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Neutral differential equations including double delays

Préparé par : DAHMANI Wafa

Soutenue devant le jury

LAIB Hafida	**MCA**	**C. U. Abdelhafid Boussouf, Mila**	**Président**
BOUZERAIEB Hayet	**MAA**	**C. U. Abdelhafid Boussouf, Mila**	**Rapporteur**
CHELLOUF Yassamine	**MCB**	**C. U. Abdelhafid Boussouf, Mila**	**Examinateur**

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DEDICATION

In the name of the Most Merciful, ALLAH, I offer this modest work as a dedication to:

- To my parents.
- To my sisters.
- To everyone who encouraged me.

Wafa

ملخص

في هذه المذكرة، تم دراسة معادلات تفاضلية خطية من الدرجة الاولى والثانية مع تاخير ثابت او تاخيرين. تم تطبيق طريقة التجميع لتايلور لإعطاء الحل التقريبي لمعادلات تفاضلية خطية حيادية من الدرجة الثانية مع تاخير ثابت وكذلك الحل التقريبي لمعادلات تفاضلية حيادية من الدرجة الاولى بتأخيرين ثابتين، كما تم دراسة تقارب الحل التقريبي للحل الدقيق باستخدام متراجحات غرونوال و إثبات أن الطريقة لها رتبة تقارب. في الأخير تم إدراج أمثلة عددية لتأكيد النتائج النظرية وتقارب الخوارزمية المقدمة.

الكلمات المفتاحية: معادلات تفاضلية خطية حيادية ذات تأخير، طريقة التجميع، كثيرات الحدود تايلور.

ABSTRACT

In this thesis, first and second-order linear differential equations with constant delay or two delays were studied. The collocation method using Taylor series was applied to provide approximate solutions for second-order neutral linear differential equations with constant delay as well as for first-order neutral differential equations with two constant delays. Additionally, the convergence of the approximate solution to the exact solution was studied using Gronwall's inequalities, and it was proven that the method has a convergence order. Finally, numerical examples were included to confirm the theoretical results and the convergence of the proposed algorithm.

Key Words: Neutral delay linear differential equations, collocation method, Taylor polynomials.

RÉSUMÉ

Dans ce mémoire, des équations différentielles linéaires du premier et du second ordre avec retard constant ou deux retards ont été étudiées. La méthode de collocation utilisant les séries de Taylor a été appliquée pour fournir des solutions approximatives aux équations différentielles linéaires neutres du second ordre avec retard constant ainsi qu'aux équations différentielles neutres du premier ordre avec deux retards constants. De plus, la convergence de la solution approximative vers la solution exacte a été étudiée en utilisant les inégalités de Gronwall, et il a été prouvé que la méthode possède un ordre de convergence. Enfin, des exemples numériques ont été inclus pour confirmer les résultats théoriques et la convergence de l'algorithme proposé.

Mots-clés: : Equations différentielles linéaires neutres à retard; Méthode de collocation; Polynômes de Taylor.

CONTENTS

INTRODUCTION

Differential equations are essential for modeling various natural and engineering phenomena. They articulate the relationship between functions and their derivatives, capturing the dynamics of systems over time. Linear differential equations are particularly important due to their broad applicability and the relative simplicity of their solutions.

This thesis delves into studying first- and second-order linear differential equations with delays. Delayed differential equations incorporating terms dependent on past states are crucial for modeling real-world systems where time delays are inherent, such as in population dynamics, control systems, and physiological processes.

The primary goal of this research is to develop and analyze methods for obtaining approximate solutions to these delayed differential equations. Specifically, the collocation method using Taylor series is employed to derive approximate solutions for second-order neutral linear differential equations with a constant delay and first-order neutral differential equations with two constant delays.

To ensure the reliability of the proposed methods, the convergence of the approximate solutions to the exact solutions is rigorously examined using Gronwall's inequalities. This study demonstrates that the methods possess a convergence order, thereby validating their effectiveness in solving delayed differential equations.

Numerical examples not only support theoretical findings but also demonstrate the practical applicability of the proposed algorithms. These examples confirm the theoretical results, illustrating the accuracy and convergence of the methods developed in this thesis, and underline their potential to be directly applied in real-world scenarios.

Through this research, we aim to contribute to differential equations by providing robust techniques for solving delayed differential equations, thereby enhancing the toolkit available to researchers and practitioners in time-delay systems.

The collocation method aims to approximate the exact solution of a differential equation by employing a suitable function from a chosen finite-dimensional space. The approximate solution must satisfy the differential equation at specific points on the interval, known as collocation points.

Key advantages of this method include:

- It is a direct method providing explicit formulas for the approximate solution. - The method has a convergence order. - Solving an algebraic system is unnecessary, making the proposed algorithm highly effective and easy to implement.

The dissertation is structured as follows:

- **Chapter One**: This chapter covers fundamental notions, definitions, and necessary theorems that will be utilized in the subsequent chapters.
- **Chapter Two**: Here, we use the Taylor collocation method to establish the numerical solution of neutral linear differential equations with a constant delay.The convergence of the approximate solution to the exact solution is proven. Numerical examples support theoretical results.
- **Chapter Three**: We present the Taylor collocation method based on Taylor polynomials. We construct a collocation solution in a piecewise polynomial spline space for first-order linear differential equations with two constant delays.

CHAPTER 1

GENERALS AND NOTIONS FUNDAMENTALS

1.1 Taylor series

The Taylor series is a representation of a function as an infinite sum of terms calculated from the values of its derivatives at a single point. If a function f is infinitely differentiable at a point a, the Taylor series of f at a is given by:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Here:
- $f^{(n)}(a)$ is the n-th derivative of f evaluated at the point a.
- $n!$ (n factorial) is the product of all positive integers up to n.
- $(x-a)^n$ is the n-th power of $(x-a)$.

If the Taylor series is centered at $a = 0$, it is also called a Maclaurin series:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n$$

Examples of the Taylor series

1. Taylor series for e^x

The exponential function e^x has the same value for all its derivatives, i.e., $f^{(n)}(x) = e^x$. Therefore, the Taylor series for e^x centered at $x = 0$ (Maclaurin series) is:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

This series converges for all x.

2. Taylor series for $\sin(x)$

The sine function $\sin(x)$ has derivatives that cycle every four terms: $\sin(x)$, $\cos(x)$, $-\sin(x)$, $-\cos(x)$. The Maclaurin series for $\sin(x)$ is:

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

This series converges for all x.

3. Taylor series for $\cos(x)$

Similarly, the cosine function $\cos(x)$ has derivatives that cycle every four terms: $\cos(x)$, $-\sin(x)$, $-\cos(x)$, $\sin(x)$. The Maclaurin series for $\cos(x)$ is:

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

This series also converges for all x.

4. Taylor series for $\ln(1 + x)$

The natural logarithm function $\ln(1 + x)$ can be represented by a Taylor series centered at $x = 0$ (Maclaurin series) for $-1 < x \leq 1$:

$$\ln(1 + x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n$$

This series converges for $|x| < 1$.

1.2 Differential equations

A differential equation is a mathematical equation that relates a function with its derivatives. Differential equations describe the rate of change of a quantity and are used to model various physical, biological, and engineering systems. There are several types of differential equations, including ordinary differential equations (ODEs) and partial differential equations (PDEs).

Ordinary differential equations (ODEs)

An ordinary differential equation (ODE) involves functions of a single variable and their derivatives. It can be expressed in the general form:

$$F\left(x, y, y', y'', \ldots, y^{(n)}\right) = 0$$

where $y = y(x)$ is the unknown function, $y', y'', \ldots, y^{(n)}$ are the derivatives of y with respect to x, and F is a given function.

Partial differential equations (PDEs)

A partial differential equation (PDE) involves functions of multiple variables and their partial derivatives. It can be expressed in the general form:

$$F\left(x_1, x_2, \ldots, x_m, u, \frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \ldots, \frac{\partial u}{\partial x_m}, \frac{\partial^2 u}{\partial x_1^2}, \frac{\partial^2 u}{\partial x_1 \partial x_2}, \ldots\right) = 0$$

where $u = u(x_1, x_2, \ldots, x_m)$ is the unknown function, and $\frac{\partial u}{\partial x_i}$ and higher-order partial derivatives are the partial derivatives of u with respect to the variables $x_1, x_2, \ldots, x_m$.

Examples

Example 1: First-Order ODE

A simple first-order ordinary differential equation is given by:

$$\frac{dy}{dx} + y = 0$$

The solution to this equation is:

$$y(x) = Ce^{-x}$$

where C is a constant determined by initial conditions.

Example 2: Second-Order ODE

A second-order ordinary differential equation is given by:

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$$

The general solution to this equation is:

$$y(x) = C_1e^{x} + C_2e^{2x}$$

where C_1 and C_2 are constants determined by initial conditions.

Example 3:

A second-order ordinary differential equation is given by :

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 2e^{3x}$$

the particular solution can have the form:

$$y_p(x) = Ae^{3x}$$

The general solution to this equation is:

$$y(x) = C_1e^x + C_2e^{2x} + \frac{1}{2}e^{3x}$$

Example 4: Partial differential equation

A common partial differential equation is the heat equation:

$$\frac{\partial u}{\partial t} = \alpha\frac{\partial^2 u}{\partial x^2}$$

where $u = u(x,t)$ represents the temperature distribution over time, and α is the thermal diffusivity constant.

1.3 Differential equations with delay

Differential equations with delay, also known as delay differential equations (DDEs), are a type of differential equation in which the derivative of the unknown function at a certain time is given in terms of the values of the function at previous times. These equations are particularly useful in modeling real-world systems where time delays are inherent, such as population dynamics, control systems, and physiological processes.

General form of delay differential equations

A general first-order delay differential equation can be written as:

$$\frac{dy(t)}{dt} = f\left(t, y(t), y(t-\tau_1), y(t-\tau_2), \ldots, y(t-\tau_n)\right)$$

where:

- $y(t)$ is the unknown function.
- $\tau_1, \tau_2, \ldots, \tau_n$ are the delay times.
- f is a given function that describes the relationship between the current state and the delayed states.

1.3.1 Types of delay differential equations

1. Constant delay

In this type, the delays $\tau_1, \tau_2, \ldots, \tau_n$ are constants. An example is:

$$\frac{dy(t)}{dt} = ay(t) + by(t-\tau)$$

where a and b are constants, and τ is a constant delay.

2. Time-varying delay

In this type, the delays $\tau_1, \tau_2, \ldots, \tau_n$ are functions of time. An example is:

$$\frac{dy(t)}{dt} = ay(t) + by(t-\tau(t))$$

where $\tau(t)$ is a time-dependent delay.

3. Neutral delay differential equations

In this type, the delays appear in the derivative terms. An example is:

$$\frac{dy(t)}{dt} = ay(t) + by'(t - \tau)$$

where $y'(t - \tau)$ is the derivative of y at $t - \tau$.

1.3.2 Application of delay differential equations

Example 1: Population dynamics

A simple model of population dynamics with a constant delay is given by:

$$\frac{dP(t)}{dt} = rP(t)\left(1 - \frac{P(t - \tau)}{K}\right)$$

where:

- $P(t)$ is the population at time t.
- r is the growth rate.
- K is the carrying capacity.
- τ is the delay representing the time taken for the population to react to changes in size.

Example 2: Control systems

A simple control system with a constant delay can be described by:

$$\frac{dx(t)}{dt} = -kx(t) + u(t - \tau)$$

where:

- $x(t)$ is the state of the system at time t.
- k is a constant.
- $u(t)$ is the control input.
- τ is the delay in the control input.

1.4 Piecewise polynomial spaces

Definition 1.4.1 *For a given mesh Π_N the piecewise polynomial space $S_\mu^{(d)}(\Pi_N)$ with $\mu \geq 0, -1 \leq d \leq \mu$, is given by*

$$S_\mu^{(d)}(\Pi_N) = \{v \in C^d(\Pi) : v|_{\sigma_n} \in \pi_\mu(0 \leq n \leq N-1)\}.$$

Here, Π_μ denotes the space of (real) polynomials of degree not exceeding μ.
It is readily verified that $S_\mu^{(d)}(\Pi_N)$ is a (real) linear vector space whose dimension is given by

$$dim S_\mu^{(d)}(\Pi_N) = N(\mu - d) + d + 1.$$

Piecewise polynomial spaces $S_m^{(1)}(\Pi_N)$

We suppose that $T = r\tau$, where $r \in \{1, 2, 3, ...\}$. Let Π_N be a uniform partition of the interval $I = [0, T]$ defined by $t_n^i = i\tau + nh, n = 0, 1, .., N,\ i = 0, 1, .., r-1$, where the stepsize is given by $h = \frac{\tau}{N}$. Define the subintervals $\sigma_n^i = [t_n^i; t_{n+1}^i), n = 0, 1, .., N-1,\ i = 0, 1, .., r-1$ and $\sigma_{N-1}^{r-1} = [t_{N-1}^{r-1}, t_N^{r-1}]$. Moreover, denote by π_m the set of all real polynomials of degree not exceeding m, with $m \geq 2$. We define the real polynomial spline space of degree m as follows:

$$S_m^{(1)}(\Pi_N) = \{u \in C^1(I, \mathbb{R}) : u_n^i = u|_{\sigma_n^i} \in \pi_m, n = 0, ..., N-1,\ i = 0, 1, .., r-1\}.$$

This is the space of piecewise polynomials of degree (at most) m. Its dimension is $rN(m-1)+2$

Piecewise polynomial spaces $S_m^{(0)}(\Pi_N)$

We suppose that $T = (r+1)\tau_2$, where $r \in \{1, 2, 3, ...\}$. Let Π_N be a uniform partition of the interval $I = [\tau_2, T]$ defined by $t_n^i = (i+1)\tau_2 + nh$, $n = 0, 1, .., N,\ i = 0, 1, .., r-1$, where the step-size is given by $h = t_{n+1}^i - t_n^i$ and assume that $h = \frac{\tau_1}{N_1} = \frac{\tau_2}{N}$ with N and N_1 positive and integer. Define the subintervals $\sigma_n^i = [t_n^i; t_{n+1}^i[$, $n = 0, 1, .., N-1,\ i = 0, 1, .., r-2$ and $\sigma_{N-1}^{r-1} = [t_{N-1}^{r-1}, t_N^{r-1}]$. Moreover, denote by π_m the set of all real polynomials of degree not exceeding m. We define the real polynomial spline space of degree $m-1$ as follows:

$$S_m^{(0)}(\Pi_N) = \{u \in C(I, \mathbb{R}) : u_n^i = u|_{\sigma_n^i} \in \pi_m, n = 0, ..., N-1,\ i = 0, 1, .., r-1\}. \tag{1.4.1}$$

This is the space of piecewise polynomials of degree (at most) m. Its dimension is rNm,

1.5 Taylor collocation method

The Taylor collocation method is a numerical technique used to approximate solutions to differential equations, particularly delay differential equations. This method combines the principles of Taylor series expansion and collocation techniques to construct an approximate solution that satisfies the differential equation at specific points within the domain.

Principles of the Taylor Collocation Method

The main idea of the Taylor collocation method is to approximate the solution of a differential equation by a Taylor series expansion and ensure that this approximation satisfies the differential equation at a set of collocation points.

Step-by-Step Procedure

1. **Taylor Series Expansion**: Assume the approximate solution $y(t)$ can be expressed as a finite Taylor series around a point t_0:

$$y(t) \approx \sum_{n=0}^{N} \frac{y^{(n)}(t_0)}{n!}(t-t_0)^n$$

where $y^{(n)}(t_0)$ is the n-th derivative of y evaluated at t_0.

2. **Collocation Points**: Choose a set of collocation points $t_1, t_2, \ldots, t_M$ within the domain of interest.

3. **differentiate the problem equation**:

By differentiate problem equation j-times, we get, for $j = 0, 1, ..., N-1$ the Taylor series expansion and enforce the equation to be satisfied at each collocation point t_i.

Advantages of the Taylor Collocation Method

The Taylor collocation method offers several advantages:

- **Direct Method**: It provides explicit formulas for the approximate solution.
- **Convergence Order**: The method has a known convergence order, which ensures the accuracy of the approximation.
- **Simplicity**: There is no need to solve an algebraic system, making the algorithm straightforward and easy to implement.

Example

Consider a first-order delay differential equation:

$$\frac{dy(t)}{dt} = -y(t) + y(t-\tau)$$

Using the Taylor collocation method, we approximate $y(t)$ around t_0 as:

$$y(t) \approx y(t_0) + y'(t_0)(t-t_0) + \frac{y''(t_0)}{2!}(t-t_0)^2 + \ldots + \frac{y^{(N)}(t_0)}{N!}(t-t_0)^N$$

Choose collocation points $t_1, t_2, \ldots, t_M$ and By differentiate problem equation j-times, we get, for $j = 0, 1, ..., N-1$ the Taylor series expansion, yielding the approximate solution.

1.6 Comparison theorems

The following three lemmas will be used in the next chapter.

Lemma 1.6.1 *(Discrete Gronwall-type inequality [1]) Let $\{k_j\}_{j=0}^{n}$ be a given non-negative sequence and the sequence $\{\varepsilon_n\}$ satisfies $\varepsilon_0 \leq p_0$ and*

$$\varepsilon_n \leq p_0 + \sum_{i=0}^{n-1} k_i \varepsilon_i, \quad n \geq 1,$$

with $p_0 \geq 0$. Then ε_n can be bounded by

$$\varepsilon_n \leq p_0 \exp\left(\sum_{j=0}^{n-1} k_j\right), \quad n \geq 1.$$

Lemma 1.6.2 *(Discrete Gronwall-type inequality [9]) If $\{f_n\}_{n\geq 0}$, $\{g_n\}_{n\geq 0}$ and $\{\varepsilon_n\}_{n\geq 0}$ are non-negative sequences and*

$$\varepsilon_n \leq f_n + \sum_{i=0}^{n-1} g_i \varepsilon_i, \quad n \geq 0,$$

Then,

$$\varepsilon_n \leq f_n + \sum_{i=0}^{n-1} f_i g_i \exp\left(\sum_{k=0}^{n-1} g_k\right), \quad n \geq 0.$$

Lemma 1.6.3 *[10] Assume that the sequence $\{\varepsilon_n\}_{n\geq 0}$ of nonnegative numbers satisfies*

$$\varepsilon_n \leq A\varepsilon_{n-1} + B\sum_{i=0}^{n-1} \varepsilon_i + K, \quad n \geq 1,$$

where $A,\ B$ and K are nonnegative constants, then

$$\varepsilon_n \leq \frac{\varepsilon_0}{R_2 - R_1}[(R_2 - 1)R_2^n + (1 - R_1)R_1^n] + \frac{K}{R_2 - R_1}[R_2^n - R_1^n],$$

where

$$\begin{aligned} R_1 &= \left(1 + A + B - \sqrt{(1-A)^2 + B^2 + 2AB + 2B}\right)/2, \\ R_2 &= \left(1 + A + B + \sqrt{(1-A)^2 + B^2 + 2AB + 2B}\right)/2, \end{aligned}$$

therefore, $0 \leq R_1 \leq 1 \leq R_2$.

CHAPTER 2

NUMERICAL SOLUTION OF SECOND ORDER DELAY DIFFERENTIAL EQUATIONS USING TAYLOR COLLOCATION METHOD

2.1 Introduction

In this chapter, we consider the second order linear differential equations with constant delay $\tau > 0$ of the form:

$$x''(t) = g(t) + A_1(t)x(t) + A_2(t)x'(t) + B_1(t)x(t-\tau) + B_2(t)x'(t-\tau), \tag{2.1.1}$$

for $t \in [0, T]$ and $x(t) = \Phi(t)$ for $t \in [-\tau, 0]$. In the following we assume that the given functions g, A_1, A_2, B_1, B_2 and Φ are sufficiently smooth. Furthermore, we suppose that

$$\Phi''(0) = g(0) + A_1(0)\Phi(0) + A_2(0)\Phi'(0) + B_1(0)\Phi(-\tau) + B_2(0)\Phi'(-\tau)$$

Existence and uniqueness results for (2.1.1) can be easily proved by comparison with the theory for differential equations (see for example[1, 2]).
This method can be used to obtain numerical solutions of second order linear delay differential equations (2.1.1), second order Initial Value Problems of ordinary differential equations ($B_1 = B_2 = 0$ in (2.1.1)).
Delay differential equations are widely used for modeling various problems from mechanics, control theory, biology, etc. (cf. [4, 3, 5]) and a detailed discussion of various classes of DDEs can be found in Chapters 1 and 2 of the monograph [6].
There have been a lot of papers concerning numerical methods for delay differential equations (cf. [1, 7, 8]). For instance, Runge-Kutta methods for delay differential equations have been investigated by Koto [7].

2.2 Description of the Method

We suppose that $T = r\tau$, where $r \in \{1, 2, 3, ...\}$. Let Π_N be a uniform partition of the interval $I = [0, T]$ defined by $t_n^i = i\tau + nh$, $n = 0, 1, .., N$, $i = 0, 1, .., r-1$, where the stepsize is given by $h = \frac{\tau}{N}$. Define the subintervals $\sigma_n^i = [t_n^i; t_{n+1}^i), n = 0, 1, .., N-1,\ i = 0, 1, .., r-1$ and $\sigma_{N-1}^{r-1} = [t_{N-1}^{r-1}, t_N^{r-1}]$. Moreover, denote by π_m the set of all real polynomials of degree not exceeding m, with $m \geq 2$. We define the real polynomial spline space of degree m as follows:

$$S_m^{(1)}(\Pi_N) = \{u \in C^1(I, \mathbb{R}) : u_n^i = u|_{\sigma_n^i} \in \pi_m, n = 0, ..., N-1,\ i = 0, 1, .., r-1\}.$$

This is the space of piecewise polynomials of degree (at most) m. Its dimension is $rN(m-1)+2$, i.e., the same as the total number of the coefficients of the polynomials $u_n^i, n = 0, ..., N-1, i = 0, 1, .., r-1$. To find these coefficients, we use Taylor polynomial on each subinterval.

2.2.1 Approximate solution in the interval σ_0^0

First, we approximate x in the interval σ_0^0 by the polynomial,

$$u_0^0(t) = \sum_{j=0}^{m} \frac{x^{(j)}(0)}{j!} t^j \ ; \quad t \in \sigma_0^0, \tag{2.2.1}$$

where $x^{(j)}(0), j = 0, ..., m$ is the exact value of $x^{(j)}$ at 0 and the function x must be differentiable around the zero point.
By differentiate equation (2.1.1) j-times, we get,

$$\begin{aligned} x^{(j+2)}(0) =& g^{(j)}(0) + \sum_{l=0}^{j} \binom{j}{l} \Big(A_1^{(j-l)}(0) x^{(l)}(0) + A_2^{(j-l)}(0) x^{(l+1)}(0) \Big) \\ &+ \sum_{l=0}^{j} \binom{j}{l} \Big(B_1^{(j-l)}(0) \Phi^{(l)}(-\tau) + B_2^{(j-l)}(0) \Phi^{(l+1)}(-\tau) \Big), \end{aligned}$$

for $j = 0, 1, ..., m-2$, $x(0) = \Phi(0)$ and $x'(0) = \Phi'(0)$.

2.2.2 Approximate solution in the interval σ_n^0

Second, for x to be approximated by u_n^0 ($n \in \{1, 2, ..., N-1\}$) on the interval σ_n^0, x must be approximated by u_k^0 ($0 \le k < n$) on each interval σ_k^0, such that

$$u_n^0(t) = \sum_{j=0}^{m} \frac{\hat{u}_{n,0}^{(j)}(t_n^0)}{j!}(t - t_n^0)^j; \quad t \in \sigma_n^0, \tag{2.2.2}$$

where $\hat{u}_{n,0}$ is the exact solution of the differential equation:

$$\hat{u}_{n,0}''(t) = g(t) + A_1(t)\hat{u}_{n,0}(t) + A_2(t)\hat{u}_{n,0}'(t) + B_1(t)\Phi(t-\tau) + B_2(t)\Phi'(t-\tau), \tag{2.2.3}$$

for $t \in \sigma_n^0$ such that $\hat{u}_{n,0}(t_n^0) = u_{n-1}^0(t_n^0)$ and $\hat{u}_{n,0}'(t_n^0) = u_{n-1}^{0'}(t_n^0)$.
Now, for all $j = 0, 1, ..., m-2$, the formula for computing the values of the coefficients $\hat{u}_{n,0}^{(j)}(t_n^0)$ can be obtained by employing similar arguments to those used for obtaining the values of $x^{(j)}(0)$ above, we get the following formula:

$$\begin{aligned}\hat{u}_{n,0}^{(j+2)}(t_n^0) &= g^{(j)}(t_n^0) + \sum_{l=0}^{j}\binom{j}{l}\left(A_1^{(j-l)}(t_n^0)\hat{u}_{n,0}^{(l)}(t_n^0) + A_2^{(j-l)}(t_n^0)\hat{u}_{n,0}^{(l+1)}(t_n^0)\right)\\ &+ \sum_{l=0}^{j}\binom{j}{l}\left(B_1^{(j-l)}(t_n^0)\Phi^{(l)}(t_n^0-\tau) + B_2^{(j-l)}(t_n^0)\Phi^{(l+1)}(t_n^0-\tau)\right),\end{aligned} \tag{2.2.4}$$

for $j = 0, ..., m-2$ such that $\hat{u}_{n,0}(t_n^0) = u_{n-1}^0(t_n^0)$ and $\hat{u}_{n,0}'(t_n^0) = u_{n-1}^{0'}(t_n^0)$.

2.2.3 Approximate solution in the interval σ_0^p

Third, for x to be approximated by u_0^p ($p \in \{1, 2, ..., r-1\}$) on the interval σ_0^p, x must be approximated by u_k^j ($0 \le k \le N-1$ and $0 \le j < p$) on each interval σ_k^j such that,

$$u_0^p(t) = \sum_{j=0}^{m} \frac{\hat{u}_{0,p}^{(j)}(t_0^p)}{j!}(t - t_0^p)^j; \quad t \in \sigma_0^p, \tag{2.2.5}$$

where $\hat{u}_{0,p}$ is the exact solution of the differential equation:

$$\hat{u}''_{0,p}(t) = g(t) + A_1(t)\hat{u}_{0,p}(t) + A_2(t)\hat{u}'_{0,p}(t) + B_1(t)\hat{u}_{0,p-1}(t-\tau) + B_2(t)\hat{u}'_{0,p-1}(t-\tau), \quad (2.2.6)$$

for $t \in \sigma_0^p$ such that $\hat{u}_{0,p}(t_0^p) = u_{N-1}^{p-1}(t_0^p)$ and $\hat{u}'_{0,p}(t_0^p) = u_{N-1}^{p-1'}(t_0^p)$.
The coefficients $\hat{u}_{0,p}^{(j)}(t_0^p)$ is given by the following formula:

$$\begin{aligned} \hat{u}_{0,p}^{(j+2)}(t_0^p) = g^{(j)}(t_0^p) &+ \sum_{l=0}^{j} \binom{j}{l} \left(A_1^{(j-l)}(t_0^p)\hat{u}_{0,p}^{(l)}(t_0^p) + A_2^{(j-l)}(t_0^p)\hat{u}_{0,p}^{(l+1)}(t_0^p) \right) \\ &+ \sum_{l=0}^{j} \binom{j}{l} \left(B_1^{(j-l)}(t_0^p)\hat{u}_{0,p-1}^{(l+1)}(t_0^{p-1}) + B_2^{(j-l)}(t_0^p)\hat{u}_{0,p-1}^{(l+1)}(t_0^{p-1}) \right), \end{aligned} \quad (2.2.7)$$

for $j = 0, \ldots, m-2$, $\hat{u}_{0,p}(t_0^p) = u_{N-1}^{p-1}(t_0^p)$ and $\hat{u}'_{0,p}(t_0^p) = u_{N-1}^{p-1'}(t_0^p)$.
Finally, for x to be approximated by u_n^p ($n \in \{1, \ldots, N-1\}$ and $p \in \{1, 2, \ldots, r-1\}$) on the interval σ_n^p, x must be approximated by u_k^j ($0 \leq k < n$ and $0 \leq j \leq p$) on each interval σ_k^j such that,

$$u_n^p(t) = \sum_{j=0}^{m} \frac{\hat{u}_{n,p}^{(j)}(t_n^p)}{j!}(t - t_n^p)^j; \quad t \in \sigma_n^p, \quad (2.2.8)$$

where $\hat{u}_{n,p}$ is the exact solution of the differential equation:

$$\hat{u}''_{n,p}(t) = g(t) + A_1(t)\hat{u}_{n,p}(t) + A_2(t)\hat{u}'_{n,p}(t) + B_1(t)\hat{u}_{n,p-1}(t-\tau) + B_2(t)\hat{u}'_{n,p-1}(t-\tau), \quad (2.2.9)$$

for $t \in \sigma_n^p$, $\hat{u}_{n,p}(t_n^p) = u_{n-1}^p(t_n^p)$ and $\hat{u}'_{n,p}(t_n^p) = u_{n-1}^{p'}(t_n^p)$.
The coefficients $\hat{u}_{n,p}^{(j)}(t_n^p)$ is given by the following formula:

$$\begin{aligned} \hat{u}_{n,p}^{(j+2)}(t_n^p) = g^{(j)}(t_n^p) &+ \sum_{l=0}^{j} \binom{j}{l} \left(A_1^{(j-l)}(t_n^p)\hat{u}_{n,p}^{(l)}(t_n^p) + A_2^{(j-l)}(t_n^p)\hat{u}_{n,p}^{(l+1)}(t_n^p) \right) \\ &+ \sum_{l=0}^{j} \binom{j}{l} \left(B_1^{(j-l)}(t_n^p)\hat{u}_{n,p-1}^{(l+1)}(t_n^{p-1}) + B_2^{(j-l)}(t_n^p)\hat{u}_{n,p-1}^{(l+1)}(t_n^{p-1}) \right), \end{aligned} \quad (2.2.10)$$

for $j = 0, \ldots, m-2$, $\hat{u}_{n,p}(t_n^p) = u_{n-1}^p(t_n^p)$ and $\hat{u}'_{n,p}(t_n^p) = u_{n-1}^{p'}(t_n^p)$.

2.3 Boundedness of the approximate polynomial's coefficients

Before starting the main result, we need the following lemma:

Lemma 2.3.1 *Let g, A_1, A_2, B_1, B_2 be $m-1$ times continuously differentiable and Φ be m times continuously differentiable on their respective domains. Then, there exists a positive number $\alpha(m)$ such that for all $n = 0, 1, ..., N-1$, $p = 0, 1, ..., r-1$, and $j = 0, 1, ..., m+1$, we have,*

$$||\hat{u}_{n,p}^{(j)}||_{L^\infty(\sigma_n^p)} \leq \alpha(m)$$

provided that h is sufficiently small, where $\hat{u}_{0,0}(t) = x(t)$ for $t \in \sigma_0^0$.

Proof. The proof is split into two steps.
Claim 1.There exists a positive constant $\alpha_1(m)$ such that $||\hat{u}_{n,0}^{(j)}||_{L^\infty(\sigma_n^0)} \leq \alpha_1(m)$ for all $n = 0, 1, ..., N-1$ and $j = 0, 1, ..., m+1$.
Let $a_n^j = ||\hat{u}_{n,0}^{(j)}||_{L^\infty(\sigma_n^0)}$, we have for all $j = 0, 1, ..., m+1$,

$$a_0^j \leq \max\{||x^{(j)}||_{L^\infty(\sigma^0)}, j = 0, 1, ..., m+1\} = \alpha_1^1(m). \tag{2.3.1}$$

On the other hand, for $n \geq 1$, by differentiating equation (2.2.3) j-times, we obtain for all $j = 1, ..., m-1$,

$$\begin{aligned} a_n^{j+2} &\leq c_1 + L \sum_{l=0}^{j} \left(a_n^l + a_n^{l+1}\right) \\ &\leq c_1 + \underbrace{2L}_{b_1} \sum_{k=0}^{j+1} a_n^k, \end{aligned} \tag{2.3.2}$$

where

$$L = \max\left\{\binom{j}{l} \left\|L_v^{(j-l)}\right\|_{L^\infty(\sigma^0)},\ j = 1, ..., m-1;\ l = 0, ..., j;\ v = 0, 1\right\},$$

$$c_1 = \max\left\{\left\| g^{(j)} + \sum_{v=0}^{1}\sum_{l=0}^{j}\binom{j}{l} M_v^{(j-l)}(t)\Phi^{(l+v)}(t-\tau)\right\|_{L^\infty(\sigma^0)},\ j = 0, ..., m-1\right\},$$

Now, for each fixed $n \geq 1$, we consider the sequence $y_j = a_n^{j+2}$ for $j = 0, ..., m-1$, then, from (2.3.2), the sequence (y_j) satisfies for $j = 1, ..., m-1$

$$y_j \leq c_1 + b_1(a_n^0 + a_n^1),$$

and for $j = 0$, we get from (2.2.3),

$$\begin{aligned} y_0 = a_n^2 &\leq c_1 + La_n^0 + La_n^1 \\ &\leq c_1 + b_1(a_n^0 + a_n^1). \end{aligned}$$

Hence, by Lemma 1.6.1, for all $j = 0, ..., m-1$

$$\begin{aligned} y_j &\leq \underbrace{c_1 \exp(b_1(m-1))}_{c_2} + \underbrace{b_1 \exp(b_1(m-1))}_{b_2}(a_n^0 + a_n^1) \\ &\leq c_2 + b_2(a_n^0 + a_n^1). \end{aligned} \tag{2.3.3}$$

Next, we consider the sequence $z_n = \sum_{j=2}^{m+1} a_n^j$ for $n = 0, ..., N-1$.
Then, from (2.3.3), z_n satisfies for $n = 1, ..., N-1$,

$$z_n = \sum_{j=0}^{m-1} y_j \leq \underbrace{mc_2}_{c_3^1} + \underbrace{mb_2}_{b_3}(a_n^0 + a_n^1).$$

Moreover, from (2.3.1), we have $z_0 \leq m\alpha_1^1(m) = c_3^2$.
Let $c_3 = \max(c_3^1, c_3^2)$, we deduce, by Lemma 1.6.2, that for all $n = 0, ..., N-1$,

$$\begin{aligned} z_n &\leq c_3 + b_3(a_n^0 + a_n^1) \\ &\leq c_4 + b_3(a_n^0 + a_n^1), \end{aligned} \tag{2.3.4}$$

where c_4 is positive number. On the other hand, by integrating $\hat{u}''_{n,0}$ in (2.2.3) from t_n^0 to $t \in \sigma_n^0$, we get,

$$\begin{aligned} a_n^1 &\leq |u_{n-1}^{0\prime}(t_n^0)| + ch + Lha_n^0 + Lha_n^1 \\ &\leq a_{n-1}^1 + h\sum_{j=2}^{m} a_{n-1}^j + ch + 2Lh(a_n^0 + a_n^1) \\ &\leq a_{n-1}^1 + hz_{n-1} + ch + b_1 h(a_n^0 + a_n^1). \end{aligned} \tag{2.3.5}$$

Now, by integrating twice $\hat{u}''_{n,0}$ in (2.2.3) from t_n^0 to $t \in \sigma_n^0$, we obtain,

$$\begin{aligned} a_n^0 &\leq |u_{n-1}^0(t_n^0)| + h|u_{n-1}^{0\prime}(t_n^0)| + ch^2 + Lh^2a_n^0 + Lh^2a_n^1 \\ &\leq a_{n-1}^0 + h\sum_{j=1}^{m} a_{n-1}^j + h\left(a_{n-1}^1 + h\sum_{j=2}^{m} a_{n-1}^j\right) + ch^2 + b_1h^2(a_n^0 + a_n^1) \\ &\leq a_{n-1}^0 + 2ha_{n-1}^1 + (h+h^2)z_{n-1} + ch^2 + b_1h^2(a_n^0 + a_n^1) + h^3 d\sum_{r=0}^{n-1}(a_r^0 + a_r^1). \end{aligned}$$

Using (2.3.4), it follows that,

$$\begin{aligned} a_n^0 \leq &\Big(1 + (h+h^2)b_3\Big)a_{n-1}^0 + \Big(2h + (h+h^2)b_3\Big)a_{n-1}^1 + hc_6 \\ &+ b_1h^2(a_n^0 + a_n^1). \end{aligned} \tag{2.3.6}$$

Where c_6 is positive number, we deduce, from (2.3.5) and (2.3.6) that,

$$\begin{aligned} a_n^0 + a_n^1 \leq &(1 + \underbrace{2(1+b_3)}_{b_4}h + b_3h^2)(a_{n-1}^0 + a_{n-1}^1) + h\underbrace{(c_5 + c_6)}_{c_7} \\ &+ b_1(h+h^2)(a_n^0 + a_n^1). \end{aligned}$$

Hence, there exists $h_1 > 0$ such that for all $h \in (0, h_1]$,

$$(a_n^0 + a_n^1) \leq \frac{1 + b_4h + b_3h^2}{1 - b_1(h+h^2)}(a_{n-1}^0 + a_{n-1}^1) + \frac{hc_7}{1 - b_1(h+h^2)}.$$

Then, by Lemma 1.6.3, we obtain for all $n \in \{0, 1, ..., N-1\}$,

$$\begin{aligned} a_n^0 + a_n^1 &\leq \frac{(a_0^0 + a_0^1)}{R_2 - R_1}[(R_2 - 1)R_2^n + (1 - R_1)R_1^n] + \frac{hc_7[R_2^n - R_1^n]}{(R_2 - R_1)(1 - b_1(h + h^2))} \\ &\leq \frac{2\alpha_1^1(m)}{R_2 - R_1}[(R_2 - 1)R_2^n + (1 - R_1)R_1^n] + \frac{hc_7[R_2^n - R_1^n]}{(R_2 - R_1)(1 - b_1(h + h^2))}, \end{aligned} \tag{2.3.7}$$

where

$$R_1 = \frac{1 + b_4 h + b_3 h^2}{1 - b_1(h + h^2)},$$
$$R_2 = 1,$$

such that ζ is a positive number.

Since, $0 \leq R_1 \leq 1 \leq R_2$, then for all $h \in (0, h_1]$, we have,

$$R_1^n \leq 1 \leq R_2^n \leq R_2^N = R_2^{\frac{\tau}{h}}, n = 0, 1, ..., N-1.$$

This implies that for all $h \in (0, h_1]$, there exists $\alpha_1^2(m) > 0$ such that,

$$a_n^0 + a_n^1 \leq \alpha_1^2(m), n = 0, 1, ..., N-1.$$

Hence, from (2.3.4), that for all $j = 2, 3, ..., m+1$ and $n = 0, 1, ..., N-1$,

$$a_n^j \leq z_n \leq c_4 + b_3\alpha_1^2(m) + \tau d_4 \alpha_1^2(m) = \alpha_1^3(m).$$

Then, the first step is completed by setting,

$$\alpha_1(m) = \max(\alpha_1^2(m), \alpha_1^3(m)).$$

Claim 2. There exists a positive constant $\alpha(m)$ such that $||\hat{u}_{n,p}^{(j)}||_{L^\infty(\sigma_n^p)} \leq \alpha(m)$ for all $n = 0, 1, ..., N-1$, $j = 0, 1, ..., m+1$ and $p = 1, ..., r-1$.

Let $a_{n,p}^j = ||\hat{u}_{n,p}^{(j)}||_{L^\infty(\sigma_n^p)}$ and $\xi_p = \max\{a_{i,p}^j, j = 0, ..., m+1, i = 0, ..., N-1\}$ for $p = 0, ..., r-1$.

Similarly to Claim 1, by differentiating equation (2.2.6) j-times, we obtain for all $j =$

$2, ..., m+1$,

$$a_{0,p}^{j} \leq c_1 + d_1 \sum_{l=0}^{j-1} a_{0,p}^{l},$$

where c_1, b_1, d_1 are positive numbers.

On the other hand, by integrating (2.2.6) from t_0^p to $t \in \sigma_0^p$, we get,

$$a_{0,p}^{1} \leq c_2 + hd_2 a_{0,p}^{0},$$

where c_2, b_2, d_2 are positive numbers.

Then, by integrating twice (2.2.6) from t_0^p to $t \in \sigma_0^p$, we obtain,

$$a_{0,p}^{0} \leq c_3 + hd_3 a_{0,p}^{0},$$

where c_3, b_3, d_3 are positive numbers.

Hence, there exists $h_2 \in (0, h_1]$ and positive numbers c_4, b_4, d_4 such that for all $h \in (0, h_2]$, we have

$$a_{0,p}^{j} \leq c_4 + d_4 \sum_{l=0}^{j-1} a_{0,p}^{l},$$

for all $j \in \{0, 1, ..., m+1\}$.

Then, by Lemma 1.6.1, for all $j \in \{0, 1, ..., m+1\}$

$$a_{0,p}^{j} \leq \underbrace{c_4 \exp(d_4(m+1))}_{c_5^1},$$

Hence, for $c_5 = \max(\alpha_1(m), c_5^1)$, we get for all $p = 0, 1, ..., r-1$, $j \in \{0, 1, ..., m+1\}$

$$a_{0,p}^{j} \leq c_5. \tag{2.3.8}$$

Next, by differentiating (2.2.9) j-times, we obtain for all $n = 1, ..., N-1$ and $j = 2, ..., m+1$,

$$a_{n,p}^{j} \leq c_6 + e_6 \sum_{l=0}^{j-1} a_{n,p}^{l},$$

where c_6, e_6 are positive numbers.

Then, by Lemma 1.6.1, for all $j \in \{2, ..., m+1\}$

$$a_{n,p}^{j} \leq \underbrace{c_6 \exp(me_6)}_{c_7} + \underbrace{e_6 \exp(me_6)}_{e_7}(a_{n,p}^{0} + a_{n,p}^{1}).$$

Consider the sequence $y_n = \sum_{j=2}^{m+1} a_{n,p}^{j}$, $n = 0, 1, ..., N-1$, hence, by the above inequality, the sequence (y_n) satisfies for all $n = 1, ..., N-1$,

$$y_n \leq \underbrace{mc_7}_{c_8^1} + \underbrace{mb_7}_{b_8^1} \sum_{i=0}^{p-1} \xi_i + \underbrace{me_7}_{e_8}(a_{n,p}^{0} + a_{n,p}^{1}) + \underbrace{md_7}_{d_8} h \sum_{i=0}^{n-1}(a_{i,p}^{0} + a_{i,p}^{1}) + \underbrace{md_7}_{d_8} h \sum_{i=0}^{n-1} y_i. \tag{2.3.9}$$

Moreover, from (2.3.8), we obtain,

$$y_0 \leq \underbrace{mc_5}_{c_8^2}. \tag{2.3.10}$$

Let $c_8 = \max\{c_8^1, c_8^2\}$

Then, from (2.3.9) and (2.3.10), we get for all $n = 0, 1, ..., N-1$,

$$y_n \leq c_8 + b_8 \sum_{i=0}^{p-1} \xi_i + e_8(a_{n,p}^{0} + a_{n,p}^{1}) + d_8 h \sum_{i=0}^{n-1}(a_{i,p}^{0} + a_{i,p}^{1}) + d_8 h \sum_{i=0}^{n-1} y_i,$$

hence, by Lemma 1.6.2, we obtain

$$y_n \leq c_9 + b_9 \sum_{i=0}^{p-1} \xi_i + e_8(a_{n,p}^{0} + a_{n,p}^{1}) + hd_9 \sum_{i=0}^{n-1}(a_{i,p}^{0} + a_{i,p}^{1}). \tag{2.3.11}$$

Where c_9, b_9, d_9 are positive numbers. On the other hand, by integrating (2.2.9) from t_n^p

to $t \in \sigma_n^p$, we get

$$\begin{aligned}
a_{n,p}^1 &\le |u_{n-1}^{p'}(t_n^p)| + hc_{10} + h^2 b_{10} \sum_{i=0}^{p-1} \xi_i + h(e_9 + L)a_{n,p}^0 + hLa_{n,p}^1 + d_{10}h^2 \sum_{i=0}^{n-1} \sum_{l=0}^{m} a_{i,p}^l \\
&\le a_{n-1,p}^1 + h \underbrace{\sum_{j=2}^{m+1} a_{n-1,p}^j}_{y_{n-1}} + hc_{10} + h^2 b_{10} \sum_{i=0}^{p-1} \xi_i + h(e_9 + L)(a_{n,p}^0 + a_{n,p}^1) \\
&+ d_{10}h^2 \sum_{i=0}^{n-1} (a_{i,p}^0 + a_{i,p}^1) + d_{10}h^2 \sum_{i=0}^{n-1} \underbrace{\sum_{l=2}^{m+1} a_{i,p}^l}_{y_i},
\end{aligned}$$

for all $n = 1, ..., N-1$, where $c_{10}, b_{10}, d_{10}, e_9$ are positive numbers.

Which implies, by using (2.3.11), that

$$\begin{aligned}
a_{n,p}^1 &\le he_8 a_{n-1,p}^0 + (1 + he_8)a_{n-1,p}^1 + hc_{11} + hb_{11} \sum_{i=0}^{p-1} \xi_i \\
&+ h(e_9 + L)(a_{n,p}^0 + a_{n,p}^1) + h^2 d_{11} \sum_{i=0}^{n-1} (a_{i,p}^0 + a_{i,p}^1),
\end{aligned} \tag{2.3.12}$$

where c_{11}, b_{11}, d_{11} are positive numbers. Moreover, by integrating twice (2.2.9) from t_n^p to $t \in \sigma_n^p$, we get

$$\begin{aligned}
a_{n,p}^0 &\le |u_{n-1}^p(t_n^p)| + h|u_{n-1}^{p'}(t_n^p)| + h^2 c_{12} + h^3 b_{12} \sum_{i=0}^{p-1} \xi_i + h^2(e_{10} + L)a_{n,p}^0 \\
&+ h^2 L a_{n,p}^1 + h^3 d_{12} \sum_{i=0}^{n-1} \sum_{l=0}^{m} a_{i,p}^l \\
&\le a_{n-1,p}^0 + 2ha_{n-1,p}^1 + (h + h^2) \underbrace{\sum_{j=2}^{m+1} a_{n-1,p}^j}_{y_{n-1}} + h^2 c_{12} + h^3 b_{12} \sum_{i=0}^{p-1} \xi_i \\
&+ h^2(e_{10} + L)(a_{n,p}^0 + a_{n,p}^1) + h^3 d_{12} \sum_{i=0}^{n-1} (a_{i,p}^0 + a_{i,p}^1) + h^3 d_{12} \sum_{i=0}^{n-1} \underbrace{\sum_{l=2}^{m+1} a_{i,p}^l}_{y_i},
\end{aligned}$$

for all $n = 1, ..., N-1$, where $c_{12}, b_{12}, d_{12}, e_{10}$ are positive numbers.
Which implies, by using (2.3.11), that

$$\begin{aligned} a^0_{n,p} \leq\; & ((h+h^2)e_8+1)a^0_{n-1,p} + (2h+(h+h^2)e_8)a^1_{n-1,p} + hc_{13} \\ & + hb_{13}\sum_{i=0}^{p-1}\xi_i + h^2(e_{10}+A_2)(a^0_{n,p}+a^1_{n,p}) + h^2 d_{13}\sum_{i=0}^{n-1}(a^0_{i,p}+a^1_{i,p}), \end{aligned} \tag{2.3.13}$$

where c_{13}, b_{13}, d_{13} are positive numbers. We deduce, from (2.3.12) and (2.3.13), that

$$\begin{aligned} a^0_{n,p} + a^1_{n,p} \leq\; & ((2h+h^2)e_8+1)a^0_{n-1,p} + (h^2 e_8 + h\underbrace{2(e_8+1)}_{e_{11}})a^1_{n-1,p} \\ & + h\underbrace{(c_{11}+c_{13})}_{c_{14}} + h\underbrace{(b_{11}+b_{13})}_{b_{14}}\sum_{i=0}^{p-1}\xi_i + (h\underbrace{(e_9+L)}_{e_{12}} + h^2\underbrace{(e_{10}+L)}_{e_{13}})(a^0_{n,p}+a^1_{n,p}) \\ & + h^2\underbrace{(d_{11}+d_{13})}_{d_{14}}\sum_{i=0}^{n-1}(a^0_{i,p}+a^1_{i,p}) \\ \leq\; & (1+e_{11}h+e_8h^2)(a^0_{n-1,p}+a^1_{n-1,p}) + hc_{14} + hb_{14}\sum_{i=0}^{p-1}\xi_i \\ & + (e_{12}h+e_{13}h^2)(a^0_{n,p}+a^1_{n,p}) + h^2 d_{14}\sum_{i=0}^{n-1}(a^0_{i,p}+a^1_{i,p}), \end{aligned}$$

hence, there exists $h_3 \in (0, h_2]$ such that for all $h \in (0, h_3]$, we have

$$\begin{aligned} a^0_{n,p} + a^1_{n,p} \leq\; & \frac{1+e_{11}h+e_8h^2}{1-e_{12}h-e_{13}h^2}(a^0_{n-1,p}+a^1_{n-1,p}) + \frac{h(c_{14}+b_{14}\sum_{i=0}^{p-1}\xi_i)}{1-e_{12}h-e_{13}h^2} \\ & + \frac{h^2 d_{14}}{1-e_{12}h-e_{13}h^2}\sum_{i=0}^{n-1}(a^0_{i,p}+a^1_{i,p}). \end{aligned}$$

Then, by Lemma 1.6.3, we get for all $n \in \{0, 1, ..., N-1\}$

$$\begin{aligned} a^0_{n,p} + a^1_{n,p} \leq\; & \frac{a^0_{0,p}+a^1_{0,p}}{R_2-R_1}[(R_2-1)R_2^n + (1-R_1)R_1^n] \\ & + \frac{h(c_{14}+b_{14}\sum_{i=0}^{p-1}\xi_i)}{(R_2-R_1)(1-e_{12}h-e_{13}h^2)}[R_2^n - R_1^n], \end{aligned}$$

where

$$R_1 = \left(1 + \frac{1 + (e_{11} - \sqrt{\zeta})h + (e_8 + d_{14})h^2}{1 - e_{12}h - e_{13}h^2}\right)/2,$$

$$R_2 = \left(1 + \frac{1 + (e_{11} + \sqrt{\zeta})h + (e_8 + d_{14})h^2}{1 - e_{12}h - e_{13}h^2}\right)/2,$$

such that,

$\zeta = e_{11} + e_{12} + 4d_{14} + 2d_{14}(e_{11} - e_{12})h + (d_{14}(1 + 2(e_8 - e_{13})) + e_{13} + e_8)h^2.$

Hence, similar as in (2.3.7), there exist $\overline{R} > 0$ such that for all $h \in (0, h_3]$, we have

$$a^0_{n,p} + a^1_{n,p} \leq (a^0_{0,p} + a^1_{0,p})\overline{R} + (c_{14} + b_{14}\sum_{i=0}^{p-1} \xi_i)\overline{R},$$

which implies, by using(2.3.8), that for all $n \in \{0, 1, ..., N - 1\}$ and $p \in \{0, 1, ..., r - 1\}$,

$$a^0_{n,p} + a^1_{n,p} \leq \underbrace{(2c_5 + c_{14})\overline{R}}_{c_{15}} + \underbrace{(2b_5 + b_{14})\overline{R}}_{b_{15}}\sum_{i=0}^{p-1} \xi_i.$$

Then, from (2.3.11), we get for all $n \in \{0, 1, ..., N-1\}$, $j \in \{2, ..., m+1\}$ and $p \in \{0, 1, ..., r-1\}$,

$$a^j_{n,p} \leq y_n \leq \underbrace{(c_9 + e_8c_{15} + \tau d_9c_{15})}_{c^1_{16}} + \underbrace{(b_9 + e_8b_{15} + \tau d_9b_{15})}_{b^1_{16}}\sum_{i=0}^{p-1} \xi_i.$$

Let $c_{16} = \max(c_{15}, c^1_{16})$ and $b_{16} = \max(b_{15}, b^1_{16})$.

We deduce that, for all $p \in \{0, 1, ..., r - 1\}$,

$$\xi_p \leq c_{16} + b_{16}\sum_{i=0}^{p-1} \xi_i.$$

Then, by Lemma 1.6.1, we get for all $p \in \{0, 1, ..., r - 1\}$, $n \in \{0, 1, ..., N - 1\}$ and $j \in$

$\{0, 1, ..., m+1\}$,

$$a^j_{n,p} \le \xi_p \le c_{16}\exp(rb_{16}) = \alpha(m).$$

This completes the proof of Lemma 2.3.1. ■

2.4 Order of convergence of the method

The following theorem describes the order of convergence of the method.

Theorem 2.4.1 *Let g, A_1, A_2, B_1, B_2 be $m-1$ times continuously differentiable and Φ be m times continuously differentiable on their respective domains. Then equations* (2.2.1), ..., (2.2.10) *define a unique approximation $u \in S^{(1)}_m(\Pi_N)$, and the resulting error function $e := x - u$ satisfies:*

$$\|e\|_{L^\infty(I)} \le Ch^{m-1},$$

where C is a finite constant independent of h.

Proof. The proof is split into two steps.
Claim 1. There exists a constant C_0 independent of h such that,

$$\|e^0\|_{L^\infty(\sigma^0)} \le C_0 h^{m-1},$$

where the error $e^0 = e|_{\sigma^0}$ which is defined on σ^0_n, by $e^0(t) = e^0_n(t) = x(t) - u^0_n(t)$ for all $n \in \{0, 1, ..., N-1\}$.
We define $e' := x' - u'$ on σ^0_n, by $e^{0'}_n(t) = x'(t) - u^{0'}_n(t)$ for all $n \in \{0, 1, ..., N-1\}$.
Let $t \in \sigma^0_0$, we have from Lemma 2.3.1, for sufficiently small h,

$$|e^0_0(t)| = |x(t) - u^0_0(t)| \le \frac{\|x^{(m+1)}\|_{L^\infty(\sigma^0_0)}}{(m+1)!}h^{m+1} \le \frac{\alpha(m)}{(m+1)!}h^{m+1},$$

and

$$|e^{0'}_0(t)| = |x'(t) - u^{0'}_0(t)| \le \frac{\|x^{(m+1)}\|_{L^\infty(\sigma^0_0)}}{m!}h^{m} \le \frac{\alpha(m)}{m!}h^{m}.$$

In general for $n = 1, 2, ..., N-1$ and $t \in \sigma_n^0$, we have from (2.2.3),

$$x''(t) - \hat{u}''_{n,0}(t) = A_1(t)(x(t) - \hat{u}_{n,0}(t)) + A_2(t)(x'(t) - \hat{u}'_{n,0}(t)),$$

this implies that,

$$\|x'' - \hat{u}''_{n,0}\|_{L^\infty(\sigma_n^0)} \leq L\left(\|x - \hat{u}_{n,0}\|_{L^\infty(\sigma_n^0)} + \|x' - \hat{u}'_{n,0}\|_{L^\infty(\sigma_n^0)}\right), \tag{2.4.1}$$

where $L = max\left\{, \|A_1\|_{L^\infty(I)}, \|A_2\|_{L^\infty(I)}\right\}$.
On the other hand, for $t \in \sigma_n^0$, we have

$$\begin{aligned} x'(t) - \hat{u}'_{n,0}(t) &= x'(t_n^0) - \hat{u}'_{n,0}(t_n^0) + \int_{t_n^0}^{t} (x''(s) - \hat{u}''_{n,0}(s))ds \\ &= e_{n-1}^{0'}(t_n^0) + \int_{t_n^0}^{t} (x''(s) - \hat{u}''_{n,0}(s))ds, \end{aligned}$$

and

$$\begin{aligned} x(t) - \hat{u}_{n,0}(t) &= x(t_n^0) - \hat{u}_{n,0}(t_n^0) + (t - t_n^0)(x'(t_n^0) - \hat{u}'_{n,0}(t_n^0)) + \int_{t_n^0}^{t}\int_{t_n^0}^{s} (x''(r) - \hat{u}''_{n,0}(r))drds \\ &= e_{n-1}^{0}(t_n^0) + (t - t_n^0)e_{n-1}^{0'}(t_n^0) + \int_{t_n^0}^{t}\int_{t_n^0}^{s} (x''(r) - \hat{u}''_{n,0}(r))drds, \end{aligned}$$

it follows that,

$$\|x' - \hat{u}'_{n,0}\|_{L^\infty(\sigma_n^0)} \leq \|e_{n-1}^{0'}\|_{L^\infty(\sigma_{n-1}^0)} + h\|x'' - \hat{u}''_{n,0}\|_{L^\infty(\sigma_n^0)}, \tag{2.4.2}$$

and

$$\|x - \hat{u}_{n,0}\|_{L^\infty(\sigma_n^0)} \leq \|e_{n-1}^{0}\|_{L^\infty(\sigma_{n-1}^0)} + h\|e_{n-1}^{0'}\|_{L^\infty(\sigma_{n-1}^0)} + h^2\|x'' - \hat{u}''_{n,0}\|_{L^\infty(\sigma_n^0)}. \tag{2.4.3}$$

Hence, by using (2.4.2) and (2.4.3), we obtain

$$\|x - \hat{u}_{n,0}\|_{L^\infty(\sigma_n^0)} + \|x' - \hat{u}'_{n,0}\|_{L^\infty(\sigma_n^0)} \leq \|e^0_{n-1}\|_{L^\infty(\sigma^0_{n-1})} + (1+h)\|e^{0'}_{n-1}\|_{L^\infty(\sigma^0_{n-1})} + h(1+h)\|x'' - \hat{u}''_{n,0}\|_{L^\infty(\sigma_n^0)},$$

Therefore, by using (2.4.1), we have

$$\|x - \hat{u}_{n,0}\|_{L^\infty(\sigma_n^0)} + \|x' - \hat{u}'_{n,0}\|_{L^\infty(\sigma_n^0)} \leq \frac{(1+h)}{1 - h(1+h)(L + k_1 h)}(\|e^0_{n-1}\|_{L^\infty(\sigma^0_{n-1})} + \|e^{0'}_{n-1}\|_{L^\infty(\sigma^0_{n-1})}) + \frac{h^2(1+h)k_1}{1 - h(1+h)(L + k_1 h)} \sum_{i=0}^{n-1} \|e^0_i\|_{L^\infty(\sigma^0_i)}.$$

Then, by using Lemma 2.3.1, we deduce that,

$$\begin{aligned}
\|e^0_n\|_{L^\infty(\sigma_n^0)} + \|e^{0'}_n\|_{L^\infty(\sigma_n^0)} &\leq \|x - \hat{u}_{n,0}\|_{L^\infty(\sigma_n^0)} + \|\hat{u}_{n,0} - u^0_n\|_{L^\infty(\sigma_n^0)} \\
&\quad + \|x' - \hat{u}'_{n,0}\|_{L^\infty(\sigma_n^0)} + \|\hat{u}'_{n,0} - u^{0'}_n\|_{L^\infty(\sigma_n^0)} \\
&\leq \|x - \hat{u}_{n,0}\|_{L^\infty(\sigma_n^0)} + \|x' - \hat{u}'_{n,0}\|_{L^\infty(\sigma_n^0)} + \frac{\alpha(m)}{(m+1)!}h^{m+1} + \frac{\alpha(m)}{m!}h^m \\
&\leq \frac{(1+h)}{1 - Lh(1+h)}\left(\|e^0_{n-1}\|_{L^\infty(\sigma^0_{n-1})} + \|e^{0'}_{n-1}\|_{L^\infty(\sigma^0_{n-1})}\right) \\
&\leq \frac{(1+h)}{1 - Lh(1+h)}\left(\|e^0_{n-1}\|_{L^\infty(\sigma^0_{n-1})} + \|e^{0'}_{n-1}\|_{L^\infty(\sigma^0_{n-1})}\right),
\end{aligned}$$

where $M = \frac{\alpha(m)(\tau+m+1)}{(m+1)!}$. Hence by Lemma 1.6.3, for all $n \in \{0, 1, ..., N-1\}$

$$\begin{aligned}
\|e^0_n\|_{L^\infty(\sigma_n^0)} + \|e^{0'}_n\|_{L^\infty(\sigma_n^0)} &\leq \frac{\|e^0_0\|_{L^\infty(\sigma_0^0)} + \|e^{0'}_0\|_{L^\infty(\sigma_0^0)}}{R_2 - R_1}\left[(R_2 - 1)R_2^n + (1 - R_1)R_1^n\right] \\
&\quad + \frac{Mh^m}{R_2 - R_1}\left[R_2^n - R_1^n\right] \\
&\leq Mh^m \frac{[(R_2 - 1)R_2^n + (1 - R_1)R_1^n] + [R_2^n - R_1^n]}{R_2 - R_1},
\end{aligned} \tag{2.4.4}$$

where

$$R_1 = \left(1 + \frac{(1+h) - h\sqrt{\zeta}}{1 - hL(1+h)}\right)/2,$$
$$R_2 = \left(1 + \frac{(1+h) + h\sqrt{\zeta}}{1 - hL(1+h)}\right)/2, \tag{2.4.5}$$

such that $\zeta = (L+1)^2 + 2hL(L+1) + h^2L^2$.

Since $0 < R_1 \leq 1 \leq R_2$, then from (2.4.4), we get

$$\|e_n^0\|_{L^\infty(\sigma_n^0)} + \|e_n^{0'}\|_{L^\infty(\sigma_n^0)} \leq Mh^{m-1}\frac{(1 - hL(1+h))\left(\left[(R_2-1)R_2^{\frac{\tau}{h}} + (1-R_1)\right] + R_2^{\frac{\tau}{h}}\right)}{\sqrt{\zeta}},$$

we deduce that, there exist C_0 and h_1 such that, for all $h \in (0, h_1]$,

$$\|e_n^0\|_{L^\infty(\sigma_n^0)} + \|e_n^{0'}\|_{L^\infty(\sigma_n^0)} \leq C_0 h^{m-1}. \tag{2.4.6}$$

Thus,

$$\|e^0\|_{L^\infty(\sigma^0)} = \max_{n=0,\ldots,N-1} \|e_n^0\|_{L^\infty(\sigma_n^0)} \leq C_0 h^{m-1}.$$

Claim 2. There exists a constant C independent of h such that $\|e\|_{L^\infty(I)} \leq Ch^{m-1}$. Define the error $e^p(t)$ on σ^p by $e^p(t) = x(t) - u^p(t)$ and on σ_n^p by $e^p(t) = e_n^p(t) = x(t) - u_n^p(t)$ for all $n \in \{0, 1, \ldots, N-1\}$ and $p \in \{0, 1, \ldots, r-1\}$.

First, let $t \in \sigma_0^p$, for all $p \in \{1, \ldots, r-1\}$. we have from (2.2.6),

$$x''(t) - \hat{u}''_{0,p}(t) = A_1(t)(x(t) - \hat{u}_{0,p}(t)) + A_2(t)(x'(t) - \hat{u}'_{0,p}(t)) + B_1(t)e_0^{p-1}(t-\tau)$$
$$+ B_2(t)e_0^{p-1'}(t-\tau),$$

such that $x(t_0^p) - \hat{u}_{0,p}(t_0^p) = x(t_0^p) - u^{p-1}(t_0^p) = e^{p-1}(t_0^p)$
and $x'(t_0^p) - \hat{u}'_{0,p}(t_0^p) = x'(t_0^p) - u^{p-1'}(t_0^p) = e^{p-1'}(t_0^p)$,

this implies that,

$$\begin{aligned}||x'' - \hat{u}''_{0,p}||_{L^\infty(\sigma_0^p)} \leq L\left(||x - \hat{u}_{0,p}||_{L^\infty(\sigma_0^p)} + ||x' - \hat{u}'_{0,p}||_{L^\infty(\sigma_0^p)}\right) \\ + \widetilde{M}\left(||e_0^{p-1}||_{L^\infty(\sigma_0^{p-1})} + ||e_0^{p-1'}||_{L^\infty(\sigma_0^{p-1})}\right)\end{aligned} \tag{2.4.7}$$

On the other hand, we have

$$||x' - \hat{u}'_{0,p}||_{L^\infty(\sigma_0^p)} \leq ||e^{p-1'}||_{L^\infty(\sigma^{p-1})} + h||x'' - \hat{u}''_{0,p}||_{L^\infty(\sigma_0^p)}, \tag{2.4.8}$$

and

$$||x - \hat{u}_{0,p}||_{L^\infty(\sigma_0^p)} \leq ||e^{p-1}||_{L^\infty(\sigma^{p-1})} + h||e^{p-1'}||_{L^\infty(\sigma^{p-1})} + h^2||x'' - \hat{u}''_{0,p}||_{L^\infty(\sigma_0^p)}, \tag{2.4.9}$$

hence, from (2.4.8), (2.4.9) and (2.4.7), we get

$$\begin{aligned}||x - \hat{u}_{0,p}||_{L^\infty(\sigma_0^p)} + ||x' - \hat{u}'_{0,p}||_{L^\infty(\sigma_0^p)} &\leq ||e^{p-1}||_{L^\infty(\sigma^{p-1})} + (1+h)||e^{p-1'}||_{L^\infty(\sigma^{p-1})} \\ &\quad + h(1+h)||x'' - \hat{u}''_{0,p}||_{L^\infty(\sigma_0^p)} \\ &\leq (1+h)\left(||e^{p-1}||_{L^\infty(\sigma^{p-1})} + ||e^{p-1'}||_{L^\infty(\sigma^{p-1})}\right) \\ &\quad + hL(1+h)\left((||x - \hat{u}_{0,p}||_{L^\infty(\sigma_0^p)} + ||x' - \hat{u}'_{0,p}||_{L^\infty(\sigma_0^p)})\right) \\ &\quad + \widetilde{M}h(1+h)\sum_{i=0}^{p-1}\left(||e^i||_{L^\infty(\sigma^i)} + ||e^{i'}||_{L^\infty(\sigma^i)}\right),\end{aligned}$$

which implies that,

$$\begin{aligned}||x - \hat{u}_{0,p}||_{L^\infty(\sigma_0^p)} + ||x' - \hat{u}'_{0,p}||_{L^\infty(\sigma_0^p)} &\leq \frac{(1+h)}{1 - hL(1+h)}\left(||e^{p-1}||_{L^\infty(\sigma^{p-1})} + ||e^{p-1'}||_{L^\infty(\sigma^{p-1})}\right) \\ &\quad + \frac{\widetilde{M}h(1+h)}{1 - hL(1+h)}\sum_{i=0}^{p-1}\left(||e^i||_{L^\infty(\sigma^i)} + ||e^{i'}||_{L^\infty(\sigma^i)}\right) \\ &\leq \frac{(1+h)(1+\widetilde{M}h)}{1 - hL(1+h)}\sum_{i=0}^{p-1}\left(||e^i||_{L^\infty(\sigma^i)} + ||e^{i'}||_{L^\infty(\sigma^i)}\right).\end{aligned}$$

Therefore, by Lemma 2.3.1, we deduce that,

$$\begin{aligned}
\|e_0^p\|_{L^\infty(\sigma_0^p)} + \|e_0^{p'}\|_{L^\infty(\sigma_0^p)} &\le \|x - \hat{u}_{0,p}\|_{L^\infty(\sigma_0^p)} + \|\hat{u}_{0,p} - u_0^p\|_{L^\infty(\sigma_0^p)} \\
&\quad + \|x' - \hat{u}'_{0,p}\|_{L^\infty(\sigma_0^p)} + \|\hat{u}'_{0,p} - u_0^{p'}\|_{L^\infty(\sigma_0^p)} \\
&\le \|x - \hat{u}_{0,p}\|_{L^\infty(\sigma_0^p)} + \|x' - \hat{u}'_{0,p}\|_{L^\infty(\sigma_0^p)} + \underbrace{\frac{\alpha(m)(\tau + m + 1)}{(m+1)!}}_{M} h^m \qquad (2.4.10) \\
&\le \frac{(1+h)(1+\widetilde{M}h)}{1 - hL(1+h)} \sum_{i=0}^{p-1} \left(\|e^i\|_{L^\infty(\sigma^i)} + \|e^{i'}\|_{L^\infty(\sigma^i)} \right) + Mh^m,
\end{aligned}$$

Next, let $t \in \sigma_n^p$ for $n \in \{1, 2, ..., N-1\}$, we have from (2.2.9),

$$\begin{aligned}
x''(t) - \hat{u}''_{n,p}(t) &= A_1(t)(x(t) - \hat{u}_{n,p}(t)) + A_2(t)(x'(t) - \hat{u}'_{n,p}(t)) + B_1(t)e_n^{p-1}(t - \tau) \\
&\quad + B_2(t)e_n^{p-1'}(t - \tau),
\end{aligned}$$

such that $x(t_n^p) - \hat{u}_{n,p}(t_n^p) = x(t_n^p) - u_{n-1}^p(t_n^p)$
and $x'(t_n^p) - \hat{u}'_{n,p}(t_n^p) = x'(t_n^p) - u_{n-1}^{p'}(t_n^p)$,
this implies that,

$$\begin{aligned}
\|x'' - \hat{u}''_{n,p}\|_{L^\infty(\sigma_n^p)} &\le L \left(\|x - \hat{u}_{n,p}\|_{L^\infty(\sigma_n^p)} + \|x' - \hat{u}'_{n,p}\|_{L^\infty(\sigma_n^p)} \right) \\
&\quad + \widetilde{M} \sum_{i=0}^{p-1} \left(\|e^i\|_{L^\infty(\sigma^i)} + \|e^{i'}\|_{L^\infty(\sigma^i)} \right),
\end{aligned} \qquad (2.4.11)$$

On the other hand, we have

$$\|x' - \hat{u}'_{n,p}\|_{L^\infty(\sigma_n^p)} \le \|e_{n-1}^{p'}\|_{L^\infty(\sigma_{n-1}^p)} + h\|x'' - \hat{u}''_{n,p}\|_{L^\infty(\sigma_n^p)}, \qquad (2.4.12)$$

and

$$\|x - \hat{u}_{n,p}\|_{L^\infty(\sigma_n^p)} \le \|e_{n-1}^p\|_{L^\infty(\sigma_{n-1}^p)} + h\|e_{n-1}^{p'}\|_{L^\infty(\sigma_{n-1}^p)} + h^2\|x'' - \hat{u}''_{n,p}\|_{L^\infty(\sigma_n^p)}, \qquad (2.4.13)$$

hence, from (2.4.12), (2.4.13) and (2.4.11), we obtain

$$\begin{aligned}||x-\hat{u}_{n,p}||_{L^\infty(\sigma_n^p)}+||x'-\hat{u}'_{n,p}||_{L^\infty(\sigma_n^p)} &\leq (1+h)\left(||e_{n-1}^p||_{L^\infty(\sigma_{n-1}^p)}+||e_{n-1}^{p'}||_{L^\infty(\sigma_{n-1}^p)}\right)\\ &+hL(1+h)\left(||x-\hat{u}_{n,p}||_{L^\infty(\sigma_n^p)}+||x'-\hat{u}'_{n,p}||_{L^\infty(\sigma_n^p)}\right)\\ &+\widetilde{M}h(1+h)\sum_{i=0}^{p-1}\left(||e^i||_{L^\infty(\sigma^i)}+||e^{i'}||_{L^\infty(\sigma^i)}\right),\end{aligned}$$

this implies that,

$$\begin{aligned}||x-\hat{u}_{n,p}||_{L^\infty(\sigma_n^p)}+||x'-\hat{u}'_{n,p}||_{L^\infty(\sigma_n^p)} &\leq \frac{(1+h)}{1-hL(1+h)}\left(||e_{n-1}^p||_{L^\infty(\sigma_{n-1}^p)}+||e_{n-1}^{p'}||_{L^\infty(\sigma_{n-1}^p)}\right)\\ &+\frac{\widetilde{M}h(1+h)}{1-hL(1+h)}\sum_{i=0}^{p-1}\left(||e^i||_{L^\infty(\sigma^i)}+||e^{i'}||_{L^\infty(\sigma^i)}\right).\end{aligned}$$

Therefore, by Lemma 2.3.1, we have

$$\begin{aligned}||e_n^p||_{L^\infty(\sigma_n^p)}+||e_n^{p'}||_{L^\infty(\sigma_n^p)} &\leq ||x-\hat{u}_{n,p}||_{L^\infty(\sigma_n^p)}+||\hat{u}_{n,p}-u_n^p||_{L^\infty(\sigma_n^p)}\\ &+||x'-\hat{u}'_{n,p}||_{L^\infty(\sigma_n^p)}+||\hat{u}'_{n,p}-u_n^{p'}||_{L^\infty(\sigma_n^p)}\\ &\leq ||x-\hat{u}_{n,p}||_{L^\infty(\sigma_n^p)}+||x'-\hat{u}'_{n,p}||_{L^\infty(\sigma_n^p)}+\underbrace{\frac{\alpha(m)(\tau+m+1)}{(m+1)!}}_{M}h^m\\ &\leq \frac{(1+h)}{1-hL(1+h)}\left(||e_{n-1}^p||_{L^\infty(\sigma_{n-1}^p)}+||e_{n-1}^{p'}||_{L^\infty(\sigma_{n-1}^p)}\right)\\ &+\frac{\widetilde{M}h(1+h)}{1-hL(1+h)}\sum_{i=0}^{p-1}||e^i||_{L^\infty(\sigma^i)}+Mh^m.\end{aligned}$$

It follows from Lemma 1.6.3, for all $n \in \{0, 1, ..., N-1\}$,

$$\begin{aligned}
||e_n^p||_{L^\infty(\sigma_n^p)} + ||e_n^{p'}||_{L^\infty(\sigma_n^p)} &\leq \frac{||e_0^p||_{L^\infty(\sigma_0^p)} + ||e_0^{p'}||_{L^\infty(\sigma_0^p)}}{R_2 - R_1}[(R_2-1)R_2^n + (1-R_1)R_1^n] \\
&+ \frac{\frac{h(h+1)\widetilde{M}}{1-hL(1+h)}\sum_{i=0}^{p-1}||e^i||_{L^\infty(\sigma^i)} + Mh^m}{R_2 - R_1}[R_2^n - R_1^n] \\
&\leq \left(||e_0^p||_{L^\infty(\sigma_0^p)} + ||e_0^{p'}||_{L^\infty(\sigma_0^p)}\right)\frac{(R_2-1)R_2^{\frac{\tau}{h}} + (1-R_1)}{R_2 - R_1} \\
&+ \frac{(1+h)\widetilde{M}\sum_{i=0}^{p-1}||e^i||_{L^\infty(\sigma^i)} + (1-hL(1+h))Mh^{m-1}}{\sqrt{\zeta}}R_2^{\frac{\tau}{h}},
\end{aligned}$$

where R_1 and R_2 are defined by (2.4.5), and $\zeta = (L+1)^2 + 2h(L(L+1)) + h^2(L^2 + 4\widetilde{M})$. So, there exist C_1 and h_2 such that, for all $h \in (0, h_2]$,

$$||e_n^p||_{L^\infty(\sigma_n^p)} + ||e_n^{p'}||_{L^\infty(\sigma_n^p)} \leq \left(||e_0^p||_{L^\infty(\sigma_0^p)} + ||e_0^{p'}||_{L^\infty(\sigma_0^p)} + \sum_{i=0}^{p-1}||e^i||_{L^\infty(\sigma^i)} + h^{m-1}\right)C_1,$$

which implies, by (2.4.10), that for all $h \leq h_2$,

$$\begin{aligned}
||e_n^p||_{L^\infty(\sigma_n^p)} + ||e_n^{p'}||_{L^\infty(\sigma_n^p)} &\leq \left(\frac{(1+h)(1+\widetilde{M}h)}{1-hL(1+h)} + 1\right)C_1\sum_{i=0}^{p-1}\left(||e^i||_{L^\infty(\sigma^i)} + ||e^{i'}||_{L^\infty(\sigma^i)}\right) \\
&\quad + (M\tau + 1)C_1h^{m-1} \\
&\leq \left(\frac{(1+h_2)(1+\widetilde{M}h_2)}{1-h_2L(1+h_2)} + 1\right)C_1\sum_{i=0}^{p-1}\left(||e^i||_{L^\infty(\sigma^i)} + ||e^{i'}||_{L^\infty(\sigma^i)}\right) \\
&\quad + (M\tau + 1)C_1h^{m-1},
\end{aligned}$$

hence, for $C_2 = \max\left\{\left(\frac{(1+h_2)(1+\widetilde{M}h_2)}{1-h_2L(1+h_2)} + 1\right)C_1, (M\tau+1)C_1\right\}$, we obtain,

$$||e_n^p||_{L^\infty(\sigma_n^p)} + ||e_n^{p'}||_{L^\infty(\sigma_n^p)} \leq C_2\sum_{i=0}^{p-1}\left(||e^i||_{L^\infty(\sigma^i)} + ||e^{i'}||_{L^\infty(\sigma^i)}\right) + C_2h^{m-1}.$$

We deduce that,

$$\|e^p\|_{L^\infty(\sigma^p)} + \|e^{p'}\|_{L^\infty(\sigma^p)} \leq C_2 \sum_{i=0}^{p-1} \left(\|e^i\|_{L^\infty(\sigma^i)} + \|e^{i'}\|_{L^\infty(\sigma^i)} \right) + C_3 h^{m-1}, \tag{2.4.14}$$

where $C_3 = \max\{C_0, C_2\}$.

Then, from (2.4.6), (2.4.14) and by using Lemma 1.6.1, we get,

$$\|e^p\|_{L^\infty(\sigma^p)} + \|e^{p'}\|_{L^\infty(\sigma^p)} \leq C_3 h^{m-1} \exp(rC_2).$$

This implies that for all $p = 0, 1, ..., r-1$,

$$\|e^p\|_{L^\infty(\sigma^p)} \leq C_3 \exp(rC_2) h^{m-1}.$$

Thus, the proof is completed by taking $C = C_3 \exp(rC_2)$. ■

2.5 Numerical Examples

To illustrate the theoretical results, we present the following examples of numerically solving some second-order linear DDEs (Examples 2.5.1-2.5.2) and an ODE (Example 2.5.3).

The examples 2.5.1-2.5.3 compare the results obtained using our method with those obtained using the Spline method [11], the Direct two-point fourth and fifth order multistep block method [12], and the Multistep method [13].

It is evident that the results produced by our method significantly outperform those obtained by the Spline method [11], the Direct two-point fourth and fifth order multistep block method [12], and the Multistep method [13].

Example 2.5.1 *([11]) Consider the second order linear delay differential equation*

$$x''(t) = -5sin(t)e^{cos(t)} - (cos(t) + sin(t))\,x(t) - (6 + sin(t))\,x'(t)$$
$$+ sin(t - \frac{\pi}{4})x(t - \frac{\pi}{4}) + x'(t - \frac{\pi}{4}),\ t \in [0, 2],$$

and $\Phi(t) = e^{cos(t)}$ *for* $t \in [-\frac{\pi}{4}, 0)$. *The exact solution is* $x(t) = e^{\cos(t)}$. *The absolute errors for* $m = 11$, $h = 0.1$, *and* $h = 0.2$ *are compared with the absolute errors of the Spline method [11] in Table 2.1.*

Table 2.1: Comparison of the absolute errors of Example 2.5.1

t	$h = 0,2$		t	$h = 0,1$	
	Spline method	Present method		Spline method	Present method
0.2	1.15×10^{-9}	3.47×10^{-12}	0.1	2.32×10^{-12}	8.51×10^{-16}
0.4	5.80×10^{-10}	1.92×10^{-11}	0.2	5.07×10^{-12}	2.95×10^{-12}
0.6	4.78×10^{-10}	9.18×10^{-11}	0.3	1.24×10^{-12}	2.10×10^{-11}
0.8	1.76×10^{-9}	8.08×10^{-11}	0.4	2.25×10^{-11}	5.70×10^{-11}
1.0	4.42×10^{-9}	6.76×10^{-11}	0.5	7.68×10^{-11}	6.21×10^{-11}
1.2	2.29×10^{-8}	5.96×10^{-11}	0.6	1.97×10^{-10}	5.80×10^{-11}
1.4	5.37×10^{-9}	4.15×10^{-11}	0.7	1.12×10^{-10}	4.58×10^{-11}
1.6	1.93×10^{-9}	1.83×10^{-10}	0.8	6.16×10^{-11}	2.90×10^{-11}
1.8	2.57×10^{-10}	9.85×10^{-9}	0.9	5.44×10^{-11}	3.22×10^{-11}
2.0	1.25×10^{-9}	7.82×10^{-9}	1.0	9.26×10^{-11}	5.90×10^{-11}

Example 2.5.2 *([12]) We consider the second-order DDEs with constant delay*

$$x''(t) = \begin{cases} -\frac{1}{2}x(t) + \dfrac{1}{2}x(t-\pi),\ t \in [0, \pi], \\ \Phi(t) = 1 - \sin(t), \qquad t \in [-\pi, 0). \end{cases} \tag{2.5.1}$$

The exact solution is $x(t) = 1 - \sin(t)$.

$$x''(t) = \begin{cases} x(t-\pi),\ t \in [0, \pi], \\ \Phi(t) = \sin(t), \qquad t \in [-\pi, 0). \end{cases} \tag{2.5.2}$$

The exact solution is $x(t) = \sin(t)$.

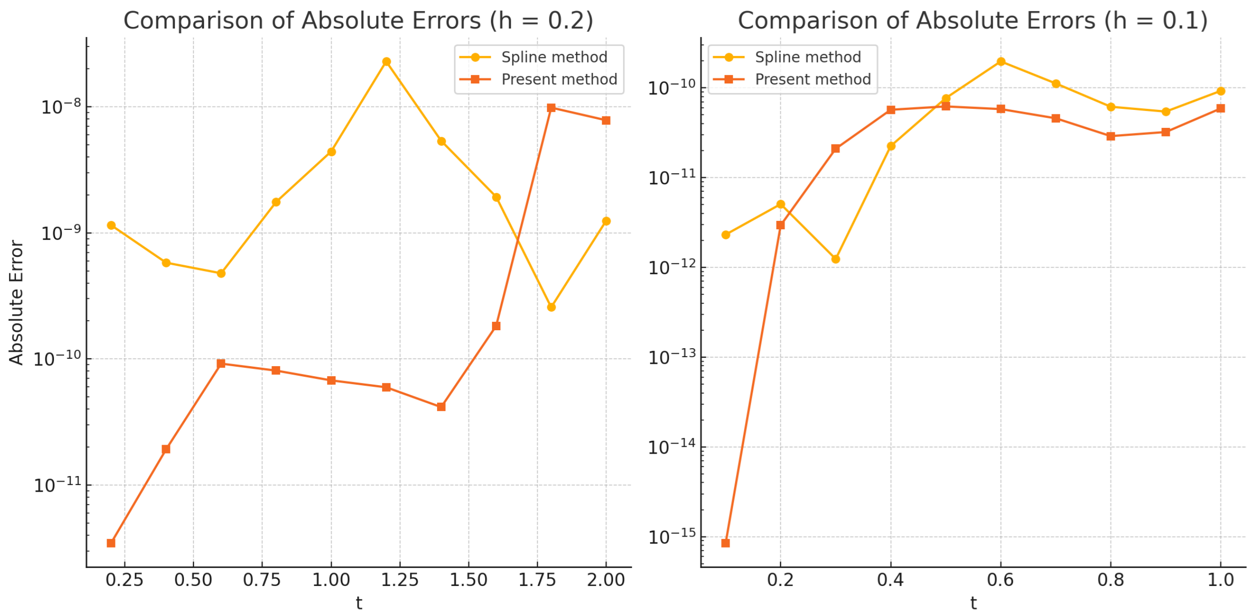

Figure 2.1: Comparison of the absolute errors of Example 2.5.1

In Table 2.2, we compare the maximum errors obtained for equations (2.5.1) *and* (2.5.2) *with* $m = 8$ *and* $h = \frac{\pi}{30}$ *against the Two Point Direct Block Method of order 4 (2PDBM4) and order 5 (2PDBM5) from [12].*

We observe that the present method outperforms the direct two-point fourth and fifth order multistep block methods [12] in terms of maximum error.

Table 2.2: Comparison of the maximum errors of Example 2.5.2

Method	Max error of eq(2.5.1)	Max error of eq(2.5.2)
2PDBM4	1.50×10^{-5}	5.83×10^{-5}
2PDBM5	2.93×10^{-6}	8.13×10^{-6}
Present Method	4.52×10^{-9}	5.38×10^{-9}

Example 2.5.3 *([13]) We consider the linear ODE for the instantaneous charge* $q(t)$ *at time* t *on the capacitor in an LRC series circuit, given by*

$$Lq'' + Rq' + \frac{1}{C}q = E(t), \quad q(0) = 0, \quad i(0) = q'(0) = 0, \quad t > 0$$

The exact solution is $q(t) = \frac{3}{4}\left(1 - e^{-10t}\left(\cos(10t) + \sin(10t)\right)\right)$*, where* L*,* C*,* R*,* $E(t)$*, and* $i(t)$

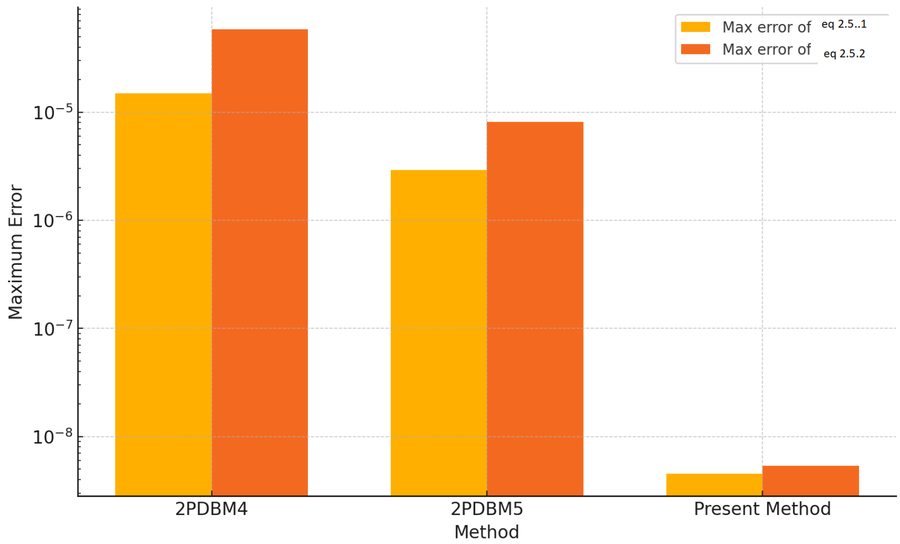

Figure 2.2: Comparison of of the maximum errors of Example 2.5.2

represent the inductance, capacitance, resistance, impressed voltage, and current, respectively. We solve the problem with $L = 1$, $R = 20$, $C = 0.005$, *and* $E(t) = 150$.

The absolute errors for $m = 7$, $m = 10$, *and* $h = 0.1$ *are compared with the absolute errors of the Multistep method [13] in Table 2.3.*

Table 2.3: Comparison of the absolute errors of Example 2.5.3

t	Multistep method[13]	Present method $m=7$	Present method $m=10$
0.0	0.0	0.0	0.0
0.1	1.61×10^{-3}	2.92×10^{-4}	1.10×10^{-6}
0.2	1.11×10^{-3}	1.00×10^{-3}	4.48×10^{-6}
0.3	3.52×10^{-4}	7.05×10^{-4}	2.22×10^{-6}
0.4	2.25×10^{-3}	1.62×10^{-4}	3.31×10^{-8}
0.5	2.81×10^{-3}	7.12×10^{-5}	5.20×10^{-7}
0.6	7.93×10^{-4}	6.86×10^{-5}	2.52×10^{-7}
0.7	1.50×10^{-5}	1.94×10^{-5}	7.57×10^{-9}
0.8	2.92×10^{-4}	3.35×10^{-6}	9.15×10^{-8}
0.9	2.43×10^{-4}	4.92×10^{-6}	4.54×10^{-8}
1.0	5.71×10^{-5}	1.63×10^{-6}	8.09×10^{-8}
1.1	9.21×10^{-6}	9.39×10^{-8}	4.59×10^{-8}
1.2	1.48×10^{-5}	3.09×10^{-7}	3.81×10^{-8}

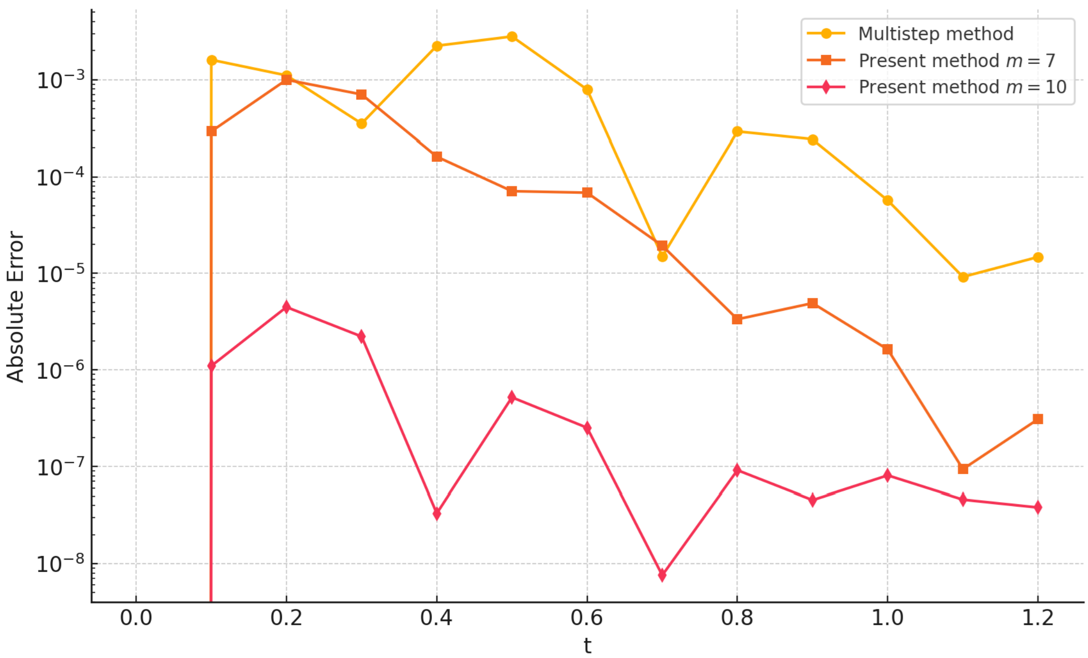

Figure 2.3: Comparison of the absolute errors of Example 2.5.3

CHAPTER 3

NUMERICAL SOLUTION OF FIRST ORDER DOUBLE DELAY DIFFERENTIAL EQUATIONS USING TAYLOR COLLOCATION METHOD

3.1 Introduction

In this chapter, we apply a direct collocation method based on the use of Taylor polynomials to approximate the solution of linear differential equations with two constant delays in the polynomial spline $S_m^{(0)}(\Pi_N)$. The approximate solution is given by using iterative formulas, and we prove the convergence of the approximate solution to the exact solution.

We consider the linear differential equation with two constant delays τ_1, τ_2 of the form:

$$x^{'}(t) = g(t) + A_1(t)x(t-\tau_1) + A_2(t)x(t-\tau_2) + B_1(t)x^{'}(t-\tau_1) + B_2(t)x^{'}(t-\tau_2), \qquad (3.1.1)$$

for $t \in [\tau_2, T]$ and $x(t) = \Phi(t)$ for $t \in [0, \tau_2]$. In the following we assume that the given functions g, A_1, A_2, B_1, B_2 and Φ are sufficiently smooth. Furthermore, we suppose that

$$\Phi^{'}(\tau_2) = g(\tau_2) + A_1(\tau_2)\Phi(\tau_2-\tau_1) + A_2(\tau_2)\Phi(0) + B_1(\tau_2)\Phi^{'}(\tau_2-\tau_1) + B_2(\tau_2)\Phi^{'}(0).$$

3.2 Description of the method

We suppose that $T = (r+1)\tau_2$, where $r \in \{1, 2, 3, ...\}$. Let Π_N be a uniform partition of the interval $I = [\tau_2, T]$ defined by $t_n^i = (i+1)\tau_2 + nh$, $n = 0, 1, .., N,\ i = 0, 1, .., r-1$, where the step-size is given by $h = t_{n+1}^i - t_n^i$ and assume that $h = \frac{\tau_1}{N_1} = \frac{\tau_2}{N}$ with N and N_1 positive and integer. Define the subintervals $\sigma_n^i = [t_n^i; t_{n+1}^i[, n = 0, 1, .., N-1,\ i = 0, 1, .., r-2$ and $\sigma_{N-1}^{r-1} = [t_{N-1}^{r-1}, t_N^{r-1}]$. Moreover, denote by π_m the set of all real polynomials of degree not exceeding m. We define the real polynomial spline space of degree $m-1$ as follows:

$$S_m^{(0)}(\Pi_N) = \{u \in C(I, \mathbb{R}) : u_n^i = u|_{\sigma_n^i} \in \pi_m, n = 0, ..., N-1,\ i = 0, 1, .., r-1\}. \qquad (3.2.1)$$

This is the space of piecewise polynomials of degree (at most) m. Its dimension is rNm, i.e., the same as the total number of the coefficients of the polynomials $u_n^p, n =$

$0, ..., N-1, p = 0, 1, .., r-1$. To find these coefficients, we use Taylor polynomial on each subinterval.

3.2.1 Approximate solution in the interval σ_0^0

First, we approximate x in the interval σ_0^0 by the polynomial

$$u_0^0(t) = \sum_{j=0}^{m} \frac{x^{(j)}(\tau_2)}{j!}(t-\tau_2)^j \, ; \quad t \in \sigma_0^0, \tag{3.2.2}$$

where $x^{(j)}(\tau_2), j = 0, ..., m$ is the exact value of $x^{(j)}$ at τ_2 and the function x must be differentiable around the τ_2 point. By differentiate equation (3.1.1) j-times, we get, for $j = 0, 1, ..., m-1$,

$$\begin{aligned} x^{(j+1)}(t) =& g^{(j)}(t) + (A_1(t)\Phi(t-\tau_1))^{(j)} + (A_2(t)\Phi(t-\tau_2))^{(j)} + \left(B_1(t)\Phi^{'}(t-\tau_1)\right)^{(j)} \\ &+ \left(B_2(t)\Phi^{'}(t-\tau_2)\right)^{(j)}(t), \end{aligned}$$

which implies,

$$\begin{aligned} x^{(j+1)}(t) =& g^{(j)}(t) + \sum_{l=0}^{j}\binom{j}{l}[A_1(t)]^{(j-l)}(t)\Phi^{(l)}(t-\tau_1) + \sum_{l=0}^{j}\binom{j}{l}[A_2(t)]^{(j-l)}(t)\Phi^{(l)}(t-\tau_2) \\ &+ \sum_{l=0}^{j}\binom{j}{l}[B_1(t)]^{(j-l)}(t)\,[\Phi]^{(l+1)}(t-\tau_1) + \sum_{l=0}^{j}\binom{j}{l}[B_2(t)]^{(j-l)}(t)\,[\Phi]^{(l+1)}(t-\tau_2), \end{aligned}$$

hence,

$$\begin{aligned} x^{(j+1)}(\tau_2) =& g^{(j)}(\tau_2) + \sum_{l=0}^{j}\binom{j}{l}[A_1(t)]^{(j-l)}(\tau_2)\Phi^{(l)}(\tau_2-\tau_1) + \sum_{l=0}^{j}\binom{j}{l}[A_2(t)]^{(j-l)}(\tau_2)\Phi^{(l)}(0) \\ &+ \sum_{l=0}^{j}\binom{j}{l}[B_1(t)]^{(j-l)}(\tau_2)\,[\Phi]^{(l+1)}(\tau_2-\tau_1) + \sum_{l=0}^{j}\binom{j}{l}[B_2(t)]^{(j-l)}(\tau_2)\,[\Phi]^{(l+1)}(0). \end{aligned}$$

3.2.2 Approximate solution in the interval σ_n^0

Second, for x to be approximated by u_n^0 ($n \in \{1, 2, ..., N-1\}$) on the interval σ_n^0, x must be approximated by u_k^0 ($0 \leq k < n$) on each interval σ_k^0, such that

$$u_n^0(t) = \sum_{j=0}^{m} \frac{\hat{u}_{n,0}^{(j)}(t_n^0)}{j!}(t - t_n^0)^j; \quad t \in \sigma_n^0, \tag{3.2.3}$$

where $\hat{u}_{n,0}$ is the exact solution of the differential equation
for $t \in \sigma_n^0$, $n \in \{1, 2, ..., N_1 - 1\}$,

$$\begin{aligned} \hat{u}_{n,0}(t) =& g(t) + A_1(t)\Phi(t-\tau_1) + A_2(t)\Phi(t-\tau_2) + B_1(t)\Phi^{'}(t-\tau_1) \\ &+ B_2(t)\Phi^{'}(t-\tau_2), \end{aligned}$$

and for $t \in \sigma_n^0$, $n \in \{N_1, N_1 + 1, ..., N-1\}$,

$$\begin{aligned} \hat{u}_{n,0}(t) =& g(t) + A_1(t)u_{n-N_1}^0(t-\tau_1) + A_2(t)\Phi(t-\tau_2) + B_1(t)\left[u_{n-N_1}^0(t-\tau_1)\right]^{'} \\ &+ B_2(t)\Phi^{'}(t-\tau_2). \end{aligned} \tag{3.2.4}$$

Now, for all $j = 0, 1, ..., m-1$, the formula for computing the values of the coefficients $\hat{u}_{n,0}^{(j)}(t_n^0)$ can be obtained by employing similar arguments to those used for obtaining the values of $x^{(j)}(\tau_2)$ above, we get the following formulas
for $n \in \{1, 2, ..., N_1 - 1\}$,

$$\begin{aligned} \hat{u}_{n,0}^{(j+1)}(t) =& g^{(j)}(t) + [A_1(t)\Phi(t-\tau_1)]^{(j)} + [A_2(t)\Phi(t-\tau_2)]^{(j)} + \left[B_1(t)\Phi^{'}(t-\tau_1)\right]^{(j)} \\ &+ \left[B_2(t)\Phi^{'}(t-\tau_2)\right]^{(j)}, \end{aligned}$$

which implies,

$$\begin{aligned}\hat{u}_{n,0}^{(j+1)}(t) =& g^{(j)}(t) + \sum_{l=0}^{j}\binom{j}{l}[A_1(t)]^{(j-l)}(t)\Phi^{(l)}(t-\tau_1) + \sum_{l=0}^{j}\binom{j}{l}[A_2(t)]^{(j-l)}(t)\Phi^{(l)}(t-\tau_2)\\ &+ \sum_{l=0}^{j}\binom{j}{l}[B_1(t)]^{(j-l)}(t)\,[\Phi]^{(l+1)}(t-\tau_1) + \sum_{l=0}^{j}\binom{j}{l}[B_2(t)]^{(j-l)}(t)\,[\Phi]^{(l+1)}(t-\tau_2),\end{aligned} \tag{3.2.5}$$

hence,

$$\begin{aligned}\hat{u}_{n,0}^{(j+1)}(t_0^n) =& g^{(j)}(t_0^n) + \sum_{l=0}^{j}\binom{j}{l}[A_1(t)]^{(j-l)}(t_0^n)\Phi^{(l)}(t_0^n-\tau_1) + \sum_{l=0}^{j}\binom{j}{l}[A_2(t)]^{(j-l)}(t_0^n)\Phi^{(l)}(t_0^n-\tau_2)\\ &+ \sum_{l=0}^{j}\binom{j}{l}[B_1(t)]^{(j-l)}(t_0^n)\,[\Phi]^{(l+1)}(t_0^n-\tau_1) + \sum_{l=0}^{j}\binom{j}{l}[B_2(t)]^{(j-l)}(t_0^n)\,[\Phi]^{(l+1)}(t_0^n-\tau_2),\end{aligned}$$

and for $n \in \{N_1, N_1+1, ..., N-1\}$,

$$\begin{aligned}\hat{u}_{n,0}^{(j+1)}(t) =& g^{(j)}(t) + \left(A_1(t)u_{n-N_1}^{0}(t-\tau_1)\right)^{(j)} + (A_2(t)\Phi(t-\tau_2))^{(j)} + \left(B_1(t)\left[u_{n-N_1}^{0}(t-\tau_1)\right]'\right)^{(j)}\\ &+ \left(B_2(t)\Phi'(t-\tau_2)\right)^{(j)}.\end{aligned} \tag{3.2.6}$$

which implies,

$$\begin{aligned}\hat{u}_{n,0}^{(j+1)}(t) =& g^{(j)}(t) + \sum_{l=0}^{j}\binom{j}{l}[A_1(t)]^{(j-l)}(t)\left[u_{n-N_1}^{0}(t-\tau_1)\right]^{(l)} + \sum_{l=0}^{j}\binom{j}{l}[A_2(t)]^{(j-l)}(t)\Phi^{(l)}(t-\tau_2)\\ &+ \sum_{l=0}^{j}\binom{j}{l}[B_1(t)]^{(j-l)}(t)\left[u_{n-N_1}^{0}(t-\tau_1)\right]^{(l+1)} + \sum_{l=0}^{j}\binom{j}{l}[B_2(t)]^{(j-l)}(t)\,[\Phi]^{(l+1)}(t-\tau_2),\end{aligned}$$

which implies,

$$\begin{aligned}\hat{u}_{n,0}^{(j+1)}(t) =& g^{(j)}(t) + \sum_{l=0}^{j}\sum_{s=0}^{m}\binom{j}{l}[A_1(t)]^{(j-l)}(t)\left[\frac{\hat{u}_{n-N_1,0}^{(s)}(t_{n-N_1}^0)}{s!}(t-\tau_1-t_{n-N_1}^0)^s\right]^{(l)}\\ &+\sum_{l=0}^{j}\binom{j}{l}[A_2(t)]^{(j-l)}(t)\Phi^{(l)}(t-\tau_2)\\ &+\sum_{l=0}^{j}\sum_{s=0}^{m}\binom{j}{l}[B_1(t)]^{(j-l)}(t)\left[\frac{\hat{u}_{n-N_1,0}^{(s)}(t_{n-N_1}^0)}{s!}(t-\tau_1-t_{n-N_1}^0)^s\right]^{(l+1)}\\ &+\sum_{l=0}^{j}\binom{j}{l}[B_2(t)]^{(j-l)}(t)\,[\Phi]^{(l+1)}(t-\tau_2),\end{aligned}$$

which implies,

$$\begin{aligned}\hat{u}_{n,0}^{(j+1)}(t) =& g^{(j)}(t) + \sum_{l=0}^{j}\sum_{s=0}^{m}\binom{j}{l}\frac{\hat{u}_{n-N_1,0}^{(s)}(t_{n-N_1}^0)}{(s-l)!}[A_1(t)]^{(j-l)}(t)(t-\tau_1-t_{n-N_1}^0)^{s-l}\\ &+\sum_{l=0}^{j}\binom{j}{l}[A_2(t)]^{(j-l)}(t)\Phi^{(l)}(t-\tau_2)\\ &+\sum_{l=0}^{j}\sum_{s=0}^{m}\binom{j}{l}\frac{\hat{u}_{n-N_1,0}^{(s)}(t_{n-N_1}^0)}{(s-l-1)!}[B_1(t)]^{(j-l)}(t)(t-\tau_1-t_{n-N_1}^0)^{s-l-1}\\ &+\sum_{l=0}^{j}\binom{j}{l}[B_2(t)]^{(j-l)}(t)\,[\Phi]^{(l+1)}(t-\tau_2),\end{aligned} \tag{3.2.7}$$

hence,

$$\begin{aligned}\hat{u}_{n,0}^{(j+1)}(t_n^0) =& g^{(j)}(t_n^0) + \sum_{l=0}^{j}\binom{j}{l}\hat{u}_{n-N_1,0}^{(l)}(t_{n-N_1}^0)\,[A_1(t)]^{(j-l)}(t_n^0)\\ &+\sum_{l=0}^{j}\binom{j}{l}[A_2(t)]^{(j-l)}(t_n^0)\Phi^{(l)}(t_n^0-\tau_2)\\ &+\sum_{l=0}^{j}\binom{j}{l}\hat{u}_{n-N_1,0}^{(l+1)}(t_{n-N_1}^0)\,[B_1(t)]^{(j-l)}(t_n^0)\\ &+\sum_{l=0}^{j}\binom{j}{l}[B_2(t)]^{(j-l)}(t_n^0)\,[\Phi]^{(l+1)}(t_n^0-\tau_2).\end{aligned}$$

3.2.3 Approximate solution in the interval σ_0^p

Third, for x to be approximated by u_0^p ($p \in \{1, 2, ..., r-1\}$) on the interval σ_0^p, x must be approximated by u_k^j ($0 \leq k \leq N-1$ and $0 \leq j < p$) on each interval σ_k^j such that,

$$u_0^p(t) = \sum_{j=0}^{m-1} \frac{\hat{u}_{0,p}^{(j)}(t_0^p)}{j!}(t - t_0^p)^j; \quad t \in \sigma_0^p, \tag{3.2.8}$$

where $\hat{u}_{0,p}$ is the exact solution of the differential equation

$$\begin{aligned} \hat{u}_{0,p}(t) =& g(t) + A_1(t)u_{N-N_1}^{P-1}(t - \tau_1) + A_2(t)u_0^{P-1}(t - \tau_2) + B_1(t)\left[u_{N-N_1}^{P-1}(t - \tau_1)\right]' \\ &+ B_2(t)\left[u_0^{P-1}(t - \tau_2)\right]'. \end{aligned} \tag{3.2.9}$$

The coefficients $\hat{u}_{0,p}^{(j+1)}(t)$ is given by the following formula

$$\begin{aligned} \hat{u}_{0,p}^{(j+1)}(t) =& g^{(j)}(t) + \left(A_1(t)u_{N-N_1}^{P-1}(t - \tau_1)\right)^{(j)} + \left(A_2(t)u_0^{P-1}(t - \tau_2)\right)^{(j)} + \left(B_1(t)\left[u_{N-N_1}^{P-1}(t - \tau_1)\right]'\right)^{(j)} \\ &+ \left(B_2(t)\left[u_0^{P-1}(t - \tau_2)\right]'\right)^{(j)}. \end{aligned} \tag{3.2.10}$$

which implies,

$$\begin{aligned} \hat{u}_{0,p}^{(j+1)}(t) =& g^{(j)}(t) + \sum_{l=0}^{j} \binom{j}{l} [A_1(t)]^{(j-l)}(t)\left[u_{N-N_1}^{P-1}(t - \tau_1)\right]^{(l)} + \sum_{l=0}^{j} \binom{j}{l} [A_2(t)]^{(j-l)}(t)\left[u_0^{P-1}(t - \tau_2)\right]^{(l)} \\ &+ \sum_{l=0}^{j} \binom{j}{l} [B_1(t)]^{(j-l)}(t)\left(u_{N-N_1}^{P-1}(t - \tau_1)\right)^{(l+1)} + \sum_{l=0}^{j} \binom{j}{l} [B_2(t)]^{(j-l)}(t)\left(u_0^{P-1}(t - \tau_2)\right)^{(l+1)}, \end{aligned}$$

which implies,

$$\begin{aligned}\hat{u}_{0,p}^{(j+1)}(t) =& g^{(j)}(t) + \sum_{l=0}^{j}\sum_{s=0}^{m}\binom{j}{l}[A_1(t)]^{(j-l)}(t)\left[\frac{\hat{u}_{N-N_1,P-1}^{(s)}(t_{N-N_1}^{P-1})}{s!}(t-\tau_1-t_{N-N_1}^{P-1})^s\right]^{(l)}\\
&+\sum_{l=0}^{j}\sum_{s=0}^{m}\binom{j}{l}[A_2(t)]^{(j-l)}(t)\left[\frac{\hat{u}_{0,P-1}^{(s)}(t_0^{P-1})}{s!}(t-\tau_2-t_0^{P-1})^s\right]^{(l)}\\
&+\sum_{l=0}^{j}\sum_{s=0}^{m}\binom{j}{l}[B_1(t)]^{(j-l)}(t)\left(\frac{\hat{u}_{N-N_1,P-1}^{(s)}(t_{N-N_1}^{P-1})}{s!}(t-\tau_1-t_{N-N_1}^{P-1})^s\right)^{(l+1)}\\
&+\sum_{l=0}^{j}\binom{j}{l}[B_2(t)]^{(j-l)}(t)\left(\frac{\hat{u}_{0,P-1}^{(s)}(t_0^{P-1})}{s!}(t-\tau_2-t_0^{P-1})^s\right)^{(l+1)},\end{aligned}$$

which implies,

$$\begin{aligned}\hat{u}_{0,p}^{(j+1)}(t) =& g^{(j)}(t) + \sum_{l=0}^{j}\sum_{s=0}^{m}\binom{j}{l}\frac{\hat{u}_{N-N_1,P-1}^{(s)}(t_{N-N_1}^{P-1})}{(s-l)!}[A_1(t)]^{(j-l)}(t)(t-\tau_1-t_{N-N_1}^{P-1})^{s-l}\\
&+\sum_{l=0}^{j}\sum_{s=0}^{m}\binom{j}{l}\frac{\hat{u}_{0,P-1}^{(s)}(t_0^{P-1})}{(s-l)!}[A_2(t)]^{(j-l)}(t)(t-\tau_2-t_0^{P-1})^{s-l}\\
&+\sum_{l=0}^{j}\sum_{s=0}^{m}\binom{j}{l}\frac{\hat{u}_{N-N_1,P-1}^{(s)}(t_{N-N_1}^{P-1})}{(s-l-1)!}[B_1(t)]^{(j-l)}(t)(t-\tau_1-t_{N-N_1}^{P-1})^{s-l-1}\\
&+\sum_{l=0}^{j}\sum_{s=0}^{m}\binom{j}{l}\frac{\hat{u}_{0,P-1}^{(s)}(t_0^{P-1})}{(s-l-1)!}[B_2(t)]^{(j-l)}(t)(t-\tau_2-t_0^{P-1})^{s-l-1},\end{aligned}$$

hence,

$$\begin{aligned}\hat{u}^{(j+1)}_{0,p}(t^P_0) =& g^{(j)}(t^P_0) + \sum_{l=0}^{j}\binom{j}{l}\hat{u}^{(l)}_{N-N_1,P-1}(t^{P-1}_{N-N_1})\,[A_1(t)]^{(j-l)}\,(t^P_0)\\ &+ \sum_{l=0}^{j}\binom{j}{l}\hat{u}^{(l)}_{0,P-1}(t^{P-1}_0)\,[A_2(t)]^{(j-l)}\,(t^P_0)\\ &+ \sum_{l=0}^{j}\binom{j}{l}\hat{u}^{(l+1)}_{N-N_1,P-1}(t^{P-1}_{N-N_1})\,[B_1(t)]^{(j-l)}\,(t^P_0)\\ &+ \sum_{l=0}^{j}\binom{j}{l}\hat{u}^{(l+1)}_{0,P-1}(t^{P-1}_0)\,[B_2(t)]^{(j-l)}\,(t^P_0).\end{aligned}$$

3.2.4 Approximate solution in the interval σ^p_n

Finally, for x to be approximated by u^p_n ($n \in \{1, ..., N-1\}$ and $p \in \{1, 2, ..., r-1\}$) on the interval σ^p_n, x must be approximated by u^j_k ($0 \le k < n$ and $0 \le j \le p$) on each interval σ^j_k such that,

$$u^p_n(t) = \sum_{j=0}^{m-1} \frac{\hat{u}^{(j)}_{n,p}(t^p_n)}{j!}(t - t^p_n)^j; \quad t \in \sigma^p_n, \tag{3.2.11}$$

where $\hat{u}_{n,p}$ is the exact solution of the differential equations
for $t \in \sigma^p_n$, $n \in \{1, 2, ..., N_1 - 1\}$,

$$\begin{aligned}\hat{u}_{n,p}(t) =& g(t) + A_1(t)u^{p-1}_{N-N_1+n}(t-\tau_1) + A_2(t)u^{p-1}_n(t-\tau_2) + B_1(t)\left[u^{p-1}_{N-N_1+n}(t-\tau_1)\right]'\\ &+ B_2(t)\left[u^{p-1}_n(t-\tau_2)\right]',\end{aligned} \tag{3.2.12}$$

and for $n \in \{N_1, N_1 + 1, ..., N-1\}$,

$$\begin{aligned}\hat{u}_{n,p}(t) =& g(t) + A_1(t)u^{p}_{n-N_1}(t-\tau_1) + A_2(t)u^{p-1}_n(t-\tau_2) + B_1(t)\left[u^{p}_{n-N_1}(t-\tau_1)\right]'\\ &+ B_2(t)\left[u^{p-1}_n(t-\tau_2)\right]',\end{aligned} \tag{3.2.13}$$

The coefficients $\hat{u}^{(j)}_{n,p}(t^p_n)$ is given by the following formula
for $t \in \sigma^p_n$, $n \in \{1, 2, ..., N_1 - 1\}$,

$$\hat{u}_{n,p}(t)^{(j+1)} = g(t)^{(j)} + \left(A_1(t)u^{p-1}_{N-N_1+n}(t-\tau_1)\right)^{(j)} + \left(A_2(t)u^{p-1}_{n}(t-\tau_2)\right)^{(j)} + \left(B_1(t)\left[u^{p-1}_{N-N_1+n}(t-\tau_1)\right]'\right)^{(j)} + \left(B_2(t)\left[u^{p-1}_{n}(t-\tau_2)\right]'\right)^{(j)}, \tag{3.2.14}$$

which implies,

$$\hat{u}_{n,p}(t)^{(j+1)}(t) = g^{(j)}(t) + \sum_{l=0}^{j}\binom{j}{l}[A_1(t)]^{(j-l)}(t)\left[u^{p-1}_{N-N_1+n}\right]^{(l)}(t-\tau_1) + \sum_{l=0}^{j}\binom{j}{l}[A_2(t)]^{(j-l)}(t)\left[u^{p-1}_{n}\right]^{(l)}(t-\tau_2) + \sum_{l=0}^{j}\binom{j}{l}[B_1(t)]^{(j-l)}(t)\left[u^{p-1}_{N-N_1+n}\right]^{(l+1)}(t-\tau_1) + \sum_{l=0}^{j}\binom{j}{l}[B_2(t)]^{(j-l)}(t)\left[u^{p-1}_{n}\right]^{(l+1)}(t-\tau_2),$$

which implies,

$$\begin{aligned}\hat{u}_{n,p}(t)^{(j+1)}(t) = {} & g^{(j)}(t) + \sum_{l=0}^{j}\sum_{s=0}^{m}\binom{j}{l}[A_1(t)]^{(j-l)}(t)\left[\frac{\hat{u}^{(s)}_{N-N_1+n,p-1}(t^{p-1}_{N-N_1+n})}{s!}(t-\tau_1-t^{p-1}_{N-N_1+n})^s\right]^{(l)} \\ & + \sum_{l=0}^{j}\sum_{s=0}^{m}\binom{j}{l}[A_2(t)]^{(j-l)}(t)\left[\frac{\hat{u}^{(s)}_{n,p-1}(t^{p-1}_{n})}{s!}(t-\tau_1-t^{p-1}_{n})^s\right]^{(l)} \\ & + \sum_{l=0}^{j}\sum_{s=0}^{m}\binom{j}{l}[B_1(t)]^{(j-l)}(t)\left[\frac{\hat{u}^{(s)}_{N-N_1+n,p-1}(t^{p-1}_{N-N_1+n})}{s!}(t-\tau_1-t^{p-1}_{N-N_1+n})^s\right]^{(l+1)} \\ & + \sum_{l=0}^{j}\sum_{s=0}^{m}\binom{j}{l}[B_2(t)]^{(j-l)}(t)\left[\frac{\hat{u}^{(s)}_{n,p-1}(t^{p-1}_{n})}{s!}(t-\tau_1-t^{p-1}_{n})^s\right]^{(l+1)},\end{aligned}$$

which implies,

$$
\begin{aligned}
\hat{u}_{n,p}(t)^{(j+1)}(t) =& g^{(j)}(t) + \sum_{l=0}^{j}\sum_{s=0}^{m}\binom{j}{l}\frac{\hat{u}^{(s)}_{N-N_1+n,p-1}(t^{p-1}_{N-N_1+n})}{(s-l)!}\,[A_1(t)]^{(j-l)}\,(t)(t-\tau_1-t^{p-1}_{N-N_1+n})^{s-l} \\
&+ \sum_{l=0}^{j}\sum_{s=0}^{m}\binom{j}{l}\frac{\hat{u}^{(s)}_{n,p-1}(t^{p-1}_{n})}{(s-l)!}\,[A_2(t)]^{(j-l)}\,(t)(t-\tau_2-t^{p-1}_{n})^{s-l} \\
&+ \sum_{l=0}^{j}\sum_{s=0}^{m}\binom{j}{l}\frac{\hat{u}^{(s)}_{N-N_1+n,p-1}(t^{p-1}_{N-N_1+n})}{(s-l-1)!}\,[B_1(t)]^{(j-l)}\,(t)(t-\tau_1-t^{p-1}_{N-N_1+n})^{s-l-1} \\
&+ \sum_{l=0}^{j}\sum_{s=0}^{m}\binom{j}{l}\frac{\hat{u}^{(s)}_{n,p-1}(t^{p-1}_{n})}{(s-l-1)!}\,[B_2(t)]^{(j-l)}\,(t)(t-\tau_2-t^{p-1}_{n})^{s-l-1},
\end{aligned}
$$

hence,

$$
\begin{aligned}
\hat{u}_{n,p}(t)^{(j+1)}(t^{p}_{n}) =& g^{(j)}(t^{p}_{n}) + \sum_{l=0}^{j}\binom{j}{l}\hat{u}^{(l)}_{N-N_1+n,p-1}(t^{p-1}_{N-N_1+n})\,[A_1(t)]^{(j-l)}\,(t^{p}_{n}) \\
&+ \sum_{l=0}^{j}\binom{j}{l}\hat{u}^{(l)}_{n,p-1}(t^{p-1}_{n})\,[A_2(t)]^{(j-l)}\,(t^{p}_{n}) \\
&+ \sum_{l=0}^{j}\binom{j}{l}\hat{u}^{(l+1)}_{n-N_1,0}(t^{0}_{N-N_1+n})\,[B_1(t)]^{(j-l)}\,(t^{p}_{n}) \\
&+ \sum_{l=0}^{j}\binom{j}{l}\hat{u}^{(l+1)}_{n,p-1}(t^{p-1}_{n})\,[B_2(t)]^{(j-l)}\,(t^{p}_{n}).
\end{aligned}
$$

And for $t \in \sigma_n^p$, $n \in \{N_1, N_1+1, ..., N-1\}$,

$$\hat{u}_{n,p}(t)^{(j+1)} = g^{(j)}(t) + \left(A_1(t)u_{n-N_1}^p(t-\tau_1)\right)^{(j)} + \left(A_2(t)u_n^{p-1}(t-\tau_2)\right)^{(j)} + \left(B_1(t)\left[u_{n-N_1}^p(t-\tau_1)\right]'\right)^{(j)} + \left(B_2(t)\left[u_n^{p-1}(t-\tau_2)\right]'\right)^{(j)}, \tag{3.2.15}$$

which implies,

$$\hat{u}_{n,p}(t)^{(j+1)}(t) = g^{(j)}(t) + \sum_{l=0}^{j}\binom{j}{l}[A_1(t)]^{(j-l)}(t)\left[u_{n-N_1}^p\right]^{(l)}(t-\tau_1) + \sum_{l=0}^{j}\binom{j}{l}[A_2(t)]^{(j-l)}(t)\left[u_n^{p-1}\right]^{(l)}(t-\tau_2) + \sum_{l=0}^{j}\binom{j}{l}[B_1(t)]^{(j-l)}(t)\left[u_{n-N_1}^p\right]^{(l+1)}(t-\tau_1) + \sum_{l=0}^{j}\binom{j}{l}[B_2(t)]^{(j-l)}(t)\left[u_n^{p-1}\right]^{(l+1)}(t-\tau_2),$$

which implies,

$$\begin{aligned}\hat{u}_{n,p}(t)^{(j+1)}(t) = & g^{(j)}(t) + \sum_{l=0}^{j}\sum_{s=0}^{m}\binom{j}{l}[A_1(t)]^{(j-l)}(t)\left[\frac{\hat{u}_{n-N_1,p}^{(s)}(t_{n-N_1}^p)}{s!}(t-\tau_1-t_{n-N_1}^p)^s\right]^{(l)} \\ & + \sum_{l=0}^{j}\sum_{s=0}^{m}\binom{j}{l}[A_2(t)]^{(j-l)}(t)\left[\frac{\hat{u}_{n,p-1}^{(s)}(t_n^{p-1})}{s!}(t-\tau_1-t_n^{p-1})^s\right]^{(l)} \\ & + \sum_{l=0}^{j}\sum_{s=0}^{m}\binom{j}{l}[B_1(t)]^{(j-l)}(t)\left[\frac{\hat{u}_{n-N_1,p}^{(s)}(t_{n-N_1}^p)}{s!}(t-\tau_1-t_{n-N_1}^p)^s\right]^{(l+1)} \\ & + \sum_{l=0}^{j}\sum_{s=0}^{m}\binom{j}{l}[B_2(t)]^{(j-l)}(t)\left[\frac{\hat{u}_{n,p-1}^{(s)}(t_n^{p-1})}{s!}(t-\tau_1-t_n^{p-1})^s\right]^{(l+1)},\end{aligned}$$

which implies,

$$
\begin{aligned}
\hat{u}_{n,p}(t)^{(j+1)}(t) =& g^{(j)}(t) + \sum_{l=0}^{j}\sum_{s=0}^{m}\binom{j}{l}\frac{\hat{u}^{(s)}_{n-N_1,p}(t^p_{n-N_1})}{(s-l)!}\left[A_1(t)\right]^{(j-l)}(t)(t-\tau_1-t^p_{n-N_1})^{s-l} \\
&+ \sum_{l=0}^{j}\sum_{s=0}^{m}\binom{j}{l}\frac{\hat{u}^{(s)}_{n,p-1}(t^{p-1}_{n})}{(s-l)!}\left[A_2(t)\right]^{(j-l)}(t)(t-\tau_2-t^{p-1}_{n})^{s-l} \\
&+ \sum_{l=0}^{j}\sum_{s=0}^{m}\binom{j}{l}\frac{\hat{u}^{(s)}_{n-N_1,p}(t^p_{n-N_1})}{(s-l-1)!}\left[B_1(t)\right]^{(j-l)}(t)(t-\tau_1-t^p_{n-N_1})^{s-l-1} \\
&+ \sum_{l=0}^{j}\sum_{s=0}^{m}\binom{j}{l}\frac{\hat{u}^{(s)}_{n,p-1}(t^{p-1}_{n})}{(s-l-1)!}\left[B_2(t)\right]^{(j-l)}(t)(t-\tau_2-t^{p-1}_{n})^{s-l-1},
\end{aligned}
$$

hence,

$$
\begin{aligned}
\hat{u}_{n,p}(t)^{(j+1)}(t^p_n) =& g^{(j)}(t^p_n) + \sum_{l=0}^{j}\binom{j}{l}\hat{u}^{(l)}_{n-N_1,p}(t^p_{n-N_1})\left[A_1(t)\right]^{(j-l)}(t^p_n) \\
&+ \sum_{l=0}^{j}\binom{j}{l}\hat{u}^{(l)}_{n,p-1}(t^{p-1}_{n})\left[A_2(t)\right]^{(j-l)}(t^p_n) \\
&+ \sum_{l=0}^{j}\binom{j}{l}\hat{u}^{(l+1)}_{n-N_1,p}(t^p_{n-N_1})\left[B_1(t)\right]^{(j-l)}(t^p_n) \\
&+ \sum_{l=0}^{j}\binom{j}{l}\hat{u}^{(l+1)}_{n,p-1}(t^{p-1}_{n})\left[B_2(t)\right]^{(j-l)}(t^p_n).
\end{aligned}
$$

Remark 3.2.1 *The proof of the convergence of the method related to equation* (3.1.1) *is in the same way as the proof in Chapter 2, relying on the theorems of Cornwall 1.6.1, 1.6.2 and 1.6.3.*

CONCLUSION

In this thesis, we investigated the numerical solutions of first and second-order linear differential equations with delays using the Taylor collocation method. The main contributions of this research can be summarized as follows:

- We formulated the Taylor collocation method for second-order neutral linear differential equations with constant delay and for first-order neutral differential equations with two constant delays.
- We derived approximate solutions using Taylor series expansions and validated their accuracy through the convergence analysis.
- We utilized Gronwall's inequalities to rigorously demonstrate that the proposed methods possess a convergence order, ensuring the reliability of the approximate solutions.
- We supported the theoretical findings with numerical examples, confirming the effectiveness and practicality of the Taylor collocation method in solving delay differential equations.

The results indicate that the Taylor collocation method is a robust and efficient technique for solving linear differential equations with delays, providing accurate and reliable approximations.

While the research presented in this thesis has made significant strides in the numerical analysis of delay differential equations, there remain several avenues for future exploration:

- **Extension to Nonlinear Equations**: The methods developed in this thesis can be extended to handle nonlinear delay differential equations, broadening their applicability to a wider range of real-world problems.
- **Higher-Order Delays**: Investigating the performance of the Taylor collocation method for differential equations with higher-order delays and exploring any necessary modifications to the algorithm.
- **Adaptive Collocation Points**: Developing adaptive strategies for selecting collocation points based on the behavior of the solution, which could enhance the accuracy and efficiency of the method.
- **Error Analysis**: Conducting a more comprehensive error analysis to better understand the limitations and potential improvements of the Taylor collocation method.
- **Applications to Other Fields**: Applying the Taylor collocation method to solve delay differential equations arising in different scientific and engineering domains, such as epidemiology, economics, and climate modeling.

By addressing these perspectives, future research can continue to enhance the capabilities and applications of the Taylor collocation method, contributing to the advancement of numerical analysis techniques for differential equations with delays.

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