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Iterative Collocation Method for Solving a class of Weakly Singular Volterra Integral Equations

Presented by: *- Yousra Djafer*

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Méthode de collocation itérative pour résoudre une classe d'équations intégrales de Volterra faiblement singulières

Préparé par: - Djafer Yousra

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DEDICATION

I would like to thank Allah first for everything, for granting us the courage and the will to pursue our studies. I extend my deepest gratitude and love to my beloved mother **Wafa**, my dear father **Djamal**, and to my sisters **Ayat, Hidaya and Takoua**.I extend my appreciation to all my family members, friends, and everyone who has encouraged me.

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ABSTRACT

In this dissertation , we presented some basic concepts, definitions, and necessary theorems for integral and integro-differential equations and their Classified has been presented.Then,we have used an iterative collocation method based on the Lagrange polynomials for solving a class of nonlinear weakly singular Volterra integral equations in the spline space$S_{m-1}^{-1}(I, \Pi_N)$. The main advantages of this method that, is easy to implement, has high order of convergence and the coefficients of approximate solution are determined by using iterative formulas without solving any system of algebraic equations. The numerical examples confirm that the method is convergent with a good accuracy.

Key words: Volterra integral equations, Collocation method, Iterative Method, Lagrange polynomials.

RÉSUMÉ

Dans ce mémoire, nous avons présenté quelques concepts de base, definitions et théor èmes nécessaires pour les equations integrales et integro-differentielles ainsi que leur classification . Ensuite,une résoudre numeriquement de quelques types déquations integrales de Volterra non lineaire au noyau faiblement singulier en utilisant une méthode de collocation iterative basée sur lútilisation de polynômes de Lagrange.

Mots-clés: Equations integrales de Volterra, Méthode de collocation, Polynomes de Lagrange.

ملخص

في هذه المذكرة قدمنا بعض المفاهيم الأساسية, التعريفات وبعض النظريات الضرورية للمعادلات التكاملية والتفاضلية- التكاملية و أصنافها. ثم قمنا بدراسة طريقة لحل فئة من المعادلات التكاملية لفولتيرا ذات النواة الشاذة. حيث يتم ايجاد الحل التقريبي لمعادلات فولتيرا التكاملية باستخدام طريقة التجميع التكرارية, بالاعتماد على كثيرات حدود لاغرانج . يتمثل الهدف في تقديم أمثلة عددية لتأكيد التقديرات النظرية وتوضيح تقارب الطريقة.

الكلمات المفتاحية:

معادلات فولتيرا التكاملية, التكاملية التفاضلية, طريقة التجميع التكرارية, كثيرات حدود لاغرانج.

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CONTENTS

INTRODUCTION

Many physical problems of science and technology which were solved with the help of theory of ordinary and partial differential equations can be solved by better methods of theory of integral equations.For example, while searching for the representation formula for the solution of linear differential equation in such a manner so as to include boundary conditions or intitial conditions explicitly, we arrive at an integral equation. The solution of the integral equation is much easier than the orginal boundary value or initial value problem. The theory of integral equations is very useful tool to deal with problems in applied mathematics, theoretical mechanis, and mathematical physics.Several situations of science lead to integral equations, e.g., neutron diffusion problem and radiation transfer problem etc.

Integral equations are one of the most useful mathematical tools in both pure and applied analysis. The first integral equation mentioned in the mathematical literature is due to Abel. He found this equation in 1823, starting from a problem in mechanics.He gave a very elegant solution that was published in 1826.

Starting in 1896, Vito Volterra built up a theory of integral equations, viewing their solutions as a problem of finding the inverses of certain integral operators. In 1900, Ivar Fredholm made his famous contribution that led to a fascinating period in the development of mathematical analysis. PoincarÃ©, FrÃ©chet, Hilbert, Schmidt, Hardy and Riesz were involved in this new area of research.Volterra integral equations belong to its owner Vito Volterra, among the most popular types of integral equations.It arises in many varieties of mathematical, scientific, and engineering problems. One such problem is the solution of parabolic differential equations with initial boundary conditions [22].

The aim of this thesis is to apply a new direct iterative collocation method based on the use of Lagrange polynomials for nonlinear Volterra integrals equations . This method is based on the idea of ing the exact solution of a given integral equation using a suitable function, belonging to a chosen finite dimensional space. The approximate solution must satisfy the integral equation on a certain subset of the interval (called the set of collocation points).
Our dissertation is organized as follows :
In the first chapter, we provide the fundamental notions, definitions and some necessary theorems , such as the classifications of integral and integro-differential equations,Leibniz rule, the linearity and the homogeneity concepts of integral equations, the conversion process of an Initial Value Problem to Volterra integral and integro-differntaille equation and discrete inequalities.

In the second chapter,It is to present a numerical method based on the use of Lagrange polynomials to solve a class of nonlinear weakly singular Volterra integral equations to approximate the solution of these equations. We prove the convergence of the approximate solution to the exact solution. Numerical examples illustrate the theoretical results. Finally, we hope that we have succeeded in presenting this topic.

CHAPTER 1

PRELIMINARY AND AUXILIARY RESULTS

An integral equation is defined as an equation in which the unknown function $u(t)$ to be determined appear under the integral sign. The subject of integral equations is one of the most useful mathematical tools in both pure and applied mathematics. It has enormous applications in many physical problems. Many initial and boundary value problems associated with ordinary differential equation (ODE) and partial differential equation (PDE) can be transformed into problems of solving some approximate integral equations. The development of science has led to the formation of many physical laws, which, when restated in mathematical form, often appear as differential equations,En gineering problems can be mathematically described by differential equations, and thus differential equations play very important roles in the solution of practical problems. For example, **Newton ś** law, stating that the rate of change of the momentum of a particle is equal to the force acting on it, can be translated into mathematical language as a differential equation. Similarly, problems arising in electric circuits, chemical kinetics,and transfer of heat in a medium can all be represented mathematically as differential equations. A typical form of an integral equation in $u(t)$ is of the form:

$$u(t) = f(t) + \lambda \int_{g(t)}^{h(t)} K(t,s)u(s)ds,$$

where$K(t,s)$ is called the kernel of the integral equation , and $g(t)$ and $h(t)$ are the limits of integration. It can be easily observed that the unknown function $u(t)$ appears under the integral sign. It is to be noted here that both the kernel $K(t,s)$ and the function $f(t)$ in equation (1.1) are given functions, λ and is a constant parameter. The prime objective of this text is to determine the unknown function $u(t)$ that will satisfy equation(1.1) using a number of solution techniques. We shall devote considerable efforts in exploring these methods to find solutions of the unknown function.

1.1 Classification of integral equations

An integral equation can be classified as a linear or nonlinear integral equation as we have seen in the ordinary and partial differential equations. In the previous section,we have noticed that the differential equation can be equivalently represented by the integral equation. Therefore, there is a good relationship between these two equations.

The most frequently used integral equations fall under two major classes, namely **Volterra** and **Fredholm** integral equations. Of course, we have to classify them as homogeneous or nonhomogeneous, and also linear or nonlinear. In some practical problems, we come across singular equations also.

In this text, we shall distinguish four major types of integral equations the two main classes and two related types of integral equations. In particular, the four types are given below:

1. Volterra integral equations.
2. Fredholm integral equations.
3. Volterra Fredholm integral equations.
4. Singular integral equations.
5. Integro-differential equations.

We shall outline these equations using basic definitions and properties of each type.

1.2 Volterra integral equations:

It is well known that linear and nonlinear Volterra integral equations arise in many scientific fields such as the population dynamics, spread of epidemics, and semi-conductor devices. Volterra started working on integral equations in 1884, but his serious study began in 1896. The name integral equation was given by du Bois-Reymond in 1888.

However, the name Volterra integral equation was first coined by Lalesco in 1908 [65].

The most standard form of a **Volterra** integral equations is of the form:

$$\phi(t)u(t) = f(t) + \lambda \int_a^t K(t,s,u(s))ds \tag{1.1}$$

where the limits of integration are function of x and the unknown function $u(t)$ appears linearly under the integral sign.

1. $\phi(t) = 0$, then equation (1.1) becomes:

   $$f(t) + \lambda \int_a^t K(t,s,u(s))ds = 0 \tag{1.2}$$

   which is known as the Volterra equation of **the first kind.**

2. If the function $\phi(t) = 1$, then equation (1.1) simply becomes:

   $$u(t) = f(t) + \lambda \int_a^t K(t,s,u(s))ds \tag{1.3}$$

   and this equation is known as the Volterra integral equation of **the second kind.**

3. If the function $\phi(t)$ vanishes on a non-empty proper subset of $[a,b]$ (for example, a finite number of points or a compact (proper) subinterval), then (1.1) becomes integral equation of **the third kind.**

Examples of the Volterra integral equations of the first kind are:

$$5t^2 + t^3 = \int_0^t (5 + 3 * t - 3s)u(s)ds \tag{1.4}$$

However, examples of the Volterra integral equations of the second kind are:

$$u(t) = t + \int_0^t (t - s)u(s)ds \tag{1.5}$$

Nonlinear Volterra Integral Equations:

The nonlinear Volterra integral equation of the second kind is represented by the form,

$$u(t) = f(t) + \lambda \int_a^t K(t, s, u(s))ds$$

The nonlinear Volterra integral equation of the first kind is expressed in the form:

$$f(t) = \lambda \int_a^t K(t, s, u(s))ds$$

Nonlinear Volterra-Hammerstein Integral Equations:

The nonlinear Volterra-Hammerstein integral equation of the second kind is represented by the form,

$$u(t) = f(t) + \lambda \int_a^t K(t, s)F(s, u(s))ds,$$

1.3 Fredholm Integral Equations:

Fredholm integral equations arise in many scientific applications. It was also shown that, this equation can be derived from boundary value problems. Erik Ivar Fredholm (1866-1927) is best remembered for his work on integral equations and spectral theory. Fredholm was a Swedish mathematician who established the theory of integral equations and his 1903 paper in Acta Mathematica played a major role in the establishment of operator theory (Wazwaz(2011)).

For Fredholm integral equations, the limits of integration are fixed. The most standard form of **Fredholm** linear integral equations is given by the following form

$$\phi(t)u(t) = f(t) + \lambda \int_a^b K(t, s, u(s))ds \tag{1.6}$$

As in Volterra equations, Fredholm integral equations fall under the following kinds,depending on the value of $\phi(t)$, namely:

1. Fredholm integral equation of **the first kind**, when $\phi(t) = 0$.
2. Fredholm integral equation of **the second kind**, when $\phi(t) = 1$.
3. If the function $\phi(t)$ vanishes on a non-empty proper subset of $[a, b]$ (for example,a finite number of points or a compact (proper) subinterval), then (1.2) becomes integral equation of **the third kind.**

Nonlinear Fredholm Integral Equations:
The nonlinear Fredholm integral equations of the second kind is given by the following form

$$u(t) = f(t) + \lambda \int_a^b K(t, s, u(s))ds, \;\; a \le t, \;\; s \le b.$$

Where the unknown function $u(t)$ occurs inside and outside the integral sign, λ is a parameter, and a and b are constants. For this type of equations, the kernel k and the function $f(t)$ are given real-valued functions.

Nonlinear Fredholm-Hammerstein Integral Equations:
Nonlinear Fredholm-Hammerstein integral equations is given by the form,

$$u(t) = f(t) + \lambda \int_a^b K(t, s)F(s, u(s))ds, \;\; a \le t, \;\; s \le b,$$

1.4 Volterra Fredholm integral equations

The Volterra-Fredholm integral equations arise from the modelling of the spatiotemporal development of an epidemic, from boundary value problems and from many physical and chemical applications [65]. The standard form of the linear **Volterra-Fredholm** integral equation is in the form:

$$u(t) = f(t) + \int_a^t K_1(t,s)u(s)ds + \int_a^b K_2(t,s)u(s)ds$$

where $k_1(t,s)$ and $k_2(t,s)$ are the kernels of the equation.

Nonlinear Volterra-Fredholm Integral Equations:
The standard form of the Nonlinear Volterra-Fredholm integral equation is in the form,

$$u(t) = f(t) + \int_a^t K_1(t,s,u(s))ds + \int_a^b K_2(t,s,u(s))ds$$

Nonlinear Volterra-Fredholm-Hammerstein Integral Equations:
The standard form of the Nonlinear Volterra-Fredholm-Hammerstein integral equation is in the form:

$$u(t) = f(t) + \int_a^t K_1(t,s)F(s,u(s))ds + \int_a^b K_2(t,s)G(s,u(s))ds$$

where $k_1(t,s)$ and $k_2(t,s)$ are the kernels of the equation.

1.5 Singular Integral Equations

Volterra integral equations of the first kind,

$$f(t) = \lambda \int_{g(t)}^{h(t)} K(t,s)u(s)ds$$

or of the second kind

$$u(t) = f(t) + \lambda \int_{g(t)}^{h(t)} K(t,s)u(s)ds$$

are called **singular** if one of the limit of integration $g(t), h(t)$ is infinite or the kernel $k(t,s)$ becomes unbounded at one or more points in the interval of integration. We focus on concern on equation of the form:

$$u(t) = f(t) + \lambda \int_{0}^{t} \frac{1}{(t-s)^{\alpha}} u(s)ds, \quad 0 \leq \alpha \leq 1 \tag{1.7}$$

or of **the second kind**

$$f(t) = \lambda \int_{0}^{t} \frac{1}{(t-s)^{\alpha}} u(s)ds, \quad 0 \leq \alpha \leq 1 \tag{1.8}$$

The Eq. (**??**) and Eq.(**??**) are called generalized Abel's integral equation and weakly singular integral equations respectively.
On the other hand, the well known weakly singular Fredholm integral equations of the form:

$$u(t) = f(t) + \int_{0}^{1} k(t,s)u(s)ds, \quad 0 \leq \alpha \leq 1$$

where the singularity of kernel may be stated in the forms $k(t,s) = \dfrac{1}{(t-s)^{\alpha}}$ or $k(t,s) = \dfrac{1}{(1-t)^{\alpha}}$.

Definition 1.5.1 ***(The homogeneity property)***
We set $f(t) = 0$ in Fredholm or Volterra integral and integro-differential equations as given in the above, the resulting equations is called a homogeneous integral and integro-differential equations, otherwise it is called nonhomogeneous or inhomogeneous integral and integro-differential equations.

Theorem 1.5.1 (Leibnits) *Let $f(x)$ be continuous $[a, b]$,so:*

$$\forall x \in [a, b], \int_0^x \int_0^{x_1} ... \int_0^{x_{n-1}} f(x_n) dx_n ... dx_1 = \frac{1}{(n-1)!} \int_a^x (x-t)^{n-1} f(t) dt.$$

1.6 Integro-differential equations

In the early 1900, Vito Volterra studied the phenomenon of population growth, and new types of equations have been developed and termed as the integro-differential equations. In this type of equations, the unknown function $u(t)$ appears as the combination of the ordinary derivative and under the integral sign.

1.7 Classification of Integro-Differential Equations

Integro-differential equations appear in many scientific applications, especially when we convert initial value problems or boundary value problems to integral equations. The integro-differential equations contain both integral and differential operators. The derivatives of the unknown functions may appear to any order. In classifying integro-differential equations.

Volterra Integro-Differential equations:

Volterra, in the early 1900, studied the population growth, where new type of equations have been developed and was termed as integro-differential equations. In this type of equations, the unknown function u(t) occurs in one side as an ordinary derivative, and appears on the other side under the integral sign. Several phenomena in physics and biology give rise to this type of integro-differential equations. Further,we point out that an integro-differential equation can be easily observed as an inter-mediate stage when we convert a differential equation to an integral equation in next section.

The Volterra integro-differential equation appeared after its establishment by Volterra.It then appeared in many physical applications such as glass forming process, nanohydrodynamics, heat transfer, diffusion process in general, neutron diffusion and biological species coexisting together with increasing and decreasing rates of generating, and wind ripple in the desert. More details about the sources where these equations arise can be found in physics, biology and engineering applications books (see, for example Brunner [10], Volterra [54]. To determine the exact solution for the integro-differential equation, the initial conditions should be given. The Volterra integro-differential equations can be converted to an integral equation by using Leibnitz rule .

Nonlinear Volterra Integro-differential Equations:

The nonlinear Volterra integro-differential equation of the second kind is in the form,

$$u^{(n)}(t) = f(t) + \int_a^t K(t, s, u(s), u'(s), \ldots, u^{n-1}(s))ds, \quad u^{(k)}(a) = b_k, \quad 0 \leq k \leq n-1$$

and the standard form of the nonlinear Volterra integro-differential equation of the first kind is given by,

$$\int_a^t K(t, s, u(s), u'(s), \ldots, u^{n-1}(s))ds = f(t),$$

Nonlinear Volterra-Hammerstein Integro-differential Equations:

The nonlinear Volterra-Hammerstein integro-differential equation of the second kind is in the form,

$$u^{(n)}(t) = f(t) + \int_a^t K(t, s)F(s, u(s), u'(s), \ldots, u^{n-1}(s))ds, \quad u^{(k)}(a) = b_k, \quad 0 \leq k \leq n-1$$

Fredholm Integro-Differential Equations :

Fredholm integro-differential equations appear when we convert differential equations to integral equations.

The Fredholm integro-differential equation contains the un-known function $u(t)$ and one of its derivatives $u^n, n \geqslant 1$ inside and outside the integral sign respectively. The limits of integration in this case are fixed as in the Fredholm integral equations. The equation is labeled as integro-differential because it contains differential and integral operators in the same equation. It is important to note that initial conditions should be given for Fredholm integro-differential equations to obtain the particular solutions. The Fredholm integro-differential equation appears in the form:

$$u^{(n)}(t) = f(t) + \lambda \int_a^b K(t, s, u(s))ds$$

where u^n indicates the nth derivative of $u(t)$. Other derivatives of less order may appear with u^n at the left side. **Nonlinear Fredholm Integro-differential Equations:**
The nonlinear Fredholm integro-differential equations is given by the following form,

$$u^n(t) = f(t) + \int_a^b K(t, s, u(s), u'(s), \dots, u^{n-1}(s))ds, \quad u^k(a) = b_k, \quad 0 \leq k \leq n-1, \qquad (1.9)$$

where $u^n(t) = \frac{d^n u}{dt^n}$ Because the resulted equation in (1.9) combines the differential operator and the integral operator, then it is necessary to define initial conditions $u(0),\ u^{'}(0), ..., u^{n-1}(0)$ for the determination of the particular solution $u(t)$ of the equation (1.9). Any Fredholm integro-differential equation is characterized by the existence of one or more of the derivatives $u'(t),\ u^{''}(t), ...$ outside the integral sign. The Fredholm integro-differential equations of the second kind appear in a variety of scientific applications such as the theory of signal processing and neural networks.

Nonlinear Fredholm-Hammerstein Integro-differential Equations:

The nonlinear Fredholm-Hammerstein integro-differential equations of the second kind is of the form:

$$u^n(t) = f(t) + \int_a^b K(t,s)F(s,u(s),u'(s),\ldots,u^{n-1}(s))ds.$$

Volterra-Fredholm Integro-Differential Equations

The Volterra-Fredholm integro-differential equation, which is a combination of disjoint Volterra and Fredholm integrals, appears in one integral equation. The Volterra-Fredholm integral equations arise from the modelling of the spatiotemporal development of an epidemic, from boundary value problems and from many physical and chemical applications [65]. The standard form of the linear Volterra-Fredholm integral equation is in the form,

$$u(t) = f(t) + \int_a^t K_1(t,s)u(s)ds + \int_a^b K_2(t,s)u(s)ds$$

where $k_1(t,s)$ and $k_2(t,s)$ are the kernels of the equation.

Volterra-Fredholm Integro-differential Equations:

The Volterra-Fredholm integro-differential equation, which is a combination of disjoint Volterra and Fredholm integrals and differential operator, may appear in one integral equation. The Volterra-Fredholm integro-differential equations arise from many physical and chemical applications similar to the Volterra-Fredholm integral equations [5], [6], [56], [55]. The standard form of the Volterra-Fredholm integro-differential equation is in the form,

$$u^{(n)}(t) = f(t) + \int_a^t K_1(t,s,u(s),u'(s),\ldots,u^{n-1}(s))ds + \int_a^b K_2(t,s,u(s),u'(s),\ldots,u^{n-1}(s))ds$$

Nonlinear Volterra-Fredholm-Hammerstein Integro-differential Equations:

$$u^{(n)}(t) = f(t) + \int_a^t K_1(t,s)F(t,s,u(s),u'(s),\ldots,u^{n-1}(s))ds + \int_a^b K_2(t,s,u(s),u'(s),\ldots,u^{n-1}(s))ds$$

1.8 Conversion of Differential equations to Integral equations

In general, the initial values problems (IVP) can be transformed to Volterra integral equations, and the boundary values problems (BVP) can be transformed to Fredholm integral equations and virse versa

IVP to Volterra Integral equations:

In this section, we will study the technique that will convert an initial value problem (IVP) to an equivalent Volterra integral equation and Volterra integro-differential equation as well [65]. For simplicity reasons, we will apply this process to a second order initial value problem given by

$$u''(t) + p(t)u'(t) + q(t)u(t) = g(t) \tag{1.10}$$

$$u(0) = \alpha, u'(0) = \beta$$

where α and β are constants. The functions $p(t)$ and $q(t)$ are analytic functions, and $g(t)$ is continuous through the interval of discussion. To achieve our goal we first set

$$u''(t) = v(t), \tag{1.11}$$

where $v(t)$ is a continuous function. Integrating both sides of (1.11) from 0 to t yields

$$u'(t) - u'(0) = \int_0^t v(s)ds$$

or equivalently

$$u'(t) = \beta + \int_0^t v(s)ds \tag{1.12}$$

Integrating both sides of (1.12) from 0 to t yields

$$u(t) - u(0) = \beta t + \int_0^t \int_0^s v(r)drds$$

or equivalently

$$u(t) = \alpha + \beta t + \int_0^t (t-s)v(s)ds \tag{1.13}$$

obtained upon using the formula that reduce double integral to a single integral that was discussed in the next section. Substituting (1.11), (1.12), and (1.13) into the initial value problem (1.10) yields the Volterra integral equation:

$$v''(t) + p(t)\left[\beta + \int_0^t v(s)ds\right] + q(t)\left[\alpha + \beta t + \int_0^t (t-s)v(t)dt\right] = g(t).$$

The last equation can be written in the standard Volterra integral equation form:

$$v(t) = f(t) + \int_0^t k(t,s)v(s)ds, \tag{1.14}$$

where

$$k(t,s) = p(t) + q(t)(t-s),$$

and

$$f(t) = g(t) - [\beta p(t) + \alpha q(t) + \beta t q(t)] .$$

It is interesting to point out that by differentiating Volterra equation (1.14) with respect to t, using Leibnitz rule, we obtain an equivalent Volterra integro-differential equation in the form:

$$u'(t) + k(t,t) = f'(t) - \int_0^t \frac{\partial k(t,s)}{\partial t} u(s)ds, \quad u(0) = f(0)$$

The technique presented above to convert initial value problems to equivalent Volterra integral equations can be generalized by considering the general initial value problem:

$$u^{(n)}(t) + a_1 u^{n-1} + ... + a_{n-1} u^{'} + a_n u = g(t) \tag{1.15}$$

subject to the initial conditions

$$u(0) = c_0, u'(0) = c_1, u''(0) = c_2, ..., u^{n-1} = c_{n-1}.$$

Let $v(t)$ be a continuous function on the interval of discussion, and we consider the transformation:

$$u^{(n)}(t) = v(t). \tag{1.16}$$

Integrating both sides with respect to t gives

$$u^{(n-1)}(t) = c_{n-1} + \int_0^t v(t)dt.$$

Integrating again both sides with respect to t yields

$$\begin{aligned} u^{(n-2)}(t) &= c_{n-2} + c_{n-1}t + \int_0^t \int_0^t u(s)dsds \\ &= c_{n-2} + c_{n-1}t + \int_0^t (t-s)u(s)ds, \end{aligned}$$

obtained by reducing the double integral to a single integral. Proceeding as before we find

$$\begin{aligned} u^{(n-3)}(t) &= c_{n-3} + c_{n-2}t + \frac{1}{2}c_{n-1}t^2 + \int_0^t \int_0^t \int_0^t v(s)dsdsds \\ &= c_{n-3} + c_{n-2}t + \frac{1}{2}c_{n-1}t^2 + \frac{1}{2}\int_0^t (t-s)^2 v(s)ds. \end{aligned}$$

Continuing the integration process leads to

$$u(t) = \sum_{k=0}^{n-1} \frac{c_k}{k!}t^k + \frac{1}{(n-1)!}\int_0^t (t-s)^{n-1} v(s)ds. \tag{1.17}$$

Substituting (1.16)?(1.17) into (1.15) gives

$$u(t) = f(t) + \int_0^t k(t,s)v(s)ds, \tag{1.18}$$

where

$$k(t,s) = \sum_{k=1}^{n} \frac{a_n}{k-1!}(t-s)^k - 1,$$

and

$$f(t) = g(t) - \sum_{j=1}^{n} a_j \left(\sum_{k=1}^{j} \frac{c_n - k}{(j-k)!} t^j \right).$$

Notice that the Volterra integro-differential equation can be obtained by differentiating (1.18).

The following examples will highlight the process to convert initial value problem to an equivalent Volterra integral equation.

BVP to Fredholm Integral equations:

In this section, we will convert a boundary value problem to an equivalent Fredholm integral equation. The method is similar to the method that was presented in the above section for converting Volterra equation to IVP, with the exception that boundary conditions will be used instead of initial values. In this case we will determine another initial condition that is not given in the problem. The technique requires more work if compared with the initial value problems when converted to Volterra integral equations. Without loss of generality, we will present two specific distinct boundary value problems (BVPs) to derive two distinct formulas that can be used for converting BVP to an equivalent Fredholm integral equation [65].

Type I: We first consider the following boundary value problem:

$$u''(t) + g(t)u(t) = h(t), \quad 0 \leq t \leq 1, \tag{1.19}$$

with the boundary conditions:

$$u(0) = \alpha, \quad u(1) = \beta.$$

We start as in the previous section and set

$$u''(t) = v(t). \tag{1.20}$$

Integrating both sides of (1.20) from 0 to t we obtain

$$\int_0^t u''(s)ds = \int_0^t v(s)ds,$$

that gives

$$u'(t) = u'(0) + \int_0^t v(s)ds, \tag{1.21}$$

where the initial condition $u'(0)$ is not given in a boundary value problem. The condition $u'(0)$ will be determined later by using the boundary condition at $t = 1$. Integrating both sides of (1.21) from 0 to t gives

$$u(t) = u(0) + tu'(0) + \int_0^t \int_0^t v(s)dsds,$$

or equivalently

$$u(t) = \alpha + tu'(0) + \int_0^t (t-s)v(s)ds, \tag{1.22}$$

obtained upon using the condition $u(0) = \alpha$ and by reducing double integral to a single integral. To determine $u'(0)$, we substitute $t = 1$ into both sides of (1.19) and using the boundary condition at $u(1) = \beta$ we find

$$u(1) = \alpha + u'(0) + \int_0^1 (1-s)v(s)ds,$$

that gives

$$\beta = \alpha + u'(0) + \int_0^1 (1-s)v(s)ds.$$

This in turn gives

$$u'(0) = \beta - \alpha - \int_0^1 (1-s)v(s)ds. \tag{1.23}$$

Substituting (1.23) into (1.22) gives

$$u(t) = \alpha + (\beta-\alpha)t - \int_0^1 t(1-s)v(s)ds + \int_0^t (t-s)v(s)ds. \tag{1.24}$$

Substituting (1.20) and (1.24) into (1.19) yields

$$u(t) + \alpha g(t) + (\beta - \alpha)tg(t) - \int_0^1 tg(t)(1-s)v(s)ds + \int_0^t g(t)(t-s)v(s)ds = h(t).$$

Hence, by using Chasles formula, we obtain

$$v(t) = h(t) - \alpha g(t) - (\beta - \alpha)tg(t) - \int_0^t g(t)(t-s)v(s)ds - tg(t)\left[\int_0^t (1-s)v(s)ds + \int_t^1 (1-s)v(s)ds\right],$$

that gives

$$v(t) = f(t) + \int_0^t s(1-t)v(s)ds + \int_t^1 t(1-s)g(t)v(s)ds, \tag{1.25}$$

that leads to the Fredholm integral equation:

$$v(t) = f(t) + \int_0^1 k(t,s)v(s)ds, \tag{1.26}$$

where

$$f(t) = h(t) - \alpha g(t) - (\beta - \alpha)tg(t),$$

and the kernel $k(t,s)$ is given by

$$k(t,s) = \begin{cases} s(1-t)g(t), \ for\ 0 \le s \le t, \\ \\ s(1-s)g(t), \ for\ t \le s \le 1. \end{cases}$$

An important conclusion can be made here. For the specific case where $u(0) = u(1) = 0$ which means that $\alpha = \beta = 0$, it is clear that $f(t) = h(t)$ in this case. This means that the resulting Fredholm equation in (1.26) is homogeneous or inhomogeneous if the boundary value problem in (1.19) is homogeneous or inhomogeneous respectively when $\alpha = \beta = 0$.

Type II: We next consider the following boundary value problem:
problem:

$$u''(t) + g(t)u(t) = h(t), \ \ 0 \leq t \leq 1 \tag{1.27}$$

with the boundary conditions:

$$u(0) = \alpha_1, \ \ u'(1) = \beta_1.$$

we again set

$$u''(t) = v(t) \tag{1.28}$$

Integrating both sides of (1.25) from 0 to t we obtain

$$\int_0^t u''(s)ds = \int_0^t v(s)ds,$$

that gives

$$u'(t) = u'(0) + \int_0^t v(s)ds \tag{1.29}$$

where the initial condition $u'(0)$ is not given in a boundary value problem. The condition $u'(0)$ will be derived later by $u'(1) = \beta_1$. Integrating both sides of (1.29) from 0 to t gives

$$u(t) = u(0) + tu'(0) + \int_0^t \int_0^t v(s)dsds,$$

or equivalently

$$u(t) = \alpha_1 + tu'(0) + \int_0^t (t-s)v(s)ds, \tag{1.30}$$

obtained upon using the condition $u(0) = \alpha_1$ and by reducing double integral to a single integral. To determine $u'(0)$, we first differentiate (1.30) with respect to t to get

$$u'(t) = u'(0) + \int_0^t v(s)ds, \tag{1.31}$$

where by substituting $t = 1$ into both sides of (1.31) and using the boundary condition at $u'(1) = \beta_1$ we find

$$u'(t) = \beta_1 + \int_0^t v(s)ds,$$

This in turn gives

$$u'(1) = u'(0) + \int_0^1 v(s)ds. \tag{1.32}$$

Using (1.32) into (1.30) gives

$$u'(0) = \beta_1 - \int_0^1 v(s)ds, \tag{1.33}$$

Substituting (1.28) and (1.33) into (1.27) yields

$$v(t) + \alpha_1 g(t) + \beta_1 t g(t) - \int_0^1 t g(s) v(s) ds + \int_0^t g(t)(t-s) v(s) ds = h(t)$$

Hence, by using Chasles formula, we obtain

$$v(t) = h(t) - (\alpha_1 + \beta_1 t) g(t) + t g(t) \left[\int_0^t v(s)ds + \int_t^1 v(s)ds \right] - g(t) \int_0^t (t-s) v(s) ds.$$

The last equation can be written as

$$v(t) = f(t) + \int_0^t s g(t)) v(s) ds + \int_t^1 t g(t) v(s) ds,$$

that leads to the Fredholm integral equation:

$$u(t) = f(t) + \int_0^1 k(t,s) u(s) ds, \tag{1.34}$$

where

$$f(t) = h(t) - (\alpha_1 + \beta_1 t)g(t),$$

and the kernel $k(t, s)$ is given by

$$k(t,s) = \begin{cases} sg(t), & for\ 0 \le s \le t, \\ \\ tg(t), & for\ t \le s \le 1. \end{cases}$$

An important conclusion can be made here. For the specific case where $u(0) = u'(1) = 0$ which means that $\alpha_1 = \beta_1 = 0$, it is clear that $f(t) = h(t)$ in this case. This means that the resulting Fredholm equation in (1.34) is homogeneous or inhomogeneous if the boundary value problem in (1.27) is homogeneous or inhomogeneous respectively.

1.9 Conversion of Volterra Integro-differential equations to Volterra Integral equation

The following Volterra integro-differential equation

$$u^{(n)}(t) = f(t) + \lambda \int_0^t K(t,s)u(s)ds, \quad u^{(k)}(0) = b_k, \quad 0 \le k \le n-1, \tag{1.35}$$

can also be solved by converting it to an equivalent Volterra integral equation. It is obvious that the Volterra integro-differential equation (1.35) involves derivatives at the left side, and integral at the right side. To perform the conversion process, we need to integrate both sides n times to convert it to a standard Volterra integral equation. Firstly, Integration of derivatives: from calculus we observe the following:

$$\int_0^t u'(s)ds = u(t) - u(0),$$

$$\int_0^t \int_0^{t_1} u''(s)dsdt_1 = u(t) - tu'(0) - u(0),$$

$$\int_0^t \int_0^{t_1} \int_0^{t_2} u'''(s)dsdt_1dt_2 = u(t) - \frac{1}{2}t^2u''(0) - tu'(0) - u(0),$$

and so on for other derivatives.

Secondly, Reducing multiple integrals to a single integral as follows,

$$\int_0^x \int_0^{x_1} u(t)dtdx_1 = \int_0^x (x-t)u(t)dt,$$

$$\int_0^x \int_0^{x_1} (x-t)u(t)dtdx_1 = \frac{1}{2}\int_0^x (x-t)^2u(t)dt,$$

$$\int_0^x \int_0^{x_1} (x-t)^2u(t)dtdx_1 = \frac{1}{3}\int_0^x (x-t)^3u(t)dt$$

$$\int_0^x \int_0^{x_1} (x-t)^3u(t)dtdx_1 == \frac{1}{4}\int_0^x (x-t)^4u(t)dt$$

and so on. This can be generalized in the form

$$\int_0^x \int_0^{x_1} ... \int_0^{x_{n-1}} (x-t)u(t)dtdx_{n-1}...dx_1 = \frac{1}{(n)!}\int_0^t (t-s)^n u(t)dt,$$

The conversion to an equivalent Volterra integral equation will be illustrated by studying the following examples.

Example 1.9.1 *Convert the following Volterra integro-differential equation to an Volterra* *integral equation:*

$$u'(x) = 1 + \int_0^x u(t)dt, \quad u(0) = 0$$

Integrating both sides from 0 *to* x*, and using the aforementioned formulas we find*

$$u(x) - u(0) = x + \int_0^x \int_0^{x_1} u(t)dtdx_1$$

Using the initial condition gives the Volterra integral equation

$$u(x) = x + \int_0^x (x-t)u(t)dt$$

1.10 Existence and uniqueness of the solution

Consider the nonlinear Volterra integro-differential equation (NVIDE)

$$y^n(x) = f(x) + \int_0^x K(x, t, y(t))ds, \quad x \in [0, b] \tag{1.36}$$

with n initial conditions

$$u^{(k)}(0) = \alpha_k, \quad 0 \leq k \leq n-1,$$

f and K are given smooth functions.

In this section, the existence and uniqueness of the solution for Eq. (1.36) are presented. First we give the following theorem from [39].

Theorem 1.10.1 *Consider the following nonlinear Volterra integral equations*

$$y(x) = f(x) + \int_0^t k(x, t, y(t))dt, \qquad (1.37)$$

Assume that

(i) - $f(x)$ *is continuous*

(ii) - $k(x, t, y(t))$ *is a continuous function for* $0 \leq t \leq s \leq b$ *and* $-\infty \leq |\, y| \ \leq \infty$,

(iii) - *the kernel satisfies the Lipschitz condition*

$$|k(x, t, y_1) - k(x, t, y_2)| \leq L|y_1 - y_2|. \qquad (1.38)$$

wherer L is independent of t, t, y_1 *and* y_2. *Then the Eq.* (1.36) *has a unique continuous solution in* $0 \leq t \leq b$.

Now we consider some cases of the integro-differential equations and investigate existence and uniqueness of the solutions of them.

Corollary 1.10.1

$$y'(x) = f(x) + \int_0^x K(x, t, y(t))dt, \qquad (1.39)$$

with initial condition $y(0) = \alpha$ *where* f *and* K *are continuous functions and* K *satisfies the Lipschitz condition*

$$|\, K(x, t, y_1) - K(x, t, y_2)| \ \leq L|\, y_1 - y_2| \,. \qquad (1.40)$$

Then this problem has a unique continuous solution.

Proof. Equation (1.39) transformed to the following Volterra integral equation

$$y(s) = \alpha + \int_0^x H(s, y(s))ds, \tag{1.41}$$

where $H(s, y(s)) = f(s) + \int_0^s K(s, t, y(t))dt$,

which is in the form of Eq.(1.37), where obviously α and $H(s, y(s))$ are continuous. Therefore, for the existence and uniqueness of a continuous solution of the Eq.(1.39) it is sufficient to show that Eq. (1.41) satisfies the Lipschitz condition. To this end, we have

$$\begin{aligned} ||H(s, y_1(s)) - H(s, y_2(s))|| &= ||\int_0^s (K(s, t, y_1(t)) - K(s, t, y_2(t)))dt|| \\ &\leq L_1||y_1 - y_2|| \int_0^s dt \\ &\leq L_1 b||y_1 - y_2||. \end{aligned}$$

So by Theorem (2.4), the Eq. (1.39) has a unique continuous solution. ■

Corollary 1.10.2

$$y'(x) + cy(x) = f(x) + \int_0^x K(x, t, y(t))dt, \tag{1.42}$$

with initial condition $y(0) = \alpha$ *,the f and K are continuous* (1.40) *then the equation* (1.42) *with given condition has a unique continuous solution.*

Proof. Equation (1.42) transformed to the following Volterra integral equation

$$y(s) = \alpha + \int_0^x H(s, y(s)), \tag{1.43}$$

where $H(s, y(s)) = f(s) + -cy(s) + \int_0^s K(s, t, y(t))dt$, similar to the previous corollary we only investigate the Lipschitz condition. To this end, we have

$$\begin{aligned}
||H(s,y_1(s)) - H(s,y_2(s))|| &= ||c[y_1(s) - y_2(s)] + \int_0^s (K(s,t,y_1(t)) - K(s,t,y_2(t)))dt|| \\
&\leq |c|||y_1 - y_2|| + L_1||y_1 - y_2|| \int_0^s dt \\
&\leq (c + bL_1)||y_1 - y_2||.
\end{aligned}$$

Again, by Theorem (2.4), the Eq. (1.42) has a unique continuous solution. ■

Corollary 1.10.3

$$y''(x) + c_1y(x) + c_2y(x) = f(x) + \int_0^x K(x,t,y(t))dt, \tag{1.44}$$

with initial condition $y(0) = \alpha$, $y'(0) = \beta$,the f and K are continuous (1.40) *Then the mentioned problem has a unique continuous solution.*

Proof. With the same manner, Volterra integro-differential equation(1.44) by converting it to the following Volterra integral equation

$$y(s) = \alpha + (\beta - c_1\alpha)z + \int_0^x H(s,y(s))dx.$$

where $H(s,y(s)) = -cy(s) + \int_0^x \left(f(s) - c_2y(s) + \int_0^s K(s,t,y(t))dt \right) ds$, then we obtain

$$
\begin{aligned}
&||H(s, y_1(s)) - H(s, y_2(s))|| \\
&= ||c_1[y_2(s) - y_1(s)] + \int_0^x \left(c_2(y_2(s) - y_{1(s)}) + \int_0^s (K(s,t,y_1(t)) - K(s,t,y_2(t))dt) \right) ds|| \\
&\leq |c_1|||y_1 - y_2|| + b|c_2|||y_1 - y_2|| + L_1||y_1 - y_2|| \int_0^x \int_0^s dtds \\
&\leq (|c_1| + b|c_2| + b^2 L_1)||y_1 - y_2||.
\end{aligned}
$$

Similar to previous cases, by Theorem (2.4), the Eq. (1.44) has a unique continuous solution. ■

The same conclusion can be drawn for the following Volterra integro-differential equation of order n

$$y^n(x) + \int_0^x K(x,t,y(t))ds = f(x), \quad x \in [0,b]$$

with conditions $y^i(0) = \alpha_i, \quad i = 0,1,...,n-1$, and similar to the previous corollaries we can convert this problem to an equation of the form (1.36).

1.11 Piecewise polynomial spaces

Let:

$$I_h = \{t_n = t_n^{(N)} : 0 = t_0^{(N)} < t_1^{(N)} < ... < t_N^{(N)} = T\}$$

denote a mesh (or: grid) on the given interval $I = [0,T]$. Define the subintervals

$$\delta_n^{(N)} = \left[t_n^{(N)}, t_{n+1}^{(N)}\right]$$

Definition 1.11.1 *For a given mesh I_h the piecewise polynomial space $S_\mu^{(d)}(I_h)$ with $\mu \geq 0, -1 \leq d \leq \mu$, is given by*

$$S_\mu^{(d)}(I_h) = \{v \in C^d(I) : v|_{\sigma_n} \in \pi_\mu(0 \leq n \leq N-1)\}$$

Here , π_μ denotes the space of (real) polynomials of degree not exceeding μ .
It is readily verified that $S_\mu^{(d)}(I_h)$ is a (real) linear vector space whose dimension is given by

$$\dim S_\mu^{(d)}(I_h) = N(\mu - d) + d + 1$$

Remark 1.11.1 *The particular piecewise polynomial space $S_{m+d}^{(d)}(I_h)$ corresponding to $\mu = m+d$ with $m \geq 1$ and $d \geq -1$ will play a central role in the chapter 2 and 3.*
Since its dimension is

$$\dim S_{m+d}^{(d)}(I_h) = Nm + (d+1), \tag{1.45}$$

it may be viewed as the ?natural? collocation space for the approximation of solutions to initial value problems for Volterra equations, the choice of the degree of regularity d will be governed by the number of prescribed initial conditions, while the term Nm suggests that m (distinct) collocation points are to be placed in each of the N subintervals σ_n . Thus, the natural choice of d in (1.45) is as follows:

- For Volterra integral equations (no initial condition) we choose $d = -1$; hence, the natural collocation space will be $S_{m-1}^{(-1)}(I_h)$. Its dimension is Nm.
- For first-order ODEs or Volterra integro-differential equations (one initial condition) we use $d = 0$, and the preferred collocation space is $S_m^{(0)}(I_h)$, with dimension equal to $Nm + 1$.
- For ODEs or VIDEs of ferst order with initial conditions the natural collocation space is $S_{m+1}^{(1)}(I_h)$, corresponding to the choice $d = 1$. The dimension of this space is $Nm + 2$.

1.12 Collocation method

A collocation method is based on the idea of approximating the exact solution of a given integral equation with a suitable function belonging to a chosen finite dimensional space such that the approximated solution satisfies the integral equation on a certain subset of the interval on which the equation has to be solved (called the set of collocation points). In our thesis, we consider the polynomial spline space as the approximating space. In order to describe the relevant collocation method for given N, let π_N be a uniform partition of a bounded interval I with grid points $t_n = nh, n = 0, 1, .., N$ and let h be the stepsize. Define the subintervals $\sigma_n = [t_n, t_{n+1}], n = 0, ..., N-1$. So, the real polynomial spline spaces of degrees m, $m + k - 1$, which will be used in this work are defined as follows:

$$S_{m-1}^{(-1)}(I, \Pi_N) = \{u : u_n = u|_{\sigma_n} \in \pi_{m-1}, n = 0, .., N-1\}.$$

CHAPTER 2

ITERATIVE COLLOCATION METHOD FOR SOLVING A CLASS OF NONLINEAR WEAKLY SINGULAR VOLTERRA INTEGRAL EQUATIONS

2.1 Introduction

In this chapter, we develop an approximation based on iterative collocation method to obtain numerical solutions of the following nonlinear weakly singular Volterra integral equations,

$$x(t) = g(t) + \int_0^t p(t,s)k(t,s,x(s))ds, t \in I = [0,T], \tag{2.1}$$

where the functions g, k are sufficiently smooth and $p(t,s) = \frac{s^{\mu-1}}{t^{\mu}}$, $\mu > 1$.

We start by the following lemma which summarizes some analytical results in the case $k(t,s,x(s)) = x(s)$.

Lemma 2.1.1 *1. - [81] .Let $\mu > 1$. If the function g belongs to $C^m[0,T]$, then the integral equation*

$$x(t) = g(t) + \int_0^t p(t,s)x(s)ds, \ \ t \in (0,T], \tag{2.2}$$

with $p(t,s) = \frac{s^{\mu-1}}{t^{\mu}}$, possesses a unique solution $x \in C^m[0,T]$.

2. - [83] . In the above case 1, the unique solution $x \in C^m[0,T]$ is given by:

$$x(t) = g(t) + t^{1-\mu}\int_0^t s^{\mu-2}g(s)ds.$$

3. - [83] .If $0 < \mu \leq 1$ and $g \in C^1[0,T]$ (with $g(0) = 0$ if $\mu = 1$), then (2.2) has a family of solutions in $C[0,T]$ of which only one has C^1 continuity.

Equations with this kind of kernel have a weak singularity at $t = 0$ and they are a particular case of the cordial equations, studied by G. Vainikko in ([64],[61],[63],[62]). Actually, as shown in [61], if the core function of a cordial operator is $\phi(s) = s^{\mu-1}$, then its kernel is $s^{\mu-1}t^{-\mu}k(t,s)$, which is the kind of kernel we are concerned with. Equations of this type are also the subject of the article [80].

The cordial integral operators have the interesting property that they are bounded but non-compact, which implies that some of the classical results for Volterra integral equations (for example, about existence and uniqueness of solution) are not applicable in this case. However an existence and uniqueness result in $C^m([0,T])$ was obtained in [90], provided that the core function satisfies $\phi(x) \in L^1([0,1])$, which is the case of our equation, when $\mu > 0$.

The application of polynomial and spline collocation methods to cordial equations was studied in ([64],[63]) and [88], respectively, where sufficient conditions for convergence were obtained and error estimates were derived. Superconvergence results for collocation methods were obtained in [80].

Equations of this type arise from heat conduction problems. They may result from boundary value problems for partial differential equations with mixed-type boundary conditions. It was shown in [79] that the following Volterra integral equation of the second kind with logarithmic singular kernel:

$$F(t) + \int_0^t P(t,s)F(s)ds = H(t), \ \ t \in [0,T], \tag{2.3}$$

where,

$$P(t,s) := \frac{1}{\sqrt{\pi}} \frac{1}{\sqrt{\ln(t/s)}} \frac{1}{s}.$$

and $H(t)$ is a given function, arises in some heat conduction problems with mixed-type boundary conditions. As an example, consider

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{a^2}\frac{\partial u}{\partial t}, \ \ 0 \le x \le l, \tag{2.4}$$

with the conditions

$$u(x,-\infty) = 0, \tag{2.5}$$

$$\frac{\partial u}{\partial x}(0,t) - u(0,t) = \phi_1(t), \tag{2.6}$$

$$-\frac{\partial u}{\partial x}(l,t) - u(l,t) = \phi_2(t). \tag{2.7}$$

The solution $u(t,x)$ can be expressed in terms of single layer potentials (see [90]) as follows:

$$u(x,t) = \frac{a}{2\sqrt{\pi}} \int_{-\infty}^{t} (t-\tau)^{-1/2} \left[\rho_1(\tau) e^{\frac{-x^2}{4a^2(t-\tau)}} + \rho_2(\tau) e^{\frac{-(x-l)^2}{4a^2(t-\tau)}} \right] d\tau. \tag{2.8}$$

Above $\rho_1(\tau)$, $\rho_2(\tau)$ are such that $u(x,t)$ satisfies conditions (2.5), (2.6) and (2.7). By imposing those conditions, the system of two integral equations is obtained:

$$\begin{aligned} &\frac{a}{\sqrt{\pi}} \int_0^u \left[\frac{l}{2a^2\sqrt{\ln^3(u/x)}} - \frac{1}{\sqrt{\ln(u/x)}} \right] \frac{1}{x} e^{\frac{-l^2}{4a^2 \ln(u/x)}} \psi_2\left(\frac{1}{x}\right) dx \\ &- \frac{a}{\sqrt{\pi}} \int_0^u \frac{1}{\sqrt{ln(u/x)}} \frac{1}{x} \psi_1\left(\frac{1}{x}\right) dx - \psi_1\left(\frac{1}{u}\right) = H_1\left(\frac{1}{u}\right), \end{aligned} \tag{2.9}$$

$$\begin{aligned} &\frac{a}{\sqrt{\pi}} \int_0^u \left[\frac{l}{2a^2\sqrt{\ln^3(u/x)}} - \frac{1}{\sqrt{\ln(u/x)}} \right] \frac{1}{x} e^{\frac{-l^2}{4a^2 \ln(u/x)}} \psi_1\left(\frac{1}{x}\right) dx \\ &- \frac{a}{\sqrt{\pi}} \int_0^u \frac{1}{\sqrt{ln(u/x)}} \frac{1}{x} \psi_2\left(\frac{1}{x}\right) dx - \psi_2\left(\frac{1}{u}\right) = H_2\left(\frac{1}{u}\right), \end{aligned} \tag{2.10}$$

where, $\psi_k(s) := \rho_k(-\ln s)$, $H_k(s) := 2\phi_k(-\ln s)$, $k = 1,2$. If l is large compared to a, then we may consider the system

$$-\frac{a}{\sqrt{\pi}} \int_0^u \frac{1}{\sqrt{\ln(u/x)}} \frac{1}{x} \psi_1\left(\frac{1}{x}\right) dx - \psi_1\left(\frac{1}{u}\right) = H_1\left(\frac{1}{u}\right), \tag{2.11}$$

$$-\frac{a}{\sqrt{\pi}} \int_0^u \frac{1}{\sqrt{\ln(u/x)}} \frac{1}{x} \psi_2\left(\frac{1}{x}\right) dx - \psi_2\left(\frac{1}{u}\right) = H_2\left(\frac{1}{u}\right). \tag{2.12}$$

We note that the above equations are independent and may be treated separately, each of them being of the form of (2.3).

We note that $\int_0^t P(t,s)ds$ is divergent. Following [84] we use the transformations

$$y(t) = t^{-\mu}F(t), \;\; f(t) = t^{-\mu}H(t),$$

where, $\mu > 0$ is a constant. From (2.3) we obtain

$$y(t) + \int_0^t q(t,s)y(s)ds = f(t), \;\; t \in [0,T],$$

with

$$q(t,s) := \frac{1}{\sqrt{\pi}} \frac{1}{\sqrt{\ln(t/s)}} \left(\frac{s}{t}\right)^{\mu} \frac{1}{s}.$$

In [28] this equation was transformed into the more tractable equation

$$y(t) - \int_0^t p(t,s)y(s)ds = g(t), \;\; t \in [0,T],$$

with

$$p(t,s) := \left(\frac{s}{t}\right)^{\mu} \frac{1}{s},$$

In [78] and [80] the authors were concerned with the numerical solution of linear cordial equations. Here we propose a computational method for a nonlinear Volterra integral equation with a weakly singular kernel of the same type.

In [49] a similar approach was proposed for nonlinear Volterra integral equations with regular kernels (when $p(t,s) \equiv 1$). This case was also well studied in the literature. In particular, Babolian and his co-authors [7] have proposed a Chebyshev approximation. In [7] and [40] numerical algorithms based on the Adomian's method were developed. In [90] an approach was proposed, based on Taylor polynomial approximation, while the homotopic perturbation method was applied to the same equation in [82]. The authors of [86] have introduced a scheme based on the fixed point method. Finally, the Haar wavelet method and the Haar rationalized functions method were proposed in [42] and [48],respectively.

In Section 2 of the present work we describe a numerical scheme for the solution of equation (2.1). In Section 3 we analyze the convergence and obtain error estimates.

2.2 Description of the collocation method

Let Π_N be a uniform partition of the interval $I = [0, T]$ defined by $t_n = nh, \quad n = 0, ..., N-1$, where the stepsize is given by $\frac{T}{N} = h$. Let the collocation parameters be $0 < c_1 < ...... < c_m \leq 1$ and the collocation points be $t_{n,j} = t_n + c_j h,\ j = 1, ..., m, n = 0, ..., N-1$. Define the subintervals $\sigma_n = [t_n, t_{n+1}[$, and $\sigma_{N-1} = [t_{N-1}, t_N]$.
Moreover, denote by π_m the set of all real polynomials of degree not exceeding m.
We define the real polynomial spline space of degree $m-1$ as follows:

$$S_{m-1}^{(-1)}(I, \Pi_N) = \{u : u_n = u|_{\sigma_n} \in \pi_{m-1}, n = 0, .., N-1\}.$$

This is the space of piecewise polynomials of degree at most $m-1$. Its dimension is Nm. We consider the space $L^\infty(I)$ with the norm

$$\left\|\varphi\right\| = \inf\left\{C \in \mathbb{R} : \left|\varphi(t)\right| \leq C \text{ for a.e. } t \in I\right\} < \infty.$$

It holds for any $y \in C^m([0, T])$ that

$$y(t_n + \tau h) = \sum_{l=1}^{m} \lambda_l(\tau) y(t_{n,l}) + \epsilon_n(\tau),\ \epsilon_n(\tau) = h^m \frac{y^{(m)}(\zeta_n(\tau))}{m!} \prod_{j=1}^{m} (\tau - c_j), \tag{2.13}$$

where $\tau \in [0, 1]$ and $\lambda_j(\tau) = \prod_{l \neq j}^{m} \frac{\tau - c_l}{c_j - c_l}$ are the Lagrange polynomials associate with the parameters c_j, $j = 1, ..., m$.
Let $\Gamma_m = \|\sum_{j=1}^{m} |\lambda_j|\|$ be the Lebesgue constants, such that

$$\|\sum_{j=1}^{m} |\lambda_j|\| = \max\left\{\sum_{j=1}^{m} |\lambda_j(s)|, s \in [0, 1]\right\}.$$

We have from (2.1) for each $j = 1, ..., m, n = 0, ..., N-1$

$$\begin{aligned} x(t_{nj}) &= g(t_{nj}) + \int_0^{t_{nj}} p(t_{nj}, s)k(t_{nj}, s, x(s))ds \\ &= g(t_{nj}) + \int_0^{t_n} p(t_{nj}, s)k(t_{nj}, s, x(s))ds + \int_{t_n}^{t_{nj}} p(t_{nj}, s)k(t_{nj}, s, x(s))ds \\ &= g(t_{nj}) + \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} p(t_{nj}, s)k(t_{nj}, s, x(s))ds + \int_{t_n}^{t_{nj}} p(t_{nj}, s)k(t_{nj}, s, x(s))ds. \end{aligned} \tag{2.14}$$

Now for $s \in [t_i, t_{i+1}]$, we use the following change of variable: $s = t_i + \tau h$ with $\tau \in [0, 1]$ and for $s \in [t_n, t_{nj}]$, we use the following change of variable: $s = t_n + \tau h$ with $\tau \in [0, c_j]$. Then, from (2.14), we have

$$\begin{aligned} x(t_{nj}) = g(t_{nj}) &+ \sum_{i=0}^{n-1} \int_0^1 hp(t_n + c_j h, t_i + \tau h)k(t_n + c_j h, t_i + \tau h, x(t_i + \tau h))d\tau \\ &+ \int_0^{c_j} hp(t_n + c_j h, t_n + \tau h)k(t_n + c_j h, t_n + \tau h, x(t_n + \tau h))d\tau. \end{aligned} \tag{2.15}$$

By substituting the expression of the function p into (2.15), we obtain

$$\begin{aligned} x(t_{nj}) = g(t_{nj}) &+ \sum_{i=0}^{n-1} \int_0^1 \frac{(i+\tau)^{\mu-1}}{(n+c_j)^{\mu}} k(t_n + c_j h, t_i + \tau h, x(t_i + \tau h))d\tau \\ &+ \int_0^{c_j} \frac{(n+\tau)^{\mu-1}}{(n+c_j)^{\mu}} k(t_n + c_j h, t_n + \tau h, x(t_n + \tau h))d\tau. \end{aligned} \tag{2.16}$$

Now, for $j = 1, ..., m$, by applying the formula (2.13) for the function $y_i(\tau) = k(t_n + c_j h, t_i + \tau h, x(t_i + \tau h))$, we have

$$k(t_n + c_j h, t_i + \tau h, x(t_i + \tau h)) = \sum_{l=1}^{m} \lambda_l(\tau) k(t_n + c_j h, t_{n,l}, x(t_{n,l})) + \epsilon_i(\tau), \tag{2.17}$$

where $\epsilon_i(\tau) = h^m \frac{y_i^{(m)}(\eta_i)}{m!} \prod_{j=1}^{m}(\tau - c_j)$.
Inserting (2.17) into (2.16), we obtain for each $j = 1, ..., m, n = 0, ..., N-1$

$$x(t_{nj}) = g(t_{nj}) + \sum_{l=1}^{m} \left(\int_0^{c_j} \frac{(n+\tau)^{\mu-1}}{(n+c_j)^{\mu}} k(t_n + c_j h, t_n + c_l h, x(t_{nl}))\lambda_l(\tau) d\tau \right) + \sum_{i=0}^{n-1} \sum_{l=1}^{m} \left(\int_0^{1} \frac{(i+\tau)^{\mu-1}}{(n+c_j)^{\mu}} k(t_n + c_j h, t_i + c_l h, x(t_{il}))\lambda_l(\tau) d\tau \right) + o(h^m), \tag{2.18}$$

where,

$$o(h^m) = \int_0^{c_j} \frac{(n+\tau)^{\mu-1}}{(n+c_j)^{\mu}} \epsilon_n(\tau) d\tau + \sum_{i=0}^{n-1} \left(\int_0^{1} \frac{(i+\tau)^{\mu-1}}{(n+c_j)^{\mu}} \epsilon_i(\tau) d\tau \right).$$

Since the function k is smooth, then there exists $\alpha_1 > 0$, such that for $i = 0, ..., N-1$, we have $\|y_i^{(m)}\| \leq \alpha_1$, which implies that

$$\| o(h^m) \| \leq h^m \frac{\alpha_1}{m!} \left(\int_0^{c_j} \frac{(n+\tau)^{\mu-1}}{(n+c_j)^{\mu}} d\tau + \sum_{i=0}^{n-1} \left(\int_0^{1} \frac{(i+\tau)^{\mu-1}}{(n+c_j)^{\mu}} d\tau \right) \right)$$

Since $i + \tau \leq n + c_j$ for all $i = 0, ..., n-1$, then for all $n = 0, ..., N-1$

$$\begin{aligned} \| o(h^m) \| &\leq h^m \frac{\alpha_1}{m!} \left(\frac{1}{(n+c_j)} + \sum_{i=0}^{n-1} \left(\frac{1}{(n+c_j)} \right) \right) \\ &\leq h^m \frac{\alpha_1}{m!} \left(\frac{1}{c_1} + \frac{n}{(n+c_j)} \right) \\ &\leq h^m \underbrace{\frac{\alpha_1}{m!} \left(\frac{1}{c_1} + 1 \right)}_{=\alpha}. \end{aligned}$$

It holds for any $u \in S_{m-1}^{(-1)}(I, \Pi_N)$ that

$$u(t_n + \tau h) = \sum_{l=1}^{m} \lambda_l(\tau) u(t_{n,l}), \tau \in [0, 1]. \tag{2.19}$$

Now, we approximate the exact solution x by $u \in S_{m-1}^{(-1)}(I, \Pi_N)$ such that $u(t_{n,j})$ satisfy the following nonlinear system,

$$\begin{aligned} u(t_{n,j}) = g(t_{nj}) + \sum_{l=1}^{m} \left(\int_0^{c_j} \frac{(n+\tau)^{\mu-1}}{(n+c_j)^{\mu}} k(t_n + c_j h, t_n + c_l h, u(t_{nl})) \lambda_l(\tau) d\tau \right) + \\ \sum_{i=0}^{n-1} \sum_{l=1}^{m} \left(\int_0^{1} \frac{(i+\tau)^{\mu-1}}{(n+c_j)^{\mu}} k(t_n + c_j h, t_i + c_l h, u(t_{il})) \lambda_l(\tau) d\tau \right). \end{aligned} \tag{2.20}$$

for $j = 1, ..., m, n = 0, ..., N-1$.

Since the above system is nonlinear, we will use an iterative collocation solution $u^q \in S_{m-1}^{(-1)}(I, \Pi_N), q \in \mathbb{N}$, to approximate the exact solution of (2.1) such that

$$u^q(t_n + \tau h) = \sum_{j=1}^{m} \lambda_j(\tau) u^q(t_{n,j}), \tau \in [0, 1] \tag{2.21}$$

where the coefficients $u^q(t_{n,j})$ are given by the following formula:

$$\begin{aligned} u^q(t_{n,j}) = g(t_{nj}) + \sum_{l=1}^{m} \left(\int_0^{c_j} \frac{(n+\tau)^{\mu-1}}{(n+c_j)^{\mu}} k(t_n + c_j h, t_n + c_l h, u^{q-1}(t_{nl})) \lambda_l(\tau) d\tau \right) + \\ \sum_{i=0}^{n-1} \sum_{l=1}^{m} \left(\int_0^{1} \frac{(i+\tau)^{\mu-1}}{(n+c_j)^{\mu}} k(t_n + c_j h, t_i + c_l h, u^q(t_{il})) \lambda_l(\tau) d\tau \right). \end{aligned} \tag{2.22}$$

such that the initial values $u^0(t_{n,j}) \in J$ (J is a bounded interval).

The above formula is explicit and the approximate solution u^q is obtained without solving any algebraic system.

In the next section, we will prove the convergence of the approximate solution u^q to the exact solution x of (2.1).

2.3 Convergence analysis

In this section, we assume that the function k satisfies the Lipschitz condition with respect to the third variable; there exists $L \geq 0$ such that

$$|k(t, s, y_1) - k(t, s, y_2)| \leq L|y_1 - y_2|, \tag{2.23}$$

for all $t, s \in I$, where L is independent of t and s.

The following result gives the existence and the uniqueness of a solution for (2.1).

Lemma 2.3.1 *Let $g \in C([0,T])$, $k(t,s,u) \in C(\Delta_T \times \mathbb{R})$, where $\Delta_T = \{(t,s) \in \mathbb{R}^2 : 0 \leq t \leq T, 0 \leq s \leq t\}$. Let*

$$\frac{\partial k}{\partial u} \in C(\Delta_T \times \mathbb{R}).$$

Assume that equation

$$\xi = k(0,0,\xi)\frac{1}{\mu} + g(0) \tag{2.24}$$

has a unique solution $\xi^* \in \mathbb{R}$, and that

$$1 \neq \frac{a^*(0,0)}{\lambda + \mu}, \forall \lambda : Re(\lambda) \geq 0, \tag{2.25}$$

where

$$a^*(0,0) = \frac{\partial k}{\partial u}(0,0,u)\bigg|_{u=\xi^*}.$$

Moreover, let k satisfy

$$|k(t,s,u)| \leq c_0 + c_1|u|, \tag{2.26}$$

with $\frac{c_1}{\mu} < 1$.

Then there is a unique solution $x^* \in C([0,T])$ of (2.1), such that $x^*(0) = \xi^*$.

Proof. The result follows from Theorems 7.1 and 7.5 of [90], taking into account that in our case $\phi(x) = x^{\mu-1}$,with $\mu > 1$, and therefore the linear integral operator V_ϕ (using the same notation as in [90]) is defined by

$$V_\phi u(t) = \int_0^t \frac{s^{\mu-1}}{t^\mu} u(s)ds;$$

hence the spectrum of this operator is

$$\sigma_0(V_\phi) = \{0\} \cup \{\frac{1}{\lambda + \mu} : \lambda \in \mathbb{C}, Re(\lambda) \geq 0\},$$

which is used to obtain condition (2.25). ■

Lemma 2.3.2 *Let the conditions of Lemma 2.3.1 be satisfied and let x^* be a solution of* (2.1). *Moreover, let $g \in C^m([0,T])$ and $k \in C^m(\Delta T \times \mathbb{R})$, for some natural m.*

Then $x^* \in C^m([0,T])$.

Proof. The result follows from Theorem 8.1 of [90]. ■

The following result gives the existence and the uniqueness of a solution for the nonlinear system (2.20).

Lemma 2.3.3 *If $\frac{L\Gamma_m}{\mu} < 1$, then the nonlinear system* (2.20) *has a unique solution $u \in S^{(-1)}_{m-1}(I, \Pi_N)$. Moreover, the function u is bounded.*

Proof. We will use the induction combined with the Banach fixed point theorem.

(i) On the interval $\sigma_0 = [t_0, t_1]$, the nonlinear system (2.20) becomes

$$u(t_{0,j}) = g(t_{0,j}) + \sum_{l=1}^{m}\left(\int_0^{c_j} \frac{(\tau)^{\mu-1}}{(c_j)^{\mu}} k(t_0 + c_j h, t_0 + c_l h, u(t_{0l}))\lambda_l(\tau)d\tau\right).$$

We consider the operator Ψ defined by

$$\Psi : \mathbb{R}^m \longrightarrow \mathbb{R}^m$$
$$x = (x_1, ..., x_m) \longmapsto \Psi(x) = (\Psi_1(x), ..., \Psi_m(x)),$$

such that for $j = 1, ..., m$, we have

$$\Psi_j(x) = g(t_{0,j}) + \sum_{l=1}^{m}\left(\int_0^{c_j} \frac{(\tau)^{\mu-1}}{(c_j)^{\mu}} k(t_0 + c_j h, t_0 + c_l h, x_l)\lambda_l(\tau)d\tau\right).$$

Hence, for all $x, y \in \mathbb{R}^m$, we have

$$\|\Psi(x) - \Psi(y)\| \leq \frac{L\Gamma_m}{\mu}\|x - y\|,$$

Since $\frac{L\Gamma_m}{\mu} < 1$, then by Banach fixed point theorem, the nonlinear system (2.20) has a unique solution u on the interval σ_0.

(ii) Suppose that u exists and is unique on the intervals $\sigma_i, i = 0, ..., n-1$ for $n \geq 1$, we show now that u exists and is unique on the interval σ_n.
On the interval σ_n, the nonlinear system (2.20) becomes

$$u(t_{n,j}) = G(t_{n,j}) + \sum_{l=1}^{m} \left(\int_0^{c_j} \frac{(n+\tau)^{\mu-1}}{(n+c_j)^{\mu}} k(t_n + c_j h, t_n + c_l h, u(t_{nl})) \lambda_l(\tau) d\tau \right) \tag{2.27}$$

where,
$G(t_{n,j}) = g(t_{n,j}) + \sum\limits_{i=0}^{n-1} \sum\limits_{l=1}^{m} \left(\int_0^1 \frac{(i+\tau)^{\mu-1}}{(n+c_j)^{\mu}} k(t_n + c_j h, t_i + c_l h, u(t_{il})) \lambda_l(\tau) d\tau \right).$
We consider the operator Ψ defined by:

$$\Psi : \mathbb{R}^m \longrightarrow \mathbb{R}^m$$
$$x = (x_1, ..., x_m) \longmapsto \Psi(x) = (\Psi_1(x), ..., \Psi_m(x)),$$

such that for $j = 1, ..., m$, we have

$$\Psi_j(x) = G(t_{n,j}) + \sum_{l=1}^{m} \left(\int_0^{c_j} \frac{(n+\tau)^{\mu-1}}{(n+c_j)^{\mu}} k(t_n + c_j h, t_n + c_l h, x_l) \lambda_l(\tau) d\tau \right).$$

Hence, for all $x, y \in \mathbb{R}^m$, we have

$$||\Psi(x) - \Psi(y)|| \leq \frac{L\Gamma_m}{\mu} ||x - y||.$$

Since $\frac{L\Gamma_m}{\mu} < 1$, then by Banach fixed point theorem, the nonlinear system (2.27) has a unique solution u on the interval σ_n.

■

Corollary 2.3.1 *Under the condition* $\frac{L\Gamma_m}{\mu} < 1$, *the following conditions of Lemma 2.3.1 are fulfilled:*

1. *Equation* (2.24) *has a unique solution* $\xi^* \in \mathbb{R}$.

2. *Inequality* (2.26) *is satisfied, moreover* $\frac{c_1}{\mu} < 1$.

Proof.

1. We consider the operator Ψ defined by

$$\begin{aligned} \Psi : \mathbb{R} &\longrightarrow \mathbb{R} \\ \xi &\longmapsto \Psi(x) = k(0,0,\xi)\frac{1}{\mu} + g(0), \end{aligned}$$

   Hence, for all $\xi_1, \xi_2 \in \mathbb{R}$, we have

$$|\Psi(\xi_1) - \Psi(\xi_2)| \leq \frac{L}{\mu}|\xi_1 - \xi_2|,$$

   Since $\frac{L}{\mu} \leq \frac{L\Gamma_m}{\mu} < 1$, then by Banach fixed point theorem, Equation (2.25) has a unique solution $\xi^* \in \mathbb{R}$.

2. We have,

$$|k(t,s,u)| \leq |k(t,s,u) - k(t,s,0)| + |k(t,s,0)| \leq \underbrace{L}_{=c_1}|u| + c_0,$$

   such that $c_0 = \max\{|k(t,s,0)|, (t,s) \in I \times I\}$.
   Hence the inequality (2.26) is satisfied, moreover $\frac{c_1}{\mu} = \frac{L}{\mu} \leq \frac{L\Gamma_m}{\mu} < 1$.

■

Remark 2.3.1 *Under our assumptions and by Lemma 2.3.1, Lemma 2.3.2 and Corollary 2.3.1, to prove the existence and uniqueness solution for equation* (2.1)*, we need only to show the condition* (2.25)*.*

The following result gives the convergence of the approximate solution u to the exact solution x.

Theorem 2.3.1 *Let g, k be m times continuously differentiable on their respective domains. If $\frac{L\Gamma_m}{\mu} < \frac{1}{2}$, then the collocation solution u converges to the exact solution x, and the resulting error function $e := x - u$ satisfies:*

$$\|e\| \leq Ch^m,$$

where C is a finite constant independent of h.

Proof. From (2.20) and (2.18), using (2.23), we obtain

$$|e(t_{nj})| \leq \alpha h^m + \frac{L\Gamma_m}{\mu} e_n + \frac{L\Gamma_m}{\mu n^\mu} \sum_{i=0}^{n-1} \left((i+1)^\mu - i^\mu\right) e_i \tag{2.28}$$

where α is a positive number and $e_n = \max\{|e(t_{n,l})|, l = 1, ..., m\}$, $n = 0, ..., N-1$.
Then, from (2.28), e_n satisfies for $n = 0, ..., N-1$,

$$e_n \leq \alpha h^m + \frac{L\Gamma_m}{\mu} e_n + \frac{L\Gamma_m}{\mu n^\mu} \sum_{i=0}^{n-1} \left((i+1)^\mu - i^\mu\right) e_i,$$

which implies that,

$$e_n \leq \frac{\alpha}{1 - \frac{L\Gamma_m}{\mu}} h^m + \frac{L\Gamma_m}{(1 - \frac{L\Gamma_m}{\mu})\mu n^\mu} \sum_{i=0}^{n-1} \left((i+1)^\mu - i^\mu\right) e_i.$$

Let $C_1 = \frac{\alpha}{1 - \frac{L\Gamma_m}{\mu}}$ and $C_2 = \frac{L\Gamma_m}{\mu(1 - \frac{L\Gamma_m}{\mu})}$. Since $\frac{L\Gamma_m}{\mu} < \frac{1}{2}$, then $C_2 < 1$.
It follows that

$$e_n \leq C_1 h^m + \frac{C_2}{n^\mu} \sum_{i=0}^{n-1} \left((i+1)^\mu - i^\mu\right) e_i.$$

Hence, for $\xi = \max\{e_n, n = 0, ..., N-1\}$, we deduce that

$$\xi \leq C_1 h^m + C_2 \xi.$$

Since $C_2 < 1$, we obtain

$$\xi \leq \frac{C_1}{1 - C_2} h^m.$$

Which implies, from (2.13) and (2.19), that there exists $C > 0$ such that

$$\begin{aligned}
\|e\| &\leq \Gamma_m \xi + h^m \frac{\|x^{(m)}\|}{m!} \prod_{j=1}^{m} (1 - c_j) \\
&\leq \Gamma_m \frac{C_1}{1 - C_2} h^m + h^m \frac{\|x^{(m)}\|}{m!} \prod_{j=1}^{m} (1 - c_j).
\end{aligned}$$

Thus, the proof is completed by setting $C = \Gamma_m \frac{C_1}{1-C_2} + \frac{\|x^{(m)}\|}{m!} \prod_{j=1}^{m} (1 - c_j)$. ∎

The following result gives the convergence of the iterative solution u^q to the exact solution x.

Theorem 2.3.2 *Consider the iterative collocation solution $u^q, q \geq 1$ defined by* (2.21) *and* (2.22). *If $\frac{L\Gamma_m}{\mu} < \frac{1}{2}$, then for any initial condition $u^0(t_{n,j}) \in J$ (bounded interval), the iterative collocation solution $u^q, q \geq 1$ converges to the exact solution x. Moreover, the following error estimate holds*

$$\|u^q - x\| \leq d\rho^q + Ch^m$$

where d, C are finite constants independent of h and $\rho < 1$.

Proof. We define the error e^q and ξ^q by $e^q(t) = u^q(t) - x(t)$ and $\xi^q(t) = u^q(t) - u(t)$, where u is defined by lemma 2.3.3. It follows that

$$e^q = \xi^q + u - x. \tag{2.29}$$

We have, from (2.20) and (2.22), for all $n = 0, ..., N-1$ and $j = 1, ..., m$

$$|\xi^q(t_{n,j})| \leq \frac{L\Gamma_m}{n^\mu \mu} \sum_{i=0}^{n-1} [(i+1)^\mu - i^\mu] \xi_i^q + \frac{L\Gamma_m}{\mu} \xi_n^{q-1}, \tag{2.30}$$

where $\xi_n^q = \max\{\left|\xi^q(t_{n,l})\right|, l = 1....m\}$ for $n = 0, ..., N-1$, it follows from (2.30) that,

$$\xi_n^q \leq \frac{L\Gamma_m}{\mu n^\mu}\sum_{i=0}^{n-1}[(i+1)^\mu - i^\mu]\xi_i^q + \frac{L\Gamma_m}{\mu}\xi_n^{q-1}.$$

We consider the sequence $\eta^q = \max\{\xi_n^q, n = 0, ..., N-1\}$ for $q \geq 1$.
Then, η^q satisfies,

$$\begin{aligned}\eta^q &\leq \frac{L\Gamma_m}{\mu n^\mu}\sum_{i=0}^{n-1}[(i+1)^\mu - i^\mu]\eta^q + \frac{L\Gamma_m}{\mu}\eta^{q-1}\\ &\leq \frac{L\Gamma_m}{\mu}\eta^q + \frac{L\Gamma_m}{\mu}\eta^{q-1}.\end{aligned}$$

Hence,

$$\eta^q \leq \rho\eta^{q-1}, \tag{2.31}$$

where $\rho = \dfrac{\frac{L\Gamma_m}{\mu}}{1-\frac{L\Gamma_m}{\mu}}$, since $\frac{L\Gamma_m}{\mu} < \frac{1}{2}$, then $\rho < 1$.
Which implies, from (2.31), that for all $q \geq 1$, that

$$\eta^q \leq \rho\eta^{q-1} \leq \rho^2\eta^{q-2} \leq ... \leq \rho^q\eta^0 \leq \rho^q\|\xi^0\|. \tag{2.32}$$

Since, $u^0(t_{n,j}) \in J$, the function u^0 is bounded.
Hence, there exists $M > 0$ such that

$$\|\xi^0\| = \|u^0 - u\| \leq \|u^0 - x\| + \|u - x\| \leq M. \tag{2.33}$$

From (2.32) and (2.33), we conclude that

$$\|\xi^q\| \leq \Gamma_m\eta^q \leq \underbrace{\Gamma_m M}_{d}\rho^q.$$

On the other hand, from Theorem (2.3.1), we have $\|u - x\| \leq Ch^m$ and therefore by (2.29) we obtain

$$\|e^q\| \le \|\xi^q\| + \|u - x\| \le d\rho^q + Ch^m.$$

Thus, the proof is completed. ■

2.4 Numerical Examples

To illustrate the theoretical results obtained in the previous section, we present the following examples with $T = 1$. All the exact solutions x are already known.
In all the examples, we have $a^*(0,0) = 0$, hence the condition (2.25) is satisfied.
In each example, we calculate the error between x and the iterative collocation solution u^q for $N = 10, 20$ and $m = 2, 3, 5$ at $t = 0, 0.1, ..., 1$. In all the examples, we choose, $q = 5$, $u^0(t_{nj}) = 1$, and we use the collocation parameters $c_j = \frac{j}{m+1}$, $j = 1, ..., m$.
Since the condition $\frac{L\Gamma_m}{\mu} < \frac{1}{2}$ is essential to guarantee the convergence of the numerical method, we checked that it is satisfied in all the numerical examples. Moreover, $\Gamma_2 = 3$, $\Gamma_3 = 7$ and $\Gamma_5 = 31$.

Example 2.4.1 *Consider the following integral equation*

$$x(t) = g(t) + \int_0^t p(t,s)k(t,s,x(s))ds, \; t \in [0,1].$$

with $k(t,s,z) = \frac{st\exp(z)}{40(1+\exp(z))}$, $\mu = 2$ *and* $g(t)$ *is chosen such that the exact solution of this equation is* $x(t) = \ln(1+t^2)$. *The absolute errors are presented in Table 2.1. The experimental orders of convergence (EOC) by using the maximum error* $\|e_N\| = \max\{|x(t_i) - u^q(t_i)|, i = 0, ..., N\}$ *given by the formula* $EOC = \frac{\ln(\frac{2^{2N}}{e_N})}{\ln(2)}$ *for* $N = 5, 10, 15, 20$ *and* $m = 1, 2, 3, 4$ *are given in Table 2.3.*

Example 2.4.2 *Consider the following integral equation*

$$x(t) = g(t) + \int_0^t p(t,s)k(t,s,x(s))ds, \; t \in [0,1].$$

Table 2.1: Absolute errors for Example 2.4.1

	$N = 10$	$N = 10$	$N = 10$	$N = 20$	$N = 20$	$N = 20$
t	$m = 2$	$m = 3$	$m = 5$	$m = 2$	$m = 3$	$m = 5$
0	2.21×10^{-3}	6.98×10^{-6}	1.66×10^{-8}	5.55×10^{-4}	4.38×10^{-7}	3.56×10^{-9}
0.1	2.10×10^{-3}	2.41×10^{-5}	3.39×10^{-8}	5.33×10^{-4}	2.65×10^{-6}	2.47×10^{-10}
0.2	1.89×10^{-3}	3.70×10^{-5}	5.27×10^{-8}	4.83×10^{-4}	4.38×10^{-6}	9.09×10^{-9}
0.3	1.60×10^{-3}	4.40×10^{-5}	5.01×10^{-8}	4.13×10^{-4}	5.39×10^{-6}	2.44×10^{-9}
0.4	1.28×10^{-3}	4.53×10^{-5}	3.02×10^{-8}	3.33×10^{-4}	5.69×10^{-6}	2×10^{-10}
0.5	9.65×10^{-4}	4.25×10^{-5}	2.23×10^{-8}	2.54×10^{-4}	5.41×10^{-6}	6.6×10^{-9}
0.6	6.79×10^{-4}	3.71×10^{-5}	9.3×10^{-9}	1.81×10^{-4}	4.78×10^{-6}	2.3×10^{-9}
0.7	4.36×10^{-4}	3.07×10^{-5}	4.5×10^{-9}	1.18×10^{-4}	3.99×10^{-6}	6×10^{-10}
0.8	2.38×10^{-4}	2.44×10^{-5}	2.6×10^{-9}	6.69×10^{-5}	3.19×10^{-6}	3.20×10^{-9}
0.9	8.37×10^{-5}	1.87×10^{-5}	2.3×10^{-9}	2.66×10^{-5}	2.46×10^{-6}	2.50×10^{-9}
1	4.06×10^{-5}	1.74×10^{-5}	1.77×10^{-7}	5.13×10^{-6}	2.05×10^{-6}	2.42×10^{-8}

with $k(t,s,z) = \frac{ts}{40(2+z^2)}$, $\mu = 2$ *and* $g(t)$ *is chosen so that the exact solution of this equation is* $x(t) = \frac{1}{5(t^3+1)}$. *The absolute errors are presented in Table 2.2. The experimental orders of convergence (EOC) by using the maximum error* $\|e_N\| = \max\{|x(t_i) - u^q(t_i)|, i = 0, ..., N\}$ *given by the formula EOC* $= \frac{\ln(\frac{2^{2N}}{e_N})}{\ln(2)}$ *for* $N = 5, 10, 15, 20$ *and* $m = 1, 2, 3, 4$ *are given in Table 2.3.*

Table 2.2: Absolute errors for Example 2.4.2

	$N = 10$	$N = 10$	$N = 10$	$N = 20$	$N = 20$	$N = 20$
t	$m = 2$	$m = 3$	$m = 5$	$m = 2$	$m = 3$	$m = 5$
0	4.44×10^{-5}	1.87×10^{-5}	5.30×10^{-9}	5.55×10^{-6}	2.34×10^{-6}	3.10×10^{-9}
0.1	1.75×10^{-4}	1.77×10^{-5}	2.39×10^{-8}	3.85×10^{-5}	2.26×10^{-6}	1.60×10^{-9}
0.2	2.91×10^{-4}	1.39×10^{-5}	3.29×10^{-8}	6.86×10^{-5}	1.87×10^{-6}	1.70×10^{-9}
0.3	3.68×10^{-4}	6.42×10^{-6}	2.22×10^{-8}	8.99×10^{-5}	1.01×10^{-6}	5.00×10^{-10}
0.4	3.81×10^{-4}	3.66×10^{-6}	5.70×10^{-9}	9.62×10^{-5}	2.13×10^{-7}	9.00×10^{-10}
0.5	3.23×10^{-4}	1.30×10^{-5}	3.50×10^{-8}	8.45×10^{-5}	1.44×10^{-6}	1.00×10^{-9}
0.6	2.10×10^{-4}	1.84×10^{-5}	4.57×10^{-8}	5.79×10^{-5}	2.23×10^{-6}	5.00×10^{-10}
0.7	7.62×10^{-5}	1.84×10^{-5}	3.15×10^{-8}	2.46×10^{-5}	2.36×10^{-6}	9.00×10^{-10}
0.8	4.41×10^{-5}	1.45×10^{-5}	1.31×10^{-8}	6.59×10^{-6}	1.94×10^{-6}	2.20×10^{-9}
0.9	1.30×10^{-4}	9.14×10^{-6}	2.90×10^{-9}	2.97×10^{-5}	1.27×10^{-6}	3.00×10^{-10}
1	1.50×10^{-4}	7.79×10^{-6}	1.52×10^{-8}	3.97×10^{-5}	8.50×10^{-7}	7.00×10^{-9}

Table 2.3: Experimental ordres of convergence (EOC) of Examples 2.4.1-2.4.2

N	$m=1$	$m=2$	$m=3$	$m=4$
5				
10	0.99	1.98	2.97	3.92
15	0.99	1.98	2.98	3.93
20	0.99	1.98	2.98	3.95

EOC of Example2.4.1

N	$m=1$	$m=2$	$m=3$	$m=4$
5				
10	0.98	1.94	3.01	4.05
15	0.99	1.96	2.99	3.92
20	0.99	1.96	2.99	3.94

EOC of Example2.4.2

Example 2.4.3 *Consider the following integral equation*

$$x(t) = g(t) + \int_0^t p(t,s)k(t,s,x(s))ds,\ t \in [0,1].$$

with $k(t,s,z) = \frac{t\cos(s+z)}{65}$, $\mu = 1.03$ *and* $g(t)$ *is chosen such that the exact solution of this equation is* $x(t) = \frac{t}{10}$. *The absolute errors are presented in Table 2.4.*

Table 2.4: Absolute errors for Example 2.4.3

	$N=10$	$N=10$	$N=10$	$N=20$	$N=20$	$N=20$
t	$m=2$	$m=3$	$m=5$	$m=2$	$m=3$	$m=5$
0	3.35×10^{-7}	1.81×10^{-10}	7×10^{-12}	4.20×10^{-8}	7×10^{-12}	2.00×10^{-12}
0.1	8.33×10^{-7}	5.40×10^{-10}	4×10^{-11}	1.67×10^{-7}	2×10^{-11}	3.00×10^{-11}
0.2	1.32×10^{-6}	9.30×10^{-10}	1.7×10^{-10}	2.89×10^{-7}	4×10^{-11}	2.3×10^{-10}
0.3	1.79×10^{-6}	1.28×10^{-9}	7.00×10^{-11}	4.08×10^{-7}	5×10^{-11}	1.3×10^{-10}
0.4	2.23×10^{-6}	1.59×10^{-9}	2.4×10^{-10}	5.22×10^{-7}	6×10^{-11}	1.3×10^{-10}
0.5	2.65×10^{-6}	1.90×10^{-9}	1.3×10^{-10}	6.30×10^{-7}	8×10^{-11}	3×10^{-11}
0.6	3.04×10^{-6}	2.21×10^{-9}	3×10^{-11}	7.29×10^{-7}	2×10^{-10}	2×10^{-11}
0.7	3.39×10^{-6}	2.39×10^{-9}	6×10^{-11}	8.19×10^{-7}	1.1×10^{-10}	3.00×10^{-11}
0.8	3.69×10^{-6}	2.65×10^{-9}	1.4×10^{-10}	8.99×10^{-7}	1.00×10^{-10}	3.00×10^{-11}
0.9	3.95×10^{-6}	2.88×10^{-9}	8×10^{-11}	9.68×10^{-7}	1.2×10^{-10}	8×10^{-11}
1	3.85×10^{-6}	2.90×10^{-9}	1.99×10^{-8}	9.85×10^{-7}	2×10^{-10}	5.12×10^{-9}

CONCLUSION

In this dissertation, we have used an iterative collocation method based on the Lagrange polynomials for solving a class of weakly singular Volterra integral equation in the spline space $S_{m-1}^{-1}(I, \Pi_N)$. The main advantages of this direct iterative collocation method are:

1. The approximate solution is given by using explicit formulas.
2. This method has a convergence order.
3. There is no algebraic system needed to be solved, which makes the proposed algorithm very effective and easy to implement.

The numerical examples confirm that the method is convergent with a good accuracy.

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