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INVERSE MATRIX PROJECTIVE SYNCHRONIZATION OF NOVEL HYPERCHAOTIC SYSTEM WITH HYPERBOLIC SINE FUNCTION NON-LINEARITY

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Abstract. In this paper, we investigate the inverse matrix projective synchronization (IMPS) of novel hyperchaotic system with hyperbolic sine function non-linearity. Recall that the studied system is generated from the modified Lü system. First, hyperchaotic attractors, symmetry, dissipation, equilibrium points and Lyapunov spectrum are the tools used to analyse this system. Moreover, this paper presents an active controller to achieve the IMPS analysis of the system. The main results are established by using Lyapunov stability theory, and finally numerical example and computer simulations are shown to illustrate all the main results.

Keywords. Hyperbolic sine function, Hyperchaotic Lü system, Active control, Lyapunov exponent, inverse matrix projective synchronization.

AMS subject classification: 34A34, 37B25, 37B55, 93C55, 37C25.

1 Introduction

Grace to the natural properties of chaotic and hyperchaotic systems, such as: sensitivities of initial conditions, boundedness and infinite recurrence, hyperchaotic systems have become good condidat for important applications in several areas such as: cryptosystems, secure communications, network signal transmission, electrical circuits and encryption [8, 12, 13, 23, 26], etc.

Chaos synchronization plays an important role in nonlinear science and must consider several aspects, such as physical systems [11], ecological systems [3] and biological systems [10], etc. For this, many different methods have been used to study the synchronization and stability of general uncertain systems, such as impulsive control, adaptive control, Adaptive fuzzy control, active control, prediction-based feedback control, sliding mode control [2, 5, 6, 17, 21, 22, 24] and so on. Different types of synchronization

have been proposed, such as complete synchronization, projective synchronization, co-existence of some chaos synchronization types, generalized synchronization, inverse generalized synchronization and hybrid synchronization [1, 4, 14, 16, 18, 19, 20].
On the other hand, most of the existing papers discus the synchronization between two chaotic or hyperchaotic systems with quadratic or cubic non-linearity. Recently, there has been increasing attention to the synchronization of hyperchaotic systems which contain a non-linear term in the form of usual functions with different characteristics and behaviour. For example, in [9] an anti synchronisation of novel hyperchaotic system with its exponential non-linearity function has been investigated.
Also, the concepts of hyperbolic functions are analogues of the usual functions. For example, hyperbolic functions occur in the solutions of many linear differential equations such as the Laplace equations, which have attracted a great deal of interest in many areas of physics, including fluid dynamics, and special relativity. Owing to the previous discussion, a novel hyperchaotic system with hyperbolic sine function non-linearity generated from the modified Lü system [7] is investigated. In addition, the IMPS analysis of the system is achieved using active control method and Lyapunov stability theory.
The remainder of this paper is structured as follows: The next Section states the novel hyperchaotic system and its dynamic analysis. The Section 3 derives an active controller for IMPS of the identical systems. Finally, the Section 4 contains the conclusion.

2 A novel hyperchaotic system and its hyperchaotic attractors

Our novel hyperchaotic system generated from the modified Lü system [7] is given as

$$\begin{cases} \dot{x_1} = b_1(x_2 - x_1) + x_4, \\ \dot{x_2} = -x_1x_3 + b_2x_2, \\ \dot{x_3} = \sinh(x_1x_2) - b_3x_3, \\ \dot{x_4} = b_4x_1, \end{cases} \tag{1}$$

where $x_1,\ x_2,\ x_3,\ x_4$ are the state variables, $b_1,\ b_2,\ b_3$ and b_4 are positive real. In addition, we shall schow that the system (1) is hyperchaotic when the parameters $b_1,\ b_2,\ b_3$ and b_4 take the values

$$(b_1,\ b_2,\ b_3,\ b_4) = (25,\ 20,\ 1.5,\ 7). \tag{2}$$

We recall that we can also find other values of the previous parameters to justify the existence of hperchaos in the propsed system by using the bufurcation diagram and also by the Wolf algorithm [25].
All numerical simulations are performed by using the fourth order Runge-Kutta algorithm. For this numerical simulations, we take the initial values

of the novel hyperchaotic (1) as

$$x_1(0) = -0.0546,\ x_2(0) = -0.0049,\ x_3(0) = 0.0427 \\ \text{and } x_4(0) = 0.9094. \tag{3}$$

The attractors of the system (1) are represented in Figure 1.

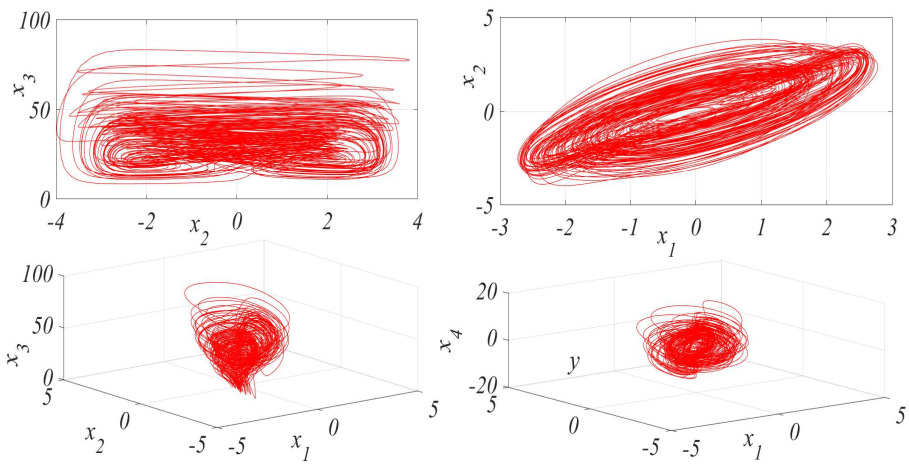

Figure 1: The attractors of the hyperchaotic system (1).

The proposed system has the following characteristics:

2.1 Symmetry

The system (1) remains invariant under the function: $(x_1,\ x_2,\ x_3,\ x_4) \rightarrow (-x_1,\ -x_2,\ x_3,\ -x_4)$. In addition, this system is symmetric with respect to the x_3 - axis.

2.2 Dissipation

The divergence of the novel $4 - D$ system (1) is

$$\nabla f = -b_1 + b_2 - b_3. \tag{4}$$

By using Liouville's theorem, we obtain

$$V(t) = V_0 \exp((-b_1 + b_2 - b_3)t) \tag{5}$$

For the system (1) to be dissipative, it is enough that $\nabla f < 0$. in this case, all the trajectories of the system tend to an attractor when $t \rightarrow +\infty$.

2.3 Equilibrium points and stability

Suppose that the system (1) has $P(x_1,\ x_2,\ x_3,\ x_4)$ as an equilibrium point, thus

$$\begin{cases} b_1(x_2 - x_1) + x_4 = 0, \\ -x_1x_3 + b_2x_2 = 0, \\ \sinh(x_1x_2) - b_3x_3 = 0, \\ b_4x_1 = 0, \end{cases} \tag{6}$$

As a result, the system (1) has only one equilibrium point $P(0, 0, 0, 0)$.
The Jacobian matrix in P is

$$\begin{pmatrix} -b_1 & b_1 & 0 & 1 \\ 0 & b_2 & 0 & 0 \\ 0 & 0 & -b_3 & 0 \\ b_4 & 0 & 0 & 0 \end{pmatrix}, \tag{7}$$

and its characteristic polynomial is

$$P(\lambda) = (\lambda - b_2)(\lambda + b_3)\left(\lambda^2 + \lambda b_1 - b_4\right). \tag{8}$$

The eigenvalues corresponding to P are

$$\lambda_1 = -b_3,\ \lambda_2 = -\frac{1}{2}b_1 - \frac{1}{2}\sqrt{4b_4 + b_1^2},\ \lambda_3 = b_2,$$
$$\text{and } \lambda_4 = -\frac{1}{2}b_1 + \frac{1}{2}\sqrt{4b_4 + b_1^2}. \tag{9}$$

Here, λ_1 and λ_2 are negative real numbers, λ_3 and λ_4 are the positive real numbers. Then the equilibrium P is an unstable saddle point.

2.4 Lyapunov exponents

Here, we assume that the parameters $b_1, b_2,\ b_4$ remain fixed and just b_3 is varied in $[0.5, 1.8]$. Using Wolf algorithm [25], the Lyapunov exponents spectrum of system (1) with $b_1 = 25, b_2 = 20$ and $b_4 = 7$ are represented in Figure 2.

In particular, for the parameter values as in (2), the values of Lyapunov exponents of non-linear system (1) are given by

$$L_1 = 1.968,\ L_2 = 0.1755,\ L_3 = -0.2914 \text{ and } L_4 = -8.367 \tag{10}$$

3 IMPS of the identical novel hyperchaotic systems

The control goal considered in this section is that the two identical hyperchaotic systems can be achieved the IMPS. The main IMPS result via active control method is proved by using Lyapunov stability theory.

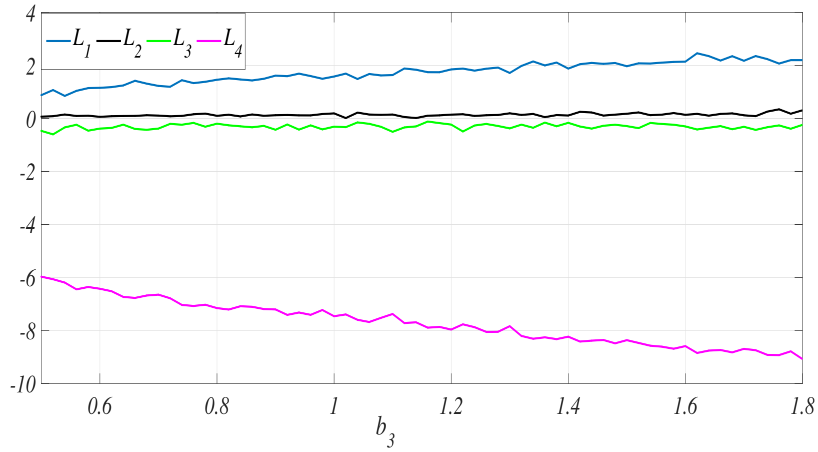

Figure 2: The Lyapunov exponents spectrum of system (1) versus b_3.

3.1 Theoretical analysis

Consider a master and a slave hyperchaotic systems, respectively, described as follows:

$$\dot{X}(t) = f(X(t)), \quad (11)$$
$$\dot{Y}(t) = AY(t) + g(Y(t)) + U, \quad (12)$$

where X, $Y \in \mathbb{R}^n$ are state variables of the master system and the slave system, respectively, $A \in \mathbb{R}^{n\times n}$, is the linear part of the systems (12), $f, g : \mathbb{R}^n \to \mathbb{R}^n$ are non linear functions and $U = (u)_{1\leq i\leq n}$ is a control input vector.

The problem of IMPS [15] for the systems (11) and (12) is to find the controller U such that the synchronization error,

$$e(t) = MY(t) - X(t), \quad (13)$$

satisfies

$$\lim_{t\to\infty} ||e(t)|| = 0, \quad (14)$$

where M is invertible scaling matrix. Hence, we have the following result.

Theorem 1 *The IMPS between the master system (11) and the slave system (12) can be achieved if the following conditions are satisfied*

(a) $U = M^{-1}((A - C)e(t) - MAY - Mg(Y(t)) + f(X(t))$, *where* C *is a control matrix.*

(b) $(A - C)^T + (A - C)$ *is a negative definite matrix.*

Proof. The error system can be derived as follow

$$\begin{aligned}\dot{e}(t) &= M\dot{Y}(t) - \dot{X}(t) \\ &= MAY(t) + Mg(Y(t)) + MU - \dot{X}(t) \quad (15)\end{aligned}$$

Substituting (a) into Eq. (15), the error system can be described as

$$\dot{e}(t) = (A - C)\, e(t). \tag{16}$$

Consider the quadratic Lyapunov function defined by

$$V(t) = e^T(t)e(t), \tag{17}$$

which is positive definite on $\mathbb{R}^n$.
Differentiating V along the trajectories of (11) and (12), we get

$$\begin{aligned}
\dot{V}(t) &= \dot{e}^T(t)e(t) + e^T(t)\dot{e}(t)\\
&= e^T(t)(A-C)^T e(t) + e^T(t)(A-C)e(t)\\
&= e^T(t)\left((A-C)^T + (A-C)\right)e(t)\\
&< 0.
\end{aligned}$$

Thus, we can conclude that all solutions of error system (13) tend towards zero exponentially as $t \to \infty$. Hence, the IMPS between the identical hyperchaotic systems (11) and (12) is achieved under the conditions (a) and (b). This completes the proof. ■

3.2 Numerical simulation

To verify the effectiveness and the feasibility of the presented synchronization method, we take the novel hyperchaotic system as a master system and its controlled system as a slave system. The master system is defined as

$$\begin{cases}
\dot{x}_1 = b_1(x_2 - x_1) + x_4,\\
\dot{x}_2 = -x_1x_2 + b_2x_2,\\
\dot{x}_3 = \sinh(x_1x_2) - b_3x_3,\\
\dot{x}_4 = b_4x_1.
\end{cases} \tag{18}$$

The slave system is described by

$$\begin{cases}
\dot{y}_1 = b_1(y_2 - y_1) + y_4 + u_1,\\
\dot{y}_2 = -y_1y_2 + b_2y_2 + u_2,\\
\dot{y}_3 = \sinh(y_1y_2) - b_3y_3 + u_3,\\
\dot{y}_4 = b_4y_1 + u_4,
\end{cases} \tag{19}$$

where u_1, u_2, u_3, u_4 are the active control functions. The linear part of the systems (18) and (19) is given by

$$A = \begin{pmatrix}
-b_1 & b_1 & 0 & 1\\
0 & b_2 & 0 & 0\\
0 & 0 & -b_3 & 0\\
b_4 & 0 & 0 & 0
\end{pmatrix}.$$

According to IMPS control technique proposed in the previous section, the matrix M and the gain matrix C are selected as

$$M = \begin{pmatrix} 2 & 0 & 0 & 2 \\ 2 & 2 & 0 & 2 \\ -1 & -1 & -1 & 0 \\ 0 & 0 & -1 & -1 \end{pmatrix}. \tag{20}$$

$$C = \begin{pmatrix} 0 & b_1 & 0 & 1 \\ 0 & 1+b_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ b_4 & 0 & 0 & 3 \end{pmatrix}. \tag{21}$$

From the condition (a) of the theorem 3.1 , the vector controller $u = (u_1, u_2, u_3, u_4)$ can be constructed as follows

$$\begin{cases} u_1 = \frac{1}{4}e_2 - \frac{3}{2}e_4 + x_4 - y_4 - \frac{1}{2}e_1b_1 + \frac{1}{2}e_3b_3 - b_1x_1 + b_1x_2 + b_1y_1 - b_1y_2, \\ u_2 = -\frac{1}{2}e_2 + \frac{1}{2}e_1b_1, \\ u_3 = \frac{1}{4}e_2 + \frac{3}{2}e_4 + \frac{1}{2}e_3b_3 - b_3x_3 + b_3y_3 + \sinh x_1x_2 - \sinh y_1y_2, \\ u_4 = -\frac{1}{4}e_2 + \frac{3}{2}e_4 - \frac{1}{2}e_3b_3 + b_4x_1 - b_4y_1. \end{cases} \tag{22}$$

With the choice control (22), the error system becomes

$$\begin{pmatrix} \dot{e}_1 \\ \dot{e}_2 \\ \dot{e}_3 \\ \dot{e}_4 \end{pmatrix} = \begin{pmatrix} -b_1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -b_3 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{pmatrix}. \tag{23}$$

For the numerical simulations, we take:
The parameter values of the master and slave systems as in the case (2).
The initial states of the master system are taken as

$$x_1(0) = 1, x_2(0) = 1, x_3(0) = 1 \text{ and } x_4(0)) = 1. \tag{24}$$

The initial states of the slave system are taken as

$$y_1(0) = 0.5, y_2(0) = 0.5, y_3(0) = 0.5 \text{ and } y_4(0) = 0.5. \tag{25}$$

.
We recall that the role of choice of the initial conditions is to justify the existence of hyperchaos and also for the uniqueness of the solutions of the proposed systems.
With the choice of the previous initial states, the error system has the initial states

$$e_1(0) = 1, e_2(0) = 2.5, e_3(0) = -0.5 \text{ and } e_4(0) = -2. \tag{26}$$

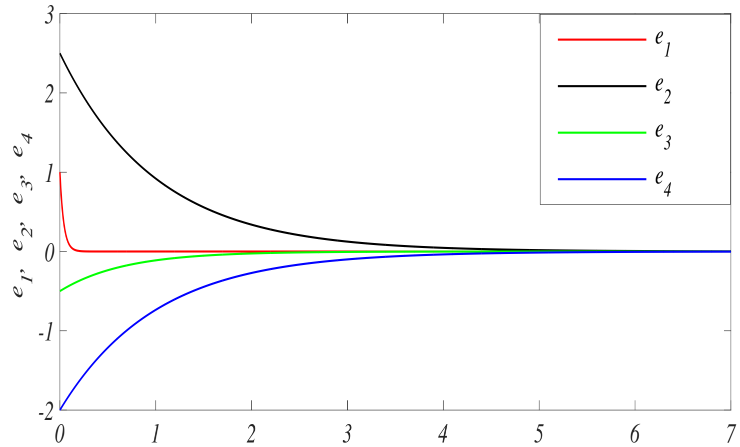

Figure 3: Time evolution of the synchronization errors (23).

The curves of synchronization error (23) are shown in Figure 3.

From Figure 3, we show that the evolution of all variables of error dynamic system (23) quickly tend towards zero as $t \to \infty$, which indicate that the IMPS between the hyperchaotic systems (18) and (19) is achieved.

4 Conclusion

In this research work, we have studied a novel hyperchaotic system with hyperbolic sine function non-linearity and their dynamic behaviours. In addition, an active controller has been proposed to ensure the possibility of IMPS of this system. Lyapunov's stability theorem has been used to provide asymptotic stability as well as convergence of synchronization errors towards zero. Finally numerical example and computer simulations are shown to illustrate all the main results of this paper.

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