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REDUCED GENERALIZED COMBINATION SYNCHRONIZATION BETWEEN TWO $n-$DIMENSIONAL INTEGER-ORDER HYPERCHAOTIC SYSTEMS AND ONE $m-$DIMENSIONAL FRACTIONAL-ORDER CHAOTIC SYSTEM

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ABSTRACT. This paper is devoted to investigate the problem of reduced generalized combination synchronization (RGCS) between two $n-$dimensional integer-order hyperchaotic drive systems and one $m-$dimensional fractional-order chaotic response system. According to the stability theorem of fractional-order linear system, an active mode controller is proposed to accomplish this end. Moreover, the proposed synchronization scheme is applied to synchronize three different chaotic systems, which are the Danca hyperchaotic system, the modified hyperchaotic Rossler system, and the fractional-order Rabinovich-Fabrikant chaotic system. Finally, numerical results are presented to fit our theoretical analysis.

Key words and phrases: Reduced generalized combination synchronization; Chaotic or hyperchaotic system; Caputo fractional derivative; Stability theorem of fractional-order linear system.

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1. INTRODUCTION

During the last decades, analysis of fractional-order differential equations have attracted importance attention [1, 2, 3, 4]. In particular, chaos synchronization of fractional-order or integer-order systems as a very interesting phenomenon has become a hot topic from many researchers [5, 6]. Many appraoches for chaos synchronization have been developed in the recent years. For example, complete synchronization have been described in [7]. Inverse matrix projective synchronization has been designed in [8, 9, 10]. Generalized synchronization has been presented in [11, 12]. And finally, $Q - S$ generalized synchronization has been designed in [13]. However, most of researches mentioned above have concentrated on the usual synchronization which is done between one drive system and one response system.

Recently, Luo et al. [14] have already extended the concept of this synchronization to the combination synchronization between two drive systems and one response system. Indeed, this kind of synchronization has stronger anti-attack ability than that of the usual synchronization in the field of secure communication, because the transmiter message can be separated into two segments and each segment can be separated into two different drive systems. The study of adaptive function projective combination synchronization of three fractional-order chaotic systems, combination synchronization of three different chaotic systems with the help of the active backstepping, and finally combination synchronization of a novel fractional-order chaotic system with two stable node-foci, are also receiving growing attention [15, 16, 17].

However, most of the previous manuscripts focused on the combination synchronization of general fractional-order or integer-order chaotic systems and also with identical dimensions. Thus, an important question is asked: does combination synchronization happen between two integer-order hyperchaotic systems and one fractional-order chaotic system with different structure and non-identical dimensions?. To the best of our knowledge, none of the previous studies investigate this kind of synchronization.

In this paper, based on the stability results of fractional-order linear system, we study the problem of reduced generalized combination synchronization between two $n-$dimensional integer-order hyperchaotic systems and one $m-$ dimensional fractional-order chaotic system. The main aim of this work are summarized as follows:

First, based on active mode controller and stability theorem of fractional-order linear system, a new approach for chaos combination synchronization between two integer-order hyperchaotic systems and one fractional-order chaotic system is presented.

Second, three chaotic systems are synchronized with different structure and non-identical dimensions.

Finaly, the theoretical results is verified by simulation example. In particular, this kind of synchronization scheme is applied to the hyperchaotic Danca system, the modified hyperchaotic Rossler system, and the fractional-order Rabinovich-Fabrikant chaotic system.

The rest of this manuscript is organized as follows: In Section 2, some basic concepts about Caputo fractional derivative and stability condition of fractional-order linear system are briefly given. The general scheme of RGCS is introduced in Section 3. In Section 4, we present an illustrative example to verify the effectiveness of the analytical results. Conclusions is given in the last Section.

2. CAPUTO'S FRACTIONAL DERIVATIVE AND STABILITY THEOREM

In this section, we present some basic concepts about Caputo fractional derivative and the stability condition of linear fractional systems.

Definition 2.1. The α^{th}-order Riemann–Liouville fractional integration of function y [18] is given by

$$I^\alpha y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\xi)^{\alpha-1} y(\xi) d\xi, \tag{2.1}$$

where

$$\Gamma(\alpha) = \int_0^{+\infty} x^{\alpha-1} \exp(-x) dx, \tag{2.2}$$

is the gamma function.

Definition 2.2. Let $n-1 < \alpha \leq n$, $n \in \mathbb{N}$; the Caputo fractional derivative of order α of function y [19] is defined as

$$D^\alpha y(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\xi)^{m-\alpha-1} y^{(n)}(\xi) d\xi \tag{2.3}$$

Theorem 2.1. [20]*Consider the following fractional-order linear system*

$$D^\alpha x = Ax, \tag{2.4}$$

where D^α is Caputo differential operator $(0 < \alpha \leq 1)$, $x \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n\times n}$.
Then the system (2.4) is asymptotically stable if

$$|\arg(\lambda_i(A))| > \frac{\alpha\pi}{2}, \ i = 1, 2, ..., n, \tag{2.5}$$

where $\arg(\lambda_i(A))$ represents the argument of the eigenvalue λ_i of A.

3. GENERAL SCHEME OF RGCS

In this section, the general scheme of RGCS between two $n-$dimensional integer-order hyperchaotic drive systems and one $m-$dimensional fractional-order chaotic response system are presented. The model can be descriebed as follows:

$$\dot{x} = f(x), \tag{3.1}$$

$$\dot{y} = g(y), \tag{3.2}$$

$$D^\alpha z = h(z) + u, \tag{3.3}$$

where D^α is Caputo differential operator $(0 < \alpha \leq 1)$, $x, y \in \mathbb{R}^n$ are the state variables of two drive systems, $z \in \mathbb{R}^m (n > m)$ is the state variable of response system, $f, g : \mathbb{R}^n \to \mathbb{R}^m$ and $h : \mathbb{R}^m \to \mathbb{R}^m$ are differentiable functions and $u \in \mathbb{R}^m$ is a controller vector which will be designed.

The definition of the proposed RGCS is given as follows:

Definition 3.1. The drive systems (1)-(2) and the response system (3) are said to achieve RGCS, if there exists three scaling matrices $Q = (Q)_{n\times m}$, $R = (R)_{n\times m}$ and $S = (S)_{m\times m}$, such that the error system:

$$e(t) = Qx(t) + Ry(t) - Sz(t), \tag{3.4}$$

satisfies:

$$\lim_{t\to+\infty} \|Qx(t) + Ry(t) - Sz(t)\| = 0, \tag{3.5}$$

where $\|.\|$ represents the matrix norm.

Remark 3.1. The scaling matrices Q, R and S can be extended to the functional matrices $Q(.), R(.)$ and $S(.)$ of the state variables x, y and z, respectively.

Remark 3.2. If $Q = I$, $R = 0$ and $S \neq 0$ (or $Q = 0$, $R = I$ and $S \neq 0$), we get the projective synchronization problem, where I represents a $n \times n$ identity matrix.

Remark 3.3. If the scaling matrix $Q = R = 0$ and $S \neq 0$, then RGCS problem will be simplified to the chaos control problem.

Remark 3.4. From definition 3.1, one can shows that the above RGCS can be extended to more $n-$dimensional integer-order hyperchaotic systems and one $m-$dimensional fractional-order chaotic systems .

In order to achieve the RGCS between the systems (1), (2) and (3), the controlled response system (3) is described as:

$$D^{\alpha}z = \Theta z + H(z) + u, \tag{3.6}$$

where $\Theta = (\Theta)_{m\times m}$ and $H : \mathbb{R}^m \to \mathbb{R}^m$ are the linear part and the nonlinear part of system (3.6), respectively.
The error dynamics system can be written as:

$$\begin{aligned} D^{\alpha}e &= QD^{\alpha}x + RD^{\alpha}y - SD^{\alpha}z \\ &= QI^{1-\alpha}\dot{x} + RI^{1-\alpha}\dot{y} - Sh(z) - Su \\ &= I^{1-\alpha}\left(Qf(x) + Rg(y)\right) - Sh(z) - Su \\ &= \Theta e + I^{1-\alpha}\left(Qf(x) + Rg(y)\right) - \left(\Theta e + Sh(z)\right) - Su. \end{aligned} \tag{3.7}$$

Assume that S^{-1} exists, then we have the following result.

Theorem 3.1. *The RGCS between systems (1), (2) and (3.6) can be achieved under the suitable control laws*

$$u = S^{-1}\left[I^{1-\alpha}\left(Qf(x) + Rg(y)\right) - \left(\Theta e + Sh(z)\right) + \Phi e\right], \tag{3.8}$$

where S^{-1} is the inverse matrix of S, and $\Phi = (\Phi)_{m\times m}$ is a feedback gain matrix which satisfy:

$$\left|\arg\left(\lambda_i(\Theta - \Phi)\right)\right| > \frac{\alpha\pi}{2},\ i = 1, 2, ..., n, \tag{3.9}$$

where $\arg\left(\lambda_i(\Theta - \Phi)\right)$ *represents the argument of the eigenvalue λ_i of $\Theta - \Phi$.*

Proof. According to (3.8), (3.7) becomes

$$D^{\alpha}e = (\Theta - \Phi)e. \tag{3.10}$$

In view of Theorem (3.1), we need to the following conditions for asymptotic stability of (3.10).

$$\left|\arg\left(\lambda_i(\Theta - \Phi)\right)\right| > \frac{\alpha\pi}{2},\ i = 1, 2, ..., m, \tag{3.11}$$

So, as long as selecting the appropriate matrix Φ, the condition of the Theorem 2.1 is satisfied and the system (3.10) asymptotically converges to zero, which means the drive systems (1) and (2) achieve the RGCS with the response system (3.6). ∎

4. NUMERICAL SIMULATIONS

In this section, we consider the hyperchaotic Danca system [21] and the modified hyperchaotic Rossler system [22] as the drive systems, and the fractional-order chaotic Rabinovich-Fabrikant system as the response system.
The hyperchaotic Danca system is described as :

$$\left\{\begin{array}{l} \dot{x_1} = -x_1 + x_2, \\ \dot{x_2} = -x_3 sgn(x_1) + x_4, \\ \dot{x_3} = |x_1| - a_1, \\ \dot{x_4} = -b_1 x_2. \end{array}\right. \tag{4.1}$$

The system (4.1) exhibits a hyperchaotic behavior as shown in Figure 1, when the parameters are given by:

$$(a_1,\ b_1) = (1,\ 0.5), \tag{4.2}$$

and the initial values are taken as:

$$(x_1(0), x_2(0), x_3(0), x_4(0))^T = (0.1, 0.1, 0.1, 0.1)^T. \tag{4.3}$$

The modified hyperchaotic Rossler system is described as:

$$\left\{\begin{array}{l} \dot{y_1} = -y_2 - y_3 + y_4, \\ \dot{y_2} = y_1 + a_2 y_2, \\ \dot{y_3} = y_1 y_3 - b_2 y_3 + c_2, \\ \dot{y_4} = d_2 y_1. \end{array}\right. \tag{4.4}$$

The system (4.4) exhibits a hyperchaotic behavior as shown in Figure 2, when the parameters are given by:

$$(a_2,\ b_2,\ c_2,\ d_2) = (0.283, 5, 0.01, 0.1), \tag{4.5}$$

and the initial values are taken as:

$$(y_1(0), y_2(0), y_3(0), y_4(0))^T = (0.2, 0.2, 0.2, 0.2)^T. \tag{4.6}$$

The controlled Rabinovich-Fabrikant fractional-order chaotic system [23] is derived as:

$$\left\{\begin{array}{l} D^\alpha z_1 = z_2(z_3 - 1 + z_1^2) + 0.1 z_1 + u_1, \\ D^\alpha z_2 = z_1(3z_3 + 1 - z_1^2) + 0.1 z_2 + u_2, \\ D^\alpha z_3 = -2z_3(p + z_1 z_2) + u_3. \end{array}\right. \tag{4.7}$$

When the fractional-order $\alpha = 0.98$, the bifurcation parameter $p = 0.98$ and the initial values are taken as:

$$(z_1(0), z_2(0), z_3(0))^T = (0.4, 0.4, 0.4)^T, \tag{4.8}$$

the system (4.1) with $u_1 = u_2 = u_3 = 0$, exhibits a chaotic behavior as shown in Figure 3.
From (4.7), one can get

$$\Theta = \begin{pmatrix} 0.1 & -1 & 0 \\ 1 & 0.1 & 0 \\ 0 & 0 & -2p \end{pmatrix}$$

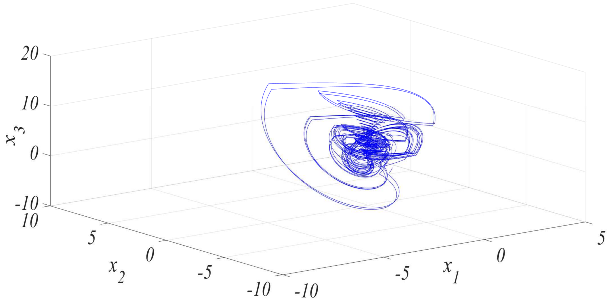

Figure 1: The hyperchaotic attractor of the system (4.1)

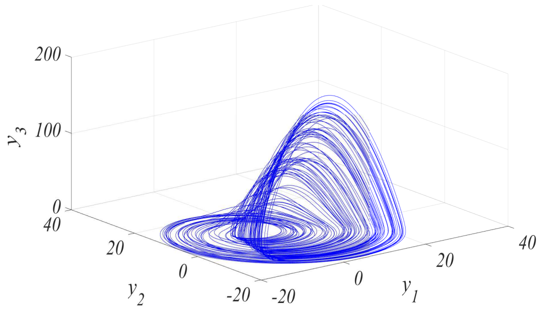

Figure 2: The hyperchaotic attractor of the system (4.4)

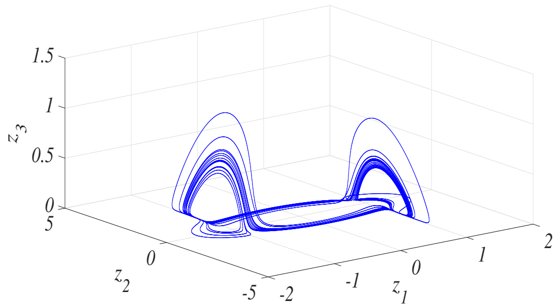

Figure 3: The chaotic attractor of the system (4.7)

To verify the effectiveness of the proposed RGCS scheme, we take

$$Q = \begin{pmatrix} 1 & 0 & 0 & 0.4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \end{pmatrix}, \; R = \begin{pmatrix} 2 & 0 & 1 & 0 \\ 0 & 4 & -0.5 & 0 \\ 0.2 & 0 & 0.2 & 0 \end{pmatrix},$$

$$S = \begin{pmatrix} 0.2 & 0 & 0 \\ 0 & 0.4 & 0 \\ 0 & 0 & 0.6 \end{pmatrix} \text{ and } \Phi = \begin{pmatrix} 0.6 & -1 & 0.5 \\ 1 & 1.1 & 0 \\ -1 & 0 & -2p+1.5 \end{pmatrix}$$

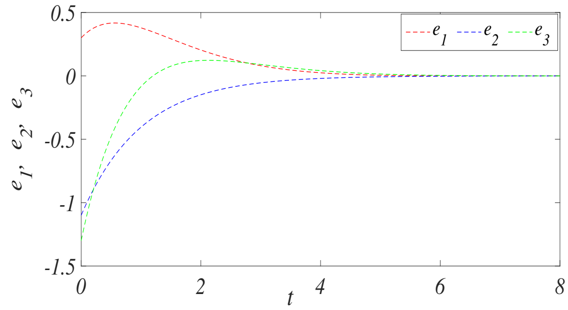

Figure 4: Time evolution of the RGCS error (4.9).

According to the active control (3.8), the error dynamics system (3.10) becomes

$$\begin{pmatrix} D^{\alpha}e_1 \\ D^{\alpha}e_2 \\ D^{\alpha}e_3 \end{pmatrix} = (\Theta - \Phi)\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} -0.5 & 0 & -0.5 \\ 0 & -1 & 0 \\ 1 & 0 & -1.5 \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}. \tag{4.9}$$

We can show that the value of $|\arg(\lambda_i(\Theta - \Phi))|$, $i = 1, 2, 3$ are

$$\lambda_{1,2} = -1 \pm 0.5i, \text{ and } \lambda_3 = -1,$$

which satisfy the condition $\min\{|\arg(\lambda_i(\Theta - \Phi))|, \ i = 1, 2, 3\} = 0.852\,4\pi > 0.49\pi$. Then the conditions of Theorem 3.1 is satisfied. Hence RGCS between systems (4.1), (4.4) and (4.7) is achieved. Figure 4 displays the time evolution of the RGCS error (4.9) with the initial conditions $(e_1(0), e_2(0), e_3(0))^T = (0.3, -1.1, -1.3)^T$.

5. CONCLUSION

In this paper, we have introduced a new appraoch to invistigate chaos combination synchronization between two integer-order hyperchaotic systems and one fractional-order chaotic system. With the help of the stability theorem of fractional-order linear system, a suitable sufficient conditions are proposed for achieving the RGCS via a suitable active controller. These chaotic systems are synchronized with different structure, different derivative orders and non-identical dimensions. Numerical simulations are also given to fit the feasibility of our theoretical analysis. Moreover, this kind of synchronization should be applied in the fields of secure communications which will be the subject of our future work.

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