الجمهورية الجزائرية الديمقراطية الشعبية
People's Democratic Republic of Algeria
وزارة التعليم العالي والبحث العلمي
Minisrty of Higher Education and Scientific Research

Nº Réf :...............

Abdelhafid Boussouf
University Center-Mila

Institute of Mathematics and Computer Sciences

Department of Mathematics

**Submitted for the degree of Master**
**In: Mathematics**
**Specialty: Fundamental Mathematics**

# Complete homogeneous symetric function for the product of certain numbers and polynomial.

**Presented by:** Allili Amira
Brioua Youssra

**In front of the jury**

| | | | |
|---|---|---|---|
| Khalfaoui Mohamed | MAA | Abdelhafid Boussouf U.C.of Mila | Chairman |
| Bouzekria Fahima | MAA | Abdelhafid Boussouf U.C.of Mila | Supervisor |
| Bourourou Siham | MCB | Abdelhafid Boussouf U.C of Mila | Examiner |

**Academic year: 2023/2024**

# *ACKNOWLEDGMENTS*

*First and foremost, we thank* **ALLAH** *who has granted us strength, health and willpower and has enabled us to undertake this work, hoping for his continued blessings.*
*We would also like to express our gratitude and our sincere thanks to our supervisor, proffessor* **Fahima Bouzekria** *for her patience, the time she devoted to us, her availability and About all her judicious Advice which contributed to Fulling our reflection.*
*We also express our gratitude to all the* **teachers**, **friends** *near and far and* ***the staff of the mathematics departement***.
*We reserve a special place to thank our* **families** *for their support and standing by us throughout our academic journey.*

**YOUSSRA, AMIRA**

# *DEDICATION*

*I dedicate this humble work to my dear father* **ELBACHIR** *through his support, i succeeded and reached where I am now thank you very muuch.*
*To my dear mother* **BERGUELLAH** *who god has placed paradise beneath her feet. you are my first and eternal supporter after allah. Ididicate this achievement to you without your sacrifices, it wouldnt have existed.*
*To the conceptes of love, friendship and support in my life and to my strong will, those who believed in my success and trusted in my abilities, no matter how weak they seemed, my soulmates* **SELWA** *and* **SALIMA***, you always gave me something that never ends my god protect then for me.*
*To those who excel in brotherhood and distinguish themselves in giving who were my support, each in their own way my* **brothers***, my god keep then for me.*
*and finally, to my dear friend and partner with whom i spent the most beautiful memori* **YOUSSRA***.*

**AMIRA**

# *DEDICATION*

*In the name of* **ALLAH**, *the most Gracious the Mercifule. All praise is due to allah, who granted us success and elevation only by his satisfaction.*
*I dedicate my graduation and the fruit of my effort and joy that have been awaiting all my life to my dear* **father** **RABIA** *my allah have mercy on him with his wide mercy and make him among the people of paradise lord.*
*To the one who adorned my name with the beautifule title, who taught me that life is a struggle and its weapon is knowledge and awareness, who planted in my soul the virtues of character and was my first supporter in my journey and my pillar.*
*To the one whom allah made paradise beneath her feet, whose heart embraced me before her hand, and who eased my hardships with her prayers, to the tender heart and the candle that was the secrte of my strenght and succes in dark nights my beloved* **mother** **AKILA** *may allah preseve her.*
*To those who supported me in life and where a source of pride and pride for me, the wellspring of my inspriration and who were my steady support after my father, may allah have mercy on him, my* **brother** **ABDELJALIL, ZITOUNI, SALAHEDDINE** *my allah protect you for me*
*To those who brought out the best in me and always encouraged me to reach my ambitions, who supported me limitessly and gave me without expecting anything in return my* **sisters** **SIHAM, SARA** *may allah protct you for me.*
*To my* **brother's wiffe** **ABDELJALIL** *and my* **brother's fiancée** **ZITOUNI**.
*To the littel chicks, my sister daughter siham* **NORHANE** *and my sister sons sara* **AMINE, ISLAM, OKBA**, *and my nephew son* **AYOUB**.
*To my sisters husbands,* **ZOUHIR, MOHAMMED**.
*To my classmate and dear me throughtout my university journey* **AMIRA**.
*To my friends* **CHOUBAILA, HAYAT, HASSNA, NESRIN, BOUCHRA, ASSIA**.

**YOUSSRA**

# CONTENTS

# INTRODUCTION

Several years ago, extensive studies were conducted on various linear recursive relations, such as the Fibonacci, Lucas, and Pell Lucas sequences, among others. The purpose of these studies was to explore their solutions, the associated generating functions, and their explicit forms. These explicit recursive relations are widely used in numerous research fields, including economics and computer science.

However, in recent years, many researchers have sought to generalize several of these recursive relations. One of the well-known is the generalization of the Fibonacci sequence. This generalization has gained significant attention, and researchers are familiar with it [14] .

Using this generalization, we can obtain numerous recursive relations and generating functions by utilizing the technique of symmetric functions [12] .

In the first chapter, we provide the necessary tools and background information to comprehend the subsequent chapters. We begin by presenting definitions and properties of linear recurrence relations for some numbers and polynomials. Additionally, we introduce ordinary generating functions for certain polynomials towards the end of the chapter.

In the second chapter, we review the elementary and complete symmetric functions along with their properties.

In the third chapter, we consider the previous theorems in order to derive new generating functions for the products of Gaussian numbers such as Gaussian Fibonacci, Gaussian Lucas, Gaussian Jacobsthal, Gaussian Jacobsthal Lucas, Gaussian Pell, Gaussian Pell Lucas numbers, and Gaussian polynomials with symmetric functions in several variables.

# CHAPTER 1

# NOTATIONS AND PRELIMINARIES

In this chapter, we solve linear recurrence relations with action of the constant form of order k with a characteristic polynomial method, and some inportant definitions and theorems related to formal series and generating functions.

## 1.1 Linear recurrence relations

**Definition 1.1.1** *[13] A linear recurrence relation of degree $k$ is a recurrence relation of the form*

$$u_n + f_1(n)u_{n-1} + f_2(n)u_{n-2} + \dots + f_k(n)u_{n-k} = g(n), \tag{1.1}$$

*where $f_1(n), f_2(n), \dots, f_k(n)$ et $g(n)$ are functions of $n$ and $f_k(n) \neq 0$.*

**Remark 1.1.1**

*If $g(n) = 0$, then the relation (1.1) is homogeneous, if not it says non-homogeneous.*

**Theorem 1.1.1** *[13] The linear recurrence relation*

$$u_n + f_1(n)u_{n-1} + f_2(n)u_{n-2} + \dots + f_k(n)u_{n-k} = g(n),$$

*with $u_0 = a_0$, $u_1$=1, ..., $u_{k-1} = a_{k-1}$, are constants has a unique solution.*

**Lemma 1.1.1** *[13] Let $u_n^{(1)}$ be the solution of the relation :*

$$u_n + f_1(n)u_{n-1} + f_2(n)u_{n-2} + ... + f_k(n)u_{n-k} = g_1(n),$$

*and $u_n^{(2)}$ the solution of the relation :*

$$u_n + f_1(n)u_{n-1} + f_2(n)u_{n-2} + ... + f_k(n)u_{n-k} = g_2(n).$$

*Then $c_1u_n^{(1)} + c_2u_n^{(2)}$ is the solution of*

$$u_n + f_1(n)u_{n-1} + f_2(n)u_{n-2} + ... + f_k(n)u_{n-k} = c_1g_1(n) + c_2g_2(n).$$

**Proof.** We have

$$[c_1u_n^{(1)} + c_2u_n^{(2)}] + f_1(n)[c_1u_{n-1}^{(1)} + c_2u_{n-1}^{(2)}] + ... + f_k(n)[c_1u_{n-k}^{(1)} + c_2u_{n-k}^{(2)}]$$

$$= c_1u_n^{(1)} + c_1f_1(n)u_{n-1}^{(1)} + ... + c_1f_k(n)u_{n-k}^{(1)} + c_2u_n^{(2)} + c_2f_1(n)u_{n-1}^{(2)} + ... + c_2f_k(n)u_{n-k}^{(2)}$$

$$= c_1[u_n^{(1)} + f_1(n)u_{n-1}^{(1)} + ... + f_k(n)u_{n-k}^{(1)}] + c_2[u_n^{(2)} + f_1(n)u_{n-1}^{(2)} + ... + f_k(n)u_{n-k}^{(2)}]$$

$$=c_1g_1(n) + c_2g_2(n).$$

This completes the proof ■

## 1.2 Linear homogeneous recurrence relations with constant coefficients

**Definition 1.2.1** *[10] A linear homogeneous recurrence relation of degree k with constant coefficients is a recurrence relation of the form*

$$u_n = c_1u_{n-1} + c_2u_{n-2} + ... + c_ku_{n-k}, \tag{1.2}$$

*where $c_1, c_2, ..., c_k$are real numbers, and $c_k \neq 0$*

**Example 1.2.1** *The recurrence relation $u_n = 6u_{n-1} + 5u_{n-2}$ is a linear homogeneous recurrence relation of degree two.*

**Example 1.2.2** *The recurrence relation $u_n = 3u_{n-1} + u_{n-2}^2$ is not linear.*

**Example 1.2.3** *The recurrence relation $u_n = 5u_{n-1} + 3$ is not homogeneous.*

**Example 1.2.4** *The recurrence relation $u_n = 4nu_{n-1} + n^3u_{n-2}^2$ does not have constant coefficients.*

**Remark 1.2.1** *[10] The basic approach for solving linear homogeneous recurrence relations is to look for solutions of the forme $u_n = t^n$ where $t$ is a constant. Note that $u_n = t^n$ is a solution of the recurrence relation $u_n = c_1u_{n-1} + c_2u_{n-2} + ... + c_ku_{n-k}$ if and only if*

$$t^n = c_1t^{n-1} + ... + c_kt^{n-k}$$

*when both sides of this equation are divided by $t^{n-k}$ and the right-hand side is subtracted from the left, we obtain the equation $t^k - c_1t^{k-1} - ... - c_k = 0$. Consequently, the sequence $u_n$ with $u_n = t^n$ is a solution if and only if $t$ a solution of this last equation. We call this the characteristic equation of the recurrence relation. The solutions of this equation are called the characteristic roots of the recurrence relation. As we will see, these characteristic roots can be used to give an explicit formula for all the solutions of the recurrence relation is complexe.*

## 1.3 Characteristic polynomial

**Definition 1.3.1** *[4] The characteristic polynomial of the reccurrence relation*

$$u_n = c_1u_{n-1} + c_2u_{n-2} + ... + c_ku_{n-k}$$

*is*

$$p(t) = t^k - c_1t^{k-1} - ... - c_k.$$

**Example 1.3.1** *The characteristic polynomial of the reccurrence relation* $u_n = 2u_{n-1} + 3u_{n-2}$ *is* $t^2 - 2t - 3 = 0$.

**Example 1.3.2** *The characteristic polynomial of the reccurrence relation* $u_n = -3u_{n-1} - 3u_{n-2} - 2u_{n-3}$ *is* $t^3 + 3t^2 + 3t + 2 = 0$.

**Theorem 1.3.1** *[10] Let* $c_1, c_2, ..., c_k$ *be real number. Suppose that the characteristic equation*

$$t^k - c_1 t^{k-1} - ... - c_k = 0,$$

*has k distinct roots* $t_1, t_2, .., t_k$*. Then a sequence* $u_n$ *is a solution of the reccurrence relation*

$$u_n = c_1 u_{n-1} + c_2 u_{n-2} + ... + c_k u_{n-k},$$

*if and only if*

$$u_n = \alpha_1 t_1^n + \alpha_2 t_1^n + \alpha_n t_n^n.$$

*for* $n = 0, 1, 2, ...,$ *where* $\alpha_1, \alpha_2, ..., \alpha_n$ *are constants.*

**Example 1.3.3** *consider the following recurrence relation :*

$$\begin{cases} u_n = 3u_{n-1} + 4u_{n-2}, n \geq 2 \\ u_0 = 0, u_1 = 1. \end{cases}$$

*The characteristic equation is* $t^2 - 3t - 4 = 0$*. This can be factored as* $(t+1)(t-4)$*, hence there are two real roots :*$-1$ *and* $4$*, then the general solution is*

$$u_n = c_1 4^n + c_2(-1)^n.$$

*The initial conditions* $u_0 = 0$*,* $u_1 = 1$ *implies that*

$$\begin{cases} c_1 + c_2 = 0 \\ 4c_1 - c_2 = 1 \end{cases}$$

$$\begin{cases} c_1 = \frac{1}{5} \\ c_2 = \frac{-1}{5} \end{cases}$$

*Thus the solution is*

$$u_n = \frac{4^n}{5} - \frac{(-1)^n}{5}.$$

**Theorem 1.3.2** *[10] Let $c_1, c_2, ..., c_k$be real numbers. suppose that the characteristic equation*

$$t^k - c_1 t^k - ... - c_k = 0,$$

*has r distenct roots $t_1, t_2, ..., t_k$ with multiplicities $m_1, m_2, ..., m_r$, respectively, so that $m_i \geq 1$ $\forall i = 1, 2, ..., r$ and $\sum_{i=1}^{r} m_i = k$. Then a sequence $u_n$ is a solution of the recurrence relation*

$$u_n = c_1 u_{n-1} + c_2 u_{n-2} + ... + c_k u_{n-k}.$$

*If and only if*

$$u_n = (\alpha_{1.0} + \alpha_{11} n + ... + \alpha_{1m-1} n^{m_1 - 1}) z_1^n + (\alpha_{2.0} + \alpha_{21} n + ... + \alpha_{2m_2-1} n^{m_r - 1}) z_2^n + ... + (\alpha_{r.0} + \alpha_{r.2} n + ... + \alpha_{rm_r-1} n^{m_r - 1}) z_r^n$$

*for n=0, 1, 2,. . . , where $\alpha_{ij}$ are constants for $1 \leq i \leq r$ and $0 \leq j \leq m_i - 1$.*

**Example 1.3.4** *consider the following recurrence relation :*

$$\begin{cases} u_n = 6u_{n-1} - 9u_{n-2}, n \geq 2 \\ u_0 = 1, u_1 = 6. \end{cases}$$

*The characteristic equation is $t^2 - 6t + 9 = 0$. admits the number 3 as the root of multiplicity 2, then the general solution is*

$$u_n = c_1 3^n + c_2 n 3^n$$

*The initial conditions $u_0 = 1$, $u_1 = 6$ implies that*

$$\begin{cases} c_1 = 1 \\ 3c_1 + 3c_2 = 6 \end{cases}$$

$$\begin{cases} c_1 = 1 \\ c_2 = 1 \end{cases}$$

*Thus the solution is*

$$u_n = 3^n + n3^n$$

### 1.3.1 Recurrence relations of some numbers and polynomials

**Definition 1.3.2** *[4] Generalized Fibonacci sequence $(G_n)_{n\in\mathbb{N}}$ is defined by the following recurrence relation*

$$\begin{cases} G_n = pG_{n-1} + qG_{n-2}, n \geq 2 \\ G_0 = \alpha, G_1 = \beta. \end{cases} \tag{1.3}$$

*with $p, q \in \mathbb{R}_+$ and $\alpha, \beta \in \mathbb{C}$.*

**Lemma 1.3.1** *[4] Let $z^2 - pz - q = 0$, the characterisic equation of the recurrence relation (1.3). Then*

1. *If the characteristic equation has two real solutions $z_1$ and $z_2$, then the general solution for (1.3) is given by:*

$$G_n = \frac{\lambda_1 z_1^n - \lambda_2 z_2^n}{z_1 - z_2},$$

*with $\lambda_1=\beta - \alpha z_2$ and $\lambda_2=\beta - \alpha z_1$.*

2. *If the characteristic equation has only one real solution $z$, then the general solution for (1.3) is given by :*

$$G_n = (c_1 + c_2 n)z^n,$$

*with $c_1 = \alpha$ and $c_2 = \dfrac{\beta - \alpha z}{z}$.*

**Definition 1.3.3** *[3] The Gaussain Fibonacci numbers are defined by the following recurrence relation:*

$$\begin{cases} GF_{n+1} = GF_n + GF_{n-1}, \forall n \geq 1 \\ GF_0 = i, GF_1 = 1. \end{cases} \tag{1.4}$$

*The first terms of the Gaussain Fibonacci numbers are given by*

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| $GF_{n+1}$ | 1 | $1+i$ | $2+i$ | $3+2i$ | $5+3i$ | $8+5i$ | $13+8i$ |

**Definition 1.3.4** *[3] The Gaussain Lucas numbers are defined by the following recurrence relation:*

$$\begin{cases} GL_{n+1} = GL_n + GL_{n-1}, \forall n \geq 1 \\ GL_0 = 2-i, GL_1 = 1+2i. \end{cases} \tag{1.5}$$

*The first terms of the Gaussain Lucas numbers are given by*

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $GL_{n+1}$ | $1+2i$ | $3+i$ | $4+3i$ | $7+4i$ | $11+7i$ | $18+11i$ |

**Definition 1.3.5** *[3] The Gaussian Jacobsthal numbers are defined by the following recurrence relation :*

$$\begin{cases} GJ_{n+1} = GJ_n + 2GJ_{n-1}, \forall n \geq 1 \\ GJ_0 = \frac{i}{2}, GJ_1 = 1. \end{cases} \tag{1.6}$$

*The first terme of the Gaussian Jacobsthal numbers are given by*

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| $GJ_{n+1}$ | 1 | $1+i$ | $3+i$ | $5+3i$ | $11+5i$ | $21+11i$ | $43+21i$ |

**Definition 1.3.6** *[3] The Gaussian Jacobsthal Lucas numbers are defined by the following recurrence relation:*

$$\begin{cases} Gj_{n+1} = Gj_n + 2Gj_{n-1}, \forall n \geq 1 \\ Gj_0 = 2-\frac{i}{2}, Gj_1 = 1+2i. \end{cases} \tag{1.7}$$

*The first terme of the Gaussian Jacobsthal Lucas numbers are given by*

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $Gj_{n+1}$ | $1+2i$ | $5+i$ | $7+5i$ | $17+7i$ | $31+17i$ | $65+31i$ |

**Definition 1.3.7** *[8] The Gaussian Pell numbers are defined by the following recurrence relation:*

$$\begin{cases} GP_{n+1} = 2GP_n + GP_{n-1}, \forall n \geq 1 \\ GP_0 = i, GP_1 = 1. \end{cases} \tag{1.8}$$

*The first terme of the Gaussian Pell numbers are given by*

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| $GP_{n+1}$ | 1 | $2+i$ | $5+2i$ | $12+5i$ | $29+12i$ | $70+29i$ | $169+70i$ |

**Definition 1.3.8** *[8]The Gaussian Pell Lucas numbers are defined by the following recurrence relation :*

$$\begin{cases} GQ_{n+1} = 2GQ_n + GQ_{n-1}, \forall n \geq 1 \\ GQ_0 = 2-2i, GQ_1 = 2+2i. \end{cases} \tag{1.9}$$

*The first terms of the Gaussian Pell Lucas numbers are given by*

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $GQ_{n+1}$ | $2+2i$ | $6+2i$ | $14+6i$ | $34+14i$ | $82+34i$ | $198+82i$ |

**Definition 1.3.9** *[3]The Gaussian Jacobsthal polynomials are defined by the following recurrence relation :*

$$\begin{cases} GJ_{n+1}(x) = GJ_n(x) + 2xGJ_{n-1}(x), \forall n \geq 1 \\ GJ_0(x) = \frac{i}{2}, GJ_1(x) = 1. \end{cases} \tag{1.10}$$

**Definition 1.3.10** *[3]The Gaussian Jacobsthal Lucas polynomials are defined by the following recurrence relation :*

$$\begin{cases} Gj_{n+1}(x) = Gj_n(x) + 2xGj_{n-1}(x), \forall n \geq 1 \\ Gj_0(x) = 2 - \frac{i}{2}, Gj_1(x) = 1+2ix. \end{cases} \tag{1.11}$$

**Definition 1.3.11** *[9]The Gaussian Pell polynomials are defined by the following recurrence relation :*

$$\begin{cases} GP_{n+1}(x) = 2xGP_n(x) + GP_{n-1}(x), \forall n \geq 1 \\ GP_0(x) = i, GP_1(x) = 1. \end{cases} \tag{1.12}$$

## 1.4 Generating functions

### 1.4.1 Formal series

Let $\mathbb{K}$ be a commutative field ( $\mathbb{K}$=$\mathbb{R}$ or $\mathbb{C}$).

**Definition 1.4.1** *[4] The elements of the set $\mathbb{K}\,[[z]] = \left\{\sum_{n=0}^{\infty} a_n z^n, a_n \in \mathbb{K}\right\}$ are called formal series with coefficients in $\mathbb{C}$. for $n \in \mathbb{N}, z^n$ called the monomial of degree $n$ and $a_n$ it's coefficient.*

**Definition 1.4.2** *[4] Let $\alpha(z) = \sum_{n=0}^{\infty} a_n z^n$ and $\beta(z) = \sum_{n=0}^{\infty} b_n z^n$ be two formal series. Then the sun of $\alpha(z)$ and $\beta(z)$ is given by*

$$\alpha(z) + \beta(z) = \sum_{n=0}^{\infty} (a_n + b_n) z^n.$$

**Definition 1.4.3** *[4] Let $\alpha(z) = \sum_{n=0}^{\infty} a_n z^n$ and $\beta(z) = \sum_{n=0}^{\infty} b_n z^n$ be two formal series. Then the product of $\alpha(z)$ and $\beta(z)$ is given by*

$$\alpha(z)\beta(z) = \sum_{n=0}^{\infty} c_n z^n,$$

*with*

$$c_n = \sum_{k=0}^{\infty} a_k b_{n-k}.$$

**Definition 1.4.4** *[4] Two formal series $\alpha(z) = \sum_{n=0}^{\infty} a_n z^n$ and $\beta(z) = \sum_{n=0}^{\infty} b_n z^n$ are equal if and only if for all $n \neq 0$, $a_n$=$b_n$.*

**Definition 1.4.5** *[4] We say that the series $\sum_{n=0}^{\infty} a_n z^n$ is the inverse of the series $\sum_{n=0}^{\infty} b_n z^n$ if :*

$$\left(\sum_{n=0}^{\infty} a_n z^n\right)\left(\sum_{n=0}^{\infty} b_n z^n\right) = 1.$$

**Proposition 1.4.1** *[4] A formal series $\sum_{n=0}^{\infty} a_n z^n$ is invertible if and only if $a_n \neq 0$.*

**Proof.** We need to determine whether or not there exists a formal series $\beta(z) = \sum_{n=0}^{\infty} a_n z^n$ , in $\mathbb{K}[[z]]$ such that $\alpha(z)\beta(z)$ =1. Expanding the product, we have

$$\begin{aligned}\alpha(z)\beta(z) &= \left(\sum_{n=0}^{\infty} a_n z^n\right)\left(\sum_{n=0}^{\infty} b_n z^n\right)\\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a_k b_{n-k}\right) z^n.\end{aligned}$$

Comparing the coefficient of $z^n$ on both sides of $\alpha(z)\beta(z)$ =1, we see that $\beta(z)$ satisfies the equation if and only if $a_0b_0$=1 and $\sum_{k=0}^{n} a_k b_{n-k} = 0$ for all $n \geq 1$.

If $a_0$ is not invertible in $\mathbb{K}$ then the equation $\alpha(z)\beta(z)$ =1 can not solved for $b_0$, so that $\beta(z)$ does not exist and $\alpha(z)$ is not invertible in $\mathbb{K}[[z]]$.

If $a_0$ is invertible in $\mathbb{K}$, then $b_0$= $a_0^{-1}$ exist. Each of the remaining equations (for $n \geq 1$) can be rewritten as $a_0b_n$=-$\sum_{k=0}^{n} a_k b_{n-k}$, or upon multiplying by $b_0$, $b_n = -b_0 \sum_{k=0}^{n} a_k b_{n-k}$. These equation can be solved by indication on $k \geq 1$, yielding a solution for $\beta(z)$ which gives the multiplicative inverse of $\alpha(z)$. Therefore $\beta(z)$ is invertible in $\mathbb{K}[[z]]$.

This completes the proof. ■

**Proposition 1.4.2** *[4] If $\alpha(z)$ and $\beta(z)$ are two nonzero formal series then $\alpha(z)\beta(z)$ is also nonzero.*

## 1.4.2 Ordinary generating functions (OGF)

**Definition 1.4.6** *[4] The OGF of the sequence $(a_n)_{n\in\mathbb{N}}$=$(a_0, a_1, a_2, ...)$, is defined by :*

$$G(z) = \sum_{n=0}^{\infty} a_n z^n.$$

**Example 1.4.1** *The generating functions for the sequences $(a_n)_{n\in\mathbb{N}}$ with $a_n = 6, a_n = n+2$ and $a_n = 4^n$ are $\sum_{n=0}^{\infty} 6z^n$, $\sum_{n=0}^{\infty}(n+2)z^n$ and $\sum_{n=0}^{\infty} 4^n z^n$.*

**Theorem 1.4.1** *[6] Let $A(z)$ the OGF of $(a_n)_{n\in\mathbb{N}}$ and $B(z)$ the OGF of $(b_n)_{n\in\mathbb{N}}$, so*

1. *$A(z) + B(z)$ is OGF of $(a_n + b_n)_{a\geq 0}$.*
2. *$zA(z)$ is the OGF of $(0, a_0, a_1, a_2, ..., a_{n-1})$*
3. *$A(z)B(z)$ is the OGF of $(a_0, a_0b_1 + a_1b_0 + a_1b_1 + a_2b_0, ...)$*
4. *$(1-z)A(z)$ is the OGF of $(a_0, a_1 - a_0, a_2 - a_1, ..., a_n - a_{n-1}, ...)$.*
5. *$\frac{A(z)}{1-t}$ is the OGF of $(a_0, a_0 + a_1, a_0 + a_1 + a_2, ..., \sum_{n=0}^{\infty} a_k, ...)$.*

**Theorem 1.4.2** *[4] Let the sequence $(G_n)_{n\in\mathbb{N}}$ defined by the following recurrence relation*

$$\begin{cases} G_n = pG_{n-1} + qG_{n-2} \\ G_0 = \alpha, G_1 = \beta \end{cases} \tag{1.13}$$

*with $p, q \in \mathbb{C}$ and $\alpha, \beta \in \mathbb{C}$. So the generating function of $(G_n)_{n\geq 0}$ is given by*

$$G(z) = \frac{\alpha + (\beta - p\alpha)z}{1 - pz - qz^2}.$$

**Proof.** We have

$$\begin{aligned} G(z) &= \sum_{n=0}^{\infty} G_n z^n = G_0 + G_1 z + \sum_{n=2}^{\infty} G_n z^n \\ &= \alpha + \beta z + \sum_{n=2}^{\infty} (pG_{n-1} + qG_{n-2}) z^n \\ &= \alpha + \beta z + pz \sum_{n=2}^{\infty} G_{n-1} z^{n-1} + qz^2 \sum_{n=2}^{\infty} G_{n-2} z^{n-2} \\ &= \alpha + \beta z + pz \sum_{n=1}^{\infty} G_n z^n + qz^2 \sum_{n=0}^{\infty} G_n z^n \\ &= \alpha + \beta z + pz \left( \sum_{n=0}^{\infty} G_n z^n - \alpha \right) + qz^2 \sum_{n=0}^{\infty} G_n z^n \\ &= \alpha + (\beta - \alpha p)z + pzG(z) + qz^2 G(z). \end{aligned}$$

Then

$$G(z)(1 - pz - qz^2) = \alpha + (\beta - \alpha p)z$$

So

$$G(z) = \frac{\alpha + (\beta - \alpha p)z}{1 - pz - qz^2}.$$

This completes the proof. ■

**Theorem 1.4.3** *[11] The generating function of the bivariate Fibonacci polynomials is given by*

$$\sum_{n1=2}^{\infty} F_n(x, y) z^n = \frac{z}{1 - xz - yz^2}.$$

**Proof.** We have

$$\begin{aligned}
G(z) &= \sum_{n=0}^{\infty} F_n(x,y)z^n = F_0(x,y) + F_1(x,y)z + \sum_{n=2}^{\infty} F_n(x,y)z^n \\
&= z + \sum_{n=2}^{\infty}(xF_{n-1}(x,y) + yF_{n-2}(x,y))z^n \\
&= z + xz\sum_{n=2}^{\infty} F_{n-1}(x,y)z^{n-1} + yz^2\sum_{n=2}^{\infty} F_{n-2}(x,y)z^{n-2} \\
&= z + xz\sum_{n=1}^{\infty} F_n(x,y)z^n + yz^2\sum_{n=0}^{\infty} F_n(x,y)z^n \\
&= z + xz\left(\sum_{n=0}^{\infty} F_n(x,y)z^n - 0\right) + yz^2\sum_{n=0}^{\infty} F_n(x,y)z^n \\
&= z + xzG(z) + yz^2G(z).
\end{aligned}$$

Then

$$G(z)(1 - xz - yz^2) = z$$

So

$$G(z) = \frac{z}{1 - xz - yz^2}.$$

This completes proof ■

**Theorem 1.4.4** *[11] The generating function of the bivariate Lucas polynomials is given by*

$$\sum_{n=2}^{\infty} L_n(x,y)z^n = \frac{2 - xz}{1 - xz - yz^2}$$

**Proof.** we have

$$\begin{aligned}
G(z) &= \sum_{n=2}^{\infty} L_n(x,y)z^n \\
&= L_0(x,y) + L_1(x,y)z + \sum_{n=2}^{\infty} L_n(x,y)z^n \\
&= 2 + xz + \sum_{n=2}^{\infty}(xL_{n-1}(x,y) + yL_{n-2}(x,y))z^n \\
&= 2 + xz + xz\sum_{n=2}^{\infty} L_{n-1}(x,y)z^{n-1} + yz^2\sum_{n=2}^{\infty} L_{n-2}(x,y)z^{n-2} \\
&= 2 + xz + xz\sum_{n=1}^{\infty} L_n(x,y)z^n + yz^2\sum_{n=0}^{\infty} L_n(x,y)z^n
\end{aligned}$$

$$= 2 + xz + xz\left(\sum_{n=0}^{\infty} L_n(x,y)z^n - 2\right) + yz^2 \sum_{n=0}^{\infty} L_n(x,y)z^n$$

$$= 2 - xz + xzG(z) + yz^2G(z)$$

Then

$$G(z)(1 - xz - yz^2) = 2 - xz.$$

So

$$G(z) = \frac{2 - xz}{1 - xz - yz^2}.$$

This competes the proof ■

**Theorem 1.4.5** *[11] The generating function of the bivariate Pell polynomials is given by*

$$\sum_{n=2}^{\infty} P_n(x,y)z^n = \frac{z}{1 - 2xyz - yz^2}$$

**Proof.** We have

$$G(z) = \sum_{n=0}^{\infty} P_n(x,y)z^n$$

$$= P_0(x,y) + P_1(x,y)z + \sum_{n=2}^{\infty} P_n(x,y)z^n$$

$$= z + \sum_{n=2}^{\infty} (2xyP_{n-1}(x,y) + yP_{n-2}(x,y))z^n$$

$$= z + 2xyz \sum_{n=2}^{\infty} P_{n-1}(x,y)z^{n-1} + yz^2 \sum_{n=2}^{\infty} P_{n-2}(x,y)z^{n-2}$$

$$= z + 2xyz \sum_{n=1}^{\infty} P_n(x,y)z^n + yz^2 \sum_{n=0}^{\infty} P_n(x,y)z^n$$

$$= z + 2xyz\left(\sum_{n=0}^{\infty} P_n(x,y)z^n - 0\right) + yz^2 \sum_{n=0}^{\infty} P_n(x,y)z^n$$

$$= z + 2xyzG(z) + yz^2G(z).$$

Then

$$G(z)(1 - 2xyz - yz^2) = z.$$

So

$$G(z) = \frac{z}{1 - 2xyz - yz^2}.$$

This completes the proof ■

**Theorem 1.4.6** *[11] The generating function of the bivariate Pell-Lucas polynomials is given by*

$$\sum_{n=0}^{\infty} Q_n(x,y)z^n = \frac{2-2xyz}{1-2xyz-yz^2}$$

**Proof.** we have

$$\begin{aligned}
G(z) &= \sum_{n=0}^{\infty} Q_n(x,y)z^n \\
&= Q_0(x,y) + Q_1(x,y)z + \sum_{n=2}^{\infty} Q_n(x,y)z^n \\
&= 2 + 2xyz + \sum_{n=2}^{\infty}(2xyQ_{n-1}(x,y) + yQ_{n-2}(x,y))z^n \\
&= 2 + 2xyz + 2xyz\sum_{n=2}^{\infty} Q_{n-1}(x,y)z^{n-1} + yz^2\sum_{n=2}^{\infty} Q_{n-2}(x,y)z^{n-2} \\
&= 2 + 2xyz + 2xyz\sum_{n=1}^{\infty} Q_n(x,y)z^n + yz^2\sum_{n=0}^{\infty} Q_n(x,y)z^n \\
&= 2 + 2xyz + 2xyz\left(\sum_{n=0}^{\infty} Q_n(x,y)z^n - 2\right) + yz^2\sum_{n=0}^{\infty} Q_n(x,y)z^n \\
&= 2 - 2xyz + 2xyzG(z) + yz^2G(z).
\end{aligned}$$

Then,

$$G(z)(1-2xyz-yz^2) = 2-2xyz.$$

So,

$$G(z) = \frac{2-2xyz}{1-2xyz-yz^2}$$

This completes the proof ■

**Theorem 1.4.7** *[11] The generating function of the bivariate Jacobsthal polynomials is given by*

$$\sum_{n=0}^{\infty} J_n(x,y)z^n = \frac{z}{1-xyz-2yz^2}$$

**Proof.** we have

$$\begin{aligned}
G(z) &= \sum_{n=0}^{\infty} J_n(x,y)z^n \\
&= J_0(x,y) + J_1(x,y)z + \sum_{n=2}^{\infty} J_n(x,y)z^n
\end{aligned}$$

$$= z + \sum_{n=2}^{\infty}(xyJ_{n-1}(x,y) + 2yJ_{n-2}(x,y))z^n$$

$$= z + xyz\sum_{n=2}^{\infty} J_{n-1}(x,y)z^{n-1} + 2yz^2\sum_{n=2}^{\infty} J_{n-2}(x,y)z^{n-2}$$

$$= z + +xyz\sum_{n=1}^{\infty} J_n(x,y)z^n + 2yz^2\sum_{n=0}^{\infty} J_n(x,y)z^n$$

$$= z + xyz\left(\sum_{n=0}^{\infty} J_n(x,y)z^n - 0\right) + 2yz^2\sum_{n=0}^{\infty} J_n(x,y)z^n$$

$$= z + xyzG(z) + 2yz^2G(z).$$

Then,

$$G(z)(1 - xyz - 2yz^2) = z.$$

So,

$$G(z) = \frac{z}{1 - xyz - 2yz^2}.$$

This completes the proof ■

**Theorem 1.4.8** *[11] The generating function of the bivariate Jacobsthal-Lucas polynomials is given by*

$$\sum_{n=0}^{\infty} j_n(x,y)z^n = \frac{2 - xyz}{1 - xyz - 2yz^2}$$

**Proof.** we have

$$G(z) = \sum_{n=0}^{\infty} j_n(x,y)z^n$$

$$= j_0(x,y) + j_1(x,y)z + \sum_{n=2}^{\infty} j_n(x,y)z^n$$

$$= 2 + xyz + \sum_{n=2}^{\infty}(xyj_{n-1}(x,y) + 2yj_{n-2}(x,y))z^n$$

$$= 2 + xyz + xyz\sum_{n=2}^{\infty} j_{n-1}(x,y)z^{n-1} + 2yz^2\sum_{n=2}^{\infty} j_{n-2}(x,y)z^{n-2}$$

$$= 2 + xyz + xyz\sum_{n=1}^{\infty} j_n(x,y)z^n + 2yz^2\sum_{n=0}^{\infty} j_n(x,y)z^n$$

$$= 2 + xyz + xyz\left(\sum_{n=0}^{\infty} j_n(x,y)z^n - 2\right) + 2yz^2\sum_{n=0}^{\infty} j_n(x,y)z^n$$

$$= 2 - xyz + xyzG(z) + 2yz^2G(z).$$

Then,

$$G(z)(1 - xyz - 2yz^2) = 2 - xyz.$$

So,

$$G(z) = \frac{2 - xyz}{1 - xyz - 2yz^2}$$

This completes the proof ■

From the previous theorems we deduce the following table [4]:

| valeus of $p, q, \alpha, \beta$ | cofficient of $z^n$ | generating function |
|---|---|---|
| $p = k, \beta = 1, q = 1, \alpha = 0$ | $F_{k.n}$ | $\frac{1}{1 - kz - z^2}$ |
| $p = \beta = k, q = 1, \alpha = 2$ | $L_{k.n}$ | $\frac{2 - kz}{1 - kz - z^2}$ |
| $\alpha = 0, p = 2, q = k, \beta = 1$ | $P_{k.n}$ | $\frac{k}{1 - 2z - kz^2}$ |
| $\alpha = \beta = p = 2, q = k$ | $Q_{k.n}$ | $\frac{2 - 2z}{1 - 2z - 2z^2}$ |
| $p = k, q = 2, \alpha = 0, \beta = 1$ | $J_{k.n}$ | $\frac{z}{1 - kz - 2z^2}$ |
| $p = \beta = k, \alpha = q = 2$ | $j_{k.n}$ | $\frac{2 - kz}{1 - kz - 2z^2}$ |

**Table1**:Generating function of some k numbers.

For k=1 in Table 1 we obtain the following table [4]

| valeus of $p, q, \alpha, \beta$ | cofficient of $z^n$ | generating function |
|---|---|---|
| $p = 1, \beta = 1, q = 1, \alpha = 0$ | $F_n$ | $\frac{1}{1 - z - z^2}$ |
| $p = \beta = 1, q = 1, \alpha = 2$ | $L_n$ | $\frac{2 - z}{1 - z - z^2}$ |
| $\alpha = 0, p = 2, q = 1, \beta = 1$ | $P_n$ | $\frac{z}{1 - 2z - z^2}$ |
| $\alpha = \beta = p = 2, q = 1$ | $Q_n$ | $\frac{2 - 2z}{1 - 2z - z^2}$ |
| $p = 1, q = 2, \alpha = 0, \beta = 1$ | $J_n$ | $\frac{z}{1 - z - 2z^2}$ |
| $p = \beta = 1, \alpha = q = 2$ | $j_n$ | $\frac{2 - z}{1 - z - 2z^2}$ |

**Table2**:Generating function of some numbers.

# CHAPTER 2

# ELEMENTARY AND COMPLETE SYMMETRIC FUNCTIONS

In this chapter, we mention some inportant definitions and properties elementary and complete symmetric functions.

## 2.1 Symmetric functions

**Definition 2.1.1** *[12] A function $f(x_1, x_2, ..., x_n)$ in n variables is symmetric if for all premutations of the index set (1, 2,. . . n) the following equality holds:*

$$f(x_1, x_2, ..., x_n) = f(x_{s(1)}, x_{s(2)}, ..., x_{s(n)}).$$

*which means, a function of several variables is symmetric if its values does not change when we swap variables.*

### 2.1.1 Elementary symmetric functions

**Definition 2.1.2** *[4] Let k and n be tow positive integer and* $(\lambda_1, \lambda_2, ...., \lambda_n)$ *are the set of given variables. Then the elementary symmetric function* $e_k(\lambda_1, \lambda_2, ...., \lambda_n)$ *is defined by*

$$e_k^{(n)} = e_k(\lambda_1, \lambda_2, ...., \lambda_n) = \sum_{i_1+i_2+...+i_n=k} \lambda_1^{i_1}\lambda_2^{i_2}...\lambda_n^{i_n}, 0 \le k \le n \tag{2.1}$$

*with* $i_1, i_2, ...., i_n = 0 \vee 1.$

**Example 2.1.1** *For an equation of degree 3 (n=3, roots:*$\lambda_1, \lambda_2, \lambda_3$*), we have*

$$\begin{cases} e_0^{(3)} = 1 \\ e_1^{(3)} = \lambda_1 + \lambda_2 + \lambda_3 \\ e_2^{(3)} = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 \\ e_3^{(3)} = \lambda_1\lambda_2\lambda_3 \end{cases}$$

**Proposition 2.1.1** *[7] The generating function of the elementary symmetric functions is given by:*

$$E(z) = \sum_{k\ge 0} e_k z^k = \prod_{i=1}^{n}(1 + \lambda_i z)$$

**Proof.** We have

$$e_k^{(n)} = e_k(\lambda_1, \lambda_2, ...., \lambda_n) = \sum_{i_1+i_2+...+i_n=k} \lambda_1^{i_1}\lambda_2^{i_2}...\lambda_n^{i_n}, e_k^{(n)} = 0 if k > n.$$

For n=2, we have

$$\begin{aligned} \prod_{i=1}^{2}(1 + \lambda_i z) &= (1 + \lambda_1 z)(1 + \lambda_2 z) \\ &= 1 + (\lambda_1 + \lambda_2)z + \lambda_1\lambda_2 z^2 \\ &= e_0 + e_1 z + e_2 z^2 \\ &= \sum_{k=0}^{2} e_k z^k. \end{aligned}$$

So the assertion is true for n=2, Assume the proposition is true for n, i. e that

$$\sum_{n\ge 0} e_k z^k = \prod_{i=1}^{n}(1 + \lambda_i z)$$

and we want prove that the proposition is true for n+1, i. e that

$$\sum_{k=0}^{n+1} e_k z^k = \prod_{i=1}^{n+1}(1+\lambda_i z)$$

We have

$$\begin{aligned}
\prod_{i=1}^{n+1}(1+\lambda_i z) &= \prod_{i=1}^{n}(1+\lambda_i z)(1+\lambda_{n+1} z) \\
&= \left(\sum_{k=0}^{n} e_k z^k\right)(1+\lambda_{n+1} z) \\
&= \sum_{k=0}^{n} e_k z^k + \lambda_{n+1}\sum_{k=0}^{n} e_k z^{k+1} \\
&= \sum_{k=0}^{n} e_k z^k + \lambda_{n+1}\sum_{k=1}^{n} e_{k-1} z^{k} \\
&= \sum_{k=0}^{n} e_k z^k + \lambda_{n+1}\sum_{k=0}^{n} e_{k-1} z^{k} \\
&= \sum_{k\geq 0}(e_k^{(n)} + \lambda_{n+1} e_{k-1}^{(n)}) z^k \\
&= \sum_{k\geq 0} e_k^{(n+1)} z^k \\
&= \sum_{k=0}^{n+1} e_k z^k.
\end{aligned}$$

Thus the proposition is true for all $n \geq 0$ ■

### 2.1.2 Complete symmetric functions

**Definition 2.1.3** *[4] Let k and n be tow positive integer and $(\lambda_1, \lambda_2, ...., \lambda_n)$ are the set of given variables. Then the complete symmetric functions $h_k(\lambda_1, \lambda_2, ...., \lambda_n)$ is defined by*

$$h_k^{(n)} = h_k(\lambda_1, \lambda_2, ...., \lambda_n) = \sum_{i_1+i_2+...+i_n=k} \lambda_1^{i_1}\lambda_2^{i_2}...\lambda_n^{i_n}, \quad (2.2)$$

*with $i_1, i_2, ...i_n \geq 0$ and $h_k^{(n)} = 0, \forall k < 0$.*

**Example 2.1.2** *For an equation of degree (n=4, roots:$\lambda_1, \lambda_2, \lambda_3, \lambda_4$), we have*

$$\begin{cases} h_0^{(4)} = 1 \\ h_1^{(4)} = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 \\ h_2^{(4)} = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_4^2 + \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_1\lambda_4 + \lambda_2\lambda_3 + \lambda_2\lambda_4 + \lambda_3\lambda_4 \\ h_3^{(4)} = \lambda_1^3 + \lambda_2^3 + \lambda_3^3 + \lambda_4^3 + \lambda_1\lambda_2\lambda_3 + \lambda_1\lambda_3\lambda_4 + \lambda_1\lambda_2\lambda_4 + \lambda_2\lambda_3\lambda_4 + \lambda_1^2\lambda_2 + \lambda_1^2\lambda_3 + \lambda_1^2\lambda_4 + ... \\ h_4^{(4)} = \lambda_1^4 + \lambda_2^4 + \lambda_3^4 + \lambda_4^4 + \lambda_1\lambda_2\lambda_3\lambda_4 + \lambda_1^2\lambda_2^2 + \lambda_1^2\lambda_3^2 + \lambda_1^2\lambda_4^2 + \lambda_2^2\lambda_3^2 + \lambda_2^2\lambda_4^2 + \lambda_3^2\lambda_4^2 + \lambda_1^3\lambda_2 + ... \end{cases}$$

**Proposition 2.1.2** *[7] The generating function of the completes symmetric function is given by*

$$H(z) = \sum_{k\geq 0} h_k z^k = \frac{1}{\prod_{i=1}^{n}(1-\lambda_i z)}$$

**Proof.** We have

$$h_k^{(n)} = h_k(\lambda_1, \lambda_2, ...., \lambda_n) = \sum_{i_1+i_2+...+i_n=k} \lambda_1^{i_1}\lambda_2^{i_2}...\lambda_n^{i_n},$$

for n=2, we have:

$$\begin{aligned} \sum_{k\geq 0} h_k^2 z^k &= h_0^2 + h_1^2 z + h_2^2 z^2 + ... \\ &= 1 + (\lambda_1 + \lambda_2)z + (\lambda_1^2 + \lambda_1\lambda_2 + \lambda_2^2)z^2 + ... \\ &= (1 + \lambda_1 z + \lambda_1^2 z^2 + ...)(1 + \lambda_2 z + \lambda_2^2 z^2 + ...) \\ &= \left(\sum_{k\geq 0}(\lambda_1 z)^k\right) + \left(\sum_{k\geq 0}(\lambda_2 z)^k\right) \\ &= \frac{1}{(1-\lambda_1 z)((1-\lambda_2 z)} \\ &= \frac{1}{\prod_{i=1}^{2}(1-\lambda_i z)}. \end{aligned}$$

So the assertion is true for n=2, Assume the proposition is true for n, i. e that

$$\sum_{k\geq 0}^{n} h_k^{(n)} z^k = \frac{1}{\prod_{i=1}^{n}(1-\lambda_i z)},$$

and we want proove that the proposition is true for n+1, i. e that

$$\sum_{k\geq 0} h_k^{(n+1)} z^k = \frac{1}{\prod_{i=1}^{n+1}(1-\lambda_i z)},$$

We have

$$h_k^{(n+1)} = \lambda_{n+1} h_{k-1}^{(n+1)} + h_k^{(n)}.$$

Thus

$$\begin{aligned}
\sum_{k\geq 0} h_k^{(n+1)} z^k &= \sum_{k\geq 0} (\lambda_{n+1} h_{k-1}^{(n+1)} + h_k^{(n)}) z^k \\
&= \lambda_{n+1} \sum_{k\geq 0} h_{k-1}^{(n+1)} z^k + \sum_{k\geq 0} h_k^{(n)} z^k \\
&= \lambda_{n+1} \sum_{k=1}^{\infty} h_{k-1}^{(n+1)} z^k + \sum_{k\geq 0} h_k^{(n)} z^k \\
&= \lambda_{n+1} z \sum_{k\geq 0} h_k^{(n+1)} z^k + \sum_{k\geq 0} h_k^{(n)} z^k
\end{aligned}$$

Which gives

$$\sum_{k\geq 0} h_k^{(n+1)} z^k - \lambda_{n+1} z \sum_{k\geq 0} h_k^{(n+1)} z^k = \sum_{k\geq 0} h_k^{(n)} z^k$$

Thus

$$\sum_{k\geq 0} h_k^{(n+1)} z^k (1-\lambda_{n+1} z) = \sum_{k\geq 0} \lambda_k^n z^k = \frac{1}{\prod_{i=1}^{n}(1-\lambda_i z)},$$

Then

$$\begin{aligned}
\sum_{k\geq 0} h_k^{(n+1)} z^k &= \frac{(1-\lambda_{n+1} z)^{-1}}{\prod_{i=1}^{n}(1-\lambda_i z)} \\
&= \frac{1}{\prod_{i=1}^{n+1}(1-\lambda_i z)}.
\end{aligned}$$

Thus the proposition is true for all $n \geq 0$ ■

**Proposition 2.1.3** *[4] For all $n \geq 0$, we have*

1. *H(z). E(-z)=1.*

**Proof.**

1. We have

$$E(z) = \sum_{k\geq 0} e_k z^k = \prod_{i\geq 1}(1+\lambda_i z)$$

$$E(-z) = \sum_{k\geq 0} e_k (-z)^k = \prod_{i\geq 1}(1-\lambda_i z)$$

$$H(z) = \sum_{k\geq 0} h_k (-z)^k = \prod_{i\geq 1}(1+\lambda_i z)^{-1}$$

Then,

$$H(z).E(-z) = \left(\prod_{i\geq 1}(1-\lambda_i z)^{-1}\right)\left(\prod_{i\geq 1}(1-\lambda_i z)\right) = 1$$

This completes the proof ■

## 2.2 Some properties on symmetric functions

**Definition 2.2.1** *[4] Let $n$ be positive integer and $A = \{a_1, a_2\}$ is set of given variables, then the symmetric function $S_n$ is defined by*

$$S_n(A) = S_n(a_1 + a_2) = \frac{a_1^{n+1} - a_2^{n+1}}{a_1 - a_2},$$

*with*

$$S_0(a_1 + a_2) = 1, S_1(a_1 + a_2) = a_1 + a_2, S_2(a_1 + a_2) = a_1^2 + a_1 a_2 + a_2^2.$$

**Definition 2.2.2** *[1] Let A and B be any two alphabets. We define $S_n(A - B)$ by the following form*

$$\sum_{j=0}^{\infty} S_j(A-B)z^j = E(-z)H(z). \tag{2.3}$$

*with*

$$H(z) = \prod_{b\in B}(1-bz)^{-1} \text{ and } E(-z) = \prod_{a\in A}(1-az)$$

**Proposition 2.2.1** *[12] By taking $A = \phi$ in (2.3), we obtain*

$$\sum_{j=0}^{\infty} S_j(-B)z^j = \prod_{b\in B}(1-bz), \tag{2.4}$$

**Proposition 2.2.2** *[12] By taking B=$\phi$ in (2.3), we obtain*

$$\sum_{j=0}^{\infty} S_j(A)z^j = \frac{1}{\prod_{a\in A}(1-az)} \tag{2.5}$$

**Lemma 2.2.1** *[4] Given two alphabet $A = \{x\}$ and $B = \{b_1, b_2, ..., b_n\}$, we have*

$$S_{n+k}(x-B) = x^k S_n(x-B),$$

*for all $k \geq 0$.*

**Proposition 2.2.3** *[4] If A is of cardinal 1 (i. e. A=x), so*

$$\frac{\prod_{b\in B}(1-bz)}{(1-xz)} = 1 + ... + z^{j-i}S_{j-1}(x-B) + z^j\frac{S_j(x-B)}{(1-xz)}. \tag{2.6}$$

**Proof.** According to (2.3) we have :

$$\frac{\prod_{b\in B}(1-bz)}{(1-xz)} = \sum_{j=0}^{\infty} S_j(x-B)z^j. \tag{2.7}$$

Thus

$$\begin{aligned}
\sum_{j=0}^{\infty} S_j(x-B)z^j &= 1 + ... + S_{j-1}(x-B)z^{j-1} + S_{j+1}(x-B)z^{j+1} + ... \\
&= 1 + ... + S_{j-1}(x-B)z^{j-1} + z^j(S_j(x-B) + S_{j+1}(x-B)z + ...) \\
&= 1 + ... + S_{j-1}(x-B)z^{j-1} + z^j(S_j(x-B) + xS_j(x-B)z + ...) \\
&= 1 + ... + S_{j-1}(x-B)z^{j-1} + z^jS_j(x-B)(1 + xz + x^2z^2 + ...) \\
&= 1 + ... + z^{j-1}S_{j-1}(x-B) + z^j\frac{S_j(x-B)}{(1-xz)}.
\end{aligned}$$

Then

$$\frac{\prod_{b\in B}(1-bz)}{(1-xz)} = 1 + ... + z^{j-1}S_{j-1}(x-B) + z^j\frac{S_j(x-B)}{(1-xz)}.$$

This completes the proof ■

**Proposition 2.2.4** *[2] Considering successively the case $A = \phi$, $B = \phi$, we get the following* *factorization*

$$\sum_{j=0}^{\infty} S_j(A-B)z^j = \sum_{j=0}^{\infty} S_j(A)z^j \sum_{j=0}^{\infty} S_j(-B)z^j.$$

*Thus*

$$S_j(A-B)z^j = \sum_{k=0}^{j} S_{j-k}(A)S_k(-B).$$

**Corollary 2.2.1** *[5] the symmetric function of bivariate Gaussian Fibonacci numbers is given by*

$$GF_n = iS_n(p_1 + [-p_2]) + (1-i)S_{n-1}(p_1 + [-p_2]).$$

**Corollary 2.2.2** *[5] the symmetric function of bivariate Gaussian Lucas numbers is given by*

$$GL_n = (2-i)S_n(p_1 + [-p_2]) + (-1+3i)S_{n-1}(p_1 + [-p_2]).$$

**Corollary 2.2.3** *[5] the symmetric function of bivariate Gaussian Jacobsthal numbers is given by*

$$GJ_n = \frac{i}{2}S_n(p_1 + [-p_2]) + (1-\frac{i}{2})S_{n-1}(p_1 + [-p_2]).$$

**Corollary 2.2.4** *[5] the symmetric function of bivariate Gaussian Jacobsthal-Lucas numbers is given by*

$$Gj_n = (2-\frac{i}{2})S_n(p_1 + [-p_2]) + (\frac{5}{2}i - 1)S_{n-1}(p_1 + [-p_2]).$$

**Corollary 2.2.5** *[5], the symmetric function of bivariate Gaussian Pell numbers is given by*

$$GP_n = iS_n(p_1 + [-p_2]) + (1-2i)S_{n-1}(p_1 + [-p_2]).$$

**Corollary 2.2.6** *[5], the symmetric function of bivariate Gaussian pell-Lucas numbers is given by*

$$GQ_n = (2-2i)S_n(p_1 + [-p_2]) + (6i-2)S_{n-1}(p_1 + [-p_2]).$$

**Corollary 2.2.7** *[5] the symmetric function of bivariate Gaussian Jacobsthal polynomial is given by*

$$GJ(x) = \frac{i}{2}S_n(p_1 + [-p_2]) + (1-\frac{i}{2})S_{n-1}(p_1 + [-p_2]).$$

**Corollary 2.2.8** *[5] the symmetric function of bivariate Gaussian Jacobsthal-Lucas polynomial is given by*

$$Gj(x) = (2 - \frac{i}{2})S_n(p_1 + [-p_2]) + (2ix + \frac{i}{2} - 1)S_{n-1}(p_1 + [-p_2]).$$

**Corollary 2.2.9** *[5] the symmetric function of bivariate Gaussian Pell polynomial is given by*

$$GP(x) = iS_n(p_1 + [-p_2]) + (1 - 2ix)S_{n-1}(p_1 + [-p_2]).$$

In the following table we give symmetric functions of some numbers and polynomial[5]

| Sequences | symmetric functions |
|---|---|
| $GF_n$ | $iS_n(p_1 + [-p_2]) + (1 - i)S_{n-1}(p_1 + [-p_2])$ |
| $GL_n$ | $(2 - i)S_n(p_1 + [-p_2])(-1 + 3i)S_{n-1}(p_1 + [-p_2])$ |
| $GJ_n$ | $\frac{i}{2}S_n(p_1 + [-p_2]) + (1 - \frac{i}{2})S_{n-1}(p_1 + [-p_2])$ |
| $Gj_n$ | $iS_n(p_1 + [-p_2]) + (1 - 2i)S_{n-1}(p_1 + [-p_2])$ |
| $GP_n$ | $= iS_n(p_1 + [-p_2]) + (1 - 2i)S_{n-1}(p_1 + [-p_2])$ |
| $GQ_n$ | $(2 - 2i)S_n(p_1 + [-p_2]) + (6i - 2)S_{n-1}(p_1 + [-p_2])$ |
| $GJ(x)$ | $\frac{i}{2}S_n(p_1 + [-p_2]) + (1 - \frac{i}{2})S_{n-1}(p_1 + [-p_2])$ |
| $GP(x)$ | $iS_n(p_1 + [-p_2]) + (1 - 2ix)S_{n-1}(p_1 + [-p_2])$ |
| $Gj(x)$ | $(2 - \frac{i}{2})S_n(p_1 + [-p_2]) + (2ix + \frac{i}{2} - 1)S_{n-1}(p_1 + [-p_2])$ |

**Table1**:Symmetric functions of some numbers and polynomial.

# CHAPTER 3

# ORDINARY GENERATING FUNCTIONS OF THE PRODUCTS OF GAUSSIAN NUMBERS WITH SYMMETRIC FUNCTIONS IN SEVERAL VARIABLES

In this chapter, we introduce some new generating functions for the products of Gaussian numbers such as Gaussian Fibonacci, Gaussian Lucas, Gaussian Jacobsthal, Gaussian Jacobsthal Lucas, Gaussian Pell, Gaussian Pell Lucas numbers, and Gaussian polynomials with symmetric functions in several variables.

## 3.1 Definitions and some properties

**Definition 3.1.1** *[4] Let f be a function on $\mathbb{R}^n$, the divided difference $\partial_{a_i,a_{i+1}}$ is defined by*

$$\partial_{a_i,a_{i+1}}(f) = \frac{f(a_1,...,a_i,a_{i+1},..,a_n) - f(a_1,...,a_{i-1},a_{i+1},a_i..,a_n)}{a_i - a_{i+1}} \tag{3.1}$$

**Definition 3.1.2** *[4] The symmetrizing operator $\delta^k_{a_1a_2}$ is defined by*

$$\delta^k_{a_1a_2} f(a_1) = \frac{a_1^k f(a_1) - a_2^k f(a_2)}{a_1 - a_2}, \forall k \in \mathbb{N}. \tag{3.2}$$

**Theorem 3.1.1** *[12] Let $P = \{p_1, p_2\}$ and $A = \{a_1, a_2, a_3, a_4\}$, be two alphabets, then we have*

$$\sum_{n=0}^{\infty} S_n (a_1 + a_2 + a_3 + a_4) S_n (p_1 + p_2) z^n = \tag{3.3}$$

$$\frac{1 - (p_1p_2) e_2^{(4)} z^2 + [(p_1p_2)(p_1 + p_2)] e_3^{(4)} z^3}{\prod_{i=1}^{4} (1 - a_ip_1z) \prod_{i=1}^{4} (1 - a_ip_2z)} - \frac{(p_1p_2) [(p_1 + p_2)^2 - p_1p_2] e_4^{(4)} z^4}{\prod_{i=1}^{4} (1 - a_ip_1z) \prod_{i=1}^{4} (1 - a_ip_2z)}.$$

**Theorem 3.1.2** *[12] Let $P = \{p_1, p_2\}$ and $A = \{a_1, a_2, a_3, a_4\}$, be two alphabets, then we have*

$$\sum_{n=0}^{\infty} S_n (a_1 + a_2 + a_3 + a_4) S_{n-1} (p_1 + p_2) z^n = \tag{3.4}$$

$$\frac{e_1^{(4)} z - (p_1 + p_2) e_2^{(4)} z^2 + [(p_1 + p_2)^2 - p_1p_2] e_3^{(4)} z^3}{\prod_{i=1}^{4} (1 - a_ip_1z) \prod_{i=1}^{4} (1 - a_ip_2z)} - \frac{(p_1 + p_2) [(p_1 + p_2)^2 - 2p_1p_2] e_4^{(4)} z^4}{\prod_{i=1}^{4} (1 - a_ip_1z) \prod_{i=1}^{4} (1 - a_ip_2z)}.$$

By changing $p_2$ by $(-p_2)$ in relationssips, we obtain

$$\sum_{n=0}^{\infty} S_n(a_1 + a_2 + a_3 + a_4) S_n(p_1 + [-p_2]) z^n = \frac{1 + (p_1p_2)e_2^{(4)} z^2 - [(p_1p_2)(p_1 - p_2)] e_3^{(4)} z^3}{\prod_{i=1}^{4} (1 - (p_1 - p_2)a_iz - p_1p_2a_i^2z^2)} + \frac{(p_1p_2)[(p_1 - p_2)^2 + p_1p_2] e_4^{(4)} z^4}{\prod_{i=1}^{4} (1 - (p_1 - p_2)a_iz - p_1p_2a_i^2z^2)} \tag{3.5}$$

$$\sum_{n=0}^{\infty} S_n(a_1+a_2+a_3+a_4)S_{n-1}(p_1+[-p_2])z^n = \frac{e_1^{(4)}z-(p_1-p_2)e_2^{(4)}z^2+[(p_1-p_2)^2+p_1p_2]e_3^{(4)}z^3}{\prod_{i=1}^{4}(1-(p_1-p_2)a_iz-p_1p_2a_i^2z^2)} - \frac{(p_1-p_2)[(p_1-p_2)^2+2p_1p_2]e_4^{(4)}z^4}{\prod_{i=1}^{4}(1-(p_1-p_2)a_iz-p_1p_2a_i^2z^2)} \tag{3.6}$$

## 3.2 Main results

### 3.2.1 On the generating functions of numbers

In this part, we now conider the previous theoremes in order to derive a new generating functions for the products of the symmetric functions in several variables with Gaussian Fibonacci, Gaussian Lucas, Gaussian Jacobsthal, Gaussian Jacobsthal Lucas, Gaussian Pell, Gaussian Pell Lucas numbers.

**Theorem 3.2.1** *Provided that n be a natural number, the novel generating function for the product of Gaussian Fibonacci numbers and symmetric function in multipe variables is*

$$\sum_{n=0}^{\infty} S_n(a_1+a_2+a_3+a_4)GF_nz^n = \frac{i+(1-i)e_1^{(4)}z-(1-2i)e_2^{(4)}z^2+(2-3i)e_3^{(4)}z^3-(3-5i)e_4^{(4)}z^4}{\prod_{i=1}^{4}(1-a_iz-a_i^2z^2)} \tag{3.7}$$

**Proof.** . We notice that

$$GF_n = iS_n(p_1+[-p_2])+(1-i)S_{n-1}(p_1+[-p_2])$$

Then,

$$\begin{aligned}
&\sum_{n=0}^{\infty} S_n(a_1+a_2+a_3+a_4)GF_nz^n \\
&= \sum_{n=0}^{\infty} S_n(a_1+a_2+a_3+a_4)\left(iS_n(p_1+[-p_2])+(1-i)S_{n-1}(p_1+[-p_2])\right)z^n \\
&= i\sum_{n=0}^{\infty} S_n(a_1+a_2+a_3+a_4)S_n(p_1+[-p_2])z^n
\end{aligned}$$

$$+ (1-i)\sum_{n=0}^{\infty} S_n(a_1+a_2+a_3+a_4)S_{n-1}(p_1+[-p_2])z^n$$
$$= i\frac{1+e_2^{(4)}z^2 - e_3^{(4)}z^3 + 2e_4^{(4)}z^4}{\prod_{i=1}^{4}(1-a_iz-a_i^2z^2)}$$
$$+ (1-i)\frac{e_1^{(4)}z - e_2^{(4)}z^2 + 2e_3^{(4)}z^3 - 3e_4^{(4)}z^4}{\prod_{i=1}^{4}(1-a_iz-a_i^2z^2)}$$
$$= \frac{i+(1-i)e_1^{(4)}z - (1-2i)e_2^{(4)}z^2 + (2-3i)e_3^{(4)}z^3 - (3-5i)e_4^{(4)}z^4}{\prod_{i=1}^{4}(1-a_iz-a_i^2z^2)}$$

This complet the proof ■

**Theorem 3.2.2** *Provided that n be a natural number, the novel generating function for the product of Gaussian Lucas numbers and symmetric function in multipe variables is*

$$\sum_{n=0}^{\infty} S_n(a_1+a_2+a_3+a_4)GL_nz^n = \frac{2-i+(-1+3i)e_1^{(4)}z + (3-4i)e_2^{(4)}z^2}{\prod_{i=1}^{4}(1-a_iz-a_i^2z^2)} + \frac{(-4+7i)e_3^{(4)}z^3 + (7-11i)e_4^{(4)}z^4}{\prod_{i=1}^{4}(1-a_iz-a_i^2z^2)} \tag{3.8}$$

**Proof.** . We notice that

$$GL_n = (2-i)S_n(p_1+[-p_2]) + (-1+3i)S_{n-1}(p_1+[-p_2])$$

Then,

$$\sum_{n=0}^{\infty} S_n(a_1+a_2+a_3+a_4)GL_nz^n$$
$$= \sum_{n=0}^{\infty} S_n(a_1+a_2+a_3+a_4)((2-i)S_n(p_1+[-p_2]) + (-1+3i)S_{n-1}(p_1+[-p_2]))z^n$$
$$= (2-i)\sum_{n=0}^{\infty} S_n(a_1+a_2+a_3+a_4)S_n(p_1+[-p_2])z^n$$
$$+ (-1+3i)\sum_{n=0}^{\infty} S_n(a_1+a_2+a_3+a_4)S_{n-1}(p_1+[-p_2])z^n$$

$$= (2-i)\frac{1+e_2^{(4)}z^2-e_3^{(4)}z^3+2e_4^{(4)}z^4}{\prod_{i=1}^{4}(1-a_iz-a_i^2z^2)}$$
$$+(-1+3i)\frac{e_1^{(4)}z-e_2^{(4)}z^2+2e_3^{(4)}z^3-3e_4^{(4)}z^4}{\prod_{i=1}^{4}(1-a_iz-a_i^2z^2)}$$
$$= \frac{2-i+(-1+3i)e_1^{(4)}z+(3-4i)e_2^{(4)}z^2}{\prod_{i=1}^{4}(1-a_iz-a_i^2z^2)}$$
$$+\frac{(-4+7i)e_3^{(4)}z^3+(7-11i)e_4^{(4)}z^4}{\prod_{i=1}^{4}(1-a_iz-a_i^2z^2)}$$

This complete the proof ■

**Theorem 3.2.3** *Provided that n be a natural number, the novel generating function for the product of Gaussian Jacobsthal numbers and symmetric function in multipe variables is*

$$\sum_{n=0}^{\infty} S_n(a_1+a_2+a_3+a_4)GJ_nz^n = \frac{\frac{i}{2}+(1-\frac{i}{2})e_1^{(4)}z+(-1+\frac{3}{2}i)e_2^{(4)}z^2}{\prod_{i=1}^{4}(1-a_iz-2a_i^2z^2)} + \frac{(3-\frac{5}{2}i)e_3^{(4)}z^3+(-5+\frac{11}{2}i)e_4^{(4)}z^4}{\prod_{i=1}^{4}(1-a_iz-2a_i^2z^2)} \tag{3.9}$$

**Proof.** . We notice that

$$GJ_n = \frac{i}{2}S_n(p_1+[-p_2])+(1-\frac{i}{2})S_{n-1}(p_1+[-p_2])$$

Then,

$$\sum_{n=0}^{\infty} S_n(a_1+a_2+a_3+a_4)GJ_nz^n$$
$$= \sum_{n=0}^{\infty} S_n(a_1+a_2+a_3+a_4)\left(\frac{i}{2}S_n(p_1+[-p_2])+(1-\frac{i}{2})S_{n-1}(p_1+[-p_2])\right)z^n$$

$$= \frac{i}{2}\sum_{n=0}^{\infty} S_n(a_1+a_2+a_3+a_4)S_n(p_1+[-p_2])z^n$$

$$+ (1-\frac{i}{2})\sum_{n=0}^{\infty} S_n(a_1+a_2+a_3+a_4)S_{n-1}(p_1+[-p_2])z^n$$

$$= \frac{i}{2}\frac{1+2e_2^{(4)}z^2-2e_3^{(4)}z^3+6e_4^{(4)}z^4}{\prod_{i=1}^{4}(1-a_iz-2a_i^2z^2)}$$

$$+ (1-\frac{i}{2})\frac{e_1^{(4)}z-e_2^{(4)}z^2+3e_3^{(4)}z^3-3e_4^{(4)}z^4}{\prod_{i=1}^{4}(1-a_iz-2a_i^2z^2)}$$

$$= \frac{\frac{i}{2}+(1-\frac{i}{2})e_1^{(4)}z+(-1+\frac{3i}{2})e_2^{(4)}z^2+(3-\frac{5}{2}i)e_3^{(4)}z^3+(-5+\frac{11}{2}i)e_4^{(4)}z^4}{\prod_{i=1}^{4}(1-a_iz-2a_i^2z^2)}$$

This complete the proof ■

**Theorem 3.2.4** *Provided that n be a natural number, the novel generating function for the product of Gaussian Jacobsthal Lucas numbers and symmetric function in multipe variables is*

$$\sum_{n=0}^{\infty} S_n(a_1+a_2+a_3+a_4)Gj_nz^n = \frac{2-\frac{i}{2}+(-1+\frac{5}{2}i)e_1^{(4)}z+(5-\frac{7}{2}i)e_2^{(4)}z^2}{\prod_{i=1}^{4}(1-a_iz-2a_i^2z^2)} + \frac{(-7+\frac{17}{2}i)e_3^{(4)}z^3+(17-\frac{31}{2}i)e_4^{(4)}z^4}{\prod_{i=1}^{4}(1-a_iz-2a_i^2z^2)} \tag{3.10}$$

**Proof.** . We notice that

$$Gj_n = (2-\frac{i}{2})S_n(p_1+[-p_2]) + (\frac{5}{2}i-1)S_{n-1}(p_1+[-p_2])$$

Then,

$$\sum_{n=0}^{\infty} S_n(a_1+a_2+a_3+a_4)Gj_nz^n$$

$$= \sum_{n=0}^{\infty} S_n(a_1+a_2+a_3+a_4)((2-\frac{i}{2})S_n(p_1+[-p_2]) + (\frac{5}{2}i-1)S_{n-1}(p_1+[-p_2]))z^n$$

$$= (2 - \frac{i}{2}) \sum_{n=0}^{\infty} S_n(a_1 + a_2 + a_3 + a_4) S_n(p_1 + [-p_2]) z^n$$
$$+ (\frac{5}{2} i - 1) \sum_{n=0}^{\infty} S_n(a_1 + a_2 + a_3 + a_4) S_{n-1}(p_1 + [-p_2]) z^n$$
$$= (2 - \frac{i}{2}) \frac{1 + 2e_2^{(4)} z^2 - 2e_3^{(4)} z^3 + 6e_4^{(4)} z^4}{\prod_{i=1}^{4} (1 - a_i z - 2a_i^2 z^2)}$$
$$+ (\frac{5}{2} i - 1) \frac{e_1^{(4)} z - e_2^{(4)} z^2 + 3e_3^{(4)} z^3 - 5e_4^{(4)} z^4}{\prod_{i=1}^{4} (1 - a_i z - 2a_i^2 z^2)}$$
$$= \frac{2 - \frac{i}{2} + (-1 + \frac{5}{2} i) e_1^{(4)} z + (5 - \frac{7}{2} i) e_2^{(4)} z^2}{\prod_{i=1}^{4} (1 - a_i z - 2a_i^2 z^2)}$$
$$+ \frac{(-7 + \frac{17}{2} i) e_3^{(4)} z^3 + (17 - \frac{31}{2} i) e_4^{(4)} z^4}{\prod_{i=1}^{4} (1 - a_i z - 2a_i^2 z^2)}$$

This complete the proof ■

**Theorem 3.2.5** *Provided that n be a natural number, the novel generating function for the product of Gaussian Pell numbers and symmetric function in multipe variables is*

$$\sum_{n=0}^{\infty} S_n(a_1 + a_2 + a_3 + a_4) GP_n z^n = \frac{i + (1 - 2i) e_1^{(4)} z + (-2 + 5i) e_2^{(4)} z^2}{\prod_{i=1}^{4} (1 - 2a_i z - a_i^2 z^2)} + \frac{(5 - 12i) e_3^{(4)} z^3 + (-12 + 29i) e_4^{(4)} z^4}{\prod_{i=1}^{4} (1 - 2a_i z - a_i^2 z^2)} \tag{3.11}$$

**Proof.** . We notice that

$$GP_n = i S_n(p_1 + [-p_2]) + (1 - 2i) S_{n-1}(p_1 + [-p_2])$$

Then,

$$\sum_{n=0}^{\infty} S_n(a_1 + a_2 + a_3 + a_4)GP_n z^n$$
$$= \sum_{n=0}^{\infty} S_n(a_1 + a_2 + a_3 + a_4)\left(iS_n(p_1 + [-p_2]) + (1-2i)S_{n-1}(p_1 + [-p_2])\right) z^n$$
$$= i\sum_{n=0}^{\infty} S_n(a_1 + a_2 + a_3 + a_4)S_n(p_1 + [-p_2])z^n$$
$$+ (1-2i)\sum_{n=0}^{\infty} S_n(a_1 + a_2 + a_3 + a_4)S_{n-1}(p_1 + [-p_2])z^n$$
$$= i\frac{1 + e_2^{(4)}z^2 - 2e_3^{(4)}z^3 + 5e_4^{(4)}z^4}{\prod_{i=1}^{4}(1 - 2a_i z - a_i^2 z^2)}$$
$$+ (1-2i)\frac{e_1^{(4)}z - 2e_2^{(4)}z^2 + 5e_3^{(4)}z^3 - 12e_4^{(4)}z^4}{\prod_{i=1}^{4}(1 - 2a_i z - a_i^2 z^2)}$$
$$= \frac{i + (1-2i)e_1^{(4)}z + (-2+5i)e_2^{(4)}z^2 + (5-12i)e_3^{(4)}z^3 + (-12+29i)e_4^{(4)}z^4}{\prod_{i=1}^{4}(1 - 2a_i z - a_i^2 z^2)}$$

This complete the proof ■

**Theorem 3.2.6** *Provided that n be a natural number, the novel generating function for the product of Gaussian Pell Lucas numbers and symmetric function in multipe variables is*

$$\sum_{n=0}^{\infty} S_n(a_1 + a_2 + a_3 + a_4)GQ_n z^n = \frac{2 - 2i + (-2+6i)e_1^{(4)}z + (6-14i)e_2^{(4)}z^2}{\prod_{i=1}^{4}(1 - 2a_i z - a_i^2 z^2)} + \frac{(-14+34i)e_3^{(4)}z^3 + (34-82i)e_4^{(4)}z^4}{\prod_{i=1}^{4}(1 - 2a_i z - a_i^2 z^2)} \tag{3.12}$$

**Proof.** . We notice that

$$GQ_n = (2-2i)S_n(p_1 + [-p_2]) + (6i-2)S_{n-1}(p_1 + [-p_2])$$

Then,

$$\sum_{n=0}^{\infty} S_n(a_1+a_2+a_3+a_4)GQ_nz^n$$

$$= \sum_{n=0}^{\infty} S_n(a_1+a_2+a_3+a_4)\left((2-2i)S_n(p_1+[-p_2]) + (6i-2)S_{n-1}\right.$$

$$\left.(p_1+[-p_2])\right)z^n$$

$$= (2-2i)\sum_{n=0}^{\infty} S_n(a_1+a_2+a_3+a_4)S_n(p_1+[-p_2])z^n$$

$$+ (6i-2)\sum_{n=0}^{\infty} S_n(a_1+a_2+a_3+a_4)S_{n-1}(p_1+[-p_2])z^n$$

$$= (2-2i)\frac{1+e_2^{(4)}z^2-2e_3^{(4)}z^3+5e_4^{(4)}z^4}{\prod_{i=1}^{4}(1-2a_iz-a_i^2z^2)}$$

$$+ (6i-2)\frac{e_1^{(4)}z-2e_2^{(4)}z^2+5e_3^{(4)}z^3-12e_4^{(4)}z^4}{\prod_{i=1}^{4}(1-2a_iz-a_i^2z^2)}$$

$$= \frac{2-2i+(-2+6i)e_1^{(4)}z+(6-14i)e_2^{(4)}z^2}{\prod_{i=1}^{4}(1-2a_iz-a_i^2z^2)}$$

$$+ \frac{(-14+34i)e_3^{(4)}z^3+(34-82i)e_4^{(4)}z^4}{\prod_{i=1}^{4}(1-2a_iz-a_i^2z^2)}$$

This complete the proof ■

**Theorem 3.2.7** *Provided that n be a natural number, the novel generating function for the product of Gaussian Jacobsthal polynomials and symmetric function in multipe variables is*

$$\sum_{n=0}^{\infty} S_n(a_1+a_2+a_3+a_4)GJ_n(x)z^n = \frac{\frac{i}{2}+(1-\frac{i}{2})e_1^{(4)}z+(-1+i(x+\frac{1}{2}))e_2^{(4)}z^2}{\prod_{i=1}^{4}(1-a_iz-2xa_i^2z^2)}$$
$$+ \frac{(1+2x-i(2x+\frac{1}{2}))e_3^{(4)}z^3+(-1-4x+i(2x^2+3x+\frac{1}{2})e_4^{(4)}z^4}{\prod_{i=1}^{4}(1-a_iz-2xa_i^2z^2)} \tag{3.13}$$

**Proof.** . We notice that

$$GJ_n(x) = \frac{i}{2}S_n(p_1 + [-p_2]) + (1 - \frac{i}{2})S_{n-1}(p_1 + [-p_2])$$

Then,

$$\sum_{n=0}^{\infty} S_n(a_1 + a_2 + a_3 + a_4)GJ_n(x)z^n$$
$$= \sum_{n=0}^{\infty} S_n(a_1 + a_2 + a_3 + a_4)\left(\frac{i}{2}S_n(p_1 + [-p_2]) + (1 - \frac{i}{2})S_{n-1}(p_1 + [-p_2])\right)z^n$$
$$= \frac{i}{2}\sum_{n=0}^{\infty} S_n(a_1 + a_2 + a_3 + a_4)S_n(p_1 + [-p_2])z^n$$
$$+ (1 - \frac{i}{2})\sum_{n=0}^{\infty} S_n(a_1 + a_2 + a_3 + a_4)S_{n-1}(p_1 + [-p_2])z^n$$
$$= \frac{i}{2}\frac{1 + 2xe_2^{(4)}z^2 - 2xe_3^{(4)}z^3 + (2x + 4x^2)e_4^{(4)}z^4}{\prod_{i=1}^{4}(1 - a_iz - 2xa_i^2z^2)}$$
$$+ (1 - \frac{i}{2})\frac{e_1^{(4)}z - e_2^{(4)}z^2 + (1 + 2x)e_3^{(4)}z^3 - (1 + 4x)e_4^{(4)}z^4}{\prod_{i=1}^{4}(1 - a_iz - 2xa_i^2z^2)}$$
$$= \frac{\frac{i}{2} + (1 - \frac{i}{2})e_1^{(4)}z + (-1 + i(x + \frac{1}{2}))e_2^{(4)}z^2}{\prod_{i=1}^{4}(1 - a_iz - 2xa_i^2z^2)}$$
$$+ \frac{(1 + 2x - i(2x + \frac{1}{2}))e_3^{(4)}z^3 + (-1 - 4x + i(2x^2 + 3x + \frac{1}{2})e_4^{(4)}z^4}{\prod_{i=1}^{4}(1 - a_iz - 2xa_i^2z^2)}$$

This complete the proof ■

**Theorem 3.2.8** *Provided that n be a natural number, the novel generating function for the product of Gaussian Jacobsthal Lucas polynomials and symmetric function in multipe variables is*

$$\sum_{n=0}^{\infty} S_n(a_1 + a_2 + a_3 + a_4)Gj_n(x)z^n = \frac{(2 - \frac{i}{2}) + (-1 + i(2x + \frac{1}{2}))e_1^{(4)}z + (4x + 1 - i(3x + \frac{1}{2}))e_2^{(4)}z^2}{\prod_{i=1}^{4}(1 - a_iz - 2xa_i^2z^2)}$$

$$+\frac{(-6x-1+i(4x^2+4x+\frac{1}{2}))e_3^{(4)}z^3+(8x^2+8x+1-i(10x^2+5x+\frac{1}{2})e_4^{(4)}z^4}{\prod_{i=1}^{4}(1-a_iz-2xa_i^2z^2)}$$

**Proof.** . We notice that

$$Gj_n(x)=(2-\frac{i}{2})S_n(p_1+[-p_2])+(2ix+\frac{i}{2}-1)S_{n-1}(p_1+[-p_2])$$

Then,

$$\sum_{n=0}^{\infty}S_n(a_1+a_2+a_3+a_4)Gj_n(x)z^n$$

$$=\sum_{n=0}^{\infty}S_n(a_1+a_2+a_3+a_4)((2-\frac{i}{2})S_n(p_1+[-p_2])+(2ix+\frac{i}{2}-1)S_{n-1}(p_1+[-p_2]))z^n$$

$$=(2-\frac{i}{2})\sum_{n=0}^{\infty}S_n(a_1+a_2+a_3+a_4)S_n(p_1+[-p_2])z^n$$

$$+(2ix+\frac{i}{2}-1)\sum_{n=0}^{\infty}S_n(a_1+a_2+a_3+a_4)S_{n-1}(p_1+[-p_2])z^n$$

$$=(2-\frac{i}{2})\frac{1+2xe_2^{(4)}z^2-2xe_3^{(4)}z^3+(2x+4x^2)}{\prod_{i=1}^{4}(1-a_iz-2xa_i^2z^2)}$$

$$+(2ix+\frac{i}{2}-1)\frac{e_1^{(4)}z-e_2^{(4)}z^21+2xe_3^{(4)}z^3+(1-4x)e_4^{(4)}z^4}{\prod_{i=1}^{4}(1-a_iz-2xa_i^2z^2)}$$

$$=\frac{(2-\frac{i}{2})+(-1+i(2x+\frac{1}{2}))e_1^{(4)}z+(4x+1-i(3x+\frac{1}{2}))e_2^{(4)}z^2}{\prod_{i=1}^{4}(1-a_iz-2xa_i^2z^2)}$$

$$+\frac{(-6x-1+i(4x^2+4x+\frac{1}{2}))e_3^{(4)}z^3+(8x^2+8x+1-i(10x^2+5x+\frac{1}{2})e_4^{(4)}z^4}{\prod_{i=1}^{4}(1-a_iz-2xa_i^2z^2)}$$

This complete the proof ■

**Theorem 3.2.9** *Provided that n be a natural number, the novel generating function for the product of Gaussian Pell polynomials and symmetric function in multipe variables is*

$$\sum_{n=0}^{\infty} S_n(a_1+a_2+a_3+a_4)GP_n(x)z^n = \frac{i+(1-2ix)e_1^{(4)}z+(-2x+i(4x^2+1))e_2^{(4)}z^2}{\prod_{i=1}^{4}(1-2xa_iz-a_i^2z^2)} + \frac{(4x^2+1-4i(2x^3+x))e_3^{(4)}z^3}{\prod_{i=1}^{4}(1-2xa_iz-a_i^2z^2)} - \frac{2x(4x^2+2)+i(16x^4+12x^2+1)e_4^{(4)}z^4}{\prod_{i=1}^{4}(1-2xa_iz-a_i^2z^2)} \quad (3.14)$$

**Proof.** . We notice that

$$GP_n(x) = iS_n(p_1+[-p_2]) + (1-2ix)S_{n-1}(p_1+[-p_2])$$

Then,

$$\begin{aligned}
&\sum_{n=0}^{\infty} S_n(a_1+a_2+a_3+a_4)GP_n(x)z^n \\
&= \sum_{n=0}^{\infty} S_n(a_1+a_2+a_3+a_4)(iS_n(p_1+[-p_2])+(1-2ix)S_{n-1}(p_1+[-p_2]))z^n \\
&= i\sum_{n=0}^{\infty} S_n(a_1+a_2+a_3+a_4)S_n(p_1+[-p_2])z^n \\
&+ (1-2ix)\sum_{n=0}^{\infty} S_n(a_1+a_2+a_3+a_4)S_{n-1}(p_1+[-p_2])z^n \\
&= i\frac{1+e_2^{(4)}z^2-2xe_3^{(4)}z^3+(4x^2+1)e_4^{(4)}z^4}{\prod_{i=1}^{4}(1-2xa_iz-a_i^2z^2)} \\
&+ (1-2ix)\frac{e_1^{(4)}z-2xe_2^{(4)}z^2+(4x^2+1)e_3^{(4)}z^3-(2x(4x^2+2))e_4^{(4)}z^4}{\prod_{i=1}^{4}(1-2xa_iz-a_i^2z^2)} \\
&= \frac{i+(1-2ix)e_1^{(4)}z+(-2x+i(4x^2+1))e_2^{(4)}z^2}{\prod_{i=1}^{4}(1-2xa_iz-a_i^2z^2)}
\end{aligned}$$

$$+\frac{(4x^2+1-4i(2x^3+x))e_3^{(4)}z^3-2x(4x^2+2)+i(16x^4+12x^2+1)e_4^{(4)}z^4}{\prod_{i=1}^{4}(1-2xa_iz-a_i^2z^2)}$$

This complete the proof ■

# CONCLUTION

In this work, the use of symmetric functions we have derived some new generating functions for the products of Gaussian numbers such as Gaussian Fibonacci, Gaussian Lucas, Gaussian Jacobsthal, Gaussian Jacobsthal Lucas, Gaussian Pell, Gaussian Pell Lucas numbers.  and Gaussian polynomials with symmetric functions in several variables.

# BIBLIOGRAPHY


[1] A. Abderrezzak, *Généralisation de la transfortmation d'Euler d'une série formelle*, Adv. Math. , 103, 180-195, 1994.

[2] A. Abderrezzak, *Généralisation d'identités de Carlitz, Howard et Lehmer*, A equationes Math. , 49, 36-46, 1995.

[3] M. Asci and E. Gurel, *Gaussian Jacobsthal and Gaussian Jacobsthal Lucas Polynomials, Notes Number Theory Discrete Math.* 19 (2013) 25-36.

[4] Kh. Boubellouta, *Fonctions symétriques et leurs applications à certains noumbres et polynômes*, (Doctoral dissertation) . Mohamed Seddik Ben Yahia University, Jijel, Algeria. 2020

[5] S. Boughaba, A. Boussayoud, S. Araci and M. Kerada, *Construction of Generating Functions of Gaussian Fibonacci Numbers and Polynomials* , Univ. J. Math. Appl. 2021.

[6] S. Fiorini, *MATH-F-307 Mathématiques discrétes*. Version 2012.

[7] D. Foata et GN. Han, *Principe de combinatoire classique, Université Louis Pasteur*, Strasbourg Département de mathématique, 2008.

[8] S. Halici and S. Oz, *On some Gaussian Pell and Pell-Lucas numbers*, Ordu Univ. J. Sci. Tech. 6 (2016) 8-18.

[9] S. Halici and S. Oz, *On Gaussian Pell polynomials and their some properties*, Palest. J. Math. 7 (2018) 251-256.

[10] K. H. Rosen, *Discrete Mathematics and its Applications*, Monmouth University(and fromerly AT et T Laboratories). 2011.

[11] N. Saba and A. Boussayoud1, *Complete homogeneous symmetric functions of Gauss Fibonacci polynomials and bivariate Pell polynomials* , Open Journal of Mathimatical Sciences, 2020.

[12] N. Saba and A. Boussayoud1, *Ordinary generating functions of binary products of (p, q)-Modified-Pell numbers and k-numbers at positive and negative indices*, 627-648, 2020.

[13] A. M. Sharari, *Introduction to Combinatorial Theory, Saudi University Publications*, Kingdom of Saudi Arabia.

[14] A. F Horadam, *Generating functions for pwers of a certain generalized sequence of numbers*, Duke Math. J. 32, 1965.

# *ABSTRACT*

In this dissertation, we present a theorems in order to calculate new generating functions for second-order recurrences relations. The presented theorem is based on symmetric functions, we gived the generating functions of the products of Gaussian numbers Gaussian Fibonacci, Gaussian Lucas, Gaussian Jacobsthal, Gaussian Jacobstal Lucas, Gaussian Pell, Gussian Pell Lucas and Gaussian Polynomials with symmetric functions in several variables.

**Key Words**: Recurrence relations, symmetric functions, generating functions

# *RÉSUMÉ*

Dans ce mémoire, nous présentons un théorème afin de calculer des nouvelles fonctions génératrices pour les relations de récurrences du second ordre, le théorème présenté est basé sur les fonctions symétriques, nous permet d'obtenir les fonctions génératrices des produits de nombres de Gauss, les nombers Gauss Fibonacci, les nombers Gauss Lucas, les nombres Gauss Jacobsthal, les nombres Gauss Jacobstal Lucas, les nombres Gauss Pell, les nombres Gauss Pell Lucas et Polynôme Gauss avec les functions symétriques plusieurs variables.

**Mots-clés**: Relations de récurrences, fonctions génératrices, functions symétriques.

# ملخص

في هذه المذكرة قمنا بعرض نظريتين تعتمد على التوابع التناظرية وذلك لحساب الدوال المولدة للعلاقات التراجعية الخطية حيث أننا قمنا بحساب الدوال المولدة لجداءات أعداد غاوس جاكوبستال،غاوس جاكوبستال لوكاس،غاوس فيبوناتشي،غاوس لوكاس،غاوس بال،غاوس بال لوكاس و كثيرات الحدود،غاوس جاكوبستال،غاوس جاكوبستال لوكاس،غاوس بال بمتغيرين مع الدوال التناظرية بعدة متغيرات



الكلمات المفتاحية : العلاقات التراجعية،التوابع التناظرية،الدوال المولدة