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Bifurcation analysis and chaos control of a discret duopoly model

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Dedication

To my dear mother **Habiba**

To my father **Habib**, who helped me become who I am today, who sacrificed so much for my studies, may Allah keep and protect him. His merits, sacrifices, and human qualities have allowed me to thrive every day. To my brothers **Moncef** and **Okba**, and my dear sister.

Finally, I dedicate this work to everyone who has helped me (my teachers, friends, familly ...)

HAMMA SOHEYB

I dedicate this work to the source of my efforts, my life, and my dear mother. May Allah reward her for all her kindness. To the one who has always sacrificed to see me succeed, my father. May Allah grant him good health and a long life. To my brother, and my sister. To my entire family, my teachers from primary school, middle school, and high school, and my friends, thank you.

ACHOURI AMMAR

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First of all, I would like to thank ALLAH who gave us the strength and the patience to accomplish this modest work.

we are immensely grateful to Prof. ABD ELOUAHAB MOHAMED SALAH, our supervisor, for granting us the incredible opportunity to engage in research and for oering invaluable guidance throughout the entire process. His dynamism, vision, sincerity, and motivation have been a profound source of inspiration for us. Prof. ABD ELOUAHAB MOHAMED SALAH has imparted upon us the essential methodologies to conduct thorough research and present our results with utmost clarity. Working and studying under his expert guidance has been a tremendous privilege and honor. we are sincerely appreciative for everything he has provided us.

Our sincere thanks to the members of jury for the honor they have bestowed upon us by accepting to review our work and enrich it with their suggestions. We also extend our thanks to all our teachers who have laid the foundations of knowledge for us. We thank also everyone who has contributed, near and far, to the completion of this research. Finally, we express our warm thanks to our parents for their support and encouragement throughout our academic journey, and we also extend our thanks to our brothers and sisters.

Abstract

In this work, we have outlined some tools to study dynamical systems, such as equilibrium points, stability, bifurcations, and chaos, as well as some control methods, the chaotic attractor, and Lyapunov exponents. As an application, we studied a duopoly model consisting of two different equations, where each equation represents a player, and each player seeks to maximize their profits and market share using different strategies. The results obtained in this work contribute to a better understanding of complex financial markets and to the improvement of the development of more accurate and efficient economic models. It is useful for financial analysts and investors to improve their strategies and reduce risks. The results indicate several important aspects: the stability of equilibrium points, bifurcations, and chaos, where the results indicate the existence of bifurcations at equilibrium solutions (Flip, Neimark Sacker), as well as the presence of a chaotic attractor that was controlled by the OGY method. In addition, numerical simulation was used to confirm the results obtained theoretically.

Key words: discrete dynamic system, bifurcation, stability, chaos, attractors, Lyapunov exponent, control.

Résumé

Dans ce mémoire, nous avons énoncé certains outils nécessaires pour étudier les systèmes dynamiques, telles que les points d'équilibre, la stabilité, les bifurcations, le chaos, ainsi que certaines méthodes de contrôle, l'attracteur chaotique et les exposants de Lyapunov. et comme application, nous avons étudié un modèle de duopole constitué de deux équations aux différences, chaque équation représente un joueur et chaque joueur cherche à maximiser ses profits et sa part de marché en utilisant des différentes stratégies. Les résultats obtenus dans ce travail contribuent à une meilleure compréhension des marchés financiers complexes et à l'amélioration du développement de modèles économiques plus précis et efficaces. Il est utile aux analystes financiers et aux investisseurs pour améliorer leurs stratégies et réduire les risques. Les résultats indiquent plusieurs aspects importants: la stabilité des points d'équilibre, les bifurcations, le chaos, où les résultats indiquent l'existence de bifurcations aux solutions d'équilibre (Flip, Neimark Sacker), ainsi que la présence d'une attracteur chaotique qui a été contrôlé par la méthode OGY. De plus, la simulation numérique a été utilisée pour confirmer les résultats obtenus théoriquement.

Les Mots Clés: système dynamique discret, bifurcation, stabilité, chaos, attracteur, exposant de Lyapunov, controle.

ملخص

في هذه المذكرة، تم تناول بعض المفاهيم الأساسية لدراسة نظام ديناميكي متقطع، بما في ذلك نقاط التوازن، الاستقرار، التشعبات، الفوضى وبعض طرق التحكم فيها، الجاذب الفوضوي، وأسس ليابونوف. كجزء من التطبيق، تمت دراسة نظام الاحتكار الثنائي حيث يتألف من معادلتين تفاضليتين (معادلتي فروق)، و تمثل كل معادلة لاعبًا يسعى إلى تعظيم الربح وتعظيم حصته السوقية باستخدام استراتيجيات مختلفة. حيث تساهم النتائج الموجودة في هذا العمل في فهم أفضل للأسواق المالية المعقدة، وتساعد في تطوير نماذج اقتصادية أكثر دقة وفعالية، وهي مفيدة للمحللين الماليين والمستثمرين لتحسين استراتيجياتهم وتقليل المخاطر. تركز النتائج على عدة جوانب مهمة: استقرار نقاط التوازن، التشعبات والفوضى، حيث تشير النتائج إلى وجود تشعبات عند حلول التوازن (فليب، نيمارك ساكر)، كما تبين وجود جاذب فوضوي وتم التحكم فيه باستخدام طريقة أوجي، كما تم استعمال المحاكاة العددية لتأكيد النتائج المحصل عليها نظريًا.

الكلمات المفتاحية: النظام الديناميكي المتقطع، التشعب، الاستقرار، الفوضى، الجاذب، أسس ليابونوف، التحكم.

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INTRODUCTION

Dynamic systems are mathematical concepts that allow the study of phenomena that develop over time [30]. This evolution can be expressed by a limited set of equations, which can be ordinary differential equations, partial differential equations, or mappings (recurence iterations). These phenomena originate from physics, chemistry, mechanics, biology, economy, or the environment [19]. Dynamical systems consist of a phase space (the space of possible states of the phenomenon suitably parameterized), equipped with an evolution law that describes the temporal variation of the system's state [19].

The theory of dynamical systems, which can be traced back to around 1665 with Isaac Newton [1], provides a mathematical framework for understanding how systems evolve over time. Newton initially described these systems using ordinary differential equations, giving rise to what we now refer to as continuous dynamical systems [1]. In these systems, also known as flows, the state of the system evolves continuously over time, with each point in the system following a smooth path or curve [4].

In the 1880s, Poincaré found it convenient to replace certain dynamical systems with discrete dynamical systems described by difference equations (or recurrences, or point transformations), that is, systems in which time evolves in regular sequence breaks [38]. Thus, for more than a hundred years, dynamical systems have been defined in two classes: continuous and discrete. Dynamic system models generally depend on one or more parameters, and variations in the parameters can cause qualitative and quantitative changes in their properties. This change produces a phenomenon that we will call bifurcation. The types of bifurcation (local or global) are determined by their effect on the system or by the way they occur, which is generally related to their causes [24].

Around 1900, mathematician Henri Poincaré discovered what is called chaos in this system, but the birth of the study of chaos dates back to 1963, thanks to meteorologist Edward Lorenz [22]. The term chaos refers to a special state of a system whose behavior never repeats, is highly sensitive to initial conditions, and is unpredictable in the long term [35]. It has been called a chaotic dynamical system [22]. From his time to the present day, chaos remains an important field of research, especially in terms of control and anti-control. This work is composed of an introduction and four chapters organized as follows:

In the first chapter, we present some important notions about dynamical systems and the concepts of stability of fixed points and periodic points.

The second chapter addresses bifurcation theory and its types, provides also various mathematical properties that help us characterize chaotic behavior, and gives some examples.

In the third chapter, we indicate the existence of methods for controlling chaotic dynamical systems, discuss some of these methods, and then provide an example.

Finally, we conclude our work with a chapter dedicated to the application of the previously discussed theoretical tools (equilibrium points, stability, bifurcations, Lyapunov exponents, and strange attractors) to analyze a discrete duopoly model.

The theoretical results obtained are confirmed by numerical simulations.

CHAPTER 1

GENERALITIES ON DISCRETE DYNAMICAL SYSTEMS

In general, a dynamic system describes phenomena that evolve in space and/or time. They were developed and specialized during the 19th century. The term "system" refers to a set of state variables (whose value evolves over time) and the interactions between these variables. Such a dynamic system has two aspects: its state and its dynamics that is, its evolution over time [4]. Dynamic systems are classified into two categories:

- **Discrete-time dynamic systems**
- **For a continuous-time dynamic system**

1.1 Discrete and continuous dynamical systems

Definition 1.1.1. ***(Discrete autonomous dynamical system)*** *[15] A discrete autonomous dynamical system of dimension n is described by the following difference equation (recurrence, iteration, or point transformation):*

$$x_{t+1} = f(x_t), \quad x_0 = x(0) \tag{1.1}$$

where $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a differentiable function, and $x_0 \in X \subset \mathbb{R}^n$ is an initial value, with $x_n \in X$ being the vector of system state. Furthermore:

- *$x_1 = f(x_0)$ is called the first iteration of x_0 by the function f.*
- *$x_2 = f(x_1) = f(f(x_0)) = f^2(x_0)$ is called the second iteration of x_0 by the function f.*
- *$x_n = f^n(x_0)$, where $f^n = f \circ f \circ \cdots \circ f$ (n times), is called the n-th iteration of x_0 by the function f.*

The triplet (X, N, φ) defines a discrete autonomous dynamical system, where φ is given by:

$$\varphi(x_0, t) = f^t(x_0) \tag{1.2}$$

Definition 1.1.2. ***(Continuous autonomous dynamical system)*** *[10] In the continuous case, a dynamical system is presented by a system of differential equations of the form:*

$$\frac{dx}{dt} = \dot{x}(t) = f(x, \mu) \tag{1.3}$$

where $x \in \mathbb{R}^n$ and $\mu \in \mathbb{R}^m$, and $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ denotes the dynamics of the system.

Example 1.1.1. *The Lorenz system is defined by the following equations:*

$$\begin{cases} \frac{dx}{dt} = \sigma(y - x), \\ \frac{dy}{dt} = x(\rho - z) - y, \\ \frac{dz}{dt} = xy - \beta z, \end{cases}$$

where x, y, z are the state variables of the system, σ, ρ, β are real parameters. The phase space is $\mathbb{R}^3$ and the parameters space is $\mathbb{R}^3$.

1.2 Transition from continuous-time to discrete-time systems

There are several techniques for discretizing (sampling) systems. Here is a simple example, often used: the Euler method. Consider a first-order differential equation:

$$\dot{x} = f(x_t) \tag{1.4}$$

We want to study the trajectory of this equation only at selected instances, equidistant $t_n = t_0 + n \times \Delta t$. If the sampling period Δt is chosen small enough, we can approximate the derivative of $x(t)$ by the difference:

$$\dot{x} \approx \frac{x(t_{n+1}) - x(t_n)}{\Delta t} \tag{1.5}$$

Then, the continuous-time dynamical system (1.4) can be approximated by the following discrete-time dynamical system:

$$x_{t+1} = x_t + \Delta t f(x_t) \tag{1.6}$$

1.3 Fixed points and orbit

In the subsequent developments, our focus will primarily be on first-order systems. The aim is to describe the evolution of system states based on initial conditions. To achieve this, it is crucial to introduce the notion of trajectory (orbit) and fuxed points of the system. Let's consider a discrete dynamical system **(DDS)** of first order defined by the iteration of a function $f(x)$:

$$x_{t+1} = f(x_t), \quad x(0) = x_0, \quad t \in \mathbb{N}. \tag{1.7}$$

Definition 1.3.1. ***(Fixed points)****[4] A fixed point (or equilibrium point) of a dynamical system is a value of the state variable x_t that remains unchanged over time according to the system's dynamics. Mathematically: it is denoted by $\bar{x} \in \mathbb{R}^n$ such that $\bar{x} = f(\bar{x})$, where $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a differentiable function. Geometrically(In one dimension): fixed points correspond to the intersections of the function $f(x)$ with the diagonal line $y = x$.*

In summary:

➤ *A fixed point $\bar{x}$ satisfies $f(\bar{x}) = \bar{x}$.*

➤ *It represents a state where the system remains unchanged under the dynamics defined by $f(x)$.*

Example 1.3.1. *the fixed points of the cubic map $f(x) = x^3$ can be obtained by solving the equation $x^3 = x$ or $x^3 - x = 0$. Hence, there are three fixed points -1 , 0 , 1 for this map.1.1*

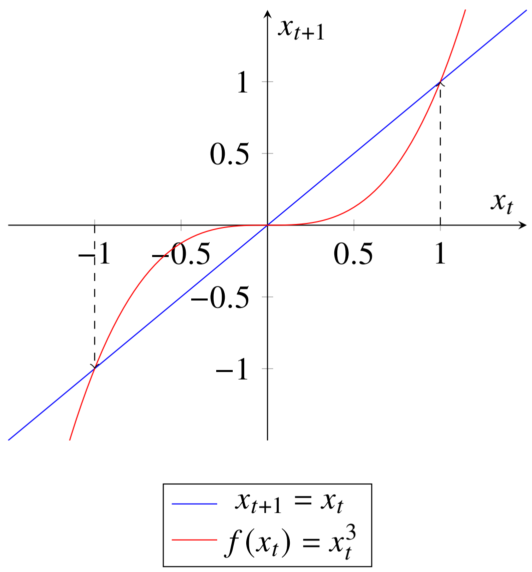


Figure 1.1: The fixed points of the cubic map

Definition 1.3.2. ***(System orbit)***

The orbit (or trajectory) *of the system* (1.7)*, given the initial point* x_0*, is defined as the sequence:*

$$\mathcal{O} = \{x(0), x(1) = f(x(0)), x(2) = f^2(x(0)), x(3) = f^3(x(0)), \ldots, x(t) = f^t(x(0)), \ldots\}.$$

Graphic representation of the orbit

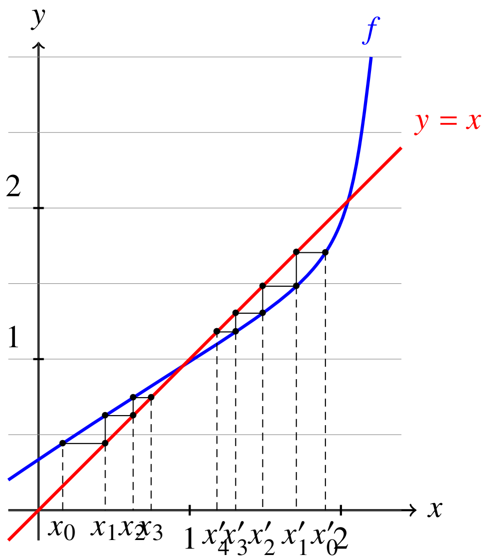


Figure 1.2: The orbit of the system $x_{t+1} = \frac{1}{4}(x_t^2 - 1)(x_t - 2) + x_t$

Definition 1.3.3. *An orbit* $\mathcal{O}(x_0)$ *is said to be periodic if there exists an integer* $p > 1$ *such that:*

$$x_{n+p} = x_n \quad (\forall n). \tag{1.8}$$

- *An orbit is said to be eventually periodic if there exist (p > 0) and (N > 0) such that equality* (1.8) *is satisfied for all (n > N)*
- ***A periodic orbit*** $\mathcal{O}(x_0)$ *is always a sequence of periodic points. All these points are called periodic points of period (p) of the system.*

Definitions 1.3.1. *A set* $A \subset X$ *is called invariant if* $f(A) \subseteq A$*, strongly invariant if* $f(A) = A$*, and completely invariant if* $f^{-1}(A) = A$*. When* f *is a homeomorphism, these notions coincide.*

Definitions 1.3.2. ***(The product of two dynamical systems)*** *((X, f)) and ((Y, g)) is the couple ((XY, fg)) where (fg) is a continuous application defined as follows:*

$$fg : X \times Y \to XY, \quad (x, y) \mapsto (f(x), g(y)).$$

Examples of discrete-time dynamical systems

Here are some examples of discrete dynamical systems :

- **Double Period System**

  The double period function, also known as the Baker function, is obtained by squaring on the unit circle.

  Let $x = \exp(2i\pi\theta)$. Squaring it yields $x^2 = \exp(2i\pi(2\theta))$. Since the rotation period is 2π, we define the Baker function as:

$$\forall x \in [0, 1[: \quad B(x) = \begin{cases} 2x, & \text{if } x \in [0, \frac{1}{2}[, \\ 2x - 1, & \text{if } x \in [\frac{1}{2}, 1[. \end{cases}$$

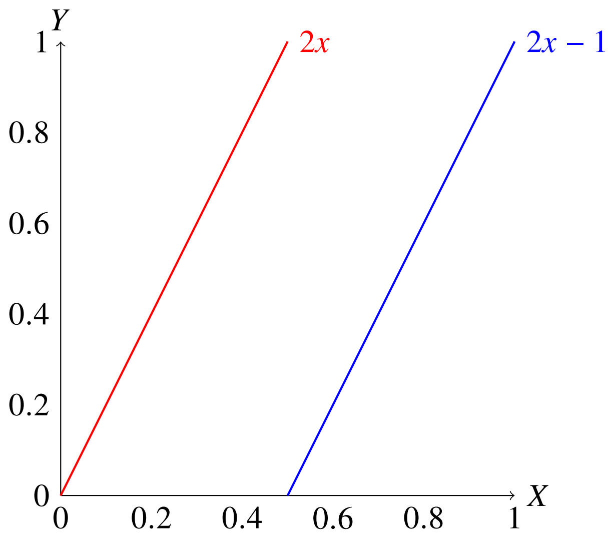


Figure 1.3: Baker Function

- **Rotation on the Circle**

  We adopt the notation $[0, 1]$ to denote the unit circle.

  $[0, 1[\times\mathbb{R}_\alpha$, where $\mathbb{R}_\alpha$ is the function called "rotation by angle α" defined as follows:

$$R_\alpha(x) = (x + \alpha) \mod 1.$$

  The rotation by angle $\theta_0 = 2\pi\alpha$ corresponds to successively applying the operation $x_0 = \exp(2\pi i\alpha)$ on the unit circle. Let $x = \exp(2\pi i\theta)$, then:

$$T_\alpha(x) = x_0 \times x = \exp(2\pi i(\alpha + \theta)).$$

  As the rotation period is 2π, we colloquially refer to the rotation by an angle α (where $\alpha \in [0, 1[$).

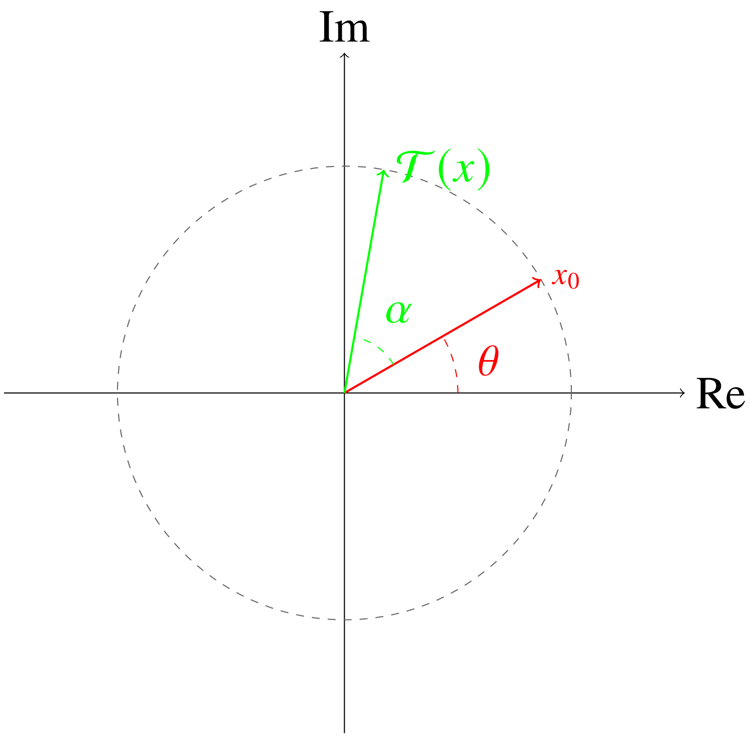


Figure 1.4: Rotation on the unit circle

- **Tent Function**

  Let $([0,1], T)$ be the function called the "tent" function defined by:

$$T(x) = \begin{cases} 2x, & \text{if } x \in [0, \frac{1}{2}[, \\ 2-2x, & \text{if } x \in [\frac{1}{2}, 1]. \end{cases}$$

  This function is illustrated graphically.

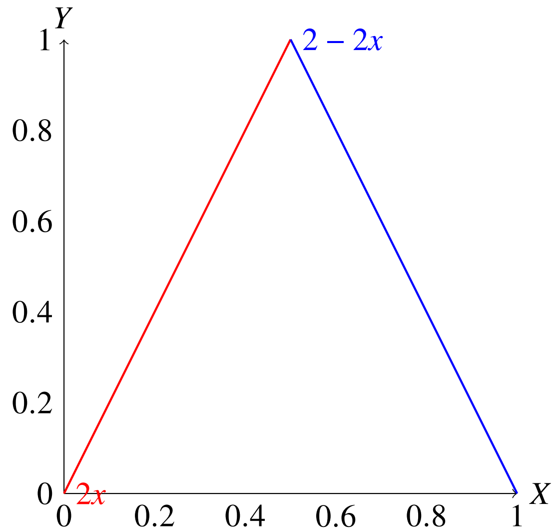


Figure 1.5: Fonction Tente

1.4 Notion of stability

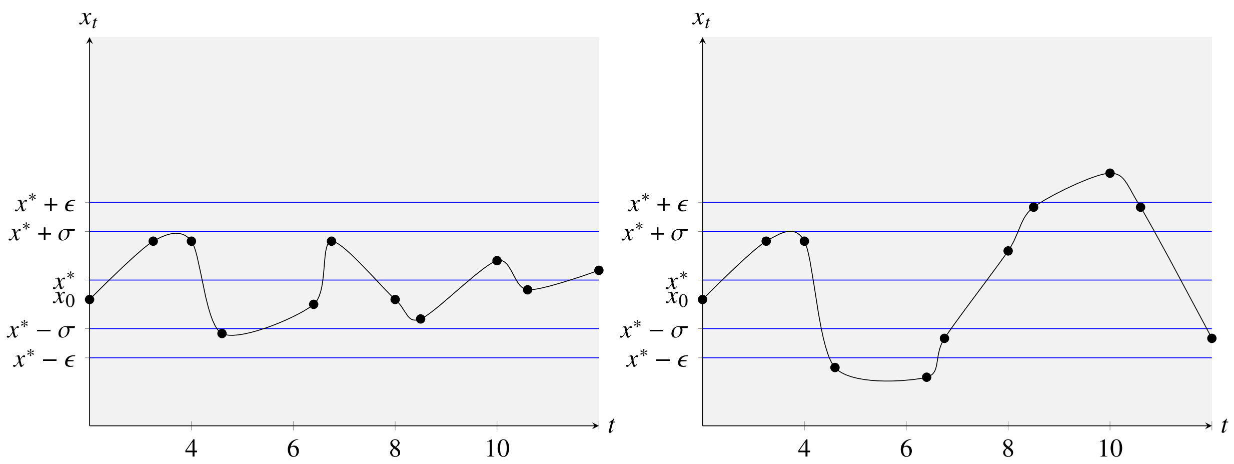
The stability analysis of the system's equilibrium points determines whether an equilibrium point is attractive or repulsive for all or at least some set of initial conditions. It facilitates the study of the local, and often the global, properties of a dynamical system, and it permits the analysis of the implications of small, and sometimes large, perturbations that occur once the system is in the vicinity of an equilibrium point.

Definition 1.4.1. *[12] let $f : I \to I$ be a map and $\bar{x}$ be an equilibrium point of f, where $I \subset \mathbb{R}^n$ then :*

1. *The equilibrium point $\bar{x}$ is stable if, given $\epsilon > 0$, there exists $\delta > 0$ such that*

$$\forall x_0 \in I \quad ||x_0 - \bar{x}|| < \delta \implies ||f^n(x_0) - \bar{x}|| < \epsilon$$

for all $n > 0$*. If* $\bar{x}$ *is not stable, then will be called unstable.*



(a) Stable fixed point $\bar{x}$. (b) Unstable fixed point $\bar{x}$.

2. *The equilibrium point* $\bar{x}$ *is a repelling fixed point if there exists* $\epsilon > 0$ *such that*

$$0 < ||x_0 - \bar{x}|| < \epsilon \implies ||f(x_0) - \bar{x}|| > ||x_0 - \bar{x}||$$

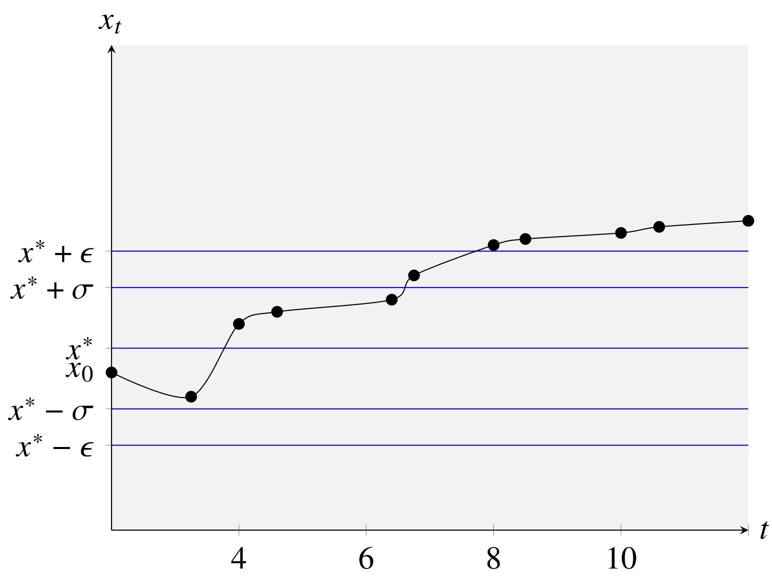
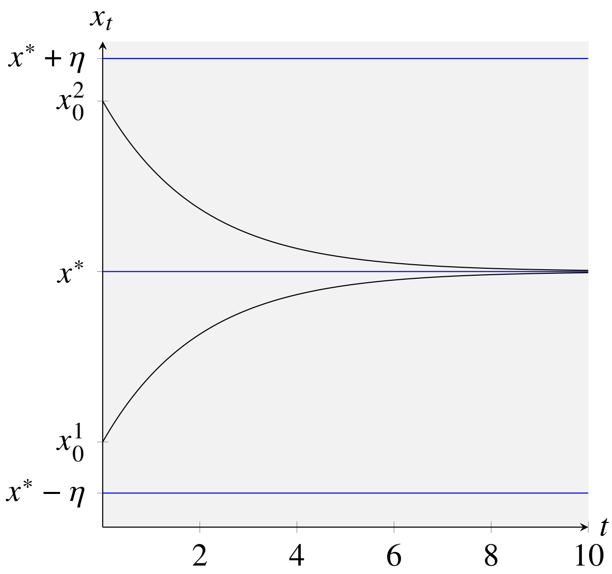


Figure 1.7: Repelling fixed point $\bar{x}$.

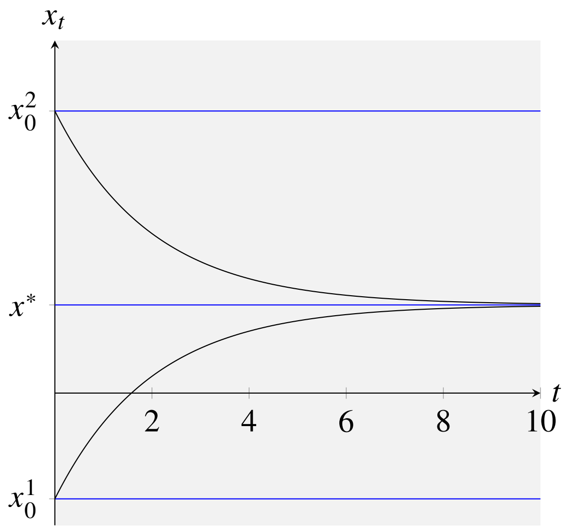
3. *The point* $\bar{x}$ *is an asymptotically stable (attracting) equilibrium point if it is stable and there exists* $\eta > 0$ *such that*

$$||x_0 - \bar{x}|| < \eta \implies \lim_{t \to \infty} x_t = \bar{x}$$

If $\eta = \infty$*, then* $\bar{x}$ *is globally asymptotically stable.*



(a) Asymptotically stable fixed point $\bar{x}$.



(b) Globally asymptotically stable fixed point $\bar{x}$.

1.4.1 Stability of linear systems

- **Linear systems of dimension 1**

Definition 1.4.2. *[16] A linear discrete dynamical system of dimension 1 is defined by the following difference equation:*

$$x_{t+1} = ax_t + b, \quad x_0 \in \mathbb{R}, \quad t \in \mathbb{N}. \tag{1.9}$$

Where:

- $a, b \in \mathbb{R}$ *are constants.*
- $x_t \in \mathbb{R}$ *is the state variable.*
- x_0 *is the initial value.*

From the initial value x_0, we can deduce from (1.9):

- *At time $t = 0$, $x_1 = a \cdot x_0 + b$.*
- *At time $t = 1$: $x_2 = a \cdot x_1 + b = a(a \cdot x_0 + b) + b = a^2 \cdot x_0 + ab + b$.*
- *At time $t = 2$: $x_3 = a \cdot x_2 + b = a(a^2 \cdot x_0 + ab + b) + b = a^3 \cdot x_0 + a^2 \cdot b + ab + b$.*

  Similarly, the value of the state variable at time 3, 4, . . . , t, is

$$
\begin{aligned}
x_3 &= a \cdot x_2 + b = a(a^2 \cdot x_0 + a \cdot b + b) + b = a^2 \cdot x_0 + a^2 \cdot b + a \cdot b + b, \\
&\vdots \qquad\qquad\qquad\qquad\qquad \vdots \\
x_t &= a^t \cdot x_0 + a^{t-1} \cdot b + a^{t-2} \cdot b + \ldots + ab + b,
\end{aligned}
$$

Hence, for $t = 1, 2, \ldots,$

$$x_t = a^t \cdot x_0 + b \cdot \sum_{i=0}^{t-1} a^i, \quad t \in \mathbb{N}^*.$$

Where $\sum_{i=0}^{t-1} a^i$ is a sum of a geometric series, so for $t \in \mathbb{N}^*$:

$$x_t = \begin{cases} a^t \cdot x_0 + b\frac{(1-a^t)}{1-a}, & \text{if } a \neq 1, \\ x_0 + bt, & \text{if } a = 1. \end{cases}$$

Theorem 1.4.1. *[12] For $f : \mathbb{R} \to \mathbb{R}$, let $\bar{x}$ be a fixed point of (1.1). and $f'(x_t) = a$ then, the nature of the fixed point is classified as follows:*

- *asymptotically stable if $|a| < 1$*
- *unstable. if $|a| > 1$*
- *Indifferent if $|a| = 1$*

Existence and Uniqueness of Fixed Point

Assuming that the system (1.9) is at equilibrium [23], i.e., for $a \neq 1$ we have: $\overline{x} = \frac{b}{1-a}$. This implies $\overline{x} = a\overline{x} + b$. Therefore, there exists a unique fixed point.

- For $a = 1$ and $b = 0$, we have: $\forall t \in \mathbb{N}.x_{k+1} = x_k$.

  This means that every initial condition is a fixed point.

- For $a = 1$ and $b \neq 0$, the fixed point does not exist.

Finally, we deduce that:

$$\overline{x} = \begin{cases} \frac{b}{1-a} & \text{for} \quad a \neq 1, \\ x_0 & \text{for} \quad a = 1 \quad \text{and} \quad b = 0. \end{cases}$$

Proposition 1.4.1. *[16] The fixed point of a discrete dynamical system $x_{t+1} = ax_t + b$ exists only if $a \neq 1$ or $(a = 1$ and $b = 0)$.*

Proposition 1.4.2. *[16] The fixed point of a dynamic system $x_{t+1} = ax_t + b$ is unique if and only if $a \neq 1$.*

Remark 1.4.2. *[16] The solution of (1.9) can be written in terms of its fixed point and the initial value x_0 as follows:*

$$x_t = \begin{cases} a^t(x_0 - \overline{x}) + \overline{x} & \text{if } a \neq 1, \\ x_0 + bt & \text{if } a = 1. \end{cases}$$

- **Linear systems of dimension n**

Definition 1.4.3. *[16] A linear discrete dynamical system of dimension n is a system of n linear first-order difference equations, i.e.:*

$$\begin{aligned} x_{1,t+1} &= a_{11}x_{1,t} + a_{12}x_{2,t} + \cdots + a_{1n}x_{n,t} + b_1 \\ x_{2,t+1} &= a_{21}x_{1,t} + a_{22}x_{2,t} + \cdots + a_{2n}x_{n,t} + b_2 \\ \vdots \quad & \quad\quad\quad , \quad\quad\quad \vdots \\ x_{n,t+1} &= a_{n1}x_{1,t} + a_{n2}x_{2,t} + \cdots + a_{nn}x_{n,t} + b_n \end{aligned}$$

where $t \in \mathbb{N}$ and $X_0 = (x_{1,0}, x_{2,0}, \ldots, x_{n,0})$ are given. The matrix representation of a linear discrete dynamical system of dimension n is:

$$\begin{cases} X_{t+1} = AX_t + B, \quad t \in \mathbb{N} \\ X_0 \quad \text{given}, \end{cases} \tag{1.10}$$

where:

- $A \in M_n(\mathbb{R})$: *a matrix of real constants of size* $n \times n$.
- $B \in \mathbb{R}^n$: *a vector of constants.*
- $X_t \in \mathbb{R}^n$: *vector of states of the system.*
- $X_0 \in \mathbb{R}^n$: *initial vector.*
- $x_t \in \mathbb{R}$: *state variable.*

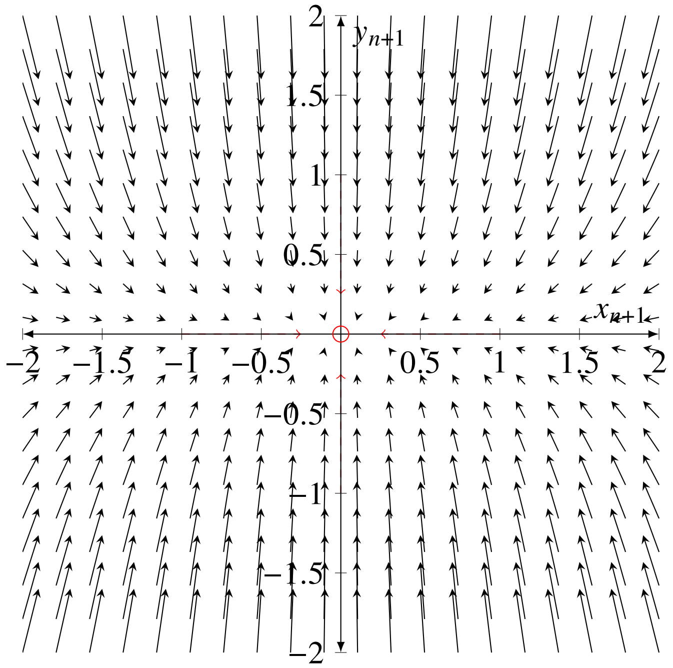
$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}, \quad X_t = \begin{pmatrix} x_{1,t} \\ x_{2,t} \\ \vdots \\ x_{n,t} \end{pmatrix}, \quad X_0 = \begin{pmatrix} x_{1,0} \\ x_{2,0} \\ \vdots \\ x_{n,0} \end{pmatrix}, \quad B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}.$$

Theorem 1.4.3. *[7] if $f : \mathbb{R}^n \to \mathbb{R}^n$, We calculate the eigenvalues λ_i, $1 \le i \le n$, of the matrix A of the system* (1.10) *.*

If the eigenvalues λ_i are real:

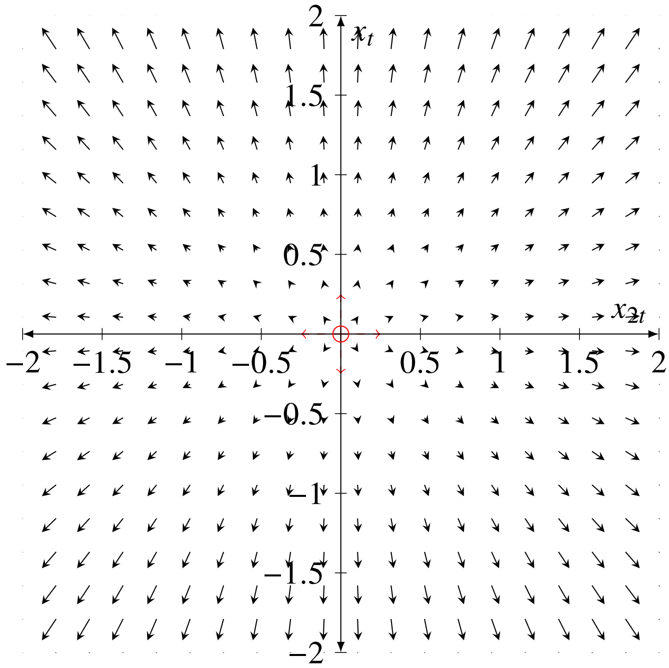
- *For $i = 1, 2, \dots, n$, if $|\lambda_i| < 1$, it is an attractive node (A sink).*

for $(f : \mathbb{R}^2 \to \mathbb{R}^2)$ A sink : $|\lambda_i| < 1$, $i = 1, 2$.



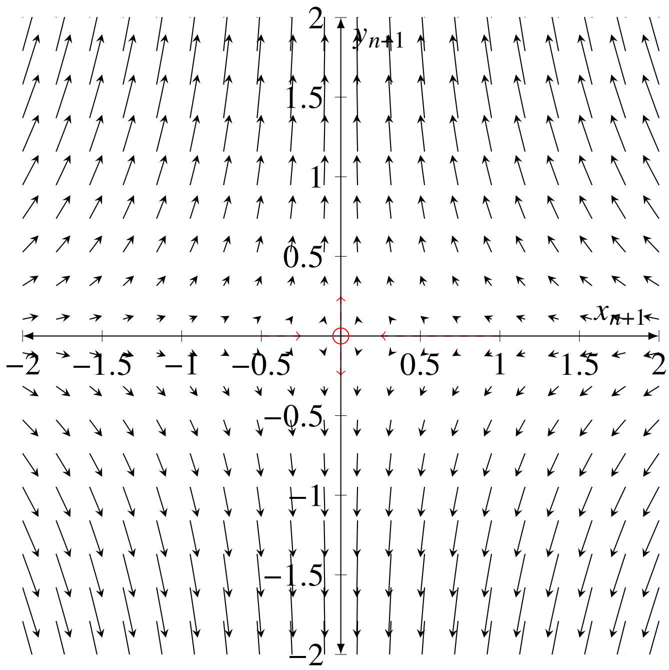
- *For $i = 1, 2, \dots, n$, if $|\lambda_i| > 1$, it is a repulsive node(A source).*

for $(f : \mathbb{R}^2 \to \mathbb{R}^2)$ A source : $|\lambda_i| > 1$, $i = 1, 2$.



- *For* i, j *such that* $1 \le i \le n$ *and* $1 \le j \le n$, *if* $|\lambda_i| < 1$ *and* $|\lambda_j| > 1$, *it is a saddle node.*

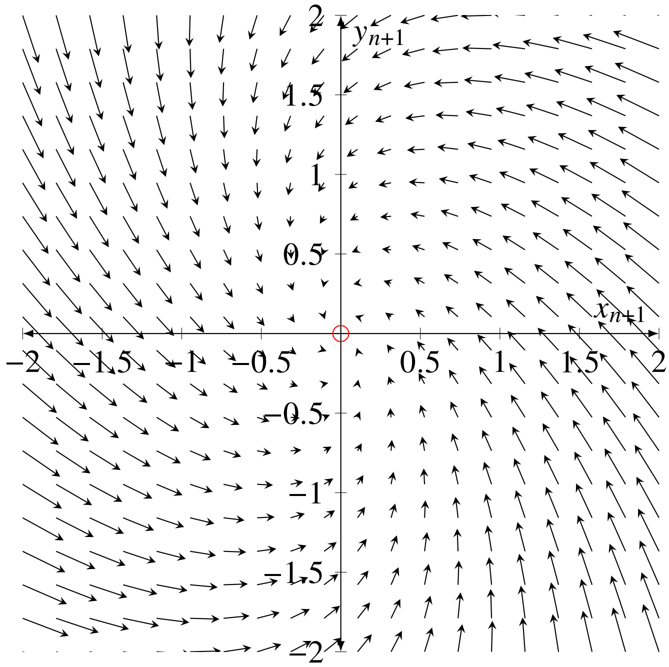
for $(f : \mathbb{R}^2 \to \mathbb{R}^2)$ A saddle: $|\lambda_i| > 1$,$i = 1, 2$.



If the eigenvalues λ_i *are complex :*

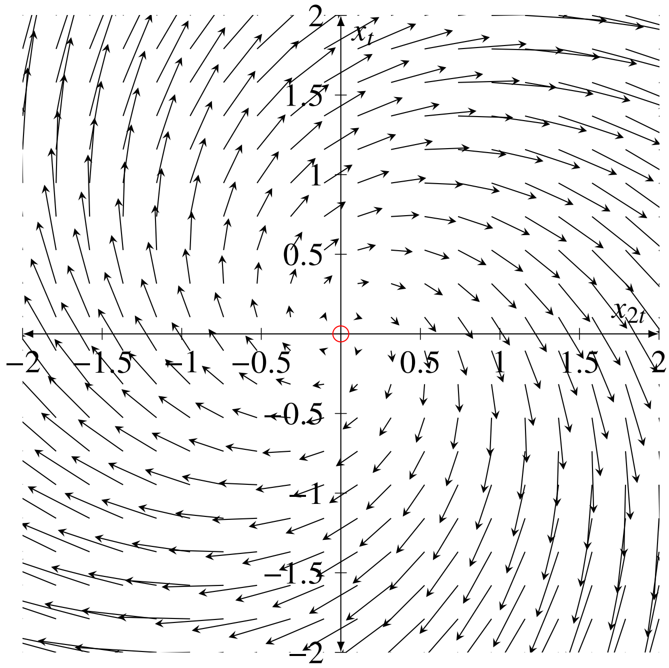
- *For* $i = 1, 2, \ldots, n$, *if* $|\lambda_i| < 1$, *it is a spiral attractor(A spiral sink).*

for ($f : \mathbb{R}^2 \to \mathbb{R}^2$) Spiral sink: $|\lambda_i| < 1, i = 1, 2.$



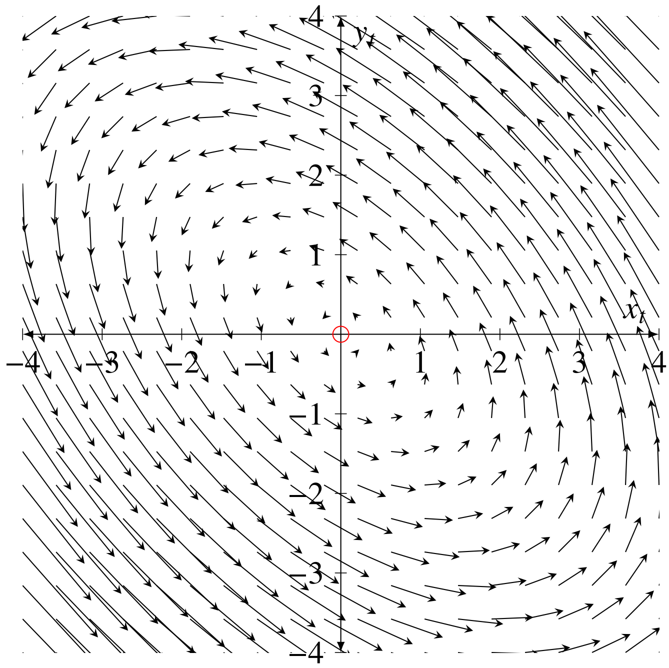
- *For* $i = 1, 2, \ldots, n$*, if* $|\lambda_i| > 1$*, it is a spiral repulsor(A spiral source).*

for ($f : \mathbb{R}^2 \to \mathbb{R}^2$) Spiral Source : $|\lambda_i| > 1, i = 1, 2.$



- *For* $i = 1, 2, \ldots, n$*, if* $|\lambda_i| = 1$*, it is a center .*

for $(f : \mathbb{R}^2 \to \mathbb{R}^2)$ Center : $|\lambda_i| = 1, i = 1, 2$



1.4.2 Stability of nonlinear systems

- **Nonlinear systems of dimension 1**

Definition 1.4.4. *[16] A discrete nonlinear dynamical system of dimension 1 is defined by the following difference equation:*

$$\begin{cases} x_{t+1} = f(x_t), & t \in \mathbb{N}, \\ x_0 \quad given , \end{cases} \tag{1.11}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function and $x_t \in \mathbb{R}$ is the state variable.

Stability of fixed points

It is not easy to find solutions to nonlinear systems. Often, these solutions do not provide enough informations about the factors that control the stability of the system. Therefore, we need analytical methods to facilitate the study of the behavior of this nonlinear system.

The linear approximation of the nonlinear system is one of the most effective ways to study the stability of nonlinear systems.

Method of Linearization

Theorem 1.4.4. *[31] Suppose that the non-linear system described by (1.11) admits a limited development in the neighborhood of the fixed point $\bar{x}$. Then:*

$$x_{t+1} = f(\bar{x}) + (x - \bar{x})f'(\bar{x}) + O((x_t - \bar{x})^2),$$

so :

$$x_{t+1} \approx f(\bar{x}) + (x - \bar{x})f'(\bar{x}) = a \cdot x_t + b, \tag{1.12}$$

where : $a = f'(\bar{x})$ and $b = f(\bar{x}) - \bar{x}f'(\bar{x})$.
Since in the vicinity of $\bar{x}$, $||x - \bar{x}|| \approx 0$, by neglecting second-order terms, the system (1.11) is linearized effectively.
The mapping $X \mapsto AX$ is called the linearized mapping of f in the vicinity of the fixed point $\bar{x}$. We say that the system (1.11) is approximated in the vicinity of the equilibrium point x by the linear system (1.12).
Now we can use the stability results of the linear system to study the stability of the nonlinear system .

Criteria for Stability

Fixed (equilibrium) points may be divided into two types: **hyperbolic** and **nonhyperbolic**.

Definition 1.4.5. *[12] A fixed point $\bar{x}$ of a map f is said to be hyperbolic if $|f'(\bar{x})| \neq 1$. Otherwise, it is nonhyperbolic. We will treat the stability of each type separately.*

Hyperbolic Fixed Points

Theorem 1.4.5. *Let $\bar{x}$ be a hyperbolic fixed point of a map f, where f is continuously differentiable at $\bar{x}$. The following statements then hold true:*

– *The equilibrium point of* (1.11) *is locally asymptotically stable if* $|f'(x)| < 1$.

– *The equilibrium point of* (1.11) *is unstable if* $|f'(x)| > 1$.

Nonhyperbolic Fixed Points

The stability criteria for nonhyperbolic fixed points are more involved. They will be summarized in the next two results, the first of which treats the case when $f'(\bar{x}) = 1$ and the second for $f'(\bar{x}) = -1$.

• 1^{st} **case** : for $f'(\bar{x}) = 1$

Theorem 1.4.6. *Let $\bar{x}$ be a fixed point of a map f such that $f'(\bar{x}) = 1$. If $f'(x)$, $f''(x)$, and $f'''(x)$ are continuous at $\bar{x}$, then :*

1. *If $f''(\bar{x}) \neq 0$, then $\bar{x}$ is unstable (semistable).*
2. *If $f''(\bar{x}) = 0$ and $f'''(\bar{x}) > 0$, then $\bar{x}$ is unstable.*
3. *If $f''(\bar{x}) = 0$ and $f'''(\bar{x}) < 0$, then $\bar{x}$ is asymptotically stable.*

• 2^{nd} **case** : for $f'(\bar{x}) = -1$

The preceding theorem may be used to establish stability criteria for the case when $f'(\bar{x}) = -1$. But before doing so, we need to introduce the notion of the Schwarzian derivative.

Definition 1.4.6. *The Schwarzian derivative, Sf, of a function f is defined by*

$$Sf(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2$$

. *And if $f'(\bar{x}) = -1$, then*

$$Sf(\bar{x}) = -f'''(\bar{x}) - \frac{3}{2}(f''(\bar{x}))^2$$

.

Theorem 1.4.7. *Let $\bar{x}$ be a fixed point of a map f such that $f'(\bar{x}) = -1$. If $f'(x)$, $f''(x)$, and $f'''(x)$ are continuous at $\bar{x}$, then:*

1. *If* $Sf(\bar{x}) < 0$*, then* $\bar{x}$ *is asymptotically stable.*

2. *If* $Sf(\bar{x}) > 0$*, then* $\bar{x}$ *is unstable.*

Example 1.4.1. *Consider the map* $f(x) = x^2 + 3x$ *on the interval* $[-3, 3]$*. The fixed points of* f *are obtained by solving the equation* $x^2 + 3x = x$*. Thus, there are two fixed points:* $\bar{x}_1 = 0$ *and* $\bar{x}_2 = -2$*. So for* $\bar{x}_1$*,* $f'(0) = 3$*, which implies by Theorem 1.4.5 that* $\bar{x}_1$ *is unstable. For* $\bar{x}_2$*, we have* $f'(-2) = -1$*, which requires the employment of Theorem 1.4.2 . We observe that*

$$Sf(-2) = -f'''(-2) - \frac{3}{2}[f''(-2)]^2 = -6 < 0.$$

Hence, $\bar{x}_2$ *is asymptotically stable.*

- **Nonlinear systems of dimension n**

Definition 1.4.7. *A nonlinear discrete dynamical system of dimension n is a system of n nonlinear first-order difference equations, i.e :*

$$\begin{aligned} x_{1,t+1} &= f_1(x_{1,t}, x_{2,t}, \ldots, x_{n,t}), \\ x_{2,t+1} &= f_2(x_{1,t}, x_{2,t}, \ldots, x_{n,t}), \\ \vdots \quad & \quad \vdots \\ x_{n,t+1} &= f_n(x_{1,t}, x_{2,t}, \ldots, x_{n,t}), \end{aligned}$$

where $t \in \mathbb{N}$ *and* $X_0 = (x_{1,0}, x_{2,0}, \ldots, x_{n,0})$ *are given. This dynamical system is written as follows:*

$$\begin{cases} X_{t+1} = f(X_t), \quad t \in \mathbb{N}. \\ X_0 \quad \text{given}, \end{cases} \tag{1.13}$$

where: - $f : \mathbb{R}^n \to \mathbb{R}^n$ *is a differentiable function.*

- $f(X_t) = (f_1(X_t), f_2(X_t), \ldots, f_n(X_t))$*.*

- $f_i : \mathbb{R}^n \to \mathbb{R}$ *for all* $i = 1, \ldots, n$ *is a differentiable function.*

- $X_t \in \mathbb{R}^n$ *is the vector of system states.*

- $X_0 \in \mathbb{R}^n$ *is the initial value.*

Local stability of fixed points

As we have seen in the case of a one-dimensional dynamical system, the dynamical system (1.13) can also be approximated by a linear system.

Method of Linearization (Indirect method of Lyapunov) [16]

Suppose the dynamical system (1.13) has a fixed point $\bar{X}$, then the first-order Taylor expansion of $f_i(X_t) = x_{i,t+1}$ in the neighborhood of $\overline{X}$ is:

$$\begin{aligned} x_{i,t+1} = f_i(X_t) &= f_i(\overline{X}) + \sum_{k=1}^{n} \frac{\partial f_i(\overline{X})}{\partial x_{k,t}}(x_{k,t} - \overline{x}_k) + o(||X_t - \overline{X}||) \\ &= \frac{\partial f_i(\overline{x})}{\partial x_{1,t}} x_{1,t} + \frac{\partial f_i(\overline{X})}{\partial x_{2,t}} x_{2,t} + \cdots + \frac{\partial f_i(\overline{X})}{\partial x_{n,t}} x_{n,t} + f_i(\overline{X}) \\ &\quad - \sum_{k=1}^{n} \frac{\partial f_i(\overline{X})}{\partial x_{k,t}} \overline{x}_k + o(||X_t - \overline{X}||) \end{aligned}$$

or $\underbrace{\left\| o(X_t - \overline{X}) \right\|}_{X_t \to \overline{X}} \to 0$ So, the first-order Taylor expansion of $f(X_t) = X_{t+1}$ in the neighborhood $\overline{X}$ is:

$$\begin{pmatrix} x_{1,t+1} \\ x_{2,t+1} \\ \vdots \\ x_{n,t+1} \end{pmatrix} = \begin{pmatrix} \frac{\partial f_1(\overline{X})}{\partial x_{1,t}} & \frac{\partial f_1(\overline{X})}{\partial x_{2,t}} & \cdots & \frac{\partial f_1(\overline{X})}{\partial x_{n,t}} \\ \frac{\partial f_2(\overline{X})}{\partial x_{1,t}} & \frac{\partial f_2(\overline{X})}{\partial x_{i,t}} & \cdots & \frac{\partial f_2(\overline{X})}{\partial x_{i,t}} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial f_n(\overline{X})}{\partial x_{n,t}} & \frac{\partial f_n(\overline{X})}{\partial x_{n,t}} & \cdots & \frac{\partial f_n(\overline{X})}{\partial x_{n,t}} \end{pmatrix} \cdot \begin{pmatrix} x_{1,t} \\ x_{2,t} \\ \vdots \\ x_{n,t} \end{pmatrix} + \begin{pmatrix} f_1(\bar{x}) - \sum_{k=1}^{n} \frac{\partial f_1(\bar{x})}{\partial x_{k,t}} \bar{x}_k \\ f_2(\bar{x}) - \sum_{k=1}^{n} \frac{\partial f_2(\bar{x})}{\partial x_{k,t}} \bar{x}_k \\ \vdots \\ f_n(\bar{x}) - \sum_{k=1}^{n} \frac{\partial f_n(\bar{x})}{\partial x_{k,t}} \bar{x}_k \end{pmatrix}. \tag{1.14}$$

The nonlinear system is approximated in the neighborhood of the fixed point $\overline{X}$ by a linear system $X_{t+1} \approx AX_t + B$ where:

$$A = \begin{pmatrix} \frac{\partial f_1(\overline{X})}{\partial x_{1,t}} & \frac{\partial f_1(\overline{X})}{\partial x_{2,t}} & \cdots & \frac{\partial f_1(\overline{X})}{\partial x_{n,t}} \\ \frac{\partial f_2(\overline{X})}{\partial x_{1,t}} & \frac{\partial f_2(\overline{X})}{\partial x_{2,t}} & \cdots & \frac{\partial f_2(\overline{X})}{\partial x_{n,t}} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial f_n(\overline{X})}{\partial x_{1,t}} & \frac{\partial f_n(\overline{X})}{\partial x_{2,t}} & \cdots & \frac{\partial f_n(\overline{X})}{\partial x_{n,t}} \end{pmatrix} \quad and \quad B = \begin{pmatrix} f_1(\bar{x}) - \sum_{k=1}^{n} \frac{\partial f_1(\bar{x})}{\partial x_{k,t}} \bar{x}_k \\ f_2(\bar{x}) - \sum_{k=1}^{n} \frac{\partial f_2(\bar{x})}{\partial x_{k,t}} \bar{x}_k \\ \vdots \\ f_n(\bar{x}) - \sum_{k=1}^{n} \frac{\partial f_n(\bar{x})}{\partial x_{k,t}} \bar{x}_k \end{pmatrix}.$$

The matrix A is called the Jacobian matrix of f at the point $\overline{X}$, denoted by $J(\overline{X})$.
We can therefore use the results of the linear system to study the stability of the nonlinear system in the neighborhood of the fixed point $\overline{X}$.

Theorem 1.4.8. *[7] Let $f : \mathbb{R}^n \to \mathbb{R}^n$, to determine the nature of the point $\bar{x}$ it is necessary to find the eigenvalues of the Jacobian matrix $J(\bar{X})$. The point $\bar{x}$ is:*

1. *is locally asymptotically stable If $|\lambda_i| < 1$ for all $i = 1, \dots, n$.*
2. *unstable If there exists an eigenvalue λ_i such that $|\lambda_i| > 1$.*
3. *If $\max_{1\leq i\leq n} |\lambda_i| = 1$, we cannot conclude anything.*

Liapunov function

In several cases, using singular points, we cannot conclude anything about stability. A new method known as Liapunov 's second method or direct method has emerged, allowing us to analyze stability or instability of critical points by constructing a suitable auxiliary function (from the recursive equations defining the system without having to calculate their solutions).
Let us consider the autonomous dynamical system

$$x_{t+1} = f(x_t), \tag{1.15}$$

where $f : G \to \mathbb{R}^n, G \subset \mathbb{R}^n$ is a continuous function.
We assume that $\overline{X}$ is a fixed point of this dynamical system.

Definition 1.4.8. *[39] Let G be an open set in $\mathbb{R}^n$. A function V from G to $\mathbb{R}$ is a Liapunov function on the set G if:*

- *V is continuous on G.*
- *$\Delta V(X_t) = V(f(X_t)) - V(X_t) = V(X_{t+1}) - V(X_t) \leq 0$ for all X_t, X_{t+1} belong to G.*

Definition 1.4.9. *The Lyapunov function V is said to be positive definite at the fixed point $\overline{X}$ if there exists an open ball $B_r(\overline{X})$ centered at $\overline{X}$ and with radius r.*
(i.e., $B_r(\overline{X}) = \{Y \in \mathbb{R}^n, ||\overline{X} - Y|| < r\}$).

Such that:

- $V(\overline{X}) = 0.$
- $V(X_t) > 0$ *for all* $X_t \in B_r(\overline{X})$, $X_t \neq \overline{X}$.

Theorem 1.4.9. *[39] If there exists a Lyapunov function V on an open ball $B_r(\overline{X})$ positive definite at $\overline{X}$ where $\overline{X}$ is the fixed point of (1.15), then $\overline{X}$ is stable.*
If additionally, $\Delta V(X_t) < 0$ for all X_t, X_{t+1} in G, $X_t \neq \overline{X}$, then $\overline{X}$ is locally asymptotically stable. Moreover if this holds true when $B_r(\overline{X})$ is extended to all $\mathbb{R}^n$ and $V(X_t) \to \infty$ as $||X_t|| \to \infty$, then $\overline{X}$ is globally asymptotically stable.

Examples 1.4.1. *Consider the dynamical system:*

$$\begin{cases} x_{t+1} = \frac{x_{2t}}{(1+x_{2t}^2)}, \\ x_{2t+1} = \frac{x_t}{(1+x_{2t}^2)}. \end{cases}$$

We have:

$$\begin{cases} \overline{x}_1 = \frac{\overline{x}_2}{(1+\overline{x}_2^2)}, \\ \overline{x}_2 = \frac{\overline{x}_1}{(1+\overline{x}_2^2)}. \end{cases}$$

Therefore, the fixed point of this system is $\overline{X} = (\overline{x}_1, \overline{x}_2) = (0,0)$. We define the function:

$$V(x_1, x_2) = x_1^2 + x_2^2$$

The function V is continuous on $\mathbb{R}^2$ and $V(X_{t+1}) = \frac{x_{2t}^2}{(1+x_{2t}^2)^2} + \frac{x_{1t}^2}{(1+x_{2t}^2)^2} = \frac{V(X_t)}{(1+x_{2t}^2)^2} \leq V(X_t)$, $\forall X_t, X_{t+1} \in \mathbb{R}^2$. Thus, V is a Liapunov function.

We also have $V(\overline{X}) = 0$ and $V(X_t) > 0$ for all $X_t \in \mathbb{R}^2, X_t \neq \overline{X}$. Therefore, V is positive definite. From Theorem (1.4.9), it follows that the fixed point $\overline{X} = (0,0)$ is asymptotically stable.

CHAPTER 2

BIFURCATIONS AND CHAOS

2.1 Bifurcations

Another set of concepts useful for the analysis of dynamic systems is the theory of "bifurcation".The name "bifurcations" was first introduced by **Henri Poincaré** in 1885 This theory focuses on families of dynamic systems (continuous or discrete) depending on a parameter $\mu \in \mathbb{R}^m$. This theory refers to the study of changes in the behavior of a system when its parameters change.

2.1.1 Definition of the bifurcation

consider a dynamical system that depends on parameters as follow :

$$x_{t+1} = f(x_t, \mu). \tag{2.1}$$

Where $x \in \mathbb{R}^n$ and $\mu \in \mathbb{R}^m$ represent phase variables and parameters, respectively. Consider the phase portrait of the system. As the parameters vary, the phase portrait also varies. There are two possibilities: either the system remains topologically equivalent to the original one, or its topology changes.

Definition 2.1.1. *[19] The appearance of a topologically nonequivalent phase portrait under variation of parameters is called a bifurcation.*
Thus ,a bifurcation is a change of the topological type of the system as its parameters pass through a bifurcation (critical) value μ_0.

Definition 2.1.2. *A bifurcation diagram of the dynamical system is a stratification of its parameter space induced by the topological equivalence, together with representative phase portraits for each stratum. It is desirable to obtain the bifurcation diagram as a result of the qualitative analysis of a given dynamical system. It classifies in a very condensed way all possible modes of behavior of the system and transitions between them (bifurcations) under parameter variations. Note that the bifurcation diagram depends in general on the considered region of phase space.*

2.1.2 Types of bifurcations

In this work,we explore briefly the bifurcation in two-dimensional system, associated to the one parameter family of maps [12]:

$$F(\mu, X) : \mathbb{R} \times \mathbb{R}^2 \longrightarrow \mathbb{R} \tag{2.2}$$

with $X = (x, y) \in \mathbb{R}^2$, $\mu \in \mathbb{R}$, and $F \in C^r$, $r \geq 5$. If $(\bar{\mu}, \bar{X})$ is a fixed point, then we make a change of variables so that our fixed point is $(0, 0)$. Let $J = D_u F(0, 0)$. Then using the center manifold theorem, we find a one-dimensional map $f_\mu(x)$. There are several types of bifurcations depending on the properties of the second derivatives of the family of functions $f_\mu(x)$. Among the different types of bifurcations observed in discrete dynamical systems we find :

1. When the Jacobian matrix J has an eigenvalue equal to 1 then we have :

   (a) **A Saddle-Node (Fold) Bifurcation :**

   This type of bifurcation is characterized by a sudden loss or acquisition of multiple stable or unstable equilibrium solutions when a parameter value crosses a critical threshold. it's satisfied the following conditions :

$$\frac{\partial f}{\partial \mu}(0, 0) \neq 0 \quad and \quad \frac{\partial^2 f}{\partial x^2}(0, 0) \neq 0$$

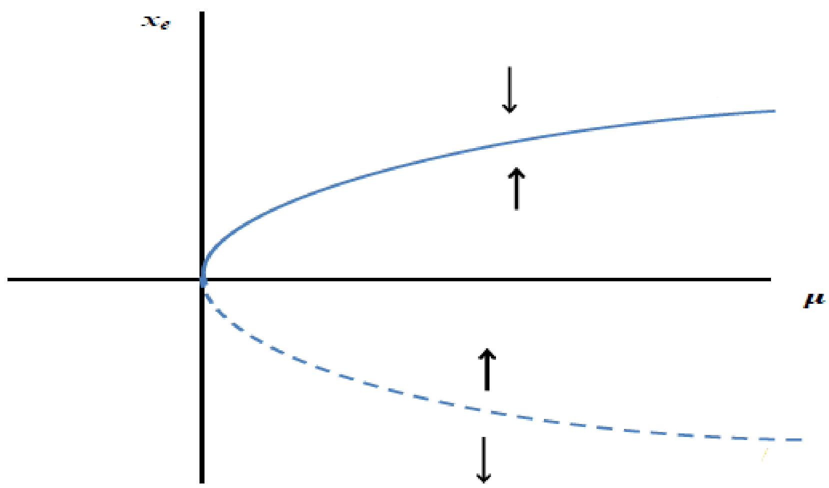


Figure 2.1: Saddle-Node Bifurcation diagram.

(b) **A Pitchfork Bifurcation :**

In general, a pitchfork bifurcation occurs near the bifurcation point (x_0, μ_0). The model has two curves of fixed points in the (x_n, μ) plane that pass through the bifurcation point, with one lying on each side of the line $\mu = \mu_0$. It is formed under the following conditions :

$$\frac{\partial f}{\partial \mu}(0,0) = 0 \quad and \quad \frac{\partial^2 f}{\partial x^2}(0,0) = 0$$

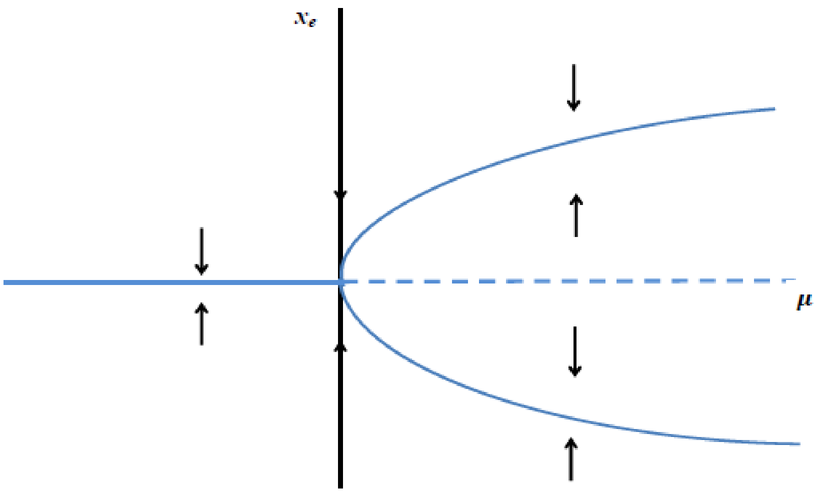


Figure 2.2: Pitchfork Bifurcation diagram.

(c) **A Transcritical Bifurcation :**

This type of bifurcation is characterized by an exchange of stability between two equilibrium solutions. Initially, the system has one stable equilibrium solution and one unstable equilibrium solution. As a parameter varies and reaches a critical value, the stable equilibrium solution becomes unstable, while the unstable equilibrium becomes stable. This type satisfied the following conditions :

$$\frac{\partial f}{\partial \mu}(0,0) = 0 \quad and \quad \frac{\partial^2 f}{\partial x^2}(0,0) \neq 0$$

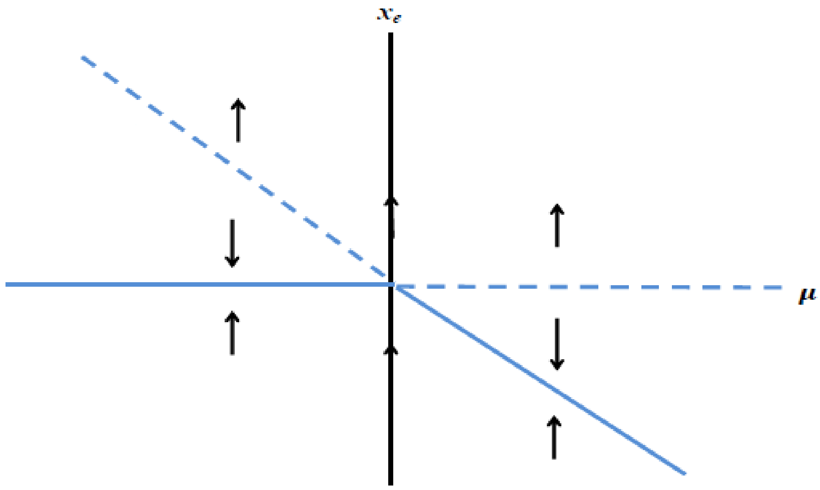


Figure 2.3: Transcritical Bifurcation diagram.

2. If J has an eigenvalue equal to -1 then we have **a period-Doubling (Flip) Bifurcation**. This bifurcation occurs when a stable k-order cycle has a multiplier that passes through the value $\lambda = -1$ At this point: The cycle becomes unstable and gives rise to a stable 2k-order cycle. it's satisfied the following conditions :
$\frac{\partial}{\partial x}[f(x,\mu) - x]\Big|_{(0,0)} \neq 0$ and $\frac{\partial^2 f}{\partial \mu \partial x} + \frac{1}{2}\frac{\partial f}{\partial \mu}\frac{\partial^2 f}{\partial x^2}\Big|_{(0,0)} \neq 0$ and $\frac{1}{3!}\frac{\partial^3 f}{\partial x^3} + \frac{1}{2!}\left(\frac{\partial f}{\partial x}\right)^2\Big|_{(0,0)} \neq 0.$

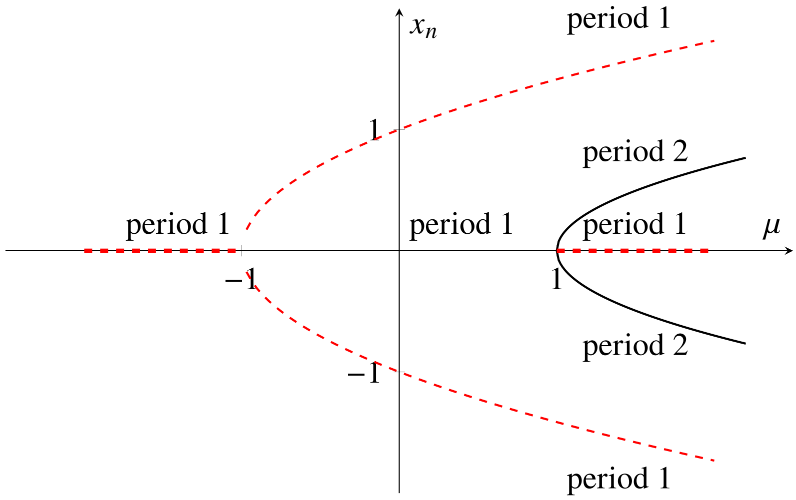


Figure 2.4: Period-Doubling Bifurcation diagram..

3. If J has a pair of complex conjugate eigenvalues of modulus 1, a new type of bifurcation called **the Neimark-Sacker bifurcation** appears.

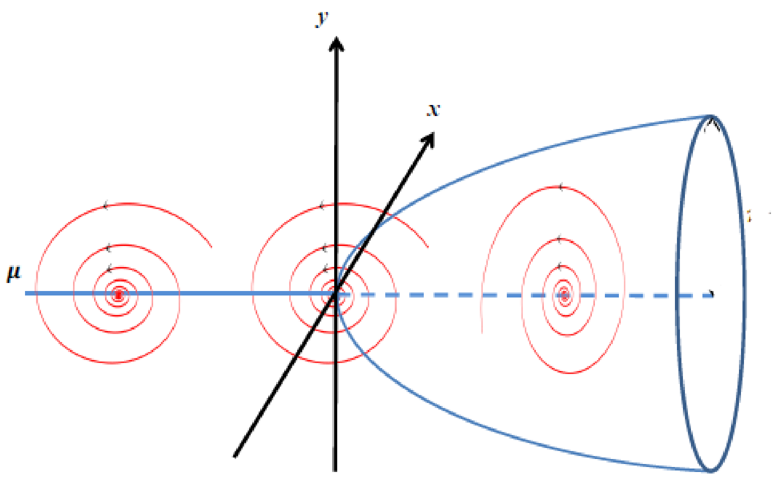


Figure 2.5: Neimark-Sacker Bifurcation diagram.

The following plot illustrates these types in two dimensional systems :

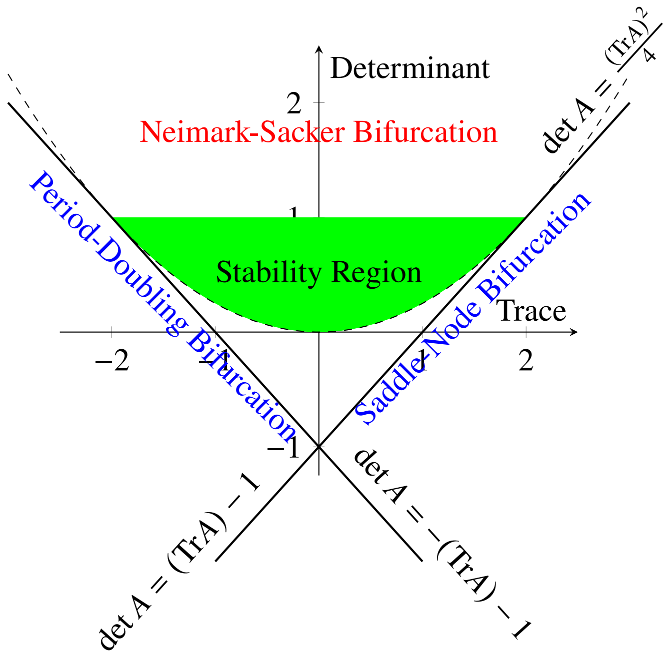


Figure 2.6: The occurrence of bifurcations in a two-dimensional discrete dynamical system.

Example 2.1.1. *consider the dynamic system defined as follows :*

$$F : \begin{cases} x_{n+1} = \mu x_n(1 - y_n), \quad \mu > 0, \\ y_{n+1} = x_n. \end{cases}$$

This system has two fixed points :

$$\begin{cases} (\bar{x}_1, \bar{y}_1) = (0, 0), \\ (\bar{x}_2, \bar{y}_2) = (1 - \frac{1}{\mu}, 1 - \frac{1}{\mu}). \end{cases}$$

The Jacobian matrix evaluated at the fixed point $(\bar{x}_2, \bar{y}_2)$ *is:*

$$J(\bar{x}_2, \bar{y}_2) = \begin{pmatrix} 1 & 1 - \mu \\ 1 & 0 \end{pmatrix}.$$

We can deduce the eigenvalues:

$$\lambda_{1,2} = \frac{1}{2} \pm \sqrt{\frac{5}{4} - \mu}.$$

If $\mu > \frac{5}{4}$*, the eigenvalues are complex with* $|\lambda_{1,2}|^2 = \mu - 1$*. For* $\mu = 2$*, the fixed point* $(\bar{x}_2, \bar{y}_2)$ *loses its stability. The eigenvalues are then* $\lambda_{1,2} = e^{\pm i \frac{\pi}{3}}$*, and the system presents a Neimark bifurcation.*

2.2 Chaos

In common usage, "chaos" means "a state of disorder" However, in chaos theory, the term is defined more precisely, it is linked to unpredictability and the inability to predict long-term development because the final state depends heavily on the initial state. Chaos theory is a field of study in mathematics, with applications in several disciplines such as physics, engineering, biology, and economics. Chaos theory examines the behavior of dynamic systems that are highly sensitive to initial conditions.

2.2.1 Definitions of chaos

Several definitions of chaos are known, although they are not mathematical until R. L. Devaney proposed a definition based on the following definitions:
Let $(I \subset \mathbb{R}, d)$ denote a compact metric space (where **d** represents a distance) and let **f** be the

function defined as :

$$f : I \longrightarrow I \ ; \ x_{k+1} = f(x_k) \ ; \ x_0 \in I. \tag{2.3}$$

Definition 2.2.1. *(Topological transitivity [21]) A map f is said to be topologically transitive on I if for any two open sets $U, V \subset I$ there exists $k \in \mathbb{N}$ such that $f^k(U) \cap V \neq \emptyset$. The function f is called totally transitive when the composition f^n is topologically transitive for all integer* $n > 1$.

Definition 2.2.2. *(Sensitive dependence on initial conditions) An attribute for a chaotic system is to exhibit exponentially fast separation of nearby trajectories for infinitesimally changed initial conditions. Mathematically, this can be expressed as follows:*
A map f is said to have sensitive dependence on initial conditions if there exists a $\delta > 0$ such that for any $x \in I$ and any neighborhood $N_\varepsilon(x) = (x - \varepsilon, x + \varepsilon)$ of x, there exists $y \in N_\varepsilon(x)$ and an integer $k > 0$ such that $|f^k(x) - f^k(y)| > \delta$.

Definition 2.2.3. *(Dence set) In a topological space (I, τ), a subset A of I is said to be a dense set (or an everywhere dense set) if $\overline{A} = X$. In other words, A is said to be a dense subset of X if for any $x \in I$, any neighborhood of x contains at least one point of A. This is equivalent to saying that A is dense in X if for every $x \in I$, we can find a sequence of points $a_n \in A$ converging to x.*

now we will state the definition of chaos, in the sense of **Devaney** [11].

Definition 2.2.4. *Let I be a set. $f : I \longrightarrow I$ is said to be choatic on I if:*

(i) The map function f has sensitive dependence on initial conditions.

(ii) f is topologically transitive .

(iii) The periodic points of f are dense in I.

2.2.2 Characteristics of chaos

There is a set of properties that summarize the characteristics observed in chaotic systems. They are considered as mathematical criteria that define chaos. The most well-known ones are:

1. **Sensitivity to initial conditions :** The sensitivity to initial conditions was first discovered at the end of the 19th century by Poincaré, and then rediscovered in 1963 by Lorenz during his work in meteorology. This discovery led to a large number of important works, mainly in the field of mathematics. This sensitivity explains the fact that, for a chaotic system, a tiny modification of the initial conditions can lead to unpredictable results in the long term. The degree of sensitivity to initial conditions quantifies the chaotic nature of the system [27].

2. **Lyapunov Exponents** The Lyapunov exponents are used to measure the potential divergence between two orbits originating from nearby initial conditions and allow quantifying the sensitivity to initial conditions of a chaotic system. The number of Lyapunov exponents is equal to the dimension of the phase space [27]. In this subsection, we will define the concept of the Lyapunov exponent and show how it is used to study chaotic systems and even detect the presence of chaos in systems.
(i) Case of one-dimensional discrete systems :

Theorem 2.2.1. *[7] Let f be a discrete function from $\mathbb{R}$ to $\mathbb{R}$ that applies x_{n+1} to x_n. The Lyapunov exponent λ indicating the average divergence rate is defined by:*

$$\lambda = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln |f'(x_i)|.$$

Proof. Let's choose two very close initial conditions x_0 and $x_0{}'$, separated by a distance d_0, and see how the trajectories arising from them behave.
we have $d_0 = |x_0{}' - x_0|$.
After one iteration, the distance between the two trajectories becomes $d_1 = |x_1{}' - x_1|$.
After n iterations, the distance evolves to $d_n = |x_n{}' - x_n|$.
The ratio $\frac{d_i}{d_{i-1}}$ describes the evolution of the error d_i in the i-th iteration, otherwise

$$\frac{d_1}{d_0} = \frac{|x_1{}' - x_1|}{|x_0{}' - x_0|} = \frac{|f(x_0{}') - f(x_0)|}{|x_0{}' - x_0|} = \frac{|f(x_0 + d_0) - f(x_0)|}{d_0}$$

and then $\lim_{d_0\to 0} \frac{d_1}{d_0} = \lim_{d_0\to 0} \frac{|f(x_0+d_0)-f(x_0)|}{d_0} = |f'(x_0)|$.

This quantity is positive, so there exists a real λ_1 such that

$$\lim_{d_0 \to 0} \frac{d_1}{d_0} = |f'(x_0)| = e^{\lambda_1}.$$

So the two trajectories diverge exponentially at the first iteration.

extract the value of $\lambda_1 = \ln |f'(x_0)|$.

The evolution of the error after n iteration:

$$\frac{d_n}{d_0} = \frac{|x_n{}' - x_n|}{|x_0{}' - x_0|} = \frac{|f(x_n{}') - f(x_n)|}{|x_0{}' - x_0|} = \frac{|f^n(x_0{}') - f^n(x_0)|}{d_0} = \frac{|f^n(x_0 + d_0) - f^n(x_0)|}{d_0}$$

When passing to the limit

$$\lim_{d_0 \to 0} \frac{d_n}{d_0} = \lim_{d_0 \to 0} \frac{|f^n(x_0 + d_0) - f^n(x_0)|}{d_0} = \frac{df^n}{dx}(x_0)$$

The error tends towards a limit $e^{\lambda n}$, then

$$\begin{aligned} \lim_{d_0 \to 0} \frac{d_n}{d_0} &= (e^{\lambda})^n = e^{n\lambda} \simeq \frac{df^n}{dx}(x_0) \\ &\Rightarrow \quad n\lambda \simeq \ln |\frac{df^n}{dx}(x_0)| \\ &\Rightarrow \quad \lambda \simeq \frac{1}{n} \ln \left| \frac{df^n}{dx}(x_0) \right| = \frac{1}{n} \ln \left| \frac{df(f^{n-1}(x_0))}{dx} \right| \end{aligned}$$

using the chain rule for differentiation $\lambda \simeq \frac{1}{n} \ln |f'(x_{n-1})||f'(x_{n-2})| \cdots |f'(x_0)| = \prod_{i=0}^{n-1} ||f'(x_i)$

when $n \longrightarrow \infty$:

$$\lambda = \lim_{n \to \infty} \frac{1}{n} \ln \left| \frac{df^n}{dx}(x_0) \right| = \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln |f'(x_i)| \,.$$

The quantity λ represents the Lyapunov exponent.

For an equilibrium point $\bar{x}$

$$\lambda = \ln |f'(\bar{x})|.$$

- If $|f'(\bar{x})| < 1$, then $\lambda < 0$, so $\bar{x}$ is asymptotically stable and the trajectory originating from an initial condition x_0 is asymptotically stable near x_0.

- If $|f'(\bar{x}))| = 1$, then $\lambda = 0$, so $\bar{x}$ is stable, and consequently the trajectory originating from x_0 is periodic, so stable.

- If $|f'(\bar{x})| > 1$, then $\lambda > 0$, so $\bar{x}$ is unstable as well as the trajectory originating from x_0. □

Example 2.2.1. *let* $f(x) = 2x$.

$$\begin{aligned} \lambda &= \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln|f'(x_i)| \\ &= \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln|2| \\ &= \ln 2 > 0 \end{aligned}$$

Therefore, the system is chaotic.

(ii) Case of multidimensional discrete systems :

Theorem 2.2.2. *[7] Let* $f : \mathbb{R}^p \to \mathbb{R}^p$ *be a discrete map such that* $x_{n+1} = f(x_n)$*. Then the* p *Lyapunov exponents are defined as follows:*

$$\lambda_i = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{p} \ln q_i(f^n(x_0)), i = 1, \cdots, p.$$

Where $q_i, i = 1, \cdots, p$ are the eigenvalues of the jacobian matrix $J^n(x_0)$.

Proof. Let's start by specifying that a p-dimensional system will have p Lyapunov exponents $\lambda_i, i = 1, 2, \ldots, p$. Each of them measures the divergence rate along one of the axes of the system. we have

$$\begin{aligned} \frac{d_n}{d_0} &= \frac{|x_n' - x_n|}{|x_0' - x_0|} = \frac{|f(x_n') - f(x_n)|}{|x_0' - x_0|} \\ &= \frac{|f^n(x_0') - f^n(x_0)|}{d_0} \\ &= \frac{|f^n(x_0 + d_0) - f^n(x_0)|}{d_0} \simeq (e^\lambda)^n \\ so \quad & f^n(x_0 + d_0) - f^n(x_0) \simeq d_0 e^{n\lambda}, \end{aligned}$$

applied the first-order Taylor expansion of the function $f_n(x_0)$ in the neighborhood of $x_0{}'$

$$\begin{aligned} x_n - x_n' &\simeq f^n(x_0) - f^n(x_0') = (x_0 - x_0')[\frac{df^n}{dx}(x_0)] \\ &= j(x_0)j(x_1)j(x_2)\cdots j(x_n) \times (x_0 - x_0{}') \\ &= \prod_{i=0}^{n} J(x_i) \times (x_0 - x_0'). \end{aligned}$$

We denote $\prod_{i=0}^{n} J(x_i)$ by $J^n(x_0)$. Then we obtain $x_n - x_n' = J^n(x_0) \times (x_0 - x_0')$, where $J^n(x_0) \in M_{n\times n}$ represents the Jacobian matrix of f^n at the point x_0.

If $J^n(x_0)$ is diagonalizable, then there exists an invertible matrix $P \in M_{n\times n}$ such that : $D_n(x_0) = P^{-1}J^n(x_0)P$.Where $D_n(x_0)$ is a diagonal matrix containing the eigenvalues of $J^n(x_0)$, then :

$$\lambda_i = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{p} \ln q_i(f^n(x_0)). \tag{2.4}$$

□

This equation gives an estimation of the Lyapunov exponents for multidimensional systems. The Lyapunov exponents are positive, zero as well as negative.

- If a Lyapunov exponent is strictly positive, then the system has a large sensitivity to initial conditions and is chaotic.

- If all Lyapunov exponents are negative or equal to zero , the system is stable or periodic.

Example 2.2.2. *The Skinny-Baker map $B(x, y)$ is defined by*

$$B(x,y) = \begin{cases} \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, & \text{if } 0 \le y \le \frac{1}{2}, \\ \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \frac{2}{3} \\ -1 \end{pmatrix}, & \text{if } \frac{1}{2} < y \le 1. \end{cases}$$

The Jacobian matrix of $B(x, y)$ at a point $(x, y) \in [0, 1] \times [0, 1]$ is given by

$$J = \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & 2 \end{pmatrix}.$$

After nth iterations, the Jacobian matrix is $J^n = \begin{pmatrix} \frac{1}{3^n} & 0 \\ 0 & 2^n \end{pmatrix}$. *The matrix* J^n *has two distinct eigenvalues, namely* $q_1(f^n(x_0)) = \frac{1}{3^n}$ *and* $q_2(f^n(x_0)) = 2^n$. *Therefore, the map has two distinct Lyapunov exponents given by*

$$\lambda_1 = \lim_{n\to\infty} \frac{1}{n} \ln\left(\frac{1}{3^n}\right) = -\ln 3 < 0.$$

$$\lambda_2 = \lim_{n\to\infty} \frac{1}{n} \ln(2^n) = \ln 2 > 0.$$

Hence, the Skinny-Baker map is chaotic.

3. **Fractal dimension :**

   Euclidean geometry is a magnificent concept to describe natural objects like roads, houses, books, etc. But the methods of Euclidean geometry and calculus are not suitable to describe all natural objects. There are objects, such as trees, coastlines, cloud boundaries, monument ranges, river meanders, coral structures, etc., for which the knowledge of Euclidean geometry is insufficient to describe them as they lacks order and are erratic in shape. Such objects are called fractal objects or simply fractals and are described by fractal geometry with fractional dimensions [21]. There are several types of fractal dimensions (capacity dimension, information dimension, correlation dimension, etc.) for chaotic attractors we can mention :

   Hausdorff dimension :

   Let A be the space. We cover the space A by means of a countable union of parts denoted by A_i, each of which has a diameter less than r. For any real non-negative number s, we consider the quantity $\sum_{i=1}^{\infty} |A_i|^s$. We introduce the quantity:

$$H_r^s(A) = \inf_{|A_i|<r} \{\sum_{i=1}^{\infty} |A_i|^s / A \subset \bigcup_{i=1}^{\infty} A_i\}$$

   where $|N_i|$ is the diameter of the non-empty set N_i.

   The function H_r^s is decreasing, which ensures the existence of a limit (possibly infinite) as

r tends to 0. Hence the definition:

$$H_s(X) = \lim_{r \to 0} H_r^s(X)$$

H^s is called the s-dimensional Hausdorff measure. The Hausdorff dimension relies on the Hausdorff measure. The Hausdorff dimension of $A \subset \mathbb{R}^n$ is defined by :

$$D_H = \sup\{s,\ H^s(A) = +\infty\} = \inf\{s,\ H^s(A) = 0\}$$

where $H^s(A)$ is the Hausdorff measure of the set A. This type of dimension depends solely on the metric properties of the space in which the set resides [8].

Lyapunov dimension :

The Lyapunov dimension is given by [27] :

$$D_L = \frac{\sum_{i=1}^{j} \lambda_i}{|\lambda_{j+1}|} + j$$

where $\lambda_n \leq \ldots \leq \lambda_1$ are the Lyapunov exponents of a dynamical system's attractor, and j is the great natural number such that $\sum_{i=1}^{j} \lambda_i \geq 0$. This type of dimension takes into account the system's dynamics.

Box dimension :

Let us consider a geometric object and ε be the length of cells which covers the space occupied by the object. The number $N(\varepsilon)$ is the minimum number of cells required to cover the space. Now, for a line segment of length L, $N(\varepsilon)$ is proportional to $\frac{L}{\varepsilon}$, and for a plane area A, $N(\varepsilon)$ is proportional to $\frac{A}{\varepsilon^2}$.

In general, we can take $N(\varepsilon)$ as $N(\varepsilon) = \frac{1}{\varepsilon^d}$.

Taking the logarithm of both sides, we get:

$$\begin{aligned} \log N(\varepsilon) &= -d \log \varepsilon \\ \Rightarrow d &= -\frac{\log N(\varepsilon)}{\log \varepsilon} \\ \Rightarrow d &= \frac{\log N(\varepsilon)}{\log(1/\varepsilon)} \end{aligned}$$

If the limit of the above expression exists for $\varepsilon \to 0$, then d is called the capacity dimension or the box dimension of the non-similar fractal. So, the box dimension (or capacity dimension) of a fractal is given by:

$$d = \lim_{\varepsilon \to 0} \frac{\log N(\varepsilon)}{\log(1/\varepsilon)}$$

The capacity or box dimension of a point in a two-dimensional space is 0, since in this case $N(\varepsilon) = 1$ for all ε [21].

2.2.3 Routes to chaos :

A route to chaos is a specific sequence of bifurcations leading from a completely predictable evolution (such as having a stable fixed point) to a chaotic evolution. A remarkable characteristic, discovered in the 1980s, is that these sequences are often qualitatively identical. This is known as the universality of routes to chaos. Three major scenarios of transition from regular dynamics to chaotic dynamics during the variation of a parameter have been identified :

Period-doubling route [34]: This route describes the passage from a situation where the system reaches an equilibrium state (a stable fixed point) to a chaotic regime through a sequence of period doublings. As the control parameter μ is increased, the fixed point is replaced in $\mu = \mu_0$ with a cycle having a certain period K (the fixed point still exists for $\mu > \mu_0$ but it is then unstable). Then, in $\mu = \mu_1$, this cycle looses its stability and is replaced by a stable cycle of period 2k . And so on, until the system becomes chaotic.

Intermittency [27]: A stable periodic motion is interspersed with bursts of turbulence. As the bifurcation parameter is increased, the bursts of turbulence become more frequent, and eventually, turbulence dominates.

The Ruelle-Takens scenario [27]: This scenario, via quasi-periodicity, was elucidated by the theoretical work of Ruelle and Takens. In a dynamical system exhibiting periodic behavior at a single frequency, if we change a parameter, a second frequency appears. If the ratio between the two frequencies is rational, the behavior is periodic. However, if the ratio is irrational, the behavior is quasi-periodic. Then, if we change the parameter again, a third frequency appears, and so on until chaos emerges.

CHAPTER 3

CHAOS CONTROL

Since the work of Ott, Grebogi, and York in 1990, there has been considerable interest in the control of chaotic systems [26]. Various studies on the subject have shown that a chaotic system can be controlled in several ways, including:

- By perturbing one of the system's parameters. This perturbation must be bounded and very small compared to the perturbed parameter.

- By stabilizing one of the system's unstable periodic orbits using a state feedback control method. This method is simple to implement but requires careful selection of gain and delay in the feedback loop.

3.1 Control methods

It is important to specify that the literature in this field is very extensive, and the methodologies described in this chapter represent only a part of it. we will introduce the principles of a few methods.

3.1.1 The OGY method

The name OGY comes from its inventors: Edward Ott, Celso Grebogi and James Yorke. In 1990 they published an article showing that it was possible to control chaos, and thereby being the first to achieve this with reasonable control efforts [26]. Ott, Grebogi and Yorke based their theory on recent articles showing that a chaotic attractor has a large number of unstable periodic orbits embedded within it. The essence of the OGY theory is simply to stabilize one (or more) of these orbits by applying small perturbations. To apply these perturbations one of the parameters of the system should be accessible, meaning that this parameter can be adjusted while the system is running. This parameter thus becomes the input of the system[36].

The Principle Of The Method :

Let F be a nonlinear dynamical system with chaotic behavior given by :

$$x_{n+1} = F(x_n, p), \tag{3.1}$$

where the vector x_n represents the system's state variables and p is a control parameter accessible from the outside for small adjustments.

Let x_f be an unstable fixed point embedded within the chaotic attractor.

$$x_f = F(x_f, p_0).$$

Since the system is ergodic the state x_i will come very close to this point at some point in time, while $p = p_0$.

Linearization of the system (3.1) around its fixed point is given by:

$$\delta x_{n+1} = A\delta x_n + B\delta p_n, \tag{3.2}$$

where :

$$A = D_x F(x) \quad / \quad B = \partial F/\partial p,$$

and :

$$\delta x_n = x_n - x_f \quad / \quad \delta p_n = p_n - p_0,$$

the Jacobian matrix (the matrix A) represents two eigendirections, one unstable (eigenvalue strictly greater than 1 in absolute value) and the other stable (eigenvalue strictly less than 1 in absolute value).

The corrections are to be applied to the unstable direction.

We introduce the following notations:

- λ_s: Eigenvalue $|\lambda_s| < 1$ (corresponding to the stable direction).
- λ_u: Eigenvalue $|\lambda_u| > 1$ (corresponding to the unstable direction).
- e_s: Eigenvector corresponding to the eigenvalue λ_s.
- e_u: Eigenvector corresponding to the eigenvalue λ_u.

Therefore, we can express the matrix A in the form:

$$A = \lambda_s e_s f_s + \lambda_u e_u f_u.$$

Where f_s and f_u represent covariance vectors.

and :

$$\begin{aligned} f_s e_s &= f_u e_u = 1, \\ f_s e_u &= f_u e_s = 0, \end{aligned}$$

so the equation (3.2) becomes :

$$\delta x_{n+1} = (\lambda_u e_u f_u + \lambda_s e_s f_s)\delta x_n + B\delta p_n \tag{3.3}$$

By multiplying (3.3) by f_u:

$$f_u \delta x_{n+1} = f_u [(\lambda_u e_u f_u + \lambda_s e_s f_s)\delta x_n + B\delta p_n]. \tag{3.4}$$

The OGY method strategy involves adjusting the control parameter p to stabilize the system at the point f_u. In other words, it is necessary for $\delta x_{n+1} = 0$.

so :

$$f_u [(\lambda_u e_u f_u + \lambda_s e_s f_s)\delta x_n + B\delta p_n] = 0, \tag{3.5}$$

and we have :

$$\begin{aligned} f_s e_s &= f_u e_u = 1, \\ f_s e_u &= f_u e_s = 0, \end{aligned}$$

so :

$$f_u \lambda_u \delta x_n + f_u B \delta p_n = 0. \tag{3.6}$$

The adjustment to the control parameter is given by:

$$\delta p_n = \frac{-f_u \lambda_u}{f_u B}\delta x_n = -K\delta x_n. \tag{3.7}$$

The perturbation of p is assumed to be small so we have the following condition :

$$|p_n - p_0| < \epsilon,$$

where ϵ : a parameter which determines the neighborhood of x_f.
so we can write :

$$|K\delta x_n| < \epsilon.$$

The control increment is thus given by:

$$\delta p_n = \begin{cases} -k(x_n - x_f), & \text{if} \quad |k(x_n - x_f)| < \epsilon. \\ 0 & elsewhere. \end{cases}$$

- **Example of control using the OGY method:**

Let's take as an example the control of the Hénon system using the OGY method.
The Hénon system is described by:

$$\begin{cases} x_{n+1} = a - x_n^2 + by_n, \\ y_{n+1} = x_n. \end{cases}$$

Where a and b represent the control parameters.

- **Fixed points** : let's set $x_{n+1} = x_n$ and $y_{n+1} = y_n$, we obtain:

$$\begin{cases} x_f = a - x_f^2 + by_f, \\ y_f = x_f, \end{cases}$$

so :

$$x_f = y_f = -\frac{(1-b)}{2} \pm \sqrt{\frac{(1-b)^2}{4} + a},$$

we set : $c = \frac{1-b}{2}$,
we obtain :

$$x_f = y_f = -c \pm \sqrt{c^2 + a}.$$

- **Application of the control algorithm :**

  The control algorithm is applied to the system (with $a = 1.4$ and $b = 0.3$ to ensure the presence of chaotic behavior).

  Control using the OGY method involves performing the following operations :

  1. **Identification of periodic orbit to stabilize:**

     we substitute a and b into equation (3.1.1), we obtain:

     $$x_{f_1} = y_{f_1} = 0.8839,$$

     $$x_{f_2} = y_{f_2} = -1.5839,$$

     In our case, we choose the point x_{f_1}.

  2. **Calculation of matrices A and B:**

     We have $A = D_x F(x)$ and $B = \partial F / \partial p$ (assuming a as the accessible parameter).

     $$A = \begin{pmatrix} -2x_{f_1} & b \\ 1 & 0 \end{pmatrix}, \quad B_a = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

     we have $x_{f_1} = 0.8839$,

     so :

     $$A = \begin{pmatrix} -1.7678 & 0.3 \\ 1 & 0 \end{pmatrix}.$$

  3. **Calculation of eigenvalues λ_u and λ_s:**

     The eigenvalues λ_u and λ_s are defined by:

     $$\begin{aligned} \lambda_{u\,,\,s} &= \frac{1}{2}(\tau \pm \sqrt{\tau^2 - 4\triangle}) \\ &= -x_{f1} \pm \sqrt{x_{f1}^2 + b}. \end{aligned}$$

     So:

     $$\lambda_s = 0.1559 \quad and \quad \lambda_u = -1.9237.$$

4. **Calculation of eigenvectors $\{e_u, e_s\}$ and the covariance vectors $\{f_u, f_s\}$:**

   The eigenvectors are calculated from the following equation:

$$[\lambda I - A]e = 0.$$

   The eigenvector is chosen in the form:

$$e = \begin{pmatrix} \lambda \\ 1 \end{pmatrix}, \quad \text{With:} \quad e_s = \begin{pmatrix} \lambda_s \\ 1 \end{pmatrix} \quad \text{and} \quad e_u = \begin{pmatrix} \lambda_u \\ 1 \end{pmatrix}.$$

   So:

$$e_s = \begin{pmatrix} 0.1559 \\ 1 \end{pmatrix} \quad \text{and} \quad e_u = \begin{pmatrix} -1.9237 \\ 1 \end{pmatrix}.$$

   Knowing that: $f_s e_s = f_u e_u = 1$ and $f_s e_u = f_u e_s = 0$

   This gives:

$$f_s = \begin{pmatrix} \frac{1}{\lambda_s - \lambda_u} & \frac{\lambda_u}{\lambda_u - \lambda_s} \end{pmatrix} \quad and \quad f_u = \begin{pmatrix} \frac{1}{\lambda_u - \lambda_s} & \frac{\lambda_s}{\lambda_s - \lambda_u} \end{pmatrix},$$

$$f_s = \begin{pmatrix} 0.4808 & 0.9250 \end{pmatrix} \quad and \quad f_u = \begin{pmatrix} -0.4787 & 0.0746 \end{pmatrix}.$$

5. **Calculation of k :**

   The parameter k is determined by:

$$k = \frac{\lambda_u f_u}{f_u B} = \frac{\lambda_u \begin{pmatrix} \frac{1}{\lambda_u - \lambda_s} & \frac{\lambda_s}{\lambda_s - \lambda_u} \end{pmatrix}}{\begin{pmatrix} \frac{1}{\lambda_u - \lambda_s} & \frac{\lambda_s}{\lambda_s - \lambda_u} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}} = \begin{pmatrix} \lambda_u - \lambda_u \lambda_s \end{pmatrix},$$

$$k = \begin{pmatrix} -1.9237 & 0.3011 \end{pmatrix}.$$

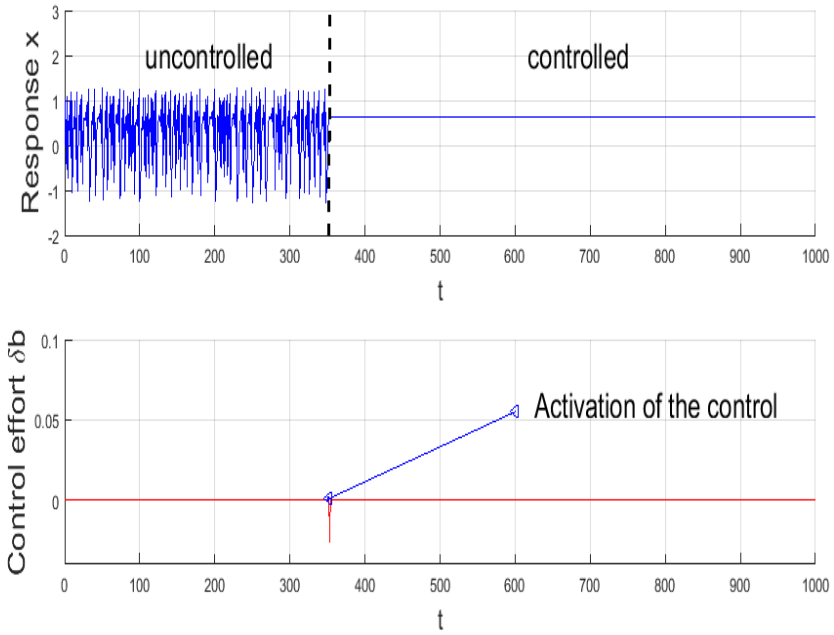


Figure 3.1: Control of the Hénon System using the OGY Method

3.1.2 The closed-loop control method (feedback) :

Feedback controls are used in many aspects of our lives, from the braking system of a car to central air conditioning. The method has been used by engineers for many years. However, the systematic study of stabilization by state feedback control is of more recent origin and dates to the 1960s. The idea of state feedback is simple: This method involves perturbing the system state variables x_n and the control u_n is adjusted based on this information to reach the target orbit. It has the advantage of ensuring robust stability and strong noise rejection capability. Generally, it is formulated as follows [13, 20]:

$$x_{n+1} = f(x_n, u_n), \tag{3.8}$$

where $f : \mathbb{R}^k \times \mathbb{R}^m \to \mathbb{R}^k$. The objective is to find a feedback control

$$u_n = h(x_n),$$

in such a way that the equilibrium point $\bar{x} = 0$ of the closed-loop system

$$x_{n+1} = f(x_n), h(x_n)),$$

is locally asymptotically stable. We make the following assumptions:
(i) $f(0, 0) = 0$
(ii) f is continuously differentiable, $A = \frac{\partial f}{\partial x}(0, 0)$ is a $k \times k$ matrix, $B = \frac{\partial f}{\partial u}(0, 0)$ is a $k \times m$ matrix.
Under the above conditions, we have the following surprising result.

Theorem 3.1.1. *If the pair $\{A, B\}$ is controllable, then the nonlinear system (3.8) is stabilizable. Moreover, if K is the gain matrix for the pair $\{A, B\}$, then the control $u(n) = -Kx(n)$ may be used to stabilize system (3.8).*

Proof. Since the pair $\{A, B\}$ is controllable, there exists a feedback control $u_n = -Kx_n$ that stabilizes the linear part of the system, namely,

$$y_{n+1} = Ay_n + Bv_n.$$

We are going to use the same control on the nonlinear system. So let $g : \mathbb{R}^k \to \mathbb{R}^k$ be a function defined by $g(x) = f(x, -Kx)$. Then system equation (3.8) becomes

$$x_{n+1} = g(x_n) \tag{3.9}$$

with

$$\left.\frac{\partial g}{\partial x}\right|_{x=0} = A - BK.$$

Since by assumption the zero solution of the linearized system

$$y_{n+1} = (A - BK)y_n,$$

is asymptotically stable, it follows that the zero solution of system (3.9) is also asymptotically stable. This completes the proof of the theorem. □

CHAPTER 4

COMPLEXITY ANALYSIS OF A 2D-PIECEWISE SMOOTH DUOPOLY MODEL

Recently, many countries have adopted the remanufacturing process in order to promote their economies [33]. This process requires the remanufacturing of used or default products, as remanufacturing is considered a friendly environmental process [25]. Indeed, the remanufacturing process may be beneficial for some companies to increase their profits, but it may be worse for other companies, especially those companies that produce new products [5]. The USA Economy magazine reveals the fact that original product manufacturers face a competitive threat from remanufacturer companies [32]. Such competition between those companies and its complex dynamic characteristics can be described and investigated by duopoly games [6]. Competition in duopoly games includes only two competitors; it has also been shown that even oligopolistic markets may become chaotic under certain conditions [2]. firms whose strategies may be quantities (as in Cournot) or prices (as in Bertrand). In a duopoly game, the behavior of each firm is closely intertwined with that of the other, as their strategic decisions such as pricing, output levels, and marketing strategies directly impact each other's outcomes and positions in the market [29]. Recently, it has also been shown that even the Cournot model markets may become chaotic under certain conditions [2]. The earliest oligopoly model was expectations del proposed by Cournot in 1838, in which the two firms competed in output and were treated with naïve expectation [9]. We considered that each player forms a different strategy in order to compute its expected output. We assume that the first player represents a boundedly rational player and the second player has local approximations [14]. The main aim of this work is to investigate the dynamic behaviors of the two players using different expectation rules.

4.1 Model building

in [32] the authors considered an economic market populated by two competed firms. the first firm is called a manufacturer and supports the market with new products, while the second firm is called a third-party remanufacturer and supports the market with differentiated remanufactured products. Customers distinguish between these two types of products based on their willingness to pay, which varies between new and remanufactured products. the demand productions sent to the market by firms are denoted by x_1 and x_2. the competition between these firms is carried out in discrete time periods, $t = 0, 1, 2, \ldots$ the first firm sends the new quantity x_1 for selling in the market at time t, while the second firm can receive returned quantity x_2 for remanufacturing and sells it again in the market at time $t + 1$ [5].The pricing equations for the new product x_1 and the remanufactured product x_2 are given by:

$$\begin{aligned} p_1 &= 1 - x_1 - \delta x_2 \\ p_2 &= \delta(1 - x_1 - x_2). \end{aligned} \quad (4.1)$$

Where δ is a parameter representes the valuation of the remanufactured products relative to the new products, and it has an important meaning in this game. if $\delta = 0$ customers are not willing to pay anything for the remanufactured product, if $\delta = 1$ customers are willing to pay the same amount for new and remanufactured products. From an economic perspective, this may not be approved. Because of the variety of customers, we restrict this parameter to $\delta \in (0, 1)$. Assuming that $C_i(x_i)$ refers to the cost of the quantity x_i and is given by the following linear form:

$$C_i(x_i) = c_i x_i, \quad i = 1, 2$$

where c_1 and c_2 refer to the marginal costs for both firms respectively. therefore, the profits of both firms are given as follows:

$$\begin{aligned} \pi_1 &= (1 - x_1 - \delta x_2 - c_1)x_1, \\ \pi_2 &= (\delta(1 - x_1 - x_2) - c_2)x_2. \end{aligned} \quad (4.2)$$

And their marginal functions become as follows:

$$\begin{aligned} \frac{\partial \pi_1}{\partial x_1} &= 1 - 2x_1 - \delta x_{2,t} - c_1, \\ \frac{\partial \pi_2}{\partial x_2} &= \delta(1 - x_1 - 2x_2) - c_2. \end{aligned} \quad (4.3)$$

Information in the game generally is deficient, so firms may utilize more complex strategies, for example, bounded rationality method. So we assume that the two firms are heterogeneous and adopt different adjustment mechanisms in order to update their productions. We presume that the first firm will behave as a bounded rational firm and consequently will maximize its profit. Firms with bounded rationality do not have the total information of the game, thus, the settling yield choices depend on a local estimate of the marginal profit. It is easy to see that for $\delta \in (0, 1)$

and $c_1 > c_2$, the marginal profit $\frac{\partial \pi_1}{\partial x_1}$ is always positive and lies within the first quadrant. When companies make use of this type of adjustments, they are to be rational players. To describe such firm's behavior based on this reasoning the first firm will update its output at period $t+1$ according to the following form [5, 32]:

$$x_{1,t+1} = x_{1,t} + kx_{1,t}\frac{\partial \pi_1}{\partial x_1} \tag{4.4}$$

where k is a positive parameter which represents the relative speed of output adjustment. On the other hand, we assume that the second firm seeks to share the market with a certain profit. It starts with assuming that it seeks a complete market share maximization. Its profit becomes zero ($\pi_2 = 0$). and then its optimum output becomes as follows:

$$\bar{x}_2 = 1 - x_1 - \frac{c_2}{\delta}. \tag{4.5}$$

But when it completely seeks profit maximization, its marginal profit will vanish and then we have the following:

$$\hat{x}_2 = \frac{1}{2}\left(1 - x_1 - \frac{c_2}{\delta}\right). \tag{4.6}$$

According to some weights, the second firm will be traded off between market share and profit as follows:

$$\begin{aligned} \tilde{x}_2 &= \omega\bar{x}_2 + (1-\omega)\hat{x}_2 \\ &= \frac{1+\omega}{2}\left(1 - x_1 - \frac{c_2}{\delta}\right). \end{aligned} \tag{4.7}$$

Where $\omega \in (0,1)$ the attitude between the profit and market share. When $\omega = 0$ it means the second firm seeks profit maximization only, while $\omega = 1$ that means it seeks market share maximization. But as it trades off between both, we have restricted the parameter ω to the interval $(0,1)$. Now we assume this firm updates its output according to the following adaptive mechanism:

$$x_{2,t+1} = (1-\beta)x_{2,t} + \beta\tilde{x}_{2,t} \tag{4.8}$$

where β is a positive parameter representes the speed of adjustment and is restricted $\beta \in (0,1)$. Since the output of the second firm at time $t+1$ should be less than or equal to those of the first firm at time t ($x_{2,t+1} \leq x_{1,t}$), therefore the equation (4.8) will be modified as follows:

$$x_{2,t+1} = \min\{(1-\beta)x_{2,t} + \beta\tilde{x}_2 \ , \ x_{1,t}\}, \tag{4.9}$$

Using (4.4) and (4.9), we will define the map that describes this game as follows:

$$(x_{1,t+1}, x_{2,t+1}) \;=\; F(x_{1,t}, x_{2,t}) = \begin{cases} x_{1,t} + kx_{1,t}(1 - 2x_1 - \delta x_{2,t} - c_1), \\ \min\{(1-\beta)x_{2,t} + \beta\tilde{x}_2 \ , \ x_{1,t}\}, \end{cases} \tag{4.10}$$

the map (4.10) is a two-dimensional piecewise smooth map and is constructed to describe the proposed duopoly game in this work. In order to study the stability of its fixed points we should study it as a piecewise-smooth map. the stability of this map depends on the second equation where the function $f = (1-\beta)x_{2,t} + \beta\tilde{x}_2$ exists. It means that any order pair (x_1, x_2) belongs to $f < x_1$ gives only one part of the map to be dynamically studied, and while $f > x_1$ the other part of the map should be studied. Hence, we get the fact that there is a borderline $f = x_1$ where the map is continuous. Furthermore, the map's phase plane will be divided by the border line into two regions (Left regio R_l and right region R_r) [5]. this borderline takes the following form:

$$x_1 = \frac{1}{2+\beta(1+\omega)}\left(2(1-\beta)x_{2,t} + \frac{\beta}{\delta}(1+\omega)(\delta - c_2)\right) = g(x_2). \tag{4.11}$$

So the map (4.10) is modified to the following:

$$\begin{aligned} x_{1,t+1} &= x_{1,t} + kx_{1,t}(1 - 2x_{1,t} - \delta x_{2,t} - c_1), \\ x_{2,t+1} &= \begin{cases} (1-\beta)x_{2,t} + \frac{1}{2}\beta(1+\omega)(1 - x_{1,t} - \frac{c_2}{\delta}), & \text{if } x_{1,t} \geq g(x_2), \\ x_{1,t}, \qquad \text{if } \; x_{1,t} \leq g(x_2). \end{cases} \end{aligned} \tag{4.12}$$

In the next sections, we will study the dynamical behaviors of the map (4.12).

4.2 Stability analysis

Consider the following discrete dynamical system:

$$(x_{1,t+1}, x_{2,t+1}) \quad = \quad F(x_{1,t}, x_{2,t}) = \begin{cases} x_{1,t} + kx_{1,t}(1 - 2x_1 - \delta x_{2,t} - c_1), \\ \min\{(1-\beta)x_{2,t} + \frac{1}{2}\beta(1+\omega)(1 - x_{1,t} - \frac{c_2}{\delta})\ ,\ x_{1,t}\}. \end{cases} \tag{4.13}$$

Fixed points

To study the system (4.13) we distinguish two cases:

(a) If $x_{1,t} \leq g(x_2)$ (**Region** R_l)

The system (4.13) is written as:

$$\begin{cases} x_{1,t+1} & = x_{1,t} + kx_{1,t}(1 - 2x_1 - \delta x_{2,t} - c_1), \\ x_{2,t+1} & = x_{1,t}. \end{cases} \tag{4.14}$$

The fixed points are solutions of the following equations:

$$\begin{cases} x_{1,t+1} = x_{1,t}, \\ x_{2,t+1} = x_{2,t}, \end{cases}$$

we obtain :

$$\begin{cases} x_{1,t} + kx_{1,t}(1 - 2x_{1,t} - \delta x_{2,t} - c_1) = x_{1,t}, \\ x_{1,t} = x_{2,t}, \end{cases}$$

so we have :

$$\begin{cases} kx_{1,t}(1 - 2x_{1,t} - \delta x_{2,t} - c_1) = 0, \\ x_{1,t} = x_{2,t}, \end{cases}$$

according to the first equation we obtain :

$kx_{1,t} = 0 \quad so \quad x_{1,t} = x_{2,t} = 0 \quad (k > 0)$

$or \quad 1 - (\delta + 2)x_{1,t} - c_1 = 0 \quad so \quad x_{1,t} = x_{2,t} = \frac{1-c_1}{\delta+2}$

so we have two fixed points :

$$\begin{cases} E_1 = (0,0), \\ E_l = (\frac{1-c_1}{\delta+2}, \frac{1-c_1}{\delta+2}). \end{cases}$$

(b) If $x_{1,t} \geq g(x_2)$ (**Region** R_r)

The system (4.13) is written as:

$$\begin{cases} x_{1,t+1} &= x_{1,t} + kx_{1,t}(1 - 2x_1 - \delta x_{2,t} - c_1), \\ x_{2,t+1} &= (1-\beta)x_{2,t} + \frac{\beta(1+\omega)}{2}(1 - x_{1,t} - \frac{c_2}{\delta}). \end{cases} \tag{4.15}$$

The fixed points are solutions of the following equations:

$$\begin{cases} x_{1,t+1} = x_{1,t}, \\ x_{2,t+1} = x_{2,t}, \end{cases}$$

we obtain :

$$\begin{cases} x_{1,t} + kx_{1,t}(1 - 2x_{1,t} - \delta x_{2,t} - c_1) = x_{1,t}, \\ (1-\beta)x_{2,t} + \frac{1}{2}\beta(1+\omega)(1 - x_{1,t} - \frac{c_2}{\delta}) = x_{2,t}, \end{cases}$$

so we have :

$$\begin{cases} kx_{1,t}(1 - 2x_{1,t} - \delta x_{2,t} - c_1) = 0, \\ -\beta x_{2,t} + \frac{1}{2}\beta(1+\omega)(1 - x_{1,t} - \frac{c_2}{\delta}) = 0, \end{cases}$$

according to the first equation we obtain :

$kx_{1,t} = 0 \quad so \quad x_{1,t} = 0 \quad (k > 0),$

$or \quad 1 - 2x_{1,t} - \delta x_{2,t} - c_1 = 0 \quad so \quad x_{1,t} = \frac{1-\delta x_{2,t} - c_1}{2}.$

For $x_{1,t} = 0$ **:**

according to the second equation we have $x_{2,t} = \frac{\beta(1+\omega)(1-\frac{c_2}{\delta})}{2\beta}$.

For $x_{1,t} = \frac{1-\delta x_{2,t}-c_1}{2}$ **:**

according to the second equation we have :

$-\beta x_{2,t} + \frac{1}{2}\beta(1+\omega)(1 - \frac{1-\delta x_{2,t}-c_1}{2} - \frac{c_2}{\delta}) = 0.$

$-\beta x_{2,t} + \frac{1}{2}\beta(1+\omega)(\frac{\delta x_{2,t}}{2}) + \frac{1}{2}\beta(1+\omega)(1 - \frac{1-c_1}{2} - \frac{c_2}{\delta}) = 0.$

$$x_{2,t}\underbrace{(\frac{\beta\delta(1+\omega)}{4}-\beta)}_{B}=\underbrace{-\frac{1}{2}\beta(1+\omega)(1-\frac{1-c_1}{2}-\frac{c_2}{\delta})}_{A},$$

where :

$$\begin{aligned} A &= -\frac{1}{2}\beta(1+\omega)(\frac{\delta(1+c_1)-2c_2}{2\delta})=\frac{\beta(1+\omega)[\delta(-1-c_1)+2c_2]}{4\delta}, \\ B &= \frac{\beta[\delta(1+\omega)-4]}{4}, \end{aligned}$$

so we obatin :

$$\begin{aligned} x_{2,t} &= \frac{\beta(1+\omega)[\delta(-1-c_1)+2c_2]}{4\delta}\times\frac{4}{\beta[\delta(1+\omega)-4]} \\ &= \frac{(1+\omega)[\delta(1+c_1)-2c_2]}{[4-(1+\omega)\delta]\delta}. \end{aligned}$$

We substitute the $x_{2,t}$ into $x_{1,t}$:

$$\begin{aligned} x_{1,t} &= \frac{1-\delta x_{2,t}-c_1}{2}=\frac{1-\frac{(1+\omega)[\delta(1+c_1)-2c_2]}{4-(1+\omega)\delta}-c_1}{2} \\ &= \frac{1}{2[4-(1+\omega)\delta]}[4-(1+\omega)\delta-(1+\omega)[(1+c_1)\delta-2c_2]-c_1[4-(1+\omega)\delta]] \\ &= \frac{1}{2[4-(1+\omega)\delta]}[(1+\omega)[-\delta-(1+c_1)\delta+2c_2c_1\delta]+4-4c_1] \\ &= \frac{1}{2[4-(1+\omega)\delta]}[(1+\omega)[\delta(-1-1-c_1+c_1)+2c_2]+4(1-c_1)] \\ &= \frac{1}{2[4-(1+\omega)\delta]}[2(1+\omega)(c_2-\delta)+4(1-c_1)] \\ &= \frac{(1+\omega)(c_2-\delta)+2(1-c_1)}{[4-(1+\omega)\delta]} \\ x_{1,t} &= \frac{2(1-c_1)-(1+\omega)(\delta-c_2)}{[4-(1+\omega)\delta]}, \end{aligned}$$

so we have two fixed points :

$$\begin{cases} E2=\left(0,\frac{\beta(1+\omega)(1-\frac{c_2}{\delta})}{2\beta}\right), \\ E_r=\left(\frac{2(1-c_1)-(1+\omega)(\delta-c_2)}{4-(1+\omega)\delta},\frac{(1+\omega)[\delta(1+c_1)-2c_2]}{[4-(1+\omega)\delta]\delta}\right). \end{cases}$$

The first point $E_2 = \left(0, \frac{\beta(1+\omega)(1-\frac{c_2}{\delta})}{2\beta}\right)$ is rejected because $x_{1,t} \geq g(x_2)$

and

$E_r = \left(\frac{2(1-c_1)-(1+\omega)(\delta-c_2)}{4-(1+\omega)\delta}, \frac{(1+\omega)[\delta(1+c_1)-2c_2]}{[4-(1+\omega)\delta]\delta}\right)$ is called the Nash equilibrium point of the game provided that :

$$\begin{cases} 2-(1+\omega)\delta-2c_1+(1+\omega)c_2 > 0, \\ \delta+\delta c_1-2c_2 > 0. \end{cases}$$

It can be reduced to:

$$\frac{2c_2}{1+c_1} < \delta < \frac{2(1-c_1)}{1+\omega} + c_2. \tag{4.16}$$

Stability

- **Stability of the points E_1 and E_l :**

To study the stability of E_1 and E_l, we use the Jacobian matrix of the map (4.14) :

$$J_l = \begin{pmatrix} 1+k(1-4x_{1,t}-\delta x_{2,t}-c_1) & -\delta k x_{1,t} \\ 1 & 0 \end{pmatrix}.$$

Stability of the point $E_1 = (0,0)$

The Jacobian matrix at The boundary equilibrium $E_1 = (0,0)$ takes the form :

$$J(E_1) = \begin{pmatrix} 1+k(1-c_1) & 0 \\ 1 & 0 \end{pmatrix},$$

which gives two eigenvalues :

$$\begin{aligned} \lambda_1 &= 1+k(1-c_1), \\ \lambda_2 &= 0. \end{aligned}$$

We have $|\lambda_1| > 1$ due to $1-c_1 > 0$, so the boundary equilibrium point E_1 is unstable (saddle point).

Stability of the point $E_l = (\frac{1-c_1}{\delta+2}, \frac{1-c_1}{\delta+2})$

the Jacobian matrix of the map (4.14) at this point can be given by :

$$\begin{aligned}
J(E_l) &= \begin{pmatrix} 1 + k(1 - (4+\delta)\frac{\alpha_1}{2+\delta} - c_1) & -\delta k \frac{\alpha_1}{2+\delta} \\ 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 1 + k\frac{2+\delta-(4+\delta)\alpha_1 - c_1(2+\delta)}{2+\delta} & -\frac{\delta\alpha_1}{2+\delta}k \\ 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 1 + k\frac{(2+\delta)\alpha_1-(4+\delta)\alpha_1}{2+\delta} & -\frac{\delta\alpha_1}{2+\delta}k \\ 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 1 + k\frac{-2\alpha_1}{2+\delta} & -\frac{\delta\alpha_1}{2+\delta}k \\ 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 1 - \frac{2\alpha_1}{2+\delta}k & -\frac{\delta\alpha_1}{2+\delta}k \\ 1 & 0 \end{pmatrix}.
\end{aligned}$$

The characteristic polynomial is given by:

$$P_1(\lambda) = \lambda^2 - \mathrm{Tr}(J(E_l))\lambda + \mathrm{Det}(J(E_l)),$$

where $\mathrm{Tr}(J(E_l))$ and $\mathrm{Det}(J(E_l))$ are respectively the trace and determinant of the Jacobian matrix, given by:

$$\begin{aligned}
tr(J(E_l)) &= 1 - \frac{2\alpha_1}{2+\delta}k, \\
Det(J(E_l)) &= \frac{\delta\alpha_1}{2+\delta}k,
\end{aligned}$$

then we have :

$$\begin{aligned}
\Delta_l &= tr(J(E_l))^2 - 4Det(J(E_l)) \\
&= (1 - \frac{2\alpha_1}{\delta+2}k)^2 - 4\frac{\delta\alpha_1}{\delta+2}k,
\end{aligned}$$

and the eigenvalues are as follows:

$$\begin{aligned}
\lambda_{1l,2l} &= \frac{1}{2}\left(tr(J(E_l)) \pm \sqrt{tr(J(E_l))^2 - 4Det(J(E_l))}\right) \\
&= \frac{1}{2}\left(1 - \frac{2\alpha_1}{2+\delta}k \pm \sqrt{(1 - \frac{2\alpha_1}{\delta+2}k)^2 - 4\frac{\delta\alpha_1}{\delta+2}k}\right) \\
&= \frac{1}{2}\left(1 - \frac{2\alpha_1}{2+\delta}k \pm \sqrt{1 + 4(\frac{\alpha_1}{\delta+2})^2 k^2 - \frac{4\alpha_1}{\delta+2}l - \frac{4\delta\alpha_1}{\delta+2}k}\right) \\
&= \frac{1}{2}\left(1 - \frac{2\alpha_1}{2+\delta}k \pm \sqrt{1 + 4(\frac{\alpha_1}{\delta+2})^2 k^2 - \frac{4\alpha_1(1+\delta)}{\delta+2}k}\right) \\
\lambda_{1l,2l} &= \frac{1}{2} - \frac{\alpha_1}{\delta+2}k \pm \sqrt{\frac{1}{4} + (\frac{\alpha_1}{\delta+2})^2 k^2 - \frac{\alpha_1(1+\delta)}{\delta+2}k}.
\end{aligned}$$

Thus, the eigenvalues are real or complex conjugated. The local stability of Nash equilibrium is given by using Jury's conditions. Furthermore, the stability region of the fixed point E_l is given by the following:

$$S_l = \{1 + tr(J(E_l)) + Det(J(E_l)) > 0,\ 1 - tr(J(E_l)) + Det(J(E_l)) > 0,\ 1 - Det(J(E_l)) > 0\}, \tag{4.17}$$

where :

$$\begin{aligned}
1 + tr(J(E_l)) + Det(J(E_l)) &= 1 + 1 - \frac{2\alpha_1}{\delta+2}k + \frac{\delta\alpha_1}{\delta+2}k \\
&= 2 - \frac{2\alpha_1 - \delta\alpha_1}{\delta+2}k \\
&= 2 - \frac{\alpha_1(2-\delta)}{\delta+2}k, \\
1 - tr(J(E_l)) + Det(J(E_l)) &= 1 - 1 + \frac{2\alpha_1}{\delta+2}k + \frac{\delta\alpha_1}{\delta+2}k \\
&= \frac{\alpha_1(2+\delta)}{2+\delta}k \\
&= \alpha_1 k, \\
1 - Det(J(E_l)) &= 1 - \frac{\delta\alpha_1}{\delta+2}k.
\end{aligned}$$

For the special case of the Jacobian matrix $J(E_l)$, the stability conditions in (4.17) can be written as follows:

$$\begin{cases} (1) : 2 - \frac{\alpha_1(2-\delta)}{\delta+2}k > 0, \\ (2) : \alpha_1 k > 0, \\ (3) : 1 - \frac{\delta\alpha_1}{\delta+2}k > 0, \end{cases} \tag{4.18}$$

the first condition is $2 - \frac{\alpha_1(2-\delta)}{\delta+2}k > 0$ which implies that :

$$k < \frac{2(2+\delta)}{\alpha_1(2-\delta)}, \tag{4.19}$$

the second condition is $\alpha_1 k > 0$, then the second condition is satisfied.

Then the third condition $1 - \frac{\delta\alpha_1}{\delta+2}k > 0$. This inequality is equivalent to :

$$k < \frac{2+\delta}{\delta\alpha_1}, \tag{4.20}$$

Lemma 4.2.1. *From (4.19) and (4.20) we obtain :*

if *$\Delta_l > 0$ the Nash equilibrium E_l of the map (4.14) is asymptotically stable provided that $k < \frac{2(2+\delta)}{\alpha_1(2-\delta)}$ and the system (4.14) undergoes a flip bifurcation at E_l when $k = \frac{2(2+\delta)}{\alpha_1(2-\delta)}$. Moreover, period-2 points bifurcate from E_l when $k > \frac{2(2+\delta)}{\alpha_1(2-\delta)}$.*

if *$\Delta_l < 0$ the Nash equilibrium E_l of the map (4.14) is asymptotically stable provided that $k < \frac{2+\delta}{\delta\alpha_1}$ and the system (4.14) undergoes a Neimark-Sacker bifurcation at E_l when $k = \frac{2+\delta}{\delta\alpha_1}$.*

- **Stability of the point** E_r :

In order to investigate the local stability of the equilibrium points E_r, we need the Jacobian matrix of the map (4.15) at this point :

$$J_r = \begin{pmatrix} A & -\delta k x_{1,t} \\ -\frac{1}{2}\beta(1+\omega) & 1-\beta \end{pmatrix}$$

where :

$$\begin{aligned}
A &= 1 + k(1 - 2x_{1,t} - \delta x_{2,t} - c_1) - 2kx_{1,t} \\
&= 1 + k(1 - \frac{8\alpha_1 - 4\alpha_2\alpha_3}{4 - \alpha_2\delta} - \frac{\alpha_2((1+c_1)\delta - 2c_2)}{4 - \alpha_2\delta} - c_1) \\
&= 1 + k(1 - \frac{8\alpha_1 - 4\alpha_2\alpha_3 + \alpha_2((1+c_1)\delta - 2c_2)}{4 - \alpha_2\delta} - c_1) \\
&= 1 + k(\frac{4 - \alpha_2\delta - 8\alpha_1 + 4\alpha_2\alpha_3 - \alpha_2((1+c_1)\delta - 2c_2) - c_1(4 - \alpha_2\delta)}{4 - \alpha_2\delta}) \\
&= 1 + k(\frac{\alpha_2(-\delta + 4\alpha_3 - (1+c_1)\delta + 2c_2 + c_1\delta) + 4 - 8\alpha_1 - 4c_1}{4 - \alpha_2\delta}) \\
&= 1 + k(\frac{\alpha_2(4\alpha_3 + \delta(-1 - 1 - c_1 + c_1) + c_2) + 4\alpha_1 - 8\alpha_1}{4 - \alpha_2\delta}) \\
&= 1 + k(\frac{\alpha_2(4\alpha_3 - 2\alpha_3) - 4\alpha_1}{4 - \alpha_2\delta}) = 1 + k\frac{2\alpha_2\alpha_3 - 4\alpha_1}{4 - \alpha_2\delta} \\
A &= 1 - \frac{4\alpha_1 - 2\alpha_2\alpha_3}{4 - \alpha_2\delta}k,
\end{aligned}$$

so we have :

$$J(E_r) = \begin{pmatrix} 1 - \frac{4\alpha_1 - 2\alpha_2\alpha_3}{4 - \alpha_2\delta}k & -\frac{2\alpha_1 - \alpha_2\alpha_3}{4 - \alpha_2\delta}\delta k \\ -\frac{\alpha_2\beta}{2} & \alpha_4 \end{pmatrix}, \tag{4.21}$$

$$\text{such :} \begin{cases} \alpha_1 = 1 - c_1, \\ \alpha_2 = 1 + \omega, \\ \alpha_3 = \delta - c_2, \\ \alpha_4 = 1 - \beta. \end{cases}$$

Eigenvalues for the above Jacobian matricx take the form :

$$\lambda_{1,2} = \frac{1}{2}\left(tr(J(E_r)) \pm \sqrt{tr(J(E_r))^2 - 4Det(J(E_r))}\right),$$

where :

$$\begin{aligned}
tr(J(E_r)) &= 1 - \frac{4\alpha_1 - 2\alpha_2\alpha_3}{4-\alpha_2\delta}k + \alpha_4 \\
&= \alpha_4 + 1 - \frac{4\alpha_1 - 2\alpha_2\alpha_3}{4-\alpha_2\delta}k, \\
Det(J(E_r)) &= \alpha_4 - 2\alpha_4\frac{2\alpha_1 - \alpha_2\alpha_3}{4-\alpha_2\delta}k - \frac{2\alpha_1 - \alpha_2\alpha_3}{4-\alpha_2\delta}\frac{\alpha_2\beta}{2}\delta k \\
&= \alpha_4 - \frac{2\alpha_1 - \alpha_2\alpha_3}{4-\alpha_2\delta}\left(2\alpha_4 + \frac{\alpha_2\beta}{2}\delta\right)k.
\end{aligned}$$

The discriminant Δ_r is calculated as follows:

$$\begin{aligned}
\Delta_r &= tr(J(E_r))^2 - 4Det(J(E_r)) \\
&= \left(\alpha_4 + 1 - \frac{4\alpha_1 - 2\alpha_2\alpha_3}{4-\alpha_2\delta}k\right)^2 - 4\left(\alpha_4 - \frac{2\alpha_1 - \alpha_2\alpha_3}{4-\alpha_2\delta}\left(2\alpha_4 + \frac{\alpha_2\beta}{2}\delta\right)k\right) \\
&= \alpha_4^2 + 1 + 2\alpha_4 + 4(\frac{2\alpha_1 - \alpha_2\alpha_3}{4-\alpha_2\delta})^2k^2 - 2(1+\alpha_4)\frac{2(2\alpha_1 - \alpha_2\alpha_3)}{4-\alpha_2\delta}k - 4\alpha_4 \\
&+ \frac{2(2\alpha_1 - \alpha_2\alpha_3)(4\alpha_4 + \alpha_2\beta\delta)}{4-\alpha_2\delta}k + 4(\frac{2\alpha_1 - \alpha_2\alpha_3}{4-\alpha_2\delta})^2k^2 \\
&= 1 + \beta^2 - 2\beta + 1 + 2\alpha_4 - 4\alpha_4 + \frac{2(2\alpha_1 - \alpha_2\alpha_3)}{4-\alpha_2\delta}k(-2 - 2\alpha_4 + 4\alpha_4 + \alpha_2\beta\delta) + 4(\frac{2\alpha_1 - \alpha_2\alpha_3}{4-\alpha_2\delta})^2k^2 \\
&= 2 + \beta^2 - 2\beta - 2(1-\beta) + \frac{2(2\alpha_1 - \alpha_2\alpha_3)}{4-\alpha_2\delta}k(-2 + 2(1-\beta) + \alpha_2\beta\delta) + 4(\frac{2\alpha_1 - \alpha_2\alpha_3}{4-\alpha_2\delta})^2k^2 \\
&= 2 + \beta^2 - 2\beta - 2 + 2\beta + \frac{2(2\alpha_1 - \alpha_2\alpha_3)}{4-\alpha_2\delta}k(-2 + 2 - 2\beta + \alpha_2\beta\delta) + 4(\frac{2\alpha_1 - \alpha_2\alpha_3}{4-\alpha_2\delta})^2k^2 \\
&= \beta^2 + \frac{2(2\alpha_1 - \alpha_2\alpha_3)}{4-\alpha_2\delta}k(-2 + \alpha_2\delta)\beta + 4(\frac{2\alpha_1 - \alpha_2\alpha_3}{4-\alpha_2\delta})^2k^2 \\
&= \beta^2 + 2\beta\frac{(2\alpha_1 - \alpha_2\alpha_3)(\alpha_2\delta - 2)}{4-\alpha_2\delta}k + 4(\frac{2\alpha_1 - \alpha_2\alpha_3}{4-\alpha_2\delta})^2k^2 \\
&= \beta^2 - 2\beta\frac{(2\alpha_1 - \alpha_2\alpha_3)(2 - \alpha_2\delta)}{4-\alpha_2\delta}k + 4(\frac{2\alpha_1 - \alpha_2\alpha_3}{4-\alpha_2\delta})^2k^2,
\end{aligned}$$

so :

$$\begin{aligned}
\lambda_{1r,2r} &= \frac{\alpha_4 + 1}{2} - \frac{2\alpha_1 - \alpha_2\alpha_3}{4-\alpha_2\delta}k \pm \sqrt{\frac{\beta^2}{4} - \frac{\beta}{2}A_2k + A_1^2k^2} \\
&= 1 - \frac{\beta}{2} - A_1k \pm \sqrt{\frac{\beta^2}{4} - \frac{\beta}{2}A_2k + A_1^2k^2},
\end{aligned}$$

such that :

$$\begin{cases} A_1 = \frac{(2\alpha_1-\alpha_2\alpha_3)}{4-\alpha_2\delta}, \\ A_2 = \frac{(2\alpha_1-\alpha_2\alpha_3)(2-\alpha_2\delta)}{4-\alpha_2\delta}. \end{cases}$$

Since $\Delta_r > 0$ (the jacobian matrices have positive discriminants), then we deduce that the eigenvalues of Nash equilibrium are real. The local stability of Nash equilibrium is given by using Jury's conditions. Furthermore, the stability region of the fixed point E_r is given by the following:

$$S_r = \{1+tr(J(E_r))+Det(J(E_r)) > 0,\ 1-tr(J(E_r))+Det(J(E_r)) > 0,\ 1-Det(J(E_r)) > 0\}, \tag{4.22}$$

where :

$$\begin{aligned} 1+tr(J(E_r))+Det(J(E_r)) &= 1+\alpha_4+1-\frac{2(2\alpha_1-\alpha_2\alpha_3)}{4-\alpha_2\delta}k+\alpha_4 \\ &- \frac{(2\alpha_1-\alpha_2\alpha_3)(4\alpha_4+\alpha_2\beta\delta)}{2(4-\alpha_2\delta)k} \\ &= 2+2\alpha_4-\frac{2\alpha_1-\alpha_2\alpha_3}{4-\alpha_2\delta}\left(2+\frac{4\alpha_4+\alpha_2\beta\delta}{2}\right)k \\ &= 2+2\alpha_4-\frac{2\alpha_1-\alpha_2\alpha_3}{4-\alpha_2\delta}\left(\frac{4+4\alpha_4+\alpha_2\beta\delta}{2}\right)k \\ &= 2+(1-\beta)-\frac{2\alpha_1-\alpha_2\alpha_3}{4-\alpha_2\delta}\left(\frac{4+4\alpha_4+\alpha_2\beta\delta}{2}\right)k \\ &= 4-2\beta-\frac{(2\alpha_1-\alpha_2\alpha_3)(\alpha_2\beta\delta+4(1+\alpha_4))}{2(4-\alpha_2\delta)}k, \\ 1-tr(J(E_r))+Det(J(E_r)) &= 1-\alpha_4-1+\frac{2(2\alpha_1-\alpha_2\alpha_3)}{4-\alpha_2\delta}k+\alpha_4 \\ &- \frac{(2\alpha_1-\alpha_2\alpha_3)(4\alpha_4+\alpha_2\beta\delta)}{2(4-\alpha_2\delta)}k \\ &= \frac{(2\alpha_1-\alpha_2\alpha_3)(4-4\alpha_4-\alpha_2\beta\delta)}{2(4-\alpha_2\delta)}k \\ &= \frac{(2\alpha_1-\alpha_2\alpha_3)(4(1-\alpha_4)-\alpha_2\beta\delta)}{2(4-\alpha_2\delta)}k \\ &= \frac{(2\alpha_1-\alpha_2\alpha_3)(\beta(4-\alpha_2\delta))}{2(4-\alpha_2\delta)}k \\ &= \frac{\beta}{2}(2\alpha_1-\alpha_2\alpha_3)k, \end{aligned}$$

$$
\begin{aligned}
1 - Det(J(E_r)) &= 1 - (1-\beta) + \frac{2\alpha_1 - \alpha_2\alpha_3}{4 - \alpha_2\delta}\left(2\alpha_4 + \frac{\alpha_2\beta}{2}\delta\right)k \\
&= \beta + \frac{2\alpha_1 - \alpha_2\alpha_3}{4 - \alpha_2\delta}\left(\frac{\alpha_2\beta\delta + 4\alpha_4}{2}\right)k \\
&= \beta + \frac{(2\alpha_1 - \alpha_2\alpha_3)(\alpha_2\beta\delta + 4\alpha_4)}{2(4 - \alpha_2\delta)}k.
\end{aligned}
$$

Proposition 4.2.1. *The Nash equilibrium point E_r is asymptotically stable if $k < \frac{2(4-2\beta)(4-\alpha_2\delta)}{(2\alpha_1-\alpha_2\alpha_3)(\alpha_2\beta\delta+4(1+\alpha_4))}$ The system (4.15) undergoes a flip bifurcation at E_r when $k = \frac{2(4-2\beta)(4-\alpha_2\delta)}{(2\alpha_1-\alpha_2\alpha_3)(\alpha_2\beta\delta+4(1+\alpha_4))}$. Moreover, period-2 points bifurcate from E_r when $k > \frac{2(4-2\beta)(4-\alpha_2\delta)}{(2\alpha_1-\alpha_2\alpha_3)(\alpha_2\beta\delta+4(1+\alpha_4))}$.*

Proof. For the special case of the Jacobian matrix J_r, the stability conditions in (4.22) can be written as follows:

$$
\begin{cases}
(1) : 4 - 2\beta - \frac{(2\alpha_1-\alpha_2\alpha_3)(\alpha_2\beta\delta+4(1+\alpha_4))}{2(4-\alpha_2\delta)}k > 0, \\
(2) : \frac{\beta}{2}(2\alpha_1 - \alpha_2\alpha_3)k > 0, \\
(3) : \beta + \frac{(2\alpha_1-\alpha_2\alpha_3)(\alpha_2\beta\delta+4\alpha_4)}{2(4-\alpha_2\delta)}k > 0,
\end{cases}
$$

the first condition is $4 - 2\beta - \frac{(2\alpha_1-\alpha_2\alpha_3)(\alpha_2\beta\delta+4(1+\alpha_4))}{2(4-\alpha_2\delta)}k > 0$ which implies that :

$$
k < \frac{2(4-2\beta)(4-\alpha_2\delta)}{(2\alpha_1 - \alpha_2\alpha_3)(\alpha_2\beta\delta + 4(1+\alpha_4))}, \tag{4.23}
$$

the second condition is $\frac{\beta}{2}(2\alpha_1 - \alpha_2\alpha_3)k > 0$, then the second condition is satisfied according to (4.16) .

the third condition $\beta + \frac{(2\alpha_1-\alpha_2\alpha_3)(\alpha_2\beta\delta+4\alpha_4)}{2(4-\alpha_2\delta)}k > 0$ this inequality is equivalent to :

$$
k > \frac{-2\beta(4-\alpha_2\delta)}{(2\alpha_1 - \alpha_2\alpha_3)(\alpha_2\beta\delta + 4\alpha_4)}. \tag{4.24}
$$

From (4.23) and (4.24) , it follows that the Nash equilibrium E_r is locally asymptotically stable if

$$
\frac{-2\beta(4-\alpha_2\delta)}{(2\alpha_1 - \alpha_2\alpha_3)(\alpha_2\beta\delta + 4\alpha_4)} < k < \frac{2(4-2\beta)(4-\alpha_2\delta)}{(2\alpha_1 - \alpha_2\alpha_3)(\alpha_2\beta\delta + 4(1+\alpha_4))}
$$

k is a positive parameter which implies that

$$k < \frac{2(4-2\beta)(4-\alpha_2\delta)}{(2\alpha_1-\alpha_2\alpha_3)(\alpha_2\beta\delta+4(1+\alpha_4))}.$$

□

4.3 Bifurcation and Lyapunov exponents diagrams

The fixed point E_1 is always unstable, so we will analyze the bifurcations through the other two points E_r and E_l.

a) Bifurcations through the point E_l

the bifurcations of the point E_l was studied through two sets of parameter values.

1) First set of parameter values

The construction of the first bifurcation diagram is done by varying the parameter $k \in [6.4, 7]$ while the other parameters are fixed as follows:

$$(c_1,\ c_2,\ \delta,\ \omega,\ \beta) = (0.5,\ 0.2,\ 0.5,\ 0.5,\ 0.5) \tag{4.25}$$

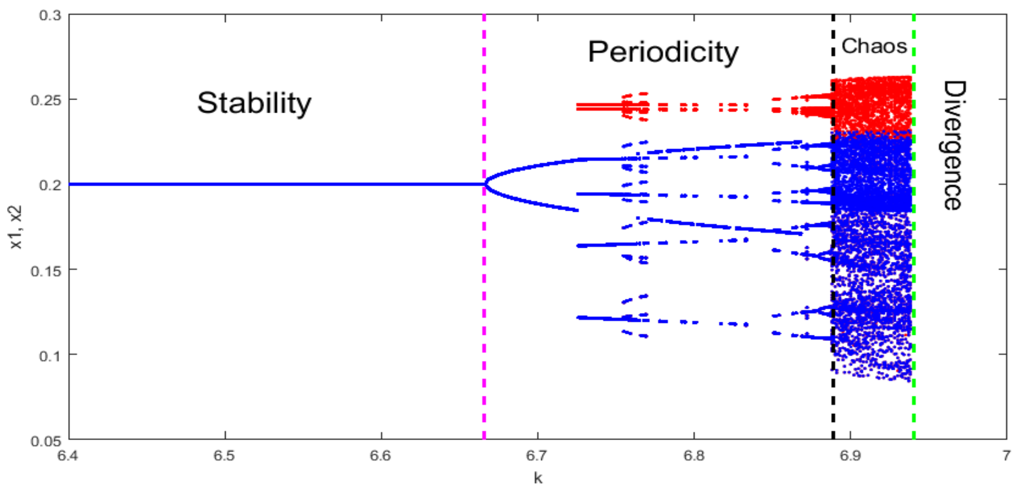


Figure 4.1: Bifircation diagram versus $k \in [6.4, 7]$ for the set (4.25).

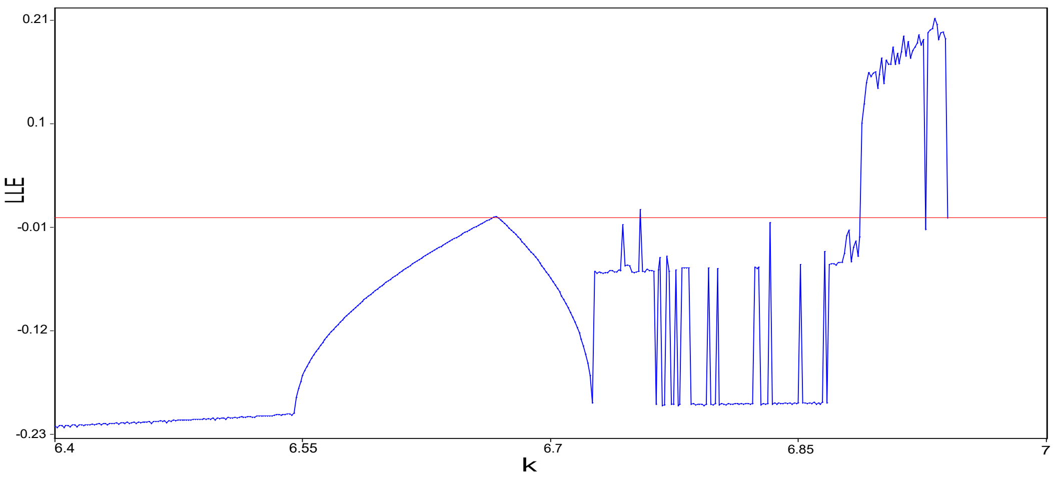


Figure 4.2: Largest Lyapunov exponent diagram versus $k \in [6.4, 7]$ for the set (4.25).

Figure (4.1) shows the bifurcation diagram of x_1 and x_2 with respect to adjustment speed k when the other parameters are set to (4.25). We can see that the system (4.14) is :

- Stable when $k < 6.666$.
- Periodic when $k \in [6.666, 6.889[$.
- Chaotique when $k \in [6.889, 6.94]$.
- Divergence when $k > 6.94$.

The results obtained is confirmed by the largest lyapunov exponent diagram in Figure (4.2).

2) Second set of parameter values

The construction of the second bifurcation diagram is done by varying the parameter $k \in [6.8, 7.1]$ while the other parameters are fixed as follows:

$$(c_1,\ c_2,\ \delta,\ \omega,\ \beta) = (0.5,\ 0.2,\ 0.8,\ 0.5,\ 0.5) \tag{4.26}$$

Figure (4.3a) shows the bifurcation diagram of x_1 and x_2 with respect to adjustment speed k when the other parameters are set to (4.26), so we can see :

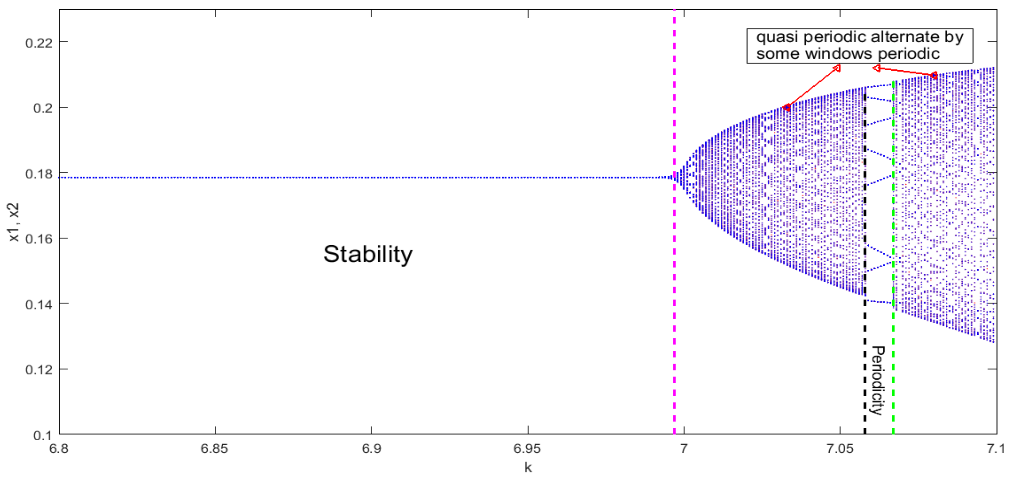
1. For $k < 6.996$, the fixed point E_l is attractive.

2. For $k = 6.996$, the eigenvalues are complex, so there is a Naimark-Sacker bifurcation.

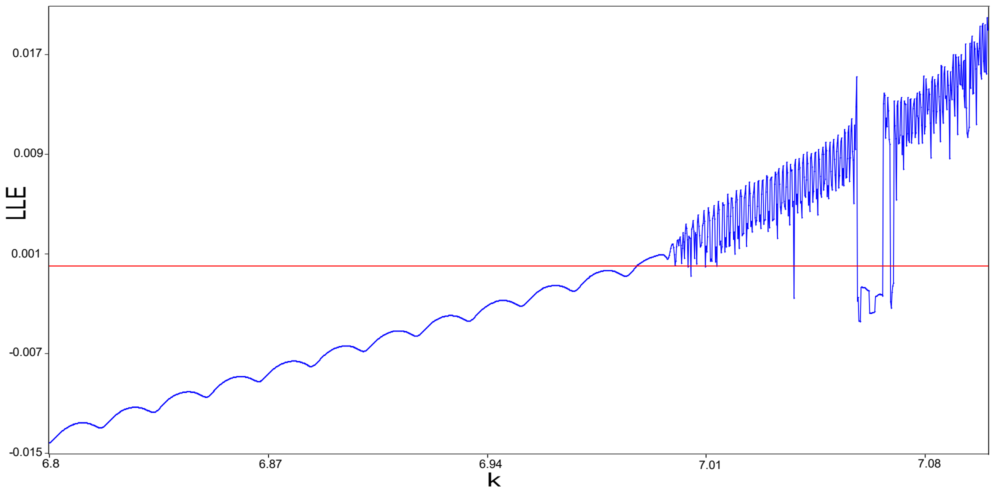
3. For $k \in]6.996, 7.058] \cup [7.076, 7.1]$, the system is quasi periodic alternate by some windows periodic.

4. For $k \in]7.058, 7.076[$ the system is periodic.

The largest lyapunov exponent diagram in Figure (4.3b) confirme all the results we obtained.



(a)



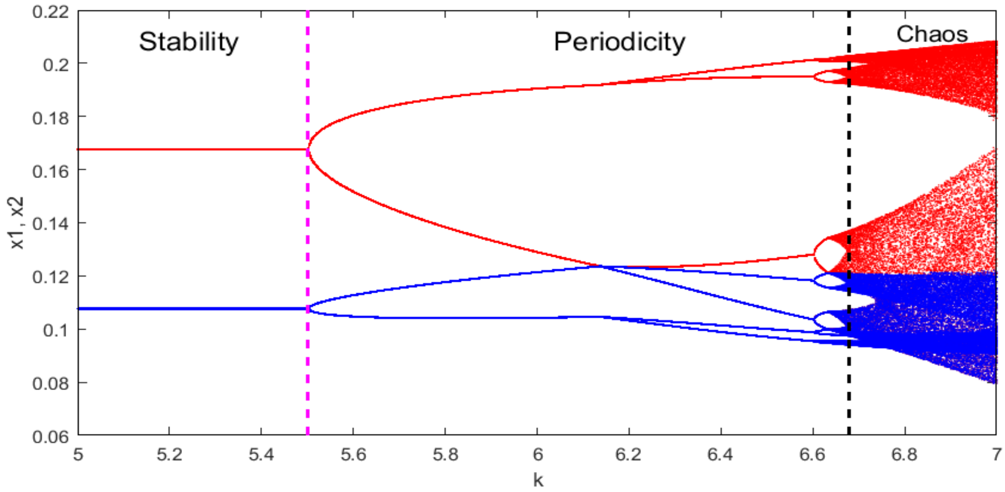
(b)

Figure 4.3: Bifurcation and lyapunov exponents diagrams versus $k \in [6.8, 7.1]$ for the set (4.26)

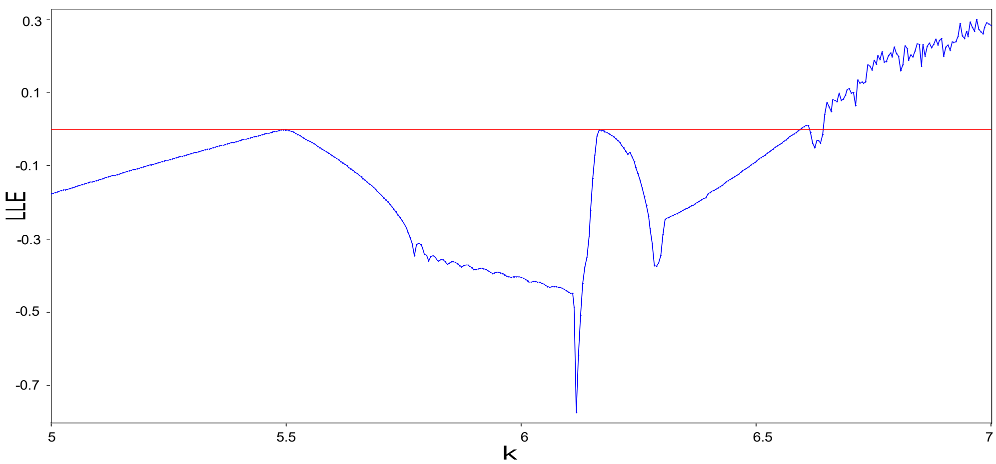
b) Bifurcations through the point E_r

The construction of the bifurcation diagram is done by varying the parameter $k \in [5, 7]$ while the other parameters are fixed as follows:

$$(c_1,\ c_2,\ \delta,\ \omega,\ \beta) = (0.6,\ 0.4,\ 0.6,\ 0.3,\ 0.6) \tag{4.27}$$



(a) Bifurcation diagram.



(b) Largest lyapunov exponent diagram.

Figure 4.4: Bifurcation and largest lyapunov exponent diagrams versus $k \in [5, 7]$ for the set (4.27).

Figure (4.4a) shows the bifurcation diagram of x_1 and x_2 with respect to adjustment speed k when the other parameters are set to (4.27), so we can see :

1. For $5 \leq k < 5.5$, the fixed point E_l is attractive (asymptotically stable).

2. For $5.5 \leq k < 6.14$, the system has an attractor which is a periodic orbit of period 2.

3. For $6.14 \leq k < 6.6$, the system has an attractor which is a periodic orbit of period 4.

4. For $6.6 \leq k < 6.68$, the system has an attractor which is a periodic orbit of period 8.

5. For $k \geq 6.68$, the system has a chaotic attractor.

The transition to chaos in this case is through the scenario of period-doubling because we notice that the period of the oscillator is doubled, then quadrupled, then octupled, and then it transitions to chaos as the parameter $k \geq 6.69$.
we confirmed the obtained results by the largest lyapunov exponent diagram in Figure (4.4b).

Attractors

The phase portrait for the set parameter values

$$(c_1,\ c_2,\ \delta,\ \omega,\ \beta) = (0.5,\ 0.2,\ 0.9,\ 0.5,\ 0.5) \tag{4.28}$$

and $k = 6.47$ is illustrated in Figure (4.5), in which we observe a closed invariant curve, indicating that the behavior of the system (4.14) changes from stable to quasiperiodic through a neimark–sacker bifurcation.

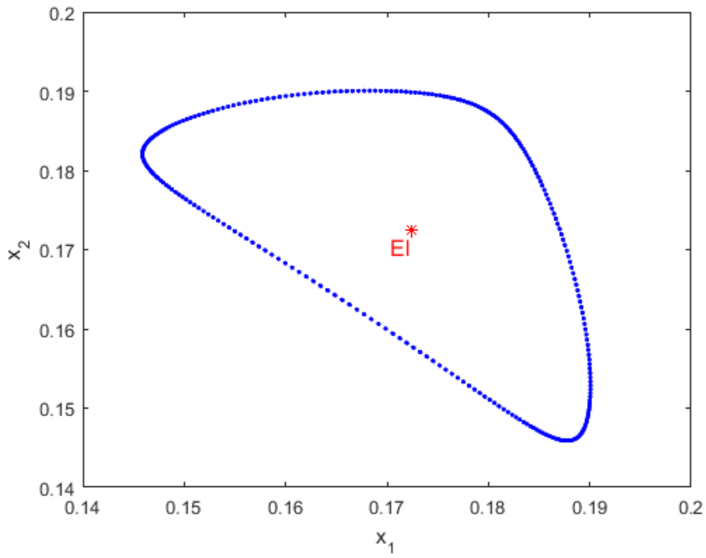


Figure 4.5: Closed invariant curve with the set parameter values (4.28) and $k = 6.47$

The diagram in Figure (4.6) illustrates a strange attractor within a 2-dimensional phase portrait with the set parameter values

$$(c_1,\ c_2,\ \delta,\ \omega,\ \beta) = (0.5,\ 0.2,\ 0.3,\ 0.5,\ 0.5) \tag{4.29}$$

for $k = 5.9$ which explain the details of the system (4.13) behavior.

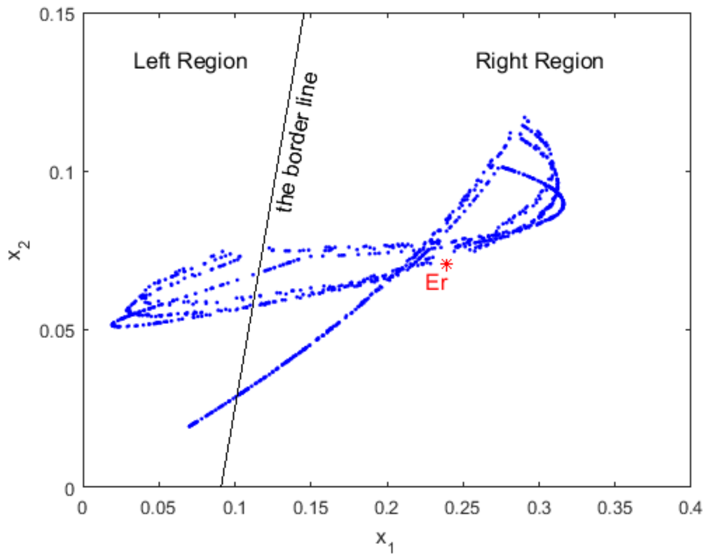


Figure 4.6: Strange attractor with the set parameter values (4.29) for $k = 5.9$

4.4 Chaos control of a duopoly game using the OGY method

Chaotic behavior in economics poses significant challenges, including diminished investor confidence, hindered economic growth, and increased operational difficulties. It amplifies volatility and financial risks, making it imperative to mitigate its effects [3]. Chaos in duopoly games results in unpredictable output decisions for both firms, stemming from their sensitivity to even the smallest errors. Consequently, there's a pressing need to devise methods to control chaos within economic systems. Several approaches have been proposed, such as feedback control, adaptive control, and the OGY method [36]. In this study, we opt for the OGY method due to its non-invasive application (very small parameter adjustment during a very small time period), ensuring that the game's evolution between these two players will be stabilized without great effort.

Stabilization of the unstable Nash equilibrium $E = (\bar{x}_1, \bar{x}_2)$

1. For $g(x_2) > x_{1,t}$ we have:

$$A_{c_1} = \frac{\partial F}{\partial X} = \begin{pmatrix} 1 - 2k\bar{x}_1 & -\delta k\bar{x}_1 \\ 1 & 0 \end{pmatrix} \quad and \quad B_{c_1} = \frac{\partial F}{\partial c_1} = \begin{pmatrix} -k\bar{x}_1 \\ 0 \end{pmatrix}.$$

The c_1 controllability matrix is:

$$P_{c_1} = \begin{pmatrix} B_{c_1} & A_{c_1}B_{c_1} \end{pmatrix} = \begin{pmatrix} -k\bar{x}_1 & k\bar{x}_1(2k\bar{x}_1 - 1) \\ 0 & -k\bar{x}_1 \end{pmatrix},$$

and its determinant is;

$$\det(P_{c_1}) = k^2\bar{x}_1^2 \neq 0.$$

Then, we conclude that the first firm can stabilize the game around the Nash equilibrium E by adjusting its marginal cost c_1.

2. If $g(x_2) < x_{1,t}$. Then

$$A_{c_2} = \frac{\partial F}{\partial X} = \begin{pmatrix} 1 - 2k\bar{x}_1 & -\delta k\bar{x}_1 \\ -\frac{(1+\omega)}{2}\beta & 1 - \beta \end{pmatrix} \quad and \quad B_{c2} = \frac{\partial F}{\partial c_2} = \begin{pmatrix} 0 \\ -(1+\omega)\frac{\beta}{\delta} \end{pmatrix}.$$

The c_2 controllability matrix is:

$$P_{c_2} = [B_{c_2}, A_{c_2}B_{c_2}] = \begin{pmatrix} 0 & k\beta\bar{x}_1(\omega+1) \\ -(1+\omega)\frac{\beta}{\delta} & \frac{\beta}{\delta}(\beta-1)(\omega+1) \end{pmatrix},$$

and its determinant is:

$$\det(P_{c_2}) = \frac{1}{\delta}k\beta^2\bar{x}_1(\omega^2+2\omega+1) \neq 0.$$

Thus, we conclude that the second firm can stabilize the Nash equilibrium E by adjusting its marginal cost c_2.

Numerical simulation

We conduct numerical simulations to confirm the theoretical findings. First, we use the marginal cost of the first firm c_1 as the control parameter, and then we use the marginal cost of the manufactring firm c_2 as the control parameter.

- **Stabilizing the unstable Nash equilibrium using the marginal cost** c_1

  Let us consider c_1 as a control parameter and the other parameters are fixed as follows:

$$(c_2,\ \delta,\ \omega,\ \beta,\ k) = (0.2,\ 0.3,\ 0.5,\ 0.5,\ 5.9). \tag{4.30}$$

For $c_1 = 0.5$ we have $E = (0.2394, 0.0704)$.

The jacobian matrix is as follow :

$$A_{c_1} = \begin{pmatrix} -1.8254 & -0.4238 \\ 1 & 0 \end{pmatrix} \quad and \quad B_{c_1} = \begin{pmatrix} -1.4127 \\ 0 \end{pmatrix},$$

the matrix A_{c_1} has two eigenvalues $\lambda_u = -1.5523 \quad and \quad \lambda_s = -0.2730$,

with two right eigenvectors

$$v_u = \begin{pmatrix} -0.84067 \\ 0.54155 \end{pmatrix} \quad and \quad v_s = \begin{pmatrix} 0.26337 \\ -0.96469 \end{pmatrix},$$

and two left eigenvectors

$$w_u = \begin{pmatrix} -0.96469 \\ -0.26337 \end{pmatrix} \quad and \quad w_s = \begin{pmatrix} -0.54155 \\ -0.84067 \end{pmatrix}.$$

So the control gain $k^T = \begin{pmatrix} 1.0989 & 0.3 \end{pmatrix}$.

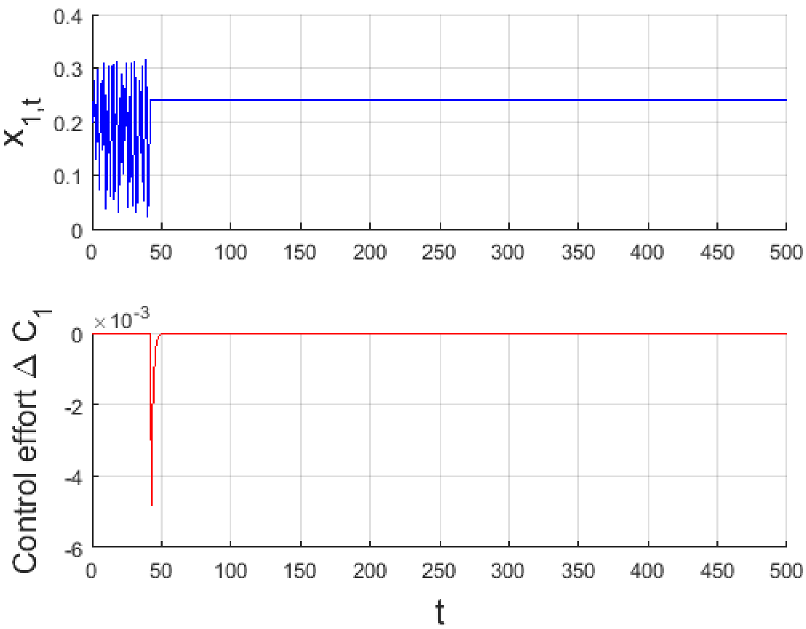


Figure 4.7: Response of the controlled duopoly game using the marginal cost c_1 with the set parameter values (4.30) for $c_1 = 0.5$.

Figure (4.7) shows the response of the controlled duopoly game along with the applied control efort.The control is activated when the system state approaches the unstable equilibrium E at $t = 42$ and the marginal cost c_1 is adjusted by a small perturbation of order 10^{-3} during the time period $t \in [42, 55]$. Subsequently, the control is established at $t = 56$, stabilizing the duopoly game to its Nash equilibrium.

- **Stabilizing the unstable Nash equilibrium point E using the marginal cost c_2**

  Let us consider c_2 as a control parameter and the other parameters are fixed as follows:

$$(c_1,\ \delta,\ \omega,\ \beta,\ k) = (0.5,\ 0.3,\ 0.5,\ 0.5,\ 5.9). \tag{4.31}$$

For $c_2 = 0.2$ we have $E = (0.2394, 0.0704)$.

The jacobian matrix is as follow :

$$A_{c_2} = \begin{pmatrix} -1.8254 & -0.4238 \\ -0.375 & 0.5 \end{pmatrix} \quad and \quad B_{c_2} = \begin{pmatrix} 0 \\ -1.25 \end{pmatrix},$$

the matrix A_{c_2} has two eigenvalues $\lambda_u = -1.8918 \quad and \quad \lambda_s = 0.5664$,

with two right eigenvectors

$$v_u = \begin{pmatrix} -0.98793 \\ 0.15489 \end{pmatrix} \quad and \quad v_s = \begin{pmatrix} 0.17447 \\ -0.98466 \end{pmatrix},$$

and two left eigenvectors

$$w_u = \begin{pmatrix} -0.98466 \\ -0.17447 \end{pmatrix} \quad and \quad w_s = \begin{pmatrix} 0.15489 \\ -0.98793 \end{pmatrix}.$$

So the control gain $k^T = \begin{pmatrix} 8.5413 & 1.5134 \end{pmatrix}$.

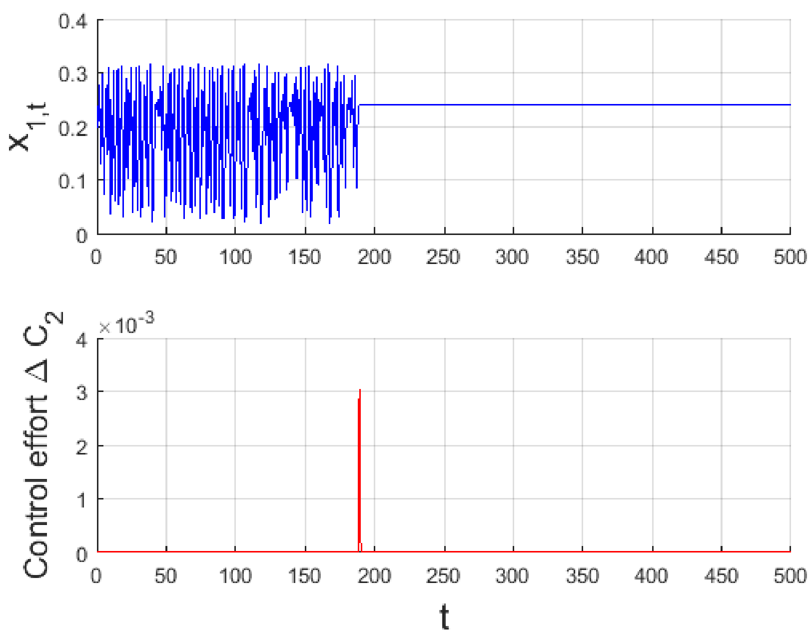


Figure 4.8: Response of the controlled duopoly game using the marginal cost c_1 with the set parameter values (4.31) for $c_2 = 0.2$.

Figure(4.8) illustrates the response of the controlled duopoly game along with the applied control efort. The control is activated when the system state approaches the unstable equilibrium E at $t = 188$ and the marginal cost c_2 is adjusted by a small perturbation of order 10^{-3} during the time period $t \in [188, 192]$. Subsequently, the control is established at $t = 192.5$, stabilizing the duopoly game to its Nash equilibrium.

Results and discussion

Compared to other control methods, the application of the **OGY** method to stabilize markets provides the advantage of achieving control with minimal effort from a single firm within a short time frame. This implies that the method can guide the market towards stability smoothly without compromising the participating firms in the game. For example, in the present study, the control was realized through adjusting the marginal cost c_1 by $|\Delta c_1| < 5\times10^{-3}$ over a short time. Additionally, it was accomplished when adjusting the marginal cost of the second firm c_2 by a perturbation $|\Delta c_2| < 3 \times 10^{-3}$ over only 4 units of time. In the counter party. In [37] a Cournot model was stabilized using a delay feedback control (Pyragas method) by adjusting the state of the first firm during approximately 50 units of time. In [17], a Cournot model was stabilized using the state variable feedback and parameter variation method during approximately 40 units of time.

GENERAL CONCLUSION

In this work, we focused on studying an economic model described by two-dimensional first-order difference equations, which simulate the competition between two heterogeneous players, the original manufacturer and a second-party remanufacturer. The stability of the equilibrium points has been analyzed. From bifurcation diagrams, the maximal Lyapunov exponents, and phase portraits, the basic properties of the game are presents. The results show that the adjustment speed of the first player has an obvious impact on the stability of the players' dynamic competition model. When it continues to increase, a series of chaotic phenomena occur. By applying a control law derived from the OGY method, the chaotic behavior is effectively eliminated, leading to the stabilization of the duopoly game at its Nash equilibrium point. This study confers significant importance on the economic market consists of: understanding economic changes, analysis of economic stability, support for strategic planning, risk management, enhancement of economic performance.

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