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A GENERAL METHOD FOR HYBRID PROJECTIVE COMBINATION SYNCHRONIZATION OF A CLASS OF NONLINEAR FRACTIONAL-ORDER CHAOTIC SYSTEMS

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Abstract: In this paper, a new suitable adaptive controller is developed to realize a general method for hybrid projective combination synchronization (HPCS)of a class of fractional-order chaotic systems. Firstly, the definition of HPCS of fractional-order uncertain chaotic systems with external disturbance is introduced. Secondly, based on fractional Lyapunov's direct method an adaptive control and parameter adaptive laws are designed to achieve HPCS of uncertain chaotic systems. Finally, a numerical example is carried out to verify the effectiveness of the proposed methods.

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Key Words: hybrid combination synchronization, adaptive control, fractional calculus, chaotic system, Lyapunov stability

1. Introduction

Chaos synchronization of fractional-order chaotic systems is one of the most important from many researchers, due to its real applications in numerous research fields, such as: secure communication and control processing [6, 15]. Several synchronization techniques to ensure the asymptotic stability of error synchronization between some types of fractional-order chaotic systems have been developed [7, 9, 10, 11, 13, 16]. Many powerful control methods have been

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reported to control or synchronize frational-order chaotic systems [2, 12, 15, 18].

Recently, observing the combination synchronization between two drive systems and one response system has been proposed by Luo et al. [14]. Some approaches have been discussed to achieve this kind of synchronization for some typical fractional-order chaotic systems [5, 8, 18].

Projective combination synchronization is characterized by the fact that the combination between two drive systems and one response system could be synchronized upto a scaling factor, whereas the HPCS can be considered as an extension of projective combination synchronization. In fact, this kind of synchronization enhances security in communication and chaotic encryptation schemes, because the signals of response system can be any proportional to the signals of the drive systems by adjusting the factors and it can be employed to extend binary digital to variety M-nary digital communications for achieving a fast communication. Thus HPCS has become a new subject of active research.

In this paper, a new suitable adaptive control is introduced to ensure the HPCS of a class of fractional-order uncertain chaotic systems with external disturbance.

The rest of the paper is organized as follows: The general method for HPCS is introduced in Section 2. In Section 3, an illustrative example is presented to verify the effectiveness of the analytical results. In the last Section 4, concluding comments are given.

2. General Method of HPCS

Consider two drive systems which are written by

$$D^p x = f(x) + F(x)\alpha, \tag{1}$$

$$D^p y = g(y) + G(y)\beta, \tag{2}$$

where D^p is the Caputo fractional differential operator, $p \in (0,1)$ is the fractional-order, $(x_1, x_2, ..., x_n)^T$, $y = (y_1, y_2, ..., y_n)^T \in \mathbb{R}^n$ are the state vectors of the drive systems, $f, g, F, G : \mathbb{R}^n \to \mathbb{R}^n$ are all continuous vector functions, $\alpha = (\alpha_1, \alpha_2, ..., \alpha_m)^T \in \mathbb{R}^m$, $\beta = (\beta_1, \beta_2, ..., \beta_s)^T \in \mathbb{R}^s$ are the uncertain parameters of the drive systems (1) and (2), respectively.

The response system with controller $u = (u_1, u_2, ..., u_n)^T \in \mathbb{R}^n$ is also given by

$$D^p z = h(z) + H(z)\gamma + E(t) + u, \tag{3}$$

where $z = (z_1, z_2, ..., z_n)^T \in \mathbb{R}^n$ is the state vector of the response system, $h, H \in \mathbb{R}^n \to \mathbb{R}^n$ are all continuous vector functions, $\gamma = (\gamma_1, \gamma_2, ..., \gamma_r)^T \in \mathbb{R}^r$

is the uncertain parameters of the response system and $E(t)$ is the external disturbance of the system (3), such that

$$\{\|E(t)\| \leq c, \text{ for all } t,$$

where c is positive constant.

Now, we give the definition of HPCS between the drive systems (1), (2) and the response system (3).

Definition 1. The two drive systems (1), (2) and response system (3) are said HPCS, if there exist a suitable adaptive controller u and a nonzero constant matrix $P = diag\,(p_1, p_2, ..., p_n)$, such that

$$\lim_{t\to+\infty} \|(z - P\,(x + y))\,(t)\| = 0,$$

i.e.

$$\lim_{t\to+\infty} |(z_i - p_i\,(x_i + y_i))\,(t)| = 0, \text{ for all } \; i = 1, 2, ..., n,$$

where $p_1, p_2,...,p_n$ are different scaling factors for HPCS.

Let us define the state error vector as

$$e = z - P\,(x + y)\,. \tag{4}$$

The error dynamical system between combination of systems (1), (2) and (3) can be written as

$$D^p e = h(z) - P\,(f(x) + g(y)) + H(z)\gamma - P\,(F(x)\alpha + G(y)\beta) + E(t) + u. \tag{5}$$

The target is now to find a controller u such that the error system (5) approaches zero, wich means HPCS of systems (1), (2) and (3).

In this section, an adaptive mode controller is introduced to ensure this goal.

Choosing the control function u as follows

$$\begin{aligned} u &= P\,(f(x) + g(y)) - h(z) + P\left(F(x)\hat{\alpha} + G(y)\hat{\beta}\right) \\ &- H(z)\hat{\gamma} - \left(k + \frac{c}{\|e\|}\right) e, \end{aligned} \tag{6}$$

where the control amplitude $k = diag(k_1, k_2, ..., k_n)$ is positive constant matrix. The update laws of unknown parameters are given by

$$D^p\hat{\alpha} = -P\,[F(x)]^T\, e, \; D^p\hat{\beta} = -P\,[G(x)]^T\, e, \; D^p\gamma = [H(x)]^T\, e, \tag{7}$$

where $\hat{\alpha}$, $\hat{\beta}$ and $\hat{\gamma}$ are estimations of the unknown parameters α, β and γ, respectively.

We have the following result.

Theorem 2. *The drive systems (1), (2) and the response system (3) are globally and asymptotically hybrid projective combination synchronized in the presence of nonlinear adaptive controller (6) and update law (7).*

Before giving the proof of our main result, we need to recall the following two classical results.

Theorem 3. ([1]) *Consider the nonlinear equation*

$$D^p x = f(x), \tag{8}$$

where D^p is Caputo's differential operator ($0 < p \leq 1$), x is the state variable and f is continuous function.

If the constructed Lyapunov function V satisfies: $V(x) \geq 0$ and $D^p V(x) < 0$, Equ. (8) is asymptotically stable.

Lemma 1. ([1]) *Consider a differentiable function $g \in \mathbb{R}^n$; then, for all $0 < p \leq 1$ and $t > t_0$*

$$\frac{1}{2} D^p g(x(t))^T g(x(t)) \leq g(x(t))^T D^p g(x(t)).$$

Proof of Theorem 2.

Construct a candidate Lyapunov function

$$V = \frac{1}{2}\left(e^2 + e_\alpha^2 + \ e_\beta^2 + \ e_\gamma^2\right),$$

which is a positive definite function.

By applying the controller (6) to (5), the error dynamics can be written as

$$D^p e = -P\left(F(x)e_\alpha + G(y)e_\beta\right) + H(z)e_\gamma + E(t) - \left(k + \frac{c}{\|e\|}\right)e, \tag{9}$$

where

$$e_\alpha = \alpha - \hat{\alpha},\ e_\beta = \beta - \hat{\beta} \text{ and } \ e_\gamma = \gamma - \hat{\gamma}.$$

By using Lemma 1, we have

$$D^p V \ \leq \ e^T\left(-P\left(F(x)e_\alpha + G(y)e_\beta\right) + H(z)e_\gamma\right)$$

$$
\begin{aligned}
&+ \quad e^T \left(E(t) - \left(k + \frac{c}{\|e\|} \right) e \right) - e_\alpha^T \left[-PF(x) \right]^T e \\
&- \quad e_\beta^T \left[-PG(y) \right]^T e + e_\gamma^T \left[H(z) \right]^T e \\
&\leq \quad -e^T k e < 0, \text{ for all } e \neq 0.
\end{aligned}
$$

In view of Theorem 3, it can be concluded that the system (9) asymptotically converges to zero, which means that the drive systems (1)-(2) achieve HPCS with the response systems (3). This completes the proof.

3. Numerical Simulation

In this section, we evaluate the performance of our proposed approach, by applying the method on two fractional-order financial systems and one modified coupled dynamos system.

The following fractional-order financial systems [3] are considered as the drive systems, which are expressed as

$$
\begin{cases}
D^p x_1 = x_3 - a x_1 + x_1 x_2, \\
D^p x_2 = 1 - b x_2 - x_1^2, \\
D^p x_3 = -x_1 - c x_3,
\end{cases} \tag{10}
$$

and

$$
\begin{cases}
D^p y_1 = y_3 - a y_1 + y_1 y_2, \\
D^p y_2 = 1 - b y_2 - y_1^2, \\
D^p y_3 = -y_1 - c y_3.
\end{cases} \tag{11}
$$

We take the modified coupled dynamos system [17] as the response system, wich is given by

$$
\begin{cases}
D^p z_1 = -\alpha z_1 + z_2 \left(z_3 + \beta \right) + E_1(t) + u_1, \\
D^p z_2 = -\alpha z_2 + z_1 \left(z_3 - \beta \right) + E_2(t) + u_2, \\
D^p z_3 = -z_1 z_2 + z_3 + E_3(t) + u_3,
\end{cases} \tag{12}
$$

where $E_1(t)$, $E_2(t)$ and $E_3(t)$ are the bounded external disturbance of the response system (12); i.e. there exist three positive constants $c_i > 0$, $i = 1.2, 3$, such that

$$
\begin{cases}
\|E_1(t)\| \leq c_1, \text{ for all } t \\
\|E_2(t)\| \leq c_2, \text{ for all } t \\
\|E_3(t)\| \leq c_3, \text{ for all } t
\end{cases}.
$$

Now, we define the synchronization errors as

$$\begin{cases} e_1 = z_1 - p_1 (x_1 + y_1), \\ e_2 = z_2 - p_2 (x_2 + y_2), \\ e_3 = z_3 - p_3 (x_3 + y_3), \end{cases}$$

where p_1, p_2 and p_3 are different scaling factors for HPCS.

The error estimate parameter systems are defined by

$$e_a = a - \hat{a}, e_b = b - \hat{b}, e_c = c - \hat{c}, e_\alpha = \alpha - \hat{\alpha} \text{ and } e_\beta = \beta - \hat{\beta},$$

where $\hat{a}$, $\hat{b}$, $\hat{c}$, $\hat{\alpha}$ and $\hat{\beta}$ are the estimate parameters of a, b, c, α, and β, respectively.

So, in view of Theorem 2, the suitable controllers are selected as

$$\begin{cases} u_1 = p_1 \left[(x_3 + y_3) - \hat{a} (x_1 + y_1) + x_1 x_2 + y_1 y_2\right] + \hat{\beta} z_1 - z_2 (z_3 + \hat{\gamma}) \\ - \left(k_1 + \dfrac{c_1}{|e_1|}\right) e_1, \\ u_2 = p_2 \left(2 - \hat{b} (x_2 + y_2) - x_1^2 - y_1^2\right) + \hat{\beta} z_2 - z_1 (z_3 - \hat{\gamma}) \\ - \left(k_2 + \dfrac{c_2}{|e_2|}\right) e_2, \\ u_3 = p_3 \left[- (x_1 + y_1) - \hat{c} (x_3 + y_3)\right] + z_1 z_2 - z_3 - \left(k_3 + \dfrac{c_3}{|e_3|}\right) e_3, \end{cases}$$

and the adaptive laws of parameters are selected as

$$\begin{cases} D^p \hat{a} = p_1 e_1 (x_1 + y_1), \ D^p \hat{b} = p_2 e_2 (x_2 + y_2) \\ D^p \hat{c} = p_3 e_3 (x_3 + y_3), \ D^p \hat{\alpha} = -e_1 z_1 - e_2 z_2, \\ D^p \hat{\beta} = e_1 z_2 - e_2 z_1. \end{cases}$$

The resulting error dynamics can be expressed as

$$\begin{cases} D^p e_1 = p_1 e_a (x_1 + y_1) - e_\alpha z_1 + e_\beta z_2 + E_1 - \left(k_1 + \dfrac{c_1}{|e_1|}\right) e_1, \\ D^p e_2 = p_2 e_b (x_2 + y_2) - e_\alpha z_2 - e_\beta z_1 + E_2 - \left(k_2 + \dfrac{c_2}{|e_2|}\right) e_2, \\ D^p e_3 = p_3 e_c (x_3 + y_3) + E_3 - \left(k_3 + \dfrac{c_3}{|e_3|}\right) e_3. \end{cases} \tag{13}$$

Now, define the following Lyapunov functional candidate

$$V = \frac{1}{2} \left(\sum_{i=1}^{3} e_i^2 + e_a^2 + e_b^2 + e_c^2 \ + \ e_\alpha^2 + e_\beta^2 \right).$$

By using Lemma 1, we have

$$\begin{aligned}
D^pV \leq\ & e_1\left(p_1e_a\left(x_1+y_1\right)-e_\alpha z_1+e_\beta z_2+E_1-\left(k_1+\frac{c_1}{|e_1|}\right)e_1\right)\\
&+e_2\left(p_2e_b\left(x_2+y_2\right)-e_\alpha z_2-e_\beta z_1+E_2-\left(k_2+\frac{c_2}{|e_2|}\right)e_2\right)\\
+\ & e_3\left(p_3e_c\left(x_3+y_3\right)+E_3-\left(k_3+\frac{c_3}{|e3|}\right)e_3\right)\\
&-\left(e_\alpha D^p\hat{a}++e_bD^p\hat{b}+e_cD^p\hat{c}\ +\ e_\alpha D^p\hat{\alpha}+e_\beta D^p\hat{\beta}\right)\\
\leq\ & -\sum_{i=1}^{3}k_ie_i^2+\sum_{i=1}^{3}\delta_i\left|e_i\right|+e_\alpha\left(p_1e_1\left(x_1+y_1\right)-D^p\hat{a}\right)\\
+\ & e_b\left(p_2e_2\left(x_2+y_2\right)-D^p\hat{b}\right)+e_c\left(p_3e_3\left(x_3+y_3\right)-D^p\hat{c}\ \right)\\
+\ & e_\alpha\left(-e_1z_1-e_2z_2-D^p\hat{\alpha}\right)+e_\beta\left(e_1z_2-e_2z_1-D^p\hat{\beta}\right)\\
-\ & \sum_{i=1}^{3}\delta_i\frac{e_i^2}{|e_i|}\leq-\sum_{i=1}^{3}k_ie_i^2<0.
\end{aligned}$$

In the numerical simulations, the Adams–Bashforth–Moulton method [18] is used to solve the systems. For this numerical simulation, we suppose that the initial conditions of the systems (10), (11) and (12) are employed, respectively as

$$x_1(0)=1,\ x_2(0)=2 \text{ and } x_3(0)=-3,$$

$$y_1(0)=2,\ y_2(0)=3 \text{ and } y_3(0)=-3,$$

$$z_1(0)=-3,\ z_2(0)=-4 \text{ and } z_3(0)=3.$$

The desired scaling factors for HPCS are selected by

$$p_1=-2,\ p_2=1 \text{ and }\ p_3=\frac{1}{2},$$

thus the initial conditions of the error system are takan as

$$e_1(0)=-9,\ e_2(0)=-9 \text{ and }\ e_3(0)=9.$$

The bounded external disturbance are selected as

$$\left(E_1(t),E_2(t),E_3(t)\right)^T=0.02\left(\cos(2t),\sin(3t)\sin(\pi t),\cos(4\pi t)\right)^T.$$

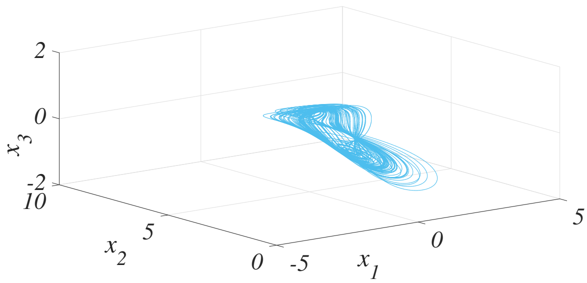

Figure 1: Chaotic attractor of the system (10).

The control inputs are selected as

$$k_1 = k_2 = k_3 = 0.5.$$

The parameters of systems (10), (11) and (12) are chosen, respectively as

$$a = 3,\ b = 0.1 \text{ and } c = 1, \alpha = 2 \text{ and } \gamma = 1.$$

The initial values of estimated parameters of system (10) and (11) are chosen as

$$\hat{a}(0) = 2,\ \hat{b}(0) = 0.2 \text{ and } \hat{c}(0) = 0.5.$$

The initial values of estimated parameters of system (12) are chosen as

$$\hat{\alpha}(0) = 1 \text{ and } \hat{\beta}(0) = 0.5.$$

The fractional order is taken as $p = 0.97$.

The financial chaotic system and the modified coupled dynamic system (12) (without bounded external disturbance $(E_1, E_2, E_3)(t)$ and controllers (u_1, u_2, u_3)) exhibit a chaotic behavior which are shown in Figures 1 and 2, respectively. Figure 3 shows the temporal evolution of synchronization error between the drive systems (10)-(11) and the response system (12). Figures 3 and 4 depict the time responses of the estimate parameters of the proposed systems. It is easy to show that the synchronization errors approach to the original values, which means the HPCS between the drive systems (10)-(11) and the response system (12) are achieved.

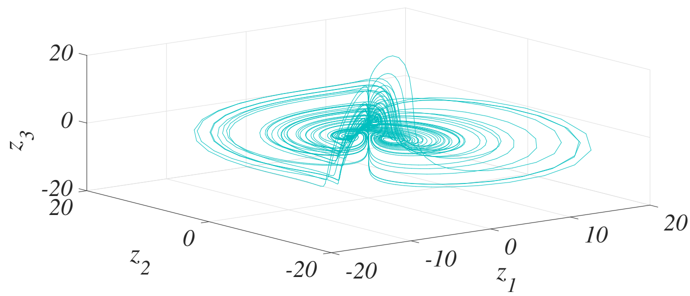

Figure 2: Chaotic attractor of the system (12).

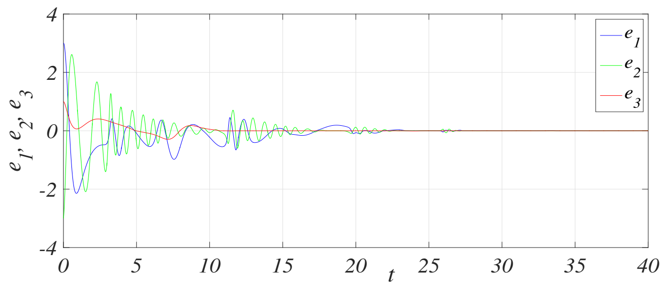

Figure 3: Temporal evolution of synchronization error system (13).

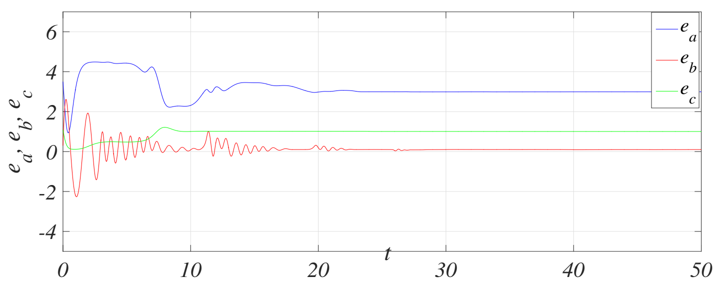

Figure 4: Temporal evolution of estimated parameters of systems (10) and 11).

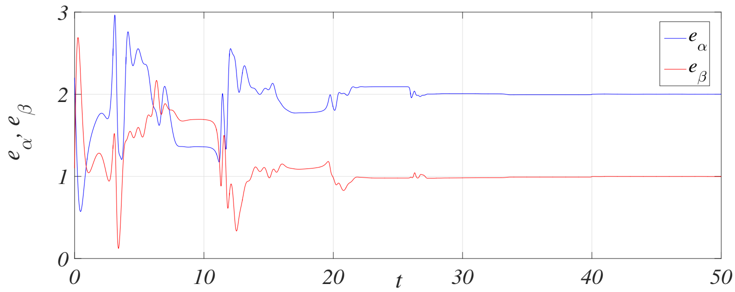

Figure 5: Temporal evolution of estimated parameters of system (12).

4. Conclusion

Based on the fractional-order extension of Lyapunov direct method, HPCS of a class of different fractional-order chaotic systems has been researched in this paper. For this reason, a new suitable adaptive control technique has been designed to realize HPCS of a class of three different fractional-order chaotic systems with unknown parameters and bounded noise. The proposed controller has simpler form and easier to implement compared with other existing controllers. Finally, the presented method has been applied to two fractional-order chaotic Financial systems and one fractional-order modified coupled dynamos system. Numerical simulations are also presented to verify the ability of the proposed method.

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