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Numerical solution of some Volterra integro-differential equations by using collocation methods

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DEDICATION

To my parents,

To those who inspired and cheered me on.

Cheima Khennaoui

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ABSTRACT

This thesis aims to present a straightforward, convergent, and readily applicable numerical method for computing approximate solutions to linear two-dimensional first-order and second-order partial Volterra integro-differential equations, along with a class of high-order linear and non-linear partial Volterra integro-differential equations in two dimensions. We construct algorithms based on Taylor polynomials of two variables to address these equations numerically, and a rigorous convergence analysis with error estimates are discussed in details to validate the convergence of the approximate solution to the exact solution. The theoretical findings are reinforced with several numerical examples to test the efficiency of the proposed scheme and to confirm the reliability of the convergence analysis of the proposed convergent algorithms.

Key words: Volterra integro-differential equation, Linear partial Volterra integro-differential equation, Two-dimensional equations, Collocation method, Taylor polynomials, Convergence analysis, Error estimation.

RÉSUMÉ

L'objectif de cette thèse est de proposer et appliquer une méthode numérique directe, convergente et facilement applicable pour calculer les solutions approchées des équations intégro-différentielles partielles linéaires de Volterra du premier et du second ordre en deux dimensions, ainsi qu'une classe d'équations intégro-différentielles partielles de Volterra linéaires et non linéaires d'ordre supérieur en deux dimensions. Nous construisons des algorithmes basés sur les polynômes de Taylor en deux dimensions pour aborder ces équations numériquement, et une analyse de convergence rigoureuse ainsi que des estimations d'erreur sont discutées en détail. Pour valider la convergence de la solution approximative à la solution exacte, les résultats théoriques sont renforcés par plusieurs exemples numériques pour confirmer l'efficacité du schémas proposés.

Mots-clés : Équation intégro-différentielle de Volterra, Équation intégro-différentielle partielle linéaire de Volterra, Équations à deux dimensions, Méthode de collocation, Polynômes de Taylor, Analyse de convergence, Estimation d'erreur.

ملخص

قدمنا في هذه الاطروحة طريقة عددية مباشرة، متقاربة، وسهلة التطبيق لحساب الحلول التقريبية لمعادلات فولتيرا التكاملية-التفاضلية الجزئية الخطية ثنائية الأبعاد من الدرجة الأولى والثانية، بالإضافة الى فئة من المعادلات التكاملية-التفاضلية الجزئية الخطية وغير الخطية ذات الدرجة العالية لفولتيرا ثنائية الأبعاد. قمنا ببناء خوارزميات باستعمال كثيرات حدود تايلور ثنائية الأبعاد وباستخدام طريقة التجميع لمعالجة هذه المعادلات عدديًا وتم ايضا مناقشة تقديرات الخطأ بالتفصيل لتأكيد تقارب الحل التقريبي إلى الحل الدقيق. يتم تعزيز النتائج النظرية بعدة أمثلة عددية لاختبار كفاءة النهج المقترحة.

الكلمات الافتتاحية:

معادلة فولتيرا التكاملية-التفاضلية، معادلة فولتيرا التكاملية-التفاضلية الجزئية الخطية، معادلات ثنائية الأبعاد، طريقة التجميع، كثيرات حدود تايلور، دراسة التقارب، تقدير الخطأ.

Contents

Introduction

The concept of integral equations (IEs) arose from the mathematical modelling of various real-life problems in physics and biology. Consequently, it stands as a vital tool in mathematical physics, astrophysics, and notably in applied mathematics. Integral equations play a significant role in addressing a variety of problems within the theory of differential equations, they also intersect with orthogonal systems theory and spectral theory. Furthermore, integral equations maintain close connections with functional analysis, a fundamental field in mathematics.

Integral equations highlights a profound and extensive history tracing back to the early developments of calculus and mathematical analysis, emerging as a natural extension of differential equations. The groundwork for integral equations was laid by mathematicians such as John Wallis (1616-1703) and James Gregory (1638-1675) in the 17^{th} century who explored problems involving continuous quantities. In the 18^{th} century, Leonhard Euler (1707-1783) introduced integral equations of the first kind, while Daniel Bernoulli (1700-1782) encountered them during his study on the oscillation of a stretched string. Mathematicians like Joseph Fourier (1768-1830) and Joseph Liouville (1809-1882) further advanced the theory through their work on solving differential equations and modelling various physical phenomena such as the heat equation in 1822. One of the earliest documented integral equations in mathematical literature dates back to Abel's problem in mechanics which involved finding the trajectory along which a material point should slide to make the time of descent an initially given function of the altitude in 1826 [1, 2].

The development of integral equations truly began towards the latter part of the 19^{th} century, largely attributed to the works of Italian mathematician Vito Volterra (1860-1940), and notably to Swedish mathematician Ivar Fredholm (1866-1927). Fredholm's seminal work [3], published in 1900, introduced a ground-breaking method for solving the Dirichlet problem. In 1902, his subsequent contributions [4] introduced the concept of compact operators. These concepts laid the foundation for Fredholm integral equations (FIEs) and established conditions for their solvability. Since then, integral equations have been the subject of continuous research by numerous mathematicians up to the present moment. Pioneers in this field include Henri Poincaré (1854-1912), Maurice Fréchet (1878-1973), David Hilbert (1862-1943), Godfrey Harold Hardy (1877-1947), Frigyes Riesz (1880-1956) and Erhard Schmidt (1876-1959), who contributed significantly to this evolving area of study.

However, in the realm of mathematical modelling, understanding complex dynamics often requires more than just integral or differential equations alone. A new class, integrating both differential and integral aspects, has emerged as a powerful tool to provide a more comprehensive framework to describe processes where both rates of change (described by differential equations) and accumulated quantities (described by integral equations) are important. These equations, known as integro-differential equations (IDEs), have extensive applications across physics, biology, chemistry, and engineering [5–9]. The comprehension and resolution of IDEs are pivotal in driving progress in these scientific fields, fostering the advancement of sophisticated mathematical methodologies. J. Fourier and Pierre-Simon Laplace (1749-1827) were among the pioneers who initially investigated their fundamental properties in the 20^{th} century. Through their integral transforms, these equations can be converted from integro-differential forms into simpler algebraic equations, thereby facilitating their analysis and resolution. Mathematicians such as Oliver Heaviside (1850-1925) and Jan Mikusinski (1913-1987) significantly contributed to operational calculus, offering a systematic approach to address complex integro-differential equations.

The rising necessity to depict more complex phenomena characterized by spatial variations (illustrated by partial derivatives) and interactions with past states (expressed through integral terms) underscores the development of partial integro-differential equations (PIDEs). Throughout the years, the advancement of PIDEs has positioned it as a pivotal area of research in applied mathematics and engineering. It has yielded significant contributions across diverse domains including stochastic processes and probability theory, control theory and optimization, and has influenced the approach to modelling biological processes with memory effects such as financial mathematics [10, 11], fluid dynamics [12], population dynamics [13, 14], and disease spread [15]. Additionally, in physics, they are instrumental in describing phenomena involving non-local interactions and memory effects, such as anomalous diffusion and viscoelastic materials [16].

The wide-ranging applications of Volterra integro-differential equations (VIDEs) and partial Volterra integro-differential equations (PVIDEs) has spurred researchers to devise advanced and efficient solution methods for this category of equations. Finding their analytical solutions has proven challenging for the majority as exact solutions are often unattainable. Therefore, with the emergence of computers in the mid-20^{th} century, numerical methods and computational techniques became essential for resolving them. Numerous studies have focused on numerical solutions for VIDEs, as exemplified by works such as [17–21]. However, research on numerical solutions for PVIDEs still remains limited [22–26], especially for multidimensional partial Volterra integro-differential equations (2D-PVIDEs), which have recently garnered increasing interest in their numerical solutions [27–31]. Zheng et al. presented a Legendre spectral method for solving multidimensional PVIDEs in their work [28]. Aziz et al. [29] introduced a collocation technique utilizing the Haar wavelet approach to obtain numerical solutions for diffusion and reaction-diffusion 2D-PVIDES. Similarly, Kumar and Vijesh [30] employed Haar wavelets to tackle nonlinear PVIDEs. In a separate study [31], Wang et al. utilized two-dimensional Bernoulli polynomials to address a subset of 2D-PVIDEs with fractional order.

The aim of this thesis is to develop direct collocation approaches using two-dimensional Taylor polynomials to obtain approximate solutions for a range of 2D-PVIDEs of first, second, and higher order, each with its own set of initial conditions. These approaches revolve around the concept of approximating the exact solution of a given PVIDE within a bounded domain using elements from the designated collocation space, i.e. the piecewise polynomial space. This approximation that conforms to the equation and the initial conditions on collocation points is referred to as the collocation solution.

Methods based on Taylor analysis have been integrated into approximation theory, facilitating the discovery of numerical solutions for various equations [32–35]. For example, Darania et al. [34] applied a differential transform method utilizing Taylor series expansion to solve two-dimensional nonlinear VIDEs. Similarly, Gurbuz and Sezer [35] utilized a matrix collocation technique incorporating Laguerre and Taylor polynomials to address specific nonlinear PVIDEs. In this thesis, our focus is on the Taylor piecewise collocation method utilizing Taylor polynomials. This proposed method offers an approximate solution through explicit formulas, providing several advantages such as the high accuracy and high convergence rate. The lack of need to solve any algebraic systems makes the process of finding numerical solutions to such problems using the Taylor collocation method easy and does not cost much computation. Therefore, in recent years, there has been significant attention on utilizing the Taylor collocation method to numerically solve various types of differential, integral, and integro-differential equations, including the complex differential equations for a rectangular domain [36], Bagley-Torvik equation [37], high-order linear differential-difference [38], Volterra-Fredholm integral [39], IDEs [40, 41], systems of IDEs [42, 43] and others [44, 45]. Moreover, Laib et al. [46] have recently investigated two-dimensional Volterra integral equations. In this pursuit, we extend the Taylor collocation method previously introduced for one-dimensional VIDEs in [41], and for two-dimensional VIEs in [46] to tackle a variety of 2D-PVIDEs.

This thesis is organized into four carefully structured chapters, outlined as follows:

Chapter 1: we provide general concepts, definitions, and auxiliary facts related to VIDEs and PVIDEs, laying the groundwork for subsequent chapters. This chapter covers topics such as the classification of integral and integro-differential equations, accompanied by real-life examples illustrating their application. Additionally, we explore the significant relation between these equations and differential equations, focusing on the conversion between the two. We also include a brief review of spline collocation methods and some discrete and integral inequalities that are necessary for later use.

Chapter 2: we expand the Taylor collocation method to two dimensions to explore a collocation solution within the piecewise polynomial spline space $S^{(-1)}_{p-1,p-1}$ for first-order linear 2D-PVIDEs. We validate the convergence of the approximate solution to the exact solution, with an order of convergence equal to p. Furthermore, we reinforce our theoretical findings with numerous numerical examples.

Chapter 3: we introduce a novel direct numerical approach designed to solve second-order linear 2D-PVIDEs within the same piecewise polynomial spline space $S^{(-1)}_{p-1,p-1}$ using Taylor polynomials in two dimensions. Our thorough analysis confirms the convergence of this algorithm, and we offer numerical results to validate the effectiveness of our proposed approach.

Chapter 4: we build upon the fundamental concepts established in the previous chapters and extend their application to a class of high order 2D-PVIDEs. Employing Taylor polynomials in two dimensions, we create a collocation solution within the piecewise polynomial spline space $S^{(-1)}_{p-1,p-1}$. Furthermore, we conduct an error analysis that demonstrates the convergence of the approximate solution and its derivatives. The inclusion of numerical examples serves to validate the theoretical results.

Ultimately, in **Conclusion and perspectives** we encapsulate the contributions made within this thesis and propose new avenues for future research, along with suggestions for improvements and perspectives in the field.

Chapter 1

Preliminary and auxiliary results

This chapter serves as an initial exploration within the framework of our thesis investigations. The focus here lies on the classifications of integral equations and their connections with differential equations. Additionally, we delve into the examination of collocation methods employed in handling integro-differential equations with a particular attention on the Taylor collocation method. Furthermore, we present specific discrete and integral inequalities that will prove essential in our theoretical analysis of the approaches we propose. To begin our exploration, let us address the main question: **what is an integral equation ?**

An integral equation is an equation in which the unknown function of one or more variables, denoted as $\mu(x)$, is situated beneath the integral sign as follows:

$$\alpha(x)\mu(x) = \hbar(x) + \int_{a(x)}^{b(x)} \mathcal{H}(x, t, \mu(t))dt, \tag{1.1}$$

where $\alpha(x)$ and $\hbar(x)$ represent known functions, and $\mathcal{H}(x, t, \mu(t))$ serves as the kernel of the integral equation over the interval $[a, b]$. Additionally, $a(x)$ and $b(x)$ denote the integration limits, and they may be either variables, constants, or a combination of both.

This general definition encompasses various specific forms that emerge from modeling diverse problems in physics and biology. In our subsequent exploration, we will concentrate on the linear case of equation (1.1), where the kernel $\mathcal{H}(x, t, \mu(t))$ fulfills

$$\mathcal{H}(x, t, \mu(t)) = H(x, t)\mu(t).$$

1.1 Classification of integral equations

Integral equations can be classified according to three main characteristics: integration bounds, the kind of the equation determined by the location of the unknown function, and the singularity adjective used in cases where one or both integration bounds are infinite, or when the equation's kernel is unbounded.

1.1.1 Fredholm integral equations

A linear Fredholm integral equation takes the following form:

$$\alpha(x)\mu(x) = \hbar(x) + \int_a^b H(x,t)\mu(t)dt, \tag{1.2}$$

where the integration bounds a and b are fixed, with $\hbar(x)$ representing a predefined function at each point x within the bounded interval $[a, b]$. The function $H(x,t)$ is also predefined and applicable to every pair of points x and t within the same interval.
Equation (1.2) with $\alpha(x) \neq 0$ represents a FIE of the second kind. In contrast, a FIE of the first kind take the form:

$$-\hbar(x) = \int_a^b H(x,t)\mu(t)dt,$$

where the unknown function $\mu(x)$ appears solely under the integral sign ($\alpha(x) = 0$).

Fredholm integral equation (1.2) is labeled singular, as previously mentioned, if its integration domain is unbounded or if its kernel is unbounded. In the latter case, the kernel can be represented as:

$$H(x,t) = \beta(x,t)K(x,t),$$

with $\beta(x,t)$ represents the singular component specifically chosen such that $K(x,t)$ remains bounded. Nevertheless, Equation (1.2) can be expressed as a weakly singular FIE if the integral of the square of the kernel's modulus exists, even if the kernel itself is unbounded. Such equations can be represented in the following form:

$$\mu(x) = \hbar(x) + \int_a^b \frac{K(x,t)}{|x-t|^\omega}\mu(t)dt,$$

where K is a predefined bounded function of two variables within the interval $[a,b]$ and $0 < \omega < 1$.

All the concepts discussed earlier concerning Fredholm integral equations can be extended to multidimensional contexts by considering the integration domain $[a, b]$ as Λ, i.e.,

$$\alpha(x)\mu(x) = \hbar(x) + \int_\Lambda H(x,t)\mu(t)dt,$$

where Λ can adopt the structure of a multidimensional domain or a combination of multiple non-overlapping multidimensional domains. For a more in-depth discussion and analysis of FIEs refer to ref. [2].

1.1.2 Volterra integral equations

A linear Volterra integral equation (VIE) takes the following form:

$$\alpha(x)\mu(x) = \hbar(x) + \int_a^x H(x,t)\mu(t)dt, \quad a \leq x \leq b, \tag{1.3}$$

where at least one of the integration limits x is a variable. Here, the functions $\hbar(x)$ and the kernel $H(x,t)$ are both predefined within the interval $[a, b]$.

Likewise, Equation (1.3) with $\alpha(x) \neq 0$ characterizes a VIE of the second kind. Moreover, a fundamentally distinct category of equations arises when $\alpha(x) = 0$, leading to the VIE of the first kind:

$$-\hbar(x) = \int_0^x H(x,t)\mu(t)dt.$$

Volterra integral equation (1.3) is also labeled singular if its kernel $H(x,t)$ is unbounded, while it is termed weakly singular if its integrands are unbounded yet integrable. One of the earliest integral equations studied, serving as a notable example of a Volterra singular integral equation of the first kind, is Abel's equation

$$\int_0^x \frac{\mu(t)dt}{(x-t)^\omega} = \hbar(x), \quad 0 < \omega < 1.$$

This equation models the problem of determining the trajectory along which a material point should slide to ensure that the time of descent becomes a specified function of the initial altitude. For a comprehensive understanding of this equation and its solution, refer to [2].

Remark 1.

- *The classical Fredholm theory is applicable to Volterra equations as they can be viewed as a particular case of Fredholm equations. However, this application diminishes some of its efficacy, as direct study of Volterra equations often yields numerous outcomes that cannot be derived through Fredholm Theory alone.*
- *In academic literature, Equation (1.3) is categorized as a third-kind Volterra integral equation if for $\omega \in [0,1)$, $\beta > 0$, $\omega + \beta \geq 1$, it is represented in the following format:*

$$x^\beta \mu(x) = x^\beta \hbar(x) + \int_a^x \frac{1}{(x-t)^\omega} H(x,t)\mu(t)dt, \; x \in [a,b], \tag{1.4}$$

where $\hbar(x)$ and $H(x,t) = t^{\omega+\beta-1}H'(x,t)$ are known continuous functions on $[a,b]$ and $\Lambda = \{(x,t)|a \leq t < x \leq b\}$ respectively. The distinctive feature of third-kind Volterra integral equations lies in the coefficient $\alpha(x) = x^\beta$ on the left-hand side of the equation. For further information on these equations refer to refs. [47–49].

Based on the aforementioned exploration, a fundamental differentiation between these two integral equations categories is established: in the Fredholm integral equation the integration bounds are constant, whereas in the Volterra integral equation one of them is a variable. However, there are scenarios where mixed equations occur, blending characteristics of both types.

Fredholm-Volterra integral equations

Fredholm-Volterra integral equations arise from boundary value problems and are employed in mathematical models that depict the spatio-temporal evolution of epidemics. They are also used in the study of predator-prey interactions within population dynamics, where population growth rates are shaped by a combination of fixed and variable factors. Additionally, these mixed integral equations have applications in diverse fields including physical, economic, and biological models [50].

Fredholm-Volterra integral equations can be encountered in one of two forms: either as separate Fredholm and Volterra integral equations

$$\mu(x) = \hbar(x) + \int_a^b H_1(x,t)\mu(t)dt + \int_a^x H_2(x,t)\mu(t)dt,$$

or as mixed Fredholm-Volterra integral equations

$$\mu(x,y) = \hbar(x,y) + \int_a^b \int_a^x H(x,y,t,s)\mu(t,s)dtds.$$

Here $\hbar(x)$, $\hbar(x,y)$, $H_1(x,t)$, $H_2(x,t)$ and $H(x,y,t,s)$ are known and well-defined analytic functions of one or multiple variables within their respective domains. Furthermore, the unknown functions $\mu(x)$ and $\mu(x,y)$ appear both inside and outside the integral signs. This characteristic feature identifies the equation as a second-kind integral equation. In contrast, if the unknown functions only appear inside the integral signs, the resulting equations are of first kind.

Systems of integral equations

More generally, the above mentioned integral equations can be extended to systems of integral equations by considering $\mu(x)$, $H(x,t)$, and $\hbar(x)$ as vector-valued functions. Using equation (1.3) with $\alpha(x) = 1$ as a model, this extension results in the following linear system of the second kind:

$$\mu(x) = \hbar(x) + \int_a^x H(x,t)\mu(t)dt, \quad x \geqslant a,$$

where

$$\hbar(x) = (\hbar_1(x), \hbar_2(x), ...\hbar_{n-1}(x), \hbar_n(x))^T,$$

$$\mu(x) = (\mu_1(x), \mu_2(x), ...\mu_{n-1}(t), \mu_n(t))^T,$$

and

$$H(x,t) = \begin{pmatrix} H_1(x,t) \\ H_2(x,t) \\ \vdots \\ H_n(x,t) \end{pmatrix}.$$

With this notation, the steps involved in proving results for systems are often formally the same as those for single equations.

1.1.3 Integro-differential equations

Integro-differential equations are equations in which the unknown function's derivatives are involved alongside the integral terms, often accompanied by associated initial or boundary conditions. A VIDE of first-order can be expressed as:

$$\mu'(x) = \hbar(x) + \int_a^x H(x,t)\mu(t)dt, \quad x \in [a,b], \tag{1.5}$$

subject to initial condition $\mu(a) = \beta$. Furthermore, a second-order VIDE can be represented as:

$$\mu''(x) = \hbar(x) + \alpha_1(x)\mu'(x) + \alpha_2(x)\mu(x) + \int_a^x H_1(x,t)\mu(t)dt + \int_a^x H_2(x,t)\mu'(t)dt + \int_a^x H_3(x,t)\mu''(t)dt,$$

for all $x \in [a,b]$, subject to initial conditions

$$\mu(a) = \beta_1, \quad \mu'(a) = \beta_2.$$

In a broader context, the Volterra integro-differential equation is expressed as

$$\mu^{(n)}(x) = \hbar(x) + \int_a^x H(x,t)\mu(t)dt, \quad x \in [a,b],$$

where $\mu^{(n)}$ indicates the nth derivative of $\mu(x)$ subject to n initial conditions. Other derivatives of less order may appear at the left side.

The combination of derivatives and integrals allows for a profusion of various forms. However, due to the absence of a widely adopted classification convention, reducing IDEs to systems of integral equations becomes a valuable approach for their analysis. For certain cases of linear VIDEs, the reduction can be made directly through integration. For example, consider the linear VIDE:

$$\mu'(x) = \hbar(x) + \int_0^x H(x,t)\mu(t)dt,$$

with $\mu(0) = \mu_0$. Through direct integration, we obtain

$$\mu(x) = \Phi(x) + \int_0^x \int_0^z H(z,t)\mu(t)dtdz,$$

where $\Phi(x) = \mu_0 + \int_0^x \hbar(z)dz$. Thus the obtained equation is a VIE of the second kind.

By employing an alternative approach, the conversion of VIDEs into systems of VIEs can be achieved through the introduction of a new function that eliminates the need for derivative terms in the equation. For instance, consider the following non-linear VIDE:

$$\mu'(x) = \hbar(x) + \int_0^x \mathcal{H}(x,t,\mu(t))dt,$$

with $\mu(0) = \mu_0$.

Introducing the new function $\mu'(x) = \omega(x)$ results in:

$$\begin{cases} \omega(x) = \hbar(x) + \int_0^x \mathcal{H}(x,t,\mu(t))dt, \\ \mu(x) = \mu_0 + \int_0^x \omega(t)dt. \end{cases}$$

Thus, the non-linear VIDE converts into a system of second-kind VIEs.

Similar reasoning can be employed to reduce integro-differential equations of higher order into integral equations.

1.1.4 Partial integro-differential equations

When the variable x is of multiple dimensions, the integro-differential equation is referred to as a partial integro-differential equation. For instance, the linear PVIDE of the first order in a two-dimensional space can be expressed as follows:

$$\frac{\partial \mu(x,y)}{\partial x} = \hbar(x,y) + \int_0^x \int_0^y H(x,y,t,s)\mu(t,s)dtds, \quad (x,y) \in [0,a_1] \times [0,a_2], \tag{1.6}$$

subject to inital condition $\mu(0,y) = \mu_0(y)$. Likewise, a second-order general linear PVIDE with two independent variables is of form

$$\sum_{0 \le i+j \le 2} \alpha_{i,j}(x,y) \frac{\partial^{i+j}\mu(x,y)}{\partial x^i \partial y^j} = \hbar(x,y) + \lambda \int_0^x \int_0^y H(x,y,t,s)\mu(t,s)dsdt, \tag{1.7}$$

subject to appropriate initial conditions.

Partial integro-differential equations (PIDEs) encompass various types, including elliptic, hyperbolic, and parabolic equations in one or multiple dimensions. Consequently, solving PIDEs poses significant challenges both analytically and numerically. In Chapter 2 of our thesis, we will utilize a novel numerical algorithm to solve the two-dimensional linear PVIDE (1.6) of first-order. Furthermore, in Chapter 3, we will tackle a similar second-order linear 2D-PVIDE analogous to Equation (1.7).

Before delving deeply into the study of VIDEs and PVIDEs, it's valuable to briefly explore some real-world scenarios where these equations are encountered. The examples provided offer a glimpse into a variety of intricate and complex history-dependent models.

1.2 Applications of Volterra integro-differential equations in real-life scenarios

VIEs, VIDEs and PVIDEs naturally emerge in specific varieties of time-dependent issues where the behavior at a particular time t is influenced by both its present and past conditions, in situations where understanding the present state alone is insufficient and it is crucial to comprehend how the state was reached in order to predict future outcomes.

Example 1.1 (**The renewal equation**).
In the realm of probability theory and stochastic processes, the study of processes involving repeated events is facilitated through the use of the renewal equation given by

$$\hbar(t) = \rho(t) + \int_0^t \hbar(t_1)\rho(t - t_1)dt_1,$$

which is a VIE of the second kind.

To illustrate its derivation, consider a scenario where a component within a machine is susceptible to failure over time. The failure time of the component is denoted by a probability density variable $\rho(t)$. *Over short time intervals, the probability of a new component failing at time* t_1 *is* $\rho(t - t_1)\Delta t$. *If each component ultimately fails, then*

$$\int_{t_1}^{\infty} \rho(t - t_1)dt = 1.$$

Now, envision a situation where a failing component is promptly replaced by a new one. This process repeats cyclically as the new component itself will be replaced upon its failure. In this context, $\hbar(t)$ *represents the renewal density signifying the probability of needing a replacement at time* t. *The probability of requiring a replacement is composed of two elements: firstly, the likelihood of the initial failure within a brief time interval, and secondly the probability of a replacement occurring at time* t_1 *accompanied by another failure after a subsequent time* $t_2 = t - t_1$. *These probabilities combined form the renewal equation.*

Example 1.2 (**Population dynamics**).
In the realm of the study of population dynamics lies a fundamental equation that may be expressed as

$$\begin{cases} \frac{dN(t)}{dt} = \alpha N(t), \ t \geq 0, \\ N(0) = N_0, \end{cases} \tag{1.8}$$

where $N(t)$ *represents the count of individuals within a population that are alive at a given time* t, *and the constant* α *represents the growth factor.*

Equation (1.8) *posits that the rate of population change is solely contingent on the number of individuals alive at a given time* t, *However this assumption proves unrealistic in many real-world scenarios. External factors like the depletion of food resources can significantly impact the population's environment. Consequently, the growth factor* α *might vary, influenced by these changing environmental conditions, which in turn are shaped by the population's past history. To accommodate this complexity, a variable growth factor is introduced incorporating a history-dependent term. For instance:*

$$\alpha(t) = \alpha_0 - \int_0^t H(t - \xi)N(\xi)d\xi.$$

Taking this variable growth factor into account, coupled with the competition among individuals within the population, Equation (1.8) *transforms into the VIDE*

$$\frac{dN(t)}{dt} = N(t)\left[\alpha_0 - \alpha_1 N(t) - \int_0^t H(t - \xi)N(\xi)d\xi\right].$$

Example 1.3 (**The Compression of Poro-Viseoelastic Media**).
Predictions regarding the compression progress of a water-saturated porous medium rely on the theory of consolidation, which posits that the porous mass behaves as a two-phase continuum. In this continuum, the fluid phase is considered incompressible, while the porous matrix follows a linear, time-dependent volumetric deformation relationship

$$e = m_v(t)\sigma^{'}.$$

In this context, e represents the dilatation of the porous matrix, m_v denotes the compressibility of the matrix, and $\sigma^{'}$ signifies the effective stress. The general linear relationship between dilatation and effective stress over time is represented by a model consisting of an infinite number of Kelvin units, all interconnected in series, as

$$e = a\sigma^{'} + \sum_{k=1}^{M} \lambda_k \int_0^t \sigma^{'} \exp\left[-\frac{-\lambda_k}{b_k}(t-\tau)\right] d\tau,$$

where a denotes the instantaneous elastic compressibility, b_k represents the retarded elastic compressibility, λ_k symbolizes the fluidity of the soil skeleton, and M stands for the number of Kelvin viscoelastic elements in the system. The corresponding governing differential equation is:

$$k\frac{\partial^2\sigma^{'}}{\partial z^2} = a\frac{\partial\sigma^{'}}{\partial t} + \sum_{k=1}^{M}\lambda_k\sigma^{'} - \sum_{k=1}^{M}\frac{\lambda_k^2}{b_k}\int_0^t \sigma^{'} \exp\left[\frac{\lambda_k}{b_k}(t-\tau)\right] d\tau.$$

This, combined with the appropriate boundary and initial conditions, constitutes a comprehensive mathematical formulation of the one-dimensional partial integro-differential equation of secondary consolidation in a compressible porous medium. For a better understanding of this modeling approach, please refer to references [51] and [52].

Example 1.4 (**Problems in heat conduction and diffusion**).
Consider a situation found in nuclear reactor dynamics where the intricate interplay between the reactor temperature, indicated as $T(y,t)$, and the generated power $\mu(t)$ can be described by a system of PIDEs:

$$\begin{cases} \frac{d\mu(t)}{dt} = \int_{-\infty}^{\infty} \alpha(y)T(y,t)dy, \\ \frac{\partial T(y,t)}{\partial t} = \frac{\partial^2 T(y,t)}{\partial y^2} + n(y)\mu(t), \quad -\infty < y < \infty, t > 0, \end{cases} \tag{1.9}$$

subject to the following conditions

$$\mu(0) = 0, \quad T(y,0) = f_0(y),$$
$$\lim_{y\to\pm\infty} T(y,t) = \lim_{y\to\pm\infty} \frac{\partial}{\partial y}T(y,t) = 0.$$

Here, the first equation represents power production in terms of temperature while the second is essentially a diffusion equation enriched with a source term originating from the reactor's generated power.

Additional examples can be found in refs. [7, 9, 53].

Differential and integral equations are universally recognized as foundational concepts in mathematics, serving as the catalyst for numerous theoretical breakthroughs in analysis. Their broad applications across various contexts in the natural and social sciences formed a seamless connection between them, a topic we will delve into further in our next sections.

1.3 The connection between integral equations and differential equations

Our primary focus is on the convertion between differential equations and integral equations. However, in the course of this process, there may arises a need to differentiate integrals with varying limits of integration. In such cases, we depend on the following rule

Theorem 1.1. *[54, 55](**Leibniz integral rule**)*
Consider a continuous function $\psi(x, y)$ with the assume that its partial derivative $\frac{\partial\psi}{\partial x}$ is continuous within the domain $[a_1, b_1] \times [a_2, b_2]$. Let

$$\Psi(x) = \int_{a(x)}^{b(x)} \psi(x, t)dt,$$

where the limits of integration $a(x)$ and $b(x)$ are defined functions having continuous derivatives for $a_1 < x < b_1$. Hence, differentiation of the integral yields its derivative as follows:

$$\Psi'(x) = \psi(x, b(x))b^{'}(x) - \psi(x, a(x))a^{'}(x) + \int_{a(x)}^{b(x)} \frac{\partial\psi(x, t)}{\partial x} dt.$$

When $a(x) = a$ and $b(x) = b$ are constant values, the Leibniz rule simplifies to

$$\Psi'(x) = \int_a^b \frac{\partial\psi(x, t)}{\partial x} dt.$$

We may also require the following lemma.

Lemma 1.1. *[54] For any function $\Psi(x)$*

$$\int_a^x \int_a^z \Psi(t)dtdz = \int_a^x (x - t)\Psi(t)dt.$$

In general, we have:

$$\int_a^x \int_a^{x_1} ... \int_a^{x_{n-1}} \Psi(x_n)dx_n dx_{n-1}...dx_1 = \frac{1}{(n-1)!} \int_a^x (x - x_1)^{(n-1)} \Psi(x_1)dx_1.$$

1.3.1 Relationship between integral equations and ordinary differential equations

The analysis of differential equations relies significantly on the use of integral equations. A fundamental observation is that any differential equation can be transformed into an integral equation (and vice versa). In fact, Fredholm and Volterra integral equations exhibit a parallel distinction to the differentiation between boundary and initial value problems in ordinary differential equations (ODEs), which often serves as the initial step in exploring their solutions.

Initial value problems

The conversion of initial value problems (IVPs) leads to Volterra integral equations, making this type of integral equations an extension of initial value problems. This feature simplifies the process of finding solutions for IVPs related to ODEs of any order. For instance, consider the linear ODE of the *nth* order:

$$\frac{d^n\mu(x)}{dx^n} + \alpha_1(x)\frac{d^{n-1}\mu(x)}{dx^{n-1}} + \cdots + \alpha_{n-1}(x)\frac{d\mu(x)}{dx} + \alpha_n(x)\mu(x) = \hbar(x), \tag{1.10}$$

where $\alpha_1(x), \alpha_2(x), ..., \alpha_n(x)$ and $\hbar(x)$ are functions defined and continuous within the closed interval $[a, b]$. We are interested in finding a solution $\mu(x)$ of Equation (1.10) that satisfies the predetermined initial conditions at a specific point x_0 within the same interval $[a, b]$

$$\mu(x_0) = \beta_0, \quad \mu'(x_0) = \beta_1, \quad \cdots, \quad \mu^{(n-1)}(x_0) = \beta_{n-1}. \tag{1.11}$$

Considering the function $\omega(x) = \frac{d^n\mu(x)}{dx^n}$ and the initial conditions, we obtain:

$$\begin{aligned}
\frac{d^{n-1}\mu(x)}{dx^{n-1}} &= \int_{x_0}^{x} \omega(t)dt + \beta_{n-1}, \\
\frac{d^{n-2}\mu(x)}{dx^{n-2}} &= \int_{x_0}^{x} (x-t)\omega(t)dt + \beta_{n-1}(x-x_0) + \beta_{n-2}, \\
&\vdots \\
\frac{d\mu(x)}{dx} &= \int_{x_0}^{x} \frac{(x-t)^{n-2}}{(n-2)!}\omega(t)dt + \frac{\beta_{n-1}}{(n-2)!}(x-x_0)^{n-2} + \cdots + \beta_2(x-x_0) + \beta_1, \\
\mu(x) &= \int_{x_0}^{x} \frac{(x-t)^{n-1}}{(n-1)!}\omega(t)dt + \frac{\beta_{n-1}}{(n-1)!}(x-x_0)^{n-1} + \cdots + \beta_1(x-x_0) + \beta_0.
\end{aligned}$$

Hence, if $\mu(x)$ fulfills Equation (1.10) with the initial conditions (1.11), then $\omega(x)$ satisfies the resulting VIE

$$\begin{aligned}
\omega(x) + &\int_{x_0}^{x} \left[\alpha_1(x) + \alpha_2(x)(x-t) + \cdots + \alpha_{n-1}(x)\frac{(x-t)^{n-2}}{(n-2)!} + \alpha_n(x)\frac{(x-t)^{n-1}}{(n-1)!}\right]\omega(t)dt \\
&+ \alpha_1(x)\beta_{n-1} + \alpha_2(x)[\beta_{n-1}(x-x_0) + \beta_{n-2}] + \cdots \\
&+ \alpha_n(x)\left[\frac{\beta_{n-1}}{(n-1)!}(x-x_0)^{n-1} + \cdots + \beta_1(x-x_0) + \beta_0\right] = \hbar(x).
\end{aligned} \tag{1.12}$$

Conversely, if $\omega(x)$ satisfies the integral equation (1.12), then $\mu(x)$ defined by

$$\mu(x) = \int_{x_0}^{x} \frac{(x-t)^{n-1}}{(n-1)!}\omega(t)dt + \frac{\beta_{n-1}}{(n-1)!}(x-x_0)^{n-1} + \cdots + \beta_1(x-x_0) + \beta_0, \tag{1.13}$$

fulfills Equation (1.10) along with initial conditions (1.11).

Boundary value problems

Converting boundary value problems (BVPs) often results in Fredholm integral equations, which simplifies the process of finding solutions for an nth-order ODE within a specific interval. At the boundaries of this interval, the solution and its derivatives up to order $n-1$ are specified at certain values or satisfy given relations. For example, consider the following boundary value problem in the theory of deformation of an elastic rod supported at two points

$$\frac{d^2\mu(x)}{dx^2} = \alpha_1(x)\mu(x) + \alpha_2(x), \tag{1.14}$$

subject to boundary conditions:

$$\mu(a) = 0, \quad \mu(b) = 0, \tag{1.15}$$

where the functions $\alpha_1(x)$ and $\alpha_2(x)$ are continuous and predefined within the interval $[a, b]$. Equation (1.14) admits a general solution in the form of:

$$\begin{aligned} \mu(x) &= \int_a^x \int_a^z \alpha_1(t)\mu(t)dtdz + \int_a^x \int_a^z \alpha_2(t)dtdz + \beta_1 x + \beta_2 \\ &= \int_a^x (x-t)\alpha_1(t)\mu(t)dt + \int_a^x (x-t)\alpha_2(t)dt + \beta_1 x + \beta_2, \end{aligned}$$

where β_1 and β_2 represent arbitrary constants. By selecting these constants in a manner that fulfills the boundary conditions (1.15), the solution to Equation (1.14) is derived as follows:

$$\mu(x) = \int_a^b G(x,t)\alpha_1(t)\mu(t)dt + \int_a^b G(x,t)\alpha_2(t)dt. \tag{1.16}$$

$G(x, t)$ is known as the Green's function and defined by

$$G(x,t) = \begin{cases} \frac{(x-a)(t-b)}{b-a}, & when\ x \leq t, \\ \frac{(x-b)(t-a)}{b-a}, & when\ x \geq t. \end{cases}$$

Therefore, we can deduce that $\mu(x)$, the solution to Equation (1.14) with boundary conditions (1.15), fulfills the Equation (1.16) and vice versa. Hence, the problem's solution is equivalent to solving the FIE of the second kind (1.16).

1.3.2 Relationship between integro-differential equations and partial differential equations

The relationship between integral equations and ordinary differential equations extends far beyond into addressing fundamental challenges in partial differential equations (PDEs) and providing significant practical benefits. Notably, this connection simplifies numerical computations by reducing the dimensionality of specific partial differential equations and enables the solution of various problems. Usually, problems linked to elliptic equations lead to FIEs, while those associated with parabolic and hyperbolic equations yield VIEs.

Here, we will delve deeply into these advantages, examining the relationship between IDEs and PDEs step by step. We will begin by exploring the connection IEs and IDEs have with PDEs of the hyperbolic type, followed by the parabolic type, and finally, the elliptic type. Throughout our study, our focus will primarily be on understanding the association between Volterra-type integro-differential equations and these differential equations, rather than the Fredholm-type. For a more comprehensive understanding, readers are encouraged to refer to refs. [2, 53, 56] for additional details.

Applications to hyperbolic type

Considering the hyperbolic linear initial value problem known as the linear form of Darboux problem:

$$\frac{\partial^2 \mu(x,y)}{\partial x \partial y} = \alpha_1(x,y)\frac{\partial \mu(x,y)}{\partial x} + \alpha_2(x,y)\frac{\partial \mu(x,y)}{\partial y} + \alpha_3(x,y)\mu(x,y) + \hbar(x,y), \tag{1.17}$$

subject to initial conditions

$$\mu(x,0) = \beta_1(x), \;\; \mu(0,y) = \beta_2(y). \tag{1.18}$$

In the defined domain $\Lambda = [0,a] \times [0,b]$, the functions $\alpha_1(x,y)$, $\alpha_2(x,y)$, $\alpha_3(x,y)$ and $\hbar(x,y)$ are continuous and known. Furthermore, the functions $\beta_1(x)$ and $\beta_2(x)$ are in $C^2([0,a])$ and $C^2([0,b])$, respectively.

Solving the Darboux problem entails finding a function $\mu(x,y)$, that is continuous along with its first and mixed derivatives within the closed region Λ and satisfies both the given equation and its initial conditions. Employing similar logic as discussed earlier, this problem can be reformulated as a linear 2D-VIDE by double integration:

$$\mu(x,y) = \Phi(x,y) + \int_0^x \int_0^y \left[\alpha_1(t,s)\frac{\partial \mu(t,s)}{\partial t} + \alpha_2(t,s)\frac{\partial \mu(t,s)}{\partial s} + \alpha_3(t,s)\mu(t,s)\right] dsdt, \tag{1.19}$$

where $\Phi(x,y)$ is defined by the formula

$$\Phi(x,y) = \beta_1(x) + \beta_2(y) - \beta_1(0) + \int_0^x \int_0^y \hbar(t,s)dsdt.$$

Clearly, if $\mu(x,y)$ satisfies Equation (1.17) in Λ along with the inital conditions (1.18), it also satisfies the VIDE (1.19) and vice versa. The resulting VIDE (1.19) can be solved through various methods, such as seeking the solution in the form of a power series sum (refer to ref. [2] for more details) or using numerical approaches such as collocation methods.

Equation (1.17) along with the initial conditions (1.18) can also be reformulated into a 2D-VIE, by defining a new function $\omega(x,y)=\frac{\partial^2\mu(x,y)}{\partial x\partial y}$, given by

$$\omega(x,y)=\Phi(x,y)+\alpha_1\int_0^y\omega(x,s)ds+\alpha_2\int_0^x\omega(t,y)dt+\alpha_3\int_0^x\int_0^y\omega(t,s)dsdt,$$

where Φ is a term obtained by using the initial values conditions.

Remark 2.

- *The discussed Darboux problem can be generalized to encompass a nonlinear hyperbolic equation expressed as:*

$$\frac{\partial^2\mu}{\partial x\partial y}=\mathcal{H}\left(x,y,\mu,\frac{\partial\mu}{\partial x},\frac{\partial\mu}{\partial y}\right), \tag{1.20}$$

where $\mathcal{H}\left(x,y,\mu,\frac{\partial\mu}{\partial x},\frac{\partial\mu}{\partial y}\right)$ is a known function in Λ and μ is described on the two characteristics $x=0$ and $y=0$. This relates to the task of ascertaining the angular orientation of a rigid body in space using its known angular velocity and initial position, expressed in quaternion form.
Several researchers have explored solutions to Equation (1.20), among whom is Dobner, who investigated it utilizing modified fixed point theorems and approximated operators. For further details, refer to ref. [57].

- *If we consider the case where the closed region Λ is a triangle enclosed by two characteristics $x=x_0$ and $y=y_0$ passing through a constant point $A_0(x_0,y_0)$, along with an arc BC of a line B_0C_0, we get the Cauchy problem*

$$\frac{\partial^2\mu(x,y)}{\partial x\partial y}=\lambda\left[\alpha_1(x,y)\frac{\partial\mu(x,y)}{\partial x}+\alpha_2(x,y)\frac{\partial\mu(x,y)}{\partial y}+\alpha_3(x,y)\mu(x,y)\right]+\hbar(x,y). \tag{1.21}$$

The task of solving involves finding a function $\mu(x,y)$ within the region Λ. This function must be continuous, along with its first and mixed derivatives, satisfying Equation (1.21) within Λ along with it's associated initial conditions at every point on the line B_0C_0.
Equation (1.21), after integrating over the triangle Λ, which defines the Cauchy problem, yields to

$$\int\int_\Lambda\frac{\partial^2\mu(t,s)}{\partial x\partial y}dsdt=\lambda\int\int_\Lambda\left[\alpha_1(t,s)\frac{\partial\mu(t,s)}{\partial x}+\alpha_2(t,s)\frac{\partial\mu(t,s)}{\partial y}+\alpha_3(t,s)\mu(t,s)\right]dsdt$$
$$+\int\int_\Lambda\hbar(t,s)dsdt,$$

Thus, it fulfills the following VIDE

$$\mu(x,y)=\Phi(x,y)+\lambda\int\int_\Lambda\left[\alpha_1(t,s)\frac{\partial\mu(t,s)}{\partial t}+\alpha_2(t,s)\frac{\partial\mu(t,s)}{\partial s}+\alpha_3(t,s)\mu(t,s)\right]dsdt, \tag{1.22}$$

where $\Phi(x,y)$ arises from the integration of $\hbar(x,y)$ and the Cauchy conditions. This reduction simplifies the problem to solving the VIDE (1.22), and thus its solution can be examined further. For in-depth information, please refer to ref. [2].

Applications to parabolic type

Reducing PDEs as IDEs extends beyond simple cases like direct integration. Often, for certain parabolic partial differential equations found in heat conduction and diffusion problems, this transformation utilizes Fourier transforms, although Laplace transforms can also be employed in certain cases.

Consider the basic heat conduction equation:

$$\begin{cases} \frac{\partial^2 \mu(x,t)}{\partial x^2} = \frac{\partial \mu(x,t)}{\partial t}, & 0 \leq x < \infty, \ t > 0 \\ \mu(x,0) = 0, & 0 \leq x < \infty \\ \frac{\partial \mu(0,t)}{\partial x} = -\beta(t), & t > 0 \end{cases} \tag{1.23}$$

with the further assumption that,

$$\lim_{x\to\infty} \mu(x,t) = 0 = \lim_{x\to\infty} \frac{\partial \mu(x,t)}{\partial x}.$$

When Fourier cosine transformation

$$\hat{\mu}_c(w) = \sqrt{\frac{2}{\pi}} \int_0^\infty \mu(x) cos(\omega x) dx,$$

is applied to Equation (1.23), we obtain

$$\int_0^\infty \frac{\partial^2 \mu(x,t)}{\partial x^2} \cos(\omega x) dx = \int_0^\infty \frac{\partial \mu(x,t)}{\partial t} \cos(\omega x) dx.$$

By performing two integration by parts and considering the relevant prescribed conditions and the assumptions at infinity, we obtain:

$$\beta(t) - \omega^2 \int_0^\infty \mu(x,t) cos(\omega x) dx = \frac{\partial}{\partial t} \int_0^\infty \mu(x,t) cos(\omega x) dx,$$

however,

$$\sqrt{\frac{\pi}{2}} \frac{\partial}{\partial t} \hat{\mu}_c(\omega,t) = \beta(t) - \sqrt{\frac{\pi}{2}} \omega^2 \hat{\mu}_c(\omega,t).$$

Furthermore, given the initial condition, Equation (1.23) meets the specified criteria and possesses a straightforward solution provided by

$$\hat{\mu}_c(\omega,t) = \sqrt{\frac{2}{\pi}} e^{-\omega^2 t} \int_0^t \beta(s) e^{\omega^2 s} ds.$$

Now, employing the inversion formula

$$\mu(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{\mu}_c(\omega) \cos(\omega x) d\omega,$$

we obtain,

$$\mu(x,t) = \frac{2}{\pi} \int_0^t \beta(s) \int_0^\infty e^{-\omega^2(t-s)} \cos \omega x d\omega ds,$$

hence, through explicit simplifications

$$\mu(x,t) = \frac{1}{\sqrt{\pi}} \int_0^t \beta(s)(t-s)^{-\frac{1}{2}} e^{\frac{-x^2}{4(t-s)}} ds,$$

which is a Volterra integral equation of the first kind.

Remark 3. *In real-world situations, the adopted boundary condition which prescribes the transfer rate as a constant function of time might not be entirely realistic, especially when μ denotes temperature or concentration. Typically, the gradient relies on the surface temperature or concentration, necessitating the substitution of $\beta(t)$ with a function that accounts for this dependency, supposily $B(\mu(0,t),t)$, and $v(t)$ for $\mu(0,t)$. As a result, this gives rise to a nonlinear weakly singular VIE of the second kind*

$$v(t) = \frac{1}{\sqrt{\pi}} \int_0^t \frac{B(v(s),s)}{\sqrt{t-s}} ds.$$

Utilizing the complete Fourier transform, Equation (1.9) can be reformulated into a direct Volterra integro-differential equation. Through integration by parts, consideration of the condition at infinity, and incorporation of the initial conditions, the resulting expression is obtained as follows:

$$\frac{d\mu(t)}{dt} = \int_0^t H(t,s)\mu(s)ds + \beta(t),$$

with

$$H(t,s) = -\int_{-\infty}^{\infty} \hat{\alpha}(-\omega)\hat{n}(\omega)e^{-\omega^2(t-s)}d\omega,$$

and

$$\beta(t) = -\int_{-\infty}^{\infty} \hat{\alpha}(-\omega)\hat{\mu}(\omega)e^{-\omega^2 t}d\omega.$$

For a more in-depth exploration, refer to [53].

Applications to elliptic type

Consider the elliptic PIDE [58]:

$$\nabla^2\mu(x,t) = \hbar(x,t) + \int_0^t \left[\nabla^2\mu(x,s)H(t,s)\right] ds, \tag{1.24}$$

where $x = (x_1, x_2)$, and

$$\nabla^2\mu(x,t) = \frac{\partial^2\mu(x,t)}{\partial x_1^2} + \frac{\partial^2\mu(x,t)}{\partial x_2^2}.$$

Equation (1.24) is subject to homogeneous Dirichlet boundary condition in a rectangular domain Λ with known smooth functions $\hbar$ and H. By defining the new function $\omega(x,t) = \frac{\partial^2\mu(x,t)}{\partial x_1^2} + \frac{\partial^2\mu(x,t)}{\partial x_2^2}$, Equation (1.24) transforms into the following integral equation:

$$\omega(x,t) = \Phi(x,t) + \int_0^t \omega(x,s)H(t,s)ds.$$

However, problems linked to elliptic equations are usually expressed as Fredholm equations, although these are not the primary focus of this thesis. An illuminating exploration of the relationship between IDEs and elliptic-type PDEs is exemplified by the Dirichlet and Neumann problems.
The essence of the Dirichlet problem lies in discovering a harmonic function within a region that is continuous in its closure, while adhering to prescribed boundary values. On the other hand, the von Neumann problem revolves around establishing a harmonic function within a region, continuous in its closure, with a pre-determined set of boundary values for its derivative. While both problems are effectively addressed through the theory of Fredholm equations, they are intricately connected and can be collectively approached. A comprehensive solution to these problems, particularly in the context of a three-dimensional region, is provided in ref. [2].

The significant relevance of integral equations, particularly VIDEs, in modelling real-life phenomena has led to the formulation of complex equations that are challenging to solve directly. Hence, numerical methods become essential for their resolution [59, 60]. In the following section, we will delve into one of the key techniques employed for solving VIDEs: "collocation methods".

1.4 Collocation methods for integral equations

A collocation method is a numerical approach employed to approximate solutions for functional equations such as VIDEs. It involves converting the given equations into a set of algebraic equations by evaluating them at designated points within the problem domain, known as collocation points. For example, Chebyshev collocation methods that provide solutions represented in Chebyshev series form, and Bernstein collocation method which utilizes Bernstein polynomials as its foundation.

The utilization of polynomial or piecewise polynomial collocation spaces for approximating solutions of functional equations traces back to the 1930s. When employing the collocation method to solve a problem within an interval Λ, it involves approximating the solution of the functional equation, such as a VIDE, with an element μ_h belonging to a designated piecewise polynomial space (the collocation space). This approximation that conforms to the equation and, if present, the initial (or boundary) conditions on collocation points is known as the collocation solution. However, what exactly is a **piecewise polynomial space**?

1.4.1 Piecewise polynomial spaces

Consider the grid (mesh) $I_h = \{x_n : 0 = x_0 < x_1 < \cdots < x_N = a\}$ of the the given interval $I = [0, a]$. Let

$$\kappa_n := (x_n, x_{n+1}), \;\; h_n := x_{n+1} - x_n, \;\; h := max\, h_n \;\; h_{min} := min_n\, h_n.$$

Piecewise spaces are defined by four types of grid sequences, as follows:

- The mesh I_h is considered uniform if it satisfies

$$h_n = h_{min} = h = \frac{a}{N}, \;\; n = 0, 1, \cdots, N-1.$$

- The mesh I_h is considered quasi-uniform if it satisfies

$$\frac{h_n}{h_{min}} \leq \frac{h}{h_{min}} \leq \gamma, \quad n = 0, 1, \cdots, N-1.$$

- The mesh I_h is considered graded if it satisfies

$$x_n := \left(\frac{n}{N}\right)^{\alpha} a, \quad n = 0, 1, \cdots, N-1, \ \ \alpha > 1.$$

  Here, α is a real number known as the grading exponent. It's important to note that when $\alpha = 1$, this mesh simplifies into a uniform one.

- The mesh I_h is considered geormetric if it satisfies

$$x_n := \gamma^{N-n} a, \quad n = 0, 1, \cdots, N-1.$$

  The parameter γ, $0 < \gamma < 1$, varies based on N.

This enables us to define the desired spaces. Given a mesh I_h, the real linear vector space $S_r^{(d)}(I_h)$, known as the piecewise polynomial space, is defined as follows:

$$S_r^{(d)}(I_h) := \{\nu \in C^d(I) : v|_{\kappa_n} \in \pi_r \ (0 \leq n \leq N-1)\}.$$

Here, π_r represents the space of real polynomials of degree not exceeding r, where $r \geq 0$ and $0 \leq d \leq r$. Furthermore,

$$dim S_r^{(d)}(I_h) = N(r-d) + d + 1.$$

In our investigation, we focus on the uniform mesh. However, different mesh types are used in specific contexts. For example, quasi-uniform meshes are utilized to assess the convergence of collocation solutions for first-kind VIEs. Graded meshes are employed to examine the achievable order of collocation solutions for VIEs with weakly singular kernels. Geometric meshes are applied to explore the optimal local superconvergence of collocation solutions for functional equations featuring vanishing proportional delays.

On the other hand, the scenario where $r = p + d$, with $p \geq 1$ and $d \geq -1$, will be considred. This results in

$$dim S_{p+d}^{(d)}(I_h) = Np + d + 1.$$

We can broaden the definition of the piecewise polynomial space to two dimensions within the interval $I = [0, a] \times [0, b]$ in the following manner:
Consider real polynomial spline space $S_{p+d,q+d}^{(d)}$ for given meshes I_h and I_k. These partitions are uniform divisions of the intervals $[0, a]$ and $[0, b]$, respectively. Here, the step sizes are denoted by $h = \frac{a}{N}$ and $k = \frac{b}{M}$. The space is defined as follows:

$$S_{p+d,q+d}^{(d)}(I_{h,k}) = \{v \in C^d(I) : \ v_{n,m} = v|_{\Lambda_{n,m}} \in \pi_{p+d,q+d}, \ n = 0, ..., N-1; \ m = 0, 1, .., M-1\},$$

where $I_{h,k} = I_h \times I_k = \{(x_n, y_m), \ 0 \leq n \leq N, \ 0 \leq m \leq M\}$.

Here, $\pi_{p+d,q+d}$ represents the set of all real polynomials of degree not exceeding $p+d$ in x and $q+d$ in y. Additionally, we define the grid

$$\Lambda_{n,m} := \kappa_n \times \delta_m \qquad (n = 0, 1, .., N-1; m = 0, 1, .., M-1),$$

with the subintervals $\kappa_n = (x_n; x_{n+1})$, and $\delta_m = (y_m; y_{m+1})$.

$S^{(d)}_{p+d,q+d}(I_{h,k})$, the space of bivariate polynomial spline functions have a dimension of

$$dim S^{(d)}_{p+d,q+d}(I_{h,k}) = (Np + d + 1)(Mq + d + 1).$$

Remark 4. *The choice of the appropriate collocation space for approximating solutions to initial-value problems for ODEs or VIEs is determined by the regularity degree* (d) *which, in turn, is influenced by the number of specified initial conditions.*

- *In situations without initial conditions, such as in Volterra integral equations, we specifically choose* $d = -1$*. This selection leads us to the designated collocation space* $S^{(-1)}_{p-1}(I_h)$*.*
- *In situations involving a single initial condition, such as in VIDEs, we set* $d = 0$*. This selection leads us to the designated collocation space* $S^{(0)}_{p}(I_h)$*.*
- *In situations involving more than one initial condition, such as in VIDEs of order* n*, we set* $d = n-1$*. This selection leads us to the designated collocation space* $S^{(n-1)}_{p+n-1}(I_h)$*.*

To explore more information on collocation methods for integral equations, consult [61].

1.4.2 Taylor collocation method

During the last decades, the Taylor collocation method has been developed rapidly for a number of advantages, such as the high accuracy and high convergence rate. The lack of need to solve any algebraic systems makes the process of finding numerical solutions to these problems by the Taylor collocation method easy and does not cost much computation.

The Taylor collocation method relies on Taylor polynomials to approximate the exact solution within a specified polynomial spline space. For example, in the real polynomial spline space $S^{(0)}_m(\Pi_N)$ of degree not exceeding m in x, we approximate the exact solution in each rectangle Λ_n, $n = 0, 1, \cdots, N-1$ of the grid by the Taylor polynomials

$$\mu_n(x) = \sum_{j=0}^{m} \frac{\hat{\mu}^{(j)}_n(x_n)}{j!}(x - x_n)^j;\ x \in \Lambda_n,$$

where $\hat{\mu}_n$ represents the precise solution of the discretized integral equation within each rectangle, and $\frac{\hat{\mu}^{(j)}_n(x_n)}{j!}$ are the unknown coefficients to be determined through differentiation.

In the upcoming chapters, we will extend the Taylor collocation method from one-dimensional to two-dimensional, aiming to approximate the exact solutions of first, second, and high-order two-dimensional PVIDEs.

1.5 Some useful discrete and integral inequalities

In the upcoming chapters, we embark on the quest for numerical solutions to carefully chosen PVIDEs using the Taylor collocation method. The obtained results will undergo rigorous theoretical validation incorporating the application of diverse discrete and integral inequalities. In this section, we present the essential inequalities in the form of lemmas while a broader range of discrete and integral inequalities for both one-dimensional and multidimensional differential and integral equations can be explored in the works of B. G. Pachpatte [62–65].

1.5.1 Discrete inequalities

Lemma 1.2. *[64] (* ***Sugiyama's inequality*** *)*

Let Ψ_i, α_i and β_i be non-negative sequences. If Ψ_i satisfies

$$\Psi_i \leq \alpha_i + \sum_{\kappa=0}^{i-1} \beta_\kappa \Psi_\kappa,$$

$\forall i \in \mathbb{N}$. Therefore

$$\Psi_i \leq \alpha_i + \sum_{\kappa=0}^{i-1} \left[\alpha_\kappa \beta_\kappa \prod_{\sigma=\kappa+1}^{i-1} [1+\beta_\sigma] \right], \quad i \in \mathbb{N}.$$

Lemma 1.3. *[65] Let β_i be a given positive sequence and the sequence Ψ_i satisfies $\Psi_0 \leq \alpha$ such that*

$$\Psi_i \leq \alpha + \sum_{\kappa=0}^{i-1} \beta_\kappa \Psi_\kappa, \quad i \geq 1,$$

with $\alpha \geq 0$. Then Ψ_i can be bounded by

$$\Psi_i \leq \alpha \exp\left(\sum_{\iota=0}^{i-1} \beta_\iota \right), \quad i \geq 1.$$

Lemma 1.4. *[66] Let Ψ_{ij} ($i = 0, 1, ..., N$; $j = 0, 1, ..., M$) be a non-negative sequence on $\Lambda = [0, T] \times [0, S]$ satisfying*

$$\Psi_{ij} \leq \alpha + h\beta_1 \sum_{\kappa=0}^{i-1} \Psi_{\kappa j} + k\beta_2 \sum_{\iota=0}^{j-1} \Psi_{i\iota} + hk\beta_3 \sum_{\kappa=0}^{i-1} \sum_{\iota=0}^{j-1} \Psi_{\kappa\iota},$$

with $h, k \geq 0$ such that $\beta_i \, (i = 1, 2, 3)$ and α are both positive and independent of h and k. Then

$$\Psi_{ij} \leq \alpha e^{\lambda(Nh+Mk)},$$

where

$$\lambda = \frac{1}{2}\left(\beta_1 + \beta_2 + \sqrt{(\beta_1+\beta_2)^2 + 4\beta_3} \right).$$

1.5.2 Integral inequalities

Lemma 1.5. *[66] (**Gronwall-type inequality**)*

Let α and $\beta_i\,(i=1,2,3)$ be a non negative constants, and Ψ be a non-negative, integrable, and bounded function defined on $\Lambda=[0,T]\times[0,S]$ where

$$\Psi(x,y)\leq\alpha+\beta_1\int_0^x\Psi(\nu,y)d\nu+\beta_2\int_0^y\Psi(x,\xi)d\xi+\beta_3\int_0^x\int_0^y\Psi(\nu,\xi)d\xi d\nu,$$

for $(x,y)\in\Lambda$. Then Ψ satisfies

$$\Psi(x,y)\leq\alpha e^{\lambda(x+y)},$$

where

$$\lambda=\frac{1}{2}\left(\beta_1+\beta_2+\sqrt{(\beta_1+\beta_2)^2+4\beta_3}\right).$$

Lemma 1.6. *[67] (**Generalization of Lemma 1.5**)*

Suppose that α and β_i $(i=1,2,3)$ are non-negative constants, and Ψ is a non-negative, integrable, and bounded function defined on $\Lambda=[a,b]\times[c,d]$ such that

$$\Psi(x,y)\leq\alpha+\beta_1\int_a^x\Psi(\nu,y)d\nu+\beta_2\int_c^y\Psi(x,\xi)d\xi+\beta_3\int_a^x\int_c^y\Psi(\nu,\xi)d\nu d\xi,\quad (x,y)\in\Lambda,\quad (1.25)$$

then it satisfies

$$\Psi(x,y)\leq\alpha e^{\lambda(x+y)},$$

where

$$\lambda=\frac{1}{2}\left(\beta_1+\beta_2+\sqrt{(\beta_1+\beta_2)^2+4\beta_3}\right).$$

We will also require the use of the following lemma:

Lemma 1.7. *[54] (**Taylor's Theorem for functions of two independent variables**)*

Set Ψ be a p times continuously differentiable function on $\Lambda=[A_1,B_1]\times[A_2,B_2]$ and let $(x_0,y_0)\in\Lambda$. Therefore, for every $(x,y)\in\Lambda$, the following holds:

$$\Psi(x,y)=\sum_{\kappa+\iota=0}^{p-1}\frac{1}{\kappa!\iota!}\frac{\partial^{\kappa+\iota}\Psi(x_0,y_0)}{\partial x^\kappa\partial y^\iota}(x-x_0)^\kappa(y-y_0)^\iota+\sum_{\kappa+\iota=p}\frac{1}{\kappa!\iota!}\frac{\partial^{\kappa+\iota}\Psi(x_1,y_1)}{\partial x^\kappa\partial y^\iota}(x-x_0)^\kappa(y-y_0)^\iota,$$

where

$$\begin{cases} x_1=\theta x+(1-\theta)x_0\in[A_1,B_1], \\ y_1=\theta y+(1-\theta)y_0\in[A_2,B_2], \end{cases}\qquad \theta\in(0,1).$$

Chapter 2

Numerical solution of first order two-dimensional partial Volterra integro-differential equations

In this chapter, our objective is to utilize the Taylor collocation method to investigate a numerical approach for the first-order two-dimensional linear PVIDE represented as:

$$\frac{\partial \mu(y_1, y_2)}{\partial y_1} = \hbar(y_1, y_2) + \int_0^{y_1} \int_0^{y_2} H(y_1, y_2, v_1, v_2)\mu(v_1, v_2)dv_2dv_1, \quad (y_1, y_2) \in \Lambda, \tag{2.1}$$

with the initial condition:

$$\mu(0, y_2) = \mu_0(y_2), \tag{2.2}$$

within the corresponding piecewise polynomial space.
Here, the function μ represents the unknown real function that needs to be determined, while the functions $\hbar$ and H are assumed to be known and sufficiently smooth, ensuring the existence and uniqueness of the solution within the domain $\Lambda := [0, A_1] \times [0, A_2] \subset \mathbb{R}^2$ and the set $S := \{(y_1, y_2, v_1, v_2) : 0 \leq v_1 \leq y_1 \leq A_1, 0 \leq v_2 \leq y_2 \leq A_2\}$, respectively.

The equation presented in (2.1) has been investigated by various researchers using diverse methodologies. Hussain et al. explored it in [68] employing an iterative variational approach. Berenguer and Gamez [69] utilized biorthogonal systems and fixed point theory in Banach spaces to address analogous equations. Singh et al. investigated PVIDEs similar to (2.1) with weakly singular kernels employing an operational matrix approach based on $2D$-shifted Legendre polynomials in [70, 71]. Furthermore, in [72], Singh et al. addressed equations resembling (2.1) using an operational matrix approach employing 2D Legendre and Chebyshev wavelets collocation method.

This chapter is structured into four main sections, each following a logical structure to introduce the proposed method, establish its theoretical foundations, and demonstrate its practical application through numerical examples. In Section 2.1, we explore the application of the proposed method to solve linear 2D-PVIDEs of the form (2.1). Section 2.2 delves into the convergence analysis and error estimates. Additionally, Section 2.3 provides numerous numerical examples and illustrations, showcasing the method's effectiveness and applicability, as well as validating the theoretical results. Finally, the chapter concludes with a summary of the research and its contributions.

2.1 Description of the method

This section is devoted to discussing the Taylor collocation approach to solve the linear first-order 2D-PVIDE (2.1). To address the provided problem using this method, we perform direct integration transforming equation (2.1) with the initial condition (2.2) into the two-dimensional Volterra integral equation:

$$\mu(y_1, y_2) = \Phi(y_1, y_2) + \int_0^{y_1} \int_0^{z} \int_0^{y_2} H(z, y_2, v_1, v_2)\mu(v_1, v_2) dv_2 dv_1 dz, \quad (y_1, y_2) \in \Lambda, \tag{2.3}$$

where $\Phi(y_1, y_2) = \mu_0(y_2) + \int_0^{y_1} \hbar(z, y_2) dz$.

According to Volterra's classical theory, Equation (2.3) has a unique solution $\mu \in C(\Lambda)$. To demonstrate its existence and uniqueness, we utilize a technique similar to the one described in [1, 73], employing Banach's fixed point theorem.

In order to use Taylor polynomials as the basis functions of the presented approach, we suppose $\Pi_N = \{y_{1,i} = ih,\ i = 0, 1, 2, \ldots, N\}$ and $\Pi_M = \{y_{2,j} = jk,\ j = 0, 1, 2, \ldots, M\}$ are two uniform partitions of the intervals $[0, A_1]$ and $[0, A_2]$, where $h = \frac{A_1}{N}$ and $k = \frac{A_2}{M}$, and define $\Pi_{N,M}$ as

$$\Pi_{N,M} = \Pi_N \times \Pi_M = \{(y_{1,n}, y_{2,m}), \quad 0 \le n \le N,\ 0 \le m \le M\}.$$

For $n = 0, 1, 2, \ldots, N-1$, set

$$\kappa_{1,n} = [y_{1,n}, y_{1,n+1}), \qquad \kappa_{1,N-1} = [y_{1,N-1}, y_{1,N}],$$

and for $m = 0, 1, 2, \ldots, M-1$, set

$$\kappa_{2,m} = [y_{2,m}, y_{2,m+1}), \qquad \kappa_{2,M-1} = [y_{2,M-1}, y_{2,M}].$$

Denote by $\pi_{p-1,p-1}$ the set of all real polynomials of degree not exceeding $p-1$ in the two dimensions y_1 and y_2. The real polynomial spline space of degree $p-1$ in y_1 and y_2 is defined by

$$S^{(-1)}_{p-1,p-1}(\Pi_{N,M}) = \{v : \Lambda \to \mathbb{R} : \ v_{n,m} = v|_{\Lambda_{n,m}} \in \pi_{p-1,p-1},\ n = 0, ..., N-1;\ m = 0, 1, .., M-1\}, \tag{2.4}$$

where

$$\Lambda_{n,m} = \kappa_{1,n} \times \kappa_{2,m}, \qquad n = 0, 1, 2, \ldots, N-1,\ m = 0, 1, 2, \ldots, M-1.$$

We approximate the unknown function $\mu(y_1, y_2)$ within each rectangle $\Lambda_{n,m}$, $n = 0, 1, 2, \ldots, N-1$, $m = 0, 1, 2, \ldots, M-1$, by the Taylor polynomial

$$\mu_{n,m}(y_1, y_2) = \sum_{i+j=0}^{p-1} \frac{1}{i!j!} \frac{\partial^{i+j}\hat{\mu}_{n,m}(y_{1,n}, y_{2,m})}{\partial y_1^i \partial y_2^j} (y_1 - y_{1,n})^i (y_2 - y_{2,m})^j \ ; \quad (y_1, y_2) \in \Lambda_{n,m},$$

where $\frac{\partial^{i+j}\hat{\mu}_{n,m}(y_{1,n}, y_{2,m})}{\partial y_1^i \partial y_2^j}$ are unknown coefficients to be determined in the sequel.

Step 1: For $n = m = 0$, we approximate the function $\mu(y_1, y_2)$ within the rectangle $\Lambda_{0,0}$ as the Taylor polynomial

$$\mu_{0,0}(y_1, y_2) = \sum_{i+j=0}^{p-1} \frac{1}{i!j!} \frac{\partial^{i+j}\mu(0,0)}{\partial y_1^i \partial y_2^j} y_1^i y_2^j \; ; \quad (y_1, y_2) \in \Lambda_{0,0}, \tag{2.5}$$

where $\frac{\partial^{i+j}\mu(0,0)}{\partial y_1^i \partial y_2^j}$ represents the precise value of $\frac{\partial^{i+j}\mu}{\partial y_1^i \partial y_2^j}$ at the point $(0,0)$. To determine it, we differentiate Eq. (2.3) $j-$times in terms of y_2

$$\begin{aligned}
\frac{\partial^j \mu(y_1, y_2)}{\partial y_2^j} &= \partial_2^{(j)}\Phi(y_1, y_2) + \int_0^{y_1}\int_0^z \sum_{r=0}^{j-1} \frac{\partial^r}{\partial y_2^r}\left[\partial_2^{(j-1-r)} H(z, y_2, v_1, y_2)\mu(v_1, y_2)\right] dv_1 dz \\
&\quad + \int_0^{y_1}\int_0^z\int_0^{y_2} \partial_2^{(j)} H(z, y_2, v_1, v_2)\mu(v_1, v_2) dv_2 dv_1 dz \\
&= \partial_2^{(j)}\Phi(y_1, y_2) + \sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\int_0^{y_1}\int_0^z \frac{\partial^{r-l}}{\partial y_2^{r-l}}\left[\partial_2^{(j-1-r)} H(z, y_2, v_1, y_2)\right] \frac{\partial^l \mu(v_1, y_2)}{\partial y_2^l} dv_1 dz \\
&\quad + \int_0^{y_1}\int_0^z\int_0^{y_2} \partial_2^{(j)} H(z, y_2, v_1, v_2)\mu(v_1, v_2) dv_2 dv_1 dz.
\end{aligned} \tag{2.6}$$

By differentiating Eq. (2.6) for y_1, we obtain

$$\begin{aligned}
\frac{\partial^{1+j}\mu(y_1, y_2)}{\partial y_1 \partial y_2^j} &= \partial_1^{(1)}\partial_2^{(j)}\Phi(y_1, y_2) + \sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\int_0^{y_1} \frac{\partial^{r-l}}{\partial y_2^{r-l}}\left[\partial_2^{(j-1-r)} H(y_1, y_2, v_1, y_2)\right] \frac{\partial^l \mu(v_1, y_2)}{\partial y_2^l} dv_1 \\
&\quad + \int_0^{y_1}\int_0^{y_2} \partial_2^{(j)} H(y_1, y_2, v_1, v_2)\mu(v_1, v_2) dv_2 dv_1.
\end{aligned} \tag{2.7}$$

Now, we differentiate Eq. (2.7) i-times in terms of y_1, we obtain

$$\begin{aligned}
&\frac{\partial^{i+1+j}\mu(y_1, y_2)}{\partial y_1^{i+1}\partial y_2^j} = \partial_1^{(i+1)}\partial_2^{(j)}\Phi(y_1, y_2) \\
&+ \sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\sum_{q=0}^{i-1}\frac{\partial^q}{\partial y_1^q}\left[\left.\frac{\partial^{i-1-q}}{\partial y_1^{i-1-q}}\right|_{v_1=y_1}\left(\frac{\partial^{r-l}}{\partial y_2^{r-l}}\left[\partial_2^{(j-1-r)} H(y_1, y_2, v_1, y_2)\right]\right)\frac{\partial^l \mu(y_1, y_2)}{\partial y_2^l}\right] \\
&+ \sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\int_0^{y_1}\frac{\partial^i}{\partial y_1^i}\left[\frac{\partial^{r-l}}{\partial y_2^{r-l}}\left[\partial_2^{(j-1-r)} H(y_1, y_2, v_1, y_2)\right]\right]\frac{\partial^l \mu(v_1, y_2)}{\partial y_2^l} dv_1 \\
&+ \int_0^{y_2}\sum_{q=0}^{i-1}\frac{\partial^q}{\partial y_1^q}\left[\partial_1^{(i-1-q)}\partial_2^{(j)} H(y_1, y_2, y_1, v_2)\mu(y_1, v_2)\right] dv_2 \\
&+ \int_0^{y_1}\int_0^{y_2}\partial_1^{(i)}\partial_2^{(j)} H(y_1, y_2, v_1, v_2)\mu(v_1, v_2) dv_2 dv_1,
\end{aligned}$$

which implies

$$\begin{aligned}
&\frac{\partial^{i+1+j}\mu(y_1, y_2)}{\partial y_1^{i+1}\partial y_2^j} = \partial_1^{(i+1)}\partial_2^{(j)}\Phi(y_1, y_2) \\
&+ \sum_{r=0}^{j-1}\sum_{l=0}^{r}\sum_{q=0}^{i-1}\sum_{\eta=0}^{q}\binom{r}{l}\binom{q}{\eta}\frac{\partial^{q-\eta}}{\partial y_1^{q-\eta}}\left[\left.\frac{\partial^{i-1-q}}{\partial y_1^{i-1-q}}\right|_{v_1=y_1}\left(\frac{\partial^{r-l}}{\partial y_2^{r-l}}\left[\partial_2^{(j-1-r)} H(y_1, y_2, v_1, y_2)\right]\right)\right]\frac{\partial^{l+\eta}\mu(y_1, y_2)}{\partial y_1^\eta \partial y_2^l} \\
&+ \sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\int_0^{y_1}\frac{\partial^i}{\partial y_1^i}\left[\frac{\partial^{r-l}}{\partial y_2^{r-l}}\left[\partial_2^{(j-1-r)} H(y_1, y_2, v_1, y_2)\right]\right]\frac{\partial^l \mu(v_1, y_2)}{\partial y_2^l} dv_1
\end{aligned}$$

$$+\int_0^{y_2}\sum_{q=0}^{i-1}\sum_{\eta=0}^{q}\binom{q}{\eta}\frac{\partial^{q-\eta}}{\partial y_1^{q-\eta}}\left[\partial_1^{(i-1-q)}\partial_2^{(j)}H(y_1,y_2,y_1,v_2)\right]\frac{\partial^{\eta}\mu(y_1,v_2)}{\partial y_1^{\eta}}dv_2$$
$$+\int_0^{y_1}\int_0^{y_2}\partial_1^{(i)}\partial_2^{(j)}H(y_1,y_2,v_1,v_2)\mu(v_1,v_2)dv_2dv_1.$$

Hence

$$\frac{\partial^{i+j+1}\mu(0,0)}{\partial y_1^{i+1}\partial y_2^{j}}=\partial_1^{(i+1)}\partial_2^{(j)}\Phi(0,0)$$
$$+\sum_{r=0}^{j-1}\sum_{l=0}^{r}\sum_{q=0}^{i-1}\sum_{\eta=0}^{q}\binom{r}{l}\binom{q}{\eta}\frac{\partial^{q-\eta}}{\partial y_1^{q-\eta}}\left[\left.\frac{\partial^{i-1-q}}{\partial y_1^{i-1-q}}\right|_{v_1=y_1}\left(\frac{\partial^{r-l}}{\partial y_2^{r-l}}\left[\partial_2^{(j-1-r)}H(y_1,y_2,v_1,y_2)\right]\right)\right]_{y_1=y_2=0}\times$$
$$\frac{\partial^{l+\eta}\mu(0,0)}{\partial y_1^{\eta}\partial y_2^{l}}.$$

Moreover, from Eqs. (2.6) and (2.7), we deduce that

$$\frac{\partial^{1+j}\mu(0,0)}{\partial y_1\partial y_2^{j}}=\partial_1^{(1)}\partial_2^{(j)}\Phi(0,0),\qquad \frac{\partial^{j}\mu(0,0)}{\partial y_2^{j}}=\partial_2^{(j)}\Phi(0,0).$$

Step 2: For $n=1,\cdots,N-1$ and $m=0$, we approximate the function $\mu(y_1,y_2)$ within the rectangles $\Lambda_{n,0}$ by the Taylor polynomial

$$\mu_{n,0}(y_1,y_2)=\sum_{i+j=0}^{p-1}\frac{1}{i!j!}\frac{\partial^{i+j}\hat{\mu}_{n,0}(y_{1,n},0)}{\partial y_1^{i}\partial y_2^{j}}(y_1-y_{1,n})^{i}y_2^{j}\ ;\quad (y_1,y_2)\in\Lambda_{n,0}. \tag{2.8}$$

Here, $\hat{\mu}_{n,0}$ represents the precise solution of the integral equation:

$$\begin{aligned}\hat{\mu}_{n,0}(y_1,y_2)=\Phi(y_1,y_2)&+\sum_{\xi=0}^{n-1}\sum_{\sigma=0}^{\xi-1}\int_{y_{1,\xi}}^{y_{1,\xi+1}}\int_{y_{1,\sigma}}^{y_{1,\sigma+1}}\int_0^{y_2}H(z,y_2,v_1,v_2)\mu_{\sigma,0}(v_1,v_2)dv_2dv_1dz\\&+\sum_{\xi=0}^{n-1}\int_{y_{1,\xi}}^{y_{1,\xi+1}}\int_{y_{1,\xi}}^{z}\int_0^{y_2}H(z,y_2,v_1,v_2)\mu_{\xi,0}(v_1,v_2)dv_2dv_1dz\\&+\sum_{\sigma=0}^{n-1}\int_{y_{1,n}}^{y_1}\int_{y_{1,\sigma}}^{y_{1,\sigma+1}}\int_0^{y_2}H(z,y_2,v_1,v_2)\mu_{\sigma,0}(v_1,v_2)dv_2dv_1dz\\&+\int_{y_{1,n}}^{y_1}\int_{y_{1,n}}^{z}\int_0^{y_2}H(z,y_2,v_1,v_2)\hat{\mu}_{n,0}(v_1,v_2)dv_2dv_1dz.\end{aligned} \tag{2.9}$$

To find $\frac{\partial^{j}\hat{\mu}_{n,0}(y_1,y_2)}{\partial y_2^{j}}$, we differentiate Eq. (2.9) j-times in terms of y_2 to get

$$\begin{aligned}\frac{\partial^{j}\hat{\mu}_{n,0}(y_1,y_2)}{\partial y_2^{j}}=\partial_2^{(j)}\Phi(y_1,y_2)&+\sum_{\xi=0}^{n-1}\sum_{\sigma=0}^{\xi-1}\int_{y_{1,\xi}}^{y_{1,\xi+1}}\int_{y_{1,\sigma}}^{y_{1,\sigma+1}}\int_0^{y_2}\partial_2^{(j)}H(z,y_2,v_1,v_2)\mu_{\sigma,0}(v_1,v_2)dv_2dv_1dz\\&+\sum_{\xi=0}^{n-1}\sum_{\sigma=0}^{\xi-1}\int_{y_{1,\xi}}^{y_{1,\xi+1}}\int_{y_{1,\sigma}}^{y_{1,\sigma+1}}\sum_{r=0}^{j-1}\frac{\partial^{r}}{\partial y_2^{r}}\left[\partial_2^{(j-1-r)}H(z,y_2,v_1,y_2)\mu_{\sigma,0}(v_1,y_2)\right]dv_1dz\\&+\sum_{\xi=0}^{n-1}\int_{y_{1,\xi}}^{y_{1,\xi+1}}\int_{y_{1,\xi}}^{z}\sum_{r=0}^{j-1}\frac{\partial^{r}}{\partial y_2^{r}}\left[\partial_2^{(j-1-r)}H(z,y_2,v_1,y_2)\mu_{\xi,0}(v_1,y_2)\right]dv_1dz\end{aligned}$$

$$
\begin{aligned}
&+\sum_{\xi=0}^{n-1}\int_{y_{1,\xi}}^{y_{1,\xi+1}}\int_{y_{1,\xi}}^{z}\int_{0}^{y_2}\partial_2^{(j)}H(z,y_2,v_1,v_2)\mu_{\xi,0}(v_1,v_2)dv_2dv_1dz\\
&+\sum_{\sigma=0}^{n-1}\int_{y_{1,n}}^{y_1}\int_{y_{1,\sigma}}^{y_{1,\sigma+1}}\sum_{r=0}^{j-1}\frac{\partial^r}{\partial y_2^r}\left[\partial_2^{(j-1-r)}H(z,y_2,v_1,y_2)\mu_{\sigma,0}(v_1,y_2)\right]dv_1dz\\
&+\sum_{\sigma=0}^{n-1}\int_{y_{1,n}}^{y_1}\int_{y_{1,\sigma}}^{y_{1,\sigma+1}}\int_{0}^{y_2}\partial_2^{(j)}H(z,y_2,v_1,v_2)\mu_{\sigma,0}(v_1,v_2)dv_2dv_1dz\\
&+\int_{y_{1,n}}^{y_1}\int_{y_{1,n}}^{z}\sum_{r=0}^{j-1}\frac{\partial^r}{\partial y_2^r}\left[\partial_2^{(j-1-r)}H(z,y_2,v_1,y_2)\hat{\mu}_{n,0}(v_1,y_2)\right]dv_1dz\\
&+\int_{y_{1,n}}^{y_1}\int_{y_{1,n}}^{z}\int_{0}^{y_2}\partial_2^{(j)}H(z,y_2,v_1,v_2)\hat{\mu}_{n,0}(v_1,v_2)dv_2dv_1dz\\
=&\partial_2^{(j)}\Phi(y_1,y_2)+\sum_{\xi=0}^{n-1}\sum_{\sigma=0}^{\xi-1}\int_{y_{1,\xi}}^{y_{1,\xi+1}}\int_{y_{1,\sigma}}^{y_{1,\sigma+1}}\int_{0}^{y_2}\partial_2^{(j)}H(z,y_2,v_1,v_2)\mu_{\sigma,0}(v_1,v_2)dv_2dv_1dz\\
&+\sum_{\xi=0}^{n-1}\sum_{\sigma=0}^{\xi-1}\sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\int_{y_{1,\xi}}^{y_{1,\xi+1}}\int_{y_{1,\sigma}}^{y_{1,\sigma+1}}\frac{\partial^{r-l}}{\partial y_2^{r-l}}\left[\partial_2^{(j-1-r)}H(z,y_2,v_1,y_2)\right]\frac{\partial^l\mu_{\sigma,0}(v_1,y_2)}{\partial y_2^l}dv_1dz\\
&+\sum_{\xi=0}^{n-1}\sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\int_{y_{1,\xi}}^{y_{1,\xi+1}}\int_{y_{1,\xi}}^{z}\frac{\partial^{r-l}}{\partial y_2^{r-l}}\left[\partial_2^{(j-1-r)}H(z,y_2,v_1,y_2)\right]\frac{\partial^l\mu_{\xi,0}(v_1,y_2)}{\partial y_2^l}dv_1dz\\
&+\sum_{\xi=0}^{n-1}\int_{y_{1,\xi}}^{y_{1,\xi+1}}\int_{y_{1,\xi}}^{z}\int_{0}^{y_2}\partial_2^{(j)}H(z,y_2,v_1,v_2)\mu_{\xi,0}(v_1,v_2)dv_2dv_1dz\\
&+\sum_{\sigma=0}^{n-1}\sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\int_{y_{1,n}}^{y_1}\int_{y_{1,\sigma}}^{y_{1,\sigma+1}}\frac{\partial^{r-l}}{\partial y_2^{r-l}}\left[\partial_2^{(j-1-r)}H(z,y_2,v_1,y_2)\right]\frac{\partial^l\mu_{\sigma,0}(v_1,y_2)}{\partial y_2^l}dv_1dz\\
&+\sum_{\sigma=0}^{n-1}\int_{y_{1,n}}^{y_1}\int_{y_{1,\sigma}}^{y_{1,\sigma+1}}\int_{0}^{y_2}\partial_2^{(j)}H(z,y_2,v_1,v_2)\mu_{\sigma,0}(v_1,v_2)dv_2dv_1dz\\
&+\sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\int_{y_{1,n}}^{y_1}\int_{y_{1,n}}^{z}\frac{\partial^{r-l}}{\partial y_2^{r-l}}\left[\partial_2^{(j-1-r)}H(z,y_2,v_1,y_2)\right]\frac{\partial^l\hat{\mu}_{n,0}(v_1,y_2)}{\partial y_2^l}dv_1dz\\
&+\int_{y_{1,n}}^{y_1}\int_{y_{1,n}}^{z}\int_{0}^{y_2}\partial_2^{(j)}H(z,y_2,v_1,v_2)\hat{\mu}_{n,0}(v_1,v_2)dv_2dv_1dz.
\end{aligned}\tag{2.10}
$$

We differentiate Eq. (2.10) for y_1, we get

$$
\begin{aligned}
&\frac{\partial^{j+1}\hat{\mu}_{n,0}(y_1,y_2)}{\partial y_1\partial y_2^j}=\partial_1^{(1)}\partial_2^{(j)}\Phi(y_1,y_2)+\sum_{\sigma=0}^{n-1}\int_{y_{1,\sigma}}^{y_{1,\sigma+1}}\int_{0}^{y_2}\partial_2^{(j)}H(y_1,y_2,v_1,v_2)\mu_{\sigma,0}(v_1,v_2)dv_2dv_1\\
&+\sum_{\sigma=0}^{n-1}\sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\int_{y_{1,\sigma}}^{y_{1,\sigma+1}}\frac{\partial^{r-l}}{\partial y_2^{r-l}}\left[\partial_2^{(j-1-r)}H(y_1,y_2,v_1,y_2)\right]\frac{\partial^l\mu_{\sigma,0}(v_1,y_2)}{\partial y_2^l}dv_1\\
&+\sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\int_{y_{1,n}}^{y_1}\frac{\partial^{r-l}}{\partial y_2^{r-l}}\left[\partial_2^{(j-1-r)}H(y_1,y_2,v_1,y_2)\right]\frac{\partial^l\hat{\mu}_{n,0}(v_1,y_2)}{\partial y_2^l}dv_1\\
&+\int_{y_{1,n}}^{y_1}\int_{0}^{y_2}\partial_2^{(j)}H(y_1,y_2,v_1,v_2)\hat{\mu}_{n,0}(v_1,v_2)dv_2dv_1.
\end{aligned}\tag{2.11}
$$

Now, we differentiate Eq. (2.11) i-times in terms of y_1

$$\begin{aligned}
\frac{\partial^{i+j+1}\hat{\mu}_{n,0}(y_1,y_2)}{\partial y_1^{i+1}\partial y_2^{j}} &= \partial_1^{(i+1)}\partial_2^{(j)}\Phi(y_1,y_2) \\
&+ \sum_{\sigma=0}^{n-1}\sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\int_{y_{1,\sigma}}^{y_{1,\sigma+1}}\frac{\partial^i}{\partial y_1^i}\left[\frac{\partial^{r-l}}{\partial y_2^{r-l}}\left[\partial_2^{(j-1-r)}H(y_1,y_2,v_1,y_2)\right]\right]\frac{\partial^l\mu_{\sigma,0}(v_1,y_2)}{\partial y_2^l}dv_1 \\
&+ \sum_{\sigma=0}^{n-1}\int_{y_{1,\sigma}}^{y_{1,\sigma+1}}\int_0^{y_2}\partial_1^{(i)}\partial_2^{(j)}H(y_1,y_2,v_1,v_2)\mu_{\sigma,0}(v_1,v_2)dv_2dv_1 \\
&+ \sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\sum_{q=0}^{i-1}\frac{\partial^q}{\partial y_1^q}\left[\left.\frac{\partial^{i-1-q}}{\partial y_1^{i-1-q}}\right|_{v_1=y_1}\left(\frac{\partial^{r-l}}{\partial y_2^{r-l}}\left[\partial_2^{(j-1-r)}H(y_1,y_2,v_1,y_2)\right]\right)\frac{\partial^l\hat{\mu}_{n,0}(y_1,y_2)}{\partial y_2^l}\right] \\
&+ \sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\int_{y_{1,n}}^{y_1}\frac{\partial^i}{\partial y_1^i}\left[\frac{\partial^{r-l}}{\partial y_2^{r-l}}\left[\partial_2^{(j-1-r)}H(y_1,y_2,v_1,y_2)\right]\right]\frac{\partial^l\hat{\mu}_{n,0}(v_1,y_2)}{\partial y_2^l}dv_1 \\
&+ \int_0^{y_2}\sum_{q=0}^{i-1}\frac{\partial^q}{\partial y_1^q}\left[\partial_1^{(i-1-q)}\partial_2^{(j)}H(y_1,y_2,y_1,v_2)\hat{\mu}_{n,0}(y_1,v_2)\right]dv_2 \\
&+ \int_{y_{1,n}}^{y_1}\int_0^{y_2}\partial_1^{(i)}\partial_2^{(j)}H(y_1,y_2,v_1,v_2)\hat{\mu}_{n,0}(v_1,v_2)dv_2dv_1,
\end{aligned}$$

which implies

$$\begin{aligned}
\frac{\partial^{i+j+1}\hat{\mu}_{n,0}(y_1,y_2)}{\partial y_1^{i+1}\partial y_2^{j}} &= \partial_1^{(i+1)}\partial_2^{(j)}\Phi(y_1,y_2) + \sum_{\sigma=0}^{n-1}\int_{y_{1,\sigma}}^{y_{1,\sigma+1}}\int_0^{y_2}\partial_1^{(i)}\partial_2^{(j)}H(y_1,y_2,v_1,v_2)\mu_{\sigma,0}(v_1,v_2)dv_2dv_1 \\
&+ \sum_{\sigma=0}^{n-1}\sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\int_{y_{1,\sigma}}^{y_{1,\sigma+1}}\frac{\partial^i}{\partial y_1^i}\left[\frac{\partial^{r-l}}{\partial y_2^{r-l}}\left[\partial_2^{(j-1-r)}H(y_1,y_2,v_1,y_2)\right]\right]\frac{\partial^l\mu_{\sigma,0}(v_1,y_2)}{\partial y_2^l}dv_1 \\
&+ \sum_{r=0}^{j-1}\sum_{l=0}^{r}\sum_{q=0}^{i-1}\sum_{\eta=0}^{q}\binom{r}{l}\binom{q}{\eta}\frac{\partial^{q-\eta}}{\partial y_1^{q-\eta}}\left[\left.\frac{\partial^{i-1-q}}{\partial y_1^{i-1-q}}\right|_{v_1=y_1}\left(\frac{\partial^{r-l}}{\partial y_2^{r-l}}\left[\partial_2^{(j-1-r)}H(y_1,y_2,v_1,y_2)\right]\right)\right]\frac{\partial^{l+\eta}\hat{\mu}_{n,0}(y_1,y_2)}{\partial y_1^{\eta}\partial y_2^l} \\
&+ \sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\int_{y_{1,n}}^{y_1}\frac{\partial^i}{\partial y_1^i}\left[\frac{\partial^{r-l}}{\partial y_2^{r-l}}\left[\partial_2^{(j-1-r)}H(y_1,y_2,v_1,y_2)\right]\right]\frac{\partial^l\hat{\mu}_{n,0}(v_1,y_2)}{\partial y_2^l}dv_1 \\
&+ \int_0^{y_2}\sum_{q=0}^{i-1}\sum_{\eta=0}^{q}\binom{q}{\eta}\frac{\partial^{q-\eta}}{\partial y_1^{q-\eta}}\left[\partial_1^{(i-1-q)}\partial_2^{(j)}H(y_1,y_2,y_1,v_2)\right]\frac{\partial^{\eta}\hat{\mu}_{n,0}(y_1,v_2)}{\partial y_1^{\eta}}dv_2 \\
&+ \int_{y_{1,n}}^{y_1}\int_0^{y_2}\partial_1^{(i)}\partial_2^{(j)}H(y_1,y_2,v_1,v_2)\hat{\mu}_{n,0}(v_1,v_2)dv_2dv_1. \qquad (2.12)
\end{aligned}$$

Hence

$$\begin{aligned}
\frac{\partial^{i+j+1}\hat{\mu}_{n,0}(y_{1,n},0)}{\partial y_1^{i+1}\partial y_2^{j}} &= \partial_1^{(i+1)}\partial_2^{(j)}\Phi(y_{1,n},0) \\
&+ \sum_{\sigma=0}^{n-1}\sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\int_{y_{1,\sigma}}^{y_{1,\sigma+1}}\frac{\partial^i}{\partial y_1^i}\left[\frac{\partial^{r-l}}{\partial y_2^{r-l}}\left[\partial_2^{(j-1-r)}H(y_1,y_2,v_1,y_2)\right]\right]_{y_1=y_{1,n},y_2=0}\frac{\partial^l\mu_{\sigma,0}(v_1,0)}{\partial y_2^l}dv_1 \\
&+ \sum_{r=0}^{j-1}\sum_{l=0}^{r}\sum_{q=0}^{i-1}\sum_{\eta=0}^{q}\binom{r}{l}\binom{q}{\eta}\frac{\partial^{q-\eta}}{\partial y_1^{q-\eta}}\left[\left.\frac{\partial^{i-1-q}}{\partial y_1^{i-1-q}}\right|_{v_1=y_1}\left(\frac{\partial^{r-l}}{\partial y_2^{r-l}}\left[\partial_2^{(j-1-r)}H(y_1,y_2,v_1,y_2)\right]\right)\right]_{y_1=y_{1,n},y_2=0}\times \\
&\qquad \frac{\partial^{l+\eta}\hat{\mu}_{n,0}(y_{1,n},0)}{\partial y_1^{\eta}\partial y_2^l}.
\end{aligned}$$

Moreover, from Eq. (2.10), we deduce that

$$\frac{\partial^j \hat{\mu}_{n,0}(y_{1,n},0)}{\partial y_2^j} = \partial_2^{(j)}\Phi(y_1,y_2) + \sum_{\xi=0}^{n-1}\sum_{\sigma=0}^{\xi-1}\sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\int_{y_{1,\xi}}^{y_{1,\xi+1}}\int_{y_{1,\sigma}}^{y_{1,\sigma+1}}\frac{\partial^{r-l}}{\partial y_2^{r-l}}\left[\partial_2^{(j-1-r)}H(z,y_2,v_1,y_2)\right]_{y_1=y_{1,n},y_2=0}\frac{\partial^l \mu_{\sigma,0}(v_1,0)}{\partial y_2^l}dv_1dz,$$

and from Eq. (2.11), we get

$$\frac{\partial^{j+1}\hat{\mu}_{n,0}(y_{1,n},0)}{\partial y_1^1\partial y_2^j} = \partial_1^{(i+1)}\partial_2^{(j)}\Phi(y_{1,n},0) + \sum_{\sigma=0}^{n-1}\sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\int_{y_{1,\sigma}}^{y_{1,\sigma+1}}\left[\frac{\partial^{r-l}}{\partial y_2^{r-l}}\left[\partial_2^{(j-1-r)}H(y_1,y_2,v_1,y_2)\right]\right]_{y_1=y_{1,n},y_2=0}\frac{\partial^l \mu_{\sigma,0}(v_1,0)}{\partial y_2^l}dv_1.$$

Step 3: For $n=0,...,N-1$ and $m=1,...,M-1$, we approximate the function $\mu(y_1,y_2)$ within each rectangle $\Lambda_{n,m}$ by the Taylor polynomial

$$\mu_{n,m}(y_1,y_2) = \sum_{i+j=0}^{p-1}\frac{1}{i!j!}\frac{\partial^{i+j}\hat{\mu}_{n,m}(y_{1,n},y_{2,m})}{\partial y_1^i\partial y_2^j}(y_1-y_{1,n})^i(y_2-y_{2,m})^j\ ; \quad (y_1,y_2)\in\Lambda_{n,m}, \tag{2.13}$$

where $\hat{\mu}_{n,m}$ stands for the exact solution of the integral equation:

$$\begin{aligned}
\hat{\mu}_{n,m}(y_1,y_2) = \Phi(y_1,y_2) &+ \sum_{\xi=0}^{n-1}\sum_{\sigma=0}^{\xi-1}\sum_{\rho=0}^{m-1}\int_{y_{1,\xi}}^{y_{1,\xi+1}}\int_{y_{1,\sigma}}^{y_{1,\sigma+1}}\int_{y_{2,\rho}}^{y_{2,\rho+1}}H(z,y_2,v_1,v_2)\mu_{\sigma,\rho}(v_1,v_2)dv_2dv_1dz\\
&+ \sum_{\xi=0}^{n-1}\sum_{\sigma=0}^{\xi-1}\int_{y_{1,\xi}}^{y_{1,\xi+1}}\int_{y_{1,\sigma}}^{y_{1,\sigma+1}}\int_{y_{2,m}}^{y_2}H(z,y_2,v_1,v_2)\mu_{\sigma,m}(v_1,v_2)dv_2dv_1dz\\
&+ \sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1}\int_{y_{1,\xi}}^{y_{1,\xi+1}}\int_{y_{1,\xi}}^{z}\int_{y_{2,\rho}}^{y_{2,\rho+1}}H(z,y_2,v_1,v_2)\mu_{\xi,\rho}(v_1,v_2)dv_2dv_1dz\\
&+ \sum_{\xi=0}^{n-1}\int_{y_{1,\xi}}^{y_{1,\xi+1}}\int_{y_{1,\xi}}^{z}\int_{y_{2,m}}^{y_2}H(z,y_2,v_1,v_2)\mu_{\xi,m}(v_1,v_2)dv_2dv_1dz\\
&+ \sum_{\sigma=0}^{n-1}\sum_{\rho=0}^{m-1}\int_{y_{1,n}}^{y_1}\int_{y_{1,\sigma}}^{y_{1,\sigma+1}}\int_{y_{2,\rho}}^{y_{2,\rho+1}}H(z,y_2,v_1,v_2)\mu_{\sigma,\rho}(v_1,v_2)dv_2dv_1dz\\
&+ \sum_{\sigma=0}^{n-1}\int_{y_{1,n}}^{y_1}\int_{y_{1,\sigma}}^{y_{1,\sigma+1}}\int_{y_{2,m}}^{y_2}H(z,y_2,v_1,v_2)\mu_{\sigma,m}(v_1,v_2)dv_2dv_1dz\\
&+ \sum_{\rho=0}^{m-1}\int_{y_{1,n}}^{y_1}\int_{y_{1,n}}^{z}\int_{y_{2,\rho}}^{y_{2,\rho+1}}H(z,y_2,v_1,v_2)\mu_{n,\rho}(v_1,v_2)dv_2dv_1dz\\
&+ \int_{y_{1,n}}^{y_1}\int_{y_{1,n}}^{z}\int_{y_{2,m}}^{y_2}H(z,y_2,v_1,v_2)\hat{\mu}_{n,m}(v_1,v_2)dv_2dv_1dz,
\end{aligned} \tag{2.14}$$

for $(y_1,y_2)\in\Lambda_{n,m}, n=0,...,N-1$ and $m=1,...,M-1$.

To find $\frac{\partial^j \hat{\mu}_{n,m}(y_1,y_2)}{\partial y_2^j}$, we differentiate Eq. (2.14) j-times in terms of y_2

$$\begin{aligned}
\frac{\partial^j \hat{\mu}_{n,m}(y_1,y_2)}{\partial y_2^j} &= \partial_2^{(j)}\Phi(y_1,y_2) + \sum_{\xi=0}^{n-1}\sum_{\sigma=0}^{\xi-1}\sum_{\rho=0}^{m-1}\int_{y_{1,\xi}}^{y_{1,\xi+1}}\int_{y_{1,\sigma}}^{y_{1,\sigma+1}}\int_{y_{2,\rho}}^{y_{2,\rho+1}} \partial_2^{(j)}H(z,y_2,v_1,v_2)\mu_{\sigma,\rho}(v_1,v_2)dv_2dv_1dz \\
&+ \sum_{\xi=0}^{n-1}\sum_{\sigma=0}^{\xi-1}\int_{y_{1,\xi}}^{y_{1,\xi+1}}\int_{y_{1,\sigma}}^{y_{1,\sigma+1}}\sum_{r=0}^{j-1}\frac{\partial^r}{\partial y_2^r}\left[\partial_2^{(j-1-r)}H(z,y_2,v_1,y_2)\mu_{\sigma,m}(v_1,y_2)\right]dv_1dz \\
&+ \sum_{\xi=0}^{n-1}\sum_{\sigma=0}^{\xi-1}\int_{y_{1,\xi}}^{y_{1,\xi+1}}\int_{y_{1,\sigma}}^{y_{1,\sigma+1}}\int_{y_{2,m}}^{y_2} \partial_2^{(j)}H(z,y_2,v_1,v_2)\mu_{\sigma,m}(v_1,v_2)dv_2dv_1dz \\
&+ \sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1}\int_{y_{1,\xi}}^{y_{1,\xi+1}}\int_{y_{1,\xi}}^{z}\int_{y_{2,\rho}}^{y_{2,\rho+1}} \partial_2^{(j)}H(z,y_2,v_1,v_2)\mu_{\xi,\rho}(v_1,v_2)dv_2dv_1dz \\
&+ \sum_{\xi=0}^{n-1}\int_{y_{1,\xi}}^{y_{1,\xi+1}}\int_{y_{1,\xi}}^{z}\sum_{r=0}^{j-1}\frac{\partial^r}{\partial y_2^r}\left[\partial_2^{(j-1-r)}H(z,y_2,v_1,y_2)\mu_{\xi,m}(v_1,y_2)\right]dv_1dz \\
&+ \sum_{\xi=0}^{n-1}\int_{y_{1,\xi}}^{y_{1,\xi+1}}\int_{y_{1,\xi}}^{z}\int_{y_{2,m}}^{y_2} \partial_2^{(j)}H(z,y_2,v_1,v_2)\mu_{\xi,m}(v_1,v_2)dv_2dv_1dz \\
&+ \sum_{\sigma=0}^{n-1}\sum_{\rho=0}^{m-1}\int_{y_{1,n}}^{y_1}\int_{y_{1,\sigma}}^{y_{1,\sigma+1}}\int_{y_{2,\rho}}^{y_{2,\rho+1}} \partial_2^{(j)}H(z,y_2,v_1,v_2)\mu_{\sigma,\rho}(v_1,v_2)dv_2dv_1dz \\
&+ \sum_{\sigma=0}^{n-1}\int_{y_{1,n}}^{y_1}\int_{y_{1,\sigma}}^{y_{1,\sigma+1}}\sum_{r=0}^{j-1}\frac{\partial^r}{\partial y_2^r}\left[\partial_2^{(j-1-r)}H(z,y_2,v_1,y_2)\mu_{\sigma,m}(v_1,y_2)\right]dv_1dz \\
&+ \sum_{\sigma=0}^{n-1}\int_{y_{1,n}}^{y_1}\int_{y_{1,\sigma}}^{y_{1,\sigma+1}}\int_{y_{2,m}}^{y_2} \partial_2^{(j)}H(z,y_2,v_1,v_2)\mu_{\sigma,m}(v_1,v_2)dv_2dv_1dz \\
&+ \sum_{\rho=0}^{m-1}\int_{y_{1,n}}^{y_1}\int_{y_{1,n}}^{z}\int_{y_{2,\rho}}^{y_{2,\rho+1}} \partial_2^{(j)}H(z,y_2,v_1,v_2)\mu_{n,\rho}(v_1,v_2)dv_2dv_1dz \\
&+ \int_{y_{1,n}}^{y_1}\int_{y_{1,n}}^{z}\sum_{r=0}^{j-1}\frac{\partial^r}{\partial y_2^r}\left[\partial_2^{(j-1-r)}H(z,y_2,v_1,y_2)\hat{\mu}_{n,m}(v_1,y_2)\right]dv_1dz \\
&+ \int_{y_{1,n}}^{y_1}\int_{y_{1,n}}^{z}\int_{y_{2,m}}^{y_2} \partial_2^{(j)}H(z,y_2,v_1,v_2)\hat{\mu}_{n,m}(v_1,v_2)dv_2dv_1dz \\
&= \partial_2^{(j)}\Phi(y_1,y_2) + \sum_{\xi=0}^{n-1}\sum_{\sigma=0}^{\xi-1}\sum_{\rho=0}^{m-1}\int_{y_{1,\xi}}^{y_{1,\xi+1}}\int_{y_{1,\sigma}}^{y_{1,\sigma+1}}\int_{y_{2,\rho}}^{y_{2,\rho+1}} \partial_2^{(j)}H(z,y_2,v_1,v_2)\mu_{\sigma,\rho}(v_1,v_2)dv_2dv_1dz \\
&+ \sum_{\xi=0}^{n-1}\sum_{\sigma=0}^{\xi-1}\sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\int_{y_{1,\xi}}^{y_{1,\xi+1}}\int_{y_{1,\sigma}}^{y_{1,\sigma+1}}\frac{\partial^{r-l}}{\partial y_2^{r-l}}\left[\partial_2^{(j-1-r)}H(z,y_2,v_1,y_2)\right]\frac{\partial^l \mu_{\sigma,m}(v_1,y_2)}{\partial y_2^l}dv_1dz \\
&+ \sum_{\xi=0}^{n-1}\sum_{\sigma=0}^{\xi-1}\int_{y_{1,\xi}}^{y_{1,\xi+1}}\int_{y_{1,\sigma}}^{y_{1,\sigma+1}}\int_{y_{2,m}}^{y_2} \partial_2^{(j)}H(z,y_2,v_1,v_2)\mu_{\sigma,m}(v_1,v_2)dv_2dv_1dz \\
&+ \sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1}\int_{y_{1,\xi}}^{y_{1,\xi+1}}\int_{y_{1,\xi}}^{z}\int_{y_{2,\rho}}^{y_{2,\rho+1}} \partial_2^{(j)}H(z,y_2,v_1,v_2)\mu_{\xi,\rho}(v_1,v_2)dv_2dv_1dz \\
&+ \sum_{\xi=0}^{n-1}\sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\int_{y_{1,\xi}}^{y_{1,\xi+1}}\int_{y_{1,\xi}}^{z}\frac{\partial^{r-l}}{\partial y_2^{r-l}}\left[\partial_2^{(j-1-r)}H(z,y_2,v_1,y_2)\right]\frac{\partial^l \mu_{\xi,m}(v_1,y_2)}{\partial y_2^l}dv_1dz \\
&+ \sum_{\xi=0}^{n-1}\int_{y_{1,\xi}}^{y_{1,\xi+1}}\int_{y_{1,\xi}}^{z}\int_{y_{2,m}}^{y_2} \partial_2^{(j)}H(z,y_2,v_1,v_2)\mu_{\xi,m}(v_1,v_2)dv_2dv_1dz
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{\sigma=0}^{n-1}\sum_{\rho=0}^{m-1}\int_{y_{1,n}}^{y_1}\int_{y_{1,\sigma}}^{y_{1,\sigma+1}}\int_{y_{2,\rho}}^{y_{2,\rho+1}}\partial_2^{(j)}H(z,y_2,v_1,v_2)\mu_{\sigma,\rho}(v_1,v_2)dv_2dv_1dz\\
&+\sum_{\sigma=0}^{n-1}\sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\int_{y_{1,n}}^{y_1}\int_{y_{1,\sigma}}^{y_{1,\sigma+1}}\frac{\partial^{r-l}}{\partial y_2^{r-l}}\left[\partial_2^{(j-1-r)}H(z,y_2,v_1,y_2)\right]\frac{\partial^l\mu_{\sigma,m}(v_1,y_2)}{\partial y_2^l}dv_1dz\\
&+\sum_{\sigma=0}^{n-1}\int_{y_{1,n}}^{y_1}\int_{y_{1,\sigma}}^{y_{1,\sigma+1}}\int_{y_{2,m}}^{y_2}\partial_2^{(j)}H(z,y_2,v_1,v_2)\mu_{\sigma,m}(v_1,v_2)dv_2dv_1dz\\
&+\sum_{\rho=0}^{m-1}\int_{y_{1,n}}^{y_1}\int_{y_{1,n}}^{z}\int_{y_{2,\rho}}^{y_{2,\rho+1}}\partial_2^{(j)}H(z,y_2,v_1,v_2)\mu_{n,\rho}(v_1,v_2)dv_2dv_1dz\\
&+\sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\int_{y_{1,n}}^{y_1}\int_{y_{1,n}}^{z}\frac{\partial^{r-l}}{\partial y_2^{r-l}}\left[\partial_2^{(j-1-r)}H(z,y_2,v_1,y_2)\right]\frac{\partial^l\hat{\mu}_{n,m}(v_1,y_2)}{\partial y_2^l}dv_1dz\\
&+\int_{y_{1,n}}^{y_1}\int_{y_{1,n}}^{z}\int_{y_{2,m}}^{y_2}\partial_2^{(j)}H(z,y_2,v_1,v_2)\hat{\mu}_{n,m}(v_1,v_2)dv_2dv_1dz. \qquad (2.15)
\end{aligned}
$$

By differentiating for y_1, we obtain

$$
\begin{aligned}
\frac{\partial^{j+1}\hat{\mu}_{n,m}(y_1,y_2)}{\partial y_1\partial y_2^j}&=\partial_1^{(1)}\partial_2^{(j)}\Phi(y_1,y_2)+\sum_{\sigma=0}^{n-1}\sum_{\rho=0}^{m-1}\int_{y_{1,\sigma}}^{y_{1,\sigma+1}}\int_{y_{2,\rho}}^{y_{2,\rho+1}}\partial_2^{(j)}H(y_1,y_2,v_1,v_2)\mu_{\sigma,\rho}(v_1,v_2)dv_2dv_1\\
&+\sum_{\sigma=0}^{n-1}\sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\int_{y_{1,\sigma}}^{y_{1,\sigma+1}}\frac{\partial^{r-l}}{\partial y_2^{r-l}}\left[\partial_2^{(j-1-r)}H(y_1,y_2,v_1,y_2)\right]\frac{\partial^l\mu_{\sigma,m}(v_1,y_2)}{\partial y_2^l}dv_1\\
&+\sum_{\sigma=0}^{n-1}\int_{y_{1,\sigma}}^{y_{1,\sigma+1}}\int_{y_{2,m}}^{y_2}\partial_2^{(j)}H(y_1,y_2,v_1,v_2)\mu_{\sigma,m}(v_1,v_2)dv_2dv_1\\
&+\sum_{\rho=0}^{m-1}\int_{y_{1,n}}^{y_1}\int_{y_{2,\rho}}^{y_{2,\rho+1}}\partial_2^{(j)}H(y_1,y_2,v_1,v_2)\mu_{n,\rho}(v_1,v_2)dv_2dv_1\\
&+\sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\int_{y_{1,n}}^{y_1}\frac{\partial^{r-l}}{\partial y_2^{r-l}}\left[\partial_2^{(j-1-r)}H(y_1,y_2,v_1,y_2)\right]\frac{\partial^l\hat{\mu}_{n,m}(v_1,y_2)}{\partial y_2^l}dv_1\\
&+\int_{y_{1,n}}^{y_1}\int_{y_{2,m}}^{y_2}\partial_2^{(j)}H(y_1,y_2,v_1,v_2)\hat{\mu}_{n,m}(v_1,v_2)dv_2dv_1. \qquad (2.16)
\end{aligned}
$$

Now, we differentiate Eq. (2.16) i-times for y_1, we obtain

$$
\begin{aligned}
\frac{\partial^{i+j+1}\hat{\mu}_{n,m}(y_1,y_2)}{\partial y_1^{i+1}\partial y_2^j}&=\partial_1^{(i+1)}\partial_2^{(j)}\Phi(y_1,y_2)\\
&+\sum_{\sigma=0}^{n-1}\sum_{\rho=0}^{m-1}\int_{y_{1,\sigma}}^{y_{1,\sigma+1}}\int_{y_{2,\rho}}^{y_{2,\rho+1}}\partial_1^{(i)}\partial_2^{(j)}H(y_1,y_2,v_1,v_2)\mu_{\sigma,\rho}(v_1,v_2)dv_2dv_1\\
&+\sum_{\sigma=0}^{n-1}\sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\int_{y_{1,\sigma}}^{y_{1,\sigma+1}}\frac{\partial^i}{\partial y_1^i}\left[\frac{\partial^{r-l}}{\partial y_2^{r-l}}\left[\partial_2^{(j-1-r)}H(y_1,y_2,v_1,y_2)\right]\frac{\partial^l\mu_{\sigma,m}(v_1,y_2)}{\partial y_2^l}\right]dv_1\\
&+\sum_{\sigma=0}^{n-1}\int_{y_{1,\sigma}}^{y_{1,\sigma+1}}\int_{y_{2,m}}^{y_2}\partial_1^{(i)}\partial_2^{(j)}H(y_1,y_2,v_1,v_2)\mu_{\sigma,m}(v_1,v_2)dv_2dv_1\\
&+\sum_{\rho=0}^{m-1}\int_{y_{2,\rho}}^{y_{2,\rho+1}}\sum_{q=0}^{i-1}\frac{\partial^q}{\partial y_1^q}\left[\partial_1^{(i-1-q)}\left(\partial_2^{(j)}H(y_1,y_2,y_1,v_2)\mu_{n,\rho}(y_1,v_2)\right)\right]dv_2\\
&+\sum_{\rho=0}^{m-1}\int_{y_{1,n}}^{y_1}\int_{y_{2,\rho}}^{y_{2,\rho+1}}\left[\partial_1^{(i)}\partial_2^{(j)}H(y_1,y_2,v_1,v_2)\mu_{n,\rho}(v_1,v_2)\right]dv_2dv_1
\end{aligned}
$$

$$+\sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\sum_{q=0}^{i-1}\frac{\partial^q}{\partial y_1^q}\left[\frac{\partial^{i-1-q}}{\partial y_1^{i-1-q}}\bigg|_{v_1=y_1}\left(\frac{\partial^{r-l}}{\partial y_2^{r-l}}\left[\partial_2^{(j-1-r)}H(y_1,y_2,v_1,y_2)\right]\frac{\partial^l\hat{\mu}_{n,m}(y_1,y_2)}{\partial y_2^l}\right)\right]$$
$$+\sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\int_{y_{1,n}}^{y_1}\frac{\partial^i}{\partial y_1^i}\left(\frac{\partial^{r-l}}{\partial y_2^{r-l}}\left[\partial_2^{(j-1-r)}H(y_1,y_2,v_1,y_2)\right]\frac{\partial^l\hat{\mu}_{n,m}(v_1,y_2)}{\partial y_2^l}\right)dv_1$$
$$+\int_{y_{2,m}}^{y_2}\sum_{q=0}^{i-1}\frac{\partial^q}{\partial y_1^q}\left[\partial_1^{(i-1-q)}\left(\partial_2^{(j)}H(y_1,y_2,y_1,v_2)\hat{\mu}_{n,m}(y_1,v_2)\right)\right]dv_2$$
$$+\int_{y_{1,n}}^{y_1}\int_{y_{2,m}}^{y_2}\partial_1^{(i)}\partial_2^{(j)}H(y_1,y_2,v_1,v_2)\hat{\mu}_{n,m}(v_1,v_2)dv_2dv_1$$
$$=\partial_1^{(i+1)}\partial_2^{(j)}\Phi(y_1,y_2)+\sum_{\sigma=0}^{n-1}\sum_{\rho=0}^{m-1}\int_{y_{1,\sigma}}^{y_{1,\sigma+1}}\int_{y_{2,\rho}}^{y_{2,\rho+1}}\partial_1^{(i)}\partial_2^{(j)}H(y_1,y_2,v_1,v_2)\mu_{\sigma,\rho}(v_1,v_2)dv_2dv_1$$
$$+\sum_{\sigma=0}^{n-1}\sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\int_{y_{1,\sigma}}^{y_{1,\sigma+1}}\frac{\partial^i}{\partial y_1^i}[\frac{\partial^{r-l}}{\partial y_2^{r-l}}\left[\partial_2^{(j-1-r)}H(y_1,y_2,v_1,y_2)\right]\frac{\partial^l\mu_{\sigma,m}(v_1,y_2)}{\partial y_2^l}\bigg]dv_1$$
$$+\sum_{\sigma=0}^{n-1}\int_{y_{1,\sigma}}^{y_{1,\sigma+1}}\int_{y_{2,m}}^{y_2}\partial_1^{(i)}\partial_2^{(j)}H(y_1,y_2,v_1,v_2)\mu_{\sigma,m}(v_1,v_2)dv_2dv_1$$
$$+\sum_{\rho=0}^{m-1}\sum_{q=0}^{i-1}\sum_{\eta=0}^{q}\binom{q}{\eta}\int_{y_{2,\rho}}^{y_{2,\rho+1}}\frac{\partial^{q-\eta}}{\partial y_1^{q-\eta}}\left[\partial_1^{(i-1-q)}\partial_2^{(j)}H(y_1,y_2,y_1,v_2)\right]\frac{\partial^\eta\mu_{n,\rho}(y_1,v_2)}{\partial y_1^\eta}dv_2$$
$$+\sum_{\rho=0}^{m-1}\int_{y_{1,n}}^{y_1}\int_{y_{2,\rho}}^{y_{2,\rho+1}}\partial_1^{(i)}\partial_2^{(j)}H(y_1,y_2,v_1,v_2)\mu_{n,\rho}(v_1,v_2)dv_2dv_1$$
$$+\sum_{r=0}^{j-1}\sum_{l=0}^{r}\sum_{q=0}^{i-1}\sum_{\eta=0}^{q}\binom{r}{l}\binom{q}{\eta}\frac{\partial^{q-\eta}}{\partial y_1^{q-\eta}}\left[\frac{\partial^{i-1-q}}{\partial y_1^{i-1-q}}\bigg|_{v_1=y_1}\left(\frac{\partial^{r-l}}{\partial y_2^{r-l}}\left[\partial_2^{(j-1-r)}H(y_1,y_2,v_1,y_2)\right]\right)\right]\times$$
$$\frac{\partial^{l+\eta}\hat{\mu}_{n,m}(y_1,y_2)}{\partial y_1^\eta\partial y_2^l}dv_2$$
$$+\sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\int_{y_{1,n}}^{y_1}\frac{\partial^i}{\partial y_1^i}\left(\frac{\partial^{r-l}}{\partial y_2^{r-l}}\left[\partial_2^{(j-1-r)}H(y_1,y_2,v_1,y_2)\right]\right)\frac{\partial^l\hat{\mu}_{n,m}(v_1,y_2)}{\partial y_2^l}dv_1$$
$$+\sum_{q=0}^{i-1}\sum_{\eta=0}^{q}\binom{q}{\eta}\int_{y_{2,m}}^{y_2}\frac{\partial^{q-\eta}}{\partial y_1^{q-\eta}}\left(\partial_1^{(i-1-q)}\partial_2^{(j)}H(y_1,y_2,y_1,v_2)\right)\frac{\partial^\eta\hat{\mu}_{n,m}(y_1,v_2)}{\partial y_1^\eta}dv_2$$
$$+\int_{y_{1,n}}^{y_1}\int_{y_{2,m}}^{y_2}\partial_1^{(i)}\partial_2^{(j)}H(y_1,y_2,v_1,v_2)\hat{\mu}_{n,m}(v_1,v_2)dv_2dv_1. \quad (2.17)$$

Hence

$$\frac{\partial^{i+j+1}\hat{\mu}_{n,m}(y_{1,n},y_{2,m})}{\partial y_1^{i+1}\partial y_2^j}=\partial_1^{(i+1)}\partial_2^{(j)}\Phi(y_{1,n},y_{2,m})$$
$$+\sum_{\sigma=0}^{n-1}\sum_{\rho=0}^{m-1}\int_{y_{1,\sigma}}^{y_{1,\sigma+1}}\int_{y_{2,\rho}}^{y_{2,\rho+1}}\partial_1^{(i)}\partial_2^{(j)}H(y_{1,n},y_{2,m},v_1,v_2)\mu_{\sigma,\rho}(v_1,v_2)dv_2dv_1$$
$$+\sum_{\sigma=0}^{n-1}\sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\int_{y_{1,\sigma}}^{y_{1,\sigma+1}}\frac{\partial^i}{\partial y_1^i}\left[\frac{\partial^{r-l}}{\partial y_2^{r-l}}\left[\partial_2^{(j-1-r)}H(y_1,y_2,v_1,y_2)\right]\right]_{y_1=y_{1,n},y_2=y_{2,m}}\frac{\partial^l\mu_{\sigma,m}(v_1,y_{2,m})}{\partial y_2^l}dv_1$$
$$+\sum_{\rho=0}^{m-1}\sum_{q=0}^{i-1}\sum_{\eta=0}^{q}\binom{q}{\eta}\int_{y_{2,\rho}}^{y_{2,\rho+1}}\frac{\partial^{q-\eta}}{\partial y_1^{q-\eta}}\left[\frac{\partial^{i-1-q}}{\partial y_1^{i-1-q}}\bigg|_{v_1=y_1}\left(\partial_2^{(j)}H(y_1,y_2,y_1,v_2)\right)\right]_{y_1=y_{1,n},y_2=y_{2,m}}\times$$
$$\frac{\partial^\eta\mu_{n,\rho}(y_{1,n},v_2)}{\partial y_1^\eta}dv_2$$

$$+\sum_{r=0}^{j-1}\sum_{l=0}^{r}\sum_{q=0}^{i-1}\sum_{\eta=0}^{q}\binom{r}{l}\binom{q}{\eta}\frac{\partial^{q-\eta}}{\partial y_1^{q-\eta}}\left[\frac{\partial^{i-1-q}}{\partial y_1^{i-1-q}}\bigg|_{v_1=y_1}\left(\frac{\partial^{r-l}}{\partial y_2^{r-l}}\left[\partial_2^{(j-1-r)}H(y_1,y_2,v_1,y_2)\right]\right)\right]_{y_1=y_{1,n},y_2=y_{2,m}}\times \frac{\partial^{l+\eta}\hat{\mu}_{n,m}(y_{1,n},y_{2,m})}{\partial y_1^{\eta}\partial y_2^{l}}dv_2.$$

Moreover, from Eq. (2.15), we deduce that

$$\frac{\partial^j\hat{\mu}_{n,m}(y_{1,n},y_{2,m})}{\partial y_2^j}=\partial_2^{(j)}\Phi(y_{1,n},y_{2,m})$$
$$+\sum_{\xi=0}^{n-1}\sum_{\sigma=0}^{\xi-1}\sum_{\rho=0}^{m-1}\int_{y_{1,\xi}}^{y_{1,\xi+1}}\int_{y_{1,\sigma}}^{y_{1,\sigma+1}}\int_{y_{2,\rho}}^{y_{2,\rho+1}}\left[\partial_2^{(j)}H(z,y_2,v_1,v_2)\right]_{y_1=y_{1,n},y_2=y_{2,m}}\mu_{\sigma,\rho}(v_1,v_2)dv_2dv_1dz$$
$$+\sum_{\xi=0}^{n-1}\sum_{\sigma=0}^{\xi-1}\sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\int_{y_{1,\xi}}^{y_{1,\xi+1}}\int_{y_{1,\sigma}}^{y_{1,\sigma+1}}\frac{\partial^{r-l}}{\partial y_2^{r-l}}\left[\partial_2^{(j-1-r)}H(z,y_2,v_1,y_2)\right]_{y_1=y_{1,n},y_2=y_{2,m}}\times \frac{\partial^l\mu_{\sigma,m}(v_1,y_{2,m})}{\partial y_2^l}dv_1dz,$$

and from Eq. (2.16), we get

$$\frac{\partial^{j+1}\hat{\mu}_{n,m}(y_{1,n},y_{2,m})}{\partial y_1\partial y_2^j}=\partial_1^{(1)}\partial_2^{(j)}\Phi(y_{1,n},y_{2,m})$$
$$+\sum_{\sigma=0}^{n-1}\sum_{\rho=0}^{m-1}\int_{y_{1,\sigma}}^{y_{1,\sigma+1}}\int_{y_{2,\rho}}^{y_{2,\rho+1}}\left[\partial_2^{(j)}H(y_1,y_2,v_1,v_2)\right]_{y_1=y_{1,n},y_2=y_{2,m}}\mu_{\sigma,\rho}(v_1,v_2)dv_2dv_1$$
$$+\sum_{\sigma=0}^{n-1}\sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\int_{y_{1,\sigma}}^{y_{1,\sigma+1}}\frac{\partial^{r-l}}{\partial y_2^{r-l}}\left[\partial_2^{(j-1-r)}H(y_1,y_2,v_1,y_2)\right]_{y_1=y_{1,n},y_2=y_{2,m}}\frac{\partial^l\mu_{\sigma,m}(v_1,y_{2,m})}{\partial y_2^l}dv_1.$$

2.2 Convergence analysis

We consider the space $L^\infty(\Lambda)$ with the norm

$$\|\mu\|_{L^\infty(\Lambda)}=\inf\{C\in\mathbb{R}:|\mu(y_1,y_2)|\le C\quad\forall(y_1,y_2)\in\Lambda\}<\infty.$$

The subsequent lemma is essential for demonstrating the convergence of the proposed method.

Lemma 2.8. *Suppose $\hbar$ and H are functions that are continuously differentiable p times within their domains. Therefore, a positive constant $\zeta(p)$ exists such that the following inequality holds:*

$$\left\|\frac{\partial^{i+j}\hat{\mu}_{n,m}}{\partial y_1^i\partial y_2^j}\right\|_{L^\infty(\Lambda_{n,m})}\le\zeta(p),$$

for all $n=0,\dots,N-1$, $m=0,\dots,M-1$, and $i+j=0,1,...,p$, where $\hat{\mu}_{0,0}(y_1,y_2)=\mu(y_1,y_2)$ for $(y_1,y_2)\in\Lambda_{0,0}$.

Proof. Let $o_{n,m}^{i,j}=\|\frac{\partial^{i+j}\hat{\mu}_{n,m}}{\partial y_1^i\partial y_2^j}\|_{L^\infty(\Lambda_{n,m})}$, we have for all $i+j=0,1,...,p$,

$$o_{0,0}^{i,j}\le\max\left\{\left\|\frac{\partial^{i+j}\mu}{\partial y_1^i\partial y_2^j}\right\|_{L^\infty(\Lambda_{0,0})},i+j=0,1,...,p\right\}=\zeta_1(p). \tag{2.18}$$

Now, considering Eq. (2.12), it follows for all $n = 1, \ldots, N-1$ and $i + j = 0, 1, ..., p$:

$$o_{n,0}^{i+1,j} \leq \gamma_1 + \gamma_1 h \sum_{\xi=0}^{n-1} \sum_{q+l=0}^{p-1} o_{\xi,0}^{q,l} + \gamma_1 hk \sum_{\xi=0}^{n-1} \sum_{q+l=0}^{p-1} o_{\xi,0}^{q,l} + \gamma_1 \sum_{r=0}^{j-1} \sum_{l=0}^{r} \sum_{q=0}^{i-1} \sum_{\eta=0}^{q} o_{n,0}^{\eta,l}$$
$$+ \gamma_1 h \sum_{r=0}^{j-1} \sum_{l=0}^{r} o_{n,0}^{0,l} + \gamma_1 k \sum_{q=0}^{i-1} \sum_{\eta=0}^{q} o_{n,0}^{\eta,0} + \gamma_1 hk o_{n,0}^{0,0},$$

where γ_1 is a positive constant unrelated to h and k. This give us

$$o_{n,0}^{i+1,j} \leq \gamma_1 + \gamma_2 h \sum_{\xi=0}^{n-1} \sum_{q+l=0}^{p-1} o_{\xi,0}^{q,l} + \gamma_2 \sum_{l=0}^{j-1} \sum_{\eta=0}^{i-1} o_{n,0}^{\eta,l} + \gamma_2 h \sum_{l=0}^{j-1} o_{n,0}^{0,l}$$
$$+ \gamma_2 k \sum_{\eta=0}^{i-1} o_{n,0}^{\eta,0} + \gamma_1 hk o_{n,0}^{0,0}. \tag{2.19}$$

Setting the sequence $\Gamma_n = \max\{o_{n,0}^{i,j}, i + j = 0, \ldots, p\}$ for all $n = 0, 1, ..., N-1$. Therefore, according to Eq. (2.19), it satisfies

$$o_{n,0}^{i+1,j} \leq \gamma_1 + \gamma_2 h \sum_{\xi=0}^{n-1} \Gamma_\xi + \gamma_2 \sum_{l=0}^{j-1} \sum_{\eta=0}^{i-1} o_{n,0}^{\eta,l} + \gamma_2 h \sum_{l=0}^{j-1} o_{n,0}^{0,l} + \gamma_2 k \sum_{\eta=0}^{i-1} o_{n,0}^{\eta,0} + \gamma_1 hk o_{n,0}^{0,0}$$
$$\leq \gamma_1 + \gamma_2 h \sum_{\xi=0}^{n-1} \Gamma_\xi + \gamma_2 \sum_{l=0}^{j-1} \sum_{\eta=0}^{i} o_{n,0}^{\eta,l} + \gamma_2 h \sum_{l=0}^{j-1} o_{n,0}^{0,l} + \gamma_2 k \sum_{\eta=0}^{i-1} o_{n,0}^{\eta,0} + \gamma_1 hk o_{n,0}^{0,0},$$

for all $i = 1, ..., p$ and $j = 0, ..., p$, this results

$$o_{n,0}^{i,j} \leq \gamma_1 + \gamma_2 h \sum_{\xi=0}^{n-1} \Gamma_\xi + \gamma_2 \sum_{l=0}^{j-1} \sum_{\eta=0}^{i-1} o_{n,0}^{\eta,l} + \gamma_2 h \sum_{l=0}^{j-1} o_{n,0}^{0,l} + \gamma_2 k \sum_{\eta=0}^{i-1} o_{n,0}^{\eta,0} + \gamma_1 hk o_{n,0}^{0,0}. \tag{2.20}$$

Furthermore, derived from Eq. (2.10), the following holds for all $j = 0, ..., p$ and $n = 0, ..., N-1$,

$$o_{n,0}^{0,j} \leq \gamma_1 + \gamma_3 h \sum_{\xi=0}^{n-1} \Gamma_\xi + \gamma_3 \sum_{l=0}^{j-1} o_{n,0}^{0,l} + \gamma_3 hk o_{n,0}^{0,0}. \tag{2.21}$$

Next, utilizing Eq. (2.9), we derive the following for all $n = 0, ..., N-1$,

$$|\hat{\mu}_{n,0}(y_1, y_2)| \leq \gamma_1 + \gamma_4 h \sum_{\xi=0}^{n-1} \Gamma_\xi + \gamma_4 \int_{y_{1,n}}^{y_1} \int_0^{y_2} |\hat{\mu}_{n,0}(v_1, v_2)| dv_2 dv_1.$$

Therefore, according to Lemma 1.6, we deduce for all $n = 0, ..., N-1$ that

$$o_{n,0}^{0,0} \leq \left(\gamma_1 + \gamma_4 h \sum_{\xi=0}^{n-1} \Gamma_\xi \right) e^{\gamma_4 (A_1 + A_2)}$$
$$\leq \gamma_1 e^{\gamma_5 (A_1 + A_2)} + \gamma_4 h \sum_{\xi=0}^{n-1} \Gamma_\xi e^{\gamma_4 (A_1 + A_2)} \tag{2.22}$$
$$\leq \gamma_5 + \gamma_5 h \sum_{\xi=0}^{n-1} \Gamma_\xi.$$

From Eqs. (2.20), (2.21) and (2.22), we infer that for all $i, j = 0, ..., p$ and $n = 0, ..., N-1$,

$$o_{n,0}^{i,j} \le \gamma_6 + \gamma_6 h \sum_{\xi=0}^{n-1} \Gamma_\xi + \gamma_6 \sum_{l=0}^{j-1} \sum_{\eta=0}^{i-1} o_{n,0}^{\eta,l} + \gamma_6 \sum_{l=0}^{j-1} o_{n,0}^{0,l} + \gamma_6 \sum_{\eta=0}^{i-1} o_{n,0}^{\eta,0}, \tag{2.23}$$

with γ_6 is positive and unrelated to N and M.
Utilizing the notations introduced in Lemma 1.4, we define

$$\Psi_{ij} = o_{n,0}^{i,j}, \alpha = \gamma_6 + \gamma_6 h \sum_{\xi=0}^{n-1} \Gamma_\xi, \beta_1 = \beta_2 = p\gamma_6, \beta_3 = p^2\gamma_6, T = S = 1.$$

Then, applying Lemma 1.4, we derive from Eq. (2.23)

$$o_{n,0}^{i,j} \le \left(\gamma_6 + \gamma_6 h \sum_{\xi=0}^{n-1} \Gamma_\xi \right) e^{2p(\gamma_6 + \sqrt{\gamma_6 + \gamma_6^2})}. \tag{2.24}$$

Thus,

$$\Gamma_n \le \gamma_7 + \gamma_7 h \sum_{\xi=0}^{n-1} \Gamma_\xi. \tag{2.25}$$

It follows, by Lemma 1.3, for all $n = 0, 1, ..., N-1$

$$\Gamma_n \le \gamma_7 e^{A_1 \gamma_7}. \tag{2.26}$$

On the other hand, we deduce from Eq. (2.17), valid for all $n = 0, ..., N-1$, $m = 1, ..., N-1$ and $i + j = 0, ..., p$, that

$$\begin{aligned} o_{n,m}^{i+1,j} \le \gamma_1' &+ \gamma_1' hk \sum_{\xi=0}^{n-1} \sum_{\rho=0}^{m-1} \sum_{s+t=0}^{p-1} o_{\xi,\rho}^{s,t} + \gamma_1' h \sum_{\xi=0}^{n-1} \sum_{r=0}^{j-1} \sum_{l=0}^{r} \sum_{s+t=0}^{p-1} o_{\xi,m}^{s,t} \\ &+ \gamma_1' hk \sum_{\xi=0}^{n-1} \sum_{s+t=0}^{p-1} o_{\xi,m}^{s,t} + \gamma_1' k \sum_{\rho=0}^{m-1} \sum_{q=0}^{i-1} \sum_{\eta=0}^{q} \sum_{s+t=0}^{p-1} o_{n,\rho}^{s,t} \\ &+ k\gamma_1' h \sum_{\rho=0}^{m-1} \sum_{s+t=0}^{p-1} o_{n,\rho}^{s,t} + \gamma_1' \sum_{r=0}^{j-1} \sum_{l=0}^{r} \sum_{q=0}^{i-1} \sum_{\eta=0}^{q} o_{n,m}^{\eta,l} \\ &+ \gamma_1' h \sum_{r=0}^{j-1} \sum_{l=0}^{r} o_{n,m}^{0,l} + \gamma_1' k \sum_{q=0}^{i-1} \sum_{\eta=0}^{q} o_{n,m}^{\eta,0} + k\gamma_1' h o_{n,m}^{0,0}. \end{aligned} \tag{2.27}$$

Consider $\Gamma_{n,m} = \max\{o_{n,m}^{i,j}, i + j = 0, \dots, p\}$ for all $n = 0, 1, ..., N-1$ and $m = 0, ..., M-1$, then by Eq. (2.27), the sequence satisfies

$$\begin{aligned} o_{n,m}^{i+1,j} \le \gamma_1' &+ \gamma_1' p^2 hk \sum_{\xi=0}^{n-1} \sum_{\rho=0}^{m-1} \Gamma_{\xi,\rho} + \gamma_1' p^4 h \sum_{\xi=0}^{n-1} \Gamma_{\xi,m} + \gamma_1' p^2 hk \sum_{\xi=0}^{n-1} \Gamma_{\xi,m} \\ &+ \gamma_1' p^4 k \sum_{\rho=0}^{m-1} \Gamma_{n,\rho} + hk\gamma_1' p^2 \sum_{\rho=0}^{m-1} \Gamma_{n,\rho} + \gamma_1' p^2 \sum_{l=0}^{j-1} \sum_{\eta=0}^{i-1} o_{n,m}^{\eta,l} \\ &+ \gamma_1' ph \sum_{l=0}^{j-1} o_{n,m}^{0,l} + \gamma_1' pk \sum_{\eta=0}^{i-1} o_{n,m}^{\eta,0} + hk\gamma_1' o_{n,m}^{0,0}. \end{aligned}$$

We get

$$\begin{aligned} o_{n,m}^{i+1,j} \leq & \gamma_1' + \gamma_2' hk \sum_{\xi=0}^{n-1} \sum_{\rho=0}^{m-1} \Gamma_{\xi,\rho} + \gamma_2' h \sum_{\xi=0}^{n-1} \Gamma_{\xi,m} + \gamma_2' k \sum_{\rho=0}^{m-1} \Gamma_{n,\rho} + \gamma_2' \sum_{l=0}^{j-1} \sum_{\eta=0}^{i-1} o_{n,m}^{\eta,l} \\ & + \gamma_2' h \sum_{l=0}^{j-1} o_{n,m}^{0,l} + \gamma_2' k \sum_{\eta=0}^{i-1} o_{n,m}^{\eta,0} + hk\gamma_2' o_{n,m}^{0,0}, \end{aligned} \tag{2.28}$$

which implies for all $i = 1, \ldots, p$ and $j = 0, \ldots, p$,

$$\begin{aligned} o_{n,m}^{i,j} \leq & \gamma_1' + \gamma_2' hk \sum_{\xi=0}^{n-1} \sum_{\rho=0}^{m-1} \Gamma_{\xi,\rho} + \gamma_2' h \sum_{\xi=0}^{n-1} \Gamma_{\xi,m} + \gamma_2' k \sum_{\rho=0}^{m-1} \Gamma_{n,\rho} + \gamma_2' \sum_{l=0}^{j-1} \sum_{\eta=0}^{i-1} o_{n,m}^{\eta,l} \\ & + \gamma_2' h \sum_{l=0}^{j-1} o_{n,m}^{0,l} + \gamma_2' k \sum_{\eta=0}^{i-1} o_{n,m}^{\eta,0} + hk\gamma_2' o_{n,m}^{0,0}. \end{aligned} \tag{2.29}$$

Furthermore, we have for all $j = 0, \ldots, p$ and $n = 0, \ldots, N-1$ and $m = 0, \ldots, M-1$,

$$o_{n,m}^{0,j} \leq \gamma_3' + \gamma_3' hk \sum_{\xi=0}^{n-1} \sum_{\rho=0}^{m-1} \Gamma_{\xi,\rho} + \gamma_3' h \sum_{\xi=0}^{n-1} \Gamma_{\xi,m} + \gamma_3' k \sum_{\rho=0}^{m-1} \Gamma_{n,\rho} + \gamma_3' \sum_{l=0}^{j-1} o_{n,m}^{0,l} + \gamma_3' o_{n,m}^{0,0}. \tag{2.30}$$

Also, from Eq. (2.14), we get for all $n = 0, \ldots, N-1$ and $m = 0, \ldots, M-1$,

$$|\hat{\mu}_{n,m}(y_1, y_2)| \leq \gamma_4' + \gamma_4' hk \sum_{\xi=0}^{n-1} \sum_{\rho=0}^{m-1} \Gamma_{\xi,\rho} + \gamma_4' h \sum_{\xi=0}^{n-1} \Gamma_{\xi,m} + \gamma_4' k \sum_{\rho=0}^{m-1} \Gamma_{n,\rho} + \gamma_4' \int_{y_{1,n}}^{y_1} \int_{y_{2,m}}^{y_2} |\hat{\mu}_{n,m}(v_1, v_2)| dv_2 dv_1.$$

Hence, according to Lemma 1.6, we have for all $n = 0, ..., N-1$ and $m = 0, ..., M-1$,

$$\begin{aligned} o_{n,m}^{0,0} & \leq \left(\gamma_4' + \gamma_4' hk \sum_{\xi=0}^{n-1} \sum_{\rho=0}^{m-1} \Gamma_{\xi,\rho} + \gamma_4' h \sum_{\xi=0}^{n-1} \Gamma_{\xi,m} + \gamma_4' k \sum_{\rho=0}^{m-1} \Gamma_{n,\rho} \right) e^{(A_1+A_2)\gamma_4'} \\ & \leq \gamma_5' + \gamma_5' hk \sum_{\xi=0}^{n-1} \sum_{\rho=0}^{m-1} \Gamma_{\xi,\rho} + \gamma_5' h \sum_{\xi=0}^{n-1} \Gamma_{\xi,m} + \gamma_5' k \sum_{\rho=0}^{m-1} \Gamma_{n,\rho}, \end{aligned} \tag{2.31}$$

then, according to Eqs. (2.29),(2.30) and (2.31) that, for all $i = 0, ..., p$ and $j = 0, ..., p$,

$$\begin{aligned} o_{n,m}^{i,j} \leq & \gamma_6' + \gamma_6' hk \sum_{\xi=0}^{n-1} \sum_{\rho=0}^{m-1} \Gamma_{\xi,\rho} + \gamma_6' h \sum_{\xi=0}^{n-1} \Gamma_{\xi,m} + \gamma_6' k \sum_{\rho=0}^{m-1} \Gamma_{n,\rho} \\ & + \gamma_7' \sum_{l=0}^{j-1} \sum_{\eta=0}^{i-1} o_{n,m}^{\eta,l} + \gamma_7' \sum_{l=0}^{j-1} o_{n,m}^{0,l} + \gamma_7' \sum_{\eta=0}^{i-1} o_{n,m}^{\eta,0}. \end{aligned} \tag{2.32}$$

Using Lemma 1.4, we set

$$\Psi_{ij} = o_{n,m}^{i,j},\ \alpha = \gamma_6' + \gamma_6' hk \sum_{\xi=0}^{n-1} \sum_{\rho=0}^{m-1} \Gamma_{\xi,\rho} + \gamma_6' h \sum_{\xi=0}^{n-1} \Gamma_{\xi,m} + \gamma_6' k \sum_{\rho=0}^{m-1} \Gamma_{n,\rho},$$

$$\beta_1 = \beta_2 = p\gamma_7', \beta_3 = p^2\gamma_7', T = S = 1.$$

Therefore, by applying Lemma 1.4 to Eq. (2.32), we get

$$o_{n,m}^{i,j} \leq \left(\gamma_6^{'} + \gamma_6^{'} hk \sum_{\xi=0}^{n-1} \sum_{\rho=0}^{m-1} \Gamma_{\xi,\rho} + \gamma_6^{'} h \sum_{\xi=0}^{n-1} \Gamma_{\xi,m} + \gamma_6^{'} k \sum_{\rho=0}^{m-1} \Gamma_{n,\rho} \right) e^{2p(\gamma_7^{'} + \sqrt{\gamma_7^{'} + \gamma_7^{2'}})}.$$

It follows that for all $n = 0, 1, ..., N-1; m = 0, ..., M-1,$

$$\Gamma_{n,m} \leq \gamma_8^{'} + hk\gamma_8^{'} \sum_{\xi=0}^{n-1} \sum_{\rho=0}^{m-1} \Gamma_{\xi,\rho} + h\gamma_8^{'} \sum_{\xi=0}^{n-1} \Gamma_{\xi,m} + k\gamma_8^{'} \sum_{\rho=0}^{m-1} \Gamma_{n,\rho}, \tag{2.33}$$

by using Lemma (1.4), we obtain

$$\Gamma_{n,m} \leq \gamma_8^{'} e^{(A_1+A_2)p(\gamma_8^{'} + \sqrt{\gamma_8^{'} + \gamma_8^{'2}})}. \tag{2.34}$$

Thus from Eqs. (2.18), (2.26) and (2.34) the proof of Lemma 2.8 is completed by setting

$$\zeta(p) = max\{\zeta_1(p), \gamma_7 e^{A_1\gamma_7}, \gamma_8^{'} e^{(A_1+A_2)p(\gamma_8^{'} + \sqrt{\gamma_8^{'} + \gamma_8^{'2}})}\}.$$

□

The following theorem establishes the convergence of the presented approach.

Theorem 2.2. *Suppose $\hbar$ and H two functions that are continuously differentiable p times within their domains. Therefore, equations (2.5), (2.8), and (2.13) establish a distinct approximation $\mu_{N,M} \in S_{p-1,p-1}^{(-1)}(\Pi_{N,M})$. Additionally, there exists a finite constant C independent of h and k such that the resulting error function $e(y_1, y_2) = \mu(y_1, y_2) - \mu_{N,M}(y_1, y_2)$ meets the condition:*

$$\|e\|_{L^\infty(\Lambda)} \leq C(h+k)^p,$$

Proof. For all $n \in \{0, \ldots, N\}$ and $m \in \{0, \ldots, M\}$, we state the error as $e_{n,m}(y_1, y_2) = \mu(y_1, y_2) - \mu_{n,m}(y_1, y_2)$ on each rectangle $\Lambda_{n,m}$.

First, for $(y_1, y_2) \in \Lambda_{0,0}$, applying Lemma 1.7 to Eq. (2.5) derives the following from

$$|e_{0,0}(y_1, y_2)| \leq \sum_{i+j=p} \frac{1}{i!j!} \left\| \frac{\partial^{i+j}\mu}{\partial y_1^i \partial y_2^j} \right\| h^i k^j.$$

Hence by Lemma 2.8, we have

$$|e_{0,0}(y_1, y_2)| \leq \zeta(p) \sum_{i+j=p} \frac{1}{i!j!} h^i k^j = \frac{\zeta(p)}{p!}(h+k)^p. \tag{2.35}$$

Thus, we put $C_1 = \frac{\zeta(p)}{p!}$.

Next, for $(y_1, y_2) \in \Lambda_{n,0}$, where $n = 1, \dots, N-1$, we derive from Eq. (2.9)

$$\begin{aligned}\mu(y_1,y_2) - \hat{\mu}_{n,0}(y_1,y_2) &= \sum_{\xi=0}^{n-1}\sum_{\sigma=0}^{\xi-1}\int_{y_{1,\xi}}^{y_{1,\xi+1}}\int_{y_{1,\sigma}}^{y_{1,\sigma+1}}\int_0^{y_2} H(z,y_2,v_1,v_2)e_{\sigma,0}(v_1,v_2)dv_2dv_1dz \\ &+ \sum_{\xi=0}^{n-1}\int_{y_{1,\xi}}^{y_{1,\xi+1}}\int_{y_{1,\xi}}^{z}\int_0^{y_2} H(z,y_2,v_1,v_2)e_{\xi,0}(v_1,v_2)dv_2dv_1dz \\ &+ \sum_{\sigma=0}^{n-1}\int_{y_{1,n}}^{y_1}\int_{y_{1,\sigma}}^{y_{1,\sigma+1}}\int_0^{y_2} H(z,y_2,v_1,v_2)e_{\sigma,0}(v_1,v_2)dv_2dv_1dz \\ &+ \int_{y_{1,n}}^{y_1}\int_{y_{1,n}}^{z}\int_0^{y_2} H(z,y_2,v_1,v_2)\left(\mu(v_1,v_2) - \hat{\mu}_{n,0}(v_1,v_2)\right)dv_2dv_1dz.\end{aligned}$$

Thus,

$$|\mu(y_1,y_2) - \hat{\mu}_{n,0}(y_1,y_2)| \le \sum_{\xi=0}^{n-1} 3A_1hk\overline{H}\|e_{\xi,0}\|_{L^\infty(\Lambda_{\xi,0})} + \overline{H}A_1\int_{y_{1,n}}^{y_1}\int_0^{y_2}|\mu(v_1,v_2) - \hat{\mu}_{n,0}(v_1,v_2)|dv_2dv_1.$$

Therefore by Lemma 1.6, we get

$$\begin{aligned}|\mu(y_1,y_2) - \hat{\mu}_{n,0}(y_1,y_2)| &\le \sum_{\xi=0}^{n-1} hk3A_1\overline{H}\|e_{\xi,0}\|_{L^\infty(\Lambda_{\xi,0})}\exp\left(\overline{H}A_1(A_1+A_2)\right) \\ &\le \sum_{\xi=0}^{n-1} 3hA_1A_2\overline{H}\exp\left(\overline{H}A_1(A_1+A_2)\right)\|e_{\xi,0}\|_{L^\infty(\Lambda_{\xi,0})} \\ &\le \sum_{\xi=0}^{n-1} h\lambda_1\|e_{\xi,0}\|_{L^\infty(\Lambda_{\xi,0})},\end{aligned}$$

using Lemma 1.7 results to

$$\begin{aligned}\|e_{n,0}\|_{L^\infty(\Lambda_{n,0})} &\le \|\mu - \hat{\mu}_{n,0}\| + \|\hat{\mu}_{n,0} - \mu_{n,0}\| \\ &\le \sum_{\xi=0}^{n-1} h\lambda_1\|e_{\xi,0}\|_{L^\infty(\Lambda_{\xi,0})} + \sum_{i+j=p}\frac{1}{i!j!}\left\|\frac{\partial^{i+j}\hat{\mu}_{n,0}}{\partial y_1^i\partial y_2^j}\right\|h^ik^j.\end{aligned}$$

Hence by Lemma 2.8, we obtain

$$\|e_{n,0}\|_{L^\infty(\Lambda_{n,0})} \le \sum_{\xi=0}^{n-1} h\lambda_1\|e_{\xi,0}\|_{L^\infty(\Lambda_{\xi,0})} + \frac{\zeta(p)}{p!}(h+k)^p,$$

then, by Lemma 1.3, we have

$$\|e_{n,0}\|_{L^\infty(\Lambda_{n,0})} \le \frac{\zeta(p)}{p!}(h+k)^p\exp(A_1\lambda_1).$$

Thus, we take $C_2 = \frac{\zeta(p)}{p!}\exp(A_1\lambda_1)$.

Finally, for $(y_1, y_2) \in \Lambda_{n,m}$, where $n = 0, \dots, N-1$ and $m = 1, \dots, M-1$, we derive from Eq. (2.14)

$$\begin{aligned}|\mu(y_1,y_2) - \hat{\mu}_{n,m}(y_1,y_2)| \le &\sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1} hkA_1\overline{H}\|e_{\xi,\rho}\| + \sum_{\xi=0}^{n-1} hkA_1\overline{H}\|e_{\xi,m}\| + \sum_{\rho=0}^{m-1} hkA_1\overline{H}\|e_{n,\rho}\| \\ &+ \overline{H}A_1\int_{y_{1,n}}^{y_1}\int_{y_{2,m}}^{y_2}|\mu(v_1,v_2) - \hat{\mu}_{n,m}(v_1,v_2)|dv_2dv_1.\end{aligned}$$

Thus, by Lemma 1.6,

$$|\mu(y_1,y_2)-\hat{\mu}_{n,m}(y_1,y_2)| \leq \left(\sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1} hkA_1\overline{H}\|e_{\xi,\rho}\| + \sum_{\xi=0}^{n-1} hkA_1\overline{H}\|e_{\xi,m}\| + \sum_{\rho=0}^{m-1} hkA_1\overline{H}\|e_{n,\rho}\|\right)\exp(A_1\overline{H}(A_1+A_2))$$
$$\leq \sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1} \lambda_2 hk\|e_{\xi,\rho}\| + \sum_{\xi=0}^{n-1} hk\lambda_2\|e_{\xi,m}\| + \sum_{\rho=0}^{m-1} hk\lambda_2\|e_{n,\rho}\|,$$

therefore, using Lemma 1.7 results to

$$\begin{aligned}\|e_{n,m}\|_{L^\infty(\Lambda_{n,0})} \leq& \|\mu-\hat{\mu}_{n,m}\| + \|\hat{\mu}_{n,m}-\mu_{n,m}\|\\ \leq& \sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1} hk\lambda_2\|e_{\xi,\rho}\| + \sum_{\xi=0}^{n-1} hk\lambda_2\|e_{\xi,m}\| + \sum_{\rho=0}^{m-1} hk\lambda_2\|e_{n,\rho}\|\\ &+ \sum_{i+j=p}\frac{1}{i!j!}\left\|\frac{\partial^{i+j}\hat{\mu}_{n,m}}{\partial y_1^i \partial y_2^j}\right\| h^i k^j.\end{aligned}$$

Hence, by Lemma 2.8, we obtain

$$\|e_{n,m}\| \leq \sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1} hk\lambda_2\|e_{\xi,\rho}\| + \sum_{\xi=0}^{n-1} hk\lambda_2\|e_{\xi,m}\| + \sum_{\rho=0}^{m-1} hk\lambda_2\|e_{n,\rho}\| + \frac{\zeta(p)}{p!}(h+k)^p. \tag{2.36}$$

Therefore, according to Lemma 1.4, we derive from Eq. (2.36)

$$\begin{aligned}\|e_{n,m}\| &\leq \left(\frac{\zeta(p)}{p!}(h+k)^p\right)\exp(\lambda_3(Nh+Mk))\\ &\leq \frac{\zeta(p)}{p!}\exp(\lambda_3(A_1+A_2))(h+k)^p.\end{aligned} \tag{2.37}$$

Thus, we take $C_3 = \frac{\zeta(p)}{p!}\exp(\lambda_3(A_1+A_2))$.
Hence, the proof concludes by selecting $C = max\{C_1, C_2, C_3\}$.

2.3 Numerical results

To evaluate the effectiveness of the proposed Taylor collocation method in solving first-order linear 2D-PVIDEs, we present five numerical examples featuring linear PVIDEs with known exact solutions. In these examples, we fix $p = 3$ and consider three distinct sets of values for N and M. The error estimation is provided to demonstrate the accuracy of the approximation.

Example 2.5. *Let us dedicate the first example to the case that the desired equation is of form*

$$\frac{\partial\mu(y_1,y_2)}{\partial y_1} = \hbar(y_1,y_2) + \int_0^{y_1}\int_0^{y_2}(v_1+v_2^2)\mu(v_1,v_2)dv_2dv_1, \qquad 0\leq y_1\leq 1,\ 0\leq y_2\leq 1,$$

with the initial condition

$$\mu_0(y_2) = sin(y_2), \qquad 0 \leq y_2 \leq 1.$$

The exact solution of this problem is given by $\mu(y_1,y_2) = (1-y_1^2)sin(y_2)$. Then, the function $\hbar(y_1,y_2)$ is *calculated using the exact solution as follows:*

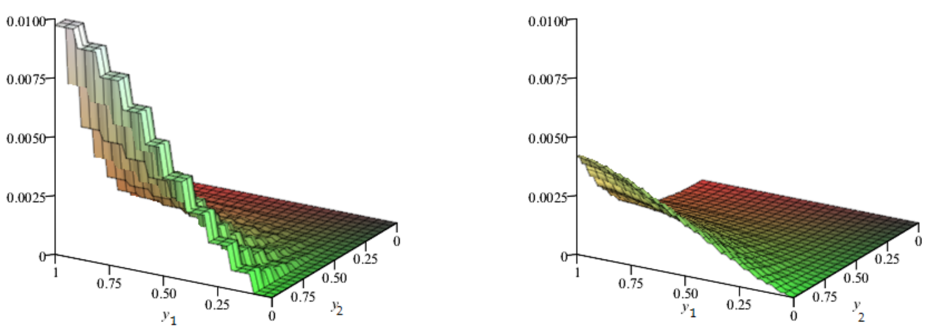
$$\hbar(y_1, y_2) = \frac{1}{4}y_1^4(1 - cos(y_2)) + \frac{2}{3}y_1^3(cos(y_2) - \frac{1}{2}y_2^2 cos(y_2) + y_2 sin(y_2) - 1) + \frac{1}{2}y_1^2(cos(y_2) - 1) + 2y_1\left(\frac{1}{2}y_2^2 cos(y_2) - y_2 sin(y_2) + sin(y_2) - cos(y_2) + 1\right).$$

The study employed the proposed Taylor collocation method to compute numerical results across various collocation points. Table 2.1 displays the absolute errors $|\mu(y_1, y_2) - \mu_{N,M}(y_1, y_2)|$ revealing a consistent decrease with an increasing number of collocation points. To provide a visual representation of the approximate solution's behavior, the absolute error function was graphed in Figure 2.1 in three dimensions for $0 \leq y_1 \leq 1$ and $0 \leq y_2 \leq 1$ using different values of N and M—specifically, $(N, M) = (10, 10)$ and $(N, M) = (20, 20)$—for comparative analysis. Additionally, Figure 2.2 illustrates both the exact and approximate functions in a 3D plot for $N = M = 20$. For a more detailed insight, Figure 2.3 showcases a visual presentation of the contrast between the precise and estimated solutions at $y_1 = 1$. Furthermore, the graphical depiction of the absolute error function at $y_1 = 1$ for $(N, M) = (20, 20)$ is presented in Figure 2.4. These visualizations serve to enhance the understanding of the proposed method's accuracy and effectiveness.

Table 2.1: *Absolute errors $|\mu(y_1, y_2) - \mu_{N,N}(y_1, y_2)|$ Example 2.5*

(y_1, y_2)	$N = 10$	$N = 20$	$N = 30$
$(0.0, 0.0)$	0	0	0
$(0.1, 0.1)$	$9.5482e-07$	$5.8051e-07$	$4.0984e-07$
$(0.2, 0.2)$	$2.0284e-05$	$1.0937e-05$	$7.4685e-06$
$(0.3, 0.3)$	$1.1512e-04$	$6.0072e-05$	$4.0605e-05$
$(0.4, 0.4)$	$3.8832e-04$	$1.9952e-04$	$1.3420e-04$
$(0.5, 0.5)$	$9.8326e-04$	$5.0065e-04$	$3.3578e-04$
$(0.6, 0.6)$	$2.0681e-03$	$1.0467e-03$	$7.0068e-04$
$(0.7, 0.7)$	$3.8101e-03$	$1.9201e-03$	$1.2835e-03$
$(0.8, 0.8)$	$6.3403e-03$	$3.1852e-03$	$2.1270e-03$
$(0.9, 0.9)$	$9.7160e-03$	$4.8698e-03$	$3.6839e-03$

Figure 2.1: *(a) Absolute error function $|\mu(y_1, y_2) - \mu_{10,10}(y_1, y_2)|$, (b) Absolute error function $|\mu(y_1, y_2) - \mu_{20,20}(y_1, y_2)|$ with $p = 3$ for Example 2.5*



(a) (b)

Figure 2.2: *(a) The exact solution* $\mu(y_1, y_2)$*, (b) The approximate solution* $\mu_{20,20}(y_1, y_2)$ *with* $p = 3$ *for Example 2.5*

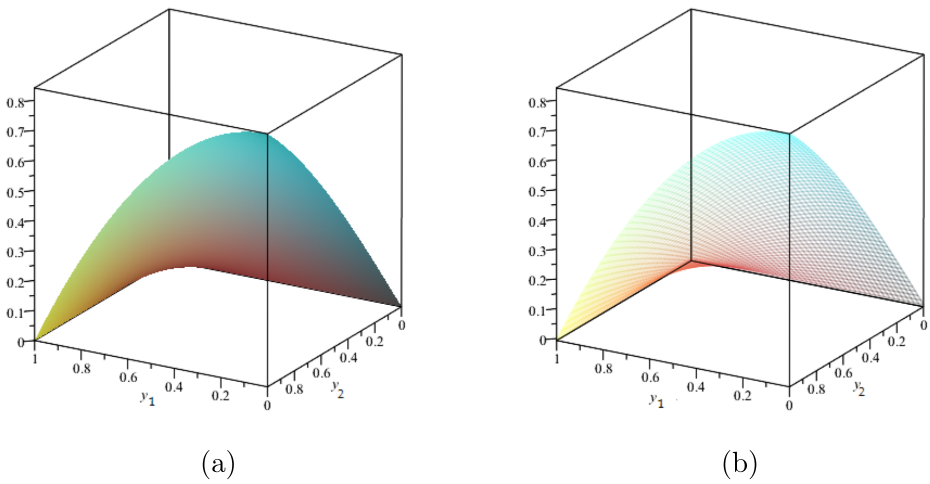


Figure 2.3: *Comparison of the exact and approximate solutions with* $N = M = 20$ *at* $y_1 = 1$ *for Example 2.5*

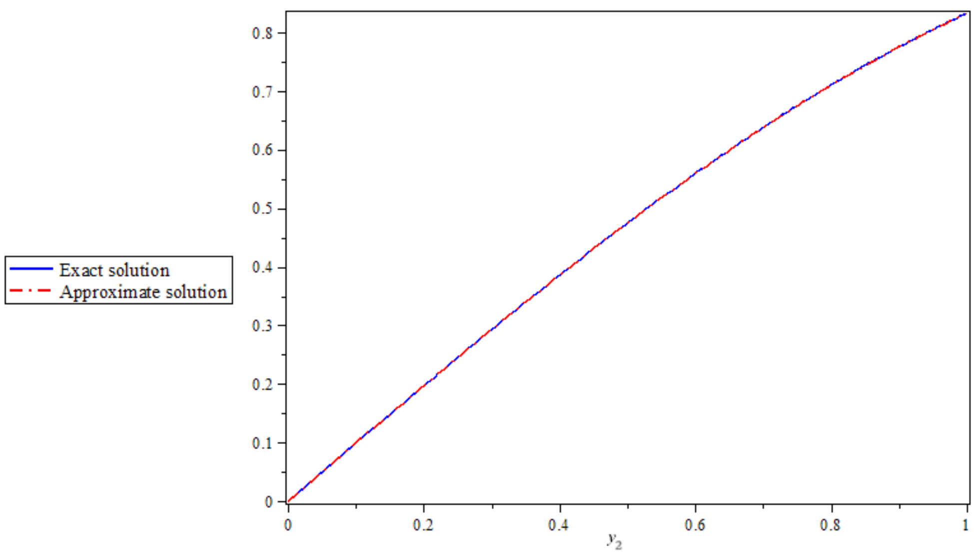
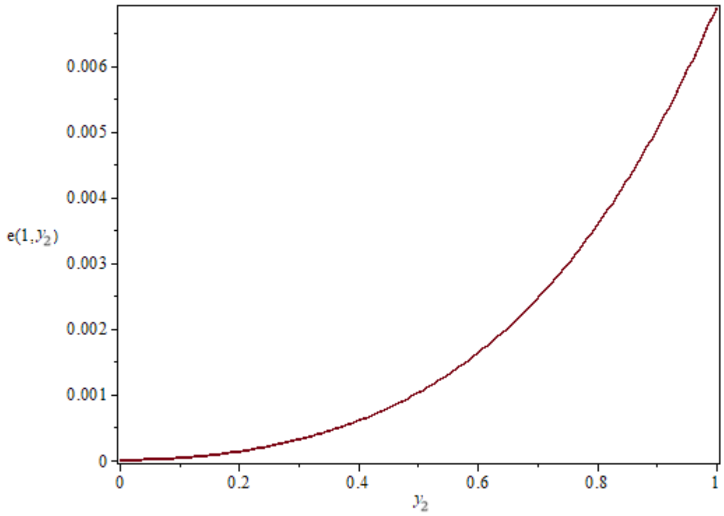


Figure 2.4: *Absolute error function for $N = M = 20$ at $y_1 = 1$ for Example 2.5*



Example 2.6. *Consider the following linear two-dimensional PVIDE*

$$\frac{\partial \mu(y_1, y_2)}{\partial y_1} = \hbar(y_1, y_2) + \int_0^{y_1} \int_0^{y_2} (v_1 cos(v_2))\mu(v_1, v_2)dv_2 dv_1, \qquad 0 \le y_1 \le 1,\ 0 \le y_2 \le 1,$$

with the initial condition

$$\mu_0(y_2) = y_2, \qquad 0 \le y_2 \le 1,$$

where the analytic solution is given by $\mu(y_1, y_2) = y_2 e^{-y_1}$, and

$$\hbar(y_1, y_2) = e^{-y_1}(y_1 y_2 sin(y_2) + y_1 cos(y_2) + y_2 sin(y_2) + cos(y_2) - y_2 - y_1 - 1) - y_2 sin(y_2) - cos(y_2) + 1.$$

Table 2.2 presents the absolute error values, while Figure 2.5-(a) illustrates the behaviors of the exact solution, and 2.5-(b) depicts the approximate solution of Example 2.6. The approximate solution graph is generated using $(N, M) = (20, 20)$ collocation points, matching the graph of the exact solution. Furthermore, Figure 2.6 compares the exact and approximate solutions at $y_1 = 0.1$. Also, Figure 2.7 outlines the absolute errors functions $|\mu(y_1, y_2) - \mu_{20,20}(y_1, y_2)|$ at $y_1 = 1$ with $p = 3$.

Table 2.2: *Absolute errors $|\mu(y_1, y_2) - \mu_{N,N}(y_1, y_2)|$ for Example 2.6*

(y_1, y_2)	$N = 10$	$N = 20$	$N = 30$
(0.0, 0.0)	0	0	0
(0.1, 0.1)	7.8981e − 07	4.8780e − 07	3.4580e − 07
(0.2, 0.2)	1.4646e − 05	7.9986e − 06	5.4816e − 06
(0.3, 0.3)	7.3101e − 05	3.8588e − 05	2.6178e − 05
(0.4, 0.4)	2.1930e − 04	1.1396e − 04	7.6933e − 05
(0.5, 0.5)	4.9957e − 04	2.5728e − 04	1.7318e − 04
(0.6, 0.6)	9.5623e − 04	4.8967e − 04	3.2902e − 04
(0.7, 0.7)	1.6220e − 03	8.2745e − 04	5.5531e − 04
(0.8, 0.8)	2.5158e − 03	1.2801e − 03	9.0808e − 04
(0.9, 0.9)	3.6403e − 03	1.8490e − 03	1.0290e − 03

Figure 2.5: *(a) The exact solution* $\mu(y_1, y_2)$*, (b) the approximate solution* $\mu_{20,20}(y_1, y_2)$ *with* $p = 3$ *for Example 2.6*

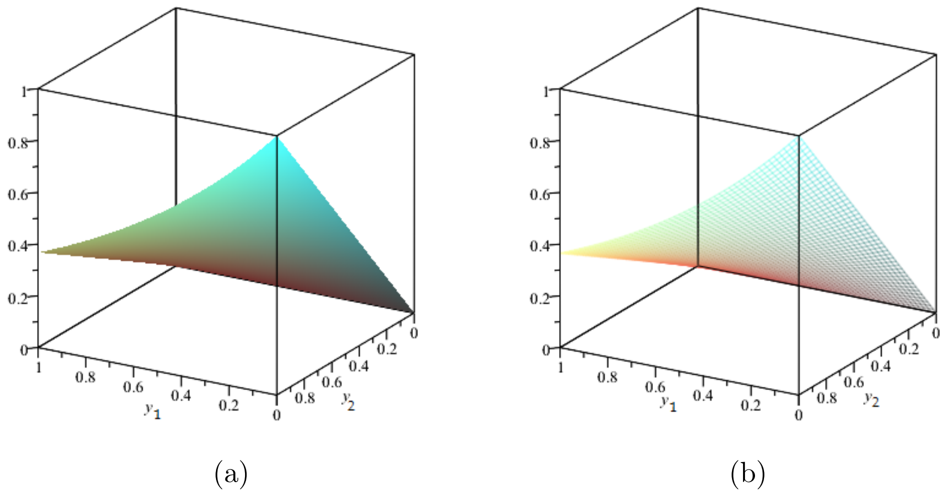


Figure 2.6: *Comparison of the exact and approximate solutions with* $N = M = 20$ *at* $y_1 = 0.1$ *for Example 2.6*

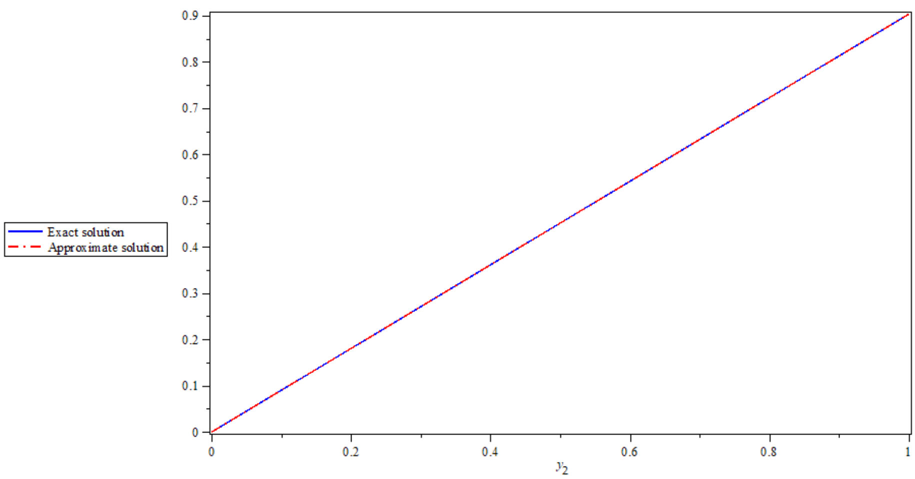
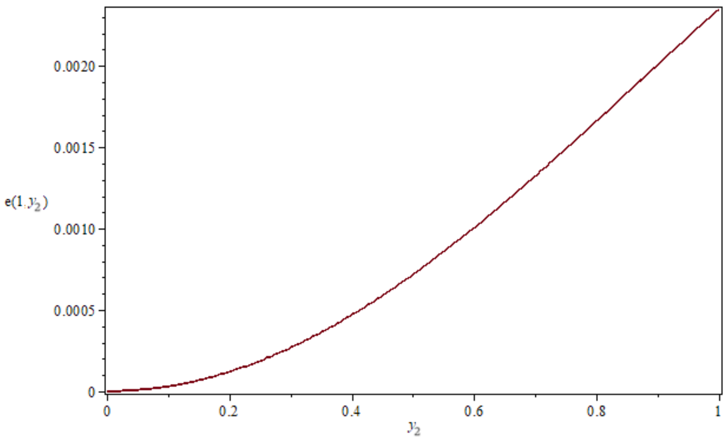


Figure 2.7: *Absolute error function for $N = M = 20$ at $y_1 = 1$ for Example 2.6*



Example 2.7. *Consider the following 2D-PVIDE*

$$\frac{\partial \mu(y_1, y_2)}{\partial y_1} = \hbar(y_1, y_2) + \int_0^{y_1} \int_0^{y_2} (y_1 v_1 + cos(v_2))\mu(v_1, v_2) dv_2 dv_1, \qquad 0 \le y_1 \le 1,\ 0 \le y_2 \le 1,$$

subject to initial condition

$$\mu_0(y_2) = 0, \qquad 0 \le y_2 \le 1,$$

along with $\mu(y_1, y_2) = y_1 \sin(y_2)$. Then, $\hbar(y_1, y_2)$ is given by

$$\hbar(y_1, y_2) = sin(y_2) + \frac{y_1^4}{3}(cos(y_2) - 1) + \frac{y_1^2}{4} sin^2(y_2).$$

Table 2.3 displays numerical results in terms of absolute errors. The proposed method exhibits excellent performance, as evident from the table. Additionally, the absolute error function is illustrated in three-dimensional space, as depicted in Figure 2.8 employing distinct combinations of N and M values, specifically $(N, M) = \{(15, 15), (20, 20)\}$, allowing for comprehensive comparative analysis. Furthermore, Figure 2.9 contrasts the exact and approximate solutions at $y_1 = 0.1$, providing a detailed snapshot of their behavior. While Figure 2.10 presents a visualization of the absolute error function $\mu_{15,15}(y_1, y_2)$ at $y_1 = 1$.

Table 2.3: *Absolute errors $|\mu(y_1, y_2) - \mu_{N,N}(y_1, y_2)|$ for Example 2.7*

(y_1, y_2)	$N = 10$	$N = 20$	$N = 30$
$(0.0, 0.0)$	0	0	0
$(0.1, 0.1)$	$8.3076e-07$	$5.2136e-07$	$3.7061e-07$
$(0.2, 0.2)$	$1.6484e-05$	$9.1714e-06$	$6.2693e-06$
$(0.3, 0.3)$	$8.8108e-05$	$4.7363e-05$	$3.1640e-05$
$(0.4, 0.4)$	$2.8382e-04$	$1.4779e-04$	$9.5703e-05$
$(0.5, 0.5)$	$6.9663e-04$	$3.4482e-04$	$1.6742e-04$
$(0.6, 0.6)$	$1.4415e-03$	$6.5519e-04$	$2.3514e-04$
$(0.7, 0.7)$	$2.6493e-03$	$1.0463e-03$	$8.1499e-04$
$(0.8, 0.8)$	$4.4561e-03$	$1.3901e-03$	$1.2109e-03$
$(0.9, 0.9)$	$6.9807e-03$	$1.4011e-03$	$1.3022e-03$

Figure 2.8: *(a) Absolute error function* $|\mu(y_1, y_2) - \mu_{15,15}(y_1, y_2)|$, *(b) Absolute error function* $|\mu(y_1, y_2) - \mu_{20,20}(y_1, y_2)|$ *with* $p = 3$ *for Example 2.7*

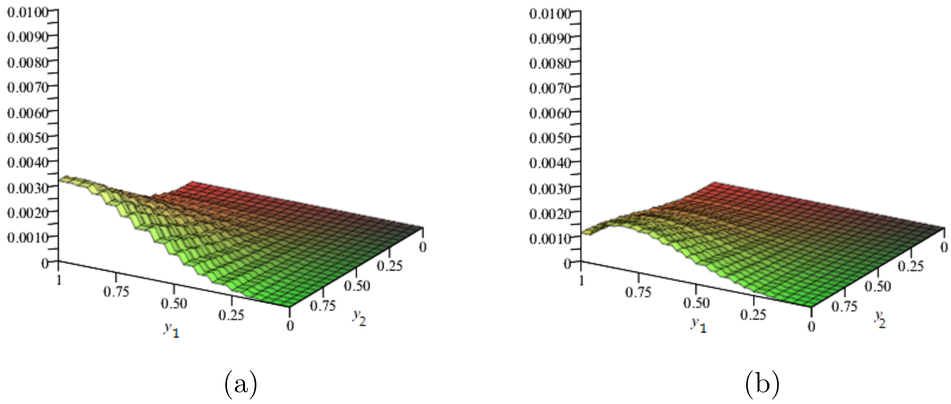


Figure 2.9: *Comparison of the exact and approximate solutions with* $N = M = 15$ *at* $y_1 = 0.1$ *for Example 2.7*

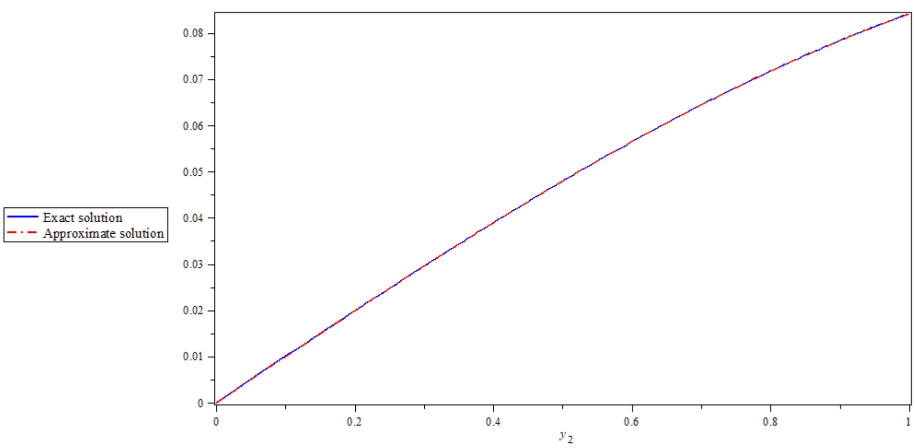
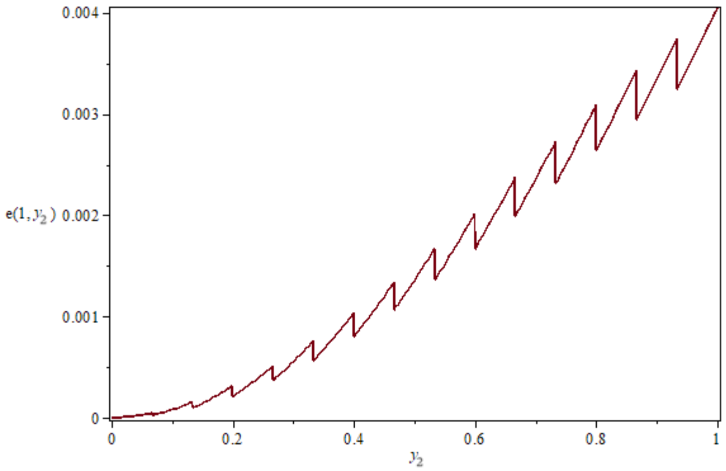


Figure 2.10: *Absolute error function for $N = M = 15$ at $y_1 = 1$ for Example 2.7*



Example 2.8. *In this example, consider the two-dimensional partial Volterra integro-differential equation discussed in [69]*

$$\frac{\partial\mu(y_1, y_2)}{\partial y_1} = \hbar(y_1, y_2) + \int_0^{y_1}\int_0^{y_2} y_1^2\mu(v_1, v_2)dv_2dv_1, \qquad 0 \le y_1 \le 1,\ 0 \le y_2 \le 1,$$

subject to initial condition

$$\mu_0(y_2) = 0, \qquad 0 \le y_2 \le 1.$$

Where $\mu(y_1, y_2) = y_2 sin(y_1)$ as the analytic solution, and $\hbar(y_1, y_2)$ is calculated using the exact solution and obtained similarly as [69]

$$\hbar(y_1, y_2) = y_2 cos(y_1) - y_1^2 y_2^2 sin^2\left(\frac{y_1}{2}\right).$$

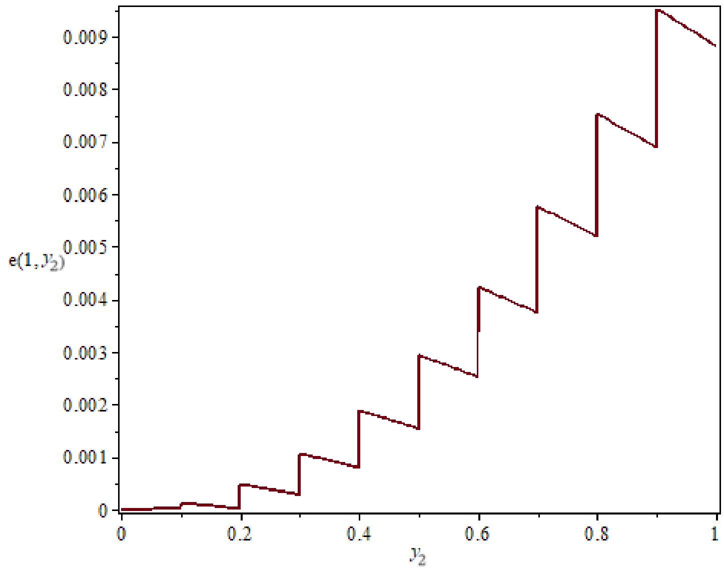
The numerical results obtained in this example for $N = M = 10$ are compared in Table 2.4 with the numerical results obtained by using the methods in [69] for different collocation points in terms of absolute error values, while Figure 2.11 illustrates the behaviors of the absolue error $|\mu(y_1, y_2) - \mu_{10,10}(y_1, y_2)|$ at $y_1 = 1$ and $p = 3$ for Example 2.8.

Table 2.4: *Absolute errors $|\mu(y_1, y_2) - \mu_{N,N}(y_1, y_2)|$ for Example 2.8*

(y_1, y_2)	Method in [69]	Our Method
$(0.0, 0.0)$	0	0
$(0.1, 0.1)$	$4.99e-09$	$4.99e-09$
$(0.2, 0.2)$	$6.38e-07$	$2.39e-07$
$(0.4, 0.4)$	$8.11e-05$	$7.94e-06$
$(0.6, 0.6)$	$1.36e-03$	$3.86e-04$
$(0.8, 0.8)$	$1.00e-02$	$3.85e-03$
$(0.9, 0.9)$	$2.27e-02$	$9.51e-03$
$(1.0, 1.0)$	$4.70e-02$	$8.82e-03$

The comparison presented in Table 2.4 indicates that the outcomes achieved through the current method exhibit significantly higher accuracy compared to those reported in [69]. The computation time for these outcomes amounted to 47.07s on a personal computer running Maple version 18.

Figure 2.11: *Absolute error function for $N = M = 10$ at $y_1 = 1$ for Example 2.8*



Example 2.9. *The final example pertains to a scenario where the provided equation assumes the following structure:*

$$\frac{\partial \mu(y_1, y_2)}{\partial y_1} = 2y_1y_2^2 - \frac{1}{12}y_2^4y_1^4 + \int_0^{y_1}\int_0^{y_2} y_2v_1\mu(v_1, v_2)dv_2dv_1, \; 0 \le y_1 \le 1, \; 0 \le y_2 \le 1,$$

with the initial condition

$$\mu(0, y_2) = 0.$$

The exact solution of this problem is given by $\mu(y_1, y_2) = y_1^2y_2^2$.
Table 2.5 illustrates a consistent decrease in absolute errors as the number of collocation points increases for $p = 3$. In Figure 2.12, three-dimensional plots of both the exact function $\mu(y_1, y_2)$ and the approximate function $\mu_{20,20}(y_1, y_2)$ are depicted over the domain $0 \le y_1 \le 1$ and $0 \le y_2 \le 1$. Additionally, Figure 2.13 offers a comparative visualization of the exact and approximate solutions at $y_1 = 0.1$, providing valuable insights into their behavior. Furthermore, Figure 2.14 presents a detailed 2D representation of the absolute error function $e_{20,20}(y_1, y_2)$ at $y_1 = 1$.

Table 2.5: *Absolute errors $|\mu(y_1, y_2) - \mu_{N,N}(y_1, y_2)|$ for Example 2.9*

(y_1, y_2)	$N = 10$	$N = 20$	$N = 30$
$(0.0, 0.0)$	0	0	0
$(0.1, 0.1)$	1.67×10^{-11}	1.47×10^{-11}	1.11×10^{-11}
$(0.2, 0.2)$	7.53×10^{-9}	4.47×10^{-9}	3.16×10^{-9}
$(0.3, 0.3)$	2.16×10^{-7}	1.21×10^{-7}	8.44×10^{-8}
$(0.4, 0.4)$	2.29×10^{-6}	1.25×10^{-6}	8.59×10^{-7}
$(0.5, 0.5)$	1.42×10^{-5}	7.59×10^{-6}	5.18×10^{-6}
$(0.6, 0.6)$	6.24×10^{-5}	3.30×10^{-5}	2.24×10^{-5}
$(0.7, 0.7)$	2.17×10^{-4}	1.14×10^{-4}	7.75×10^{-5}
$(0.8, 0.8)$	6.42×10^{-4}	3.35×10^{-4}	2.26×10^{-4}
$(0.9, 0.9)$	1.66×10^{-3}	8.65×10^{-4}	5.84×10^{-4}

Figure 2.12: *(a) The exact solution $\mu(y_1, y_2)$, (b) the approximate solution $\mu_{20,20}(y_1, y_2)$ with $p = 3$ for Example 2.9*

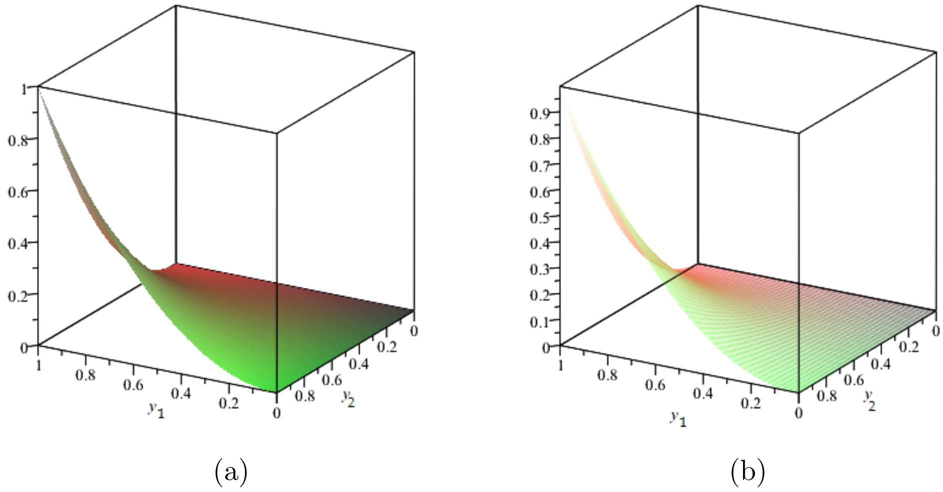


Figure 2.13: *Comparison of the exact and approximate solutions with $N = M = 20$ at $y_1 = 0.1$ for Example 2.9*

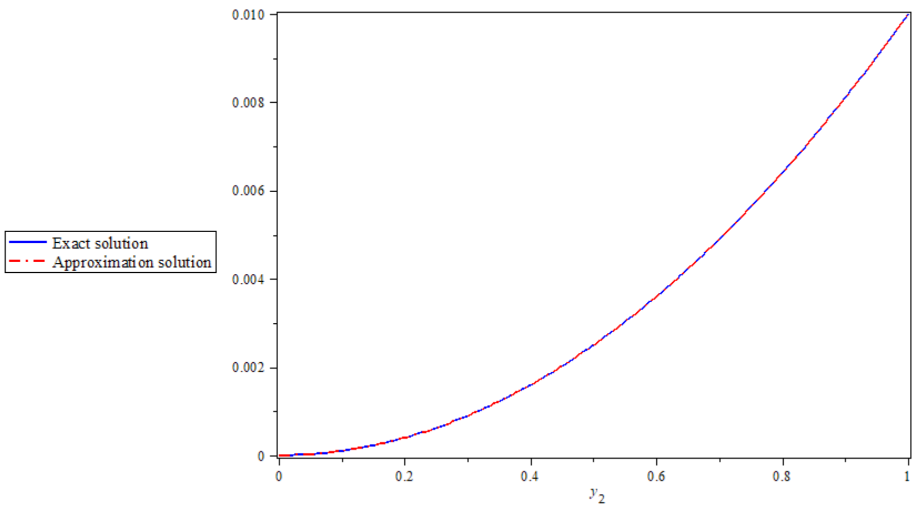
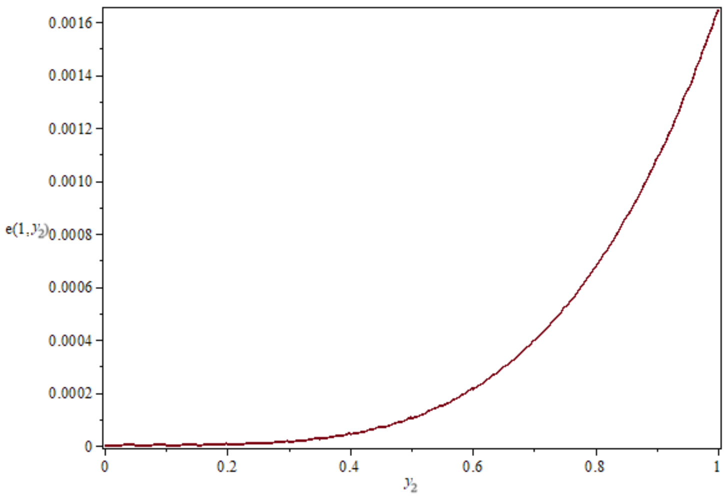


Figure 2.14: *Absolute error function for $N = M = 20$ at $y_1 = 1$ for Example 2.9*



Based on the numerical experiments, it is evident that the Taylor collocation method serves as an effective tool for approximating solutions to linear PVIDEs, aligning with our convergence analysis in Section 2.2. Furthermore, we observe that the error diminishes and tends toward zero as M and N increase.

2.4 Concluding remarks

In this chapter, we introduce a novel numerical algorithm for solving linear 2D-PVIDEs represented by form (2.1). The method utilizes a Collocation approach grounded on Taylor polynomials in two dimensions. The iterative formulas directly yield the approximate solutions, eliminating the need to solve algebraic systems. In Section 2.2, we perform convergence and error analysis, shedding light on our theoretical findings, while Section 2.3 showcases several test examples aimed at evaluating the method's efficiency and confirming the theoretical estimates.

Chapter 3

Numerical solution of second order two-dimensional partial Volterra integro-differential equations

The purpose of this chapter is to provide a numerical solution to the following hyperbolic 2D-PVIDE:

$$\frac{\partial^2 \mu(y_1, y_2)}{\partial y_1 \partial y_2} = \alpha_1 \frac{\partial \mu(y_1, y_2)}{\partial y_1} + \alpha_2 \frac{\partial \mu(y_1, y_2)}{\partial y_2} + \alpha_3 \mu(y_1, y_2) + \hbar(y_1, y_2) + \int_0^{y_1} H(y_1, y_2, v_1)\mu(v_1, y_2)dv_1, \tag{3.1}$$

along with the appropriate associated initial value conditions. $(y_1, y_2) \in \Lambda = [0, A_1] \times [0, A_2] \subset \mathbb{R}^2$, and $\hbar$ and H are sufficiently smooth functions to ensure the existence and uniqueness of the solution on Λ and the region $\mathrm{S} := \{(y_1, y_2, v_1) : 0 \leq v_1 \leq y_1 \leq A_1, 0 \leq y_2 \leq A_2\}$ accordingly.

An initial investigation into second-order 2D-PVIDEs in a similar form to Eq. (3.1) has been pursued by several researchers. For instance, Rivaz et al. [74] undertook a study where they transformed the linear 2D-PVIDE into a system of linear algebraic equations through the application of two-dimensional Chebyshev polynomials and their operational matrix of integration. Similarly, Rostami and Maleknejad [75] explored the solution to a related but mixed 2D-PVFIDEs using two-dimensional hybrid Taylor polynomials and Block-Pulse functions. The analysis of the singular case of the 2D-PVIDEs in (3.1) has been explored in prior works such as [76–78]. These studies employed two-dimensional orthonormal Bernstein polynomials, two-dimensional wavelets approximations and their operational matrices of integration, as well as two-dimensional Bernoulli wavelets along with their corresponding operational matrices, respectively. Additionally, Mirazee et al. [79] employed Bernstein polynomials to solve the fractional order case.

To the best of our knowledge, no prior endeavors have been undertaken to solve the second-order 2D-PVIDE (3.1) using the Taylor collocation method, and therefore applying it to address these significant problems stands as a major challenge. Our main objective here is to extend and generalize the numerical method introduced in Chapter 2 of our thesis to effectively solve the second-order 2D-PVIDE (3.1). In this regard, we reformulate the 2D-PVIDE (3.1) into another problem involving the solution of a two-dimensional Volterra integral equation. Using the two-dimensional Taylor polynomials as the basis function of the piecewise collocation approach, we get an explicit form of the approximate solution to the main problem.

The rest of this work is as follows: Section 3.1 obtains the neccessary background and notations and constructs the Taylor collocation approach to solve the second-order 2D-PVIDE (3.1), while

Section 3.2 presents details of the error estimates and convergence analysis of the proposed method. Section 3.3 introduces several numerical examples and illustrations to test the applicability of the suggested method and the theoretical results. Finally, the last section gives the concluding remarks.

3.1 Description of the method

This section is devoted to constructing the Taylor collocation approach to solve the second-order 2D-PVIDE (3.1). For simplicity and without loss of generality, we assume that $A_1 = A_2 = 1$.

If we define a new function $\omega(y_1, y_2)$, such that $\omega(y_1, y_2) = \frac{\partial^2 \mu(y_1,y_2)}{\partial y_1 \partial y_2}$, then

$$\mu(y_1, y_2) = \int_0^{y_1} \int_0^{y_2} \omega(v_1, v_2) dv_2 dv_1 - \mu(0, 0) + \mu(0, y_2) + \mu(y_1, 0), \tag{3.2}$$

and the second-order 2D-PVIDE (3.1) may be transformed to the two-dimensional VIE

$$\begin{aligned}\omega(y_1, y_2) =& \Phi(y_1, y_2) + \alpha_1 \int_0^{y_2} \omega(y_1, v_2) dv_2 + \alpha_2 \int_0^{y_1} \omega(v_1, y_2) dv_1 + \alpha_3 \int_0^{y_1} \int_0^{y_2} \omega(v_1, v_2) dv_2 dv_1 \\ &+ \int_0^{y_1} \int_0^{v_1} \int_0^{y_2} H(y_1, y_2, v_1) \omega(z, v_2) dv_2 dz dv_1, \qquad (y_1, y_2) \in [0, 1] \times [0, 1].\end{aligned} \tag{3.3}$$

It follows from the classical theory of Volterra that (3.3) possesses a unique solution $\omega \in C(\Lambda)$, with Φ is a term obtained by using the initial values conditions as follows:

$$\begin{aligned}\Phi(y_1, y_2) = \hbar(y_1, y_2) &+ \alpha_1 \left(\frac{\partial \mu(y_1, y_2)}{\partial y_2}\right)_{y_2=0} + \alpha_2 \left(\frac{\partial \mu(y_1, y_2)}{\partial y_1}\right)_{y_1=0} + \alpha_3 \mu(0, y_2) \\ &+ \alpha_3 \mu(y_1, 0) - \alpha_3 \mu(0, 0) + \int_0^{y_1} H(y_1, y_2, v_1) \mu(v_1, 0) dv_1 \\ &+ \int_0^{y_1} H(y_1, y_2, v_1) \mu(0, y_2) dv_1 - \int_0^{y_1} H(y_1, y_2, v_1) \mu(0, 0) dt.\end{aligned}$$

We examine the numerical solutions within the real polynomial spline space $S^{(-1)}_{p-1,p-1}(\Pi_{N,M})$ of degree $p - 1$ in both y_1 and y_2, as defined by (2.4)

$$S^{(-1)}_{p-1,p-1}(\Pi_{N,M}) = \{\omega : \ \omega_{n,m} = \omega|_{\Lambda_{n,m}} \in \pi_{p-1,p-1}, \ n = 0, \cdots, N - 1, \ m = 0, \cdots, M - 1\},$$

where we approximate the unknown function $\omega(y_1, y_2)$ within the rectangle $\Lambda_{n,m}$, $n = 0, 1, 2, \ldots, N - 1$, $m = 0, 1, 2, \ldots, M - 1$, as

$$\omega_{n,m}(y_1, y_2) = \sum_{i+j=0}^{p-1} \frac{1}{i!j!} \frac{\partial^{i+j} \hat{\omega}_{n,m}(y_{1,n}, y_{2,m})}{\partial y_1^i \partial y_2^j} (y_1 - y_{1,n})^i (y_2 - y_{2,m})^j, \tag{3.4}$$

where $\dfrac{\partial^{i+j} \hat{\omega}_{n,m}(y_{1,n}, y_{2,m})}{\partial y_1^i \partial y_2^j}$ are unknown coefficients to be determined in the sequel.

1^{st} **step:** For $n = m = 0$, we approximate the function $\omega(y_1, y_2)$ within the rectangle $\Lambda_{0,0}$ as

$$\omega_{0,0}(y_1, y_2) = \sum_{i+j=0}^{p-1} \frac{1}{i!j!} \left(\frac{\partial^{i+j}\omega(y_1, y_2)}{\partial y_1^i \partial y_2^j} \right)_{y_1=0, y_2=0} y_1^i y_2^j, \quad (y_1, y_2) \in \Lambda_{0,0}. \tag{3.5}$$

We differentiate Eq. (3.3) j-times in terms of y_2,

$$\begin{aligned}
\frac{\partial^j \omega(y_1, y_2)}{\partial y_2^j} &= \partial_2^{(j)}\Phi(y_1, y_2) + \alpha_1 \partial_2^{(j-1)}\omega(y_1, y_2) + \alpha_2 \int_0^{y_1} \partial_2^{(j)}\omega(v_1, y_2)dv_1 \\
&\quad + \alpha_3 \int_0^{y_1} \partial_2^{(j-1)}\omega(v_1, y_2)dv_1 \\
&\quad + \int_0^{y_1}\int_0^{v_1} \sum_{r=0}^{j-1} \frac{\partial^r}{\partial y_2^r}\left[\partial_2^{(j-1-r)}H(y_1, y_2, v_1)\omega(z, y_2)\right] dz dv_1 \\
&\quad + \int_0^{y_1}\int_0^{v_1}\int_0^{y_2} \partial_2^{(j)}H(y_1, y_2, v_1)\omega(z, v_2)dv_2 dz dv_1 \\
&= \partial_2^{(j)}\Phi(y_1, y_2) + \alpha_1 \partial_2^{(j-1)}\omega(y_1, y_2) + \alpha_2 \int_0^{y_1} \partial_2^{(j)}\omega(v_1, y_2)dv_1 \\
&\quad + \alpha_3 \int_0^{y_1} \partial_2^{(j-1)}\omega(v_1, y_2)dv_1 \\
&\quad + \sum_{r=0}^{j-1}\sum_{l=0}^{r} \binom{r}{l} \int_0^{y_1}\int_0^{v_1} \frac{\partial^{r-l}}{\partial y_2^{r-l}}\left[\partial_2^{(j-1-r)}H(y_1, y_2, v_1)\right] \frac{\partial^l \omega(z, y_2)}{\partial y_2^l} dz dv_1 \\
&\quad + \int_0^{y_1}\int_0^{v_1}\int_0^{y_2} \partial_2^{(j)}H(y_1, y_2, v_1)\omega(z, v_2)dv_2 dz dv_1.
\end{aligned}$$

Thus, differentiating Eq. (3.3) $i-$ and $j-$times in terms of y_1 and y_2, respectively, we get

$$\begin{aligned}
&\frac{\partial^{i+j}\omega(y_1, y_2)}{\partial y_1^i \partial y_2^j} = \partial_1^{(i)}\partial_2^{(j)}\Phi(y_1, y_2) \\
&+ \alpha_1 \partial_1^{(i)}\partial_2^{(j-1)}\omega(y_1, y_2) + \alpha_2 \partial_1^{(i-1)}\partial_2^{(j)}\omega(y_1, y_2) + \alpha_3 \partial_1^{(i-1)}\partial_2^{(j-1)}\omega(y_1, y_2) \\
&+ \sum_{r=0}^{j-1}\sum_{l=0}^{r}\sum_{q=0}^{i-1}\sum_{\eta=0}^{q} \binom{r}{l}\binom{q}{\eta} \frac{\partial^{q-\eta}}{\partial y_1^{q-\eta}} \left[\left.\frac{\partial^{i-1-q}}{\partial y_1^{i-1-q}}\right|_{v_1=y_1} \left(\frac{\partial^{r-l}}{\partial y_2^{r-l}}\left[\partial_2^{(j-1-r)}H(y_1, y_2, v_1)\right]\right)\right] \times \\
&\qquad\qquad \frac{\partial^{l+\eta}}{\partial y_1^\eta \partial y_2^l}\left(\int_0^{y_1}\omega(z, y_2)dz\right) \\
&+ \sum_{r=0}^{j-1}\sum_{l=0}^{r} \binom{r}{l} \int_0^{y_1}\int_0^{v_1} \frac{\partial^i}{\partial y_1^i}\left[\frac{\partial^{r-l}}{\partial y_2^{r-l}}\left[\partial_2^{(j-1-r)}H(y_1, y_2, v_1)\right]\right] \frac{\partial^l \omega(z, y_2)}{\partial y_2^l} dz dv_1 \\
&+ \sum_{q=0}^{i-1}\sum_{\eta=0}^{q} \binom{q}{\eta} \int_0^{y_2} \frac{\partial^{q-\eta}}{\partial y_1^{q-\eta}}\left[\partial_1^{(i-1-q)}\partial_2^{(j)}H(y_2, y_1, y_2)\right] \frac{\partial^\eta}{\partial y_1^\eta}\left(\int_0^{y_1}\omega(z, v_2)dz\right) dv_2 \\
&+ \int_0^{y_1}\int_0^{v_1}\int_0^{y_2} \partial_1^{(i)}\partial_2^{(j)}H(y_1, y_2, v_1)\omega(z, v_2)dv_2 dz dv_1,
\end{aligned}$$

where

$$\frac{\partial^\eta}{\partial y_1^\eta}\left(\int_0^{y_1}\omega(z, v_2)dz\right) = \int_0^{y_1}\omega(z, v_2)dz, \quad if\ \eta = 0.$$

Hence,

$$\frac{\partial^{i+j}\omega(0,0)}{\partial y_1^i \partial y_2^j} = \partial_1^{(i)}\partial_2^{(j)}\Phi(0,0)$$
$$+\alpha_1\partial_1^{(i)}\partial_2^{(j-1)}\omega(0,0)+\alpha_2\partial_1^{(i-1)}\partial_2^{(j)}\omega(0,0)+\alpha_3\partial_1^{(i-1)}\partial_2^{(j-1)}\omega(0,0)$$
$$+\sum_{r=0}^{j-1}\sum_{l=0}^{r}\sum_{q=0}^{i-1}\sum_{\nu=0}^{q}\binom{r}{l}\binom{q}{\eta}\frac{\partial^{q-\eta}}{\partial y_1^{q-\eta}}\left[\frac{\partial^{i-1-q}}{\partial y_1^{i-1-q}}\bigg|_{v_1=y_1}\left(\frac{\partial^{r-l}}{\partial y_2^{r-l}}\left[\partial_2^{(j-1-r)}H(y_1,y_2,v_1)\right]\right)\right]_{y_1=0,y_2=0}$$
$$\times\frac{\partial^{l+\eta-1}\omega(0,0)}{\partial y_1^{\eta-1}\partial y_2^{l}}. \tag{3.6}$$

2^{nd} **step:** For $n=1,2,\ldots,N-1$ and $m=0$, we approximate the function $\omega(y_1,y_2)$ within the rectangles $\Lambda_{n,0},\ n=0,1,2,\ldots,N-1$, as

$$\omega_{n,0}(y_1,y_2)=\sum_{i+j=0}^{p-1}\frac{1}{i!j!}\frac{\partial^{i+j}\hat{\omega}_{n,0}(y_{1,n},0)}{\partial y_1^i\partial y_2^j}(y_1-y_{1,n})^i y_2^j,\quad (y_1,y_2)\in\Lambda_{n,0}, \tag{3.7}$$

where $\hat{\omega}_{n,0}(y_1,y_2)$ is the precise solution to the VIE

$$\hat{\omega}_{n,0}(y_1,y_2)=\Phi(y_1,y_2)+\alpha_1\int_0^{y_2}\hat{\omega}_{n,0}(y_1,v_2)dv_2+\alpha_2\sum_{\xi=0}^{n-1}\int_{y_{1,\xi}}^{y_{1,\xi+1}}\omega_{\xi,0}(v_1,y_2)dv_1+\alpha_2\int_{y_{1,n}}^{y_1}\hat{\omega}_{n,0}(v_1,y_2)dv_1$$
$$+\alpha_3\sum_{\xi=0}^{n-1}\int_{y_{1,\xi}}^{y_{1,\xi+1}}\int_0^{y_2}\omega_{\xi,0}(v_1,v_2)dv_2dv_1+\alpha_3\int_{y_{1,n}}^{y_1}\int_0^{y_2}\hat{\omega}_{n,0}(v_1,v_2)dv_2dv_1$$
$$+\sum_{\xi=0}^{n-1}\sum_{\sigma=0}^{\xi-1}\int_{y_{1,\xi}}^{y_{1,\xi+1}}\int_{y_{1,\sigma}}^{y_{1,\sigma+1}}\int_0^{y_2}H(y_1,y_2,v_1)\omega_{\sigma,0}(z,v_2)dv_2dzdv_1$$
$$+\sum_{\xi=0}^{n-1}\int_{y_{1,\xi}}^{y_{1,\xi+1}}\int_{y_{1,\xi}}^{v_1}\int_0^{y_2}H(y_1,y_2,v_1)\omega_{\xi,0}(z,v_2)dv_2dzdv_1$$
$$+\sum_{\sigma=0}^{n-1}\int_{y_{1,n}}^{y_1}\int_{y_{1,\sigma}}^{y_{1,\sigma+1}}\int_0^{y_2}H(y_1,y_2,v_1)\omega_{\sigma,0}(z,v_2)dv_2dzdv_1$$
$$+\int_{y_{1,n}}^{y_1}\int_{y_{1,n}}^{v_1}\int_0^{y_2}H(y_1,y_2,v_1)\hat{\omega}_{n,0}(z,v_2)dv_2dzdv_1. \tag{3.8}$$

Similarly, we differentiate Eq. (3.8) j-times in terms of y_2, we obtain

$$\frac{\partial^j\hat{\omega}_{n,0}(y_1,y_2)}{\partial y_2^j}=\partial_2^{(j)}\Phi(y_1,y_2)+\alpha_1\partial_2^{(j-1)}\hat{\omega}_{n,0}(y_1,y_2)$$
$$+\alpha_2\sum_{\xi=0}^{n-1}\int_{y_{1,\xi}}^{y_{1,\xi+1}}\partial_2^{(j)}\omega_{\xi,0}(v_1,y_2)dv_1+\alpha_2\int_{y_{1,n}}^{y_1}\partial_2^{(j)}\hat{\omega}_{n,0}(v_1,y_2)dv_1$$

$$+\alpha_3\sum_{\xi=0}^{n-1}\int_{y_{1,\xi}}^{y_{1,\xi+1}}\partial_2^{(j-1)}\omega_{\xi,0}(v_1,y_2)dv_1+\alpha_3\int_{y_{1,n}}^{y_1}\partial_2^{(j-1)}\hat{\omega}_{n,0}(v_1,y_2)dv_1$$

$$+\sum_{\xi=0}^{n-1}\sum_{\sigma=0}^{\xi-1}\sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\int_{y_{1,\xi}}^{y_{1,\xi+1}}\int_{y_{1,\sigma}}^{y_{1,\sigma+1}}\frac{\partial^{r-l}}{\partial y_2^{r-l}}\left[\partial_2^{(j-1-r)}H(y_1,y_2,v_1)\right]\frac{\partial^l\omega_{\sigma,0}}{\partial y_2^l}(z,y_2)dzdv_1$$

$$+\sum_{\xi=0}^{n-1}\sum_{\sigma=0}^{\xi-1}\int_{y_{1,\xi}}^{y_{1,\xi+1}}\int_{y_{1,\sigma}}^{y_{1,\sigma+1}}\int_0^{y_2}\partial_2^{(j)}H(y_1,y_2,v_1)\omega_{\sigma,0}(z,v_2)dv_2dzdv_1$$

$$+\sum_{\xi=0}^{n-1}\sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\int_{y_{1,\xi}}^{y_{1,\xi+1}}\int_{y_{1,\xi}}^{v_1}\frac{\partial^{r-l}}{\partial y_2^{r-l}}\left[\partial_2^{(j-1-r)}H(y_1,y_2,v_1)\right]\frac{\partial^l\omega_{\xi,0}}{\partial y_2^l}(z,y_2)dzdv_1$$

$$+\sum_{\xi=0}^{n-1}\int_{y_{1,\xi}}^{y_{1,\xi+1}}\int_{y_{1,\xi}}^{v_1}\int_0^{y_2}\partial_2^{(j)}H(y_1,y_2,v_1)\omega_{\xi,0}(z,v_2)dv_2dzdv_1$$

$$+\sum_{\sigma=0}^{n-1}\sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\int_{y_{1,n}}^{y_1}\int_{y_{1,\sigma}}^{y_{1,\sigma+1}}\frac{\partial^{r-l}}{\partial y_2^{r-l}}\left[\partial_2^{(j-1-r)}H(y_1,y_2,v_1)\right]\frac{\partial^l\omega_{\sigma,0}}{\partial y_2^l}(z,y_2)dzdv_1$$

$$+\sum_{\sigma=0}^{n-1}\int_{y_{1,n}}^{y_1}\int_{y_{1,\sigma}}^{y_{1,\sigma+1}}\int_0^{y_2}\partial_2^{(j)}H(y_1,y_2,v_1)\omega_{\sigma,0}(z,v_2)dv_2dzdv_1$$

$$+\sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\int_{y_{1,n}}^{y_1}\int_{y_{1,n}}^{v_1}\frac{\partial^{r-l}}{\partial y_2^{r-l}}\left[\partial_2^{(j-1-r)}H(y_1,y_2,v_1)\right]\frac{\partial^l\hat{\omega}_{n,0}}{\partial y_2^l}(z,y_2)dzdv_1$$

$$+\int_{y_{1,n}}^{y_1}\int_{y_{1,n}}^{v_1}\int_0^{y_2}\partial_2^{(j)}H(y_1,y_2,v_1)\hat{\omega}_{n,0}(z,v_2)dv_2dzdv_1.$$

Thus, the differentiation of Eq. (3.8) $i-$ and $j-$times in terms of y_1 and y_2, respectively, gives

$$\frac{\partial^{i+j}\hat{\omega}_{n,0}(y_1,y_2)}{\partial y_1^i\partial y_2^j}=\partial_1^{(i)}\partial_2^{(j)}\Phi(y_1,y_2)$$

$$+\alpha_1\partial_1^{(i)}\partial_2^{(j-1)}\hat{\omega}_{n,0}(y_1,y_2)+\alpha_2\partial_1^{(i-1)}\partial_2^{(j)}\hat{\omega}_{n,0}(y_1,y_2)+\alpha_3\partial_1^{(i-1)}\partial_2^{(j-1)}\hat{\omega}_{n,0}(y_1,y_2)$$

$$+\sum_{\xi=0}^{n-1}\sum_{\sigma=0}^{\xi-1}\sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\int_{y_{1,\xi}}^{y_{1,\xi+1}}\int_{y_{1,\sigma}}^{y_{1,\sigma+1}}\frac{\partial^i}{\partial y_1^i}\left[\frac{\partial^{r-l}}{\partial y_2^{r-l}}\left[\partial_2^{(j-1-r)}H(y_1,y_2,v_1)\right]\right]\frac{\partial^l\omega_{\sigma,0}}{\partial y_2^l}(z,y_2)dzdv_1$$

$$+\sum_{\xi=0}^{n-1}\sum_{\sigma=0}^{\xi-1}\int_{y_{1,\xi}}^{y_{1,\xi+1}}\int_{y_{1,\sigma}}^{y_{1,\sigma+1}}\int_0^{y_2}\partial_1^{(i)}\partial_2^{(j)}H(y_1,y_2,v_1)\omega_{\sigma,0}(z,v_2)dv_2dzdv_1$$

$$+\sum_{\xi=0}^{n-1}\sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\int_{y_{1,\xi}}^{y_{1,\xi+1}}\int_{y_{1,\xi}}^{v_1}\frac{\partial^i}{\partial y_1^i}\left[\frac{\partial^{r-l}}{\partial y_2^{r-l}}\left[\partial_2^{(j-1-r)}H(y_1,y_2,v_1)\right]\right]\frac{\partial^l\omega_{\xi,0}}{\partial y_2^l}(z,y_2)dzdv_1$$

$$+\sum_{\xi=0}^{n-1}\int_{y_{1,\xi}}^{y_{1,\xi+1}}\int_{y_{1,\xi}}^{v_1}\int_0^{y_2}\partial_1^{(i)}\partial_2^{(j)}H(y_1,y_2,v_1)\omega_{\xi,0}(z,v_2)dv_2dzdv_1$$

$$+\sum_{\sigma=0}^{n-1}\sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\int_{y_{1,n}}^{y_1}\int_{y_{1,\sigma}}^{y_{1,\sigma+1}}\frac{\partial^i}{\partial y_1^i}\left[\frac{\partial^{r-l}}{\partial y_2^{r-l}}\left[\partial_2^{(j-1-r)}H(y_1,y_2,v_1)\right]\right]\frac{\partial^l\omega_{\sigma,0}(z,y_2)}{\partial y_2^l}dzdv_1$$

$$+\sum_{\sigma=0}^{n-1}\sum_{r=0}^{j-1}\sum_{l=0}^{r}\sum_{q=0}^{i-1}\binom{r}{l}\int_{y_{1,\sigma}}^{y_{1,\sigma+1}}\frac{\partial^q}{\partial y_1^q}\left[\frac{\partial^{i-1-q}}{\partial y_1^{i-1-q}}\bigg|_{v_1=y_1}\left(\frac{\partial^{r-l}}{\partial y_2^{r-l}}\left[\partial_2^{(j-1-r)}H(y_1,y_2,v_1)\right]\right)\right]\frac{\partial^l\omega_{\sigma,0}(z,y_2)}{\partial y_2^l}dz$$

$$
\begin{aligned}
&+\sum_{\sigma=0}^{n-1}\sum_{q=0}^{i-1}\int_{y_{1,\sigma}}^{y_{1,\sigma+1}}\int_{0}^{y_2}\frac{\partial^q}{\partial y_1^q}\left[\partial_1^{(i-1-q)}\partial_2^{(j)}H(y_2,y_1,y_2)\right]\omega_{\sigma,0}(z,v_2)dv_2dz\\
&+\sum_{\sigma=0}^{n-1}\int_{y_{1,n}}^{y_1}\int_{y_{1,\sigma}}^{y_{1,\sigma+1}}\int_{0}^{y_2}\partial_1^{(i)}\partial_2^{(j)}H(y_1,y_2,v_1)\omega_{\sigma,0}(z,v_2)dv_2dzdv_1\\
&+\sum_{r=0}^{j-1}\sum_{l=0}^{r}\sum_{q=0}^{i-1}\sum_{\eta=0}^{q}\binom{r}{l}\binom{q}{\eta}\frac{\partial^{q-\eta}}{\partial y_1^{q-\eta}}\left[\frac{\partial^{i-1-q}}{\partial y_1^{i-1-q}}\bigg|_{v_1=y_1}\left(\frac{\partial^{r-l}}{\partial y_2^{r-l}}\left[\partial_2^{(j-1-r)}H(y_1,y_2,v_1)\right]\right)\right]\times\\
&\qquad\qquad\frac{\partial^{\eta+l}}{\partial y_1^{\eta}\partial y_2^{l}}\left(\int_{y_{1,n}}^{y_1}\hat{\omega}_{n,0}(z,y_2))dz\right)\\
&+\sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\int_{y_{1,n}}^{y_1}\int_{y_{1,n}}^{v_1}\frac{\partial^i}{\partial y_1^i}\left[\frac{\partial^{r-l}}{\partial y_2^{r-l}}\left[\partial_2^{(j-1-r)}H(y_1,y_2,v_1)\right]\right]\frac{\partial^l\hat{\omega}_{n,0}(z,y_2)}{\partial y_2^l}dzdv_1\\
&+\sum_{q=0}^{i-1}\sum_{\eta=0}^{q}\binom{q}{\eta}\int_{0}^{y_2}\frac{\partial^{q-\eta}}{\partial y_1^{q-\eta}}\left[\partial_1^{(i-1-q)}\partial_2^{(j)}H(y_2,y_1,y_2)\right]\frac{\partial^\eta}{\partial y_1^\eta}\left(\int_{y_{1,n}}^{y_1}\hat{\omega}_{n,0}(z,v_2)dz\right)dv_2\\
&+\int_{y_{1,n}}^{y_1}\int_{y_{1,n}}^{v_1}\int_{0}^{y_2}\partial_1^{(i)}\partial_2^{(j)}H(y_1,y_2,v_1)\hat{\omega}_{n,0}(z,v_2)dv_2dzdv_1.
\end{aligned}\tag{3.9}
$$

Hence, for $n=0,1,2,\dots,N-1$,

$$
\begin{aligned}
&\frac{\partial^{i+j}\hat{\omega}_{n,0}(y_{1,n},0)}{\partial y_1^i\partial y_2^j}=\partial_1^{(i)}\partial_2^{(j)}\Phi(y_{1,n},0)\\
&+\alpha_1\partial_1^{(i)}\partial_2^{(j-1)}\hat{\omega}_{n,0}(y_{1,n},0)+\alpha_2\partial_1^{(i-1)}\partial_2^{(j)}\hat{\omega}_{n,0}(y_{1,n},0)+\alpha_3\partial_1^{(i-1)}\partial_2^{(j-1)}\hat{\omega}_{n,0}(y_{1,n},0)\\
&+\sum_{r=0}^{j-1}\sum_{l=0}^{r}\sum_{q=0}^{i-1}\sum_{\eta=0}^{q}\binom{r}{l}\binom{q}{\eta}\frac{\partial^{q-\eta}}{\partial y_1^{q-\eta}}\left[\frac{\partial^{i-1-q}}{\partial y_1^{i-1-q}}\bigg|_{v_1=y_1}\left(\frac{\partial^{r-l}}{\partial y_2^{r-l}}\left[\partial_2^{(j-1-r)}H(y_1,y_2,v_1)\right]\right)\right]_{y_1=y_{1,n},y_2=0}\times\\
&\qquad\qquad\frac{\partial^{\eta+l-1}\hat{\omega}_{n,0}(y_{1,n},0)}{\partial y_1^{\eta-1}\partial y_2^{l}}\\
&+\sum_{\xi=0}^{n-1}\sum_{\sigma=0}^{\xi-1}\sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\int_{y_{1,\xi}}^{y_{1,\xi+1}}\int_{y_{1,\sigma}}^{y_{1,\sigma+1}}\frac{\partial^i}{\partial y_1^i}\left[\frac{\partial^{r-l}}{\partial y_2^{r-l}}\left[\partial_2^{(j-1-r)}H(y_1,y_2,v_1)\right]\right]_{y_1=y_{1,n},y_2=0}\times\\
&\qquad\qquad\frac{\partial^l\omega_{\sigma,0}(z,0)}{\partial y_2^l}dzdv_1\\
&+\sum_{\xi=0}^{n-1}\sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\int_{y_{1,\xi}}^{y_{1,\xi+1}}\int_{y_{1,\xi}}^{v_1}\frac{\partial^i}{\partial y_1^i}\left[\frac{\partial^{r-l}}{\partial y_2^{r-l}}[\partial_2^{(j-1-r)}H(y_1,y_2,v_1)]\right]_{y_1=y_{1,n},y_2=0}\frac{\partial^l\omega_{\xi,0}(z,0)}{\partial y_2^l}dzdv_1\\
&+\sum_{\sigma=0}^{n-1}\sum_{r=0}^{j-1}\sum_{l=0}^{r}\sum_{q=0}^{i-1}\binom{r}{l}\int_{y_{1,\sigma}}^{y_{1,\sigma+1}}\frac{\partial^q}{\partial y_1^q}\left[\frac{\partial^{i-1-q}}{\partial y_1^{i-1-q}}\bigg|_{v_1=y_1}\left(\frac{\partial^{r-l}}{\partial y_2^{r-l}}\left[\partial_2^{(j-1-r)}H(y_1,y_2,v_1)\right]\right)\right]_{y_1=y_{1,n},y_2=0}\times\\
&\qquad\qquad\frac{\partial^l\omega_{\sigma,0}(z,0)}{\partial y_2^l}dz.
\end{aligned}\tag{3.10}
$$

3^{rd} **step:** For $n=0,1,2,\dots,N-1,\ m=1,2,\dots,M-1$, the function $\omega(y_1,y_2)$ is approximated in the rectangles $\Lambda_{n,m}$ as

$$\omega_{n,m}(y_1,y_2)=\sum_{i+j=0}^{p-1}\frac{1}{i!j!}\frac{\partial^{i+j}\hat{\omega}_{n,m}(y_{1,n},y_{2,m})}{\partial y_1^i\partial y_2^j}(y_1-y_{1,n})^i(y_2-y_{2,m})^j, \tag{3.11}$$

where $\hat{\omega}_{n,m}(y_1,y_2)$ refers to the precise solution to the VIE

$$\begin{aligned}
\hat{\omega}_{n,m}(y_1,y_2)=\Phi(y_1,y_2)&+\alpha_1\sum_{\rho=0}^{m-1}\int_{y_{2,\rho}}^{y_{2,\rho+1}}\omega_{n,\rho}(y_1,v_2)dv_2+\alpha_1\int_{y_{2,m}}^{y_2}\hat{\omega}_{n,m}(y_1,v_2)dv_2\\
&+\alpha_2\sum_{\xi=0}^{n-1}\int_{y_{1,\xi}}^{y_{1,\xi+1}}\omega_{\xi,m}(v_1,y_2)dv_1+\alpha_2\int_{y_{1,n}}^{y_1}\hat{\omega}_{n,m}(v_1,y_2)dv_1\\
&+\alpha_3\sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1}\int_{y_{1,\xi}}^{y_{1,\xi+1}}\int_{y_{2,\rho}}^{y_{2,\rho+1}}\omega_{\xi,\rho}(v_1,v_2)dv_2dv_1+\alpha_3\int_{y_{1,n}}^{y_1}\int_{y_{2,m}}^{y_2}\hat{\omega}_{n,m}(v_1,v_2)dv_2dv_1\\
&+\alpha_3\sum_{\rho=0}^{m-1}\int_{y_{1,n}}^{y_1}\int_{y_{2,\rho}}^{y_{2,\rho+1}}\omega_{n,\rho}(v_1,v_2)dv_2dv_1+\alpha_3\sum_{\xi=0}^{n-1}\int_{y_{1,\xi}}^{y_{1,\xi+1}}\int_{y_{2,m}}^{y_2}\omega_{\xi,m}(v_1,v_2)dv_2dv_1\\
&+\sum_{\xi=0}^{n-1}\sum_{\sigma=0}^{\xi-1}\sum_{\rho=0}^{m-1}\int_{y_{1,\xi}}^{y_{1,\xi+1}}\int_{y_{1,\sigma}}^{y_{1,\sigma+1}}\int_{y_{2,\rho}}^{y_{2,\rho+1}}H(y_1,y_2,v_1)\omega_{\sigma,\rho}(z,v_2)dv_2dzdv_1\\
&+\sum_{\xi=0}^{n-1}\sum_{\sigma=0}^{\xi-1}\int_{y_{1,\xi}}^{y_{1,\xi+1}}\int_{y_{1,\sigma}}^{y_{1,\sigma+1}}\int_{y_{2,m}}^{y_2}H(y_1,y_2,v_1)\omega_{\sigma,m}(z,v_2)dv_2dzdv_1\\
&+\sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1}\int_{y_{1,\xi}}^{y_{1,\xi+1}}\int_{y_{1,\xi}}^{v_1}\int_{y_{2,\rho}}^{y_{2,\rho+1}}H(y_1,y_2,v_1)\omega_{\xi,\rho}(z,v_2)dv_2dzdv_1\\
&+\sum_{\xi=0}^{n-1}\int_{y_{1,\xi}}^{y_{1,\xi+1}}\int_{y_{1,\xi}}^{v_1}\int_{y_{2,m}}^{y_2}H(y_1,y_2,v_1)\omega_{\xi,m}(z,v_2)dv_2dzdv_1\\
&+\sum_{\sigma=0}^{n-1}\sum_{\rho=0}^{m-1}\int_{y_{1,n}}^{y_1}\int_{y_{1,\sigma}}^{y_{1,\sigma+1}}\int_{y_{2,\rho}}^{y_{2,\rho+1}}H(y_1,y_2,v_1)\omega_{\sigma,\rho}(z,v_2)dv_2dzdv_1\\
&+\sum_{\sigma=0}^{n-1}\int_{y_{1,n}}^{y_1}\int_{y_{1,\sigma}}^{y_{1,\sigma+1}}\int_{y_{2,m}}^{y_2}H(y_1,y_2,v_1)\omega_{\sigma,m}(z,v_2)dv_2dzdv_1\\
&+\sum_{\rho=0}^{m-1}\int_{y_{1,n}}^{y_1}\int_{y_{1,n}}^{v_1}\int_{y_{2,\rho}}^{y_{2,\rho+1}}H(y_1,y_2,v_1)\omega_{n,\rho}(z,v_2)dv_2dzdv_1\\
&+\int_{y_{1,n}}^{y_1}\int_{y_{1,n}}^{v_1}\int_{y_{2,m}}^{y_2}H(y_1,y_2,v_1)\hat{\omega}_{n,m}(z,v_2)dv_2dzdv_1,
\end{aligned} \tag{3.12}$$

Thus, the differentiation of Eq. (3.12) $i-$ and $j-$times in terms of y_1 and y_2, respectively, gives

$$\begin{aligned}
&\frac{\partial^{i+j}\hat{\omega}_{n,m}(y_1,y_2)}{\partial y_1^i\partial y_2^j}=\partial_1^{(i)}\partial_2^{(j)}\Phi(y_1,y_2)\\
&+\alpha_2\partial_1^{(i)}\partial_2^{(j-1)}\hat{\omega}_{n,m}(y_1,y_2)+\alpha_3\partial_1^{(i-1)}\partial_2^{(j)}\hat{\omega}_{n,m}(y_1,y_2)+\alpha_4\partial_1^{(i-1)}\partial_2^{(j-1)}\hat{\omega}_{n,m}(y_1,y_2)\\
&+\sum_{\xi=0}^{n-1}\sum_{\sigma=0}^{\xi-1}\sum_{\rho=0}^{m-1}\int_{y_{1,\xi}}^{y_{1,\xi+1}}\int_{y_{1,\sigma}}^{y_{1,\sigma+1}}\int_{y_{2,\rho}}^{y_{2,\rho+1}}\partial_1^{(i)}\partial_2^{(j)}H(y_1,y_2,v_1)\omega_{\sigma,\rho}(z,v_2)dv_2dzdv_1
\end{aligned}$$

$$+\sum_{\xi=0}^{n-1}\sum_{\sigma=0}^{\xi-1}\sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\int_{y_{1,\xi}}^{y_{1,\xi+1}}\int_{y_{1,\sigma}}^{y_{1,\sigma+1}}\frac{\partial^i}{\partial y_1^i}\left[\frac{\partial^{r-l}}{\partial y_2^{r-l}}\left[\partial_2^{(j-1-r)}H(y_1,y_2,v_1)\right]\right]\frac{\partial^l\omega_{\sigma,m}(z,y_2)}{\partial y_2^l}dzdv_1$$

$$+\sum_{\xi=0}^{n-1}\sum_{\sigma=0}^{\xi-1}\int_{y_{1,\xi}}^{y_{1,\xi+1}}\int_{y_{1,\sigma}}^{y_{1,\sigma+1}}\int_{y_{2,m}}^{y_2}\partial_1^{(i)}\partial_2^{(j)}H(y_1,y_2,v_1)\omega_{\sigma,m}(z,v_2)dv_2dzdv_1$$

$$+\sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1}\int_{y_{1,\xi}}^{y_{1,\xi+1}}\int_{y_{1,\xi}}^{v_1}\int_{y_{2,\rho}}^{y_{2,\rho+1}}\partial_1^{(i)}\partial_2^{(j)}H(y_1,y_2,v_1)\omega_{\xi,\rho}(z,v_2)dv_2dzdv_1$$

$$+\sum_{\xi=0}^{n-1}\sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\int_{y_{1,\xi}}^{y_{1,\xi+1}}\int_{y_{1,\xi}}^{v_1}\frac{\partial^i}{\partial y_1^i}\left[\frac{\partial^{r-l}}{\partial y_2^{r-l}}\left[\partial_2^{(j-1-r)}H(y_1,y_2,v_1)\right]\right]\frac{\partial^l\omega_{\xi,m}(z,y_2)}{\partial y_2^l}dzdv_1$$

$$+\sum_{\xi=0}^{n-1}\int_{y_{1,\xi}}^{y_{1,\xi+1}}\int_{y_{1,\xi}}^{v_1}\int_{y_{2,m}}^{y_2}\partial_1^{(i)}\partial_2^{(j)}H(y_1,y_2,v_1)\omega_{\xi,m}(z,v_2)dv_2dzdv_1$$

$$+\sum_{\sigma=0}^{n-1}\sum_{\rho=0}^{m-1}\sum_{q=0}^{i-1}\int_{y_{1,\sigma}}^{y_{1,\sigma+1}}\int_{y_{2,\rho}}^{y_{2,\rho+1}}\frac{\partial^q}{\partial y_1^q}\left[\partial_1^{(i-1-q)}\partial_2^{(j)}H(y_2,y_1,y_2)\omega_{\sigma,\rho}(z,v_2)\right]dv_2dz$$

$$+\sum_{\sigma=0}^{n-1}\sum_{\rho=0}^{m-1}\int_{y_{1,n}}^{y_1}\int_{y_{1,\sigma}}^{y_{1,\sigma+1}}\int_{y_{2,\rho}}^{y_{2,\rho+1}}\partial_1^{(i)}\partial_2^{(j)}H(y_1,y_2,v_1)\omega_{\sigma,\rho}(z,v_2)dv_2dzdv_1$$

$$+\sum_{\sigma=0}^{n-1}\sum_{r=0}^{j-1}\sum_{l=0}^{r}\sum_{q=0}^{i-1}\binom{r}{l}\int_{y_{1,\sigma}}^{y_{1,\sigma+1}}\frac{\partial^q}{\partial y_1^q}\left[\left.\frac{\partial^{i-1-q}}{\partial y_1^{i-1-q}}\right|_{v_1=y_1}\left(\frac{\partial^{r-l}}{\partial y_2^{r-l}}\left[\partial_2^{(j-1-r)}H(y_1,y_2,v_1)\right]\right)\right]\times$$

$$\frac{\partial^l\omega_{\sigma,m}(z,y_2)}{\partial y_2^l}dzdv_1$$

$$+\sum_{\sigma=0}^{n-1}\sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\int_{y_{1,n}}^{y_1}\int_{y_{1,\sigma}}^{y_{1,\sigma+1}}\frac{\partial^i}{\partial y_1^i}\left[\frac{\partial^{r-l}}{\partial y_2^{r-l}}[\partial_2^{(j-1-r)}H(y_1,y_2,v_1)\right]\frac{\partial^l\omega_{\sigma,m}(z,y_2)}{\partial y_2^l}dzdv_1$$

$$+\sum_{\sigma=0}^{n-1}\sum_{q=0}^{i-1}\int_{y_{1,\sigma}}^{y_{1,\sigma+1}}\int_{y_{2,m}}^{y_2}\frac{\partial^q}{\partial y_1^q}\left[\partial_1^{(i-1-q)}\partial_2^{(j)}H(y_2,y_1,y_2)\right]\omega_{\sigma,m}(z,v_2)dv_2dz$$

$$+\sum_{\sigma=0}^{n-1}\int_{y_{1,n}}^{y_1}\int_{y_{1,\sigma}}^{y_{1,\sigma+1}}\int_{y_{2,m}}^{y_2}\partial_1^{(i)}\partial_2^{(j)}H(y_1,y_2,v_1)\omega_{\sigma,m}(z,v_2)dv_2dzdv_1$$

$$+\sum_{\rho=0}^{m-1}\sum_{q=0}^{i-1}\sum_{\eta=0}^{q}\binom{q}{\eta}\int_{y_{2,\rho}}^{y_{2,\rho+1}}\frac{\partial^{q-\eta}}{\partial y_1^{q-\eta}}\left[\partial_1^{(i-1-q)}\partial_2^{(j)}H(y_2,y_1,y_2)\right]\frac{\partial^\eta}{\partial y_1^\eta}\left(\int_{y_{1,n}}^{y_1}\omega_{n,\rho}(z,v_2)dz\right)dv_2$$

$$+\sum_{\rho=0}^{m-1}\int_{y_{1,n}}^{y_1}\int_{y_{1,n}}^{v_1}\int_{y_{2,\rho}}^{y_{2,\rho+1}}\partial_1^{(i)}\partial_2^{(j)}H(y_1,y_2,v_1)\omega_{n,\rho}(z,v_2)dv_2dzdv_1$$

$$+\sum_{r=0}^{j-1}\sum_{l=0}^{r}\sum_{q=0}^{i-1}\sum_{\eta=0}^{q}\binom{r}{l}\binom{q}{\eta}\frac{\partial^{q-\eta}}{\partial y_1^{q-\eta}}\left[\left.\frac{\partial^{i-1-q}}{\partial y_1^{i-1-q}}\right|_{v_1=y_1}\left(\frac{\partial^{r-l}}{\partial y_2^{r-l}}\left[\partial_2^{(j-1-r)}H(y_1,y_2,v_1)\right]\right)\right]\times$$

$$\frac{\partial^{\eta+l}}{\partial y_1^\eta\partial y_2^l}\left(\int_{y_{1,n}}^{y_1}\hat{\omega}_{n,m}(z,y_2)dz\right)$$

$$+\sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\int_{y_{1,n}}^{y_1}\int_{y_{1,n}}^{v_1}\frac{\partial^i}{\partial y_1^i}\left[\frac{\partial^{r-l}}{\partial y_2^{r-l}}[\partial_2^{(j-1-r)}H(y_1,y_2,v_1)\right]\frac{\partial^l\hat{\omega}_{n,m}(z,y_2)}{\partial y_2^l}dzdv_1$$

$$+\sum_{q=0}^{i-1}\sum_{\eta=0}^{q}\binom{q}{\eta}\int_{y_{2,m}}^{y_2}\frac{\partial^{q-\eta}}{\partial y_1^{q-\eta}}\left[\left(\partial_1^{(i-1-q)}\partial_2^{(j)}H(y_2,y_1,y_2)\right)\right]\frac{\partial^{\eta}}{\partial y_1^{\eta}}\left(\int_{y_{1,n}}^{y_1}\hat{\omega}_{n,m}(z,v_2)dz\right)dv_2$$
$$+\int_{y_{1,n}}^{y_1}\int_{y_{1,n}}^{v_1}\int_{y_{2,m}}^{y_2}\partial_1^{(i)}\partial_2^{(j)}H(y_1,y_2,v_1)\hat{\omega}_{n,m}(z,v_2)dv_2dzdv_1. \tag{3.13}$$

which leads to, for $n=0,1,2,\ldots,N-1$ and $m=1,2,\ldots,M-1$,

$$\frac{\partial^{i+j}\hat{\omega}_{n,m}(y_{1,n},y_{2,m})}{\partial y_1^i\partial y_2^j}=\partial_1^{(i)}\partial_2^{(j)}\Phi(y_{1,n},y_{2,m})$$
$$+\alpha_1\partial_1^{(i)}\partial_2^{(j-1)}\hat{\omega}_{n,m}(y_{1,n},y_{2,m})+\alpha_2\partial_1^{(i-1)}\partial_2^{(j)}\hat{\omega}_{n,m}(y_{1,n},y_{2,m})+\alpha_3\partial_1^{(i-1)}\partial_2^{(j-1)}\hat{\omega}_{n,m}(y_{1,n},y_{2,m})$$
$$+\sum_{\xi=0}^{n-1}\sum_{\sigma=0}^{\xi-1}\sum_{\rho=0}^{m-1}\int_{y_{1,\xi}}^{y_{1,\xi+1}}\int_{y_{1,\sigma}}^{y_{1,\sigma+1}}\int_{y_{2,\rho}}^{y_{2,\rho+1}}\partial_1^{(i)}\partial_2^{(j)}H(y_1,y_2,v_1)\omega_{\sigma,\rho}(z,v_2)dv_2dzdv_1$$
$$+\sum_{\xi=0}^{n-1}\sum_{\sigma=0}^{\xi-1}\sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\int_{y_{1,\xi}}^{y_{1,\xi+1}}\int_{y_{1,\sigma}}^{y_{1,\sigma+1}}\frac{\partial^i}{\partial y_1^i}\left[\frac{\partial^{r-l}}{\partial y_2^{r-l}}\left[\partial_2^{(j-1-r)}H(y_1,y_2,v_1)\right]\right]_{y_1=y_{1,n},y_2=y_{2,m}}\times$$
$$\frac{\partial^l\omega_{\sigma,m}(z,y_{2,m})}{\partial y_2^l}dzdv_1$$
$$+\sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1}\int_{y_{1,\xi}}^{y_{1,\xi+1}}\int_{y_{1,\xi}}^{v_1}\int_{y_{2,\rho}}^{y_{2,\rho+1}}\partial_1^{(i)}\partial_2^{(j)}H(y_1,y_2,v_1)\omega_{\xi,\rho}(z,v_2)dv_2dzdv_1$$
$$+\sum_{\xi=0}^{n-1}\sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\int_{y_{1,\xi}}^{y_{1,\xi+1}}\int_{y_{1,\xi}}^{v_1}\frac{\partial^i}{\partial y_1^i}\left[\frac{\partial^{r-l}}{\partial y_2^{r-l}}\left[\partial_2^{(j-1-r)}H(y_1,y_2,v_1)\right]\right]_{y_1=y_{1,n},y_2=y_{2,m}}\times$$
$$\frac{\partial^l\omega_{\xi,m}(z,y_{2,m})}{\partial y_2^l}dzdv_1$$
$$+\sum_{\sigma=0}^{n-1}\sum_{\rho=0}^{m-1}\sum_{q=0}^{i-1}\int_{y_{1,\sigma}}^{y_{1,\sigma+1}}\int_{y_{2,\rho}}^{y_{2,\rho+1}}\frac{\partial^q}{\partial y_1^q}\left[\partial_1^{(i-1-q)}\partial_2^{(j)}H(y_2,y_1,y_2)\right]\omega_{\sigma,\rho}(z,v_2)dv_2dz$$
$$+\sum_{\sigma=0}^{n-1}\sum_{r=0}^{j-1}\sum_{l=0}^{r}\sum_{q=0}^{i-1}\binom{r}{l}\int_{y_{1,\sigma}}^{y_{1,\sigma+1}}\frac{\partial^q}{\partial y_1^q}\left[\left.\frac{\partial^{i-1-q}}{\partial y_1^{i-1-q}}\right|_{v_1=y_1}\left(\frac{\partial^{r-l}}{\partial y_2^{r-l}}[\partial_2^{(j-1-r)}H(y_1,y_2,v_1)]\right)\right]_{y_1=y_{1,n},y_2=y_{2,m}}$$
$$\times\frac{\partial^l\omega_{\sigma,m}(z,y_{2,m})}{\partial y_2^l}dz$$
$$+\sum_{\rho=0}^{m-1}\sum_{q=0}^{i-1}\sum_{\eta=0}^{q}\binom{q}{\eta}\int_{y_{2,\rho}}^{y_{2,\rho+1}}\frac{\partial^{q-\eta}}{\partial y_1^{q-\eta}}\left[\partial_1^{(i-1-q)}\partial_2^{(j)}H(y_2,y_1,y_2)\right]\frac{\partial^{\eta-1}\omega_{n,\rho}(y_{1,n},v_2)}{\partial y_1^{\eta-1}}dv_2$$
$$+\sum_{r=0}^{j-1}\sum_{l=0}^{r}\sum_{q=0}^{i-1}\sum_{\eta=0}^{q}\binom{r}{l}\binom{q}{\eta}\frac{\partial^{q-\eta}}{\partial y_1^{q-\eta}}\left[\left.\frac{\partial^{i-1-q}}{\partial y_1^{i-1-q}}\right|_{v_1=y_1}\left(\frac{\partial^{r-l}}{\partial y_2^{r-l}}[\partial_2^{(j-1-r)}H(y_1,y_2,v_1)]\right)\right]_{y_1=y_{1,n},y_2=y_{2,m}}$$
$$\times\frac{\partial^{\eta+l-1}\hat{\omega}_{n,m}(y_{1,n},y_{2,m})}{\partial y_1^{\eta-1}\partial y_2^l}, \tag{3.14}$$

Finally, in virtue of Eqs. (3.6), (3.10) and (3.14), the approximate solution $\omega_{N,M}(y_1,y_2)$ of the 2D-VIE (3.3) can be determined, and therefore the approximate solution $\mu_{N,M}(y_1,y_2)$ of the

2D-PVIDE (3.1) may be given by

$$\mu_{N,M}(y_1,y_2)=\int_0^{y_1}\int_0^{y_2}\omega_{N,M}(v_1,v_2)dv_2dv_1-\mu(0,0)+\mu(0,y_2)+\mu(y_1,0). \tag{3.15}$$

3.2 Convergence analysis

The current section deals with the convergence analysis of the numerical approach described above. In this regard, two main results are stated and proven to obtain the error bounds of the approximate solution (3.15) computed using the Taylor collocation method applied to the 2D-PVIDE (3.1). In the sequel, the following lemma is needed.

Lemma 3.9. *Suppose $\hbar$ and H are two $p-$times continuously differentiable functions defined on their respective domains. Then there exists a positive number $\zeta(p)$, such that*

$$\left\|\frac{\partial^{i+j}\hat{\omega}_{n,m}}{\partial y_1^i\partial y_2^j}\right\|_{L^\infty(\Lambda_{n,m})}\leq\zeta(p),$$

for $n=0,1,\dots,N-1$, $m=0,1,\dots,M-1$, $i+j=0,1,\dots,p$, with $\hat{\omega}_{0,0}(y_1,y_2)=\omega(y_1,y_2)$ and $(y_1,y_2)\in\Lambda_{0,0}$.

Proof. Let $o_{n,m}^{i,j}=\|\frac{\partial^{i+j}\hat{\omega}_{n,m}}{\partial y_1^i\partial y_2^j}\|_{L^\infty(\Lambda_{n,m})}$, we have for all $i+j=0,1,...,p$,

$$o_{0,0}^{i,j}\leq\max\left\{\left\|\frac{\partial^{i+j}\omega}{\partial y_1^i\partial y_2^j}\right\|_{L^\infty(\Lambda_{0,0})},i+j=0,1,...,p\right\}=\zeta_1(p). \tag{3.16}$$

Now, from Eq. (3.9), we have for all $n=1,\dots,N-1$ and $i+j=0,1,...,p$

$$\begin{aligned}o_{n,0}^{i,j}\leq{}&\gamma_1+\alpha_1o_{n,0}^{i,j-1}+\alpha_2o_{n,0}^{i-1,j}+\alpha_3o_{n,0}^{i-1,j-1}+\gamma_1h\sum_{\xi=0}^{n-1}\sum_{q+l=0}^{p-1}o_{\xi,0}^{q,l}+\gamma_1hk\sum_{\xi=0}^{n-1}\sum_{q+l=0}^{p-1}o_{\xi,0}^{q,l}\\&+\gamma_1\sum_{r=0}^{j-1}\sum_{l=0}^{r}\sum_{q=0}^{i-1}\sum_{\eta=0}^{q}o_{n,0}^{\eta,l}+\gamma_1\sum_{r=0}^{j-1}\sum_{l=0}^{r}o_{n,0}^{0,l}+\gamma_1\sum_{q=0}^{i-1}\sum_{\eta=0}^{q}o_{n,0}^{\eta,0}+\gamma_1hko_{n,0}^{0,0},\end{aligned}$$

with γ_1 is positive and unrelated to N and M. Hence

$$\begin{aligned}o_{n,0}^{i,j}\leq{}&\gamma_1+\alpha_1o_{n,0}^{i,j-1}+\alpha_2o_{n,0}^{i-1,j}+\alpha_3o_{n,0}^{i-1,j-1}+\gamma_2h\sum_{\xi=0}^{n-1}\sum_{q+l=0}^{p-1}o_{\xi,0}^{q,l}+\gamma_2\sum_{l=0}^{j-1}\sum_{\eta=0}^{i-1}o_{n,0}^{\eta,l}\\&+\gamma_2\sum_{r=0}^{j-1}o_{n,0}^{0,r}+\gamma_2\sum_{q=0}^{i-1}o_{n,0}^{q,0}+\gamma_1hko_{n,0}^{0,0}.\end{aligned} \tag{3.17}$$

Taking into consideration the sequence $\Gamma_n=\max\{o_{n,0}^{i,j},i+j=0,\dots,p\}$ for all $n=0,1,...,N-1$, then by Eq. (3.17), the sequence Γ_n fulfills

$$o_{n,0}^{i,j} \leq \gamma_1 + \alpha_1 o_{n,0}^{i,j-1} + \alpha_2 o_{n,0}^{i-1,j} + \alpha_3 o_{n,0}^{i-1,j-1} + \gamma_2 (p)^2 h \sum_{\xi=0}^{n-1} \Gamma_\xi + \gamma_2 \sum_{l=0}^{j-1} \sum_{\eta=0}^{i-1} o_{n,0}^{\eta,l}$$
$$+ \gamma_2 \sum_{r=0}^{j-1} o_{n,0}^{0,r} + \gamma_2 \sum_{q=0}^{i-1} o_{n,0}^{q,0} + \gamma_1 hk o_{n,0}^{0,0},$$

which implies for all $i = 1, ..., p$ and $j = 0, ..., p$,

$$o_{n,0}^{i,j} \leq \gamma_1 + \alpha_1 o_{n,0}^{i,j-1} + \alpha_2 o_{n,0}^{i-1,j} + \gamma_3 h \sum_{\xi=0}^{n-1} \Gamma_\xi + \gamma_3 \sum_{l=0}^{j-1} \sum_{\eta=0}^{i-1} o_{n,0}^{\eta,l} + \gamma_2 \sum_{r=0}^{j-1} o_{n,0}^{0,r}$$
$$+ \gamma_2 \sum_{q=0}^{i-1} o_{n,0}^{q,0} + \gamma_1 hk o_{n,0}^{0,0}. \tag{3.18}$$

On the other hand, from Eq. (3.8), we obtain for all $n = 0, ..., N-1$,

$$|\hat{\omega}_{n,0}(y_1, y_2)| \leq \gamma_1 + \gamma_4 h \sum_{\xi=0}^{n-1} \Gamma_\xi + \alpha_1 \int_0^{y_2} |\hat{\omega}_{n,0}(y_1, v_2)| dv_2 + \alpha_2 \int_{y_{1,n}}^{y_1} |\hat{\omega}_{n,0}(v_1, y_2)| dv_1$$
$$+ \gamma_4 \int_{y_{1,n}}^{y_1} \int_0^{y_2} |\hat{\omega}_{n,0}(v_1, v_2)| dv_2 dv_1.$$

Hence, by using Lemma 1.6, we are able to derive for all $n = 0, ..., N-1$ that

$$\begin{aligned} o_{n,0}^{0,0} &\leq \left(\gamma_1 + \gamma_4 h \sum_{\xi=0}^{n-1} \Gamma_\xi \right) e^{\lambda_1 (A_1 + A_2)} \\ &\leq \gamma_1 e^{\lambda_1 (A_1 + A_2)} + \gamma_4 h \sum_{\xi=0}^{n-1} \Gamma_\xi e^{\lambda_1 (A_1 + A_2)} \\ &\leq \gamma_5 + \gamma_5 h \sum_{\xi=0}^{n-1} \Gamma_\xi, \end{aligned} \tag{3.19}$$

with

$$\lambda_1 = \frac{1}{2} \left(\alpha_1 + \alpha_2 + \sqrt{(\alpha_1 + \alpha_2)^2 + 4\gamma_4} \right).$$

From Eqs. (3.18) and (3.19), we deduce that for all $i, j = 0, ..., p$ and $n = 0, ..., N-1$,

$$\begin{aligned} o_{n,0}^{i,j} \leq \gamma_6 + \alpha_1 o_{n,0}^{i,j-1} + \alpha_2 o_{n,0}^{i-1,j} + \gamma_6 h \sum_{\xi=0}^{n-1} \Gamma_\xi + \gamma_6 \sum_{l=0}^{j-1} \sum_{\eta=0}^{i-1} o_{n,0}^{\eta,l} \\ + \gamma_6 \sum_{r=0}^{j-1} o_{n,0}^{0,r} + \gamma_6 \sum_{q=0}^{i-1} o_{n,0}^{q,0}, \end{aligned} \tag{3.20}$$

with γ_6 is positive and unrelated to N and M. It follows from Eq. (3.20), that

$$\begin{aligned} o_{n,0}^{i,j} \leq \gamma_6 + \alpha_1 \sum_{r=0}^{j-1} o_{n,0}^{i,r} + \alpha_2 o_{n,0}^{i-1,j} + \gamma_6 h \sum_{\xi=0}^{n-1} \Gamma_\xi + \gamma_6 \sum_{l=0}^{j-1} \sum_{\eta=0}^{i-1} o_{n,0}^{\eta,l} \\ + \gamma_6 \sum_{r=0}^{j-1} o_{n,0}^{0,r} + \gamma_6 \sum_{q=0}^{i-1} o_{n,0}^{q,0}, \end{aligned} \tag{3.21}$$

using the notations of Lemma 1.2, we put $\Psi_j = o_{n,0}^{i,j}$, $\beta_j = \alpha_1$ and

$$\alpha_j^{'} = \gamma_6 + \alpha_2 o_{n,0}^{i-1,j} + \gamma_6 h \sum_{\xi=0}^{n-1} \Gamma_\xi + \gamma_6 \sum_{l=0}^{j-1} \sum_{\eta=0}^{i-1} o_{n,0}^{\eta,l} + \gamma_6 \sum_{r=0}^{j-1} o_{n,0}^{0,r} + \gamma_6 \sum_{q=0}^{i-1} o_{n,0}^{q,0},$$

therefore, by Lemma 1.2, we get from Eq. (3.21)

$$\begin{aligned} o_{n,0}^{i,j} &\leq \alpha_j^{'} \alpha_1 \sum_{s=0}^{j-1} \alpha_s^{'} \prod_{\sigma=s+1}^{j-1} (1+\alpha_1) \\ &\leq \alpha_j^{'} + \alpha_1 (1+\alpha_1)^p \sum_{s=0}^{j-1} \alpha_s^{'} \\ &\leq \gamma_7 + \alpha_2 o_{n,0}^{i-1,j} + \gamma_7 h \sum_{\xi=0}^{n-1} \Gamma_\xi + \gamma_7 \sum_{l=0}^{j-1} \sum_{\eta=0}^{i-1} o_{n,0}^{\eta,l} + \gamma_7 \sum_{r=0}^{j-1} o_{n,0}^{0,r} + \gamma_7 \sum_{q=0}^{i-1} o_{n,0}^{q,0} \\ &\leq \gamma_7 + \alpha_2 \sum_{q=0}^{i-1} o_{n,0}^{q,j} + \gamma_7 h \sum_{\xi=0}^{n-1} \Gamma_\xi + \gamma_7 \sum_{l=0}^{j-1} \sum_{\eta=0}^{i-1} o_{n,0}^{\eta,l} + \gamma_7 \sum_{r=0}^{j-1} o_{n,0}^{0,r} + \gamma_7 \sum_{q=0}^{i-1} o_{n,0}^{q,0}, \end{aligned} \tag{3.22}$$

with γ_7 is positive and unrelated to N and M.
Once more, by application of Lemma 1.2, we put $\Psi_i = o_{n,0}^{i,j}$, $\beta_i = \alpha_2$ and

$$\alpha_i^{''} = \gamma_7 + \gamma_7 h \sum_{\xi=0}^{n-1} \Gamma_\xi + \gamma_7 \sum_{l=0}^{j-1} \sum_{\eta=0}^{i-1} o_{n,0}^{\eta,l} + \gamma_7 \sum_{r=0}^{j-1} o_{n,0}^{0,r} + \gamma_7 \sum_{q=0}^{i-1} o_{n,0}^{q,0},$$

and as a result, we obtain from Eq. (3.22)

$$\begin{aligned} o_{n,0}^{i,j} &\leq \alpha_i^{''} \alpha_2 \sum_{s=0}^{i-1} \alpha_s^{''} \prod_{\sigma=s+1}^{j-1} (1+\alpha_2) \\ &\leq \alpha_i^{''} + \alpha_2 (1+\alpha_2)^p \sum_{s=0}^{j-1} \alpha_s^{''} \\ &\leq \gamma_8 + \gamma_8 h \sum_{\xi=0}^{n-1} \Gamma_\xi + \gamma_8 \sum_{l=0}^{j-1} \sum_{\eta=0}^{i-1} o_{n,0}^{\eta,l} + \gamma_8 \sum_{r=0}^{j-1} o_{n,0}^{0,r} + \gamma_8 \sum_{q=0}^{i-1} o_{n,0}^{q,0}. \end{aligned} \tag{3.23}$$

Now, for $i = 0$ in Eq. (3.23), we get

$$o_{n,0}^{0,j} \leq \gamma_8 + \gamma_8 h \sum_{\xi=0}^{n-1} \Gamma_\xi + \gamma_8 \sum_{r=0}^{j-1} o_{n,0}^{0,r},$$

Lemma 1.3 implies that

$$\begin{aligned} o_{n,0}^{0,j} &\leq \left(\gamma_8 + \gamma_8 h \sum_{\xi=0}^{n-1} \Gamma_\xi \right) exp \left(\sum_{s=0}^{j-1} \gamma_8 \right) \\ &\leq \gamma_8 exp(p\gamma_8) + \gamma_8 exp(p\gamma_8) h \sum_{\xi=0}^{n-1} \Gamma_\xi \\ &\leq \gamma_9 + \gamma_9 h \sum_{\xi=0}^{n-1} \Gamma_\xi, \end{aligned} \tag{3.24}$$

with γ_9 is positive and unrelated to N and M.
Moreover, for $j = 0$ in Eq. (3.23), we obtain

$$o_{n,0}^{i,0} \leq \gamma_8 + \gamma_8 h \sum_{\xi=0}^{n-1} \Gamma_\xi + \gamma_8 \sum_{q=0}^{i-1} o_{n,0}^{q,0},$$

according to Lemma 1.3, it results

$$\begin{aligned} o_{n,0}^{i,0} &\leq \left(\gamma_8 + \gamma_8 h \sum_{\xi=0}^{n-1} \Gamma_\xi \right) exp \left(\sum_{s=0}^{i-1} \gamma_8 \right) \\ &\leq \gamma_9 + \gamma_9 h \sum_{\xi=0}^{n-1} \Gamma_\xi. \end{aligned} \tag{3.25}$$

Hence, from Eqs. (3.23), (3.24) and (3.25), we derive that

$$o_{n,0}^{i,j} \leq \gamma_{10} + \gamma_{10} h \sum_{\xi=0}^{n-1} \Gamma_\xi + \gamma_{10} \sum_{l=0}^{j-1} \sum_{\eta=0}^{i-1} o_{n,0}^{\eta,l}. \tag{3.26}$$

Using the notations of Lemma 1.4, we put

$$\Psi_{ij} = o_{n,0}^{i,j}, \alpha = \gamma_{10} + \gamma_{10} h \sum_{\xi=0}^{n-1} \Gamma_\xi, \beta_1 = \beta_2 = 0, \beta_3 = p^2 \gamma_{10}, T = S = 1, \; N = M = p,$$

then, by Lemma 1.4, we obtain from Eq. (3.26)

$$o_{n,0}^{i,j} \leq \left(\gamma_{10} + \gamma_{10} h \sum_{\xi=0}^{n-1} \Gamma_\xi \right) e^{2p\sqrt{\gamma_{10}}},$$

which implies that,

$$\Gamma_n \leq \gamma_{11} + \gamma_{11} h \sum_{\xi=0}^{n-1} \Gamma_\xi,$$

with γ_{11} is positive and unrelated to N and M.
It follows, by Lemma 1.3, for all $n = 0, 1, ..., N-1$

$$\Gamma_n \leq \gamma_{11} e^{a\gamma_{11}}. \tag{3.27}$$

Next, we have from Eq. (3.13), for all $n = 0, ..., N-1$, $m = 1, ..., M-1$ and $i + j = 0, ..., p$

$$\begin{aligned} o_{n,m}^{i,j} \leq \gamma_1^{'} &+ \alpha_1 o_{n,m}^{i,j-1} + \alpha_2 o_{n,m}^{i-1,j} + \alpha_3 o_{n,m}^{i-1,j-1} \\ &+ \gamma_1^{'} hk \sum_{\xi=0}^{n-1} \sum_{\rho=0}^{m-1} \sum_{s+t=0}^{p-1} o_{\xi,\rho}^{s,t} + \gamma_1^{'} h \sum_{\xi=0}^{n-1} \sum_{r=0}^{j-1} \sum_{l=0}^{r} \sum_{s+t=0}^{p-1} o_{\xi,m}^{s,t} \\ &+ \gamma_1^{'} hk \sum_{\xi=0}^{n-1} \sum_{s+t=0}^{p-1} o_{\xi,m}^{s,t} + \gamma_1^{'} k \sum_{\rho=0}^{m-1} \sum_{q=0}^{i-1} \sum_{\eta=0}^{q} \sum_{s+t=0}^{p-1} o_{n,\rho}^{s,t} \\ &+ k\gamma_1^{'} h \sum_{\rho=0}^{m-1} \sum_{s+t=0}^{p-1} o_{n,\rho}^{s,t} + \gamma_1^{'} \sum_{r=0}^{j-1} \sum_{l=0}^{r} \sum_{q=0}^{i-1} \sum_{\eta=0}^{q} o_{n,m}^{\eta,l} \\ &+ \gamma_1^{'} \sum_{r=0}^{j-1} \sum_{l=0}^{r} o_{n,m}^{0,l} + \gamma_1^{'} \sum_{q=0}^{i-1} \sum_{\eta=0}^{q} o_{n,m}^{\eta,0} + \gamma_1^{'} hk o_{n,m}^{0,0}. \end{aligned} \tag{3.28}$$

Defining the sequence $\Gamma_{n,m} = \max\{o_{n,m}^{i,j}, i+j = 0, \ldots, p\}$ for all $n = 0, 1, ..., N-1$ and $m = 0, ..., M-1$, then by Eq. (3.28), we have

$$\begin{aligned} o_{n,m}^{i,j} \leq \gamma_1^{'} &+ \alpha_1 o_{n,m}^{i,j-1} + \alpha_2 o_{n,m}^{i-1,j} + \alpha_3 o_{n,m}^{i-1,j-1} \\ &+ \gamma_1^{'} p^2 hk \sum_{\xi=0}^{n-1} \sum_{\rho=0}^{m-1} \Gamma_{\xi,\rho} + \gamma_1^{'} p^4 h \sum_{\xi=0}^{n-1} \Gamma_{\xi,m} \\ &+ \gamma_1^{'} p^2 hk \sum_{\xi=0}^{n-1} \Gamma_{\xi,m} + \gamma_1^{'} p^4 k \sum_{\rho=0}^{m-1} \Gamma_{n,\rho} \\ &+ hk\gamma_1^{'} p^2 \sum_{\rho=0}^{m-1} \Gamma_{n,\rho} + \gamma_1^{'} p^2 \sum_{l=0}^{j-1} \sum_{\eta=0}^{i-1} o_{n,m}^{\eta,l} \\ &+ \gamma_1^{'} p \sum_{l=0}^{j-1} o_{n,m}^{0,l} + \gamma_1^{'} p \sum_{\eta=0}^{i-1} o_{n,m}^{\eta,0} + hk\gamma_1^{'} o_{n,m}^{0,0}, \end{aligned}$$

we obtain

$$\begin{aligned} o_{n,m}^{i,j} \leq& \gamma_1^{'} + \alpha_1 o_{n,m}^{i,j-1} + \alpha_2 o_{n,m}^{i-1,j} + \gamma_2^{'} hk \sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1} \Gamma_{\xi,\rho} + \gamma_2^{'} h \sum_{\xi=0}^{n-1} \Gamma_{\xi,m} \\ &+ \gamma_2^{'} k \sum_{\rho=0}^{m-1} \Gamma_{n,\rho} + \gamma_2^{'} \sum_{l=0}^{j-1}\sum_{\eta=0}^{i-1} o_{n,m}^{\eta,l} + \gamma_2^{'} \sum_{l=0}^{j-1} o_{n,m}^{0,l} + \gamma_2^{'} \sum_{\eta=0}^{i-1} o_{n,m}^{\eta,0} + hk\gamma_2^{'} o_{n,m}^{0,0}. \end{aligned} \tag{3.29}$$

Also, from Eq. (3.12), we obtain for all $n = 0, \ldots, N-1$ and $m = 0, \ldots, M-1$,

$$\begin{aligned} |\hat{\omega}_{n,m}(y_1, y_2)| \leq& \gamma_3^{'} + \gamma_3^{'} hk \sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1} \Gamma_{\xi,\rho} + \gamma_3^{'} h \sum_{\xi=0}^{n-1} \Gamma_{\xi,m} + \gamma_3^{'} k \sum_{\rho=0}^{m-1} \Gamma_{n,\rho} + \alpha_1 \int_{y_{2,m}}^{y_2} |\hat{\omega}_{n,m}(y_1, v_2)| dv_2 \\ &+ \alpha_2 \int_{y_{1,n}}^{y_1} |\hat{\omega}_{n,m}(v_1, y_2)| dv_1 + \gamma_3^{'} \int_{y_{1,n}}^{y_1}\int_{y_{2,m}}^{y_2} |\hat{\omega}_{n,m}(v_1, v_2)| dv_2 dv_1. \end{aligned}$$

Therefore, according to Lemma 1.6, we get for all $n = 0, ..., N-1$ and $m = 0, ..., M-1$,

$$\begin{aligned} o_{n,m}^{0,0} \leq& \left(\gamma_3^{'} + \gamma_3^{'} hk \sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1} \Gamma_{\xi,\rho} + \gamma_3^{'} h \sum_{\xi=0}^{n-1} \Gamma_{\xi,m} + \gamma_3^{'} k \sum_{\rho=0}^{m-1} \Gamma_{n,\rho} \right) e^{\lambda_2 (A_1 + A_2)} \\ \leq& \gamma_4^{'} + \gamma_4^{'} hk \sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1} \Gamma_{\xi,\rho} + \gamma_4^{'} h \sum_{\xi=0}^{n-1} \Gamma_{\xi,m} + \gamma_4^{'} k \sum_{\rho=0}^{m-1} \Gamma_{n,\rho}, \end{aligned} \tag{3.30}$$

with

$$\lambda_2 = \frac{1}{2}\left(\alpha_1 + \alpha_2 + \sqrt{(\alpha_1 + \alpha_2)^2 + 4\gamma_3^{'}} \right),$$

which gives from Eqs. (3.29), and (3.30) that, for all $i = 0, ..., p$ and $j = 0, ..., p$,

$$\begin{aligned} o_{n,m}^{i,j} \leq& \gamma_5^{'} + \alpha_1 o_{n,m}^{i,j-1} + \alpha_2 o_{n,m}^{i-1,j} + \gamma_5^{'} hk \sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1} \Gamma_{\xi,\rho} + \gamma_5^{'} h \sum_{\xi=0}^{n-1} \Gamma_{\xi,m} \\ &+ \gamma_5^{'} k \sum_{\rho=0}^{m-1} \Gamma_{n,\rho} + \gamma_5^{'} \sum_{l=0}^{j-1}\sum_{\eta=0}^{i-1} o_{n,m}^{\eta,l} + \gamma_5^{'} \sum_{l=0}^{j-1} o_{n,m}^{0,l} + \gamma_5^{'} \sum_{\eta=0}^{i-1} o_{n,m}^{\eta,0}. \end{aligned} \tag{3.31}$$

with $\gamma_5^{'}$ is positive and unrelated to N and M. Eq. (3.31) shows that

$$\begin{aligned} o_{n,m}^{i,j} \leq& \gamma_5^{'} + \alpha_1 \sum_{r=0}^{j-1} o_{n,m}^{i,r} + \alpha_2 o_{n,m}^{i-1,j} + \gamma_5^{'} hk \sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1} \Gamma_{\xi,\rho} + \gamma_5^{'} h \sum_{\xi=0}^{n-1} \Gamma_{\xi,m} \\ &+ \gamma_5^{'} k \sum_{\rho=0}^{m-1} \Gamma_{n,\rho} + \gamma_5^{'} \sum_{l=0}^{j-1}\sum_{\eta=0}^{i-1} o_{n,m}^{\eta,l} + \gamma_5^{'} \sum_{l=0}^{j-1} o_{n,m}^{0,l} + \gamma_5^{'} \sum_{\eta=0}^{i-1} o_{n,m}^{\eta,0}. \end{aligned} \tag{3.32}$$

Using the notations of Lemma 1.2, we put $\Psi_j = o_{n,m}^{i,j}$, $\beta_j = \alpha_1$ and

$$\alpha_j^{'} = \gamma_5^{'}+\alpha_2 o_{n,m}^{i-1,j} + \gamma_5^{'} hk \sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1}\Gamma_{\xi,\rho} + \gamma_5^{'} h\sum_{\xi=0}^{n-1}\Gamma_{\xi,m} + \gamma_5^{'} k\sum_{\rho=0}^{m-1}\Gamma_{n,\rho} + \gamma_5^{'}\sum_{l=0}^{j-1}\sum_{\eta=0}^{i-1} o_{n,m}^{\eta,l} + \gamma_5^{'}\sum_{l=0}^{j-1} o_{n,m}^{0,l} + \gamma_5^{'}\sum_{\eta=0}^{i-1} o_{n,m}^{\eta,0},$$

thus, we obtain from Eq. (3.32)

$$\begin{aligned} o_{n,m}^{i,j} &\leq \alpha_j^{'}\alpha_1\sum_{s=0}^{j-1}\alpha_s^{'}\prod_{\sigma=s+1}^{j-1}(1+\alpha_1) \\ &\leq \alpha_j^{'} + \alpha_1(1+\alpha_1)^p\sum_{s=0}^{j-1}\alpha_s^{'} \\ &\leq \gamma_6^{'} + \alpha_2 o_{n,m}^{i-1,j} + \gamma_6^{'} hk\sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1}\Gamma_{\xi,\rho} + \gamma_6^{'} h\sum_{\xi=0}^{n-1}\Gamma_{\xi,m} + \gamma_6^{'} k\sum_{\rho=0}^{m-1}\Gamma_{n,\rho} \\ &+ \gamma_6^{'}\sum_{l=0}^{j-1}\sum_{\eta=0}^{i-1} o_{n,m}^{\eta,l} + \gamma_6^{'}\sum_{l=0}^{j-1} o_{n,m}^{0,l} + \gamma_6^{'}\sum_{\eta=0}^{i-1} o_{n,m}^{\eta,0} \\ &\leq \gamma_6^{'} + \alpha_2\sum_{q=0}^{i-1} o_{n,m}^{q,j} + \gamma_6^{'} hk\sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1}\Gamma_{\xi,\rho} + \gamma_6^{'} h\sum_{\xi=0}^{n-1}\Gamma_{\xi,m} + \gamma_6^{'} k\sum_{\rho=0}^{m-1}\Gamma_{n,\rho} \\ &+ \gamma_6^{'}\sum_{l=0}^{j-1}\sum_{\eta=0}^{i-1} o_{n,m}^{\eta,l} + \gamma_6^{'}\sum_{l=0}^{j-1} o_{n,m}^{0,l} + \gamma_6^{'}\sum_{\eta=0}^{i-1} o_{n,m}^{\eta,0} \end{aligned} \tag{3.33}$$

with $\gamma_6^{'}$ is positive and unrelated to N and M.
Again, by using the notations of Lemma 1.2, we put $\Psi_i = o_{n,m}^{i,j}$, $\beta_i = \alpha_2$ and

$$\alpha_i^{''} = \gamma_6^{'} + \gamma_6^{'} hk\sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1}\Gamma_{\xi,\rho} + \gamma_6^{'} h\sum_{\xi=0}^{n-1}\Gamma_{\xi,m} + \gamma_6^{'} k\sum_{\rho=0}^{m-1}\Gamma_{n,\rho} + \gamma_6^{'}\sum_{l=0}^{j-1}\sum_{\eta=0}^{i-1} o_{n,m}^{\eta,l} + \gamma_6^{'}\sum_{l=0}^{j-1} o_{n,m}^{0,l} + \gamma_6^{'}\sum_{\eta=0}^{i-1} o_{n,m}^{\eta,0}.$$

Thus, we obtain from Eq. (3.33)

$$\begin{aligned} o_{n,m}^{i,j} &\leq \alpha_i^{''}\alpha_2\sum_{s=0}^{i-1}\alpha_s^{''}\prod_{\sigma=s+1}^{j-1}(1+\alpha_2) \\ &\leq \alpha_i^{''} + \alpha_2(1+\alpha_2)^p\sum_{s=0}^{j-1}\alpha_s^{''} \\ &\leq \gamma_7^{'} + \gamma_7^{'} hk\sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1}\Gamma_{\xi,\rho} + \gamma_7^{'} h\sum_{\xi=0}^{n-1}\Gamma_{\xi,m} + \gamma_7^{'} k\sum_{\rho=0}^{m-1}\Gamma_{n,\rho} + \gamma_7^{'}\sum_{l=0}^{j-1}\sum_{\eta=0}^{i-1} o_{n,m}^{\eta,l} + \gamma_7^{'}\sum_{l=0}^{j-1} o_{n,m}^{0,l} + \gamma_7^{'}\sum_{\eta=0}^{i-1} o_{n,m}^{\eta,0} \end{aligned} \tag{3.34}$$

For $i = 0$ in Eq. (3.34), we obtain

$$o_{n,m}^{0,j} \leq \gamma_7' + \gamma_7' hk \sum_{\xi=0}^{n-1} \sum_{\rho=0}^{m-1} \Gamma_{\xi,\rho} + \gamma_7' h \sum_{\xi=0}^{n-1} \Gamma_{\xi,m} + \gamma_7' k \sum_{\rho=0}^{m-1} \Gamma_{n,\rho} + \gamma_7' \sum_{l=0}^{j-1} o_{n,m}^{0,l},$$

which implies by using lemma 1.3, that

$$\begin{aligned} o_{n,m}^{0,j} &\leq \left(\gamma_7' + \gamma_7' hk \sum_{\xi=0}^{n-1} \sum_{\rho=0}^{m-1} \Gamma_{\xi,\rho} + \gamma_7' h \sum_{\xi=0}^{n-1} \Gamma_{\xi,m} + \gamma_7' k \sum_{\rho=0}^{m-1} \Gamma_{n,\rho} \right) exp \left(\sum_{s=0}^{j-1} \gamma_7' \right) \\ &\leq \gamma_7' exp(p\gamma_7') + \gamma_7' exp(p\gamma_7') hk \sum_{\xi=0}^{n-1} \sum_{\rho=0}^{m-1} \Gamma_{\xi,\rho} + \gamma_7' exp(p\gamma_7') h \sum_{\xi=0}^{n-1} \Gamma_{\xi,m} + \gamma_7' exp(p\gamma_7') k \sum_{\rho=0}^{m-1} \Gamma_{n,\rho} \\ &\leq \gamma_8' + \gamma_8' hk \sum_{\xi=0}^{n-1} \sum_{\rho=0}^{m-1} \Gamma_{\xi,\rho} + \gamma_8' h \sum_{\xi=0}^{n-1} \Gamma_{\xi,m} + \gamma_8' k \sum_{\rho=0}^{m-1} \Gamma_{n,\rho}, \end{aligned} \tag{3.35}$$

with γ_8' is positive and unrelated to N and M. On the other hand, for $j = 0$ in Eq. (3.34), we get

$$o_{n,m}^{i,0} \leq \gamma_7' + \gamma_7' hk \sum_{\xi=0}^{n-1} \sum_{\rho=0}^{m-1} \Gamma_{\xi,\rho} + \gamma_7' h \sum_{\xi=0}^{n-1} \Gamma_{\xi,m} + \gamma_7' k \sum_{\rho=0}^{m-1} \Gamma_{n,\rho} + \gamma_7' \sum_{\eta=0}^{i-1} o_{n,m}^{\eta,0},$$

which implies, by using Lemma 1.3, that

$$o_{n,m}^{i,0} \leq \gamma_8' + \gamma_8' hk \sum_{\xi=0}^{n-1} \sum_{\rho=0}^{m-1} \Gamma_{\xi,\rho} + \gamma_8' h \sum_{\xi=0}^{n-1} \Gamma_{\xi,m} + \gamma_8' k \sum_{\rho=0}^{m-1} \Gamma_{n,\rho}. \tag{3.36}$$

As a result, from Eqs. (3.34), (3.35) and (3.36), we deduce that

$$o_{n,m}^{i,j} \leq \gamma_8' + \gamma_8' hk \sum_{\xi=0}^{n-1} \sum_{\rho=0}^{m-1} \Gamma_{\xi,\rho} + \gamma_8' h \sum_{\xi=0}^{n-1} \Gamma_{\xi,m} + \gamma_8' k \sum_{\rho=0}^{m-1} \Gamma_{n,\rho} + \gamma_8' \sum_{l=0}^{j-1} \sum_{\eta=0}^{i-1} o_{n,m}^{\eta,l}. \tag{3.37}$$

Using the notations of Lemma 1.4, we put

$$\Psi_{ij} = o_{n,m}^{i,j},\ \alpha = \gamma_8' + \gamma_8' hk \sum_{\xi=0}^{n-1} \sum_{\rho=0}^{m-1} \Gamma_{\xi,\rho} + \gamma_8' h \sum_{\xi=0}^{n-1} \Gamma_{\xi,m} + \gamma_8' k \sum_{\rho=0}^{m-1} \Gamma_{n,\rho},$$
$$\beta_1 = \beta_2 = 0, \beta_3 = p^2 \gamma_8', T = S = 1.$$

Then, by Lemma 1.4, we obtain

$$o_{n,m}^{i,j} \leq \left(\gamma_8' + \gamma_8' hk \sum_{\xi=0}^{n-1} \sum_{\rho=0}^{m-1} \Gamma_{\xi,\rho} + \gamma_8' h \sum_{\xi=0}^{n-1} \Gamma_{\xi,m} + \gamma_8' k \sum_{\rho=0}^{m-1} \Gamma_{n,\rho} \right) e^{2p\sqrt{\gamma_8'}}.$$

Consequently, for all $n = 0, 1, ..., N-1; m = 0, ..., M-1$,

$$\Gamma_{n,m} \leq \gamma_9' + hk\gamma_9' \sum_{\xi=0}^{n-1} \sum_{\rho=0}^{m-1} \Gamma_{\xi,\rho} + h\gamma_9' \sum_{\xi=0}^{n-1} \Gamma_{\xi,m} + k\gamma_9' \sum_{\rho=0}^{m-1} \Gamma_{n,\rho}, \tag{3.38}$$

Lemma (1.4) gives

$$\Gamma_{n,m} \leq \gamma_9' e^{(A_1+A_2)p(\gamma_9'+\sqrt{\gamma_9'+\gamma_{10}'^2})}. \tag{3.39}$$

Therefore from Eqs. (3.16), (3.27) and (3.39) the proof of Lemma 3.9 is completed by setting

$$\zeta(p) = max\{\zeta_1(p), \gamma_{10}e^{a\gamma_{10}}, \gamma_9' e^{(A_1+A_2)p(\gamma_9'+\sqrt{\gamma_9'+\gamma_{10}'^2})}\}.$$

□

Theorem 3.3. *Assume that $\omega(y_1, y_2)$ is the exact solution of the two-dimensional Volterra integral equation (3.3) and $\omega_{N,M}(y_1, y_2)$ is the approximate solution of the same problem computed using the Taylor collocation method. Then, there is a finite constant C independent of h and k, such that*

$$\|\omega - \omega_{N,M}\|_{L^\infty(\Lambda)} \leq C(h+k)^p,$$

where $\hbar$ and H are two $p-$times continuously differentiable on their respective domains.

Proof. For $(y_1, y_2) \in \Lambda_{0,0}$, making use of Lemma 1.7 and Eq. (3.5), we have

$$|\omega(y_1, y_2) - \omega_{0,0}(y_1, y_2)| \leq \sum_{i+j=p} \frac{1}{i!j!} \left\| \frac{\partial^{i+j}\omega}{\partial y_1^i \partial y_2^j} \right\| h^i k^j.$$

Therefore, according to Lemma 3.9, we obtain

$$\|\omega - \omega_{0,0}\|_{L^\infty(\Lambda_{0,0})} \leq \zeta(p) \sum_{i+j=p} \frac{1}{i!j!} h^i k^j = \frac{\zeta(p)}{p!}(h+k)^p. \tag{3.40}$$

Also, for $(y_1, y_2) \in \Lambda_{n,0},\ n = 1, 2, \ldots, N-1$, it follows from Eqs. (3.3) and (3.8), that

$$\begin{aligned}
\omega(y_1, y_2) - \hat{\omega}_{n,0}(y_1, y_2) &= \alpha_1 \int_0^{y_2} (\omega - \hat{\omega}_{n,0})(y_1, v_2)dv_2 + \alpha_2 \sum_{\xi=0}^{n-1} \int_{y_{1,\xi}}^{y_{1,\xi+1}} (\omega(v_1, y_2) - \omega_{\xi,0}(v_1, y_2))dv_1 \\
&+ \alpha_2 \int_{y_{1,n}}^{y_1} (\omega - \hat{\omega}_{n,0})(v_1, y_2)dv_1 + \alpha_3 \int_{y_{1,n}}^{y_1} \int_0^{y_2} (\omega - \hat{\omega}_{n,0})(v_1, y_2)dv_1 \\
&+ \alpha_3 \sum_{\xi=0}^{n-1} \int_{y_{1,\xi}}^{y_{1,\xi+1}} \int_0^{y_2} (\omega(v_1, v_2) - \omega_{\xi,0}(v_1, v_2))dv_2 dv_1 \\
&+ \sum_{\xi=0}^{n-1} \sum_{\sigma=0}^{\xi-1} \int_{y_{1,\xi}}^{y_{1,\xi+1}} \int_{y_{1,\sigma}}^{y_{1,\sigma+1}} \int_0^{y_2} H(y_1, y_2, v_1, v_2)(\omega(z, v_2) - \omega_{\sigma,0}(z, v_2))dv_2 dz dv_1
\end{aligned}$$

$$+\sum_{\xi=0}^{n-1}\int_{y_{1,\xi}}^{y_{1,\xi+1}}\int_{y_{1,\xi}}^{v_1}\int_0^{y_2} H(y_1,y_2,v_1,v_2)(\omega(z,v_2)-\omega_{\xi,0}(z,v_2))dv_2dzdv_1$$
$$+\sum_{\sigma=0}^{n-1}\int_{y_{1,n}}^{y_1}\int_{y_{1,\sigma}}^{y_{1,\sigma+1}}\int_0^{y_2} H(y_1,y_2,v_1,v_2)(\omega(z,v_2)-\omega_{\sigma,0}(z,v_2))dv_2dzdv_1$$
$$+\int_{y_{1,n}}^{y_1}\int_{y_{1,n}}^{v_1}\int_0^{y_2} H(y_1,y_2,v_1,v_2)\left(\omega(z,v_2)-\hat{\omega}_{n,0}(z,v_2)\right)dv_2dzdv_1.$$

which leads to

$$|\omega(y_1,y_2)-\hat{\omega}_{n,0}(y_1,y_2)|\le \beta_1 h\overline{H}\sum_{\xi=0}^{n-1}\|\omega-\omega_{\xi,0}\|+\alpha_1\int_0^{y_2}|\omega(y_1,v_2)-\hat{\omega}_{n,0}(y_1,v_2)|dv_2$$
$$+\alpha_2\int_{y_{1,n}}^{y_1}|\omega(v_1,y_2)-\hat{\omega}_{n,0}(v_1,y_2)|dv_1+\beta_1\overline{H}\int_{y_{1,n}}^{y_1}\int_0^{y_2}|\omega(v_1,v_2)-\hat{\omega}_{n,0}(v_1,v_2)|dv_2dv_1,$$

where $\overline{H}=\max\{\|H\|_{L^\infty(\Lambda)}\}$, and β_1 is a positive constant independent of h and k.
Using Lemma 1.6, we get that there exists a positive constant β_2 independent of h and k, such that

$$\|\omega-\hat{\omega}_{n,0}\|\le \beta_2 h\overline{H}\sum_{\xi=0}^{n-1}\|\omega-\omega_{\xi,0}\|.$$

Now, we use Lemmas 1.7 and 3.9 to deduce that

$$\begin{aligned}\|\omega-\omega_{n,0}\|_{L^\infty(\Lambda_{n,0})}&\le\|\omega-\hat{\omega}_{n,0}\|_{L^\infty(\Lambda_{n,0})}+\|\hat{\omega}_{n,0}-\omega_{n,0}\|_{L^\infty(\Lambda_{n,0})}\\&\le\beta_2 h\sum_{\xi=0}^{n-1}\|\omega-\omega_{\xi,0}\|_{L^\infty(\Lambda_{n,0})}+\sum_{i+j=p}\frac{1}{i!j!}\left\|\frac{\partial^{i+j}\hat{\omega}_{n,0}}{\partial y_1^i\partial y_2^j}\right\|h^ik^j\\&\le\beta_2 h\sum_{\xi=0}^{n-1}\|\omega-\omega_{\xi,0}\|_{L^\infty(\Lambda_{n,0})}+\frac{\zeta(p)}{p!}(h+k)^p.\end{aligned}$$

Thus, based on Lemma 1.3, one can conclude that

$$\|\omega-\omega_{n,0}\|_{L^\infty(\Lambda_{n,0})}\le\frac{\zeta(p)}{p!}(h+k)^p\exp(A_1\beta_2). \tag{3.41}$$

For $(y_1,y_2)\in\Lambda_{n,m},\ n=0,1,\dots,N-1,\ m=1,2,\dots,M-1$, one can use Eqs. (3.3) and (3.12) to get

$$|\omega(y_1,y_2)-\hat{\omega}_{n,m}(y_1,y_2)|\le\overline{\beta_1}k\sum_{\rho=0}^{m-1}\|\omega-\omega_{n,\rho}\|_{L^\infty(\Lambda_{n,\rho})}+\overline{\beta_1}h\sum_{\xi=0}^{n-1}\|\omega-\omega_{\xi,m}\|_{L^\infty(\Lambda_{\xi,m})}$$
$$+\overline{\beta_1}hk\sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1}\|\omega-\omega_{\xi,\rho}\|_{L^\infty(\Lambda_{\xi,\rho})}+\alpha_1\int_{y_{2,m}}^{y_2}|\omega(y_1,v_2)-\hat{\omega}_{n,m}(y_1,v_2)|dv_2$$
$$+\alpha_2\int_{y_{1,n}}^{y_1}|\omega(v_1,y_2)-\hat{\omega}_{n,m}(v_1,y_2)|dv_1+\overline{\beta_1}\int_{y_{1,n}}^{y_1}\int_{y_{2,m}}^{y_2}|\omega(v_1,v_2)-\hat{\omega}_{n,m}(v_1,v_2)|dv_2dv_1,$$

such that $\overline{\beta_1}$ is independent of h and k. Which in turn, with the help of Lemma 1.5, leads to

$$\begin{aligned}|\omega(y_1,y_2)-\hat{\omega}_{n,m}(y_1,y_2)| \leq & \overline{\beta_2}k\sum_{\rho=0}^{m-1}\|\omega-\omega_{n,\rho}\|_{L^\infty(\Lambda_{n,\rho})}+\overline{\beta_2}h\sum_{\xi=0}^{n-1}\|\omega-\omega_{\xi,m}\|_{L^\infty(\Lambda_{\xi,m})}\\ &+\overline{\beta_2}hk\sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1}\|\omega-\omega_{\xi,\rho}\|_{L^\infty(\Lambda_{\xi,\rho})}.\end{aligned}$$

Also, using Lemmas 1.7 and 3.9, we get

$$\begin{aligned}\|\omega-\omega_{n,m}\|_{L^\infty(\Lambda_{n,m})} &\leq \|\omega-\hat{\omega}_{n,m}\|_{L^\infty(\Lambda_{n,m})}+\|\hat{\omega}_{n,m}-\omega_{n,m}\|_{L^\infty(\Lambda_{n,m})}\\ &\leq \overline{\beta_2}k\sum_{\rho=0}^{m-1}\|\omega-\omega_{n,\rho}\|_{L^\infty(\Lambda_{n,\rho})}+\overline{\beta_2}h\sum_{\xi=0}^{n-1}\|\omega-\omega_{\xi,m}\|_{L^\infty(\Lambda_{\xi,m})}\\ &+\overline{\beta_2}hk\sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1}\|\omega-\omega_{\xi,\rho}\|_{L^\infty(\Lambda_{\xi,\rho})}+\sum_{i+j=p}\frac{1}{i!j!}\left\|\frac{\partial^{i+j}\hat{v}_{n,m}}{\partial y_1^i\partial y_2^j}\right\|h^ik^j\\ &\leq \overline{\beta_2}k\sum_{\rho=0}^{m-1}\|\omega-\omega_{n,\rho}\|_{L^\infty(\Lambda_{n,\rho})}+\overline{\beta_2}h\sum_{\xi=0}^{n-1}\|\omega-\omega_{\xi,m}\|_{L^\infty(\Lambda_{\xi,m})}\\ &+\overline{\beta_2}hk\sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1}\|\omega-\omega_{\xi,\rho}\|_{L^\infty(\Lambda_{\xi,\rho})}+\frac{\zeta(p)}{p!}(h+k)^p,\end{aligned}$$

Now, we utilize Lemma 1.4 to get

$$\|\omega-\omega_{n,m}\|_{L^\infty(\Lambda_{n,m})}\leq \beta_3\frac{\zeta(p)}{p!}(h+k)^p. \tag{3.42}$$

Finally, with the aid of Eqs. (3.40), (3.41) and (3.42), we have

$$\|\omega-\omega_{N,M}\|_{L^\infty(\Lambda)}\leq C(h+k)^p,$$

□

We now study the main theorem of convergence analysis of the suggested numerical method.

Theorem 3.4. *Assume that $\mu(y_1,y_2)$ is the exact solution of the 2D-PVIDE (3.1) and $\mu_{N,M}(y_1,y_2)$, $n=0,1,\dots,N-1,\ m=0,1,\dots,M-1$, is the approximate solution of the same problem computed using the Taylor collocation method (3.15). Then, there is a finite constant L independent of h and k, such that*

$$\|\mu-\mu_{N,M}\|_{L^\infty(\Lambda)}\leq L(h+k)^p,$$

where $\hbar$ and H are two $p-$times continuously differentiable on their respective domains.

Proof. Referring to Eqs. (3.2) and (3.15), one can deduce that for $n=0,1,\dots,N-1$, $m=0,1,\dots,M-1$,

$$\mu(y_1,y_2)-\mu_{N,M}(y_1,y_2)=\int_0^{y_1}\int_0^{y_2}\left(\omega(v_1,v_2)-\omega_{N,M}(v_1,v_2)\right)dv_2dv_1.$$

Therefore, we obtain

$$|\mu(y_1, y_2) - \mu_{N,M}(y_1, y_2)| \leq \int_0^{y_1} \int_0^{y_2} |\omega(v_1, v_2) - \omega_{N,M}(v_1, v_2)|\, dv_2 dv_1.$$

Theorem 3.3 implies that

$$\|\mu - \mu_{N,M}\|_{L^\infty(\Lambda)} \leq L(h+k)^p.$$

□

3.3 Numerical results

In the current section, we provide some numerical examples to test the accuracy of the proposed numerical method and validate the convergence analysis demonstrated in the previous section. We define the absolute error, the maximum absolute error and the convergence rates for temporal and spatial sizes, respectively, by

$$e_{N,M}(y_1, y_2) = |\mu(y_1, y_2) - \mu_{N,M}(y_1, y_2)|,$$

$$E(N, M) = \|\mu - \mu_{N,M}\|_\infty,$$

and

$$\text{Rate} = \log_2\left(\frac{E(N/2, M/2)}{E(N, M)}\right).$$

Example 3.10. *Taking into consideration the following second-order 2D-PVIDE as a starting example*

$$\frac{\partial^2\mu(y_1, y_2)}{\partial y_1 \partial y_2} = \frac{\partial\mu(y_1, y_2)}{\partial y_1} + \frac{\partial\mu(y_1, y_2)}{\partial y_2} + \mu(y_1, y_2) + \hbar(y_1, y_2) + \int_0^{y_1} y_2 y_1^2 \mu(v_1, y_2) dv_1, \;\; 0 \leq y_1, y_2 \leq 1,$$

with the initial conditions

$$\mu(y_1, 0) = \frac{\partial\mu(y_1, 0)}{\partial y_1} = \mu(0, y_2) = 0,$$

where $\hbar(y_1, y_2) = -(1+y_1)(\cos y_2 + \sin y_2) - \frac{1}{4} y_2 sin(y_2) y_1^4$*, and the exact solution is* $\mu(y_1, y_2) = y_1 \sin y_2$.

In Table 3.1, we display the absolute errors $e_{N,M}(y_1, y_2)$ *at* $p = 3$ *with* $N = M = \{4, 8, 16, 32\}$*, while Table 3.2 presents the maximum absolute errors* $E(N, N)$ *with* $p = 3$ *at different choices of* N *and obtains the convergence order of the solution. These tables collectively demonstrate the reliability of our results across various nodes.*
Furthermore, Figure 3.1 showcases the absolute error functions $|\mu(y_1, y_2) - \mu_{N,M}(y_1, y_2)|$ *plotted in three dimensions using* $(N, M) = \{(4, 4), (16, 16)\}$ *and* $p = 3$ *as parameters. Figures 3.2 and 3.3 provides a visual presentations of the contrast between the precise and estimated solutions at* $y_1 = 0.1$*, and a graphical depiction of the absolute error function at* $y_1 = 1$*, respectfully for* $(N, M) = (16, 16)$*. Finally, Figure 3.4 exhibits the logarithmic graph of* $log_{10}(L^\infty - errors)$ *with* $N = 5$ *for diverse values of* M*.*

Table 3.1: *Absolute errors* $|\mu(y_1, y_2) - \mu_{N,N}(y_1, y_2)|$ *for Example 3.10*

(y_1, y_2)	$N = 4$	$N = 8$	$N = 16$	$N = 32$
$(0.000, 0.000)$	0	0	0	0
$(0.125, 0.125)$	1.17×10^{-6}	3.18×10^{-8}	7.03×10^{-9}	1.07×10^{-9}
$(0.250, 0.250)$	2.03×10^{-6}	4.55×10^{-7}	7.17×10^{-8}	9.93×10^{-9}
$(0.500, 0.500)$	3.02×10^{-5}	5.02×10^{-6}	7.22×10^{-7}	4.65×10^{-8}
$(0.750, 0.750)$	1.37×10^{-4}	2.21×10^{-5}	3.13×10^{-6}	4.13×10^{-7}
$(1.000, 1.000)$	4.57×10^{-4}	7.41×10^{-5}	1.05×10^{-5}	1.38×10^{-6}

Table 3.2: *Maximum absolute error* $E(N, M)$ *with* $N = M$ *and* $p = 3$ *for Example 3.10*

N	$E(N, N)$	Rate
2	0.0021974180	
4	0.0004570776	2.2653
8	0.0000741240	2.6244
16	0.0000105107	2.8181
32	0.0000013867	2.9220

Figure 3.1: *(a) Absolute error function* $|\mu(y_1, y_2) - \mu_{4,4}(y_1, y_2)|$, *(b) Absolute error function* $|\mu(y_1, y_2) - \mu_{16,16}(y_1, y_2)|$ *for* $p = 3$ *for Example 3.10*

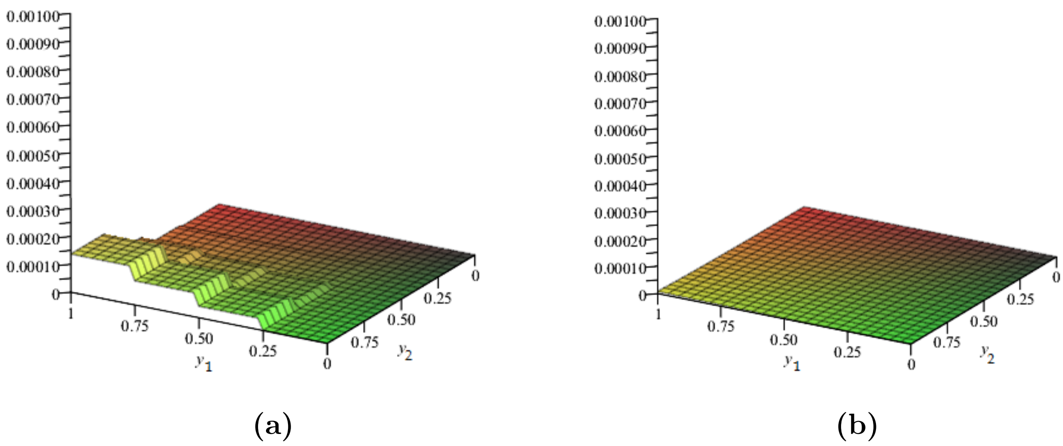


Figure 3.2: *Comparison of the exact and approximate solutions with* $N = M = 16$ *at* $y_1 = 0.1$ *for Example* 3.10

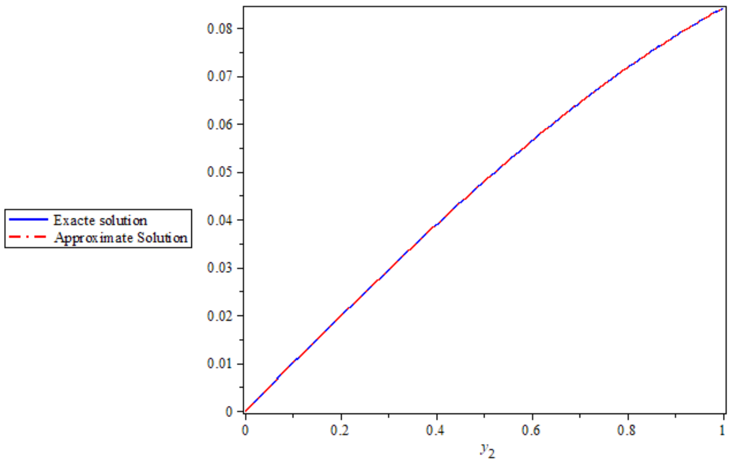


Figure 3.3: *Absolute error function for* $N = M = 16$ *at* $y_1 = 1$ *for Example* 3.10

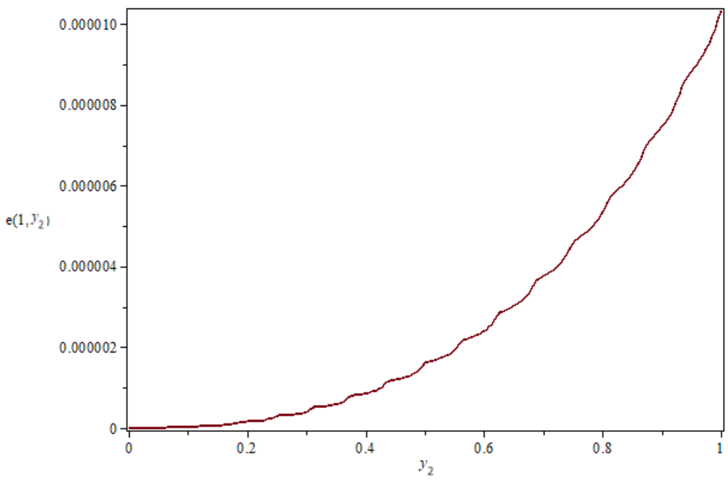
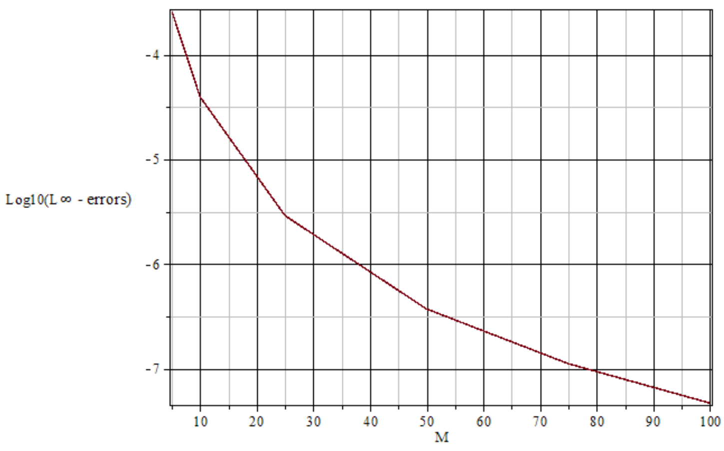


Figure 3.4: $Log_{10}(L^{\infty} - errors)$ *of* $\mu_{N,M}(y_1, y_2)$ *for* $N = 5$ *versus* M *for Example* 3.10



Example 3.11. *We examine the linear second-order 2D-PVIDE:*

$$\frac{\partial^2 \mu(y_1,y_2)}{\partial y_1 \partial y_2} = 2\frac{\partial \mu(y_1,y_2)}{\partial y_2} + \hbar(y_1,y_2) + \int_0^{y_1} y_1 \cos(y_1+y_2)\mu(v_1,y_2)dv_1, \quad 0 \le y_1 \le 1,\ 0 \le y_2 \le 1,$$

with the initial conditions

$$\mu(y_1,0) = y_1^2, \quad \frac{\partial \mu(0,y_2)}{\partial y_2} = 0,$$

where $\hbar(y_1,y_2) = 2y_1e^{y_2} - 2y_1^2e^{y_2} - \frac{1}{4}e^{y_2}y_1^4\cos(y_1+y_2)$, *and the exact solution is* $\mu(y_1,y_2) = y_1^2e^{y_2}$.

Table 3.3 provides a detailed account of the absolute errors $e_{N,M}(y_1,y_2)$ *at* $p=3$ *with* $(N,M) = \{(4,4),(8,8),(16,16),(32,32)\}$, *while Table 3.4 offers insights into the maximum absolute errors* $E(N,N)$ *observed for* $p=3$ *at different choices of* N *along with the convergence order of the solution.*

To further elucidate the efficacy of our technique, we present corresponding graphical representations. Figures 3.5 and 3.6 display three-dimensional plots of the absolute error functions $|\mu(y_1,y_2) - \mu_{N,M}(y_1,y_2)|$ *and a comparison between exact and approximate solutions, respectively, using* $(N,M) = \{(4,4),(16,16)\}$ *and* $p=3$ *as parameters. Figure 3.7 provides a detailed comparison between the exact and approximate solutions at* $y_1 = 0.1$, *while Figure 3.8 plots the absolute errors at* $y_1 = 1$, *respectively for* $(N,M) = (16,16)$ *affirming the precision of our method even at specific points. Lastly, Figure 3.9 offers a logarithmic representation* $log_{10}(L^\infty - errors)$ *for varying values of* M, *while maintaining* $N = 5$.

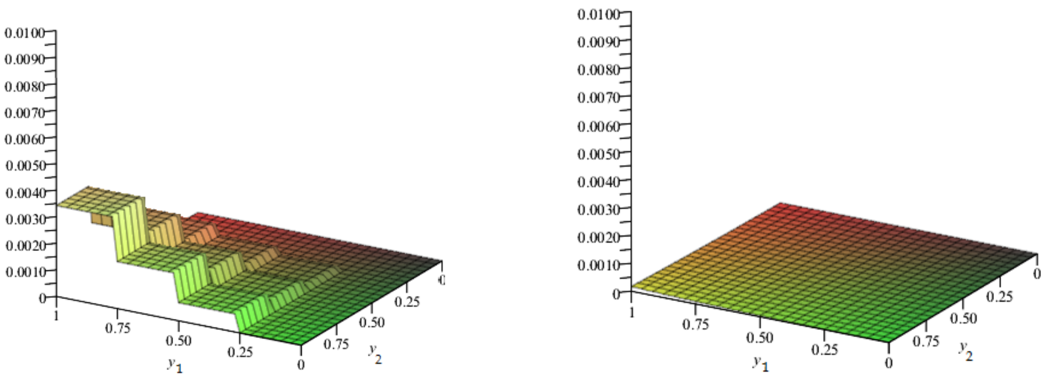
Table 3.3: *Absolute errors* $|\mu(y_1,y_2) - \mu_{N,N}(y_1,y_2)|$ *for Example 3.11*

(y_1,y_2)	$N=4$	$N=8$	$N=16$	$N=32$
$(0.000, 0.000)$	0	0	0	0
$(0.125, 0.125)$	5.611×10^{-5}	5.249×10^{-6}	7.180×10^{-7}	1.022×10^{-7}
$(0.250, 0.250)$	1.735×10^{-4}	2.580×10^{-5}	3.547×10^{-6}	5.049×10^{-7}
$(0.500, 0.500)$	1.027×10^{-3}	1.556×10^{-4}	2.165×10^{-5}	3.081×10^{-6}
$(0.750, 0.750)$	3.424×10^{-3}	5.307×10^{-4}	7.494×10^{-5}	1.066×10^{-5}
$(1.000, 1.000)$	9.121×10^{-3}	1.454×10^{-3}	2.089×10^{-4}	2.974×10^{-5}

Table 3.4: *Maximum absolute error* $E(N,M)$ *with* $N=M$ *and* $p=3$ *for Example 3.11*

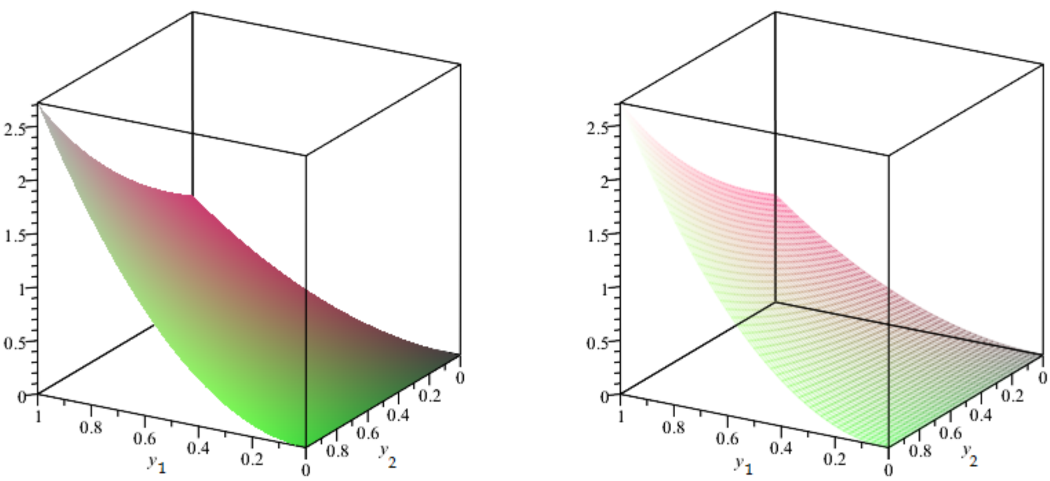
N	$E(N,N)$	*Rate*
2	0.049213546	
4	0.009121234	2.4317
8	0.001454353	2.6488
16	0.000208986	2.7985
32	0.000029749	2.8124

Figure 3.5: *(a) Absolute error function* $|\mu(y_1,y_2) - \mu_{4,4}(y_1,y_2)|$*, (b) Absolute error function* $|\mu(y_1,y_2) - \mu_{16,16}(y_1,y_2)|$ *for* $p = 3$ *for Example 3.11*



(a) **(b)**

Figure 3.6: *(a) the exact solution* $\mu(y_1,y_2)$*, (b) the approximate solution* $\mu_{16,16}$ *with* $p = 3$ *for Example 3.11*



(a) **(b)**

Figure 3.7: *Comparison of the exact and approximate solutions with* $N = M = 16$ *at* $y_1 = 0.1$ *for Example 3.11*

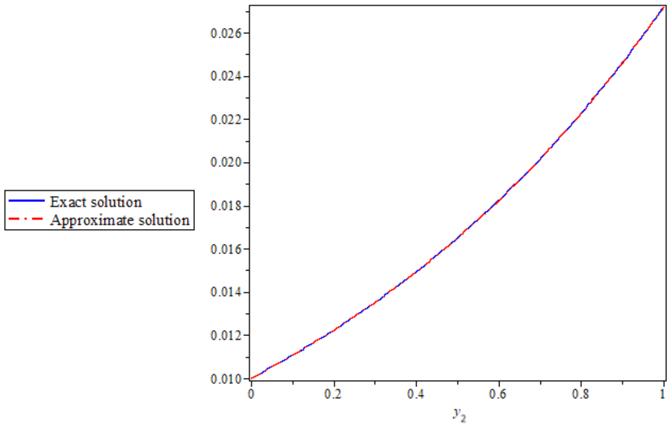


Figure 3.8: *Absolute error function for* $N = M = 16$ *at* $y_1 = 1$ *for Example 3.11*

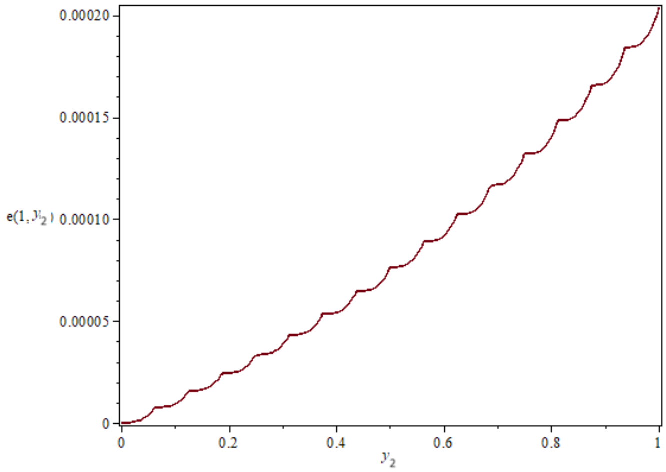
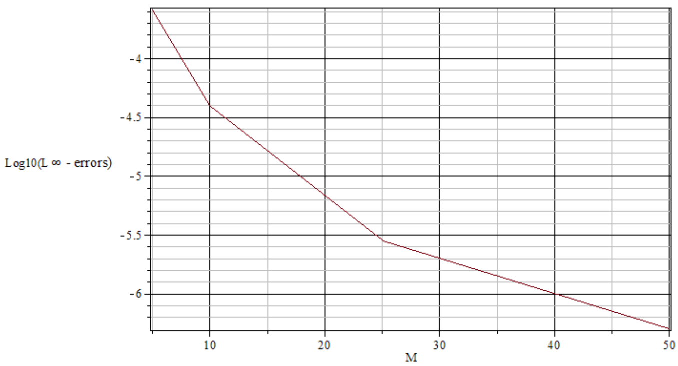


Figure 3.9: $Log_{10}(L^{\infty} - errors)$ *of* $\mu_{N,M}(y_1, y_2)$ *for* $N = 5$ *versus* M *for Example 3.11*



Example 3.12. *Consider the 2D-PVIDE*

$$\frac{\partial^2\mu(y_1,y_2)}{\partial y_1\partial y_2} = \frac{\partial\mu(y_1,y_2)}{\partial y_1} - 3\mu(y_1,y_2) + \hbar(y_1,y_2) + \int_0^{y_1} e^{y_2}(1-v_1)y_2v_1^2\mu(v_1,y_2)dv_1, \quad 0 \le y_1, y_2 \le 1,$$

with the initial conditions

$$\mu(0,y_2) = \frac{\partial\mu(y_1,0)}{\partial y_1} = 0,$$

where $\hbar(y_1,y_2) = 4y_1y_2 + 2y_2 + 5y_1y_2^2 - y_2^2 - 3y_1^2y_2^2 - y_2^3e^{y_2}\left[\frac{1}{4}y_1^4 - \frac{2}{5}y_1^5 + \frac{1}{6}y_1^6\right]$ *, and the exact solution is* $\mu(y_1,y_2) = y_2^2(1-y_1)y_1$.

Table 3.5 outlines the absolute errors $e_{N,M}(y_1,y_2)$ *obtained using our proposed method in solving Example 3.12 with* $p = 3$ *and* $(N,M) = \{(8,8),(16,16),(32,32)\}$*. Complementing this, Figure 3.10 compares the exact and approximate solutions at* $y_1 = 0.1$*, providing a targeted examination of our method's precision at specific points within the domain. Additionally, Figure 3.11 presents the function* $\log_{10}(L^\infty$*-errors) of* $\mu_{N,M}(y_1,y_2)$ *with* $N = 5$ *versus* M*.*

Table 3.5: *Absolute errors* $|\mu(y_1,y_2) - \mu_{N,N}(y_1,y_2)|$ *for Example 3.12*

(y_1,y_2)	$N=8$	$N=16$	$N=32$
$(0.000, 0.000)$	0	0	0
$(0.125, 0.125)$	4.884×10^{-14}	1.058×10^{-13}	1.799×10^{-14}
$(0.250, 0.250)$	2.073×10^{-12}	6.850×10^{-13}	1.165×10^{-13}
$(0.500, 0.500)$	2.251×10^{-14}	1.002×10^{-12}	1.704×10^{-13}
$(0.750, 0.750)$	1.928×10^{-10}	1.136×10^{-11}	1.932×10^{-12}
$(1.000, 1.000)$	6.962×10^{-10}	1.189×10^{-10}	2.022×10^{-11}

Figure 3.10: *Comparison of the exact and approximate solutions with* $N = M = 16$ *at* $y_1 = 0.1$ *for Example 3.12*

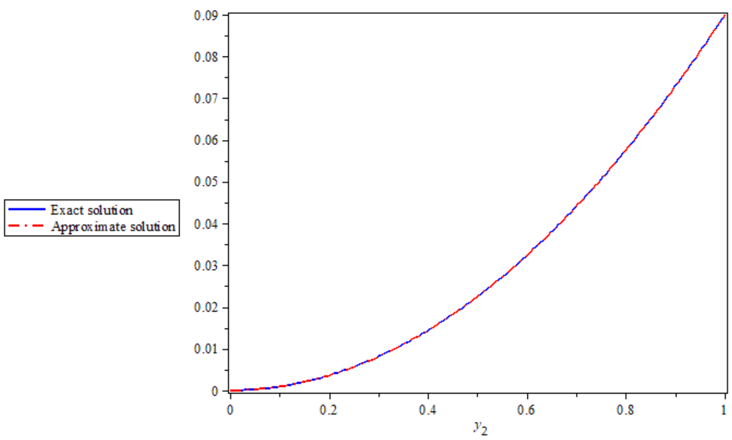
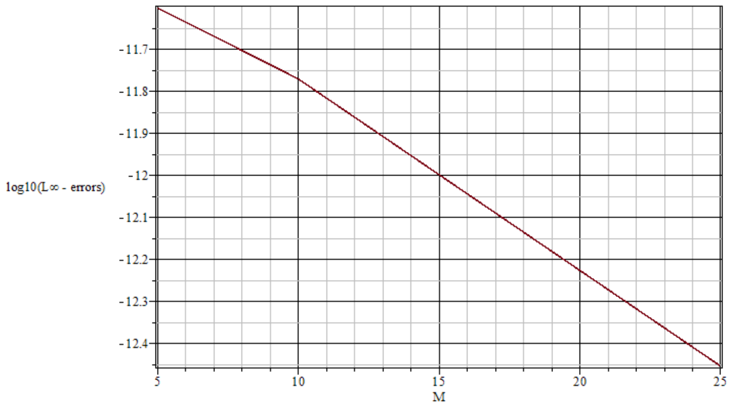


Figure 3.11: *$Log_{10}(L^{\infty} - errors)$ of $\mu_{N,M}(y_1, y_2)$ for $N = 5$ versus M for Example 3.12*



Example 3.13. *As a concluding example, our attention is directed towards the following 2D-PVIDE:*

$$\frac{\partial^2 \mu(y_1, y_2)}{\partial y_1 \partial y_2} = \frac{\partial \mu(y_1, y_2)}{\partial y_2} + \frac{\partial \mu(y_1, y_2)}{\partial y_1} + \mu(y_1, y_2) + \hbar(y_1, y_2) + \int_0^{y_1} \mu(v_1, y_2) dv_1, \quad 0 \leq y_1, y_2 \leq 1,$$

with the initial conditions

$$\mu(y_1, 0) = \mu(0, y_2) = \frac{\partial \mu(0, y_2)}{\partial y_2} = 0,$$

where $\hbar(y_1, y_2) = \left(2y_2 + y_2^2 - 2y_2 y_1 - y_2^2 y_1 - \frac{1}{2} y_2^2 y_1^2\right) \ln(y_1+2) + \frac{2y_2 y_1}{y_1+2} - 2y_2^2 ln(2) - \frac{y_2^2 y_1}{y_1+2} + \frac{1}{4} y_2^2 - y_1 y_2^2,$ *and the exact solution is* $\mu(y_1, y_2) = y_2^2 y_1 \ln(y_1 + 2)$.

Table 3.6 compiles the absolute errors $e_{N,M}(y_1, y_2) = |\mu(y_1, y_2) - \mu_{N,M}(y_1, y_2)|$ calculated for $(N, M) = \{(4, 4), (8, 8), (16, 16), (32, 32)\}$ and $p = 3$. Concurrently, Table 3.13 provides an overview of the maximum errors $E(N, N)$ observed for various values of N, along with the computational order of convergence.

On a different note, Figure 3.12 presents three-dimensional plots of the absolute error functions $e_{N,M}(y_1, y_2)$ with $(N, M) = \{(4, 4), (16, 16)\}$ and $p = 3$. Figure 3.13 conducts a detailed comparison between the exact and approximate solutions at $y_1 = 0.1$, while Figure 3.14 outlines the absolute errors $e_{16,16}(y_1, y_2)$ at $y_1 = 1$. Lastly, Figure 3.15 provides a logarithmic representation of $\mu_{N,M}(y_1, y_2)$ for varying values of M with a constant $N = 5$.

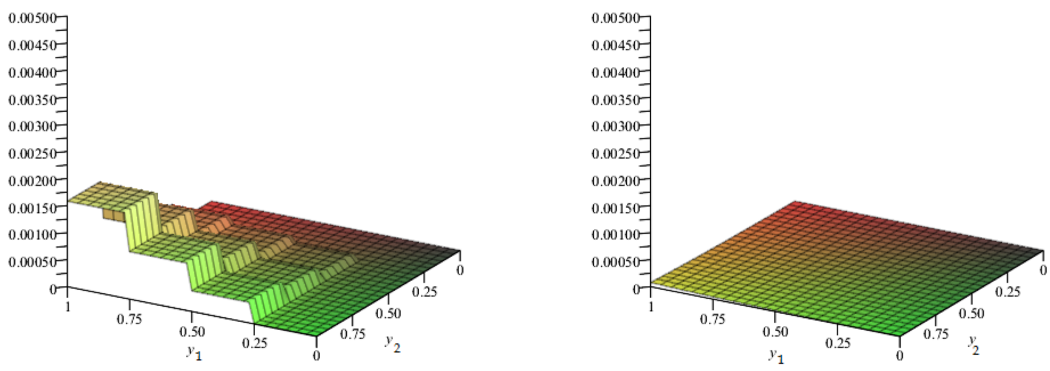
Table 3.6: *Absolute errors $|\mu(y_1, y_2) - \mu_{N,M}(y_1, y_2)|$ for Example 3.13*

(y_1, y_2)	$N = 4$	$N = 8$	$N = 16$	$N = 32$
$(0.000, 0.000)$	0	0	0	0
$(0.125, 0.125)$	3.662×10^{-6}	3.662×10^{-6}	4.709×10^{-7}	5.974×10^{-8}
$(0.250, 0.250)$	1.127×10^{-4}	1.514×10^{-5}	1.968×10^{-6}	2.511×10^{-7}
$(0.500, 0.500)$	5.160×10^{-4}	7.300×10^{-5}	9.719×10^{-6}	1.253×10^{-6}
$(0.750, 0.750)$	1.582×10^{-3}	2.361×10^{-4}	3.213×10^{-5}	4.185×10^{-6}
$(1.000, 1.000)$	4.577×10^{-3}	7.160×10^{-4}	9.920×10^{-5}	1.307×10^{-5}

Table 3.7: *Maximum absolute error $E(N, M)$ with $N = M$ and $p = 3$ for Example 3.13*

N	$E(N, N)$	*Rate*
2	0.023113101	
4	0.004577720	2.33600
8	0.000716013	2.67657
16	0.000099202	2.85154
32	0.000013073	2.92368

Figure 3.12: *(a) Absolute error function $|\mu(y_1, y_2) - \mu_{4,4}(y_1, y_2)|$, (b) Absolute error function $|\mu(y_1, y_2) - \mu_{16,16}(y_1, y_2)|$ for $p = 3$ for Example 3.13*



(a) **(a)**

Figure 3.13: *Comparison of the exact and approximate solutions with $N = M = 16$ at $y_1 = 0.1$ for Example 3.13*

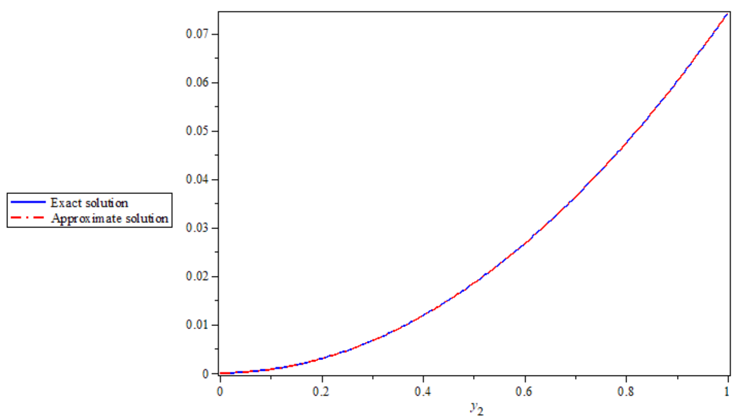


Figure 3.14: *Absolute error function for* $N = M = 16$ *at* $y_1 = 1$ *for Example 3.13*

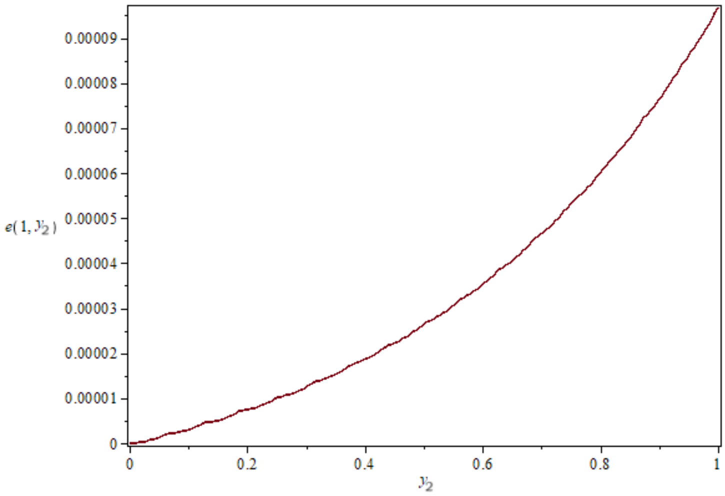
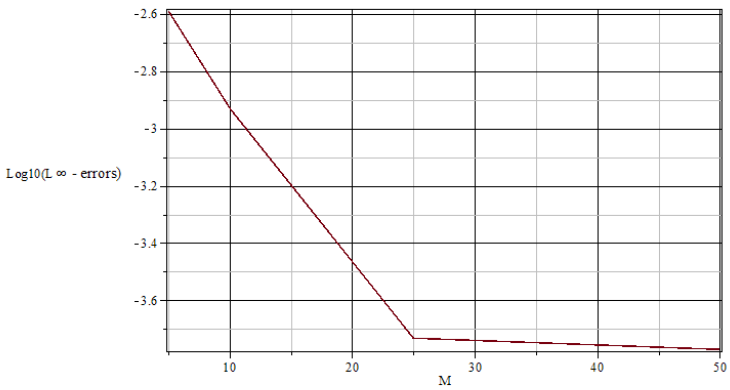


Figure 3.15: $Log_{10}(L^{\infty} - errors)$ *of* $\mu_{N,M}(y_1, y_2)$ *for* $N = 5$ *versus* M *for Example 3.13*



3.4 Concluding remarks

This chapter introduces a novel approach for handling two-dimensional second-order PVIDEs by employing a piecewise collocation approach based on two-dimensional Taylor polynomials. To the best of our knowledge, this marks the inaugural effort in utilizing two-dimensional Taylor polynomials to address two-dimensional second-order PVIDEs. Furthermore, we derive a new theorem that guarantees the convergence of the proposed approach. The effectiveness of the developed method is underscored through the presentation of numerical results from various examples, validating the theoretical findings.

Chapter 4

Numerical solution of two-dimensional partial Volterra integro-differential equations of high-order

In the present chapter, we are interested in solving the high-order linear and nonlinear 2D-PVIDEs:

$$\frac{\partial^{r_1+r_2}\mu(y_1,y_2)}{\partial y_1^{r_1}\partial y_2^{r_2}} = \hbar(y_1,y_2) + \int_0^{y_1}\int_0^{y_2} H\left(y_1,y_2,v_1,v_2\right)\frac{\partial^{r_1+r_2}\mu(v_1,v_2)}{\partial y_1^{r_1}\partial y_2^{r_2}}dv_2dv_1, \tag{4.1}$$

and

$$\frac{\partial^{r_1+r_2}\mu(y_1,y_2)}{\partial y_1^{r_1}\partial y_2^{r_2}} = \hbar(y_1,y_2) + \int_0^{y_1}\int_0^{y_2} H\left(y_1,y_2,v_1,v_2,\frac{\partial^{r_1+r_2}\mu(v_1,v_2)}{\partial y_1^{r_1}\partial y_2^{r_2}}\right)dv_2dv_1, \tag{4.2}$$

respectively, with r_1+r_2 appropriate initial conditions where $(y_1,y_2)\in\Lambda=[0,A_1]\times[0,A_2]$ and $\hbar$ and H are sufficiently smooth functions.

An initial exploration of these equations was conducted by Babaaghaie and Maleknejad in [80] where they estimated the kernel of the original equation through the application of Haar wavelets, consequently formulating a nonlinear system awaiting resolution. Subsequently, Wang et al. [81] utilized a combination of shifted Jacobi polynomials and a collocation method to convert the 2D-PVIDE (4.2) into a set of algebraic equations.

To our knowledge, no attempts have been made to solve the high-order 2D-PVIDE (4.1) nor the 2D-NPVIDE (4.2) using the Taylor collocation method, and therefore applying the Taylor collocation method to these important problems represents a major challenge. Our main objective here is to generalize the numerical methods introduced in the previous chapters to solve the high-order 2D-PVIDE (4.1) and 2D-NPVIDE (4.2), respectively. In this regard, we transform the 2D-PVIDE (4.1) and the 2D-NPVIDE (4.2) to another problems of solving two-dimensional Volterra integral equations. Using the two-dimensional Taylor polynomials as the basis functions of the piecewise collocation approach, we get an explicit form of the approximate solution to the main problem.

The rest of this work is as follows: Section 4.1 constructs the Taylor collocation approach to solve the 2D-PVIDE (4.1) and the 2D-NPVIDE (4.2), while Section 4.2 presents details of the error estimates and convergence analysis of the proposed method. Section 4.3 introduces several numerical examples and illustrations to test the applicability of the suggested method and the theoretical results. Finally, the last section describes the concluding remarks.

4.1 Description of the method

4.1.1 Two-dimensional linear partial Volterra integro-differential equations

This part is devoted to discussing the Taylor collocation approach to solve the high-order 2D-PVIDE (4.1). For simplicity, we use the notation

$$\int_0^{y_{1,q}} \cdots \int_0^{y_{1,2}} \int_0^{y_1} \omega(y_1) dy_1 dy_{1,2} \cdots dy_{1,q} = \mathcal{I}_{y_1}^q \omega(y_1),$$

and without loss of generality, we assume that $A_1 = A_2 = 1$.

If we define a new function $\omega(y_1, y_2)$, such that $\omega(y_1, y_2) = \frac{\partial^{r_1+r_2}\mu(y_1,y_2)}{\partial y_1^{r_1} \partial y_2^{r_2}}$, then [81]

$$\begin{aligned}\mu(y_1, y_2) = \mathcal{I}_{y_1}^{(r_1)} \mathcal{I}_{y_2}^{(r_2)} \omega(y_1, y_2) - \sum_{i=0}^{r_1-1} \sum_{j=0}^{r_2-1} \frac{y_1^i y_2^j}{i! j!} \left(\frac{\partial^{i+j} \mu(y_1, y_2)}{\partial y_1^i \partial y_2^j} \right)_{y_1=0, y_2=0} \\ + \sum_{i=0}^{r_1-1} \frac{y_1^i}{i!} \left(\frac{\partial^i \mu(y_1, y_2)}{\partial y_1^i} \right)_{y_1=0} + \sum_{j=0}^{r_2-1} \frac{y_2^j}{j!} \left(\frac{\partial^j \mu(y_1, y_2)}{\partial y_2^j} \right)_{y_2=0},\end{aligned} \tag{4.3}$$

and the high-order 2D-PVIDE (4.1) may be transformed to the two-dimensional VIE

$$\omega(y_1, y_2) = \hbar(y_1, y_2) + \int_0^{y_1} \int_0^{y_2} H(y_1, y_2, v_1, v_2) \omega(v_1, v_2) dv_2 dv_1, \ \ (y_1, y_2) \in [0, 1] \times [0, 1]. \tag{4.4}$$

We examine the numerical solutions within the previously defined real polynomial spline space $S_{p-1,p-1}^{(-1)}(\Pi_{N,M})$ of degree $p-1$ in both y_1 and y_2, where we approximate the unknown function $\omega(y_1, y_2)$ within each rectangle $\Lambda_{n,m}$, $n = 0, 1, 2, \ldots, N-1$, $m = 0, 1, 2, \ldots, M-1$, by the Taylor polynomials (3.4)

$$\omega_{n,m}(y_1, y_2) = \sum_{i+j=0}^{p-1} \frac{1}{i! j!} \frac{\partial^{i+j} \hat{\omega}_{n,m}(y_{1,n}, y_{2,m})}{\partial y_1^i \partial y_2^j} (y_1 - y_{1,n})^i (y_2 - y_{2,m})^j,$$

where $\dfrac{\partial^{i+j} \hat{\omega}_{n,m}(y_{1,n}, y_{2,m})}{\partial y_1^i \partial y_2^j}$ are unknown coefficients to be determined in the sequel.

Step 1: For $n = m = 0$, we approximate the function $\omega(y_1, y_2)$ within the rectangle $\Lambda_{0,0}$ by the polynomials (3.5):

$$\omega_{0,0}(y_1, y_2) = \sum_{i+j=0}^{p-1} \frac{1}{i! j!} \left(\frac{\partial^{i+j} \omega(y_1, y_2)}{\partial y_1^i \partial y_2^j} \right)_{y_1=0, y_2=0} y_1^i y_2^j, \quad (y_1, y_2) \in \Lambda_{0,0}.$$

Differentiating Eq. (4.4) $i-$ and $j-$times in terms of y_1 and y_2, respectively, we get

$$\frac{\partial^{i+j}\omega(y_1,y_2)}{\partial y_1^i \partial y_2^j} = \partial_{y_1}^{(i)}\partial_{y_2}^{(j)}\hbar(y_1,y_2) + \int_0^{y_1}\int_0^{y_2} \partial_{y_1}^{(i)}\partial_{y_2}^{(j)} H(y_1,y_2,v_1,v_2)\omega(v_1,v_2)dv_2dv_1$$
$$+\sum_{r=0}^{j-1}\sum_{l=0}^{r}\sum_{q=0}^{i-1}\sum_{\eta=0}^{q}\binom{r}{l}\binom{q}{\eta}\frac{\partial^{q-\eta}}{\partial y_1^{q-\eta}}\left[\frac{\partial^{i-1-q}}{\partial y_1^{i-1-q}}\bigg|_{v_1=y_1}\left(\frac{\partial^{r-l}}{\partial y_2^{r-l}}\left[\partial_{y_2}^{(j-1-r)}H(y_1,y_2,v_1,y_2)\right]\right)\right]\frac{\partial^{l+\eta}\omega(y_1,y_2)}{\partial y_1^{\eta}\partial y_2^{l}}$$
$$+\sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\int_0^{y_1}\frac{\partial^i}{\partial y_1^i}\left[\frac{\partial^{r-l}}{\partial y_2^{r-l}}\left[\partial_{y_2}^{(j-1-r)}H(y_1,y_2,v_1,y_2)\right]\right]\frac{\partial^l\omega(v_1,y_2)}{\partial y_2^l}dv_1$$
$$+\int_0^{y_2}\sum_{q=0}^{i-1}\sum_{\eta=0}^{q}\binom{q}{\eta}\frac{\partial^{q-\eta}}{\partial y_1^{q-\eta}}\left[\partial_{y_1}^{(i-1-q)}\partial_{y_2}^{(j)}H(y_1,y_2,y_1,v_2)\right]\frac{\partial^{\eta}\omega(y_1,v_2)}{\partial y_1^{\eta}}dv_2.$$

Hence,

$$\frac{\partial^{i+j}\omega(0,0)}{\partial y_1^i\partial y_2^j} = \partial_{y_1}^{(i)}\partial_{y_2}^{(j)}\hbar(0,0)$$
$$+\sum_{r=0}^{j-1}\sum_{l=0}^{r}\sum_{q=0}^{i-1}\sum_{\eta=0}^{q}\binom{r}{l}\binom{q}{\eta}\frac{\partial^{q-\eta}}{\partial y_1^{q-\eta}}\left[\frac{\partial^{i-1-q}}{\partial y_1^{i-1-q}}\bigg|_{v_1=y_1}\left(\frac{\partial^{r-l}}{\partial y_2^{r-l}}\left[\partial_2^{(j-1-r)}H(y_1,y_2,v_1,y_2)\right]\right)\right]_{y_1=y_2=0}\times$$
$$\frac{\partial^{l+\eta}\omega(0,0)}{\partial y_1^{\eta}\partial y_2^{l}}. \tag{4.5}$$

Step 2: For $n=1,2,\ldots,N-1$ and $m=0$, we approximate the function $\omega(y_1,y_2)$ within the rectangles $\Lambda_{n,0},\ n=0,1,2,\ldots,N-1$, by the polynomials (3.7)

$$\omega_{n,0}(y_1,y_2) = \sum_{i+j=0}^{p-1}\frac{1}{i!j!}\frac{\partial^{i+j}\hat{\omega}_{n,0}(y_{1,n},0)}{\partial y_1^i\partial y_2^j}(y_1-y_{1,n})^i y_2^j, \quad (y_1,y_2)\in\Lambda_{n,0},$$

where $\hat{\omega}_{n,0}(y_1,y_2)$ is the precise solution to the VIE

$$\hat{\omega}_{n,0}(y_1,y_2) = \hbar(y_1,y_2) + \sum_{\xi=0}^{n-1}\int_{y_{1,\xi}}^{y_{1,\xi+1}}\int_0^{y_2} H(y_1,y_2,v_1,v_2)\omega_{\xi,0}(v_1,v_2)dv_2dv_1$$
$$+\int_{y_{1,n}}^{y_1}\int_0^{y_2} H(y_1,y_2,v_1,v_2)\hat{\omega}_{n,0}(v_1,v_2)dv_2dv_1. \tag{4.6}$$

Similarly, we differentiate Eq. (4.6) $i-$ and $j-$times in terms of y_1 and y_2, respectively, to get

$$\frac{\partial^{i+j}\hat{\omega}_{n,0}(y_1,y_2)}{\partial y_1^i\partial y_2^j} = \partial_{y_1}^{(i)}\partial_{y_2}^{(j)}\hbar(y_1,y_2) + \sum_{\xi=0}^{n-1}\int_{y_{1,\xi}}^{y_{1,\xi+1}}\int_0^{y_2}\partial_{y_1}^{(i)}\partial_{y_2}^{(j)}H(y_1,y_2,v_1,v_2)\omega_{\xi,0}(v_1,v_2)dv_2dv_1$$
$$+\sum_{\xi=0}^{n-1}\sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\int_{y_{1,\xi}}^{y_{1,\xi+1}}\frac{\partial^i}{\partial y_1^i}\left[\frac{\partial^{r-l}}{\partial y_2^{r-l}}\left[\partial_{y_2}^{(j-1-r)}H(y_1,y_2,v_1,y_2)\right]\right]\frac{\partial^l\omega_{\xi,0}(v_1,y_2)}{\partial y_2^l}dv_1$$

$$+\sum_{r=0}^{j-1}\sum_{l=0}^{r}\sum_{q=0}^{i-1}\sum_{\eta=0}^{q}\binom{r}{l}\binom{q}{\eta}\frac{\partial^{q-\eta}}{\partial y_1^{q-\eta}}\left[\frac{\partial^{i-1-q}}{\partial y_1^{i-1-q}}\bigg|_{v_1=y_1}\left(\frac{\partial^{r-l}}{\partial y_2^{r-l}}\left[\partial_{y_2}^{(j-1-r)}H(y_1,y_2,v_1,y_2)\right]\right)\right]\times \frac{\partial^{l+\eta}\hat{\omega}_{n,0}(y_1,y_2)}{\partial y_1^{\eta}\partial y_2^{l}}$$
$$+\sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\int_{y_{1,n}}^{y_1}\frac{\partial^i}{\partial y_1^i}\left[\frac{\partial^{r-l}}{\partial y_2^{r-l}}\left[\partial_{y_2}^{(j-1-r)}H(y_1,y_2,v_1,y_2)\right]\right]\frac{\partial^l\hat{\omega}_{n,0}(v_1,y_2)}{\partial y_2^l}dv_1$$
$$+\int_0^{y_2}\sum_{q=0}^{i-1}\sum_{\eta=0}^{q}\binom{q}{\eta}\frac{\partial^{q-\eta}}{\partial y_1^{q-\eta}}\left[\partial_{y_1}^{(i-1-q)}\partial_{y_2}^{(j)}H(y_1,y_2,y_1,v_2)\right]\frac{\partial^{\eta}\hat{\omega}_{n,0}(y_1,v_2)}{\partial y_1^{\eta}}dv_2$$
$$+\int_{y_{1,n}}^{y_1}\int_0^{y_2}\partial_{y_1}^{(i)}\partial_{y_2}^{(j)}H(y_1,y_2,v_1,v_2)\hat{\omega}_{n,0}(v_1,v_2)dv_2dv_1. \tag{4.7}$$

Hence, for $n=0,1,2,\dots,N-1$,

$$\frac{\partial^{i+j}\hat{\omega}_{n,0}(y_{1,n},0)}{\partial y_1^i\partial y_2^j}=\partial_{y_1}^{(i)}\partial_{y_2}^{(j)}\hbar(y_{1,n},0)$$
$$+\sum_{\xi=0}^{n-1}\sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\int_{y_{1,\xi}}^{y_{1,\xi+1}}\frac{\partial^i}{\partial y_1^i}\left[\frac{\partial^{r-l}}{\partial y_2^{r-l}}\left[\partial_{y_2}^{(j-1-r)}H(y_1,y_2,v_1,y_2)\right]\right]_{y_1=y_{1,n},y_2=0}\frac{\partial^l\omega_{\xi,0}(v_1,0)}{\partial y_2^l}dv_1$$
$$+\sum_{r=0}^{j-1}\sum_{l=0}^{r}\sum_{q=0}^{i-1}\sum_{\eta=0}^{q}\binom{r}{l}\binom{q}{\eta}\frac{\partial^{q-\eta}}{\partial y_1^{q-\eta}}\left[\frac{\partial^{i-1-q}}{\partial y_1^{i-1-q}}\bigg|_{v_1=y_1}\left(\frac{\partial^{r-l}}{\partial y_2^{r-l}}\left[\partial_2^{(j-1-r)}H(y_1,y_2,v_1,y_2)\right]\right)\right]_{y_1=y_{1,n},y_2=0}\times \frac{\partial^{l+\eta}\hat{\omega}_{n,0}(y_{1,n},0)}{\partial y_1^{\eta}\partial y_2^{l}}. \tag{4.8}$$

Step 3: For $n=0,1,2,\dots,N-1,\ m=1,2,\dots,M-1$, the function $\omega(y_1,y_2)$ is approximated in the rectangles $\Lambda_{n,m}$ by the polynomials (3.11)

$$\omega_{n,m}(y_1,y_2)=\sum_{i+j=0}^{p-1}\frac{1}{i!j!}\frac{\partial^{i+j}\hat{\omega}_{n,m}(y_{1,n},y_{2,m})}{\partial y_1^i\partial y_2^j}(y_1-y_{1,n})^i(y_2-y_{2,m})^j,$$

where $\hat{\omega}_{n,m}(y_1,y_2)$ refers to the precise solution of the VIE

$$\begin{aligned}\hat{\omega}_{n,m}(y_1,y_2)=\hbar(y_1,y_2)&+\sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1}\int_{y_{1,\xi}}^{y_{1,\xi+1}}\int_{y_{2,\rho}}^{y_{2,\rho+1}}H(y_1,y_2,v_1,v_2)\omega_{\xi,\rho}(v_1,v_2)dv_2dv_1\\&+\sum_{\xi=0}^{n-1}\int_{y_{1,\xi}}^{y_{1,\xi+1}}\int_{y_{2,m}}^{y_2}H(y_1,y_2,v_1,v_2)\omega_{\xi,m}(v_1,v_2)dv_2dv_1\\&+\sum_{\rho=0}^{m-1}\int_{y_{1,n}}^{y_1}\int_{y_{2,\rho}}^{y_{2,\rho+1}}H(y_1,y_2,v_1,v_2)\omega_{n,\rho}(v_1,v_2)dv_2dv_1\\&+\int_{y_{1,n}}^{y_1}\int_{y_{2,m}}^{y_2}H(y_1,y_2,v_1,v_2)\hat{\omega}_{n,m}(v_1,v_2)dv_2dv_1,\end{aligned}\tag{4.9}$$

and by differentiating $i-$times with respect to y_1 and $j-$times with respect to y_2, we obtain

$$\begin{aligned}
&\frac{\partial^{i+j}\hat{\omega}_{n,m}(y_1,y_2)}{\partial y_1^i \partial y_2^j} = \partial_{y_1}^{(i)}\partial_{y_2}^{(j)}\hbar(y_1,y_2)\\
&+\sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1}\int_{y_{1,\xi}}^{y_{1,\xi+1}}\int_{y_{2,\rho}}^{y_{2,\rho+1}}\partial_{y_1}^{(i)}\partial_{y_2}^{(j)}H(y_1,y_2,v_1,v_2)\omega_{\xi,\rho}(v_1,v_2)dv_2dv_1\\
&+\sum_{\xi=0}^{n-1}\sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\int_{y_{1,\xi}}^{y_{1,\xi+1}}\frac{\partial^i}{\partial y_1^i}\left[\frac{\partial^{r-l}}{\partial y_2^{r-l}}\left[\partial_{y_2}^{(j-1-r)}H(y_1,y_2,v_1,y_2)\right]\right]\frac{\partial^l\omega_{\xi,m}(v_1,y_2)}{\partial y_2^l}dv_1\\
&+\sum_{\xi=0}^{n-1}\int_{y_{1,\xi}}^{y_{1,\xi+1}}\int_{y_{2,m}}^{y_2}\partial_{y_1}^{(i)}\partial_{y_2}^{(j)}H(y_1,y_2,v_1,v_2)\omega_{\xi,m}(v_1,v_2)dv_2dv_1\\
&+\sum_{\rho=0}^{m-1}\sum_{q=0}^{i-1}\sum_{\eta=0}^{q}\binom{q}{\eta}\int_{y_{2,\rho}}^{y_{2,\rho+1}}\frac{\partial^{q-\eta}}{\partial y_1^{q-\eta}}\left[\partial_{y_1}^{(i-1-q)}\partial_{y_2}^{(j)}H(y_1,y_2,y_1,v_2)\right]\frac{\partial^\eta\omega_{n,\rho}(y_1,v_2)}{\partial y_1^\eta}dv_2\\
&+\sum_{\rho=0}^{m-1}\int_{y_{1,n}}^{y_1}\int_{y_{2,\rho}}^{y_{2,\rho+1}}\partial_{y_1}^{(i)}\partial_{y_2}^{(j)}H(y_1,y_2,v_1,v_2)\omega_{n,\rho}(v_1,v_2)dv_2dv_1\\
&+\sum_{r=0}^{j-1}\sum_{l=0}^{r}\sum_{q=0}^{i-1}\sum_{\eta=0}^{q}\binom{r}{l}\binom{q}{\eta}\frac{\partial^{q-\eta}}{\partial y_1^{q-\eta}}\left[\frac{\partial^{i-1-q}}{\partial y_1^{i-1-q}}\bigg|_{v_1=y_1}\left(\frac{\partial^{r-l}}{\partial y_2^{r-l}}\left[\partial_{y_2}^{(j-1-r)}H(y_1,y_2,v_1,y_2)\right]\right)\right]\frac{\partial^{l+\eta}\hat{\omega}_{n,m}(y_1,y_2)}{\partial y_1^\eta\partial y_2^l}\\
&+\sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\int_{y_{1,n}}^{y_1}\frac{\partial^i}{\partial y_1^i}\left[\frac{\partial^{r-l}}{\partial y_2^{r-l}}\left[\partial_{y_2}^{(j-1-r)}H(y_1,y_2,v_1,y_2)\right]\right]\frac{\partial^l\hat{\omega}_{n,m}(v_1,y_2)}{\partial y_2^l}dv_1\\
&+\sum_{q=0}^{i-1}\sum_{\eta=0}^{q}\binom{q}{\eta}\int_{y_{2,m}}^{y_2}\frac{\partial^{q-\eta}}{\partial y_1^{q-\eta}}\left[\partial_{y_1}^{(i-1-q)}\partial_{y_2}^{(j)}H(y_1,y_2,y_1,v_2)\right]\frac{\partial^\eta\hat{\omega}_{n,m}(y_1,v_2)}{\partial y_1^\eta}dv_2\\
&+\int_{y_{1,n}}^{y_1}\int_{y_{2,m}}^{y_2}\partial_{y_1}^{(i)}\partial_{y_2}^{(j)}H(y_1,y_2,v_1,v_2)\hat{\omega}_{n,m}(v_1,v_2)dv_2dv_1, \qquad (4.10)
\end{aligned}$$

which leads to

$$\begin{aligned}
&\frac{\partial^{i+j}\hat{\omega}_{n,m}(y_{1,n},y_{2,m})}{\partial y_1^i \partial y_2^j} = \partial_{y_1}^{(i)}\partial_{y_2}^{(j)}\hbar(y_{1,n},y_{2,m})\\
&+\sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1}\int_{y_{1,\xi}}^{y_{1,\xi+1}}\int_{y_{2,\rho}}^{y_{2,\rho+1}}\partial_{y_1}^{(i)}\partial_{y_2}^{(j)}H(y_{1,n},y_{2,m},v_1,v_2)\omega_{\xi,\rho}(v_1,v_2)dv_2dv_1\\
&+\sum_{\xi=0}^{n-1}\sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\int_{y_{1,\xi}}^{y_{1,\xi+1}}\frac{\partial^i}{\partial y_1^i}\left[\frac{\partial^{r-l}}{\partial y_2^{r-l}}\left[\partial_{y_2}^{(j-1-r)}H(y_1,y_2,v_1,y_2)\right]\right]_{y_1=y_{1,n},y_2=y_{2,m}}\frac{\partial^l\omega_{\xi,m}(v_1,y_{2,m})}{\partial y_2^l}dv_1\\
&+\sum_{\rho=0}^{m-1}\sum_{q=0}^{i-1}\sum_{\eta=0}^{q}\binom{q}{\eta}\int_{y_{2,\rho}}^{y_{2,\rho+1}}\frac{\partial^{q-\eta}}{\partial y_1^{q-\eta}}\left[\partial_{y_1}^{(i-1-q)}\partial_{y_2}^{(j)}H(y_1,y_2,y_1,v_2)\right]_{y_1=y_{1,n},y_2=y_{2,m}}\frac{\partial^\eta\omega_{n,\rho}(y_{1,n},v_2)}{\partial y_1^\eta}dv_2\\
&+\sum_{r=0}^{j-1}\sum_{l=0}^{r}\sum_{q=0}^{i-1}\sum_{\eta=0}^{q}\binom{r}{l}\binom{q}{\eta}\frac{\partial^{q-\eta}}{\partial y_1^{q-\eta}}\left[\frac{\partial^{i-1-q}}{\partial y_1^{i-1-q}}\bigg|_{v_1=y_1}\left(\frac{\partial^{r-l}}{\partial y_2^{r-l}}\left[\partial_{y_2}^{(j-1-r)}H(y_1,y_2,v_1,y_2)\right]\right)\right]_{y_1=y_{1,n},y_2=y_{2,m}}\times\\
&\qquad\frac{\partial^{l+\eta}\hat{\omega}_{n,m}(y_{1,n},y_{2,m})}{\partial y_1^\eta\partial y_2^l}, \qquad (4.11)
\end{aligned}$$

for $n = 0, 1, 2, \dots, N-1$ and $m = 1, 2, \dots, M-1$.

Finally, in virtue of Eqs. (4.5), (4.8) and (4.11), the approximate solution $\omega_{N,M}(y_1, y_2)$ of the two-dimensional Volterra integral equation (4.4) can be determined, and therefore the approximate solution $\mu_{N,M}(y_1, y_2)$ of the 2D-PVIDE (4.1) may be given by

$$\begin{aligned}\mu_{N,M}(y_1,y_2) &= \mathcal{I}_{y_1}^{(r_1)}\mathcal{I}_{y_2}^{(r_2)}\omega_{N,M}(y_1,y_2) - \sum_{i=0}^{r_1-1}\sum_{j=0}^{r_2-1}\frac{y_1^i y_2^j}{i!j!}\left(\frac{\partial^{i+j}\mu(y_1,y_2)}{\partial y_1^i \partial y_2^j}\right)_{y_1=0,y_2=0} \\ &+ \sum_{i=0}^{r_1-1}\frac{y_1^i}{i!}\left(\frac{\partial^i\mu(y_1,y_2)}{\partial y_1^i}\right)_{y_1=0} + \sum_{j=0}^{r_2-1}\frac{y_2^j}{j!}\left(\frac{\partial^j\mu(y_1,y_2)}{\partial y_2^j}\right)_{y_2=0}.\end{aligned} \tag{4.12}$$

4.1.2 Two-dimensional nonlinear partial Volterra integro-differential equations

In the current section, we present a numerical approach to solve the high-order 2D-NPVIDE (4.2). First, we define the function $\omega(y_1, y_2) = \frac{\partial^{r_1+r_2}\mu(y_1,y_2)}{\partial y_1^{r_1}\partial y_2^{r_2}}$ to transform the 2D-NPVIDE (4.2) to the 2D-NVIE:

$$\omega(y_1,y_2) = \hbar(y_1,y_2) + \int_0^{y_1}\int_0^{y_2} H\left(y_1,y_2,v_1,v_2,\omega(v_1,v_2)\right)dv_2dv_1,\ \ (y_1,y_2)\in[0,1]\times[0,1]. \tag{4.13}$$

Similarly, as in the previous section, we search for the numerical solution in the space $S_{p-1,p-1}^{(-1)}(\Pi_{N,M})$, then we may approximate the solution of the two-dimensional nonlinear Volterra integral equation(4.13).

Step 1: We approximate the function $\omega(y_1, y_2)$ within the rectangle $\Lambda_{0,0}$ as in polynomials (3.5), we have

$$\begin{aligned}\frac{\partial^{i+j}\omega(0,0)}{\partial y_1^i\partial y_2^j} &= \partial_{y_1}^{(i)}\partial_{y_2}^{(j)}\hbar(0,0) \\ &+ \sum_{r=0}^{j-1}\sum_{q=0}^{i-1}\frac{\partial^q}{\partial y_1^q}\left[\frac{\partial^{i-1-q}}{\partial y_1^{i-1-q}}\bigg|_{v_1=y_1}\left(\frac{\partial^r}{\partial y_2^r}\left[\partial_{y_2}^{(j-1-r)}H(y_1,y_2,v_1,y_2,\omega(y_1,y_2))\right]\right)\right]_{y_1=y_2=0}.\end{aligned} \tag{4.14}$$

Step 2: We approximate $\omega(y_1, y_2)$ within the rectangles $\Lambda_{n,0}$, $n = 0, 1, 2, \dots, N-1$, as in polynomials (3.7), where $\hat{\omega}_{n,0}(y_1, y_2)$ is the precise solution to the VIE

$$\begin{aligned}\hat{\omega}_{n,0}(y_1,y_2) = \hbar(y_1,y_2) &+ \sum_{\xi=0}^{n-1}\int_{y_{1,\xi}}^{y_{1,\xi+1}}\int_0^{y_2} H(y_1,y_2,v_1,v_2,\omega_{\xi,0}(v_1,v_2))dv_2dv_1 \\ &+ \int_{y_{1,n}}^{y_1}\int_0^{y_2} H(y_1,y_2,v_1,v_2,\omega_{n,0}(v_1,v_2))dv_2dv_1,\end{aligned}$$

then, we get

$$\begin{aligned}\frac{\partial^{i+j}\hat{\omega}_{n,0}(y_{1,n},0)}{\partial y_1^i \partial y_2^j} &= \partial_{y_1}^{(i)}\partial_{y_2}^{(j)}\hbar(y_{1,n},0)\\ &+\sum_{\xi=0}^{n-1}\sum_{r=0}^{j-1}\int_{y_{1,\xi}}^{y_{1,\xi+1}}\frac{\partial^i}{\partial y_1^i}\left[\frac{\partial^r}{\partial y_2^r}[\partial_{y_2}^{(j-1-r)}H(y_1,y_2,v_1,y_2,\omega_{\xi,0}(v_1,0))]\right]_{y_1=y_{1,n},y_2=0}dv_1\\ &+\sum_{r=0}^{j-1}\sum_{q=0}^{i-1}\frac{\partial^q}{\partial y_1^q}\left[\frac{\partial^{i-1-q}}{\partial y_1^{i-1-q}}\bigg|_{v_1=y_1}\left(\frac{\partial^r}{\partial y_2^r}\left[\partial_{y_2}^{(j-1-r)}H(y_1,y_2,v_1,y_2,\omega_{n,0}(y_{1,n},0))\right]\right)\right]_{y_1=y_{1,n},y_2=0}.\end{aligned} \tag{4.15}$$

Step 3: We approximate $\omega(y_1, y_2)$ within the rectangles $\Lambda_{n,m}$ for $n = 0, 1, 2, \ldots, N-1,\ m = 1, 2, \ldots, M-1$, as in polynomials (3.11), where $\hat{\omega}_{n,m}(y_1, y_2)$ is the precise solution to the VIE

$$\begin{aligned}\hat{\omega}_{n,m}(y_1,y_2) = \hbar(y_1,y_2) &+ \sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1}\int_{y_{1,\xi}}^{y_{1,\xi+1}}\int_{y_{2,\rho}}^{y_{2,\rho+1}}H(y_1,y_2,v_1,v_2,\omega_{\xi,\rho}(v_1,v_2))dv_2dv_1\\ &+\sum_{\xi=0}^{n-1}\int_{y_{1,\xi}}^{y_{1,\xi+1}}\int_{y_{2,m}}^{y_2}H(y_1,y_2,v_1,v_2,\omega_{\xi,m}(v_1,v_2))dv_2dv_1\\ &+\sum_{\rho=0}^{m-1}\int_{y_{1,n}}^{y_1}\int_{y_{2,\rho}}^{y_{2,\rho+1}}H(y_1,y_2,v_1,v_2,\omega_{n,\rho}(v_1,v_2))dv_2dv_1\\ &+\int_{y_{1,n}}^{y_1}\int_{y_n}^{y_2}H(y_1,y_2,v_1,v_2,\hat{\omega}_{n,m}(v_1,v_2))dv_2dv_1.\end{aligned}$$

Hence, for $n = 0, 1, 2, \ldots, N-1$ and $m = 1, 2, \ldots, M-1$,

$$\begin{aligned}\frac{\partial^{i+j}\hat{\omega}_{n,m}(y_{1,n},y_{2,m})}{\partial y_1^{i+1}\partial y_2^j} &= \partial_{y_1}^{(i)}\partial_{y_2}^{(j)}\hbar(y_{1,n},y_{2,m})\\ &+\sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1}\int_{y_{1,\xi}}^{y_{1,\xi+1}}\int_{y_{2,\rho}}^{y_{2,\rho+1}}\partial_{y_1}^{(i)}\partial_{y_2}^{(j)}H\left(y_{1,n},y_{2,m},v_1,v_2,\omega_{\xi,\rho}(v_1,v_2)\right)dv_2dv_1\\ &+\sum_{\xi=0}^{n-1}\sum_{r=0}^{j-1}\int_{y_{1,\xi}}^{y_{1,\xi+1}}\frac{\partial^i}{\partial y_1^i}\left[\frac{\partial^r}{\partial y_2^r}\left[\partial_{y_2}^{(j-1-r)}H\left(y_1,y_2,v_1,y_2,\omega_{\xi,m}(v_1,y_{2,m})\right)\right]\right]_{y_1=y_{1,n},y_2=y_{2,m}}dv_1\\ &+\sum_{\rho=0}^{m-1}\sum_{q=0}^{i-1}\int_{y_{2,\rho}}^{y_{2,\rho+1}}\frac{\partial^q}{\partial y_1^q}\left[\partial_{y_1}^{(i-1-q)}\partial_{y_2}^{(j)}H\left(y_1,y_2,y_1,v_2,\omega_{n,\rho}(y_{1,n},v_2)\right)\right]_{y_1=y_{1,n},y_2=y_{2,m}}dv_2\\ &+\sum_{r=0}^{j-1}\sum_{q=0}^{i-1}\frac{\partial^q}{\partial y_1^q}\left[\frac{\partial^{i-1-q}}{\partial y_1^{i-1-q}}\bigg|_{v_1=y_1}\left(\frac{\partial^r}{\partial y_2^r}\left[\partial_{y_2}^{(j-1-r)}H\left(y_1,y_2,v_1,y_2,\hat{\omega}_{n,m}(y_{1,n},y_{2,m})\right)\right]\right)\right]_{y_1=y_{1,n},y_2=y_{2,m}}.\end{aligned} \tag{4.16}$$

The approximate solution $\omega_{N,M}(y_1, y_2)$ can be determined using Eqs. (4.14), (4.15) and (4.16), which in turn leads to the approximate solution $\mu_{N,M}(y_1, y_2)$ of the 2D-NPVIDE (4.2).

4.2 Convergence analysis

The current section deals with the convergence analysis of the numerical approach described above. In this regard, two new theorems are stated and proven to obtain the error bounds of the approximate solution $\mu_{N,M}$ (4.12) computed using the Taylor collocation method applied to the 2D-PVIDE (4.1). In the sequel, the following lemma is needed.

Lemma 4.10. *[46] Suppose $\hbar$ and H are two $p-$times continuously differentiable functions defined on their respective domains. Then there exists a positive number $\alpha(p)$, such that*

$$\left\| \frac{\partial^{i+j}\hat{\omega}_{n,m}}{\partial y_1^i \partial y_2^j} \right\|_{L^\infty(\Lambda_{n,m})} \leq \alpha(p),$$

for $n = 0, 1, \ldots, N-1$, $m = 0, 1, \ldots, M-1$, $i + j = 0, 1, \ldots, p$, with $\hat{\omega}_{0,0}(y_1, y_2) = \omega(y_1, y_2)$ and $(y_1, y_2) \in \Lambda_{0,0}$.

Theorem 4.5. *Assume that $\omega(y_1, y_2)$ is the exact solution of the two-dimensional Volterra integral equation (4.4) and $\omega_{N,M}(y_1, y_2)$ is the approximate solution of the same problem computed using the Taylor collocation method. Then, there is a finite constant C independent of h and k, such that*

$$\|\omega - \omega_{N,M}\|_{L^\infty(\Lambda)} \leq C(h + k)^p,$$

where $\hbar$ and H are two $p-$times continuously differentiable functions on their respective domains.

Proof. For $(y_1, y_2) \in \Lambda_{0,0}$, making use of Lemma 1.7 and Eq. (3.5), we have

$$|\omega(y_1, y_2) - \omega_{0,0}(y_1, y_2)| \leq \sum_{i+j=p} \frac{1}{i!j!} \left\| \frac{\partial^{i+j}\omega}{\partial y_1^i \partial y_2^j} \right\| h^i k^j.$$

Therefore, according to Lemma 4.10, we obtain

$$\|\omega - \omega_{0,0}\|_{L^\infty(\Lambda_{0,0})} \leq \alpha(p) \sum_{i+j=p} \frac{1}{i!j!} h^i k^j = \frac{\alpha(p)}{p!}(h + k)^p = C_1(h + k)^p. \tag{4.17}$$

Also, for $(y_1, y_2) \in \Lambda_{n,0}$, $n = 1, 2, \ldots, N-1$, it follows from Eqs. (4.4) and (4.6), that

$$\begin{aligned}\omega(y_1, y_2) - \hat{\omega}_{n,0}(y_1, y_2) &= \sum_{\xi=0}^{n-1} \int_{y_{1,\xi}}^{y_{1,\xi+1}} \int_0^{y_2} H(y_1, y_2, v_1, v_2) \left(\omega(v_1, v_2) - \omega_{\xi,0}(v_1, v_2)\right) dv_2 dv_1 \\ &+ \int_{y_{1,n}}^{y_1} \int_0^{y_2} H(y_1, y_2, v_1, v_2) \left(\omega(v_1, v_2) - \hat{\omega}_{n,0}(v_1, v_2)\right) dv_2 dv_1,\end{aligned}$$

which leads to

$$|\omega(y_1, y_2) - \hat{\omega}_{n,0}(y_1, y_2)| = \sum_{\xi=0}^{n-1} \overline{H} hk \|\omega - \omega_{\xi,0}\|_{L^\infty(\Lambda_{\xi,0})} + \int_{y_{1,n}}^{y_1} \int_0^{y_2} \overline{H} |\omega(v_1, v_2) - \hat{\omega}_{n,0}(v_1, v_2)| dv_2 dv_1,$$

where $\overline{H} = \max\{\|H\|_{L^\infty(\Lambda)}\}$

Using Lemma 1.6, we get

$$\begin{aligned}|\omega(y_1,y_2)-\hat{\omega}_{n,0}(y_1,y_2)| &\le \sum_{\xi=0}^{n-1}\overline{H}hk\|\omega-\omega_{\xi,0}\|_{L^\infty(\Lambda_{\xi,0})}\exp\left(2\sqrt{\overline{H}}\right)\\ &\le \sum_{\xi=0}^{n-1}\overline{H}\exp\left(2\sqrt{\overline{H}}\right)\|\omega-\omega_{\xi,0}\|_{L^\infty(\Lambda_{\xi,0})}.\end{aligned}$$

Now, we use Lemmas 1.7 and 4.10 to deduce that

$$\begin{aligned}\|\omega-\omega_{n,0}\|_{L^\infty(\Lambda_{n,0})} &\le \|\omega-\hat{\omega}_{n,0}\|_{L^\infty(\Lambda_{n,0})}+\|\hat{\omega}_{n,0}-\omega_{n,0}\|_{L^\infty(\Lambda_{n,0})}\\ &\le \sum_{\xi=0}^{n-1}\overline{H}\|\omega-\omega_{\xi,0}\|_{L^\infty(\Lambda_{\xi,0})}\exp\left(2\sqrt{\overline{H}}\right)+\sum_{i+j=p}\frac{1}{i!j!}\left\|\frac{\partial^{i+j}\hat{\omega}_{n,0}}{\partial y_1^i\partial y_2^j}\right\|h^ik^j\\ &\le \sum_{\xi=0}^{n-1}\overline{H}\|\omega-\omega_{\xi,0}\|_{L^\infty(\Lambda_{\xi,0})}\exp\left(2\sqrt{\overline{H}}\right)+\frac{\alpha(p)}{p!}(h+k)^p.\end{aligned}$$

Thus, based on Lemma 1.3, one can conclude that

$$\|\omega-\omega_{n,0}\|_{L^\infty(\Lambda_{n,0})}\le\frac{\alpha(p)}{p!}\exp\left(\overline{H}\exp\left(2\sqrt{\overline{H}}\right)\right)(h+k)^p=C_2(h+k)^p. \tag{4.18}$$

For $(y_1,y_2)\in\Lambda_{n,m},\ n=0,1,\dots,N-1,\ m=1,2,\dots,M-1$, one can use Eqs. (4.4) and (4.9) to get

$$\begin{aligned}\omega(y_1,y_2)-\hat{\omega}_{n,m}(y_1,y_2)=&\sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1}\int_{y_{1,\xi}}^{y_{1,\xi+1}}\int_{y_{2,\rho}}^{y_{2,\rho+1}}H(y_1,y_2,v_1,v_2)\left(\omega(v_1,v_2)-\omega_{\xi,\rho}(v_1,v_2)\right)dv_2dv_1\\ &+\sum_{\xi=0}^{n-1}\int_{y_{1,\xi}}^{y_{1,\xi+1}}\int_{y_{2,m}}^{y_2}H(y_1,y_2,v_1,v_2)\left(\omega(v_1,v_2)-\omega_{\xi,m}(v_1,v_2)\right)dv_2dv_1\\ &+\sum_{\rho=0}^{m-1}\int_{y_{1,n}}^{y_1}\int_{y_{2,\rho}}^{y_{2,\rho+1}}H(y_1,y_2,v_1,v_2)\left(\omega(v_1,v_2)-\omega_{n,\rho}(v_1,v_2)\right)dv_2dv_1\\ &+\int_{y_{1,n}}^{y_1}\int_{y_{2,m}}^{y_2}H(y_1,y_2,v_1,v_2)\left(\omega(v_1,v_2)-\hat{\omega}_{n,m}(v_1,v_2)\right)dv_2dv_1.\end{aligned}$$

Hence,

$$\begin{aligned}|\omega(y_1,y_2)-\hat{\omega}_{n,m}(y_1,y_2)|\le&\sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1}hk\overline{H}\|\omega-\omega_{\xi,\rho}\|_{L^\infty(\Lambda_{\xi,\rho})}+\sum_{\xi=0}^{n-1}hk\overline{H}\|\omega-\omega_{\xi,m}\|_{L^\infty(\Lambda_{\xi,m})}\\ &+\sum_{\rho=0}^{m-1}hk\overline{H}\|\omega-\omega_{n,\rho}\|_{L^\infty(\Lambda_{n,\rho})}+\int_{y_{1,n}}^{y_1}\int_{y_{2,m}}^{y_2}\overline{H}\left|\omega(v_1,v_2)-\hat{\omega}_{n,m}(v_1,v_2)\right|dv_2dv_1\\ \le&\sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1}hk\overline{H}\|\omega-\omega_{\xi,\rho}\|_{L^\infty(\Lambda_{\xi,\rho})}+\sum_{\xi=0}^{n-1}h\overline{H}\|\omega-\omega_{\xi,m}\|_{L^\infty(\Lambda_{\xi,m})}\\ &+\sum_{\rho=0}^{m-1}k\overline{H}\|\omega-\omega_{n,\rho}\|_{L^\infty(\Lambda_{n,\rho})}+\int_{y_{1,n}}^{y_1}\int_{y_{2,m}}^{y_2}\overline{H}\left|\omega(v_1,v_2)-\hat{\omega}_{n,m}(v_1,v_2)\right|dv_2dv_1,\end{aligned}$$

which in turn, with the help of Lemma 1.6, leads to

$$|\omega(y_1,y_2)-\hat{\omega}_{n,m}(y_1,y_2)| \le \sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1} hk\overline{H}\exp\left(2\sqrt{\overline{H}}\right)\|\omega-\omega_{\xi,\rho}\|_{L^\infty(\Lambda_{\xi,\rho})}$$
$$+\sum_{\xi=0}^{n-1} h\overline{H}\exp\left(2\sqrt{\overline{H}}\right)\|\omega-\omega_{\xi,m}\|_{L^\infty(\Lambda_{\xi,m})}+\sum_{\rho=0}^{m-1} k\overline{H}\exp\left(2\sqrt{\overline{H}}\right)\|\omega-\omega_{n,\rho}\|_{L^\infty(\Lambda_{n,\rho})}.$$

Also, using Lemmas 1.7 and 4.10, we get

$$\begin{aligned}
\|\omega-\omega_{n,m}\|_{L^\infty(\Lambda_{n,m})} &\le \|\omega-\hat{\omega}_{n,m}\|_{L^\infty(\Lambda_{n,m})}+\|\hat{\omega}_{n,m}-\omega_{n,m}\|_{L^\infty(\Lambda_{n,m})}\\
&\le \sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1} hk\overline{H}\exp\left(2\sqrt{\overline{H}}\right)\|\omega-\omega_{\xi,\rho}\|_{L^\infty(\Lambda_{\xi,\rho})}+\sum_{\xi=0}^{n-1} h\overline{H}\exp\left(2\sqrt{\overline{H}}\right)\|\omega-\omega_{\xi,m}\|_{L^\infty(\Lambda_{\xi,m})}\\
&+\sum_{\rho=0}^{m-1} k\overline{H}\exp\left(2\sqrt{\overline{H}}\right)\|\omega-\omega_{n,\rho}\|_{L^\infty(\Lambda_{n,\rho})}+\sum_{i+j=p}\frac{1}{i!j!}\left\|\frac{\partial^{i+j}\hat{\omega}_{n,m}}{\partial y_1^i\partial y_2^j}\right\|h^ik^j\\
&\le \sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1} hk\overline{H}\exp\left(2\sqrt{\overline{H}}\right)\|\omega-\omega_{\xi,\rho}\|_{L^\infty(\Lambda_{\xi,\rho})}+\sum_{\xi=0}^{n-1} h\overline{H}\exp\left(2\sqrt{\overline{H}}\right)\|\omega-\omega_{\xi,m}\|_{L^\infty(\Lambda_{\xi,m})}\\
&+\sum_{\rho=0}^{m-1} k\overline{H}\exp\left(2\sqrt{\overline{H}}\right)\|\omega-\omega_{n,\rho}\|_{L^\infty(\Lambda_{n,\rho})}+\frac{\alpha(p)}{p!}(h+k)^p.
\end{aligned}$$

Now, we utilize Lemma 1.4 to get

$$\begin{aligned}
\|\omega-\omega_{n,m}\|_{L^\infty(\Lambda_{n,m})} &\le \left(\frac{\alpha(p)}{p!}(h+k)^p\right)\exp(2\lambda)\\
&= C_3(h+k)^p,
\end{aligned} \tag{4.19}$$

where

$$\lambda=\overline{H}\exp\left(2\sqrt{\overline{H}}\right)+\sqrt{\overline{H}^2\exp\left(4\sqrt{\overline{H}}\right)+\overline{H}\exp\left(2\sqrt{\overline{H}}\right)}.$$

Finally, with the aid of (4.17), (4.18) and (4.19), we have

$$\|\omega-\omega_{N,M}\|_{L^\infty(\Lambda)} \le C(h+k)^p,$$

where $C=\max\{C_1,C_2,C_3\}$ □

We now study the main theorem of convergence analysis of the suggested numerical method.

Theorem 4.6. *Assume that $\mu(y_1,y_2)$ is the exact solution of the 2D-VIDE (4.1) and $\mu_{N,M}(y_1,y_2)$, $n=0,1,\ldots,N-1$, $m=0,1,\ldots,M-1$, is the approximate solution of the same problem computed using the Taylor collocation method (4.12). Then, there is a finite constant L independent of h and k, such that*

$$\|\mu-\mu_{N,M}\|_{L^\infty(\Lambda)} \le L(h+k)^p,$$

where $\hbar$ and H are two $p-$times continuously differentiable on their respective domains.

Proof. Referring to Eqs. (4.3) and (4.12), one can deduce that for $n = 0, 1, \ldots, N-1$, $m = 0, 1, \ldots, M-1$,

$$\mu(y_1, y_2) - \mu_{N,M}(y_1, y_2) = \mathcal{I}_{y_1}^{(r_1)} \mathcal{I}_{y_2}^{(r_2)} \left(\omega(y_1, y_2) - \omega_{N,M}(y_1, y_2)\right).$$

Therefore, we obtain

$$\begin{aligned} |\mu(y_1, y_2) - \mu_{N,M}(y_1, y_2)| &\leq \mathcal{I}_{y_1}^{(r_1)} \mathcal{I}_{y_2}^{(r_2)} |\omega(y_1, y_2) - \omega_{N,M}(y_1, y_2)| \\ &\leq \mathcal{I}_{y_1}^{(r_1)} \mathcal{I}_{y_2}^{(r_2)} \|\omega - \omega_{N,M}\|_{L^\infty(\Lambda)}. \end{aligned}$$

Theorem 4.5 implies that

$$\|\mu - \mu_{N,M}\|_{L^\infty(\Lambda)} \leq L(h+k)^p.$$

□

Remark 5. *We can estimate the error of all derivatives* $\frac{\partial^{k_1+k_2}\mu(y_1,y_2)}{\partial y_1^{k_1} \partial y_2^{k_2}}$, $0 \leq k_1 < r_1$, $0 \leq k_2 < r_2$, *by applying technique similar to that applied in Theorem 4.6 with the help of [81]*

$$\begin{aligned} \frac{\partial^{k_1+k_2}\mu(y_1,y_2)}{\partial y_1^{k_1} \partial y_2^{k_2}} &= \mathcal{I}_{y_1}^{(r_1-k_1)} \mathcal{I}_{y_2}^{(r_2-k_2)} \omega(y_1, y_2) - \sum_{i=k_1}^{r_1-1} \sum_{j=k_2}^{r_2-1} \frac{y_1^{i-k_1} y_2^{j-k_2}}{(i-k_1)!(j-k_2)!} \left(\frac{\partial^{i+j}\mu(y_1,y_2)}{\partial y_1^i \partial y_2^j} \right)_{y_1=0, y_2=0} \\ &+ \sum_{i=k_1}^{r_1-1} \frac{y_1^{i-k_1}}{(i-k_1)!} \left(\frac{\partial^{i+k_2}\mu(y_1,y_2)}{\partial y_1^i \partial y_2^{k_2}} \right)_{y_1=0} + \sum_{j=k_2}^{r_2-1} \frac{y_2^{j-k_2}}{(j-k_2)!} \left(\frac{\partial^{j+k_1}\mu(y_1,y_2)}{\partial y_1^{k_1} y_2^j} \right)_{y_2=0}. \end{aligned}$$

4.3 Numerical results

In the current section, we provide some numerical examples to test the accuracy of the proposed numerical method and validate the convergence analysis demonstrated in the previous section. We define the maximum absolute error and the convergence rates for temporal and spatial sizes, respectively, by

$$E(N, M) = \|\mu - \mu_{N,M}\|_\infty,$$

and

$$\text{Rate} = \log_2 \left(\frac{E(N/2, M/2)}{E(N, M)} \right).$$

Example 4.14. *Consider the 2D-PVIDE [82–84]*

$$\mu(y_1, y_2) = \hbar(y_1, y_2) + \int_0^{y_2} \int_0^{y_1} \mu^2(v_1, v_2) dv_1 dv_2, \qquad 0 \leq y_1 \leq 1,\ 0 \leq y_2 \leq 1,$$

where $\hbar(y_1, y_2) = y_1^2 + y_2^2 - \frac{1}{5} y_1 y_2^5 - \frac{2}{9} y_1^3 y_2^3 - \frac{1}{5} y_1^5 y_2$, *and the exact solution is* $\mu(y_1, y_2) = y_1^2 + y_2^2$.

For the solution of this problem, Fazli et al. [82] applied the reproducing kernel function method (RKFM), Nemati et al. [83] introduced an operational approach based on shifted Legendre function with the collocation method (SLFCM), while Ray and Behera [84] applied the Gegenbauer wavelet method (GWM).

In Table 4.1, we display the absolute errors of $\mu_{N,M}(y_1, y_2)$ *at* $p = 4$ *with* $N = M = \{4, 8, 16\}$ *and we compare the new results versus those given using the RKFM [82] and SLFCM [83], while in Table 4.2, we compare our results with* $(N, M) = (10, 10)$ *versus those given using the GWM [84]. Furthermore, Figure 4.1 obtains the function* $\log_{10}(L^\infty - errors)$ *of* $\mu_{N,M}(y_1, y_2)$ *with* $N = 5$ *versus* M*. Figure 4.2 visually contrasts the precise and estimated solutions at* $y_1 = 0.2$*. Subsequently, Figure 4.3 provides a graphical representation of the absolute error function for* $N = M = 16$ *with* $p = 4$ *at* $y_1 = 0.2$.

Table 4.1: *Comparing the absolute error of* $\mu_{N,M}(y_1, y_2)$ *against the RKFM [82] and SLFCM [83] for Example 4.14*

(y_1, y_2)	RKFM [82]	SLFCM [83]	Our method ($p = 4$)		
	$N = M = 30$	$N = M = 3$	$N = M = 4$	$N = M = 8$	$N = M = 16$
$(\frac{1}{2}, \frac{1}{2})$	$2.64e-03$	$2.80e-06$	$2.85e-05$	$1.48e-05$	$1.04e-05$
$(\frac{1}{4}, \frac{1}{4})$	$2.4e-04$	$1.70e-04$	$1.29e-06$	$1.28e-06$	$1.29e-06$
$(\frac{1}{8}, \frac{1}{8})$	$1.50e-04$	$1.30e-05$	$6.69e-07$	$7.66e-09$	$7.65e-09$
$(\frac{1}{16}, \frac{1}{16})$	$9.45e-05$	$3.50e-05$	$2.87e-07$	$2.62e-09$	$3.55e-11$

Table 4.2: *Comparing the absolute error of* $\mu_{N,M}(y_1, y_2)$ *against the GWM [84] for Example 4.14*

(y_1, y_2)	GWM ($N = M = 8$) [84]		Our method
	$\lambda = 0.75$	$\lambda = 1.75$	$N = M = 10$
$(0.2, 0.2)$	$1.06e-02$	$4.39e-02$	$2.61e-07$
$(0.2, 0.4)$	$1.84e-03$	$1.01e-02$	$2.74e-05$
$(0.4, 0.4)$	$7.49e-03$	$3.96e-02$	$2.09e-05$
$(0.4, 0.2)$	$1.84e-02$	$1.01e-02$	$1.93e-05$
$(0.6, 0.2)$	$1.00e-02$	$4.81e-02$	$5.10e-04$
$(0.6, 0.6)$	$6.23e-03$	$1.32e-02$	$6.56e-04$

Figure 4.1: $\log_{10}(L^\infty - errors)$ *of* $\mu_{N,M}(y_1, y_2)$ *for* $N = 5$ *versus* M *for Example 4.14*

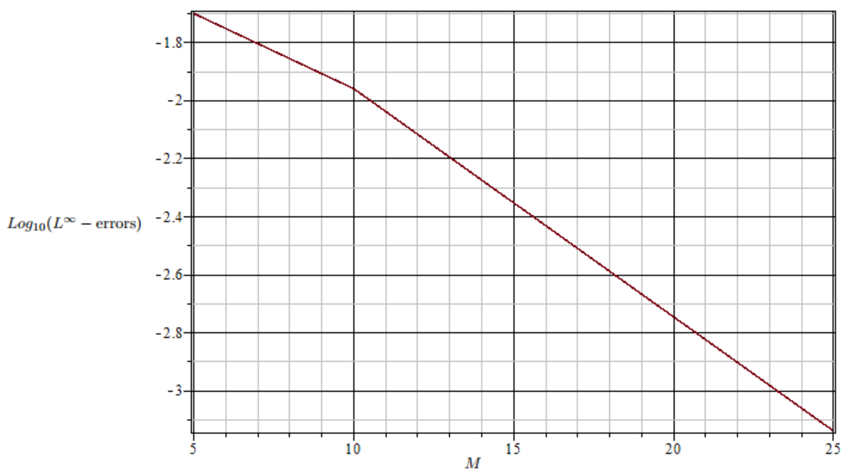


Figure 4.2: *Comparison of the exact and approximate solutions for $N = M = 16$ at $y_1 = 0.2$ for Example 4.14*

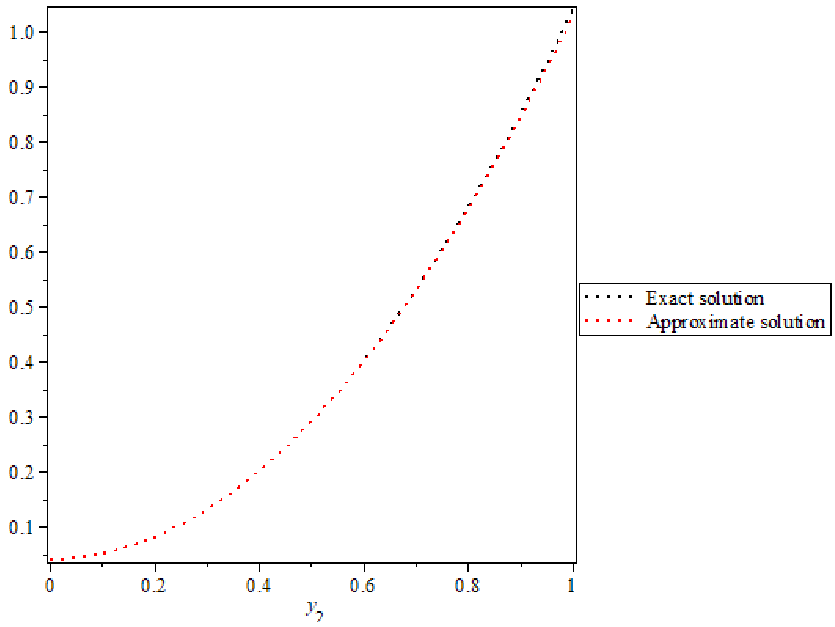
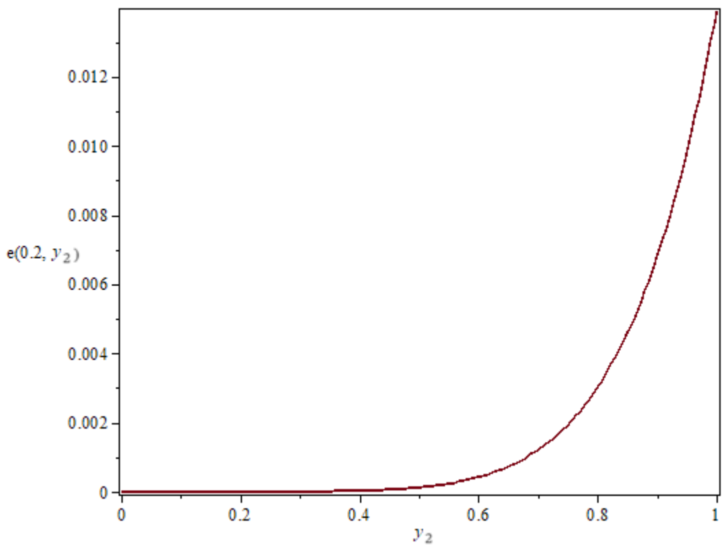


Figure 4.3: *Absolute error function for $N = M = 16$ at $y_1 = 0.2$ for Example 4.14*



Example 4.15. *Consider the 2D-NPVIDE [80] by*

$$\frac{\partial^3\mu(y_1,y_2)}{\partial y_1\partial y_2^2} = \hbar(y_1,y_2) + \int_0^{y_2}\int_0^{y_1} y_1\frac{\partial^3\mu(v_1,v_2)}{\partial y_1\partial^2 y_2} + y_2^3\left(\frac{\partial^3\mu(v_1,v_2)}{\partial y_1\partial^2 y_2}\right)^5 dv_2dv_1, \quad 0\le y_1,y_2\le 1,$$

with the initial conditions

$$\mu(y_1,0) = \frac{\partial\mu(y_1,0)}{\partial y_1} = \left.\frac{\partial^2\mu(y_1,y_2)}{\partial y_1\partial y_2}\right|_{y_2=0} = 0,$$

where $\hbar(y_1,y_2) = y_1^2y_2^2 - \frac{y_1^4y_2^3}{9} - \frac{y_1^{11}y_2^{14}}{121}$*, and the exact solution is* $\mu(y_1,y_2) = \frac{1}{36}y_1^3y_2^4$.

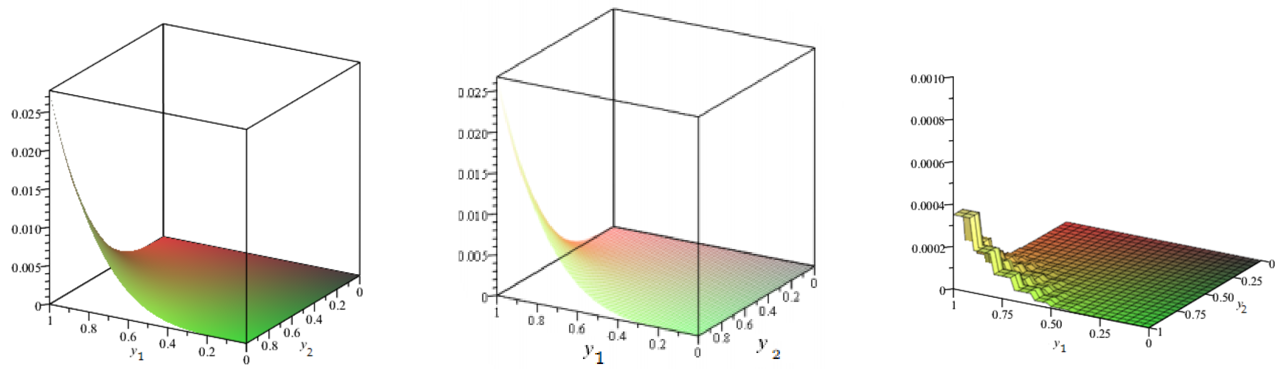
Babaaghaie and Maleknejad [80] considered the current problem and used the Haar wavelets method (HWM) with the mesh nodes $y_{1,N} = \dfrac{N-0.5}{2N}, N = 1,2,\ldots,2N,\ y_{2,M} = \dfrac{M-0.5}{2M}, M = 1,2,\ldots,2M,$ *to get its numerical solution.*

Table 4.3 compares the maximum absolute errors $E(N,M)$ *with* $p = 4$ *against these results introduced using the HWM [80]. Figure 4.4 obtains the exact solution* $\mu(y_1,y_2)$*, approximate solution* $\mu_{N,M}(y_1,y_2)$ *and the absolute error function* $|\mu(y_1,y_2) - \mu_{N,M}(y_1,y_2)|$ *with* $(N,M) = (16,16)$ *and* $p = 4$.

Table 4.3: *Comparing the* $E(N,M)$ *versus the HWM [80] of Example 4.15*

HWM [80]		Our method	
$2N$	$E(2N,2N)$	N	$E(N,N)$
2	$7.1e-03$	1	$1.5e-03$
4	$2.9e-03$	2	$6.2e-04$
8	$9.8e-04$	4	$7.1e-04$

Figure 4.4: *(a) Exact solution* $\mu(y_1,y_2)$*, (b) Approximate solution* $\mu_{16,16}(y_1,y_2)$*, (c) Absolute error function* $|\mu(y_1,y_2) - \mu_{16,16}(y_1,y_2)|$ *with* $p = 4$ *for Example 4.15*



(a) **(b)** **(c)**

Figure 4.5: *Comparison of the exact and approximate solutions for* $N = M = 16$ *at* $y_1 = 1$ *for Example 4.15*

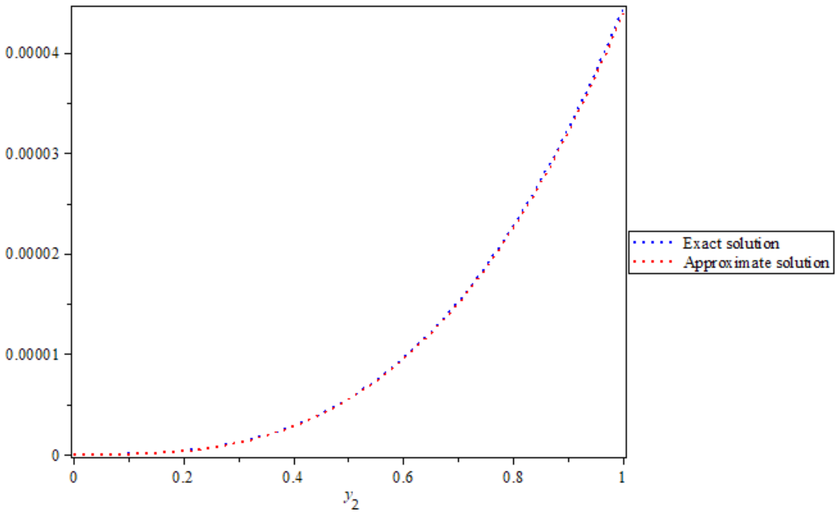
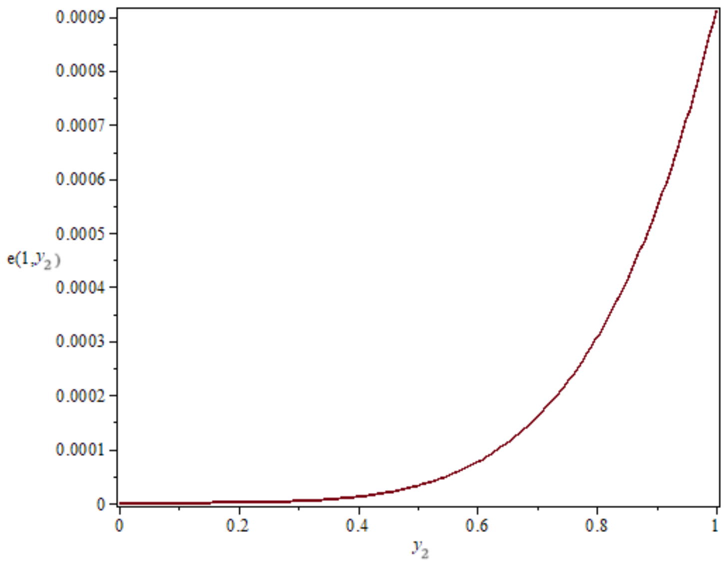


Figure 4.6: *Absolute error function for* $N = M = 16$ *at* $y_1 = 1$ *for Example 4.15*



Example 4.16. *Consider the 2D-PVIDE*

$$\frac{\partial^2\mu(y_1,y_2)}{\partial y_1\partial y_2} = \cos(y_2) - \frac{1}{2}(y_1+\cos(y_2))\,y_1\sin(y_2) - \frac{y_1y_2}{2} + \int_0^{y_1}\int_0^{y_2}(v_1+\cos(v_2))\frac{\partial^2\mu(v_1,v_2)}{\partial y_1\partial y_2}dv_2dv_1,$$

with the initial conditions

$$\mu(0,y_2) = \frac{\partial\mu(y_1,0)}{\partial y_1} = 0,$$

where $0 \le y_1 \le 1,\ 0 \le y_2 \le 1$, *and the exact solution is* $\mu(y_1,y_2) = y_1\sin(y_2)$.

Table 4.4 provides a detailed account of the absolute errors $|\mu(y_1,y_2) - \mu_{N,N}(y_1,y_2)|$ *at* $p = 3$ *with* $(N,M) = \{(4,4),(8,8),(16,16),(32,32)\}$, *while Table 4.5 presents the maximum absolute errors* $E(N,M)$ *with* $p = 3$ *and different choices of* N *and* M, *and obtains the convergence order of the solution. On the other hand, Figure 4.7 provides a visual representation of the contrast between the precise and estimated solutions at* $y_1 = 0.2$. *Figure 4.8 offers a graphical depiction of the absolute error function for* $N = M = 16$ *with* $p = 3$ *at* $y_1 = 1$ *for Example 4.16. Finally, Figure 4.9 exhibits the logarithmic graph of* $\log_{10}(L^\infty - errors)$ *with* $N = 5$ *for diverse values of* M.

Table 4.4: *Absolute errors* $|\mu(y_1,y_2) - \mu_{N,N}(y_1,y_2)|$ *for Example 4.16*

(y_1,y_2)	$N=4$	$N=8$	$N=16$	$N=32$
$(0.00,0.00)$	0	0	0	0
$(0.25,0.25)$	$2.24e-06$	$5.59e-06$	$1.06e-06$	$1.56e-07$
$(0.50,0.50)$	$1.86e-04$	$3.69e-05$	$5.48e-06$	$7.41e-07$
$(0.75,0.75)$	$6.03e-04$	$1.01e-04$	$1.44e-05$	$1.91e-06$
$(1.00,1.00)$	$9.71e-04$	$1.55e-04$	$2.16e-05$	$2.82e-06$

Table 4.5: *Maximum absolute error* $E(N,M)$ *with* $N = M$ *and* $p = 3$ *for Example 4.16*

N	$E(N,M)$	Rate
2	0.0042218339	
4	0.0009709566	2.1204
8	0.0001547186	2.6497
16	0.0000216087	2.8399
32	0.0000028152	2.9403

Figure 4.7: *Comparison of the exact and approximate solutions with $N = M = 16$ at $y_1 = 1$ for Example 4.16*

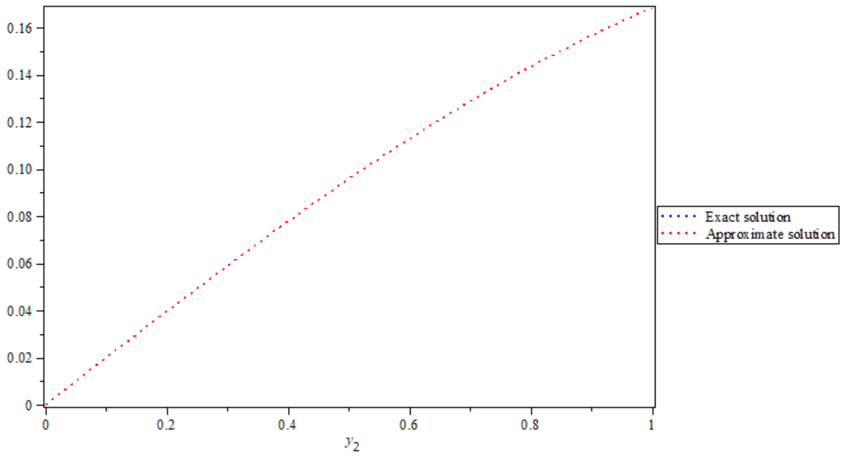


Figure 4.8: *Absolute error function for $N = M = 16$ at $y_1 = 1$ for Example 4.16*

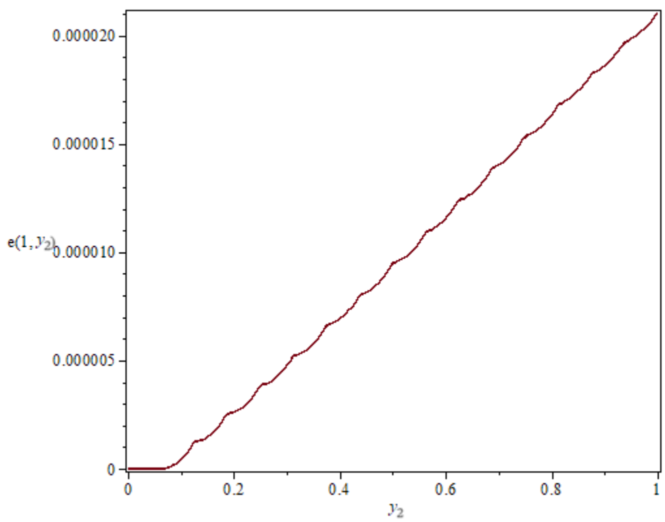
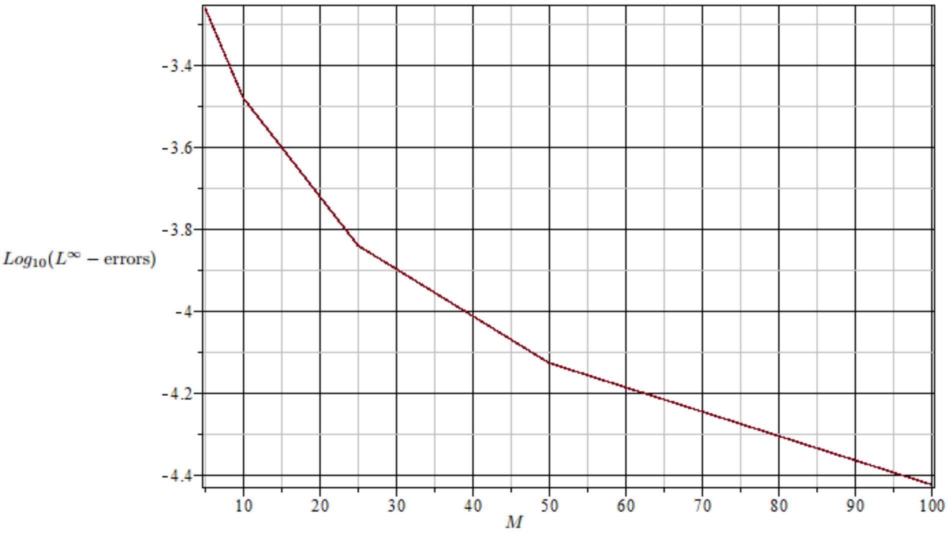


Figure 4.9: $\log_{10}(L^\infty - errors)$ *of* $\mu_{N,M}(y_1, y_2)$ *for* $N = 5$ *versus* M *for Example 4.16*



Example 4.17. *As the final example, we consider the following 2D-NPVIDE*

$$\frac{\partial^2 \mu(y_1, y_2)}{\partial y_1 \partial y_2} = \hbar(y_1, y_2) + \int_0^{y_1} \int_0^{y_2} (y_1 v_1^2 + \cos(v_2)) \left(\frac{\partial^2 \mu(v_1, v_2)}{\partial y_1 \partial y_2} \right)^2 dv_2 dv_1, \quad 0 \leq y_1, y_2 \leq 1,$$

subjected to the initial conditions

$$\mu(0, y_2) = \frac{\partial \mu(y_1, 0)}{\partial y_1} = 0,$$

with $\hbar(y_1, y_2) = y_1 \sin(y_2) + \frac{1}{10} y_1^6 \cos(y_2) \sin(y_2) - \frac{1}{10} y_1^6 y_2 - \frac{1}{9} \sin^3(y_2) y_1^3$, *and the exact solution is* $\mu(y_1, y_2) = \frac{-y_1^2}{2} (\cos(y_2) - 1)$.

Numerical results in terms of absolute errors are reported in Table 4.6. Table 4.7 displays the maximum absolute errors $E(N, M)$ *at* $p = 3$ *with various choices of* N *and* M, *and obtains the convergence order of the solution. Also, the absolute error function is depicted in three-dimensional space as shown in Figure 4.10 employing* $(N, M) = \{(4, 4), (16, 16)\}$. *Figure 4.17 plots the* $Log_{10}(L^\infty - errors)$ *of* $\mu_{N,M}(y_1, y_2)$ *for different values of* M *while maintaining* $N = 5$.

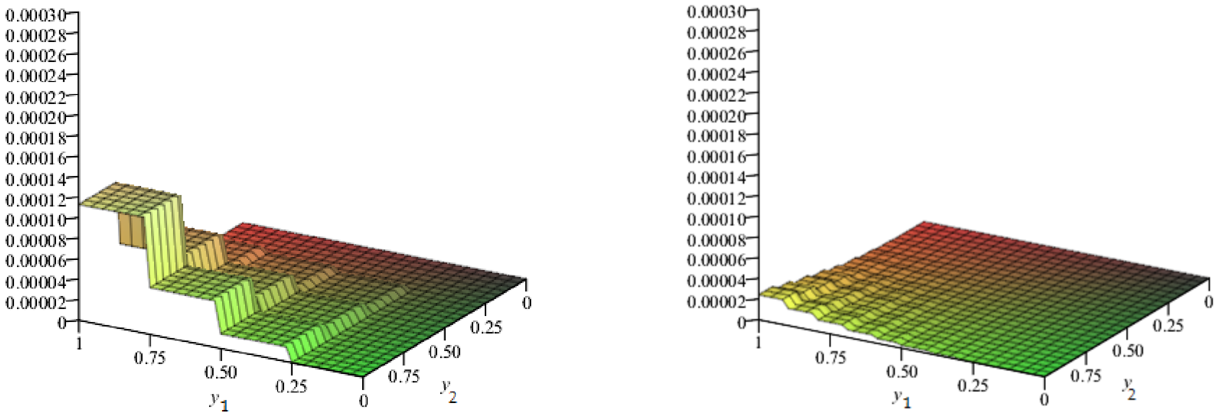
Table 4.6: *Absolute errors* $|\mu(y_1, y_2) - \mu_{N,N}(y_1, y_2)|$ *for Example 4.17*

(y_1, y_2)	$N = 4$	$N = 8$	$N = 16$
$(0.00, 0.00)$	0	0	0
$(0.25, 0.25)$	$3.327e-07$	$1.637e-08$	$2.309e-09$
$(0.50, 0.50)$	$5.312e-06$	$6.591e-07$	$9.297e-08$
$(0.75, 0.75)$	$7.059e-05$	$1.128e-05$	$1.591e-06$
$(1.00, 1.00)$	$4.649e-04$	$7.571e-05$	$1.068e-05$

Table 4.7: *Maximum absolute error* $E(N, M)$ *with* $N = M$ *and* $p = 3$ *for Example 4.17*

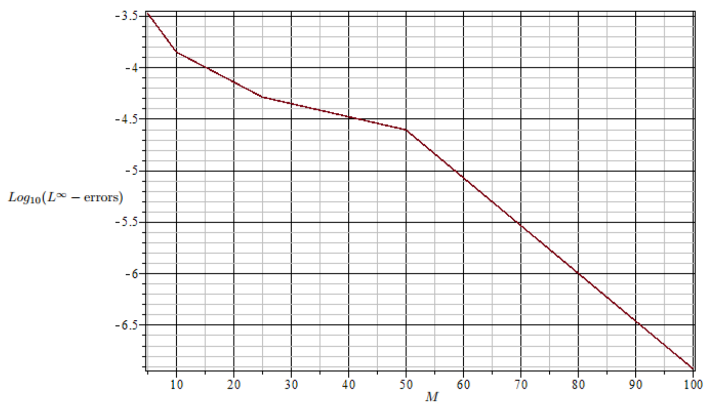
N	$E(N,M)$	Rate
2	0.0023618004	
4	0.0004648727	2.3449
8	0.0000757084	2.6183
16	0.0000106795	2.8256

Figure 4.10: *(a) Absolute error function* $|\mu(y_1, y_2) - \mu_{4,4}(y_1, y_2)|$*, (b) Absolute error function* $|\mu(y_1, y_2) - \mu_{16,16}(y_1, y_2)|$ *with* $p = 3$ *for Example 4.17*



(a) **(b)**

Figure 4.11: $\log_{10}(L^\infty - errors)$ *of* $\mu_{N,M}(y_1, y_2)$ *for* $N = 5$ *versus* M *for Example 4.17*



4.4 Concluding remarks

This chapter outlined a novel approach for approximating the solution of the two-dimensional high-order PVIDEs through the application of piecewise collocation approach based on the two-dimensional Taylor polynomials. The proposed method offered the advantage of obtaining approximate solutions directly through iterative formulas, eliminating the requirement of solving any algebraic system. As far as we know, it is the first attempt to solve two-dimensional high-order PVIDEs using two-dimensional Taylor polynomials. In addition, a new theorem was derived that ensures the convergence of the presented approach. The efficacy of the developed approach stands out as demonstrated through a comparative analysis of numerical results across various examples in existing literature.

Conclusion and perspectives

In this thesis, a new numerical approach utilizing two-dimensional Taylor polynomials to approximate solutions for linear and certain nonlinear Volterra 2D-PIDEs was developed. These equations encompass first, second, and arbitrary orders, each with its set of initial conditions within the real polynomial spline space $S^{(-1)}_{p-1,p-1}$. The employed Taylor collocation method exhibits high accuracy and convergence rates. Its primary advantage lies in providing approximate solutions directly through iterative formulas, eliminating the need to solve algebraic systems. This feature renders the numerical solution process for such problems straightforward and computationally efficient.

In each of the three studies, the derived approaches have been subjected to theoretical validation. The error analysis has confirmed their convergence to order p, where $p-1$ denotes the degree of the Taylor polynomials used in both directions. Moreover, the incorporation of various numerical examples serves to illustrate the method's convergence and validate the theoretical estimates. These numerical results were presented through different tables and figures, each highlighting that even with a small number of parameters, high accuracy is achievable. The computational effort required was minimal and executed on a personal computer. The obtained results attest to the practical reliability, speed, and ease of implementation of the proposed method.

Finally, further investigations into such problems will involve expanding the current numerical method utilized to approximate the second-order 2D-PVIDE of the form (3.1) to approximate solutions for similar and more specific equations. For example:

- Development of a novel approach to solve the general second-order linear 2D-PVIDE under appropriate initial conditions.
- Development of an innovative approach to solve the linear fractional 2D-PVIDE under appropriate initial conditions.
- Development of a pioneering approach to solve the linear weakly singular 2D-PVIDEs under appropriate initial conditions.
- Development of an innovative approach to solve the linear two-dimensional mixed Volterra-Fredholm PIDEs under appropriate initial conditions.

Some of these equations are currently under study, with further considerations for generalizing the use of this method.

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