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Numerical solution of some Volterra nonlinear integral equations by using spline iterative Collocation method

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″ they are the other soul that inhabit my soul, and my shadow that prevents me from falling ″. FATIHA, TAHAR.

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ملخص

قدمنا في هذه المذكرة طريقة مقترحة مع خوارزمية جديدة لحل معادلات فولتيرا التكاملية غير الخطية مع تاخير زمني حيث يتم إيجاد الحل التقريبي لهذه المعدلات باستخدام طريقة التجميع التكرارية ,بالاعتماد على كثيرات حدود لاغرانج .

كما من الممكن ملاحظة كفاءة الطريقة و سهولة الحسابات فيها من خلال بعض الأمثلة التوضيحية لحل معادلات فولتيرا التكاملية الغير خطية مع تأخير زمني وقد تم الحصول على نتائج جيدة .

الكلمات المفتاحية

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RÉSUMÉ

L'objectif essentiel de ce travail consiste à résoudre numériquement des équations intégrales de Volterra non linéaires avec retard par la méthode de " collocation itérative" en utilisant les polynômes de Lagrange. Des exemples numériques sont présentés pour confirmer les estimations théoriques et illustrer la convergence de la méthode.

Mots-clés : Équations intégrales de volterra non linéaires avec retard, Méthode de collocation, Polynômes de Lagrange.

ABSTRACT

The main purpose of this thesis is to provide a direct, convergent and easy to implement numerical method to obtain the approximate solution for nonlinear delay Volterra integral equations. Algorithms based on iterative collocation method is developed for the numerical solution of these kinds of equations. We also provide a rigorous error analysis. A theoretical proof is given and we present some numerical results which illustrate the performance of the methods.
Key Words: Nonlinear delay Volterra integral equations, Collocation method, Iterative Method, Lagrange polynomials, Convergence analysis,Error estimation.

INTRODUCTION

The mathematical modeling of the phenomena from the population dynamics gives models that are discrete or continuous. The discrete models are represented by matrix equations or difference equations. The continuous models are represented, in general, by differential equations, partial differential equations, integral equations or intego-differential equations with delay arguments.

The first integral equation mentioned in the mathematical literature is due to Abel and can be found in almost any book on this subject (see, for instance, [9]). Abel found this equation in 1812, starting from a problem in mechanics. He gave a very elegant solution that was published in 1826.

Starting in 1896, Vito Volterra built up a theory of integral equations, viewing their solutions as a problem of finding the inverses of certain integral operators. In 1900, Ivar Fredholm made his famous contribution that led to a fascinating period in the development of mathematical analysis. Poincaré, Fréchet, Hilbert, Schmidt, Hardy and Riesz were involved in this new area of research. The impact of Fred- holm s theory on the foundation and development of functional analysis has also been outstanding. These facts could explain why Volterra s equations, whose role in the investigation of some dynamical processes (mainly in biology) had been em- phasized by Volterra himself, took a place of secondary importance. Actually, the integral equations of Volterra type are present any time we deal with a differential equation. According to J. Dieudonné," ... differential equations constitute a swin- dling. In fact, there exist no differential equations. The only interesting equations are the integral ones"(Nico, 1969).

Volterra integral equations arise in a wide variety of mathematical, scientific, and engineering problems. One such problem is the solution of parabolic differential equations with initial boundary conditions [[31] , p.68]. Another application deals with the temperature in nuclear reactors [[31] ,Chap.IV, Sect.8] where the delayed neutron is ignored. For more physical applications of Volterra integral equations see the references in [33]. Many physical problems are better represented by a delay Volterra integral equation rather than a Volterra integral equation; that is, the prob- lem has a delay in which cannot be ignored [18, 27, 30]. Delay integral equation is a model for the spread of some infectious disease (cf.[34, 36]) and arise in the pop- ulation dynamics with a finite life span (see [17, 23, 22, 36, 40]). Moreover, integral equations with constant delays are frequently encountered in physical and biological modeling processes (e.g. [7, 23]). On the other hand, the delay integro-differential equations have become important in the mathematical modeling of biological and physical phenomena (see, for example, [8, 26, 29]).

The monograph [14] presents a historical survey of mathematical models in biology, which can be described by Volterra integral and integro-differential equations with constant delays.

In the following examples we will briefly describe three such classes of delay Volterra integral and integro-differential equations.

Example 0.1. We present in the following, the first model gotten by Cooke and Kaplan [21]. In the goal to express the spasmodic character of the apparitions of some epidemics in the big concentrations of populations. Cooke and Kaplan presented the following model: We suppose here that the total population is constant, equal to N. The evolution of the proportions of this population, $I(t) + S(t) = \alpha(\alpha$ is a positive constant), such that $I(t)$ is the proportion of sick at the time t, $S(t)$ is the proportion of people susceptible to become sick at the time t.

The proposed hypotheses are: there is not any natural immunity or acquired. The illness takes a determined time τ , the same for every individual. All sick is infectious during the time of its illness. We suppose that there is no period of incubation. Finally, the population is enough important and homogeneous to consider I and S as continuous functions. we modeling by a function $a(t)$ the rate of meeting between the susceptible and infected populations at the instant t by unit of time of a sick with other peoples in these conditions:

- $a(t)S(t)$ is the proportion of the susceptible having a contact by sick and by unit of time.

- $I(t)Na(t)NS(t)dt$ represents the fraction of an individual reached by the ill- ness in the interval of time [t, t + dt]: NI(t)a(t)S(t)dt is the corresponding proportion (by report to the total population).

- $NI(t-\tau)a(t-\tau)S(t-\tau)dt$ represents the proportion that returns in the compartment of the susceptible individuals between the times t and t + dt.

Therefore, the variation of the rate of the individuals infected between t and t + dt is given by the following relation:

$$\frac{dI(t)}{dt} = a(t)I(t)(1-I(t)) - a(t-\tau)I(t-\tau)(1-I(t-\tau))$$

which could be integrated as follows:

$$I(t) = \int_{t-\tau}^{t} a(s)I(s)(1-I(s))ds + c \qquad (*).$$

Since the infection has a duration less than τ , the proportion of infected individuals at the instant t is given exactly by the integral term $(*)$. Therefore $c = 0$.

Example 0.2. Many basic mathematical models in epidemiology and population growth (see, e.g. [23, 22, 36, 40]) are described by nonlinear Volterra integral equations of the second kind with (constant) delay $\tau > 0$, namely,

$$x(t) = g(t) + \int_{t-\tau}^{t} P(t-s)G(s, x(s))ds, t \geq t_0.$$

The function g is usually assumed to be such that $\lim_{t \to +\infty} g(t) = g(+\infty)$ exists. These delay integral equations model the deterministic growth of a population y (e.g. of animals, or cells) or the spread of an epidemic with immigration into the population; it also has applications in economics.

This work is organized as follows:

In the first chapter, we provide some notations, definitions and auxiliary facts which will be needed for stating our results, such as the existence and the uniqueness of the smooth solution for the second kind Volterra integral and integro-differential equations and basic Discrete Gronwall-type inequalities.

As for the second chapter, we discussed the iterative collocation method to solve the nonlinear Delay Volterra integral equation.

CHAPTER 1

PRELIMINARY AND AUXILIARY RESULTS

Integral Equation

An integral equation is defined as an equation in which the unknown function $u(t)$ to be determined appear under the integral sign. The subject of integral equations is one of the most useful mathematical tools in both pure and applied mathematics. It has enormous applications in many physical problems. Many initial and boundary value problems associated with ordinary differential equation (ODE) and partial differential equation (PDE) can be transformed into problems of solving some approximate integral equations. The development of science has led to the formation of many physical laws, which, when restated in mathematical form, often appear as differential equations [41].
An integral equation is an equation in which the unknown function $u(t)$ appears under an integral sign. A standard integral equation in $u(t)$ is of the form:

$$u(t) = f(t) + \lambda \int_{g(t)}^{h(t)} K(t,s)u(s)ds,$$

where $g(t)$ and $h(t)$ are the limits of integration, λ is a constant parameter, and $k(t,s)$ is a function of two variables t and s called the kernel of the integral equation. The function $u(t)$ that will be determined appears under the integral sign, and it appears inside the integral sign and outside the integral sign as well. The functions $f(t)$ and $k(t,s)$ are given in advance. It is to be noted that the limits of integration $g(t)$ and $h(t)$ may be both variables, constants, or mixed.
An integro-differential equation is an equation in which the unknown function $u(t)$ appears under an integral sign and contains an ordinary derivative $u^{(n)}(t)$ as well. A standard integro-differential equation is of the form:

$$u^{(n)}(t) = f(t) + \lambda \int_{g(t)}^{h(t)} K(t,s)u(s)ds,$$

where $g(t)$, $h(t)$, $f(t)$, λ and the kernel $k(t,s)$ are as prescribed before. Integral equations and integro-differential equations will be classified into distinct types according to the limits of integration and the kernel $k(t,s)$. [41].

1.1 Classifications of integral and integro differential equations

The most integral and integro-differential equations fall under two main classes namely Fredholm and Volterra integral and integro differential equations.

1.1.1 Fredholm integral and integro differential equations

Fredholm integral equations: Fredholm integral equations arise in many scientific applications. It was also shown that, this equation can be derived from boundary value problems. Erik Ivar Fredholm (1866-1927) is best remembered for his work on integral equations and spectral theory. Fredholm was a Swedish mathematician who established the theory of integral equations and his 1903 paper in Acta Mathematica played a major role in the establishment of operator theory (Wazwaz (2011)). The most standard form of Fredholm linear integral equations is given by the following form

$$v(t)u(t) = f(t) + \lambda \int_a^b K(t,s)u(s)ds, \quad a \leq t, \quad s \leq b, \tag{1.1}$$

where the limit of integration a and b are constants and the unknown function $u(t)$ appears under the integral sign. Where $k(t,s)$ is the kernel of the integral equation and λ is a parameter. The Eq. (1.1) is called linear because the unknown function $u(t)$ under the integral sign occurs linearly, i.e. the power of $u(t)$ is one.
The value of $v(t)$ will give the following kinds of Fredholm integral equations:
If $v(t) = 0$, then Eq. (1.1) yields

$$f(t) = \lambda \int_a^b K(t,s)y(s)ds, \quad a \leq t, \quad s \leq b,$$

which is called Fredholm integral equation of the first kind.

If the function $v(t) = 1$, then Eq. (1.1) becomes simply

$$u(t) = f(t) + \lambda \int_a^b K(t,s)u(s)ds, \quad a \leq t, \quad s \leq b,$$

and this equation is called Fredholm integral equation of second kind.

If $v(t) \neq 0$, then Eq.(1.1) becomes Fredholm integral equations of third kind. Fredholm integral equation is of the first kind if the unknown function $u(t)$ appears only under the integral sign.

Nonlinear Fredholm integral equations:

The nonlinear Fredholm integral equations of the second kind is given by the following form

$$u(t) = f(t) + \lambda \int_a^b K(t,s,u(s))ds, \quad a \leq t, \quad s \leq b.$$

Where the unknown function $u(t)$ occurs inside and outside the integral sign, λ is a parameter, and a and b are constants. For this type of equations, the kernel k and the function $f(t)$ are given real-valued functions.

Nonlinear Fredholm-Hammerstein integral equations:

Nonlinear Fredholm-Hammerstein integral equations is given by the form,

$$u(t) = f(t) + \lambda \int_a^b K(t,s)F(s,u(s))ds, \quad a \leq t, \quad s \leq b,$$

Nonlinear Fredholm integro-differential equations:

The nonlinear Fredholm integro-differential equations is given by the following form,

$$u^{(n)}(t) = f(t) + \int_a^b K(t,s,u(s),u'(s),\ldots,u^{(n-1)}(s))ds, \quad u^{(k)}(a) = b_k, \quad 0 \leq k \leq n-1, \tag{1.2}$$

where $u^n(t) = \frac{d^n u}{dt^n}$ Because the resulted equation in (1.2) combines the differential operator and the integral operator, then it is necessary to define initial conditions $u(0),\ u^{'}(0), ..., u^{n-1}(0)$ for the determination of the particular solution $u(t)$ of the equation (1.2). Any Fredholm integro-differential equation is characterized by the existence of one or more of the derivatives $u'(t),\ u^{''}(t), ...$ outside the integral sign. The Fredholm integro-differential equations of the second kind appear in a variety of scientific applications such as the theory of signal processing and neural networks.

Nonlinear Fredholm-Hammerstein integro-differential equations:
The nonlinear Fredholm-Hammerstein integro-differential equations of the second kind is of the form,

$$u^{(n)}(t) = f(t) + \int_a^b K(t,s)F(s, u(s), u'(s), \dots, u^{(n-1)}(s))ds,$$

1.1.2 Volterra integral and integro-differential equations

It is well known that linear and nonlinear Volterra integral equations arise in many scientific fields such as the population dynamics, spread of epidemics, and semi-conductor devices. Volterra started working on integral equations in 1884, but his serious study began in 1896. The name integral equation was given by du Bois-Reymond in 1888. However, the name Volterra integral equation was first coined by Lalesco in 1908 [41].
Volterra integral equations:
The standard form of linear Volterra integral equations, where the limits of integration are functions of t rather than constants, are of the form,

$$v(t)u(t) = f(t) + \lambda \int_a^t K(t,s)u(s)ds, \ \ a \leq t, \ \ s \leq b, \tag{1.3}$$

where the unknown function $u(t)$ under the integral sign occurs linearly as stated before. It is worth noting that (1.3) can be viewed as a special case of Fredholm integral equation when the kernel $k(t,s)$ vanishes for $s > t$, t is in the range of integration $[a,b]$. As in Fredholm equations, Volterra integral equations fall under the following kinds, depending on the value of $v(t)$, namely:

First, when $v(t) = 0$, Eq. (1.3) becomes,

$$0 = f(t) + \lambda \int_a^t K(t,s)u(s)ds, \quad a \le t, \quad s \le b,$$

and in this case the integral equation is called Volterra integral equation of the first kind.

Secondly, when $v(t) = 1$, Eq. (1.3) becomes,

$$u(t) = f(t) + \lambda \int_a^t K(t,s)u(s)ds, \quad a \le t, \quad s \le b,$$

and in this case the integral equation is called Volterra integral equation of the second kind.

Thirdly, when $v(t) \neq 0$, Eq. (1.3) becomes Volterra integral equations of third kind.

Nonlinear Volterra integral equations:

The nonlinear Volterra integral equation of the second kind is represented by the form,

$$u(t) = f(t) + \lambda \int_a^t K(t,s,u(s))ds$$

The nonlinear Volterra integral equation of the first kind is expressed in the form,

$$f(t) = \lambda \int_a^t K(t,s,u(s))ds$$

Nonlinear Volterra-Hammerstein integral equations:

The nonlinear Volterra-Hammerstein integral equation of the second kind is represented by the form,

$$u(t) = f(t) + \lambda \int_a^t K(t,s)F(s,u(s))ds,$$

Volterra Integro-differential equations:

Volterra, in the early 1900, studied the population growth, where new type of equations have been developed and was termed as integro-differential equations. In this type of equations, the unknown function $u(t)$ occurs in one side as an ordinary derivative, and appears on the other side under the integral sign. Several phenomena in physics and biology give rise to this type of integro-differential equations. Further, we point out that an integro-differential equation can be easily observed as an intermediate stage when we convert a differential equation to an integral equation in next section.
The Volterra integro-differential equation appeared after its establishment by Volterra. It then appeared in many physical applications such as glass forming process, nanohydrodynamics, heat transfer, diffusion process in general, neutron diffusion and biological species coexisting together with increasing and decreasing rates of generating, and wind ripple in the desert. More details about the sources where these equations arise can be found in physics, biology and engineering applications books (see, for example Brunner [14], Volterra [39]. To determine the exact solution for the integro-differential equation, the initial conditions should be given. The Volterra integro-differential equations can be converted to an integral equation by using Leibnitz rule .
Nonlinear Volterra integro-differential equations:
The nonlinear Volterra integro-differential equation of the second kind is in the form,

$$u^{(n)}(t) = f(t) + \int_a^t K(t,s,u(s),u'(s),\ldots,u^{(n-1)}(s))ds, \quad u^{(k)}(a) = b_k, \quad 0 \leq k \leq n-1$$

and the standard form of the nonlinear Volterra integro-differential equation of the first kind is given by,

$$\int_a^t K(t,s,u(s),u'(s),\ldots,u^{(n-1)}(s))ds = f(t),$$

Nonlinear Volterra-Hammerstein integro-differential equations:

The nonlinear Volterra-Hammerstein integro-differential equation of the second kind is in the form,

$$u^{(n)}(t) = f(t) + \int_a^t K(t,s)F(s,u(s),u'(s),\dots,u^{(n-1)}(s))ds, \quad u^{(k)}(a) = b_k, \quad 0 \leq k \leq n-1$$

1.1.3 Volterra-Fredholm integral and integro-differential equations

Volterra-Fredholm integral equations: The Volterra-Fredholm integral equation, which is a combination of disjoint Volterra and Fredholm integrals, appears in one integral equation. The Volterra-Fredholm integral equations arise from the modelling of the spatiotemporal development of an epidemic, from boundary value problems and from many physical and chemical applications [41]. The standard form of the linear Volterra-Fredholm integral equation is in the form,

$$u(t) = f(t) + \int_a^t K_1(t,s)u(s)ds + \int_a^b K_2(t,s)u(s)ds$$

where $k_1(t,s)$ and $k_2(t,s)$ are the kernels of the equation.

Nonlinear Volterra-Fredholm integral equations:

The standard form of the Nonlinear Volterra-Fredholm integral equation is in the form,

$$u(t) = f(t) + \int_a^t K_1(t,s,u(s))ds + \int_a^b K_2(t,s,u(s))ds$$

Nonlinear Volterra-Fredholm-Hammerstein integral equations:

The standard form of the Nonlinear Volterra-Fredholm-Hammerstein integral equation is in the form,

$$u(t) = f(t) + \int_a^t K_1(t,s)F(s,u(s))ds + \int_a^b K_2(t,s)G(s,u(s))ds$$

where $k_1(t,s)$ and $k_2(t,s)$ are the kernels of the equation.

Volterra-Fredholm integro-differential equations:

The Volterra-Fredholm integro-differential equation, which is a combination of disjoint Volterra and Fredholm integrals and differential operator, may appear in one integral equation. The Volterra-Fredholm integro-differential equations arise from many physical and chemical applications similar to the Volterra-Fredholm integral equations [4, 5, 37, 38]. The standard form of the Volterra-Fredholm integro-differential equation is in the form,

$$u^{(n)}(t) = f(t) + \int_a^t K_1(t,s,u(s),u'(s),\ldots,u^{(n-1)}(s))ds + \int_a^b K_2(t,s,u(s),u'(s),\ldots,u^{(n-1)}(s))ds$$

Nonlinear Volterra-Fredholm-Hammerstein integro-differential equations:

$$u^{(n)}(t) = f(t)+\int_a^t K_1(t,s)F(t,s,u(s),u'(s),\ldots,u^{(n-1)}(s))ds+\int_a^b K_2(t,s,u(s),u'(s),\ldots,u^{(n-1)}(s))ds$$

1.1.4 Singular integral equations

Volterra integral equations of the first kind,

$$f(t) = \lambda \int_{g(t)}^{h(t)} K(t,s)u(s)ds$$

or of the second kind

$$u(t) = f(t) + \lambda \int_{g(t)}^{h(t)} K(t,s)u(s)ds$$

are called singular if one of the limit of integration $g(t), h(t)$ is infinite or the kernel $k(t,s)$ becomes unbounded at one or more points in the interval of integration. We focus on concern on equation of the form:

$$u(t) = f(t) + \lambda \int_{0}^{t} \frac{1}{(t-s)^{\alpha}} u(s)ds, \quad 0 \leq \alpha \leq 1 \tag{1.4}$$

or of the second kind

$$f(t) = \lambda \int_{0}^{t} \frac{1}{(t-s)^{\alpha}} u(s)ds, \quad 0 \leq \alpha \leq 1 \tag{1.5}$$

The Eq. (1.4) and Eq.(1.5) are called generalized Abel's integral equation and weakly singular integral equations respectively.
On the other hand, the well known weakly singular Fredholm integral equations of the form,

$$u(t) = f(t) + \int_{0}^{1} k(t,s)u(s)ds, \quad 0 \leq \alpha \leq 1$$

where the singularity of kernel may be stated in the forms $k(t,s) = \frac{1}{(t-s)^{\alpha}}$ or $k(t,s) = \frac{1}{(1-t)^{\alpha}}$.

Definition 1.1.1 ***(The homogeneity property)***
We set $f(t) = 0$ in Fredholm or Volterra integral and integro-differential equations as given in the above, the resulting equations is called a homogeneous integral and integro-differential equations, otherwise it is called nonhomogeneous or inhomogeneous integral and integro-differential equations.

Theorem 1.1.1 (Leibnits) *Let $f(x)$ be continuous $[a,b]$,so:*

$$\forall x \in [a,b], \int_0^x \int_0^{x_1} ... \int_0^{x_{n-1}} f(x_n)dx_n...dx_1 = \frac{1}{(n-1)!} \int_a^x (x-t)^{n-1} f(t)dt.$$

1.2 Systems integral and integro-differential equations

The most systems integral and integro-differential equations fall under two main classes namely Fredholm and Volterra systems integral and integro-differential equations.

1.2.1 Systems of Fredholm integral and integro-differential Equations

Systems of Fredholm integral equations:

The systems of Fredholm integral equations appear in two kinds. The system of Fredholm integral equations of the first kind reads

$$\begin{cases} f_1(x) = \int_a^b (K_1(x,t)u(t) + k_1(x,t)v(t))\, dt \\ f_2(x) = \int_a^b (K_2(x,t)u(t) + k_2(x,t)v(t))\, dt \end{cases} \tag{1.6}$$

Where the unknown functions $u(x)$ and $v(x)$ appear only ender the integral sign, and a and b are constants. However,for systems of Fredholm integral equations of the second kind,the unknown functions $u(x)$ and $v(x)$ appear inside and outside the integral sign. The second kind represented by the form:

$$\begin{cases} u(x) = f_1(x) + \int_a^b (K_1(x,t)u(t) + k_1(x,t)v(t))\, dt \\ v(x) = f_2(x) + \int_a^b (K_2(x,t)u(t) + k_2(x,t)v(t))\, dt \end{cases} \tag{1.7}$$

Systems of nonlinear Fredholm integral equations:

Systems of nonlinear Fredholm integral equations of the second kind is given by the following form:

$$\begin{cases} u(x) = f_1(x) + \int\limits_a^b (K_1(x,t)F_1(u(t)) + k_1(x,t)(F_1(v(t))))dt \\ v(x) = f_2(x) + \int\limits_a^b (K_2(x,t)F_2(u(t)) + k_2(x,t)v(t))(F_2(v(t)))dt \end{cases} \tag{1.8}$$

Systems of Fredholm integro-differential equations:

Systems of Fredholm integro-differential equations of the second kind given by

$$\begin{cases} u^{(i)}(x) = f_1(x) + \int\limits_a^b (K_1(x,t)(u(x)) + k_1(x,t)(v(x)))\,dt \\ v^{(i)}(x) = f_2(x) + \int\limits_a^b (K_2(x,t)(u(x)) + k_2(x,t)v(x))\,dt \end{cases} \tag{1.9}$$

The unknown functions u(x),v(x),....,that will be determined,occur inside the integral sign whereases the derivatives of u(x),v(x),....appear mostly outside the integral sign.The kernels $k_i(x,t)$ and $k_i(x,t)$,and the function $f_i(x)$ are given real-valued functions.

Systems of nonlinear Fredholm integro-differential equations:

Systems of Nonlinear Fredholm integro-differential equations of the second kind given by

$$\begin{cases} u^{(i)}(x) = f_1(x) + \int\limits_a^b (K_1(x,t)F_1(u(x)) + k_1(x,t)(F_1(v(x))))dt \\ v^{(i)}(x) = f_2(x) + \int\limits_a^b (K_2(x,t)F_2(u(x)) + k_2(x,t)v(t))(F_2(v(x)))dt \end{cases} \tag{1.10}$$

1.2.2 Systems of Volterra integral and integro-differential equations

Systems of Volterra integral equations:

The systems of Volterra integral equations appear in two kinds. For systems of Volterra integral equations of the first kind, the unknown functions appear only under the integral sign in the form:

$$\begin{cases} f_1(x) = \int\limits_0^x \Big(K_1(x,t)u(t) + \tilde{K_1}(x,t)v(t)\Big)\,dt \\ f_2(x) = \int\limits_0^x \Big(K_2(x,t)u(t) + \tilde{K_2}(x,t)v(t)\Big)\,dt \end{cases} \tag{1.11}$$

However,systems of Volterra integral equations of the second kind, the unknown functions appear inside and outside the integral sign in the form:

$$\begin{cases} u(x) = f_1(x) + \int\limits_0^x \Big(K_1(x,t)u(t) + \tilde{K_1}(x,t)v(t)\Big)\,dt \\ v(x) = f_2(x) + \int\limits_0^x \Big(K_2(x,t)u(t) + \tilde{K_2}(x,t)v(t)\Big)\,dt \end{cases} \tag{1.12}$$

The kernels $K_i(x,t)$ and $k_i(x,t)$,and the functions $f_i(x)$,$i = 1, 2, \ldots, n$ are given real-valued functions.

Systems of nonlinear Volterra integral equations:

Systems of nonlinear Volterra integral equations of the second kind are given by

$$\begin{cases} u(x) = f_1(x) + \int\limits_0^x (K_1(x,t)F_1(u(t)) + K_1(x,t)g_1(v(t)))\,dt \\ v(x) = f_2(x) + \int\limits_0^x (K_2(x,t)F_2(u(t)) + K_2(x,t)g_2(v(t)))\,dt \end{cases} \tag{1.13}$$

The unknown functions $u(x)$ and $v(x)$, that will be determined, occur inside and outside the integral sign. The kernels $K_i(x,t)$ and $K_i(x,t)$, and the functions $f_i(x)$ are given real-valued functions, for $i = 1, 2$. The functions F_i and g_i, for $i = 1, 2$ are nonlinear functions of $u(x)$ and $v(x)$.

And of the first kind are given by

$$\begin{cases} f_1(x) = \int\limits_0^x (K_1(x,t)F_1(u(t)) + K_1(x,t)(F_1(v(t))))dt \\ f_2(x) = \int\limits_0^x (K_2(x,t)F_2(u(t)) + K_2(x,t)F_2(v(t)))\,dt \end{cases} \tag{1.14}$$

Systems of Volterra integro-differential equations:

The systems of Volterra integro-differential equations the second kind by

$$\begin{cases} u^{(i)}(x) = f_1(x) + \int\limits_0^x (K_1(x,t)u(t) + k_1(x,t)v(t))\,dt \\ v^{(i)}(x) = f_2(x) + \int\limits_0^x (K_2(x,t)u(t) + K_2(x,t)v(t))\,dt \end{cases} \tag{1.15}$$

The unknown functions $u(x)$ and $v(x)$,..., that will be determined, occur inside the integral sign whereas derivatives of $u(x)$ and $v(x)$,... appear mostly outside the integral sign. The kernels $K_i(x,t)$ and $K_i(x,t)$, and the functions $f_i(x)$ are given real-valued functions.

And the standard form of the nonlinear Volterra integro-differential equations of the first inside given by

$$\begin{cases} f_1(x) = \int\limits_0^x (K_1(x,t)u(t) + K_1(x,t)v(t))\,dt \\ f_2(x) = \int\limits_0^x (K_2(x,t)u(t) + K_2(x,t)v(t))\,dt \end{cases} \tag{1.16}$$

Systems of nonlinear Volterra integro-differential equations:

Systems of nonlinear Volterra integro-differential equations of the second kind are given by

$$\begin{cases} u^{(i)}(x) = f_1(x) + \int\limits_0^x (K_1(x,t)F_1(u(t)) + K_1(x,t)F_1(v(t)))\,dt, \\ v^{(i)}(x) = f_2(x) + \int\limits_0^x (K_2(x,t)F_2(u(t)) + K_2(x,t)F_2(v(t)))\,dt, \end{cases} \tag{1.17}$$

And of the first kind are given by

$$\begin{cases} f_1(x) = \int\limits_0^x (K_1(x,t)F_1(u(t)) + K_1(x,t)F_1(v(t)))\,dt, \\ f_2(x) = \int\limits_0^x (K_2(x,t)F_2(u(t)) + K_2(x,t)F_2(v(t)))\,dt, \end{cases} \tag{1.18}$$

1.2.3 Systems of singular integral equations

Systems of singular integral equations of the first kind,

$$\begin{cases} f_1(x) = \int\limits_0^x (K_{11}(x,t)u(t) + K_{12}(x,t)v(t))\,dt, \\ f_2(x) = \int\limits_0^x (K_{21}(x,t)u(t) + K_{22}(x,t)v(t))\,dt, \end{cases} \tag{1.19}$$

or of the second kind

$$\begin{cases} u(x) = f_1(x) + \int\limits_0^x (K_{11}(x,t)u(t) + K_{12}(x,t)v(t))\,dt, \\ v(x) = f_2(x) + \int\limits_0^x (K_{21}(x,t)u(t) + K_{22}(x,t)v(t))\,dt, \end{cases} \tag{1.20}$$

Were the kernels K_{ij} are singular kernrls given by

$$K_{ij} = \frac{1}{(x-t)^{\alpha_{ij}}}, \quad 1 \le i, \quad j \le 2.$$

The system (1.19) and the system (1.20) are called the system of the generalized Abel singular integral equations and the system of the weakly generalized singular integral equations respectively. For $\alpha_{ij} = \frac{1}{2}$, the system (1.19) is called the system of the Abel singular integral equations.

1.3 Conversion of differential equations to integral equations

In general, the initial values problems (IVP) can be transformed to Volterra integral equations, and the boundary values problems (BVP) can be transformed to Fredholm integral equations and virse versa

1.3.1 IVP to Volterra integral equations:

In this section, we will study the technique that will convert an initial value problem (IVP) to an equivalent Volterra integral equation and Volterra integro-differential equation as well [41]. For simplicity reasons, we will apply this process to a second order initial value problem given by

$$u''(t) + p(t)u'(t) + q(t)u(t) = g(t) \tag{1.21}$$

$$u(0) = \alpha, u'(0) = \beta$$

where α and β are constants. The functions $p(t)$ and $q(t)$ are analytic functions, and $g(t)$ is continuous through the interval of discussion. To achieve our goal we first set

$$u''(t) = v(t), \tag{1.22}$$

where $v(t)$ is a continuous function. Integrating both sides of (1.22) from 0 to t yields

$$u'(t) - u'(0) = \int_0^t v(s)ds$$

or equivalently

$$u'(t) = \beta + \int_0^t v(s)ds \tag{1.23}$$

Integrating both sides of (1.23) from 0 to t yields

$$u(t) - u(0) = \beta t + \int_0^t \int_0^s v(r)drds$$

or equivalently

$$u(t) = \alpha + \beta t + \int_0^t (t-s)v(s)ds \tag{1.24}$$

obtained upon using the formula that reduce double integral to a single integral that was discussed in the next section. Substituting (1.22), (1.23), and (1.24) into the initial value problem (1.21) yields the Volterra integral equation:

$$v''(t) + p(t)\left[\beta + \int_0^t v(s)ds\right] + q(t)\left[\alpha + \beta t + \int_0^t (t-s)v(t)dt\right] = g(t).$$

The last equation can be written in the standard Volterra integral equation form:

$$v(t) = f(t) + \int_0^t k(t,s)v(s)ds, \tag{1.25}$$

where

$$k(t,s) = p(t) + q(t)(t-s),$$

and

$$f(t) = g(t) - \left[\beta p(t) + \alpha q(t) + \beta t q(t)\right].$$

It is interesting to point out that by differentiating Volterra equation (1.25) with respect to t, using Leibnitz rule, we obtain an equivalent Volterra integro-differential equation in the form:

$$u'(t) + k(t,t) = f'(t) - \int_0^t \frac{\partial k(t,s)}{\partial t} u(s)ds, \quad u(0) = f(0)$$

The technique presented above to convert initial value problems to equivalent Volterra integral equations can be generalized by considering the general initial value problem:

$$u^{(n)}(t) + a_1 u^{n-1} + ... + a_{n-1}u^{'} + a_n u = g(t) \tag{1.26}$$

subject to the initial conditions

$$u(0) = c_0, u'(0) = c_1, u''(0) = c_2, ..., u^{n-1} = c_{n-1}.$$

Let $v(t)$ be a continuous function on the interval of discussion, and we consider the transformation:

$$u^{(n)}(t) = v(t). \tag{1.27}$$

Integrating both sides with respect to t gives

$$u^{(n-1)}(t) = c_{n-1} + \int_0^t v(t)dt.$$

Integrating again both sides with respect to t yields

$$\begin{aligned} u^{(n-2)}(t) &= c_{n-2} + c_{n-1}t + \int_0^t \int_0^t u(s)dsds \\ &= c_{n-2} + c_{n-1}t + \int_0^t (t-s)u(s)ds, \end{aligned}$$

obtained by reducing the double integral to a single integral. Proceeding as before we find

$$\begin{aligned} u^{(n-3)}(t) &= c_{n-3} + c_{n-2}t + \frac{1}{2}c_{n-1}t^2 + \int_0^t \int_0^t \int_0^t v(s)dsdsds \\ &= c_{n-3} + c_{n-2}t + \frac{1}{2}c_{n-1}t^2 + \frac{1}{2}\int_0^t (t-s)^2 v(s)ds. \end{aligned}$$

Continuing the integration process leads to

$$u(t) = \sum_{k=0}^{n-1} \frac{c_k}{k!}t^k + \frac{1}{(n-1)!}\int_0^t (t-s)^{n-1}v(s)ds. \tag{1.28}$$

Substituting (1.27) (1.28) into (1.26) gives

$$u(t) = f(t) + \int_0^t k(t,s)v(s)ds, \tag{1.29}$$

where

$$k(t,s) = \sum_{k=1}^{n} \frac{a_n}{k-1!}(t-s)^k - 1,$$

and

$$f(t) = g(t) - \sum_{j=1}^{n} a_j \left(\sum_{k=1}^{j} \frac{c_n - k}{(j-k)!} t^j \right).$$

Notice that the Volterra integro-differential equation can be obtained by differentiating (1.29).

The following examples will highlight the process to convert initial value problem to an equivalent Volterra integral equation.

1.3.2 BVP to Fredholm integral equations:

In this section, we will convert a boundary value problem to an equivalent Fredholm integral equation. The method is similar to the method that was presented in the above section for converting Volterra equation to IVP, with the exception that boundary conditions will be used instead of initial values. In this case we will determine another initial condition that is not given in the problem. The technique requires more work if compared with the initial value problems when converted to Volterra integral equations. Without loss of generality, we will present two specific distinct boundary value problems (BVPs) to derive two distinct formulas that can be used for converting BVP to an equivalent Fredholm integral equation [41].

Type I: We first consider the following boundary value problem:

$$u''(t) + g(t)u(t) = h(t), \quad 0 \leq t \leq 1, \tag{1.30}$$

with the boundary conditions:

$$u(0) = \alpha, \quad u(1) = \beta.$$

We start as in the previous section and set

$$u''(t) = v(t). \tag{1.31}$$

Integrating both sides of (1.31) from 0 to t we obtain

$$\int_0^t u''(s)ds = \int_0^t v(s)ds,$$

that gives

$$u'(t) = u'(0) + \int_0^t v(s)ds, \tag{1.32}$$

where the initial condition $u'(0)$ is not given in a boundary value problem. The condition $u'(0)$ will be determined later by using the boundary condition at $t = 1$. Integrating both sides of (1.32) from 0 to t gives

$$u(t) = u(0) + tu'(0) + \int_0^t \int_0^t v(s)dsds,$$

or equivalently

$$u(t) = \alpha + tu'(0) + \int_0^t (t-s)v(s)ds, \tag{1.33}$$

obtained upon using the condition $u(0) = \alpha$ and by reducing double integral to a single integral. To determine $u'(0)$, we substitute $t = 1$ into both sides of (1.30) and using the boundary condition at $u(1) = \beta$ we find

$$u(1) = \alpha + u'(0) + \int_0^1 (1-s)v(s)ds,$$

that gives

$$\beta = \alpha + u'(0) + \int_0^1 (1-s)v(s)ds.$$

This in turn gives

$$u'(0) = \beta - \alpha - \int_0^1 (1-s)v(s)ds. \tag{1.34}$$

Substituting (1.34) into (1.33) gives

$$u(t) = \alpha + (\beta - \alpha)t - \int_0^1 t(1-s)v(s)ds + \int_0^t (t-s)v(s)ds. \tag{1.35}$$

Substituting (1.31) and (1.35) into (1.30) yields

$$u(t) + \alpha g(t) + (\beta - \alpha)tg(t) - \int_0^1 tg(t)(1-s)v(s)ds + \int_0^t g(t)(t-s)v(s)ds = h(t).$$

Hence, by using Chasles formula, we obtain

$$v(t) = h(t) - \alpha g(t) - (\beta - \alpha)tg(t) - \int_0^t g(t)(t-s)v(s)ds - tg(t)\left[\int_0^t (1-s)v(s)ds + \int_t^1 (1-s)v(s)ds\right],$$

that gives

$$v(t) = f(t) + \int_0^t s(1-t)v(s)ds + \int_t^1 t(1-s)g(t)v(s)ds, \tag{1.36}$$

that leads to the Fredholm integral equation:

$$v(t) = f(t) + \int_0^1 k(t,s)v(s)ds, \tag{1.37}$$

where

$$f(t) = h(t) - \alpha g(t) - (\beta - \alpha)tg(t),$$

and the kernel $k(t,s)$ is given by

$$k(t,s)=\begin{cases} s(1-t)g(t), \; for\ 0\leq s\leq t, \\ \\ s(1-s)g(t), \; for\ t\leq s\leq 1. \end{cases}$$

An important conclusion can be made here. For the specific case where $u(0)=u(1)=0$ which means that $\alpha=\beta=0$, it is clear that $f(t)=h(t)$ in this case. This means that the resulting Fredholm equation in (1.37) is homogeneous or inhomogeneous if the boundary value problem in (1.30) is homogeneous or inhomogeneous respectively when $\alpha=\beta=0$.

Type II: We next consider the following boundary value problem:
problem:

$$u''(t)+g(t)u(t)=h(t), \; 0\leq t\leq 1 \tag{1.38}$$

with the boundary conditions:

$$u(0)=\alpha_1, \; u'(1)=\beta_1.$$

we again set

$$u''(t)=v(t) \tag{1.39}$$

Integrating both sides of (1.36) from 0 to t we obtain

$$\int_0^t u''(s)ds=\int_0^t v(s)ds,$$

that gives

$$u'(t)=u'(0)+\int_0^t v(s)ds \tag{1.40}$$

where the initial condition $u'(0)$ is not given in a boundary value problem. The condition $u'(0)$ will be derived later by $u'(1) = \beta_1$. Integrating both sides of (1.40) from 0 to t gives

$$u(t) = u(0) + tu'(0) + \int_0^t \int_0^t v(s)dsds,$$

or equivalently

$$u(t) = \alpha_1 + tu'(0) + \int_0^t (t-s)v(s)ds, \tag{1.41}$$

obtained upon using the condition $u(0) = \alpha_1$ and by reducing double integral to a single integral. To determine $u'(0)$, we first differentiate (1.41) with respect to t to get

$$u'(t) = u'(0) + \int_0^t v(s)ds, \tag{1.42}$$

where by substituting $t = 1$ into both sides of (1.42) and using the boundary condition at $u'(1) = \beta_1$ we find

$$u'(t) = \beta_1 + \int_0^t v(s)ds,$$

This in turn gives

$$u'(1) = u'(0) + \int_0^1 v(s)ds. \tag{1.43}$$

Using (1.43) into (1.41) gives

$$u'(0) = \beta_1 - \int_0^1 v(s)ds, \tag{1.44}$$

Substituting (1.39) and (1.44) into (1.38) yields

$$v(t)+\alpha_1 g(t)+\beta_1 t g(t)-\int_0^1 t g(s)v(s)ds+\int_0^t g(t)(t-s)v(s)ds=h(t)$$

Hence, by using Chasles formula, we obtain

$$v(t)=h(t)-(\alpha_1+\beta_1 t)g(t)+tg(t)\left[\int_0^t v(s)ds+\int_t^1 v(s)ds\right]-g(t)\int_0^t (t-s)v(s)ds.$$

The last equation can be written as

$$v(t)=f(t)+\int_0^t sg(t))v(s)ds+\int_t^1 tg(t)v(s)ds,$$

that leads to the Fredholm integral equation:

$$u(t)=f(t)+\int_0^1 k(t,s)u(s)ds, \tag{1.45}$$

where

$$f(t)=h(t)-(\alpha_1+\beta_1 t)g(t),$$

and the kernel $k(t,s)$ is given by

$$k(t,s)=\begin{cases} sg(t), \ for\ 0\leq s\leq t, \\ \\ tg(t), \ for\ t\leq s\leq 1. \end{cases}$$

An important conclusion can be made here. For the specific case where $u(0)=u'(1)=0$ which means that $\alpha_1=\beta_1=0$, it is clear that $f(t)=h(t)$ in this case. This means that the resulting Fredholm equation in (1.45) is homogeneous or inhomogeneous if the boundary value problem in (1.38) is homogeneous or inhomogeneous respectively.

1.4 Conversion of Volterra integro-differential equations to Volterra integral equation

The following Volterra integro-differential equation

$$u^{(n)}(t) = f(t) + \lambda \int_0^t K(t,s)u(s)ds, \quad u^{(k)}(0) = b_k, \quad 0 \leq k \leq n-1, \tag{1.46}$$

can also be solved by converting it to an equivalent Volterra integral equation. It is obvious that the Volterra integro-differential equation (1.46) involves derivatives at the left side, and integral at the right side. To perform the conversion process, we need to integrate both sides n times to convert it to a standard Volterra integral equation. Firstly, Integration of derivatives: from calculus we observe the following:

$$\int_0^t u'(s)ds = u(t) - u(0),$$

$$\int_0^t \int_0^{t_1} u''(s)dsdt_1 = u(t) - tu'(0) - u(0),$$

$$\int_0^t \int_0^{t_1} \int_0^{t_2} u'''(s)dsdt_1dt_2 = u(t) - \frac{1}{2}t^2u''(0) - tu'(0) - u(0),$$

and so on for other derivatives.

Secondly, Reducing multiple integrals to a single integral as follows,

$$\int_0^x \int_0^{x_1} u(t)dtdx_1 = \int_0^x (x-t)u(t)dt,$$
$$\int_0^x \int_0^{x_1} (x-t)u(t)dtdx_1 = \frac{1}{2}\int_0^x (x-t)^2 u(t)dt,$$
$$\int_0^x \int_0^{x_1} (x-t)^2 u(t)dtdx_1 = \frac{1}{3}\int_0^x (x-t)^3 u(t)dt$$
$$\int_0^x \int_0^{x_1} (x-t)^3 u(t)dtdx_1 == \frac{1}{4}\int_0^x (x-t)^4 u(t)dt$$

and so on. This can be generalized in the form

$$\int_0^x \int_0^{x_1} ... \int_0^{x_{n-1}} (x-t)u(t)dtdx_{n-1}...dx_1 = \frac{1}{(n)!}\int_0^t (t-s)^n u(t)dt,$$

The conversion to an equivalent Volterra integral equation will be illustrated by studying the following examples.

Example 1.4.1 *Convert the following Volterra integro-differential equation to an Volterra integral equation:*

$$u'(x) = 1 + \int_0^x u(t)dt, \quad u(0) = 0$$

Integrating both sides from 0 *to* x*, and using the aforementioned formulas we find*

$$u(x) - u(0) = x + \int_0^x \int_0^{x_1} u(t)dtdx_1$$

Using the initial condition gives the Volterra integral equation

$$u(x) = x + \int_0^x (x-t)u(t)dt$$

1.5 Existence and uniqueness of the solution

Consider the nonlinear Volterra integro-differential equation (NVIDE)

$$y^{(n)}(x) = f(x) + \int_0^x K(x,t,y(t))ds, \quad x \in [0,b] \tag{1.47}$$

with n initial conditions

$$u^{(k)}(0) = \alpha_k, \quad 0 \le k \le n-1,$$

f and K are given smooth functions.

In this section, the existence and uniqueness of the solution for Eq. (1.47) are presented. First we give the following theorem from [**?**].

Theorem 1.5.1 *Consider the following nonlinear Volterra integral equations*

$$y(x) = f(x) + \int_0^t k(x,t,y(t))dt, \tag{1.48}$$

Assume that

(i) $f(x)$ *is continuous ,*

(ii) $k(x,t,y(t))$ *is a continuous function for* $0 \le t \le s \le b$ *and* $-\infty \le |\, y| \ \le \infty$*,*

(iii) the kernel satisfies the Lipschitz condition

$$|k(x,t,y_1) - k(x,t,y_2)| \le L|y_1 - y_2|. \tag{1.49}$$

wherer L is independent of t, t, y_1 and y_2. Then the Eq. (1.47) *has a unique continuous solution in* $0 \leq t \leq b$.

Now we consider some cases of the integro-differential equations and investigate existence and uniqueness of the solutions of them.

Corollary 1.5.1

$$y'(x) = f(x) + \int_0^x K(x, t, y(t))dt, \tag{1.50}$$

with initial condition $y(0) = \alpha$ where f and K are continuous functions and K satisfies the Lipschitz condition

$$| K(x, t, y_1) - K(x, t, y_2)| \leq L| y_1 - y_2| . \tag{1.51}$$

Then this problem has a unique continuous solution.

Proof. Equation (1.50) transformed to the following Volterra integral equation

$$y(s) = \alpha + \int_0^x H(s, y(s))ds, \tag{1.52}$$

where $H(s, y(s)) = f(s) + \int_0^s K(s, t, y(t))dt$,

which is in the form of Eq.(1.48), where obviously α and $H(s, y(s))$ are continuous. Therefore, for the existence and uniqueness of a continuous solution of the Eq.(1.50) it is sufficient to show that Eq. (1.52) satisfies the Lipschitz condition. To this end, we have

$$\|H(s,y_1(s)) - H(s,y_2(s))\| = \|\int_0^s (K(s,t,y_1(t)) - K(s,t,y_2(t)))dt\|$$
$$\leq L_1\|y_1 - y_2\| \int_0^s dt$$
$$\leq L_1 b\|y_1 - y_2\|.$$

So by Theorem (1.5.1), the Eq. (1.50) has a unique continuous solution. ■

Corollary 1.5.2

$$y'(x) + cy(x) = f(x) + \int_0^x K(x,t,y(t))dt, \tag{1.53}$$

with initial condition $y(0) = \alpha$ *,the* f *and* K *are continuous* (1.51) *then the equation* (1.53) *with given condition has a unique continuous solution.*

Proof. Equation (1.53) transformed to the following Volterra integral equation

$$y(s) = \alpha + \int_0^x H(s,y(s)), \tag{1.54}$$

where $H(s,y(s)) = f(s) + -cy(s) + \int_0^s K(s,t,y(t))dt$, similar to the previous corollary we only investigate the Lipschitz condition. To this end, we have

$$\|H(s,y_1(s)) - H(s,y_2(s))\| = \|c[y_1(s) - y_2(s)] + \int_0^s (K(s,t,y_1(t)) - K(s,t,y_2(t)))dt\|$$
$$\leq |c|\|y_1 - y_2\| + L_1\|y_1 - y_2\| \int_0^s dt$$
$$\leq (c + bL_1)\|y_1 - y_2\|.$$

Again, by Theorem (1.5.1), the Eq. (1.53) has a unique continuous solution. ■

Corollary 1.5.3

$$y''(x) + c_1 y(x) + c_2 y(x) = f(x) + \int_0^x K(x, t, y(t))dt, \tag{1.55}$$

with initial condition $y(0) = \alpha$, $y'(0) = \beta$ *,the* f *and* K *are continuous* (1.51) *Then the mentioned problem has a unique continuous solution.*

Proof. With the same manner, Volterra integro-differential equation(1.55) by converting it to the following Volterra integral equation

$$y(s) = \alpha + (\beta - c_1\alpha)z + \int_0^x H(s, y(s))dx.$$

where $H(s, y(s)) = -cy(s) + \int_0^x \left(f(s) - c_2 y(s) + \int_0^s K(s, t, y(t))dt \right) ds$, then we obtain

$$\begin{aligned}
&\|H(s, y_1(s)) - H(s, y_2(s))\| \\
&= \|c_1[y_2(s) - y_1(s)] + \int_0^x \left(c_2(y_2(s) - y_{1(s)}) + \int_0^s (K(s, t, y_1(t)) - K(s, t, y_2(t))dt) \right) ds\| \\
&\leq |c_1|\|y_1 - y_2\| + b|c_2|\|y_1 - y_2\| + L_1\|y_1 - y_2\| \int_0^x \int_0^s dtds \\
&\leq (|c_1| + b|c_2| + b^2 L_1)\|y_1 - y_2\|.
\end{aligned}$$

Similar to previous cases, by Theorem (1.5.1), the Eq. (1.55) has a unique continuous solution. ■

The same conclusion can be drawn for the following Volterra integro-differential equation of order n

$$y^n(x) + \int_0^x K(x, t, y(t))ds = f(x), \quad x \in [0, b]$$

with conditions $y^i(0) = \alpha_i, \;\; i = 0, 1, ..., n-1$, and similar to the previous corollaries we can convert this problem to an equation of the form (1.47).

1.6 Piecewise polynomial spaces

Let:

$$I_h = \{t_n = t_n^{(N)} : 0 = t_0^{(N)} < t_1^{(N)} < ... < t_N^{(N)} = T\}$$

denote a mesh (or: grid) on the given interval $I = [0, T]$. Define the subintervals

$$\delta_n^{(N)} = \left[t_n^{(N)}, t_{n+1}^{(N)}\right]$$

Definition 1.6.1 *For a given mesh I_h the piecewise polynomial space $S_\mu^{(d)}(I_h)$ with $\mu \geq 0, -1 \leq d \leq \mu$, is given by*

$$S_\mu^{(d)}(I_h) = \{v \in C^d(I) : v|_{\sigma_n} \in \pi_\mu(0 \leq n \leq N-1)\}$$

Here , π_μ denotes the space of (real) polynomials of degree not exceeding μ .
It is readily verified that $S_\mu^{(d)}(I_h)$ is a (real) linear vector space whose dimension is given by

$$\dim S_\mu^{(d)}(I_h) = N(\mu - d) + d + 1$$

1.7 Review of basic discrete Gronwall-type inequalities

We give general results of discrete Gronwall-type inequalities. We will need the following discrete Gronwall-type inequalities.

Lemma 1.7.1 *[14] Let $\{k_j\}_{j=0}^n$ be a given non-negative sequence and the sequence $\{\varepsilon_n\}$ satisfies $\varepsilon_0 \leq p_0$ and*

$$\varepsilon_n \leq p_0 + \sum_{i=0}^{n-1} k_i \varepsilon_i, \quad n \geq 1,$$

with $p_0 \geq 0$. Then ε_n can be bounded by

$$\varepsilon_n \leq p_0 \exp\left(\sum_{j=0}^{n-1} k_j\right), \quad n \geq 1.$$

Lemma 1.7.2 *[1] If $\{f_n\}_{n\geq 0}$, $\{g_n\}_{n\geq 0}$ and $\{\varepsilon_n\}_{n\geq 0}$ are nonnegative sequences and*

$$\varepsilon_n \leq f_n + \sum_{i=0}^{n-1} g_i \varepsilon_i, \quad n \geq 0.$$

Then,

$$\varepsilon_n \leq f_n + \sum_{i=0}^{n-1} f_i g_i \exp\left(\sum_{k=0}^{n-1} g_k\right), \quad n \geq 0.$$

CHAPTER 2

COLLOCATION ITERATIVE METHOD FOR SOLVING NONLINEAR DELAY VOLTERRA INTEGRAL EQUATION

2.1 Introduction

In this chapter, we study a numerical method for the solution of Volterra integral nonlinear equations with (constant)delay $\tau > 0$,

$$x(t) = f(t) + \int_0^t k_1(t, s, x(s))ds + \int_0^{t-\tau} k_2(t, s, x(s))ds, t \in I = [0, T], \tag{2.1}$$

with $x(t) = \Phi(t)$, $t \in [-\tau, 0]$. where the functions f, k_1, k_2 and Φ are sufficiently smooth. Equation (2.1) is frequently encountered in physical and biological modeling processes (e.g. [7, 23]). The monograph [13] presents a historical survey of mathematical models in biology, which can be described by Volterra integral equations with constant delays. The numerical solutions of Volterra integral equations have been investigated by many authors (see, for example, [3, 16, 15, 13, 12, 11, 19, 20, 25, 32]). Ali et al. [3] proposed a spectral method for pantograph-type delay integral equations by using Legendre collocation method. Brunner [13] applied the polynomial collocation method to approximate the solution of (2.1). Caliò et al. [19, 20] proposed a deficient spline collocation method and Horvat [25] used the spline collocation method to find a numerical solution of (2.1) in the spline space $S^{(d)}_{m+d}(\Pi_N)$.

The Taylor polynomial method for approximating the solution of integral equations has been proposed. Bellour and Rawashdeh [6] used Taylor method to find an approximate solution for first kind integral equations. Darania and Ivaz [24], Maleknejad and Mahmoudi [28], Sezer and Gülsu [35] applied Taylor method to certain linear and nonlinear Volterra integral equations.

This chapter is concerned with piecewise polynomial collocation method based on the use of Lagrange polynomials. Our goal is to develop an iterative explicit solution to approximate the solution of Volterra integral equation with a constant delay (2.1).

The main advantages of the current collocation method are:

1) A more direct and convergent algorithm is introduced to compute the approximation solution and this provides an explicit numerical solution of the equation (2.1) which is a basic motivation for using an iterative collocation method.

2) In the current method, there is no algebraic system needed to be solved, which makes the proposed algorithm very effective, easy to implement and the calculation cost low. The chapter is organized as follows. In section 2, we divide the interval $[0, T]$ into subintervals, and we approximate the solution of (2.1) in each interval by using Lagrange polynomials. Global convergence is established in section 3, and three numerical examples are provided in section 4. In the last section, we give a conclusion.

2.2 Description of the collocation method

Let Π_N be a uniform partition of the interval $I = [0, T]$ defined by $t_n^i = i\tau + nh, \quad i = 0, ..., r-1, \quad n = 0, ..., N-1$, where the stepsize is given by $\dfrac{\tau}{N} = h$ and $\dfrac{T}{r} = \tau$. Let the collocation parameters be $0 \le c_1 < ...... < c_m \le 1$ and the collocation points be $t_{n,j}^i = t_n^i + c_j h, \ j = 1, ..., m, i = 0, ..., r-1, n = 0, ..., N-1$. Define the subintervals $\sigma_n^i = [t_n^i, t_{n+1}^i]$, and $\sigma_{N-1}^i = [t_{N-1}^i, t_N^i]$.

Moreover, denote by π_m the set of all real polynomials of degree not exceeding m.

We define the real polynomial spline space of degree m as follows:

$$S_{m-1}^{(-1)}(I, \Pi_N) = \{u : u_n = u|_{\sigma_n^i} \in \pi_{m-1}, n = 0, .., N-1, i = 0, ..., r-1\}.$$

This is the space of piecewise polynomials of degree at most m. Its dimension is rNm. It holds for any $y \in C^m([0, T])$ that

$$y(t_n^i + sh) = \sum_{j=1}^{m} L_j(s) y(t_{n,j}^i) + \epsilon_n(s), \ \epsilon_n(s) = h^m \frac{y^m(\zeta_n)(s)}{m!} \prod_{j=1}^{m} (s - c_j), \tag{2.2}$$

where $s \in [0, 1]$ and $L_j(v) = \prod_{l \neq j}^{m} \dfrac{v - c_l}{c_j - c_l}$ are the Lagrange polynomials associate with the parameters $c_j, j = 1, ..., m$.

Inserting (2.2) into (2.1), we obtain for each $j = 1, ..., m, i = 0, ..., r-1, n = 0, ..., N-1$

$$\begin{aligned} x(t^i_{n,j}) = f(t^i_{n,j}) &+ h\sum_{l=0}^{i-1}\sum_{p=0}^{N-1}\sum_{v=1}^{m} b_v k_1(t^i_{n,j}, t^l_{pv}, x(t^l_{p,v})) + h\sum_{p=0}^{n-1}\sum_{v=1}^{m} b_v k_1(t^i_{n,j}, t^i_{pv}, x(t^i_{pv})) \\ &+ h\sum_{v=1}^{m} a_{j,v} k_1(t^i_{n,j}, t^i_{n,v}, x(t^i_{n,v})) + h\sum_{l=0}^{i-2}\sum_{p=0}^{N-1}\sum_{v=1}^{m} b_v k_2(t^i_{n,j}, t^l_{p,v}, x(t^l_{p,v})) \\ &+ h\sum_{p=0}^{n-1}\sum_{v=1}^{m} b_v k_2(t^i_{n,j}, t^{i-1}_{p,v}, x(t^{i-1}_{p,v})) + h\sum_{v=1}^{m} a_{j,v} k_2(t^i_{n,j}, t^{i-1}_{n,v}, x(t^{i-1}_{n,v})) \\ &+ o(h^m), \end{aligned} \tag{2.3}$$

such that $a_{j,v} = \int_0^{c_j} L_v(\eta)d\eta$ and $b_v = \int_0^1 L_v(\eta)d\eta$.
It holds for any $u \in S^{-1}_{m-1}(I, \Pi_N)$ that

$$u(t^i_n + sh) = \sum_{j=1}^{m} L_j(s)u(t^i_{n,j}), s \in [0, 1]. \tag{2.4}$$

Now, we approximate the exact solution x by $u \in S^{-1}_{m-1}(I, \Pi_N)$ such that $u(t^i_{n,j})$ satisfy the following nonlinear system,

$$\begin{aligned} u(t^i_{n,j}) = f(t^i_{n,j}) &+ h\sum_{l=0}^{i-1}\sum_{p=0}^{N-1}\sum_{v=1}^{m} b_v k_1(t^i_{n,j}, t^l_{pv}, u(t^l_{p,v})) + h\sum_{p=0}^{n-1}\sum_{v=1}^{m} b_v k_1(t^i_{n,j}, t^i_{pv}, u(t^i_{pv})) \\ &+ h\sum_{v=1}^{m} a_{j,v} k_1(t^i_{n,j}, t^i_{n,v}, u(t^i_{n,v})) + h\sum_{l=0}^{i-2}\sum_{p=0}^{N-1}\sum_{v=1}^{m} b_v k_2(t^i_{n,j}, t^l_{p,v}, u(t^l_{p,v})) \\ &+ h\sum_{p=0}^{n-1}\sum_{v=1}^{m} b_v k_2(t^i_{n,j}, t^{i-1}_{p,v}, u(t^{i-1}_{p,v})) + h\sum_{v=1}^{m} a_{j,v} k_2(t^i_{n,j}, t^{i-1}_{n,v}, u(t^{i-1}_{n,v})), \end{aligned} \tag{2.5}$$

for $v = 1, ..., m$, $n = 0, ..., N-1$, $i = 0, ..., r-1$, with $u(t) = \phi(t)$ on $[-\tau, 0]$.
Since the above system is nonlinear, we will use an iterative collocation solution $u^q \in S^{-1}_{m-1}(I, \Pi_N), q \in \mathbb{N}$, to approximate the exact solution of (2.1) such that

$$u^q(t^i_n + sh) = \sum_{j=1}^{m} L_j(s)u^q(t^i_{n,j}), s \in [0, 1]. \tag{2.6}$$

where the coefficients $u^q(t^i_{n,j})$ are given by the following formula:

$$\begin{aligned} u^q(t^i_{n,j}) &= f(t^i_{n,j}) + h\sum_{l=0}^{i-1}\sum_{p=0}^{N-1}\sum_{v=1}^{m} b_v k_1(t^i_{n,j}, t^l_{pv}, u^q(t^l_{p,v})) \\ &+ h\sum_{p=0}^{n-1}\sum_{v=1}^{m} b_v k_1(t^i_{n,j}, t^i_{pv}, u^q(t^i_{pv})) + h\sum_{v=1}^{m} a_{j,v} k_1(t^i_{n,j}, t^i_{n,v}, u^{q-1}(t^i_{n,v})) \\ &+ h\sum_{l=0}^{i-2}\sum_{p=0}^{N-1}\sum_{v=1}^{m} b_v k_2(t^i_{n,j}, t^l_{p,v}, u^q(t^l_{p,v})) + h\sum_{p=0}^{n-1}\sum_{v=1}^{m} b_v k_2(t^i_{n,j}, t^{i-1}_{p,v}, u^q(t^{i-1}_{p,v})) \\ &+ h\sum_{v=1}^{m} a_{j,v} k_2(t^i_{n,j}, t^{i-1}_{n,v}, u^q(t^{i-1}_{n,v})) \end{aligned} \tag{2.7}$$

such that the initial values $u^0(t^i_{n,j}) \in J$ (J is a bounded interval).
The above formula is explicit and the approximate solution u^q is given without needed to solve any algebraic system.
In the next section, we will prove the convergence of the approximate solution u^q to the exact solution x of (2.1), moreover, the order of convergence is m for all $q \geq m$.

2.3 Convergence analysis

In this section, we assume that the functions k_1 and k_2 satisfy the Lipschitz condition with respect to the third variable: there exist $L_i \geq 0$ ($i = 1, 2$) such that

$$|k_i(t, s, y_1) - k_i(t, s, y_2)| \leq L_i |y_1 - y_2|.$$

The following result gives the existence and the uniqueness of a solution for the nonlinear system (2.5), moreover this solution is bounded.

Lemma 2.3.1 *For sufficiently small h, the nonlinear system* (2.5) *has a unique solution $u \in S^{-1}_{m-1}$. Moreover, the function u is bounded.*

Proof. Claim 1. The nonlinear system (2.5) has a unique solution in $\in S^{-1}_{m-1}$.

We will use the induction combined with the Banach fixed point theorem.

(i) On the interval $\sigma_0^0 = [t_0^0, t_1^0]$,
for $j = 1...m$,where $x(t) = \Phi(t)$ for $t \in [-\tau, 0]$,.

$$u(t_{0,j}^0) = f(t_{0,j}^0) + h\sum_{v=1}^{m} a_{j,v}k_1(t_{0,j}^0, t_{0,v}^0, u(t_{0,v}^0)) + h\sum_{v=1}^{m} a_{j,v}k_2(t_{0,j}^0, t_{0,v}^0 - \tau, \phi(t_{0,v}^0))$$

We put :$F_0^0 : R^m \to R^m \quad for \quad j = 1...m$ so

$$F_{0,j}^0(x) = f(t_{0,j}^0) + h\sum_{v=1}^{m} a_{j,v}k_1(t_{0,j}^0, t_{0,v}^0, x_v) + h\sum_{v=1}^{m} a_{j,v}k_2(t_{0,j}^0, t_{0,v}^0 - \tau, \phi(t_{0,v}^0)),$$

from Banach fixed point theorem, we have
$\|F_0^0(x) - F_0^0(y)\| \leq hL_1\|x - y\|$ so u is exists and unique on σ_0^0 for h is sufficiently small

(ii) Suppose that u exists and unique on each interval $\sigma_k^l, l = 0, ..., i-1, k = 0, ..., N-1$ and we show that u exists and unique on $\sigma_n^i = [t_n^i, t_{n+1}^i]$, $j = 1, ..., m$, due to (2,5). Hence,

$$\begin{aligned} F_{n,j}^i(x) = f(t_{n,j}^i) &+ h\sum_{l=0}^{i-1}\sum_{p=0}^{N-1}\sum_{v=1}^{m} b_v k_1(t_{n,j}^i, t_{pv}^l, u(t_{p,v}^l)) + h\sum_{p=0}^{n-1}\sum_{v=1}^{m} b_v k_1(t_{n,j}^i, t_{pv}^i, u(t_{pv}^i)) \\ &+ h\sum_{v=1}^{m} a_{j,v}k_1(t_{n,j}^i, t_{n,v}^i, x_v) + h\sum_{l=0}^{i-2}\sum_{p=0}^{N-1}\sum_{v=1}^{m} b_v k_2(t_{n,j}^i, t_{p,v}^l, u(t_{p,v}^l)) \\ &+ h\sum_{p=0}^{n-1}\sum_{v=1}^{m} b_v k_2(t_{n,j}^i, t_{p,v}^{i-1}, u(t_{p,v}^{i-1})) + h\sum_{v=1}^{m} a_{j,v}k_2(t_{n,j}^i, t_{n,v}^{i-1}, u(t_{n,v}^{i-1})) \end{aligned}$$

whenever, $i = 0....r-1, n = 0...N-1, j = 1...m$, we have

$$\|F_{n,j}^i(x) - F_{n,j}^i(y)\| \leq hL_1\|x - y\|$$

so u exists and unique for all σ_n^i and h is sufficiently small.

Claim 2. The solution u is bounded.

by using (2.5), with the functions k_1, k_2 satisfies the Lipschitz condition with respect to the third variable, we obtain

$$\begin{aligned}\left|u(t^i_{n,j})\right| \leq \alpha + hL_1 \sum_{l=0}^{i-1}\sum_{p=0}^{N-1}\sum_{v=1}^{m} b_v \left|u(t^l_{p,v})\right| + hL_2 \sum_{l=0}^{i-2}\sum_{p=0}^{N-1}\sum_{v=1}^{m} b_v \left|u(t^l_{p,v})\right| \\ + hL_1 \sum_{p=0}^{n-1}\sum_{v=1}^{m} b_v \left|u(t^i_{pv})\right| + hL_2 \sum_{p=0}^{n-1}\sum_{v=1}^{m} b_v \left|u(t^{i-1}_{p,v}))\right| \\ + hL_1 \sum_{v=1}^{m} a_{j,v} \left|u(t^i_{n,v})\right| + hL_2 \sum_{v=1}^{m} a_{j,v} \left|u(t^{i-1}_{n,v}))\right|,\end{aligned}$$

we put $\alpha = \left\|f\right\| + (\tau + T + h)(||k_1|| + ||k_2||)$ let $y^i_n = max\{u(t^i_{n,P}), p = 1....m\}$, we have

$$\begin{aligned}y^i_n - hL_1 y^i_n &\leq \alpha + hL_1 \sum_{l=0}^{i-1}\sum_{p=0}^{N-1} y^l_p + hL_2 \sum_{l=0}^{i-2}\sum_{p=0}^{N-1} y^l_p \\ &+ hL_2 \sum_{p=0}^{n-1} y^{i-1}_p + hL_2 y^{i-1}_n + hL_1 \sum_{p=0}^{n-1} y^i_p \\ &\leq \alpha + h(L_1 + 3L_2) \sum_{l=0}^{i-1}\sum_{p=0}^{N-1} y^l_p + hL_1 \sum_{p=0}^{n-1} y^i_p\end{aligned}$$

Hence, for all $h \in (0, \frac{1}{2L_1}]$, we have

$$y^i_n \leq 2\alpha + hL_3 \sum_{l=0}^{i-1}\sum_{p=0}^{N-1} y^l_p + hL_4 \sum_{p=0}^{n-1} y^i_p,$$

where $L_3 = 6L_2 + 2L_1$ and $L_4 = 2L_1$ Then, by Lemma 1.7.1, we obtain

$$\begin{aligned}y^i_n &\leq (2\alpha + hL_3 \sum_{l=0}^{i-1}\sum_{p=0}^{N-1} y^l_p) \exp(\tau L_4) \\ &\leq \alpha_2 + hL_5 \sum_{l=0}^{i-1}\sum_{p=0}^{N-1} y^l_p\end{aligned}$$

We put $z^i_n = max\{y^i_n, n = 0....N-1\}$,we have

$$z^i \leq \alpha_2 + hL_5 \sum_{l=0}^{i-1} Nz^l$$

Therefore, by Lemma (1.7.1), we obtain

$$z^i \leq \alpha_2 \exp(TL_5)$$

So $(u(t^i_{n,j}))$ is bounded ■ The following result gives the convergence of the approximate solution u to the exact solution x.

Theorem 2.3.1 *Let f, k_1, k_2 and Φ be m times continuously differentiable on their respective domains. Then for sufficiently small h, the collocation solution u converge to the exact solution x, and the resulting error function $e := x - u$ satisfies:*

$$||e||_{L^\infty(I)} \leq Ch^m,$$

where C is a finite constant independent of h.

Proof. we calculate the error between x and the approximate solution u for $v = 1.2...m$, $n = 0.1.2......N-1$, $i = 0...r-1$

Using the expression ((2,3)) and ((2,5)), and setting $e := x - u$ is the collocation error then

$$\begin{aligned}
|e(t^i_{n,j})| \leq{} & hL_1 \sum_{l=0}^{i-1} \sum_{p=0}^{N-1} \sum_{v=1}^{m} b_v \left|e(t^l_{p,v})\right| + hL_2 \sum_{l=0}^{i-2} \sum_{p=0}^{N-1} \sum_{v=1}^{m} b_v \left|e(t^l_{p,v})\right| \\
& + hL_1 \sum_{p=0}^{n-1} \sum_{v=1}^{m} b_v \left|e(t^i_{pv})\right| + hL_2 \sum_{p=0}^{n-1} \sum_{v=1}^{m} b_v \left|e(t^{i-1}_{p,v}))\right| \\
& + hL_1 \sum_{v=1}^{m} a_{j,v} \left|e(t^i_{n,v})\right| + hL_2 \sum_{v=1}^{m} a_{j,v} \left|e(t^{i-1}_{n,v}))\right|,
\end{aligned}$$

let $e_n^i = max\{e(t_{n,v}^i), v = 1...m\}$,we have

$$e_n^i \leq hL_3 \sum_{l=0}^{i-1} \sum_{p=0}^{N-1} e_p^l + hL_1 \sum_{p=0}^{n-1} e_p^i + o(h^m), L_3 = 3L_2 + L_1$$

Hence, for all $h \in (0, \frac{1}{2L_1}]$, we have

$$e_n^i \leq 2hL_3 \sum_{l=0}^{i-1} \sum_{p=0}^{N-1} e_p^l + 2hL_1 \sum_{p=0}^{n-1} e_p^i + 2o(h^m)$$

Then, by Lemma 1.7.1, we obtain

$$\begin{aligned} e_n^i &\leq (2hL_3 \sum_{l=0}^{i-1} \sum_{p=0}^{N-1} e_p^l + 2o(h^m)) \exp(2hL_1 N) \\ &\leq h\alpha \sum_{l=0}^{i-1} \sum_{p=0}^{N-1} e_p^l + ch^m, \alpha = 2hL_3 \exp(2hL_1 N) \end{aligned}$$

let $e^i = max\{e_n^i, n = 0...N-1\}$,

$$\begin{aligned} e^i &\leq \tau\alpha \sum_{l=0}^{i-1} e_p^l + ch^m \\ &\leq ch^m \exp(T\alpha) \\ &\leq Ch^m. \end{aligned}$$

Thus, the proof is completed by taking $C = c\exp(T\alpha)$ ■ The following result gives the convergence of the iterative solution u^q to the exact solution x.

Theorem 2.3.2 *Consider the iterative collocation solution $u^q(t_{n,j}^i)$ defined by (2,6), then for any initial condition $u^0(t_{n,j}^i) \in J$, the sequence $u^q(t_{n,j}^i)$ converges to the exact solution x. Moreover, the following error estimates hold*

$$\left|u^q(t_{n,j}^i) - x\right| \leq (hd)^q \left|(u)^0 - x\right| + Cd^q h^{m+q} + Ch^m$$

where $d = L_1 \exp(\tau L_1) + rL_1 \exp(\tau L_1)\tau(L_1 + 2L_2)\exp(\tau L_1)\exp(r\tau(L_1 + 2L_2)\exp(\tau L_1))$,

Proof. Let $(e_n^i)^q = max \left| u^{q+1}(t_{p,v}^l) - u(t_{p,v}^l) \right| v = 1....m$

$$\begin{aligned}(e_n^i)^{q+1} &\leq hL_1 \sum_{l=0}^{i-1} \sum_{p=0}^{N-1} (e_p^l)^{q+1} + hL_1 \sum_{p=0}^{n-1} (e_p^i)^{q+1} + hL_1 (e_n^i)^q \\ &+ hL_2 \sum_{l=0}^{i-2} \sum_{p=0}^{N-1} (e_p^l)^{q+1} + hL_2 \sum_{p=0}^{n-1} (e_p^{i-1})^{q+1} + hL_2 (e_n^{i-1})^{q+1} \\ &\leq hL_3 \sum_{l=0}^{i-1} \sum_{p=0}^{N-1} (e_p^l)^{q+1} + hL_1 (e_n^i)^q + hL_1 \sum_{p=0}^{n-1} (e_p^i)^{q+1}, L_3 = L_1 + 3L_2\end{aligned}$$

We put $e^i = max\{e_n^i, n = 0....N-1\}$, This implies,

$$(e_n^i)^{q+1} \leq hL_1 (e^i)^q + \tau L_3 \sum_{l=0}^{i-1} (e^l)^{q+1} + hL_1 \sum_{p=0}^{n-1} (e_p^i)^{q+1}$$

a well-known result on discrete Gronwall inequalities (see, e.g, Lemma (1.7.1)) leads to

$$\begin{aligned}(e_n^i)^{q+1} &\leq (hL_1 (e^i)^q + \tau L_3 \sum_{l=0}^{i-1} (e^l)^{q+1}) \exp(\tau L_1) \\ &\leq hL_4 (e^i)^q + L_5 \sum_{l=0}^{i-1} (e^l)^{q+1}, L_4 = L_1 \exp(\tau L_1), L_5 = \tau L_3 \exp(\tau L_1),\end{aligned}$$

Then, by Lemma (1.7.2), we obtain

$$(e^i)^{q+1} \leq hL_4 (e^i)^q + \sum_{l=0}^{i-1} hL_4 (e^i)^q L_5 \exp(rL_5),$$

let $(e)^q = max\{(e^i)^q, i = 0....r-1\}$, we have

$$\begin{aligned}(e)^{q+1} &\leq hL_4 (e)^q + hrL_4 (e)^q L_5 \exp(rL_5), \\ &\leq hd(e)^q \qquad d = L_4 + rL_4 L_5 \exp(rL_5), \\ &\leq (hd)^q \left| (u)^0 - x \right| + Cd^q h^{m+q}\end{aligned}$$

for all $j = 1...m, i = 0, ...r-1, n = 0, ...N-1, q \in N^*$ This implies,

$$\begin{aligned}\left|u^q(t^i_{n,j}) - x(t^i_{n,j})\right| &\leq \left|u^q(t^i_{n,j}) - u(t^i_{n,j})\right| + \left|u(t^i_{n,j}) - x(t^i_{n,j})\right| \\ &\leq (hd)^q \left|(u)^0 - x\right| + Cd^q h^{m+q} + Ch^m\end{aligned}$$

■

2.4 Numerical examples

To illustrate the theoretical results, we present the following three examples with $\tau = 0.5$ and $T = 1$. All the exact solutions x are already known. In each example, we calculate the error between x and the iterative collocation solution u^m.

The results in these examples confirm the theoretical results; moreover, the absolute error decreases as N or m increases.

Example 2.4.1 *Here, the functions characterizing equation* (2.1) *are given by* $k_1(t,s,z) = \sin(t-s)\cos(2z-s) + 3$, $k_2(t,s,z) = \frac{t\cos(s-z)}{1+st}$, *and f is chosen so that the exact solution is* $x(t) = t+1$

The absolute errors for $(m, N) = \{(3,5), (4,5), (4,10), (5,10)\}$ *at* $t = 0, 0.2, ..., 1$ *are presented in Table 2.1.*

Table 2.1: Absolute errors of Example 4.1

t	$m = 3, N = 5$	$m = 4, N = 5$	$m = 4, N = 10$	$m = 5, N = 10$
0.0	0.267×10^{-5}	0.136×10^{-6}	0.6×10^{-8}	0.12×10^{-7}
0.2	0.684×10^{-5}	0.109×10^{-5}	0.56×10^{-7}	0.2×10^{-7}
0.4	0.159×10^{-4}	0.198×10^{-5}	0.122×10^{-6}	0.7×10^{-8}
0.6	0.239×10^{-4}	0.269×10^{-5}	0.177×10^{-6}	0.9×10^{-8}
0.8	0.304×10^{-4}	0.324×10^{-5}	0.193×10^{-6}	0.35×10^{-7}
1.0	0.109×10^{-4}	0.316×10^{-5}	0.22×10^{-6}	0.4×10^{-7}

Example 2.4.2 *Consider the nonlinear Volterra delay integral equations*

$$x(t) = f(t) + \int_0^t k_1(t,s,x(s))ds + \int_0^{t-\tau} k_2(t,s,x(s))ds, \; t \in [0,1],$$

with $k_1(t,s,z) = s\sin(t+2z-s)$, $k_2(t,s,z) = \frac{se^{t-z}}{1+t}$ *and* f *is chosen so that the exact solution is* $x(t) = 2t+1$.

The absolute errors for $(m,N) = \{(4,4),(5,5),(6,6),(8,8)\}$ *at* $t = 0, 0.2, ..., 1$ *are presented in Table 2.2.*

Table 2.2: Absolute errors of Example 4.2

t	$m=4, N=5$	$m=5, N=5$	$m=7, N=5$	$m=7, N=10$
0.0	0.146×10^{-5}	0.69×10^{-7}	0.25×10^{-7}	0.33×10^{-7}
0.2	0.145×10^{-4}	0.342×10^{-6}	0.68×10^{-7}	0.6×10^{-8}
0.4	0.272×10^{-4}	0.75×10^{-7}	0.34×10^{-7}	0.6×10^{-8}
0.6	0.256×10^{-4}	0.82×10^{-6}	0.27×10^{-7}	0.4×10^{-8}
0.8	0.193×10^{-4}	0.20×10^{-5}	0.59×10^{-7}	0.56×10^{-7}
1.0	0.147×10^{-3}	0.13×10^{-4}	0.39×10^{-6}	0.7×10^{-7}

Example 2.4.3 *The given functions in equation* (2.1) *are*
$k_1(t,s,z) = 2\cos(t+z-s)s^2$, $k_2(t,s,z) = \frac{s^{tz}}{1+t^2}$ *and* f *is such that equation* (2.1) *possesses the solution* $x(t) = \sin(t)+1$.

The absolute errors for $(m,N) = \{(2,5),(4,5),(5,5),(6,10)\}$ *at* $t = 0, 0.2, ..., 1$ *are presented in Table 2.3.*

Table 2.3: Absolute errors of Example 4.3

t	$m=2, N=5$	$m=4, N=5$	$m=5, N=5$	$m=6, N=10$
0.2	0.123×10^{-3}	0.81×10^{-7}	0.14×10^{-7}	0.1×10^{-8}
0.2	0.472×10^{-3}	0.16×10^{-6}	0.17×10^{-7}	0.2×10^{-8}
0.4	0.794×10^{-3}	0.34×10^{-6}	0.4×10^{-8}	0.41×10^{-7}
0.6	0.232×10^{-2}	0.58×10^{-5}	0.19×10^{-7}	0.12×10^{-7}
0.8	0.813×10^{-2}	0.83×10^{-4}	0.312×10^{-6}	0.52×10^{-7}
1	0.599×10^{-2}	0.22×10^{-4}	0.245×10^{-6}	0.1×10^{-7}

CONCLUSION AND PERSPECTIVE

In this dissertation, we have used a an iterative collocation method based on the use Lagrange polynomials for the numerical solution of nonlinear Volterra delay integral equations (2.1) in the spline space $S_{m-1}^{(-1)}(\Pi_N)$. The main advantages of this method is the study of the convergence, this method is easy to implement and the coefficients of the approximation solution are determined by using iterative formulas without the need to solve any system of algebraic equations. Numerical examples showing that the method is convergent with a good accuracy and the numerical results confirmed the theoretical estimates.

Further researches will be conducted by generalizing this method to approximate Nonlinear Delay Volterra integro-differential equations :

$$x'(t) = f(t) + \int_0^t k_1(t, s, x(s))ds + \int_0^{t-\tau} k_2(t, s, x(s))ds, t \in I = [0, T],$$

with $x'(0) = x_0$ $x(t) = \Phi(t)$, $t \in [-\tau, 0]$.

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