Aust. J. Math. Anal. Appl.
Vol. **20** (2023), No. 2, Art. 12, 9 pp.
AJMAA

A GENERAL FRACTIONAL CONTROL SCHEME FOR COMPOUND COMBINATION SYNCHRONIZATION BETWEEN DIFFERENT FRACTIONAL-ORDER IDENTICAL CHAOTIC SYSTEMS

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Received 17 May, 2023; accepted 14 November, 2023; published 22 December, 2023.

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ABSTRACT. In this paper, we aim to investigate the problem of compound combination synchronization (CCS) between four different fractional-order identical chaotic systems. Based on Laplace transformation and stability theory of linear dynamical systems, a new control law is proposed to assure the achievement of this kind of synchronization. Secondly, this control scheme is applied to realised CCS between four identical unified chaotic systems. Recall, that the proposed control scheme can be applied to wide classes of chaotic and hyperchaotic systems. Numerical simulations are given to show the effectiveness of the proposed method.

Key words and phrases: Compound combination synchronization; Chaotic system; Caputo fractional derivative; Stability theorem of linear system, Laplace transformation.

2010 *Mathematics Subject Classification.* 34A34, 37B25, 37B55, 93C55, 37C25.

ISSN (electronic): 1449-5910

This research was supported by the Algerian General Directorate for Scientific Research and Technological Development (DG-RSDT)..

1. INTRODUCTION

The use of fractional calculus theory provides an excellent tool for describing the memories and hereditary properties of various materials and processes [1, 2, 3, 4].

Recently, chaos synchronization of dynamical systems has become a hot topic from many researchers [5, 6, 7]. Until now, different synchronization types have been proposed between chaotic systems. For example, in [8], a complete synchronization have been described. in [9], an inverse matrix projective synchronization has been designed. In [10], a generalized synchronization has been presented. In [11], a modified projective synchronization has been studied. Also, $Q-S$ generalized synchronization has been designed in [12].

On the other hand, combination synchronizations between two drive systems and one response system are also receiving growing attention [13, 14, 15, 16, 17].

Meanwhile, in [18], J. Sun et all. have introduced a new CCS scheme which realized chaos synchronization by multiplication and addition of various chaotic systems, while the usual or combination synchronization scheme was realized by simple addition of two or three chaotic systems. Indeed, this kind of synchronization has stronger anti-attack ability than that of the usual or combination synchronization in the field of secure communication. As a result of the complexity of the compound system of three drive systems, many kinds of CCS schemes have been developed and investigated [19, 20, 21].

Motivated by the previous discussion, CCS promises to be a meaningful expansion in this way. In this paper, based on the stability results of linear system, we study the problem of CCS between four different fractional-order identical chaotic systems. The main aims of this work are summarized as follows:

Firstly, based on active mode controller and stability theorem of fractional-order linear system, a new approach for CCS between four different fractional-order chaotic system is presented.

Secondly, the theoretical results are verified by an illustrative example and numerical simulations. In particular, this kind of synchronization scheme is applied to the fractional-order unified chaotic systems.

The rest of this manuscript is organized as follows: In Section 2, some basic concepts about Caputo fractional derivative are briefly given. The general scheme of CCS is introduced in Section 3. In Section 4, an illustrative example and numerical simulations are presented to verify the effectiveness of the analytical results. Conclusions and perspectives are given in the last section.

2. PRELIMINARIES

Fractional calculus is an extension of integration and differentiation to non-integer-order fundamental operator D^p, which is defined by

$$D^p = \begin{cases} \dfrac{d^p}{(dt)^p}, \text{ si } p > 0, \\ 1, \text{ si } p = 0, \\ \displaystyle\int_0^t (ds)^{-p}, \text{ si } p < 0. \end{cases} \tag{2.1}$$

One of the commonly used definitions for the fractional-order differential operator is the Caputo definition [22], which is defined as

$$D^p f(t) = I^{m-p} D^m f(t), \tag{2.2}$$

where $m = \lceil p \rceil$ is the first integer greater then p, and I^p is the Riemann–Liouville fractional integral of order $p > 0$, which is given by

$$I^p f(t) = \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} f(s) ds, \tag{2.3}$$

where Γ is the gamma function.
The Laplace transform of a function $f(t)$ is the function $F(s)$ defined as follows

$$F(s) = L\{f(t), s\} = \int_0^{+\infty} \exp(-st) f(t) dt, \tag{2.4}$$

$f(t)$ is called original which can be reconstituted from the inverse Laplace transform

$$f(t) = L^{-1}\{F(s), t\} = \int_{c-i\infty}^{c+i\infty} \exp(st) F(s) ds, \quad c = \Re(s) > 0. \tag{2.5}$$

Taking into account that the Laplace transform of the convolution is

$$L\{f(t) * g(t), s\} = F(s).G(s), \tag{2.6}$$

where $f(t)$ and $g(t)$ are two causal functions for $t < 0$, $F(s)$ and $G(s)$ are their Laplace transforms.
Using the Laplace transform formula of the Riemann-Liouville integral, the Laplace transform of the Caputo fractional derivative is

$$L\{D^p f(t), s\} = s^p F(s) - \sum_{k=0}^{m-1} s^{p-k-1} f^{(k)}(0) \tag{2.7}$$

with $m - 1 \leq p < m$.
Particularly, when $p \in (0, 1]$, we have

$$L\{D^p f(t), s\} = s^p F(s) - s^{p-1} f(0) \tag{2.8}$$

3. GENERAL CONTROL SCHEME OF CCS

In this section, CCS between four different fractional-order chaotic systems is designed. The first system which is called the scaling drive system is given by

$$\dot{x} = f(x), \tag{3.1}$$

The second and the third systems which are called the base drive systems are given by

$$\dot{y} = g(y), \tag{3.2}$$

$$\dot{z} = h(z), \tag{3.3}$$

where $x = (x_1, x_2, ..., x_n)^T$, $y = (y_1, y_2, ..., y_n)^T$, $z = (z_1, z_2, ..., z_n)^T \in \mathbb{R}^n$ are the state vectors of the drive systems, $f(x) = (f_1(x), f_2(x), ..., f_n(x))^T$, $g(y) = (g_1(y), g_2(y), ..., g_n(y))^T$ and $h(z) = (h_1(z), h_2(z), ..., h_n(z))^T : \mathbb{R}^n \to \mathbb{R}^n$ are continuous vector functions.
The response system with a controller $u = (u_1, u_2, ..., u_n)^T \in \mathbb{R}^n$ is given by

$$D^p w = k(w) + u, \tag{3.4}$$

where p is a rational number between 0 and 1, D^p is the Caputo fractional derivative of order p, $w = (w_1, w_2, ..., w_n)^T \in \mathbb{R}^n$ is the state vector of the response system (3.4) and the vector $k = (k_1(w), k_2(w), ..., k_n(w))^T : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function.

Now, we give the definition of CCS between the drive systems (3.1)-(3.3) and the response system (3.4).

Definition 3.1. The three drive systems (3.1)-(3.3) and response system (3.4) are said to be CCS, if there exist a suitable controller u and diagonal matrices $A = diag\,(\alpha_1, \alpha_2, ..., \alpha_n)$, $B = diag\,(\beta_1, \beta_2, ..., \beta_n)\,, C = diag\,(\gamma_1, \gamma_2, ..., \gamma_n)$ and $D = diag\,(\theta_1, \theta_2, ..., \theta_n)$, such that

$$\lim_{t \to +\infty} \left\| \left(DW + AX\,(CZ - BY) \right)(t) \right\| = 0,$$

where $X = diag\,(x_1, x_2, ..., x_n)$, $Y = diag\,(y_1, y_2, ..., y_n)$, $Z = diag\,(z_1, z_2, ..., z_n)$ and $W = diag\,(w_1, w_2, ..., w_n)\,.$

Let us define the state error vector as

$$e = DW + AX\,(CZ - BY) \tag{3.5}$$

The error dynamical system (3.5) can be derived as

$$\begin{aligned} \dot{e} &= D\dot{W} + A\dot{X}\,(CZ - BY) + AX\left(C\dot{Z} - B\dot{Y}\right) \\ &= D\dot{W} + AF(x)\,(CZ - BY) + AX\,(CH(z) - BG(y))\,, \end{aligned} \tag{3.6}$$

where $F(x) = diag\,(f_1(x), f_2(x), ..., f_n(x))$, $G(y) = diag\,(g_1(y), g_2(y), ..., g_n(y))$ and $H(z) = diag\,(h_1(z), h_2(z), ..., h_n(z))$.

The objective of this work is now to find a controller u such that the error system (3.6) approaches zero, which means the combination of systems (3.1), (3.2) and system (3.4) achieve CCS. The synchronization criterion is obtained by controlling the linear part of the proposed system.

Here, a new control law is proposed to ensure this goal.

Choose the control function u as follows:

$$u = -k(w) + I^{1-p}\left[k(w) + v\right], \tag{3.7}$$

where the vector quantity v is given by

$$v = -D^{-1}\chi, \tag{3.8}$$

where D^{-1} is the inverse of matrix D and

$$\chi = -\,(P + M)\,e + DK(w) + AF(x)\,(CZ - BY) + AX\,(CH(z) - BG(y))\,, \tag{3.9}$$

where $K(w) = diag\,(k_1(w), k_2(w), ..., k_n(w))$, P is the linear part of the system studied and M is an unknown control matrix to be determined.
Hence, we have the following result.

Theorem 3.1. *The drive systems (3.1)- (3.3) will achieve CCS with the response system (3.4) under the nonlinear active controller (3.7) .*

Proof. By substituting the control law described by (3.7) into (3.4), we can rewrite the response system as follows:

$$D^p w = I^{1-p}\left[k(w) + v\right]. \tag{3.10}$$

Applying the Laplace transform (2.8) to (3.10) and letting $F(s) = L(w)$, we obtain

$$s^p F(s) - s^{p-1} w(0) = s^{p-1} L(k(w) + v). \tag{3.11}$$

Multiplying both the left-hand and right-hand sides of (3.11) by s^{1-p} and applying the inverse Laplace transform to the result, we get

$$\dot{w} = k(w) + v. \tag{3.12}$$

By substituting Equ. (3.12) into (3.6), we get

$$\dot{e} = (P + M)\, e + \chi + Dv. \tag{3.13}$$

where χ is defined by (3.8).
Hence, by substituting (3.9) into (3.13), we get

$$\dot{e} = (P + M)\, e. \tag{3.14}$$

In view of stability property of linear dynamical systems, if the feedback gain matrix M is selected such that all eigenvalues of $(P + M)$ are strictly negative, it can be concluded that the system (3.14) asymptotically converges to zero, which means that the drive systems (3.1)-(3.3) will reach CCS with the response systems (3.4). This completes the proof. ∎

4. RESULTS OF NUMERICAL SIMULATIONS

To validate the effectiveness of the synchronization method, three integer-order unified chaotic systems are taken as the drive systems and one fractional-order unified chaotic system is selected as the response system.

The integer-order unified chaotic systems (Lorenz, Chen and Lu systems) [23] can be described as

$$\begin{cases} \dot{x}_1 = (25\delta + 10)(x_2 - x_1), \\ \dot{x}_2 = (-35\delta + 28)x_1 - x_1x_3 + (29\delta - 1)x_2, \\ \dot{x}_3 = x_1x_2 - (\dfrac{\delta + 8}{3})x_3. \end{cases} \tag{4.1}$$

$$\begin{cases} \dot{y}_1 = (25\delta + 10)(y_2 - y_1), \\ \dot{y}_2 = (-35\delta + 28)y_1 - y_1y_3 + (29\delta - 1)y_2, \\ \dot{y}_3 = y_1y_2 - (\dfrac{\delta + 8}{3})y_3, \end{cases} \tag{4.2}$$

and

$$\begin{cases} \dot{z}_1 = (25\delta + 10)(z_2 - z_1) \\ \dot{y}_2 = (-35\delta + 28)z_1 - z_1z_3 + (29\delta - 1)z_2, \\ \dot{y}_3 = z_1z_2 - (\dfrac{\delta + 8}{3})z_3. \end{cases} \tag{4.3}$$

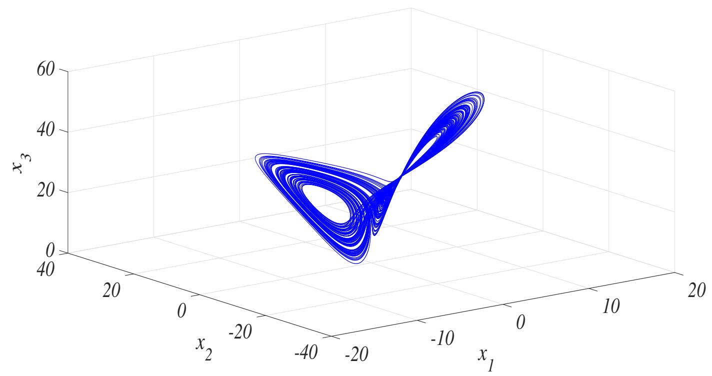

Figure 1: The chaotic attractor of the Lorenz system (4.1), when $p = 1$ and $\delta = 0$.

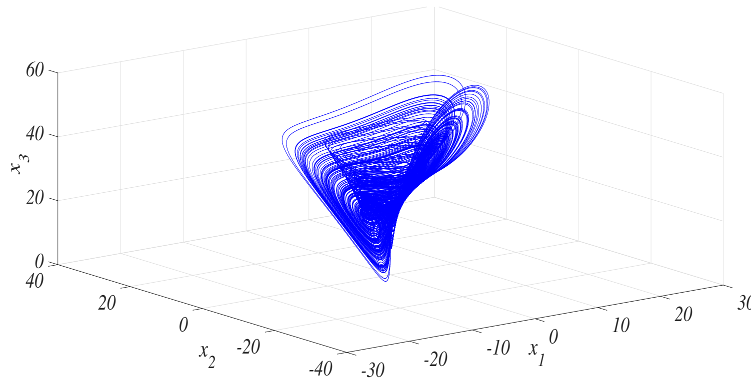

Figure 2: The chaotic attractor of the Lü system (4.1), when $p = 1$ and $\delta = 0.8$.

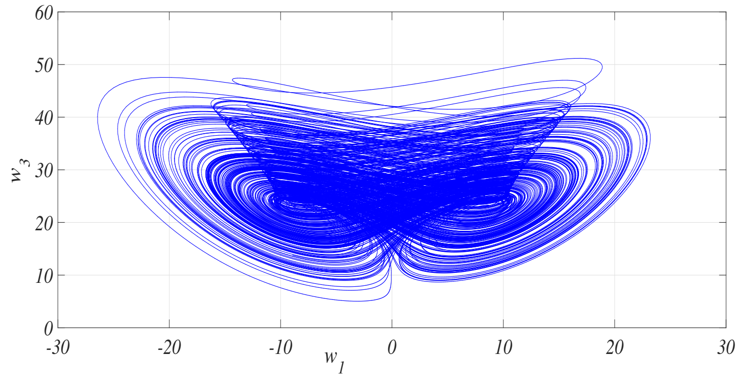

Figure 3: The chaotic attractor of the fractional-order Chen system (4.4), when $\delta = 1$ and $p = 0.98$.

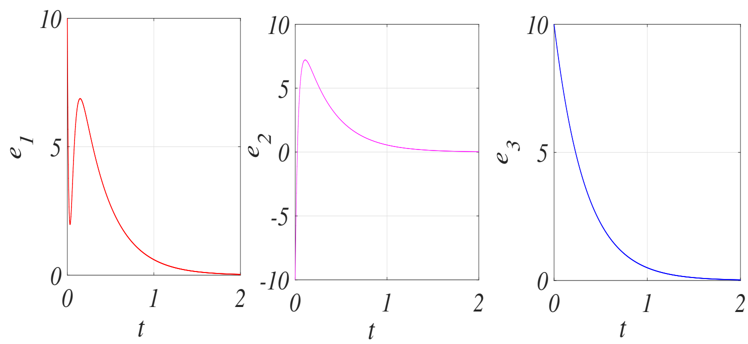

Figure 4: The temporal evolution of synchronization errors (4.5), when $\delta = 1$.

The controlled fractional-order unified chaotic system can be described by

$$(4.4)\qquad \begin{cases} D_1^p w_1 = (25\delta + 10)(w_2 - w_1) + u_1, \\ D_1^p w_2 = (-35\delta + 28)w_1 - w_1 w_3 + (29\delta - 1)w_2 + u_2, \\ D_1^p w_3 = w_1 w_2 - (\dfrac{\delta + 8}{3})w_3 + u_3, \end{cases}$$

where x_i, y_i , z_i and w_i ,$i = 1, 2, 3$ are the state variables of the drive and response systems, respectively, $\delta \in [0; 1]$ and u_1, u_2, u_3 are the active controllers to be designed.
Then the linear part of the proposed system is given by

$$P = \begin{pmatrix} -(25\delta+10) & (25\delta+10) & 0 \\ (-35\delta+28) & (29\delta-1) & 0 \\ 0 & 0 & -(\dfrac{\delta+8}{3}) \end{pmatrix}$$

The diagonal matrices A, B, C and D are selected as follows

$$A = diag\,(\alpha_1, \alpha_2, \alpha_3)$$

$$B = diag\,(\beta_1, \beta_2, \beta_3)$$
$$C = diag\,(\gamma_1, \gamma_2, \gamma_3)$$
$$D = diag\,(\theta_1, \theta_2, \theta_3)$$

According to the CCS control technique proposed in the previous section, we define the error states as

$$\begin{cases} e_1 = \theta_1 w_1 + \alpha_1 x_1 (\gamma_1 z_1 - \beta_1 y_1)\,, \\ e_2 = \theta_2 w_2 + \alpha_2 x_2 (\gamma_2 z_2 - \beta_2 y_2)\,, \\ e_3 = \theta_3 w_3 + \alpha_3 x_3 (\gamma_3 z_3 - \beta_3 y_3)\,. \end{cases}$$

To ensure the stability condition, we choose the feedback gain matrix M as

$$M = \begin{pmatrix} 0 & 0 & 0 \\ -(-35\delta+28) & -58\delta & (29\delta-1) \\ 0 & 0 & -(\dfrac{\delta+8}{3}) \end{pmatrix}$$

Then the resulting error dynamics can be expressed as

$$\begin{cases} \dot{e}_1 = -(25\delta+10)\,(e_1 - e_2)\,, \\ \dot{e}_2 = -(29\delta+1)\,(e_2 - e_3)\,, \\ \dot{e}_3 = -(\dfrac{\delta+8}{3})e_3. \end{cases} \tag{4.5}$$

The eigenvalues of the system are: $-(25\delta+10)$, $-(29\delta+1)$ and $-(\dfrac{\delta+8}{3})$, which are strictly negative. By using stability property of integer-order linear systems, all solutions of the error system (4.5) go to zero. Therefore, systems (4.1)-(4.4) are globally synchronized.
In the numerical simulation, we suppose that the initial conditions of the drive systems are employed, respectively as

$$x_1(0) = 0,\ x_2(0) = 1 \text{ and } x_3(0) = -5,$$
$$y_1(0) = 2,\ y_2(0) = 3 \text{ and } y_3(0) = -3,$$
$$z_1(0) = 3,\ z_2(0) = 3 \text{ and } z_3(0) = -3,$$

the initial conditions of the response system are taken as

$$w_1(0) = 5,\ w_2(0) = -5 \text{ and } w_3(0) = 5.$$

We take

$$\alpha_i = \beta_i = \gamma_i = \theta_i = 2, i = 1, 2, 3,$$

thus the initial conditions of the error system are taken as

$$e_1(0) = 10,\ e_2(0) = -10 \text{ and } e_3(0) = 10,$$

The attractors of integer-order unified chaotic system and the fractional-order unified chaotic system are shown in Figures 1, 2 and 3, respectively. Figure 4 shows the temporal evolution of synchronization errors between the drive systems (4.1)-(4.3) and the response system (4.4) with $\delta = 1$. Obviously, all solutions of the error system (4.5) approach to the origine values, which shows that the CCS between the proposed systems is realized.

5. CONCLUSIONS AND PERSPECTIVES

In this paper, the problem of CCS between different fractional-order chaotic systems has been introduced by using Laplace transformation and stability theory of linear dynamical system. A fractional control technique has been designed to robustly synchronize four chaotic systems. The synchronization criterion was obtained by controlling the linear part of the response system. Our proposed controller is very simple and could be easily realized experimentally. In addition, the present method is universal and applicable for wide classes of chaotic and hyperchaotic systems. Finally, one applies the presented method to the integer-order unified chaotic systems and fractional-order unified system. Numerical simulation results were used to verify the effectiveness of the proposed control scheme. Recall that this kind of synchronization has advantages stronger antidecode and antiattack ability than that of the other types of synchronization in secure communication, because the origin message can be divided into three segments and each segment can be separated into three distinct drive systems. This will be studied in the near future.

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