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MODIFIED HYBRID SYNCHRONIZATION OF IDENTICAL FRACTIONAL HYPERCHAOTIC SYSTEMS WITH INCOMMENSURATE ORDER

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Abstract. In this research work, we introduce a new approach for hybrid synchronization called modified hybrid synchronization (MHS). Specifically, we focus on the MHS of identical fractional hyperchaotic systems with incommensurate order, which is a mixture between complete synchronization, anti-synchronization, projective synchronization and modified projective synchronization. To start, we propose a novel hyperchaotic system and also we analyse some its dynamic behaviors. In addition, we prove the MHS approach for identical fractional-order hyperchaotic systems by using a suitable nonlinear controller and stability theory of fractional-order systems. Finally, we take our fractional system as an example to confirm the effectiveness of the analytical results.

Keywords. Hybrid synchronization, Hyperchaotic system, Caputo fractional derivative, Lyapunov exponents, Active control.

AMS (MOS) subject classification: 34A34, 35B35, 37C25, 37N30

1 Introduction

Dynamics of fractional-order nonlinear systems have become the focus in scientific research. They are more suited and better approach than the usual dynamics of integer-order systems for the description of nonlinear phenomena's memory in many fields of science and technology, such as diffusion modeling [1], viscoelasticity [2], control processing [3], signal transmission [4] and so forth.

Chaos synchronization of fractional-order chaotic systems is a fundamental concept of dynamical system and has applications in several fields of science, such as secure communication [5]. Different synchronization types have been proposed for chaotic systems, such as complete synchronization [6], anti-synchronization [7], generalized synchronization [8], $Q-S$ synchronization [9], projective synchronization [10, 11], generalized projective synchronization [12], modified projective synchronization [13] and hybrid projective synchronization [14].

The phenomenon of hybrid synchronization is one of the most noticeable. In practical application, the property of hybrid synchronization is generated from the co-existence of complete synchronization and anti-synchronization.

Recently, Ouannas [15] consider a new approach of hybrid synchronization between hyperchaotic maps which a coexistance between projective synchronization, full state hybrid projective synchronization and generalized synchronization. Hence, realization of MHS for fractional-order hyperchaotic systems is an challenging work and lead to a dynamics richer than hybrid synchronization. However, this kind of synchronization have not been explored.

Motivated by the above reasons, in this work, we investigate the MHS for fractional-order hyperchaotic systems in continuous-time which is a mixture between complete synchronization, anti-synchronization, projective synchronization and modified projective synchronization. First, we construct a novel hyperchaotic system and we analyse its dynamic properties. Then, we prove the generalized MHS approach for identical fractional-order hyperchaotic systems by using an active control technique and stability theory of fractional-order systems. Finally, we use our novel fractional-order hyperchaotic system as an example to confirm the effectiveness of the analytical results.

This paper is structured as follows: In Sect. 2, some properties of fractional derivatives are introduced. A novel hyperchaotic system and its dynamic properties are studied in Sect. 3. According to the stability criterion of fractional-order linear system, an active control [16] is proposed in Sect. 4 to realize the MHS of identical fractional-order hyprchaotic systems. This synchronization scheme is applied on the fractional version of our novel hyperchaotic system. Finally, conclusions is given in Sect. 5.

2 Basic concepts of fractional derivatives

The fractional derivatives are a class of differential systems with non integer-order of derivatives in many different sense as Riemann-Liouville, Caputo, Grunwald-Letnikov [17] and so on. The Caputo fractional derivative is the smooth fractional derivative, it is defined as follows:

$$D^{\alpha} f(t) = J^{m-\alpha} f^{m}(t), \tag{1}$$

where $\alpha \in (0, m)$, m is the first integer which is not less than α, J^{β} $(\beta > 0)$ is the β-order Riemann– Liouville fractional derivative, with expression:

$$J^{\beta}\zeta(t) = \frac{1}{\Gamma(\beta)} \int_{0}^{t} (t-s)^{\beta-1} \zeta(s) ds, \tag{2}$$

and Γ is the gamma function defined by:

$$\Gamma(y) = \int_{0}^{+\infty} t^{y-1} \exp(-t) dt. \tag{3}$$

The following stability results play an important role in studying the existence of chaotic attractors and the synchronization of fractional order systems.

Theorem 1 *[18]For the linear incommensurate fractional-order system:*

$$D^{\alpha} x = Ax, x(0) = x_0, \tag{4}$$

where $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$ is the state vector, $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)^T$ is the fractional orders, with $0 < \alpha_i \leq 2, i = 1, 2, ..., n$. Assume that $\alpha_i = \frac{v_i}{u_i}$ with $(v_i, u_i) = 1, v_i, u_i \in \mathbb{Z}_+^*$, for $i = 1, 2, ..., n$ and let s be the least common multiple of the denominators u_i's of α_i's.
Define the following characteristic equation:

$$\det \left[diag(\lambda^{s\alpha_1}, \lambda^{s\alpha_2}, ..., \lambda^{s\alpha_n}) - A \right] = 0, \tag{5}$$

then, the zero solution to the system (4) is globally asymptotically stable if all roots λ_i of the characteristic equation (5) satisfy :

$$|\arg(\lambda_i)| > \frac{\pi}{2s}, i = 1, 2, 3, ..., n. \tag{6}$$

3 Novel hyperchaotic system

In this section, a novel hyperchaotic system with only one equilibrium point generated from a Chen system [19] is proposed. The dynamic equation of the system is as follow:

$$\begin{cases} \dot{x_1} = a(x_2 - x_1), \\ \dot{x_2} = bx_1 - x_1 x_3 + cx_2 - x_4, \\ \dot{x_3} = x_1 x_2 - dx_3, \\ \dot{x_4} = hx_2, \end{cases} \tag{7}$$

where x_i, $i = 1, 2, 3, 4$ are state variables, a, b, c, d and h are positive real parameters. In this paper, we shall show that the system (7) exhibits hyperchaos, when the parameters a, b, c, d and h take the values:

$$(a,\ b,\ c,\ d,\ h) = (30, 10, 10, 3.8, 10). \tag{8}$$

Now, the novel hyperchaotic system (7) has the following properties:

3.1 Dissipativity

The divergence of the vector field V is:

$$\nabla V = -a + c - d.$$

When $-a + c - d < 0$, the system (7) is dissipative and it meets:

$$V(t) = V_0 \exp(-(a - c + d)t). \tag{9}$$

This means the hyperchaotic system in Eq. (7) is able to model a physical system.

3.2 Equilibrium points and stability

Suppose that the system (7) having equilibrium points, then:

$$a(x_2 - x_1) + x_4 = bx_1 - x_1x_3 + cx_2 - x_4 = x_1x_2 - dx_3 = hx_2 = 0. \tag{10}$$

Hence, the system has only one equilibrium point $O(0, 0, 0, 0)$.
When the parameters sytem are taken as in the hyperchaotic case (8), the eigenvalues of equilibrium point O are:

$$15.90,\ 0.51,\ -3.8 \text{ and } -36.42. \tag{11}$$

Thus O is an unstable saddle point.

3.3 Hyperchaotic attractors

In the numerical simulations, the fourth-order Runge–Kutta method is used to solve the systems with time step size 0.002. For this numerical simulation, the initial condition are taken as:

$$x_1(0) = 2,\ x_2(0) = 0.2,\ x_3(0) = 0.5 \text{ and } x_4(0) = 0.1. \tag{12}$$

The phase portraits of the system (7) in different $4 - D$ projection planes for the parameter values given in (8) is illustrated in Figure 1.

3.4 Lyapunov exponents spectrum

Here, by using the Wolf algorithm method [20], with time step size 0.002, the variation of two largest Lyapunov exponents spectrum for the initial condition are taken as in (12), the parameters value $(a, b, c, h) = (30, 10, 10, 10)$ and $d \in [0, 4.5]$ is given in Figure 2. Particularly, when the parameters sytem are taken as in (8), our novel system exhibits hyperchaos with Lyapunov exponents:

$$L_1 = 0.54,\ L_2 = 0.24, L_3 = 0.00,\ L_4 = -24.58, \tag{13}$$

and the Kaplan-Yorke dimension [21] is obtained as:

$$D = 3 + \frac{L_1 + L_2 + L_3}{|L_4|} = 3.03. \tag{14}$$

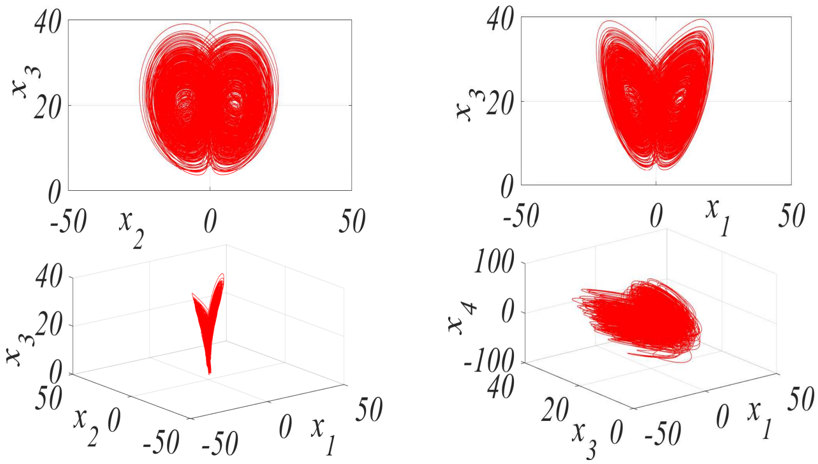

Figure 1: The phase portraits of the system (7) in different $4-D$ projection planes

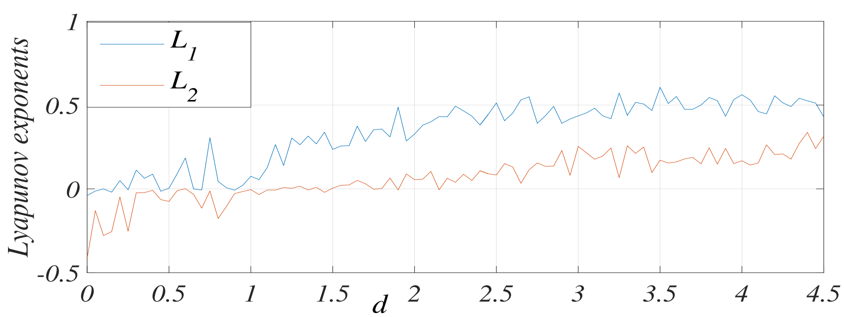

Figure 2: The variation of two largest Lyapunov exponents of the system (7).

4 MHS of incommensurate fractional-order hyprchaotic systems

In this section, we construct an active control technique for the MHS of incommensurate fractional-order hyprchaotic systems. This technique is carried out using stability theorems of fractional-order linear system.

4.1 Theoretical analysis

Consider the two identical fractional hyperchaotic systems described by:

$$D^{\alpha}x = f(x), \tag{15}$$

$$D^{\alpha}y = g(y) + U, \tag{16}$$

where α is rational numbers between 0 and 2, $x,\ y \in \mathbb{R}^4$ are the states vector of the drive system (15) and the response system (16), respectively, $f,\ g : \mathbb{R}^4 \to\in \mathbb{R}^4$ are the continuous vectors function and $U \in \mathbb{R}^4$ is control input vector to be designed later. The error state vector is defined as

$$e(t) = By(t) - Cx(t), \tag{17}$$

where $e(t) \in \mathbb{R}^4$, $B = diag(1,\ 1,\ 1,\ \sigma)$ and $C = diag(1,\ -1,\ \beta,\ 1)$ denote diagonal matrices.

Definition 2 *The drive system (15) and the response system (16) are defined to be MHS if there are two constant matrices $B = diag(1,\ 1,\ 1,\ \sigma)$ and $C = diag(1,\ -1,\ \beta,\ 1)$ such that*

$$\lim_{t\to\infty} ||By(t) - Cx(t)|| = 0, \tag{18}$$

where $||.||$ stands for the matrix norm.

Remark 3 *If we consider $B = C = I$, where I is an 4×4 identity matrix, then the MHS problem will be simplified to the complette synchronization.*

Remark 4 *If we consider $B = I$ and $C = -I$, where I is an 4×4 identity matrix, then the MHS problem will be simplified to the anti-synchronization.*

Remark 5 *If we consider $B = I$ and $C = diag(\beta, \beta, \beta, \beta)$, then the MHS problem will be reduced to the projective synchronization.*

Remark 6 *If we consider $B = diag(\sigma_1, \sigma_2, \sigma_3, \sigma_4)$ and $C = I$, then the MHS problem will be reduced to the modified projective synchronization.*

Now, from equations (15), (16) and (17) , we can get the following error system:

$$\begin{aligned} D^{\alpha}e &= BD_t^{\alpha}y - CD_t^{\alpha}x \\ &= Ae + (BA - AB)y - (CA - AC)x + \\ &+ BG(y) - CF(x) + BU, \end{aligned} \quad (19)$$

where $A \in \mathbb{R}^{4\times4}$, is the linear part of the system, F and $G : \mathbb{R}^4 \rightarrow\in \mathbb{R}^4$ are the nonlinear parts.
To achieve the desired MHS between the above systems, the nonlinear active controller $U = (u_1, u_2, u_3, u_4)^T$ is constructed as:

$$U = B^{-1}((CA - AC)x - BA - AB)y + CF(x) - BG(y) + Me, \quad (20)$$

where $M \in \mathbb{R}^{4\times4}$ is a feedback gain matrix to be determined.
So, when we use the controller (20) to control the fractional-order response system (16), the HMS problem of the fractional-order drive system (15) and fractional-order response system (16) is changed into the analysis of the asymptotical stability of the following system:

$$D^{\alpha}e = (A + M)e. \quad (21)$$

Then, we have the following result.

Theorem 7 *If the matrix M is selected such that all roots λ_i of the characteristic equation:*

$$\det(diag(\lambda^{s\alpha_1},\ \lambda^{s\alpha_2},\ \lambda^{s\alpha_3},\ \lambda^{s\alpha_4}) - (A + M)) = 0, \quad (22)$$

satisfy $|\arg(\lambda_i)| > \frac{\pi}{2s}, i = 1, 2, 3, 4$, where s is the least common multiple of the denominators of λ_i, then the drive system (15) and response system (16) can be synchronized in the sens of HMS under the controller (20).

Proof. Immediately, by using theorem (1). ■

4.2 Numerical example and simulation results

In order to confirm the feasibility of the theoretical analysis presented in above section, we consider the novel fractional-order hyprchaotic system (NFOHS) as the drive system and the the controlled NFOHS as the response system. They are described as follows:
The NFOHS is given by:

$$\begin{cases} D^{\alpha_1}x_1 = a(x_2 - x_1), \\ D^{\alpha 2}x_2 = bx_1 - x_1x_3 + cx_2 - x_4, \\ D^{\alpha 3}x_3 = x_1x_2 - dx_3, \\ D^{\alpha 4}x_4 = hx_2, \end{cases} \quad (23)$$

and the controlled NFOHS is given by:

$$\begin{cases} D^{\alpha_1}y_1 = a(y_2 - y_1) + u_1, \\ D^{\alpha_2}y_2 = by_1 - y_1y_3 + cy_2 - y_4 + u_2, \\ D^{\alpha_3}y_3 = y_1y_2 - dy_3 + u_3, \\ D^{\alpha_4}y_4 = hy_2 + u_4, \end{cases}, \tag{24}$$

where $0 < \alpha_1, \alpha_2, \alpha_3, \alpha_4 \leq 2$ and $U = (u_1, u_2, u_3, u_4) \in \mathbb{R}^4$ is the active control function to be determined later. The linear part A of system is given by:

$$A = \begin{pmatrix} -a & a & 0 & 0 \\ b & c & 0 & -1 \\ 0 & 0 & -d & 0 \\ 0 & h & 0 & 0 \end{pmatrix}. \tag{25}$$

The suitable gain matrix M is selected as:

$$M = \begin{pmatrix} -1+a & -a & 0 & 0 \\ -b & -2-c & 0 & -1 \\ 0 & 0 & -3+d & 1 \\ -2 & -h & 0 & -4 \end{pmatrix}. \tag{26}$$

Using the method presented in above section, the error system can be rewritten as:

$$\begin{aligned} \begin{pmatrix} D^{\alpha_1}e_1 \\ D^{\alpha_2}e_2 \\ D^{\alpha_3}e_3 \\ D^{\alpha_4}e_4 \end{pmatrix} &= (A+M)\begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{pmatrix} \\ &= \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -2 & 0 & -2 \\ 0 & 0 & -3 & 1 \\ -2 & 0 & 0 & -4 \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{pmatrix}, \end{aligned} \tag{27}$$

and so, the characteristic equation:

$$\det(diag(\lambda^{s\alpha_1}, \lambda^{s\alpha_2}, \lambda^{s\alpha_3}, \lambda^{s\alpha_4}) - (B+M)) = 0, \tag{28}$$

can be rewritten as:

$$(\lambda^{s\alpha_1} + 1)(\lambda^{s\alpha_2} + 2)(\lambda^{s\alpha_3} + 3)(\lambda^{s\alpha_4} + 4) = 0, \tag{29}$$

where s is the least common multiple of the denominators of α_i, for $i = 1, 2, 3$ and 4.

According to the stability results, the drive system (23) and the response system (24) are synchronized if all roots λ of (29) satisfy $|\arg(\lambda)| > \dfrac{\pi}{2s}$. Let us take:

$$(a, b, c, d, h\ ) = (30, 10, 10, 3, 10), \tag{30}$$

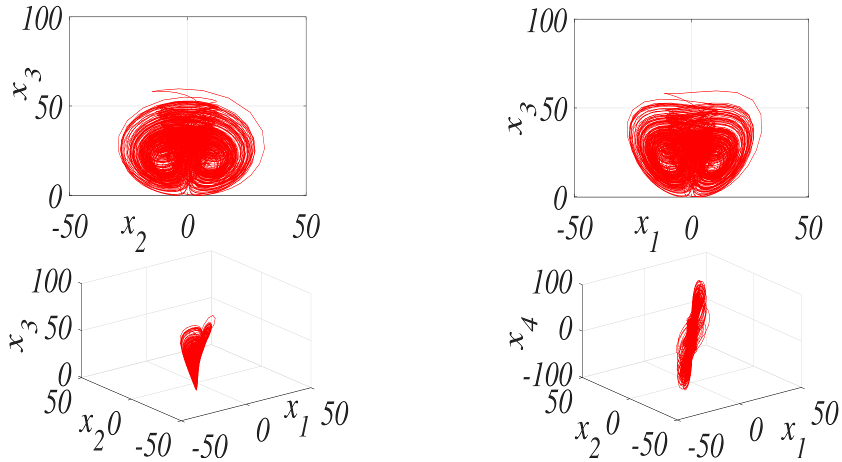

Figure 3: The hyperchaotic attractors of the NFOHS (23).

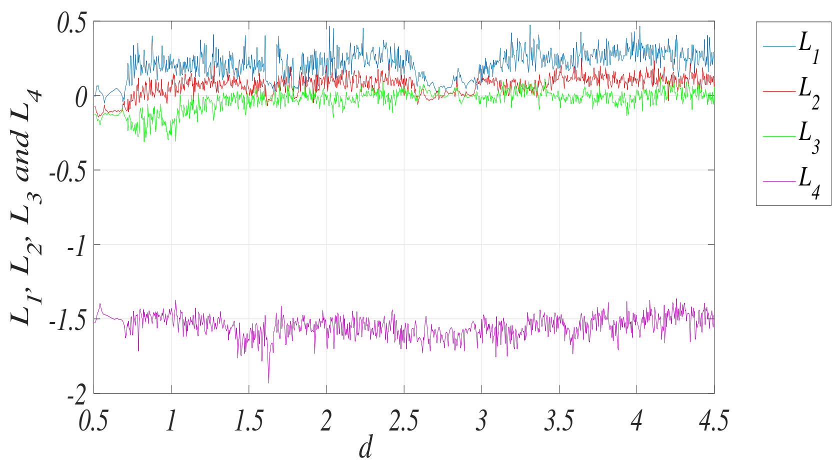

Figure 4: The variation of the Lyapunov exponents of the NFOHS system (23).

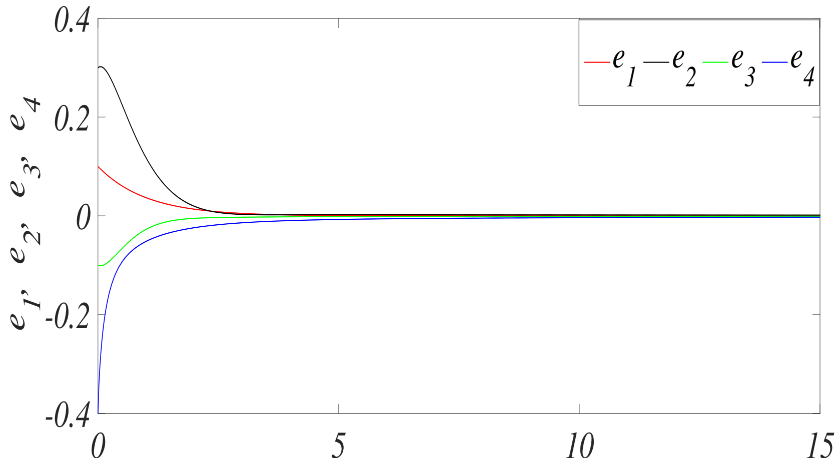

Figure 5: Time evolution of MHS errors between systems (23) and (24).

and

$$(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (0.9, 1, 0.9, 0.8), \tag{31}$$

then the Equ. (29) becomes:

$$\left(\lambda^9 + 1\right)\left(\lambda^{10} + 2\right)\left(\lambda^9 + 3\right)\left(\lambda^8 + 4\right) = 0. \tag{32}$$

It is easy to see that all the roots λ of (32) satisfy the condition $|\arg(\lambda)| > \frac{\pi}{20}$. Therefore, under the controller:

$$U = B^{-1}((CA - AC)x - (BA - AB)y + CF(x) - BG(y) + Me), \tag{33}$$

i.e.,

$$\begin{cases} u_1 = -e_1 + a(x_2 - x_1 + y_1 - y_2), \\ u_2 = -e_1 + a(x_2 - x_1 + y_1 - y_2), \\ u_3 = e_4 - 3e_3 - \beta(dx_3 - x_1 x_2) + dy_3 - y_1 y_2, \\ u_4 = -\frac{1}{\delta}\left(2e_1 + 4e_4 - h(x_2 - \delta y_2)\right), \end{cases} \tag{34}$$

the drive system (23) and the response system (24) are synchronized.
In the numerical simulations, the Adams method [22] is used to solve the systems with time step size 0.004. For this numerical simulation, the initial conditions of the drive system (23) and the response system (24) are respectively taken as:

$$x_1(0) = 0.2,\ x_2(0) = 0.2,\ x_3(0) = 0.2 \text{ and } x_4(0) = 0.2, \tag{35}$$

$$y_1(0) = 0.3,\ y_2(0) = 0.1,\ y_3(0) = 0.3 \text{ and } y_4(0) = -1. \tag{36}$$

So, when the gain parameters β and σ are taken as $\beta = 2$ and $\sigma = \frac{1}{5}$, the error system has the initial values:

$$e_1(0) = 0.1,\ e_2(0) = 0.3,\ e_3(0) = -0.1 \text{ and } e_4(0) = -0.4. \quad (37)$$

In order to justify that (23) is hyperchaotic system, its parameter values are taken as in (30) and the different fractional-order derivatives are taken as in (31).
Figure 3 describes the hyperchaotic attractors of the NFOHS (23).
By using Matlab code for Lyapunov exponents of fractional-order systems [23], the variation of the Lyapunov exponents spectrum of the NFOHS (23) is given in Figure 4.
Particularly, when the parameters sytem are taken as in (30), the NFOHS exhibits hyperchaos with Lyapunov exponents:

$$L_1 = 0.178,\ L_2 = 0.10, L_3 = 0.00, \text{and } L_4 = -1.50. \quad (38)$$

In the other hand, Figure 5 describes the time evolution of MHS errors between systems (23) and (24). From this figure, for the given parameters, we can numerically see that the errors converge to zero, and so the desired synchronization of the systems (23) and (24) is sufficiently achieved under the controller (34).

5 Conclusion

In this paper, we have investigated the MHS between two identical fractional-order hyperchaotic systems. A novel hyperchaotic system is also proposed and its dynamics has been analysed. With the stability criterion of linear fractional-order systems, analysis of MHS is performed for the proposed method by using a suitable nonlinear controller. Numerical example and simulations results have been given to confirm the effectiveness of the results.

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