1.18}$$

by the integration of $(-1)^m \mathrm{k}_x^{(2m)}(y) = \delta(x-y)$ from $x-\xi$ to $x+\xi$ with respect to y and let $\xi \longrightarrow 0$, we get the jump degree of $\mathrm{k}_x^{(2m-1)}(y)$ at $y = x$ given by

$$(-1)^m(\mathrm{k}_x^{(2m-1)}(x+0) - \mathrm{k}_x^{(2m-1)}(x-0)) = 1. \tag{1.19}$$

For $i = 0, 1, \cdots, 2m-1$, we have $2m$ equations: equations (1.18) and (1.19) provide $2m$ conditions for solving the cofficients $\mathcal{P}_i(x)$ and $\mathcal{Q}_i(x)$ in (1.18). Additionally, $2m$ boundary conditions were provided by the equation (1.16). These $4m$ equations, which have the variables $\mathcal{P}_i(x)$ and $\mathcal{Q}_i(x)$ with unknown coefficients, are obviously linear equations. Calculating $\mathcal{P}_i(x)$ and $\mathcal{Q}_i(x)$ in equation (1.17) can be done with the Mathematica 11.0 software.

The following corollary provides some necessary properties of the reproducing kernel $\mathrm{k}_x(y)$.

Corollary 1.4.1. *If* $\mathrm{k}_x(y)$ *is the reproducing kernel of the space* $\mathscr{W}_2^m[a,b]$*, then for any fixed* $x \in [a,b]$*,* $\mathrm{k}_x(y)$ *is symmetric, unique, and* $\mathrm{k}_x(y) \geq 0$*.*

Proof : Using the reproducing property, we have

$$\mathrm{k}_x(y) = \langle \mathrm{k}_x(.), \mathrm{k}_y(.) \rangle = \langle \mathrm{k}_y(.), \mathrm{k}_x(.) \rangle = \mathrm{k}_y(x).$$

Now, let that $\mathrm{k}_x(y)$ and $\widetilde{\mathrm{k}}_x(y)$ be all the reproducing kernel of the space $\mathscr{W}_2^m[a,b]$, then

$$\mathrm{k}_x(y) = \langle \mathrm{k}_x(.), \widetilde{\mathrm{k}}_y(.) \rangle = \langle \widetilde{\mathrm{k}}_y(.), \mathrm{k}_x(.) \rangle = \widetilde{\mathrm{k}}_y(x).$$

since $\widetilde{\mathrm{k}}_x(y)$ is symmetric, we have unique representation of $\mathrm{k}_x(y)$. For the last condition, we observe that

$$\mathrm{k}_x(x) = \langle \mathrm{k}_x(.), \mathrm{k}_x(.) \rangle = \|\mathrm{k}_x(.)\|^2 \geq 0.$$

□

We now provide some formulas for recreating kernel functions with respect to various norms in the space $\mathscr{W}_2^1[a,b]$ using the method provided in this section.

Example 1.4.1. *Given the space* $\mathscr{W}_2^1[a,b] = \{u : [a,b] \longrightarrow \mathbb{R} : \ u(x)$ *is Abc.C and* $u'(x) \in L^2[a,b]\}$, *in this space, the inner product and the norm are defined by*

$$\langle u, v \rangle_{\mathscr{W}_2^1} = u(a)v(a) + \int_a^b u'(y)v'(y)dy, \ \text{and} \ \|u\|_{\mathscr{W}_2^1} = \langle u, u \rangle^{\frac{1}{2}}, \ \forall u(x), v(x) \in \mathscr{W}_2^1[a,b].$$

To find the reproducing kernel function $\mathrm{k}_x(y)$, *we use the integration by parts, we get*

$$\langle u, \mathrm{k}_x \rangle_{\mathscr{W}_2^1[a,b]} = u(a)\mathrm{k}_x(a) + u(y)\mathrm{k}_x'(y)|_{y=a}^{b} - \int_a^b u(y)\mathrm{k}_x''(y)dy.$$

Since $u(y), \mathrm{k}_x(y) \in \mathscr{W}_2^1[a,b]$, *we get* $\mathrm{k}_x(a) - \mathrm{k}_x'(a) = 0$ *and* $\mathrm{k}_x'(b) = 0$. *So, we must solve the BVP*

$$\begin{cases} -\mathrm{k}_x''(y) & = & \delta(x-y), \\ \mathrm{k}_x(a) - \mathrm{k}_x'(a) & = & 0, \\ \mathrm{k}_x'(b) & = & 0. \end{cases}$$

$\lambda^2 = 0$ is characteristic equation of the differential equation $-\mathrm{k}''_x(y) = 0$, moreover, the characteristic value is $\lambda = 0$ with 2 multiple roots. So

$$\mathrm{k}_x(y) = \begin{cases} \mathcal{P}_1(x) + \mathcal{P}_2(x)y, & y \le x, \\ \mathcal{Q}_1(x) + \mathcal{Q}_2(x)y, & y > x. \end{cases}$$

Additionally, by using the equations (1.18) and (1.19), we obtain $\mathrm{k}_x(x+0) = \mathrm{k}_x(x-0)$ and $\mathrm{k}'_x(x+0) - \mathrm{k}'_x(x-0) = -1$. Thus, the coefficients $\mathcal{P}_i(x)$, and $\mathcal{Q}_i(x)$, $i = 1, 2$ can be calculated by solving the following equations.

1) $\mathrm{k}_x(x+0) = \mathrm{k}_x(x-0)$,

2) $\mathrm{k}'_x(x+0) - \mathrm{k}'_x(x-0) = -1$.

3) $\mathrm{k}_x(a) - \mathrm{k}'_x(a) = 0$,

4) $\mathrm{k}'_x(b) = 0$,

Then, the kernel function is given by

$$\mathrm{k}_x(y) = \begin{cases} y - a + 1, & y \le x, \\ x - a + 1, & y > x. \end{cases}$$

Example 1.4.2. *Consider the space $\mathscr{W}_2^1[a,b] = \{u : u(x)$ is Abs.C, $u'(x) \in L^2[a,b]$ and $u(a) = u(b) = 0\}$. The inner product and the norm in this space are given respectively by*

$$\begin{cases} \langle u, v\rangle_{\mathscr{W}_2^1} = \int_a^b u'(y)v'(y)dy, \\ \|u\| = (\langle u, u\rangle)^{\frac{1}{2}}, \end{cases} \quad \textit{where } u(x),\ v(x) \in \mathscr{W}_2^1. \tag{1.20}$$

Similarly, as in Example (1.4.1) we have

$$\mathrm{R}_x(y) = \begin{cases} c_1(x) + c_2(x)y, \ \ y \leq x \ , \\ \\ d_1(x) + d_2(x)y, \ \ y > x \ . \end{cases}$$

We can obtain the unknown coefficients $c_i(x)$, *and* $d_i(x)$, $i = 1, 2$ *by solving the following equations*

$$1) \ \mathrm{R}_x(b) = 0,$$

$$2) \ \mathrm{R}_x(a) = 0,$$

$$3) \ \mathrm{R}_x(x+0) = \mathrm{R}_x(x-0),$$

$$4) \ \mathrm{R}'_x(x+0) - \mathrm{R}'_x(x-0) = -1.$$

Hence, the reproducing kernel function $\mathrm{R}_x(y)$ *is given by*

$$\mathrm{R}_x(y) = \begin{cases} \dfrac{(b-x)(a-y)}{a-b}, \ \ y \leq x, \\ \dfrac{(a-x)(b-y)}{a-b}, \ \ y > x. \end{cases}$$

Example 1.4.3. *Consider the space* $\mathscr{W}_2^1[a,b]$ *defi*

ned as the same set of functions in Example (1.4.1), and provide a new inner product in the space $\mathscr{W}_2^1[a,b]$ *by* $\langle u, v\rangle_{\mathscr{W}_2^1} = \int_a^b (u(y)v(y) + u'(y)v'(y))dy$, *such that* $u(x), v(x) \in \mathscr{W}_2^1[a,b]$, *and considering the same norm. Through the integration by part of* $\langle u, \mathrm{P}_x\rangle_{\mathscr{W}_2^1} = \int_a^b (u(y)\mathrm{P}_x(y) + u'(y)\mathrm{P}'_x(y))dy$, *we get*

$$\langle u, \mathrm{P}_x\rangle_{\mathscr{W}_2^1} = u(y)\mathrm{P}'_x(y)|_{y=a}^b + \int_a^b u(y)(\mathrm{P}_x(y) - \mathrm{P}''_x(y))dy.$$

Because $u(y), \mathrm{P}_x(y) \in \mathscr{W}_2^1[a,b]$, *so* $\mathrm{P}'_x(a) = \mathrm{P}'_x(b) = 0$. *and* $\mathrm{P}_x(y) - \mathrm{P}''_x(y) = \delta(x-y)$. *The characteristic equation is* $1 - \lambda^2 = 0$, *and the characteristic values are* $\lambda = -1, 1$.

Then

$$P_x(y) = \begin{cases} a_1(x)e^{-y} + a_2(x)e^{y}, & y \leq x, \\ b_1(x)e^{-y} + b_2(x)e^{y}, & y > x. \end{cases}$$

By solving the following equations

$$1)\ P'_x(a) = 0,$$

$$2)\ P'_x(b) = 0,$$

$$3)\ P_x(x+0) = P_x(x-0),$$

$$4)\ P'_x(x+0) - P'_x(x-0) = -1,$$

we obtain the unknown coefficients $a_i(x),\ b_i(x),\ i = 1, 2,$ *hence, the reproducing kernel function is given by*

$$P_x(y) = \begin{cases} -\dfrac{e^{-(x+y)}(e^{2b} + e^{2x})(e^{2a} + e^{2y})}{2(e^{2a} - e^{2b})}, & y \leq x, \\ -\dfrac{e^{-(x+y)}(e^{2a} + e^{2x})(e^{2b} + e^{2y})}{2(e^{2a} - e^{2b})}, & y > x\ . \end{cases}$$

1.5 Description of Reproducing Kernel Method

An iterative procedure for developing and calculating the general $m^{th}-$order BVP solution is shown in this section. The RKHS provides the computation formula as well as a representation of the exact solution. The approximate solution can be obtained by truncating the $n-$term of the exact solution. We take the general $m^{th}-$order BVP of the following type to give an overview of the RKHS method.

$$u^{(m)}(x) + a_1(x)u^{(m-1)}(x) + \cdots + a_{m-1}(x)u'(x) = \mathscr{F}(x, u(x)),\ a \leq x \leq b, \quad (1.21)$$

subject to the BCs

$$\begin{cases} u^{(i)}(a) = \alpha_i, & i = 0, 1, 2, \cdots, r-1, \\ u^{(i)}(b) = \beta_i, & i = r, r+1, \cdots, m-1. \end{cases} \tag{1.22}$$

Where $a_i(x), i = 1, 2, \cdots, m-1$, are continuous real-valued functions, $\alpha_i, 0 \leq i \leq r-1,\ and\ \beta_i,\ r \leq i \leq m-1$ are real constants, and $u(x)$ is unknown function to be determined, $u^{(m)}(x)$ indicates the m^{th} derivative of $u(x)$, and $\mathscr{F}(x, u(x))$ is a linear or nonlinear function depending on the problem discussed.

We use the RKHS method to solve the BVP (1.21) and (1.22), first of all, we construct a reproducing kernel space $\mathscr{W}_2^{m+1}[a,b]$ in which every function satisfies the homogeneous BC's (1.22), and then utilize the space $\mathscr{W}_2^1[a,b]$. The inner product and the norm in the space $\mathscr{W}_2^{m+1}[a,b]$ are given previously in equations (1.11), and (1.12) respectively.

Let $\mathrm{K}_x(y)\ and\ \mathrm{R}_x(y)$ be the reproducing kernel functions of the spaces $\mathscr{W}_2^{m+1}[a,b]$ and $\mathscr{W}_2^1[a,b]$ respectively, such that $\mathrm{K}_x(y)$ is a piecewise polynomial with 2m+1 degrees. Define a linear bounded differential operator $\mathcal{L} : \mathscr{W}_2^{m+1}[a,b] \to \mathscr{W}_2^1[a,b]$, such that

$$\mathcal{L}u(x) = u^{(m)}(x) + a_1(x)u^{(m-1)}(x) + \cdots + a_{m-1}(x)u'(x),$$

After homogenization of the BCs (1.22), the BVP(1.21)and (1.22) can be converted into the equivalent form as follows:

$$\mathcal{L}u = \mathscr{F}(x, u(x)),\ a \leq x \leq b \tag{1.23}$$

$$u^{(i)}(a) = 0,\ i = 0,1,2,\cdots,r-1,\ u^{(i)}(b) = 0,\ i = r, r+1,\cdots,m-1, \qquad (1.24)$$

where $u(x) \in \mathscr{W}_2^{m+1}[a,b]$ and $\mathscr{F}(x,u) \in \mathscr{W}_2^1[a,b]$.

Now, we construct an orthogonal function system of the space $\mathscr{W}_2^{m+1}[a,b]$.

Consider a countable dense set$\{x_i\}_{i=1}^{\infty}$ of $[a,b]$, set $\varphi_i(x) = \mathrm{R}_{x_i}(x)$. So, from the properties of $\mathrm{R}_x(y)$, for every $u(x) \in \mathscr{W}_2^1[a,b]$, it follows that $\langle u(x), \varphi_i(x)\rangle_{\mathscr{W}_2^1[a,b]} = \Big\langle u(x), \mathrm{R}_{x_i}(x)\rangle_{\mathscr{W}_2^1[a,b]} = u(x_i)$.

Additionally, let $\psi_i(x) = \mathcal{L}^*\varphi_i(x)$, where $\mathcal{L}^*$ is the adjoint operator of $\mathcal{L}$, and $\psi_i(x) \in \mathscr{W}_2^{m+1}[a,b]$. According to the property of the $\mathrm{K}_x(y)$, we obtains

$$\begin{aligned}
\langle u(x), \psi_i(x)\rangle_{\mathscr{W}_2^{m+1}} &= \langle u(x), \mathcal{L}^*\varphi_i(x)\rangle_{\mathscr{W}_2^{m+1}}, \\
&= \langle \mathcal{L}u(x), \varphi_i(x)\rangle_{\mathscr{W}_2^1}, \\
&= \mathcal{L}u(x_i).
\end{aligned}$$

where i=1,2...

Lemma 1.5.1. *$\psi_i(x)$ can be written on the following form $\psi_i(x) = \mathcal{L}_y \mathrm{K}_x(y)\,|_{y=x_i}$.*

Proof : From the above assumption, it is clear that

$$\begin{aligned}
\psi_i(x) = \mathcal{L}^*\varphi_i(x) &= \langle \mathcal{L}^*\varphi_i(y), \mathrm{k}_x(y)\rangle_{\mathscr{W}_2^{m+1}}, \\
&= \langle \varphi_i(y), \mathcal{L}\mathrm{k}_x(y)\rangle_{\mathscr{W}_2^1}, \\
&= \mathcal{L}_y \mathrm{k}_x(y)|_{y=x_i}.
\end{aligned}$$

□

Lemma 1.5.2. *$\psi_i(a) = \psi_i(b) = 0, i = 1,2,...$*

Proof :

$$\begin{aligned}\psi_i(a) &= \langle \psi_i(y), \mathrm{K_a}(y)\rangle_{\mathscr{W}_2^{m+1}} \\ &= \langle \mathcal{L}^*\varphi_i(y), \mathrm{K_a}(y)\rangle_{\mathscr{W}_2^{m+1}} \\ &= \langle \varphi_i(y), \mathcal{L}_y\mathrm{K_a}(y)\rangle_{\mathscr{W}_2^1}.\end{aligned}$$

By the symmetry of $\mathrm{K}_a(y)$, we arrive at $\mathrm{K}_a(y) = \mathrm{K}_y(a) = 0$. thus $\psi_i(a) = 0$. Similarly, we can obtain $\psi_i(b) = 0$. □

Theorem 1.5.1. *Assume that the inverse operator $\mathcal{L}^{-1}$ in Equation (1.23) exists, and $\{x_i\}_{i=1}^{\infty}$ is dense on $[a,b]$, then $\{\psi_i(x)\}_{i=1}^{\infty}$ is the complete function system of the space $\mathscr{W}_2^{m+1}[a,b]$.*

Proof : For each fixed $u(x) \in \mathscr{W}_2^{m+1}[a,b]$, let $\langle u(x), \psi_i(x)\rangle = 0, \forall i = 1, 2, \cdots$, that is

$$\begin{aligned}\langle u(x), \psi_i(x)\rangle_{\mathscr{W}_2^{m+1}} &= \langle u(x), \mathcal{L}^*\phi_i(x)\rangle_{\mathscr{W}_2^{m+1}}, \\ &= \langle \mathcal{L}u(x), \phi_i(x)\rangle_{\mathscr{W}_2^1}, \\ &= \mathcal{L}u(x_i) = 0.\end{aligned}$$

Note that $\{x_i\}_{i=1}^{\infty}$ is dense on $[a,b]$, so $\mathcal{L}u(x) = 0$. Then $u(x) = 0$ from the existence of $\mathcal{L}^{-1}$ and the continuity of $u(x)$. □

We can obtained the orthonormal function system $\{\overline{\psi_i}(x)\}_{i=1}^{\infty}$ of the space $\mathscr{W}_2^{m+1}[a,b]$ from Gram-Schmidt orthogonalization process of $\{\psi_i(x)\}_{i=1}^{\infty}$ as follows:

$$\overline{\psi}_i(x) = \sum_{k=1}^{i} \beta_{ik}\psi_k(x), i = 1, 2... \tag{1.25}$$

where β_{ik} are orthogonalization coefficients and are given by

$$\beta_{11} = \frac{1}{\|\psi_1\|}, \ \beta_{ii} = \frac{1}{\sqrt{\|\psi_i\|^2 - \sum_{k=1}^{i-1} C_{ik}^2}}, \ \beta_{ij} = \frac{-\sum_{k=1}^{i-1} C_{ik}\beta_{kj}}{\sqrt{\|\psi_i\|^2 - \sum_{k=1}^{i-1} C_{ik}^2}} j < i, \quad (1.26)$$

such that $C_{ik} = \left\langle \psi_i, \overline{\psi_k} \right\rangle_{\mathscr{W}_2^{m+1}}$ and $\left\{ \overline{\psi_i}\,(x) \right\}_{i=1}^{\infty}$ is the orthonormal system in the space $\mathscr{W}_2^{m+1}\,[a,b]$.

Theorem 1.5.2. *For all* $u(x) \in \mathscr{W}_2^{m+1}[a,b]$*, the series* $\sum_{i=1}^{\infty} \langle u(x), \overline{\psi}_i \rangle \overline{\psi}_i(x)$ *are convergent in the sense of the norm of* $\mathscr{W}_2^{m+1}[a,b]$*. In contrast if* $\{x_i\}_{i=1}^{\infty}$ *is dense subset on* $[a,b]$*, then the unique solution of the BVP (1.21) and (1.22) satisfies the form:*

$$u(x) = \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} \mathscr{F}(x_k, u(x_k), \overline{\psi}_i(x)). \quad (1.27)$$

Proof : According the Theorem (1.5.1), it is clear that $\{\overline{\psi}_i(x)\}_{i=1}^{\infty}$ is the complete orthonormal basis of the space $\mathscr{W}_2^{m+1}[a,b]$. Thus, $u(x)$ can be expanded in the Fourier series about the orthonormal system $\{\overline{\psi}_i(x)\}_{i=1}^{\infty}$ as $u(x) = \sum_{i=1}^{\infty} \left\langle u(x), \overline{\psi}_i(x) \right\rangle \overline{\psi}_i(x)$. Additionally, the space $\mathscr{W}_2^{m+1}[a,b]$ is Hilbert space, then the series $\sum_{i=1}^{\infty} \left\langle u(x), \overline{\psi}_i(x) \right\rangle \overline{\psi}_i(x)$ is convergent in the sense of the norm of $\mathscr{W}_2^{m+1}[a,b]$.

Since $\langle v(x), \phi_i(x) \rangle = v(x_i), \forall v(x) \in \mathscr{W}_2^{1}[a,b]$, we have

$$\begin{aligned}
u(x) &= \textstyle\sum_{i=1}^{\infty} \langle u(x), \overline{\psi}_i(x) \rangle_{\mathscr{W}_2^{m+1}} \overline{\psi}_i(x), \\
&= \textstyle\sum_{i=1}^{\infty} \langle u(x), \sum_{k=1}^{i} \beta_{ik}\psi_k(x) \rangle_{\mathscr{W}_2^{m+1}} \overline{\psi}_i(x), \\
&= \textstyle\sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} \langle u(x), \psi_k(x) \rangle_{\mathscr{W}_2^{m+1}} \overline{\psi}_i(x), \\
&= \textstyle\sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} \langle u(x), \mathcal{L}^* \phi_k(x) \rangle_{\mathscr{W}_2^{m+1}} \overline{\psi}_i(x), \\
&= \textstyle\sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} \langle \mathcal{L} u(x), \phi_k(x) \rangle_{\mathscr{W}_2^{1}} \overline{\psi}_i(x), \\
&= \textstyle\sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} \langle \mathscr{F}(x, u(x)), \phi_k(x) \rangle_{\mathscr{W}_2^{1}} \overline{\psi}_i(x), \\
&= \textstyle\sum_{i=0}^{\infty} \sum_{k=1}^{i} \beta_{ik} \mathscr{F}(x_k, u(x_k)) \overline{\psi}_i(x).
\end{aligned}$$

□

Remark 1.5.1. *If Equation (1.21) is linear, then the analytical and the approximate solution to Equation (1.21) can be obtained directly from Equation (1.27).*

We denote the n-term approximate solution to $u(x)$ by

$$u_n(x) = \sum_{i=1}^{n} \sum_{k=1}^{i} \beta_{ik} \mathscr{F}(x_k, u(x_k)) \overline{\psi}_i(x) \tag{1.28}$$

where $\overline{\psi_i}(x)$ and β_{ik} are given in Equations (1.25) and (1.26) , respectively.

Remark 1.5.2. *If Equation (1.21) is nonlinear: Then, the approximation solution to Equation (1.21) can be obtained using the following iteration method.*

From the Equation (1.27), the solution of Equation (1.21) can be obtained by

$$u(x) = \sum_{i=1}^{\infty} \mathcal{A}_i \overline{\psi_i}(x) \tag{1.29}$$

where $\mathcal{A}_i = \sum_{k=1}^{i} \beta_{ik} \mathscr{F}(x_k, u(x_k))$. In fact , $\mathcal{A}_i, i = 1, 2, ...,$ are unknown, we will approximate $\mathcal{A}_i$ using known $\mathcal{B}_i$. let $u(x_1) = 0$, so $\mathscr{F}(x_1, u(x_1))$ is known. Then, by a numerical computation, we set $u_0(x_1) = u(x_1)$ and define the n-term approximation to $u(x)$ by

$$u_n(x) = \sum_{i=1}^{n} \mathcal{B}_i \overline{\psi_i}(x) \tag{1.30}$$

where the coefficients $\mathcal{B}_i, i = 1, 2, ..., n$*, are given as*

$$\begin{cases} \mathcal{B}_1 = \beta_{11}\mathscr{F}\left(x_1, u_0\left(x_1\right)\right), \\ u_1\left(x\right) = \mathcal{B}_1\overline{\psi_1}\left(x\right); \\ \mathcal{B}_2 = \sum_{k=1}^{2}\beta_{2k}\mathscr{F}\left(x_k, u_{k-1}\left(x_k\right)\right), \\ u_2\left(x\right) = \mathcal{B}_1\overline{\psi_1}\left(x\right) + \mathcal{B}_2\overline{\psi_2}\left(x\right); \\ \vdots \\ \mathcal{B}_N = \sum_{k=1}^{n}\beta_{nk}\mathscr{F}\left(x_k, u_{k-1}\left(x_k\right)\right). \end{cases} \tag{1.31}$$

Remark 1.5.3. *We can make sure that the approximation* $u_n(x)$ *satisfies the BCs of Equation (1.21) in the iteration process of Equation (1.29).*

In fact, with the proper choosing of the initial term $u_0\left(x\right)$, the solution to Equation (1.21) is regarded as the fixed point of the following functional:

$$\begin{aligned} u_{n+1}\left(x\right) &= \mathcal{L}^{-1}\mathscr{F}\left(x, u_n\left(x\right)\right) \\ &= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}\mathscr{F}\left(x_k, u_n\left(x_k\right)\right)\overline{\psi_i}\left(x\right). \end{aligned}$$

As a well known powerful tool, we have the Banach's fixed point theorem .

Theorem 1.5.3. *Assume that* X *is a Banach space and* $A : X \longrightarrow X$ *is a nonlinear mapping, and suppose that*

$$\| A\left[u\right] - A\left[v\right] \| \leq \alpha \| u - v \|, \; u, v \in X$$

for some constants $\alpha < 1$*. Then* A *has a unique fixed point. Furthermore, the sequence* $u_{n+1}\left(x\right) = A\left[u_n\right]$*, with an arbitrary choice of* $u_0 \in X$*, converges to the fixed point of* A *.*

According to above Theorem, for the nonlinear mapping

$$\begin{aligned}\mathcal{A}[u(x)] &= \mathcal{L}^{-1}\mathscr{F}(x, u(x)) \\ &= \textstyle\sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}\mathscr{F}(x_k, u(x_k))\overline{\psi_i}(x),\end{aligned}$$

a sufficient condition for convergence of the present iteration method is strictly contraction of A. Furthermore, the sequence (1.29) converges to the fixed point of A which is also the solution of Equation (1.21) .

However, the approximate solution $u_n^N(x)$ can be obtained by taking finitely many terms in the series representation of $u_n(x)$ and

$$u_n^N(x) = \sum_{i=1}^{N}\sum_{k=1}^{i}\beta_{ik}\mathscr{F}(x_k, u_{n-1}(x_k))\overline{\psi_i}(x).$$

Theorem 1.5.4. *For every $u(x) \in \mathscr{W}_2^{m+1}[a, b]$, $u_n^{(i)}(x)$ are uniformly convergent to $u^{(i)}(x)$, $i = \overline{0, m}$.*

Proof : For any $x \in [a, b]$, we get

$$\begin{aligned}\left|u_n^{(i)}(x) - u^{(i)}(x)\right| &= \left|\left\langle u_n^{(i)}(x) - u^{(i)}(x), K_x(x)\right\rangle_{\mathscr{W}^+}\right| \\ &= \left|\left\langle u_n(x) - u(x), K_x^{(i)}(x)\right\rangle_{\mathscr{W}^{m+1}}\right| \\ &\leq \left\|K_x^{(i)}(x)\right\|_{\mathscr{W}^{m+1}}\|u_n(x) - u(x)\|_{\mathscr{W}^{m+1}} \\ &\leq M_i\|u_n(x) - u(x)\|_{\mathscr{W}^{m+1}} \longrightarrow 0, \ as \ n \longrightarrow \infty\end{aligned}$$

□

Chapter 2

First-order integro-differential equations

Functional equations are typically the outcome of mathematical modeling of physics and engineering problems encountered in everyday life. For instance, stochastic equations, PDEs, integral and IDEs, and others. IDEs are found in many mathematical formulations of physical processes, these equations arise in chemical kinetics, fluid dynamics, and biological models.

The most integral and integrodifferential equations fall under two main classes namely Fredholm and Volterra integro differential equations. Since IDEs are typically challenging to solve analytically, an effective approximation solution must be found. As a result, several writers have shown a great deal of interest in them. Using the RKHS, Yang and Cui (2006) were able to solve a class of IDEs and convert them into linear equations. A novel approach for providing the analytical and approximation solutions to the Fredholm-Volterra IDE in the RKHS was given by the authors (Yulan, et al., 2009).

In this chapter, the RKHS method is applied to approximate the solution of a general form of first-order IDE. It is a relatively new analytical technique.

Software packages have great capabilities for solving IDEs. Sometimes, it is difficult to solve them analytically so it is required to obtain an efficient approximate solution. Thus, some software mathematical packages such as Mathematica or MathCad can be helpful in visualizing the behavior of the solutions of IDEs. Indeed, throughout the whole thesis we used Mathematica 7.0 software package for numerical experiment. In this section, based on RKHS method, we will introduce an effective algorithm for the following nonlinear IDE

$$\begin{cases} u' = F(x, u(x)),\ 0 \leq x \leq 1; \\ u(0) = 0, \end{cases} \tag{2.1}$$

where $F(x, u(x)) = f(x) + \int_0^x \ h(x,t)N(u(t))\ dt \in \mathscr{W}_2^1[a,b],\ f(x)$ and $h(x,t)$ are known functions, $N(u(x))$ is a nonlinear function of $u,\ u(x) \in \mathscr{W}_2^1[a,b]$ is unknown function to be determined. We suppose that IDE (2.1) has a unique solution.

The analytical and approximate solutions are represented in terms of series in the RKHS. The $n-$term approximation is obtained and is proved to converge to the analytical solution. In the meantime, an iterative method of obtaining the solution is presented in the RKHS. Further, implementations of the method on a nonlinear IDE of Volterra type

$$u'(x) = f(x) + \int_a^x F(x, t, u(t), u'(t))\ dt,\ x \in [a, b], \tag{2.2}$$

and Fredholm type

$$u'(x) = f(x) + \int_a^b G(x, t, u(t), u'(t))\ dt,\ x \in [a, b], \tag{2.3}$$

will be given in the space $\mathscr{W}_2^2[a, b]$.

The aim of the next algorithm is to implement a procedure to solve IDE (2.1) based on RKHS method that described in previous chapter.

Algorithm 1: To approximate the solution of the IDE (2.1) based on RKHS method, there are five main steps:

Input: integer n, the functions $f(x), k_1(x, y)$and$k_2(x, y)$; the differential operator $\mathcal{L}$; the inner product $\langle u(x), v(x)\rangle_{\mathscr{W}_2^2}$.

Output: Approximate solutions $u_n(x)$ of the IDE (2.1).

- **Stepe A:** Fixed x and set $x, y \in [0, 1]$;

  For $i = 1, 2, ..., n$ do stepes$(1, 2\ \&3)$;

  - **stepe 1:** set $x_i = \frac{i-1}{n-1}$;
  - **stepe 2:** if $y \leq x$ then set $\mathrm{K}(x, y) = k_1(x, y)$ else set $\mathrm{K}(x, y) = k_2(x, y)$;
  - **stepe 3:** Set $\psi_i(x) = \mathcal{L}_y[\mathrm{K}(x, y)]|_{y=x_i}$;

  Output the orthogonal functions system $\psi_i(x)$.

- **Stepe B:** For $i = 1, 2, ..., n$;

  For $j = 1, 2, ..., i$ set $C_{ij} = \langle\psi_j, \psi_i\rangle_{\mathscr{W}_2^2}$, set $\beta_{11} = \frac{1}{Sqrt(C_{11})}$;

  Output C_{ij} and β_{11}.

- **Stepe C:** For $i = 2, 3, ..., n$, do stepes $(1^\circ\ \&\ 2^\circ)$;

  - **stepe** 1°**:** For $k = 1, 2, ..., i-1$ set $CC_{ik} = \sum_{m=1}^{k} \beta_{km} C_{im}$;
  - **stepe** 2°**:** For $j = 1, 2, ..., i$, if $j \neq i$;

    then set $\beta_{ij} = -(\sum_{k=j}^{i-1} CC_{ik}\beta_{kj}).(C_{ii} - \sum_{k=1}^{i-1} CC_{ik}^2)^{\frac{-1}{2}}$;

else set $\beta_{ii} = (C_{ii} - \sum_{k=1}^{i-1} CC_{ik}^2)^{-\frac{1}{2}}$;

Output the orthogonalization coefficients β_{ij}.

- **Stepe D:** For $i = 1, 2, ..., n$ set $\overline{\psi}_i(x) = \sum_{k=1}^{i} \beta_{ik}\psi_i(x)$;

  Output the orthonormal functions system $\overline{\psi}_i(x)$;

- **Stepe E:** Set $u_0(x_1) = 0$;

  For $i = 1, 2, ..., n$ do stepes$(1^*, 2^*$ & $3^*)$;

  - **stepe** 1^***:** Set $u(x_i) = u_{i-1}(x_i)$;
  - **stepe** 2^***:** Set $\mathrm{B}_i = \sum_{k=1}^{i} \beta_{ik}\mathrm{F}(x_k, u_{k-1}(x_k))$;
  - **stepe** 3^***:** Set $u_i(x) = \sum_{k=1}^{i} \mathrm{B}_k\overline{\psi}_k(x)$.

The n-term approximate solution $u_n(x)$ of IDE (2.1) is obtained.

2.1 Volterra Integro-Differential Equation

This section studies the RKHS method solutions for the first-order Volterra IDEs. In the space $\mathscr{W}_2^2[a,b]$, the analytical solution $u(x)$ and the approximation solution $u_n(x)$ are represented as a series. The existence of solutions for Equation (2.2) based on RKHS is then explained using this technique. Additionally, we provide an iterative solution for a first-order nonlinear Volterra IDE. In terms of the norm of $\mathscr{W}_2^2[a,b]$, the approximation solution's error is monotone decreasing. At the end of this section, a number of numerical examples are provided to show the accuracy and computing efficiency of the suggested method.

Consider the following Volterra Integro-Differential Equation:

$$\begin{cases} u'(x) = F(x, u(x), \mathcal{T}u(x)), \ a \le x \le b; \\ u(a) = u_0, \end{cases} \tag{2.4}$$

such the $\mathcal{T}u(x) = \int_a^x h(x,t)u(t)\, dt$, a, b, u_0 are real finite constants, $h(x,t)$ is known continuous function, $u(x)$ is unknown function to be determined and $F(x,t,u(t),\mathcal{T}u(x))$ is a linear or nonlinear function depending on the problem discussed. We suppose that IDE (2.4) have a unique solution.

Now, we construct several reproducing kernels of the space

$$\mathscr{W}_2^2[a,b] = \{u|\ u,\ u'\ is\ Abs.C,\ u,\ u',\ u'' \in L^2[a,b],\ u(a) = 0\}$$

in order to solve IDE (2.4). Taking the following inner product and the norm of the space $\mathscr{W}_2^2[a,b]$

$$\begin{aligned} \langle u, v \rangle &= u(a)v(a) + u'(a)v'(a) + \int_a^x u''(t)v''(t)\, dt; \\ \|u\|_{\mathscr{W}_2^2} &= \sqrt{\langle u, u \rangle_{\mathscr{W}_2^2}} \end{aligned} \tag{2.5}$$

and by applying the same steps in chaper one we get the following kernel function of the space $\mathscr{W}_2^2[a,b]$

$$K_x(y) = \begin{cases} \frac{1}{6}(y-a)(2a^2 - y^2 + 3x(2+y) - a(6+3x+y)),\ y \le x; \\ \frac{1}{6}(x-a)(2a^2 - x^2 + 3y(2+x) - a(6+3y+x)),\ x < y, \end{cases} \tag{2.6}$$

It is clear that the kernel function $K_x(y)$ is symetric that is $K_x(y) = K_y(x),\ \forall x,\ and\, y$, and $K_x(y) \ge 0$, for any fixed $x \in [a,b]$. Moreover, in case we are redefining the inner product in (2.5) as follow

$$\langle u, v \rangle = u(a)v(a) + u(b)v(b) + \int_a^b u''(t)v''(t)\, dt,\ u,\ v \in \mathscr{W}_2^2[a,b]$$

then the space $\mathscr{W}_2^2[a,b]$ is a RKHS, and its reproducing kernel is given by

$$R_x(y) = \begin{cases} \begin{pmatrix} -2a^3(b-x)(b-y)+a^2(6+2b^3+x^3+3xy^2 \\ -3b(x^2+y^2))+y(-3b^2x^2+bx^3-b^2y^2+x \\ (6+2b^3+by^2))-a((-3bx^2+x^3)(b+y)+y \\ (6+2b^3-3b^2y-by^2)+x(6+2b^3+3by^2+y^3)) \end{pmatrix} & , \quad y \leq x \\ \\ \begin{pmatrix} -2a^3(b-y)(b-x)+a^2(6+2b^3+y^3+3yx^2 \\ -3b(y^2+x^2))+x(-3b^2y^2+by^3-b^2x^2+y \\ (6+2b^3+bx^2))-a((-3by^2+y^3)(b+x)+x \\ (6+2b^3-3b^2x-bx^2)+y(6+2b^3+3bx^2+x^3)) \end{pmatrix} & , \quad y > x \end{cases}$$

2.1.1 The Analytical Solution and Theoretical Basis

The solution of equation (2.4) is given in the space $\mathscr{W}_2^2[a,b]$, we define a differential operator $\mathcal{L} : \mathscr{W}_2^2 \longrightarrow \mathscr{W}_2^1$ *such that* $\mathcal{L}u(x) = u'(x)$. After homogenization of the initial condition of Equation (2.4), the IDE (2.4) can be converted into the equivalent form as follows:

$$\begin{cases} \mathcal{L}u(x) & = F(x, u(x), \mathcal{T}u(x)); \\ u(a) & = 0, \end{cases} \tag{2.7}$$

It is obvious that the operator L is bounded, By reproducing property of $K_x(y)$ and Schwarz inequality, we obtain

$$\begin{aligned} |u(x)| &= |\langle u(y), K_x(y)\rangle|_{\mathscr{W}_2^2} =\leq \|K_x(y)\|_{\mathscr{W}_2^2} \|u(y)\|_{\mathscr{W}_2^2} \leq M_0 \|u(y)\|_{\mathscr{W}_2^2} \\ |u'(x)| &= |\langle u(y), K'_x(y)\rangle|_{\mathscr{W}_2^2} =\leq \|K'_x(y)\|_{\mathscr{W}_2^2} \|u(y)\|_{\mathscr{W}_2^2} \leq M_1 \|u(y)\|_{\mathscr{W}_2^2} \end{aligned}$$

Since $K_x^{(i)}(y),\ i=1,\ 2$ is uniformly bounded about x and y, we have $\left|u^{(i)}(x)\right| \leq M_i \|u(y)\|_{\mathscr{W}_2^2}$. Hence $\|\mathcal{L}u(x)\|^2_{\mathscr{W}_2^1} = \|u'(x)\|^2_{\mathscr{W}_2^1} = \int_a^b (u'(t))^2 + (u''(t))^2 dt \leq M \|u(x)\|^2_{\mathscr{W}_2^2}$, where $M = (b-a)(M_1^2 + M_2^2)$.

In order to solve the equation (2.4), we applying the steps of algorithm 1.

Theorem 2.1.1. *If $\{x_i\}_{i=1}^{\infty}$ is dense on $[a,b]$, then for IDE (2.4)*

1. *the exact solution of Equation (2.4) could be represented by*

$$u(x) = \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} F\left(x_k, u(x_k), \mathcal{T}u(x_k)\right)\ \overline{\psi}_i(x),$$

2. *the approximate solution:*

$$u_n(x) = \sum_{i=1}^{n} \sum_{k=1}^{i} \beta_{ik} F\left(x_k, u(x_k), \mathcal{T}u(x_k)\right)\ \overline{\psi}_i(x),$$

and $u_n'(x)$ are converging uniformly to the exact solution $u(x)$ and its derivative $u'(x)$ respectively, as $n \to \infty$:

Proof : See (5) □

Theorem 2.1.2. *Assume $u(x)$ is the solution of IDE (2.4) and r_n is the approximate error of $u_n(x)$. Then the error $r_n(x)$ is monotone decreasing in sense of the norm of $\mathscr{W}_2^2[a,b]$.*

Proof : See (5) □

2.1.2 Numerical Examples

Some numerical examples are examined to show the accuracy of the present approach and to provide a clear overview of it. When the method's results are compared to each example's analytical solution, they are found to be in good agreement with each other.

Example 2.1.1. *Consider the following linear Volterra IDE*

$$u'(x) = 1 - \int_0^x u(t)\,dt,\ 0 \le x \le 1$$
$$u(0) = 0$$

The exact solution is $u(x) = \sin x$*. By using the RKHSM, and taking* $x_i = \dfrac{i-1}{n-1}$, $i = 1, \cdots, n$ *with the reproducing kernel* $\mathrm{k}_x(y)$ *on* $[0, 1]$*, for* $n = 51$*, and* $n = 101$*, we get the following results:*

Table 2.1: Numerical results for Example 2.1.1.

x	Exact Solution	Absolute Error for $n = 51$	Absolute Error for $n = 101$
0.	0	0	0
0.1	0.099833	3.3278×10^{-6}	8.31947×10^{-7}
0.2	0.198669	6.62236×10^{-6}	1.65558×10^{-6}
0.3	0.295520	9.85074×10^{-6}	2.46267×10^{-6}
0.4	0.389418	1.29807×10^{-5}	3.24516×10^{-6}
0.5	0.479426	1.5981×10^{-5}	3.99522×10^{-6}
0.6	0.564642	1.88215×10^{-5}	4.70536×10^{-6}
0.7	0.644218	2.14741×10^{-5}	5.36849×10^{-6}
0.8	0.717356	2.3912×10^{-5}	5.97798×10^{-6}
0.9	0.783327	2.61111×10^{-5}	6.52774×10^{-6}
1.	0.841471	2.80492×10^{-5}	7.01227×10^{-6}

Example 2.1.2. *Consider the nonlinear Volterra IDE*

$$u'(x) = -1 + \int_0^x u^2(t)\,dt,\ 0 \le x \le 1$$
$$u(0) = 0$$

The exact solution is $u(x) = \dfrac{\frac{1}{28}x^4 - x}{\frac{1}{21}x^3 + 1}$*, using RKHSM, taking* $x_i = \dfrac{i-1}{n-1}$, $i = 1, \cdots, n$*, we get the following results for* $n = 51$

Table 2.2: Numerical results for Example 2.1.2.

x	Exact Solution	Aprroximate solution	Absolute Error	Relative Error
0.	0	0	0	Indeterminate
0.1	−0.0999917	−0.0999913	3.33242×10^{-7}	-3.3327×10^{-6}
0.2	−0.199867	−0.199865	1.33047×10^{-6}	-6.6568×10^{-6}
0.3	−0.299326	−0.299323	2.97841×10^{-6}	-9.95038×10^{-6}
0.4	−0.397873	−0.397868	5.24281×10^{-5}	-1.31771×10^{-5}
0.5	−0.494822	−0.494814	8.0591×10^{-6}	-1.62868×10^{-5}
0.6	−0.58931	−0.589299	1.13246×10^{-5}	-1.92167×10^{-5}
0.7	−0.680313	−0.680298	1.48938×10^{-5}	-2.18926×10^{-5}
0.8	−0.766679	−0.76666	1.85778×10^{-5}	-2.42315×10^{-5}
0.9	−0.847159	−0.847137	2.21478×10^{-5}	-2.61436×10^{-5}
1.	−0.920455	−0.920429	2.53429×10^{-5}	-2.7533×10^{-5}

Example 2.1.3. *Consider the nonlinear Volterra IDE*

$$u'(x) = 2\sin x\cos x - \int_0^x 3\cos(x-t)u^2(t)\,dt,\ 0 \le x \le 1$$
$$u(0) = 1$$

The exact solution is $u(x) = \cos(x)$. *Using RKHS method, taking* $x_i = \dfrac{i-1}{n-1}$, $i = 1, \cdots, n$, *the following table shows the numerical results for n = 11 and we compare its with results obtained by using other methods*

Table 2.3: Numerical results for Example 2.1.3.

x	Exact Sol	RKHSM	Method (15)	BPFs method (38)	Adomian's M (39)
0.	1.000000	1.000000	1.000000	1.000000	1.000000
0.1	0.995004	0.995058	0.994555	0.995141	0.994951
0.2	0.980067	0.98028	0.979825	0.975784	0.980303
0.3	0.955336	0.955796	0.955174	0.960386	0.955685
0.4	0.921061	0.921795	0.920861	0.918443	0.921165
0.5	0.877583	0.878479	0.877921	0.862193	0.877048
0.6	0.825336	0.826005	0.825397	0.828963	0.822596
0.7	0.764842	0.764416	0.765164	0.752929	0.755333
0.8	0.696707	0.693569	0.697142	0.710418	0.667739
0.9	0.621610	0.613055	0.622057	0.617232	0.547241
1.	0.540302	0.522129	0.541102	0.566917	0.364798

2.2 Fredholm Integro-Differential Equation

In this part, the replicating kernel approach is used to analyze the numerical solutions to the first-order Fredholm IDEs. It's a somewhat recent analytical method. The analytical solution $u(x)$ and approximate solution $u_n(x)$ are represented in the form of series in $\mathscr{W}_2^2[a,b]$.

We consider the Fredholm IDE of the following form:

$$\begin{cases} u'(x) = F(x, u(x), \mathcal{T}u(x)),\ a \le x \le b; \\ u(a) = u_0, \end{cases} \tag{2.8}$$

such that $\mathcal{T}u(x) = \int_a^b h(x,t)u(t)dt$, a, b and u_0 are real finite constants, $h(x,t)$ is known continuous function, $u(x)$ is unknown function to be determined and $F(x,t,u(t),\mathcal{T}u(x))$ is a linear or nonlinear function depending on the problem discussed. We assume that IDE (2.8) have a unique solution.

We also construct the reproducing kernel spaces $\mathscr{W}_2^1[a,b]$ and $\mathscr{W}_2^2[a,b]$ in a similar way. The IDE (2.8), once the starting condition has been homogenized, may be transformed into the corresponding form:

$$\begin{cases} \mathcal{L}u(x) = F\left(x, u\left(x\right), \mathcal{T}u\left(x\right)\right), \ a \leq x \leq b; \\ u(a) = 0. \end{cases} \tag{2.9}$$

where $x \in [a,b]\,, u\left(x\right) \in \mathscr{W}_2^2[a,b]$ and $F\left(x, u\left(x\right), \mathcal{T}u\left(x\right)\right) \in \mathscr{W}_2^1[a,b]$.

Consider the bounded linear operator $\mathcal{L} : \mathscr{W}_2^2[a,b] \longrightarrow \mathscr{W}_2^1[a,b]$ such that $\mathcal{L}u\left(x\right) = u'\left(x\right)$. let $\left\{\overline{\psi}_i\left(x\right)\right\}_{i=1}^{\infty}$ be a complete orthonormal system in $\mathscr{W}_2^2[a,b]$ and it's given by $\overline{\psi}_i\left(x\right) = \sum_{k=1}^{i} \beta_{ik}\psi_k\left(x\right),$ where β_{ik} are the coefficients of the Gram-Schmidt orthonormalization.

The analytical solution $u\left(x\right)$ and approximate solution $u_n\left(x\right)$ are represented in the form of series in $\mathscr{W}_2^2[a,b]$. That means, the analytical solution $u\left(x\right)$ and approximate solution $u_n\left(x\right)$ are given respectively by :

$$u\left(x\right) = \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} \left[F\left(x_k, u\left(x_k\right), \mathcal{T}u\left(x_k\right)\right)\right] \overline{\psi}_i\left(x\right).$$

$$u\left(x\right) = \sum_{i=1}^{n} \sum_{k=1}^{i} \beta_{ik} \left[F\left(x_k, u\left(x_k\right), \mathcal{T}u\left(x_k\right)\right)\right] \overline{\psi}_i\left(x\right).$$

Moreover, Theorem (2.1.1) hold. Therefore, if Equation (2.8) is nonlinear, the approximate solution $u_n^N(x)$ can be obtained by taking finitely many terms in the series representation of $u_n(x)$, using the iterative method, and

$$u_n^N(x) = \sum_{i=1}^{n}\sum_{k=1}^{i} \beta_{ik} F(x_k, u_{n-1}(x_k), \mathcal{T}u_{n-1}(x_k))\overline{\psi_i}(x). \tag{2.10}$$

2.2.1 Numerical Experiment

Several numerical examples are provided in this part to demonstrate the method's excellent accuracy. The outcomes of the cases demonstrate how helpful our technique is for computing the numerical solution of a nonlinear IDE.

Example 2.2.1. *Consider the following nonlinear Fredholm IDE*

$$\begin{aligned} u'(x) &= 1 - \frac{1}{3}x^3 + \int_0^1 x^3 u^2(t)\ dt,\ 0 \leq x \leq 1 \\ u(0) &= 0 \end{aligned}$$

The exact solution is $u(x) = x$, *by using RKHS method, and taking* $x_i = \dfrac{i-1}{n-1}$, $i = 1, \cdots n$, *we get the following results for* $n = 11$

Table 2.4: Numerical results for Example 2.2.1.

x	Exact Solution	Aprroximate solution	Absolute Error	Relative Error
0.	0	0	0	Indeterminate
0.1	0.1	0.1	8.27199×10^{-10}	8.27199×10^{-9}
0.2	0.2	0.2	8.27199×10^{-9}	4.13599×10^{-8}
0.3	0.3	0.3	3.72239×10^{-8}	1.2408×10^{-7}
0.4	0.4	0.4	1.12499×10^{-7}	2.81248×10^{-7}
0.5	0.5	0.5	2.6884×10^{-7}	5.37679×10^{-7}
0.6	0.6	0.599999	5.50914×10^{-7}	9.18191×10^{-7}
0.7	0.7	0.699999	1.01332×10^{-6}	1.4476×10^{-6}
0.8	0.8	0.799998	1.72057×10^{-6}	2.15072×10^{-6}
0.9	0.9	0.899997	2.74713×10^{-6}	3.05236×10^{-6}
1.	1	0.999996	4.17735×10^{-6}	4.17735×10^{-6}

Example 2.2.2. *Consider the linear Fredholm IDE*

$$\begin{aligned} u'(x) &= xe^x + e^x - x + \int_0^1 xu(t)\ dt,\ 0 \leq x \leq 1 \\ u(0) &= 0 \end{aligned}$$

The exact solution is $u(x) = xe^x$, *by using RKHS method, and taking* $x_i = \dfrac{i-1}{n-1}$, $i = 1, \cdots n$, *we get the following results for* $n = 11$ *and we compare its with results obtained by using other methods*

Table 2.5: Numerical results for Example 2.2.2.

x	Exact Sol	Appr sol	RKHSM	Method in (32)	Method in (42)
0.	0.000000	0.000000	0.000000	0.000000	0.000000
0.1	0.110517	0.110366	1.50879×10^{-4}	1.001183×10^{-2}	$1.34917637 \times 10^{-3}$
0.2	0.244281	0.244068	2.12891×10^{-4}	2.786514×10^{-2}	$1.15960044 \times 10^{-3}$
0.3	0.404958	0.404777	1.8104×10^{-4}	5.087309×10^{-2}	$5.67152531 \times 10^{-3}$
0.4	0.59673	0.59668	4.97023×10^{-5}	7.553563×10^{-2}	$5.93105645 \times 10^{-2}$
0.5	0.824361	0.824548	1.8746×10^{-4}	9.718886×10^{-2}	$1.32330751 \times 10^{-2}$
0.6	1.09327	1.09381	5.37579×10^{-4}	1.095517×10^{-1}	$4.39287720 \times 10^{-2}$
0.7	1.40963	1.41064	1.00868×10^{-3}	1.041332×10^{-1}	$1.41201624 \times 10^{-2}$
0.8	1.78043	1.78204	1.6098×10^{-3}	0.945127×10^{-2}	$1.34514117 \times 10^{-2}$
0.9	2.21364	2.21599	2.3511×10^{-3}	1.000343×10^{-2}	$1.32045209 \times 10^{-2}$
1.	2.71828	2.72153	3.244×10^{-3}	1.551477×10^{-1}	$---$

Conclusion

Analytically solving the majority of IDEs is generally difficult. Obtaining the approximate solutions is often necessary. Our aim is to create a new RKHS and provide a method for expressing the reproducing kernel function $K_x(y)$. by adding the initial and BC's to the space $\mathscr{W}_2^m[a,b]$, we also utilized a reproducing kernel and its conjugate operator to build the complete orthonormal basis in that space.

The ability to build a global approximation over the whole solution domain and uniform convergence are the primary characteristics of the RKHS approach. On the basis of this, a novel numerical approach is introduced and solved analytically for a class of IDEs in the space $\mathscr{W}_2^m[a,b]$. The analytical solution $u(x)$ and approximate solution $u_n(x)$ are represented in the form of series in the space $\mathscr{W}_2^m[a,b]$. Moreover, the approximate solution and its derivatives converge uniformly to the exact solution and its derivatives, respectively. Meanwhile, the error of the approximate solution is monotone decreasing in the sense of the norm of $\mathscr{W}_2^m[a,b]$.

The experimental findings demonstrate that a high degree of precision can only be achieved by using a small number of repetition steps. As such, the current approach represents a precise and trustworthy analytical method for first IDEs.

As a follow-up to our study, we plan to do more research and investigations in the near future with the goal of solving mixed Fredholm-Volterra IDEs. A few of these studies are highlighted here: second, third and fourth order of integro-differential equations, mixed integro-differential equations, system of first and second-order IDEs, integro-differential equations of fractional order.

References

[1] Abu Arqub, O. (2017), Fitted reproducing kernel Hilbert space method for the solutions of some certain classes of time-fractional partial differential equations subject to initial and Neumann boundary conditions. **Computers and Mathematics with Applications**, 73(6), 1243-1261.

[2] AlHabees, A., Maayah, B., Bushnaq, S. (2016), Solving fractional proportional delay integro differential equations of first order by reproducing kernel hilbert space method. **Global Journal of Pure and Applied Mathematics**, 12(4), 3499-3516.

[3] Alpay D., Shapiro M., and Volok D. (2005), Reproducing Kernel Spaces of Series of Fueter Polynomials. **Advances and Applications**, 162, 19-45.

[4] Alpay D. (2003), Reproducing Kernel Spaces and Applications. **1.st edition**, Birkhuser Basel, Switzerland.

[5] Al-Smadi, M. (2011), Numerical solution of integro-differential equations using reproducing kernel Hilbert space method. Ph.D. Thesis, university of Jordan, Amman, Jordan.

[6] Al-Smadi, M. Freihat, A. Khalil, H. Momani, S. Khan, R. A. (2017), Numerical Multistep Approach for Solving Fractional Partial Differential Equations. **Inter- national Journal of Computational Methods**, 14(3).

[7] Al-Smadi, M. Abu Arqub, O. Momani, S. (2013), A computational method for twopoint boundary value problems of fourth-order mixed integrodifferential equations. **Mathematical Problems in Engineering**, 2013.

[8] Altawallbeh, Z. Al-Smadi, M. Komashynska, I. Ateiwi, A. (2018), Numerical Solutions of Fractional Systems of Two-Point BVPs by Using the Iterative Reproducing Kernel Algorithm. **Ukrainian Mathematical Journal**, 70(5), 687-701.

[9] Arikoglu A., and Ozkol I. (2005), Solution of boundary value problems for integro differential equations by using differential transform method. **Applied Mathematics and Computation**, 168, 1145-1158.

[10] Arnoszajn N. (1950), Theory of reproducing kernels. **Trans. Amer. Math. Soc**, 68, 337-404.

[11] Arqub, O. A. Al-Smadi, M. Shawagfeh, N. (2013), Solving Fredholm integrodifferential equations using reproducing kernel Hilbert space method. **Applied Mathematics and Computation**, 219(17), 8938-8948.

[12] Arqub, O. A. Maayah, B. (2018), Numerical solutions of integrodifferential equations of Fredholm operator type in the sense of the Atangana-Baleanu fractional operator. **Chaos, Solitons and Fractals**, 117, 117-124.

[13] Avudainayagam A., and Vani C. (2000), Wavelet-Galerkin method for integro-differential equations, **Applied Numerical Mathematics**, 32, 247-254.

[14] Babolian E., Masouri Z., and Hatamzadeh-Varmazyar S. (2009), Numerical solution of nonlinear Volterra Fredholm integro-differential equations via direct method using triangular functions, **Computers and Mathematics with Applications**, 58, 239-247.

[15] Babolian E., Masouri Z., and Hatamzadeh-Varmazyar S. (2008), New direct

method to solve nonlinear Volterra Fredholm integral and integro differential equation using operational matrix with Block-pulse functions, **Progress In Electromagnetics Research B 8**, 59-76.

[16] Bahuguna D., Ujlayan A., and Pandey D.N. (2009), A comparative study of numerical methods for solving an integro-differential equation, **Computers and Mathematics with Applications**, 57, 1485-1493.

[17] Berlinet, A. Thomas-Agnan, C. (2004), Reproducing Kernel Hilbert Spaces in Probability and Statistics. https://doi.org/10.1007/978-1-4419-9096-9

[18] Beyrami, H. Lot, T. Mahdiani, K. (2017), Stability and error analysis of the reproducing kernel Hilbert space method for the solution of weakly singular Volterra integral equation on graded mesh. **Applied Numerical Mathematics**, 120, 197-214.

[19] Bushnaq, S, Maayah, B. Ahmad, M. (2016), Reproducing kernel hilbert space method for solving fredholm integro-differential equations of fractional order. **Italian Journal of Pure and Applied Mathematics**, 36, 307-318.

[20] Bushnaq, S. Maayah, B. AlHabees, A. (2016), Application of multistep reproducing kernel Hilbert space method for solving giving up smoking model. **International Journal of Pure and Applied Mathematics**, 109(2), 311-324.

[21] Chellouf, Y., Maayah, B., Momani, S., Alawneh, A., Alnabulsi, S. (2021). Numerical solution of fractional differential equations with temporal two-point bvps using reproducing kernel Hilbert space method. **AIMS Mathematics**, 6(4), 3465-3485.

[22] Cui M.G., and Geng F.Z. (2007), Solving singular two-point boundary value problem in reproducing kernel space, **Journal of Computational and Applied Mathematics**, 205 , 6-15.

[23] Cui M.G., and Li Ch.L. (2003), The exact solution for solving a class non- linear operator equations in the reproducing kernel space, **Applied Math- ematics and Computation**,143, 393-399.

[24] Cui M.G., and Du H.G. (2006), Representation of exact solution for the nonlinear Volterra-Fredholm integral equations, **Applied Mathematics and Computation**, 182 (2), 1795-1802.

[25] Danciu E. (2009), On Applications of the reproducing kernel method for construction of cubature formulas, Studia University. .Babes-Bolyai., Math- ematica, LIV (4).

[26] Darania P., and Ebadian A. (2007), A method for the numerical solution of the integro-differential equations, **Applied Mathematics and Computation**, 188, 657-668.

[27] Darania P., and Ebadian A. (2006), Development of the Taylor expansion approach for nonlinear integro-differential equations, **Int. J. Contemp. Math. Sciences**, 1 (14), 651-664.

[28] Du H.G., and Cui M.G. (2006), Representation of the exact solution and a stability analysis on the Fredholm integral equation of the first kind in reproducing kernel space, **Applied Mathematics and Computation**, 182 (2), 1608-1614.

[29] El-Sayed S. M., and Abdel-Aziz M.R. (2003), A comparison of Adomian.s de-

composition method and Wavelet-Galerkin method for integro-differential equations, **Applied Mathematics and Computation**, 136,151-159.

[30] Geng F.Z. (2009), Solving singular second order three-point boundary value problems using reproducing kernel Hilbert space method, **Applied Mathematics and Computation**, 215, 2095-2102.

[31] Geng F.Z., and Cui M.G. (2007), Solving singular nonlinear second-order periodic boundary value problems in the reproducing kernel space, **Applied Mathematics and Computation**, 192, 389-398.

[32] Ghasemi M., Tavassoli Kajani M., and Babolian E. (2007), Application of He's homotopy perturbation method to nonlinear integro-differential equations, **Applied Mathematics and Computation**, 188, 538-548.

[33] Hashim I. (2006), Adomian decomposition method for solving BVPs for fourth-order integro-differential equations, **Journal of Computational and Applied Mathematics**, 193, 658-664.

[34] Hofmann, T. Scholkopf, B. Smola, A. J. (2005), A Tutorial Review of RKHS Methods in Machine Learning. **Machine Learning, January**, 2006, 1-59.

[35] Javan, S. F. Abbasbandy, S. Araghi, M. A. F. (2017), Application of Reproducing Kernel Hilbert Space Method for Solving a Class of Nonlinear Integral Equations. **Mathematical Problems in Engineering**, 2017.

[36] Koppel, A. Zhang, K. Zhu, H. Basar, T. (2019), Projected stochastic primal-dual method for constrained online learning with kernels. **IEEE Transactions on Signal Processing**, 67(10), 2528-2542.

[37] Kreyszig, E. (1978), **Introductory functional analysis**. New York: John Wiley and Sons.

[38] Maleknejad K., and Mahmoudi Y. (2003), Taylor polynomial solution of high-order nonlinear Volterra Fredholm integro-differential equations, **Applied Mathematics and Computation**, 145, 641-653.

[39] Maleknejad K., and Hadizadeh M. (1997), The numerical analysis of Adomian's decomposition method for nonlinear Volterra integral and integro-differential equations, **International Journal of Engineering Science** , 33-48.

[40] Moaddy, K. Freihat, A. Al-Smadi, M. Abuteen, E. Hashim, I. (2018), Numerical investigation for handling fractional-order Rabinovich-Fabrikant model using the multistep approach. **Soft Computing**, 22(3), 773-782.

[41] Sunder, V. S. (2008), The Riesz representation theorem. **Indian Journal of Pure and Applied Mathematics**, 39(6), 467-481.

[42] Wazwaz A. M. (2006), A comparison study between the modified decomposition method and the traditional methods for solving nonlinear integral equations, **Applied Mathematics and Computation**, 181, 1703-1712.

[43] Zhao, Z., Lin, Y., Niu, J. (2016), Convergence order of the reproducing kernel method for solving boundary value problems. **Mathematical Modelling and Analysis**, 21(4), 466 477.