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On a system of difference equations of third order solved in closed form

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Abstract. In this work, we show that the system of difference equations

$$x_{n+1} = \frac{ay_{n-2}x_{n-1}y_n + bx_{n-1}y_{n-2} + cy_{n-2} + d}{y_{n-2}x_{n-1}y_n},$$

$$y_{n+1} = \frac{ax_{n-2}y_{n-1}x_n + by_{n-1}x_{n-2} + cx_{n-2} + d}{x_{n-2}y_{n-1}x_n},$$

where $n \in \mathbb{N}_0$, x_{-2}, x_{-1}, x_0, y_{-2}, y_{-1} and y_0 are arbitrary nonzero real numbers and a, b, c and d are arbitrary real numbers with $d \neq 0$, can be solved in a closed form.

We will see that when $a = b = c = d = 1$ the solutions are expressed using the famous Tetranacci numbers. In particular, the results obtained here extend those in our recent work.

Keywords: System of difference equations, general solution, Tetranacci numbers.

2020 Mathematics Subject Classification: 39A05, 39A06, 39A10.

1 Introduction

Nonlinear difference equations and their systems are hot topics that attract the attention of several researchers. A significant number of papers are devoted to this field of research. One can consult, for example, the papers [3, 5–18, 20–23, 26, 27, 30, 31, 36–44, 46], where one can find concrete models of such equations and systems, as well as understand the techniques used to solve them and investigate the behavior of their solutions. Recently, in [1] and as a

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generalization of the equations and systems studied in [4,19,32,45], we have solved in a closed form the system of difference equations

$$
\begin{cases}
x_{n+1} = \dfrac{a y_n x_{n-1} + b x_{n-1} + c}{x_{n-1} y_n}, \\[2mm]
y_{n+1} = \dfrac{a x_n y_{n-1} + b y_{n-1} + c}{y_{n-1} x_n}.
\end{cases}
\tag{1.1}
$$

Here, and motivated by the above papers, one shows that one can express in closed form the well-defined solutions of the following system of difference equations

$$
\begin{cases}
x_{n+1} = \dfrac{a y_{n-2} x_{n-1} y_n + b x_{n-1} y_{n-2} + c y_{n-2} + d}{y_{n-2} x_{n-1} y_n}, \\[2mm]
y_{n+1} = \dfrac{a x_{n-2} y_{n-1} x_n + b y_{n-1} x_{n-2} + c x_{n-2} + d}{x_{n-2} y_{n-1} x_n},
\end{cases}
\tag{1.2}
$$

where $n \in \mathbb{N}_0$, the initial values x_{-2}, x_{-1}, x_0, y_{-2}, y_{-1} and y_0 are arbitrary nonzero real numbers and the parameters a, b, c and d are arbitrary real numbers with $d \neq 0$.

Clearly if $d = 0$, then System (1.2) is nothing other than system (1.1). For the readers interested in the solutions of this system, one refers to [1], where the system (1.1) has been completely solved.

Noting also that the system (1.2) can be seen as a generalization of the equation

$$
x_{n+1} = \frac{a x_{n-2} x_{n-1} x_n + b x_{n-1} x_{n-2} + c x_{n-2} + d}{x_{n-2} x_{n-1} x_n}, \ n \in \mathbb{N}_0.
\tag{1.3}
$$

In fact, the solutions of (1.3) can be obtained from the solutions of (1.2) by choosing $y_{-i} = x_{-i}$, $i = 0,1,2$. The equation (1.3) was the subject of a substantial part of the paper [4], which also motivated our present study. The same equation was studied in complex numbers by Stevic in [29].

We will see that the explicit formulas of the well defined solutions of system (1.2) are expressed using the terms of the sequence $(J_n)_{n=0}^{+\infty}$ which are the solutions of the fourth-order linear homogeneous difference equation defined by the relation

$$
J_{n+4} = a J_{n+3} + b J_{n+2} + c J_{n+1} + d J_n, \quad n \in \mathbb{N}_0,
\tag{1.4}
$$

and the special initial values

$$
J_0 = 0, \quad J_1 = 0, \quad J_2 = 1 \text{ and } J_3 = a.
\tag{1.5}
$$

In this article one solves in closed form the equation (3.3). This well-known equation (with the same or different initial values and parameters) was the subject of some papers in the literature, see for example [25,29,47].

The characteristic equation associated to (3.3) is

$$
\lambda^4 - a\lambda^3 - b\lambda^2 - c\lambda - d = 0,
\tag{1.6}
$$

and let α, β, γ and δ its four roots, then

$$
\begin{cases}
\alpha + \beta + \gamma + \delta = a \\
\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = -b \\
\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = c \\
\alpha\beta\gamma\delta = -d
\end{cases}
\tag{1.7}
$$

One has:

Case 1: If all roots are real and equal. In this case,

$$J_n = \left(c_1 + c_2 n + c_3 n^2 + c_4 n^3\right) \alpha^n.$$

Now using (1.7) and the fact that $J_0 = 0$, $J_1 = 0$, $J_2 = 1$ and $J_3 = a$, one obtains

$$J_n = \left(\frac{-n + n^3}{6\alpha^2}\right) \alpha^n. \tag{1.8}$$

Case 2: If three roots are real and equal, i.e. $\beta = \gamma = \delta$. In this case

$$J_n = c_1 \alpha^n + \left(c_2 + c_3 n + c_4 n^2\right) \beta^n.$$

Now using (1.7) and the fact that $J_0 = 0$, $J_1 = 0$, $J_2 = 1$ and $J_3 = a$, one obtains

$$J_n = \frac{-\alpha}{(\beta - \alpha)^3} \alpha^n + \left(\frac{\alpha}{(\beta - \alpha)^3} - \frac{n(\alpha + \beta)}{2\beta(\beta - \alpha)^2} + \frac{n^2}{2\beta(\beta - \alpha)}\right) \beta^n, \tag{1.9}$$

Case 3: If two real roots are equal, i.e. $\gamma = \delta$. In this case

$$J_n = c_1 \alpha^n + c_2 \beta^n + \left(c_3 + c_4 n\right) \gamma^n.$$

Now using (1.7) and the fact that $J_0 = 0$, $J_1 = 0$, $J_2 = 1$ and $J_3 = a$, one obtains

$$J_n = \frac{-\alpha}{(\gamma - \alpha)^2(\beta - \alpha)} \alpha^n + \frac{\beta}{(\gamma - \beta)^2(\beta - \alpha)} \beta^n + \left(\frac{\alpha\beta - \gamma^2}{(\gamma - \alpha)^2(\gamma - \beta)^2} + \frac{n}{(\gamma - \alpha)(\gamma - \beta)}\right) \gamma^n, \tag{1.10}$$

Case 4: If two double real roots are equal, i.e. $\alpha = \beta \neq \gamma = \delta$. In this case

$$J_n = \left(c_1 + c_2 n\right) \alpha^n + \left(c_3 + c_4 n\right) \gamma^n.$$

Now using (1.7) and the fact that $J_0 = 0$, $J_1 = 0$, $J_2 = 1$ and $J_3 = a$, one obtains

$$J_n = \left(\frac{\gamma + \alpha}{(\gamma - \alpha)^3} + \frac{n}{(\gamma - \alpha)^2}\right) \alpha^n + \left(-\frac{\gamma + \alpha}{(\gamma - \alpha)^3} + \frac{n}{(\gamma - \alpha)^2}\right) \gamma^n, \tag{1.11}$$

Case 5: If all the roots are real and different. In this case

$$J_n = c_1 \alpha^n + c_2 \beta^n + c_3 \gamma^n + c_4 \delta^n.$$

Again, using (1.7) and the fact that $J_0 = 0$, $J_1 = 0$, $J_2 = 1$ and $J_3 = a$, one obtains

$$
\begin{aligned}
J_n &= \frac{-\alpha}{(\delta - \alpha)(\gamma - \alpha)(\beta - \alpha)} \alpha^n + \frac{\beta}{(\delta - \beta)(\gamma - \beta)(\beta - \alpha)} \beta^n + \frac{-\gamma}{(\delta - \gamma)(\gamma - \beta)(\gamma - \alpha)} \gamma^n \\
&\quad + \frac{\delta}{(\delta - \gamma)(\delta - \beta)(\delta - \alpha)} \delta^n.
\end{aligned}
\tag{1.12}
$$

Case 6: If two real roots are equal, i.e. $\alpha = \beta$ and two roots are complex conjugate, i.e. $\delta = \overline{\gamma}$. In this case

$$J_n = \left(c_1 + c_2 n\right) \alpha^n + c_3 \gamma^n + c_4 \overline{\gamma}^n.$$

Again, using (1.7) and the fact that $J_0 = 0$, $J_1 = 0$, $J_2 = 1$ and $J_3 = a$, one obtains

$$J_n = \left(\frac{\overline{\gamma}\gamma - \alpha^2}{(\overline{\gamma} - \alpha)^2(\gamma - \alpha)^2} + \frac{n}{(\overline{\gamma} - \alpha)(\gamma - \alpha)} \right) \alpha^n + \frac{-\gamma}{(\overline{\gamma} - \gamma)(\gamma - \alpha)^2}\gamma^n$$
$$+ \frac{\overline{\gamma}}{(\overline{\gamma} - \gamma)(\overline{\gamma} - \alpha)^2}\overline{\gamma}^n. \tag{1.13}$$

Case 7: If two real roots α, β are different and two roots are complex conjugate, i.e. $\delta = \overline{\gamma}$. In this case

$$J_n = c_1\alpha^n + c_2\beta^n + c_3\gamma^n + c_4\overline{\gamma}^n.$$

Again, using (1.7) and the fact that $J_0 = 0$, $J_1 = 0$, $J_2 = 1$ and $J_3 = a$, one obtains

$$J_n = \frac{-\alpha}{(\overline{\gamma} - \alpha)(\gamma - \alpha)(\beta - \alpha)}\alpha^n + \frac{\beta}{(\overline{\gamma} - \beta)(\gamma - \beta)(\beta - \alpha)}\beta^n + \frac{-\gamma}{(\overline{\gamma} - \gamma)(\gamma - \beta)(\gamma - \alpha)}\gamma^n$$
$$+ \frac{\overline{\gamma}}{(\overline{\gamma} - \gamma)(\overline{\gamma} - \beta)(\overline{\gamma} - \alpha)}\overline{\gamma}^n. \tag{1.14}$$

Case 8: If two complex roots are equal, i.e. $\alpha = \gamma$ and $\beta = \delta = \overline{\alpha}$. In this case

$$J_n = (c_1 + c_2 n)\alpha^n + (c_3 + c_4 n)\overline{\alpha}^n.$$

Again, using (1.7) and the fact that $J_0 = 0$, $J_1 = 0$, $J_2 = 1$ and $J_3 = a$, one obtains

$$J_n = \left(\frac{\overline{\alpha} + \alpha}{(\overline{\alpha} - \alpha)^3} + \frac{n}{(\overline{\alpha} - \alpha)^2} \right) \alpha^n + \left(\frac{-\overline{\alpha} - \alpha}{(\overline{\alpha} - \alpha)^3} + \frac{n}{(\overline{\alpha} - \alpha)^2} \right) \overline{\alpha}^n. \tag{1.15}$$

Case 9: If the roots are all complex and different, i.e. $\beta = \overline{\alpha}$ and $\delta = \overline{\gamma}$. In this case

$$J_n = c_1\alpha^n + c_2\overline{\alpha}^n + c_3\gamma^n + c_4\overline{\gamma}^n.$$

Again, using (1.7) and the fact that $J_0 = 0$, $J_1 = 0$, $J_2 = 1$ and $J_3 = a$, one obtains

$$J_n = \frac{-\alpha}{(\overline{\gamma} - \alpha)(\gamma - \alpha)(\overline{\alpha} - \alpha)}\alpha^n + \frac{\overline{\alpha}}{(\overline{\gamma} - \overline{\alpha})(\gamma - \overline{\alpha})(\overline{\alpha} - \alpha)}\overline{\alpha}^n + \frac{-\gamma}{(\overline{\gamma} - \gamma)(\gamma - \overline{\alpha})(\gamma - \alpha)}\gamma^n$$
$$+ \frac{\overline{\gamma}}{(\overline{\gamma} - \gamma)(\overline{\gamma} - \overline{\alpha})(\overline{\gamma} - \alpha)}\overline{\gamma}^n. \tag{1.16}$$

2   The main theorem and some particular cases

Here, one gives a closed form for the well defined solutions of the system (1.2) with $d \neq 0$. One will use the same change of variables as in [1] to transform the system (1.2) to a linear one and then follows the same procedure as in [1] to obtain the closed-form of the solutions. To get the solutions of the corresponding linear system, one needs to solve some fourth-order linear difference equations. In particular, one derives from the main result (Main Theorem), for which one leaves the proof to the next section, the solutions of some particular systems and equations where their solutions are related to the famous Tetranacci numbers.

One recalls that by a well defined solution of system (1.2), one means a solution that satisfies $x_n y_n \neq 0$, $n \geq -2$. The set of well defined solutions is not empty. In fact, it suffices to choose the initial values and the parameters a, b, c and d positive, to see that every solution of (1.2) will be well defined.

2.1 Closed form of well defined solutions of the system (1.2)

The following result gives an explicit formula for well-defined solutions of the system (1.2).

Theorem 2.1. *(Main Theorem) Let $\{x_n, y_n\}_{n \geq -2}$ be a well defined solution of (1.2). Then, for $n \in \mathbb{N}_0$, one has*

$$x_{2n+1} = \frac{dJ_{2n+2} + (cJ_{2n+2} + dJ_{2n+1}) y_{-2} + (J_{2n+4} - aJ_{2n+3}) x_{-1}y_{-2} + J_{2n+3}y_0 x_{-1}y_{-2}}{dJ_{2n+1} + (cJ_{2n+1} + dJ_{2n}) y_{-2} + (J_{2n+3} - aJ_{2n+2}) x_{-1}y_{-2} + J_{2n+2}y_0 x_{-1}y_{-2}},$$

$$x_{2n+2} = \frac{dJ_{2n+3} + (cJ_{2n+3} + dJ_{2n+2}) x_{-2} + (J_{2n+5} - aJ_{2n+4}) y_{-1}x_{-2} + J_{2n+4}x_0 y_{-1}x_{-2}}{dJ_{2n+2} + (cJ_{2n+2} + dJ_{2n+1}) x_{-2} + (J_{2n+4} - aJ_{2n+3}) y_{-1}x_{-2} + J_{2n+3}x_0 y_{-1}x_{-2}},$$

$$y_{2n+1} = \frac{dJ_{2n+2} + (cJ_{2n+2} + dJ_{2n+1}) x_{-2} + (J_{2n+4} - aJ_{2n+3}) y_{-1}x_{-2} + J_{2n+3}x_0 y_{-1}x_{-2}}{dJ_{2n+1} + (cJ_{2n+1} + dJ_{2n}) x_{-2} + (J_{2n+3} - aJ_{2n+2}) y_{-1}x_{-2} + J_{2n+2}x_0 y_{-1}x_{-2}},$$

$$y_{2n+2} = \frac{dJ_{2n+3} + (cJ_{2n+3} + dJ_{2n+2}) y_{-2} + (J_{2n+5} - aJ_{2n+4}) x_{-1}y_{-2} + J_{2n+4}y_0 x_{-1}y_{-2}}{dJ_{2n+2} + (cJ_{2n+2} + dJ_{2n+1}) y_{-2} + (J_{2n+4} - aJ_{2n+3}) x_{-1}y_{-2} + J_{2n+3}y_0 x_{-1}y_{-2}},$$

where the initial values $x_{-2}, x_{-1}, x_0, y_{-2}, y_{-1}$ and $y_0 \in (\mathbb{R} - \{0\}) - F$, with F is the Forbidden set of system (1.2) given by

$$F = \bigcup_{n=0}^{\infty} \{(x_{-2}, x_{-1}, x_0, y_{-2}, y_{-1}, y_0) \in (\mathbb{R} - \{0\}) : A_n = 0 \text{ or } B_n = 0\},$$

where

$$A_n = dJ_{n+1} + (cJ_{n+1} + dJ_n) y_{-2} + (J_{n+3} - aJ_{n+2}) x_{-1}y_{-2} + J_{n+2}y_0 x_{-1}y_{-2},$$
$$B_n = dJ_{n+1} + (cJ_{n+1} + dJ_n) x_{-2} + (J_{n+3} - aJ_{n+2}) y_{-1}x_{-2} + J_{n+2}x_0 y_{-1}x_{-2}.$$

2.2 Particular cases

Now, we focus our study on some particular cases of system (1.2).

2.2.1 The solutions of the equation $x_{n+1} = (ax_{n-2}x_{n-1}x_n + bx_{n-1}x_{n-2} + cx_{n-2} + d) / (x_{n-2}x_{n-1}x_n)$

If one chooses $y_{-2} = x_{-2}$, $y_{-1} = x_{-1}$ and $y_0 = x_0$, then system (1.2) is reduced to the equation

$$x_{n+1} = \frac{ax_{n-2}x_{n-1}x_n + bx_{n-1}x_{n-2} + cx_{n-2} + d}{x_{n-2}x_{n-1}x_n}, \quad n \in \mathbb{N}_0. \tag{2.1}$$

So, it follows from the **Main Theorem**

Corollary 2.2. *Let $\{x_n\}_{n \geq -2}$ be a well defined solution of the equation (2.1). Then for $n \in \mathbb{N}_0$, one has*

$$x_{2n+1} = \frac{dJ_{2n+2} + (cJ_{2n+2} + dJ_{2n+1}) x_{-2} + (J_{2n+4} - aJ_{2n+3}) x_{-1}x_{-2} + J_{2n+3}x_0 x_{-1}x_{-2}}{dJ_{2n+1} + (cJ_{2n+1} + dJ_{2n}) x_{-2} + (J_{2n+3} - aJ_{2n+2}) x_{-1}x_{-2} + J_{2n+2}x_0 x_{-1}x_{-2}},$$

$$x_{2n+2} = \frac{dJ_{2n+3} + (cJ_{2n+3} + dJ_{2n+2}) x_{-2} + (J_{2n+5} - aJ_{2n+4}) x_{-1}x_{-2} + J_{2n+4}x_0 x_{-1}x_{-2}}{dJ_{2n+2} + (cJ_{2n+2} + dJ_{2n+1}) x_{-2} + (J_{2n+4} - aJ_{2n+3}) x_{-1}x_{-2} + J_{2n+3}x_0 x_{-1}x_{-2}}.$$

It is worth noting that this equation was studied in [4, 29].

2.3 The solutions of the system (1.2) with $a = b = c = d = 1$

Consider the system

$$\begin{cases} x_{n+1} = \dfrac{y_{n-2}x_{n-1}y_n + x_{n-1}y_{n-2} + y_{n-2} + 1}{y_{n-2}x_{n-1}y_n}, \\ y_{n+1} = \dfrac{x_{n-2}y_{n-1}x_n + y_{n-1}x_{n-2} + x_{n-2} + 1}{x_{n-2}y_{n-1}x_n}, \quad n \in \mathbb{N}_0, \end{cases} \tag{2.2}$$

which is a particular case of the system (1.2) with $a = b = c = d = 1$. In this case the sequence $\{J_n\}$ is nothing other than the sequence of Tetranacci numbers $\{T_n\}$, that is

$$T_{n+4} = T_{n+3} + T_{n+2} + T_{n+1} + T_n, \ n \in \mathbb{N}_0, \ \text{where} \ T_0 = T_1 = 0, \ T_2 = 1 \ \text{and} \ T_3 = 1,$$

and one has

$$\begin{aligned} T_n &= \frac{-\alpha}{(\overline{\gamma} - \alpha)(\gamma - \alpha)(\beta - \alpha)}\alpha^n + \frac{\beta}{(\overline{\gamma} - \beta)(\gamma - \beta)(\beta - \alpha)}\beta^n + \frac{-\gamma}{(\overline{\gamma} - \gamma)(\gamma - \beta)(\gamma - \alpha)}\gamma^n \\ &\quad + \frac{\overline{\gamma}}{(\overline{\gamma} - \gamma)(\overline{\gamma} - \beta)(\overline{\gamma} - \alpha)}\overline{\gamma}^n, \qquad n \in \mathbb{N}_0, \end{aligned}$$

with

$$\alpha = \frac{1}{4} + \frac{1}{2}\omega + \frac{1}{2}\sqrt{\frac{11}{4} - \omega^2 + \frac{13}{4}\omega^{-1}}, \ \beta = \frac{1}{4} + \frac{1}{2}\omega - \frac{1}{2}\sqrt{\frac{11}{4} - \omega^2 + \frac{13}{4}\omega^{-1}},$$

$$\gamma = \frac{1}{4} - \frac{1}{2}\omega + \frac{1}{2}\sqrt{\frac{11}{4} - \omega^2 - \frac{13}{4}\omega^{-1}}, \ \delta = \frac{1}{4} - \frac{1}{2}\omega - \frac{1}{2}\sqrt{\frac{11}{4} - \omega^2 - \frac{13}{4}\omega^{-1}},$$

$$\omega = \sqrt{\frac{11}{12} + \left(\frac{-65}{54} + \sqrt{\frac{563}{108}}\right)^{\frac{1}{3}} + \left(\frac{-65}{54} - \sqrt{\frac{563}{108}}\right)^{\frac{1}{3}}}.$$

The 1-dimensional version of the system (2.2), is the equation

$$x_{n+1} = \frac{x_{n-2}x_{n-1}x_n + x_{n-1}x_{n-2} + x_{n-2} + 1}{x_{n-2}x_{n-1}x_n}, \quad n \in \mathbb{N}_0. \tag{2.3}$$

From the main theorem it follows respectively.

Corollary 2.3. *Let $\{x_n, y_n\}_{n \geq -2}$ be a well defined solution of (2.2). Then, for $n \in \mathbb{N}_0$, one has*

$$x_{2n+1} = \frac{T_{2n+2} + (T_{2n+2} + T_{2n+1})\,y_{-2} + (T_{2n+4} - T_{2n+3})\,x_{-1}y_{-2} + T_{2n+3}y_0 x_{-1}y_{-2}}{T_{2n+1} + (T_{2n+1} + T_{2n})\,y_{-2} + (T_{2n+3} - T_{2n+2})\,x_{-1}y_{-2} + T_{2n+2}y_0 x_{-1}y_{-2}},$$

$$x_{2n+2} = \frac{T_{2n+3} + (T_{2n+3} + T_{2n+2})\,x_{-2} + (T_{2n+5} - T_{2n+4})\,y_{-1}x_{-2} + T_{2n+4}x_0 y_{-1}x_{-2}}{T_{2n+2} + (T_{2n+2} + T_{2n+1})\,x_{-2} + (T_{2n+4} - T_{2n+3})\,y_{-1}x_{-2} + T_{2n+3}x_0 y_{-1}x_{-2}},$$

$$y_{2n+1} = \frac{T_{2n+2} + (T_{2n+2} + T_{2n+1})\,x_{-2} + (T_{2n+4} - T_{2n+3})\,y_{-1}x_{-2} + T_{2n+3}x_0 y_{-1}x_{-2}}{T_{2n+1} + (T_{2n+1} + T_{2n})\,x_{-2} + (T_{2n+3} - T_{2n+2})\,y_{-1}x_{-2} + T_{2n+2}x_0 y_{-1}x_{-2}},$$

$$y_{2n+2} = \frac{T_{2n+3} + (T_{2n+3} + T_{2n+2})\,y_{-2} + (T_{2n+5} - T_{2n+4})\,x_{-1}y_{-2} + T_{2n+4}y_0 x_{-1}y_{-2}}{T_{2n+2} + (T_{2n+2} + T_{2n+1})\,y_{-2} + (T_{2n+4} - T_{2n+3})\,x_{-1}y_{-2} + T_{2n+3}y_0 x_{-1}y_{-2}}.$$

Corollary 2.4. *Let* $\{x_n\}_{n\geq-2}$ *be a well defined solution of the equation* (2.3). *Then for* $n \in \mathbb{N}_0$, *one has*

$$x_{2n+1} = \frac{T_{2n+2} + (T_{2n+2} + T_{2n+1})\, x_{-2} + (T_{2n+4} - T_{2n+3})\, x_{-1}x_{-2} + T_{2n+3}x_0x_{-1}x_{-2}}{T_{2n+1} + (T_{2n+1} + T_{2n})\, x_{-2} + (T_{2n+3} - T_{2n+2})\, x_{-1}x_{-2} + T_{2n+2}x_0x_{-1}x_{-2}},$$

$$x_{2n+2} = \frac{T_{2n+3} + (T_{2n+3} + T_{2n+2})\, x_{-2} + (T_{2n+5} - T_{2n+4})\, x_{-1}x_{-2} + T_{2n+4}x_0x_{-1}x_{-2}}{T_{2n+2} + (T_{2n+2} + T_{2n+1})\, x_{-2} + (T_{2n+4} - T_{2n+3})\, x_{-1}x_{-2} + T_{2n+3}x_0x_{-1}x_{-2}}.$$

Remark 2.5. When $a = d = 0$, the system (1.2) takes the form

$$x_{n+1} = \frac{bx_{n-1} + c}{y_n x_{n-1}}, \quad y_{n+1} = \frac{by_{n-1} + c}{x_n y_{n-1}} \quad n \in \mathbb{N}_0. \tag{2.4}$$

As it is noted in [1], the solutions are expressed using Padovan numbers. This system, and some particular cases of it, were the subject of the papers [19, 45].

If $d = c = 0$, the system (1.2) becomes

$$x_{n+1} = \frac{ay_n + b}{y_n}, \quad y_{n+1} = \frac{ax_n + b}{x_n}, \quad n \in \mathbb{N}_0. \tag{2.5}$$

Again, it is noted in [1] that:
- The system (2.5) is a particular case of the more general system

$$x_{n+1} = \frac{ay_n + b}{cy_n + d}, \quad y_{n+1} = \frac{\alpha x_n + \beta}{\gamma x_n + \lambda}, \quad n \in \mathbb{N}_0, \tag{2.6}$$

which was completely solved by Stevic in [33] and the solutions are expressed using a generalized Fibonacci sequence.
- Also, particular cases of System (2.6) were studied in [24, 28, 34, 35].

- If also $b = 0$, then the solutions of the system (2.5) are given by

$$\{(x_0, y_0), (a, a), (a, a), \dots, \}.$$

3 Proof of the Main Theorem

In order to solve the system (1.2), one needs first to solve the following two homogeneous fourth-order linear difference equations

$$R_{n+1} = aR_n + bR_{n-1} + cR_{n-2} + dR_{n-3}, \; n \in \mathbb{N}_0, \tag{3.1}$$

$$S_{n+1} = -aS_n + bS_{n-1} - cS_{n-2} + dS_{n-3}, \; n \in \mathbb{N}_0, \tag{3.2}$$

where the initial values $R_0, R_{-1}, R_{-2}, R_{-3}, S_0, S_{-1}, S_{-2}$ and S_{-3} and the constant coefficients a, b, c and d are real numbers with $d \neq 0$. In fact, one will express the terms of the sequences $(R_n)_{n=-3}^{+\infty}$ and $(S_n)_{n=-3}^{+\infty}$ using the sequence $(J_n)_{n=0}^{+\infty}$.

The difference equation (3.1) has the same characteristic equation as $(J_n)_{n=0}^{+\infty}$, that is the equation (1.6).

To solve the difference equation (3.2) using terms of (3.3), one needs the following fourth-order linear difference equation defined by

$$j_{n+4} = -aj_{n+3} + bj_{n+2} - cj_{n+1} + dj_n, \quad n \in \mathbb{N}_0, \tag{3.3}$$

and the special initial values

$$j_0 = 0, \quad j_1 = 0, \quad j_2 = 1 \text{ and } j_3 = -a. \tag{3.4}$$

The characteristic equation of (3.2) and (3.3) is

$$\lambda^4 + a\lambda^3 - b\lambda^2 + c\lambda - d = 0. \tag{3.5}$$

Clearly the roots of (3.5) are $-\alpha$, $-\beta$, $-\gamma$ and $-\delta$.
Now following the same procedure in solving $\{J_n\}$, it is not hard to see that

$$j_n = (-1)^n J_n.$$

Now, it is possible to prove the following result.

Lemma 3.1. *One has for all $n \in \mathbb{N}_0$,*

$$R_n = dJ_{n+1}R_{-3} + (cJ_{n+1} + dJ_n) R_{-2} + (J_{n+3} - aJ_{n+2}) R_{-1} + J_{n+2}R_0, \tag{3.6}$$

$$S_n = (-1)^{n+1} \left[dJ_{n+1}S_{-3} - (cJ_{n+1} + dJ_n) S_{-2} + (J_{n+3} - aJ_{n+2}) S_{-1} - J_{n+2}S_0 \right]. \tag{3.7}$$

Proof. Assume that α, β, γ and δ are the distinct roots of the characteristic equation (1.6), so

$$R_n = c_1'\alpha^n + c_2'\beta^n + c_3'\gamma^n + c_4'\delta^n, \, n \geq -3.$$

Using the initial values R_0, R_{-1}, R_{-2} and R_{-3}, one get

$$\begin{cases} \dfrac{1}{\alpha^3}c_1' + \dfrac{1}{\beta^3}c_2' + \dfrac{1}{\gamma^3}c_3' + \dfrac{1}{\delta^3}c_4' &= R_{-3} \\ \dfrac{1}{\alpha^2}c_1' + \dfrac{1}{\beta^2}c_2' + \dfrac{1}{\gamma^2}c_3' + \dfrac{1}{\delta^2}c_4' &= R_{-2} \\ \dfrac{1}{\alpha}c_1' + \dfrac{1}{\beta}c_2' + \dfrac{1}{\gamma}c_3' + \dfrac{1}{\delta}c_4' &= R_{-1} \\ c_1' + c_2' + c_3' + c_4' &= R_0, \end{cases} \tag{3.8}$$

after some calculations using the Cramer method one get

$$
\begin{aligned}
c_1' &= \frac{\beta\gamma\delta\alpha^3}{(\delta-\alpha)(\gamma-\alpha)(\beta-\alpha)}R_{-3} - \frac{(\gamma\beta+\gamma\delta+\beta\delta)\alpha^3}{(\delta-\alpha)(\gamma-\alpha)(\beta-\alpha)}R_{-2} \\
&\quad + \frac{(\beta+\gamma+\delta)\alpha^3}{(\delta-\alpha)(\gamma-\alpha)(\beta-\alpha)}R_{-1} - \frac{\alpha^3}{(\delta-\alpha)(\gamma-\alpha)(\beta-\alpha)}R_0 \\
c_2' &= -\frac{\alpha\gamma\delta\beta^3}{(\delta-\beta)(\gamma-\beta)(\beta-\alpha)}R_{-3} + \frac{(\gamma\alpha+\gamma\delta+\alpha\delta)\beta^3}{(\delta-\beta)(\gamma-\beta)(\beta-\alpha)}R_{-2} \\
&\quad - \frac{(\alpha+\gamma+\delta)\beta^3}{(\delta-\beta)(\gamma-\beta)(\beta-\alpha)}R_{-1} + \frac{\beta^3}{(\delta-\beta)(\gamma-\beta)(\beta-\alpha)}R_0 \\
c_3' &= \frac{\alpha\beta\delta\gamma^3}{(\delta-\gamma)(\gamma-\beta)(\gamma-\alpha)}R_{-3} - \frac{(\alpha\beta+\alpha\delta+\beta\delta)\gamma^3}{(\delta-\gamma)(\gamma-\beta)(\gamma-\alpha)}R_{-2} \\
&\quad + \frac{(\alpha+\beta+\delta)\gamma^3}{(\delta-\gamma)(\gamma-\beta)(\gamma-\alpha)}R_{-1} - \frac{\gamma^3}{(\delta-\gamma)(\gamma-\beta)(\gamma-\alpha)}R_0 \\
c_4' &= -\frac{\alpha\beta\gamma\delta^3}{(\delta-\gamma)(\delta-\beta)(\delta-\alpha)}R_{-3} + \frac{(\gamma\alpha+\gamma\beta+\alpha\beta)\delta^3}{(\delta-\gamma)(\delta-\beta)(\delta-\alpha)}R_{-2} \\
&\quad - \frac{(\alpha+\beta+\gamma)\delta^3}{(\delta-\gamma)(\delta-\beta)(\delta-\alpha)}R_{-1} + \frac{\delta^3}{(\delta-\gamma)(\delta-\beta)(\delta-\alpha)}R_0
\end{aligned}
$$

that is,

$$
\begin{aligned}
R_n &= \left(\frac{\beta\gamma\delta\alpha^3}{(\delta-\alpha)(\gamma-\alpha)(\beta-\alpha)}\alpha^n - \frac{\alpha\gamma\delta\beta^3}{(\delta-\beta)(\gamma-\beta)(\beta-\alpha)}\beta^n + \frac{\alpha\beta\delta\gamma^3}{(\delta-\gamma)(\gamma-\beta)(\gamma-\alpha)}\gamma^n \right. \\
&\quad \left. - \frac{\alpha\beta\gamma\delta^3}{(\delta-\gamma)(\delta-\beta)(\delta-\alpha)}\delta^n \right) R_{-3} \\
&\quad + \left(-\frac{(\gamma\beta+\gamma\delta+\beta\delta)\alpha^3}{(\delta-\alpha)(\gamma-\alpha)(\beta-\alpha)}\alpha^n + \frac{(\gamma\alpha+\gamma\delta+\alpha\delta)\beta^3}{(\delta-\beta)(\gamma-\beta)(\beta-\alpha)}\beta^n - \frac{(\alpha\beta+\alpha\delta+\beta\delta)\gamma^3}{(\delta-\gamma)(\gamma-\beta)(\gamma-\alpha)}\gamma^n \right. \\
&\quad \left. + \frac{(\gamma\alpha+\gamma\beta+\alpha\beta)\delta^3}{(\delta-\gamma)(\delta-\beta)(\delta-\alpha)}\delta^n \right) R_{-2} \\
&\quad + \left(\frac{(\beta+\gamma+\delta)\alpha^3}{(\delta-\alpha)(\gamma-\alpha)(\beta-\alpha)}\alpha^n - \frac{(\alpha+\gamma+\delta)\beta^3}{(\delta-\beta)(\gamma-\beta)(\beta-\alpha)}\beta^n + \frac{(\alpha+\beta+\delta)\gamma^3}{(\delta-\gamma)(\gamma-\beta)(\gamma-\alpha)}\gamma^n \right. \\
&\quad \left. - \frac{(\alpha+\beta+\gamma)\delta^3}{(\delta-\gamma)(\delta-\beta)(\delta-\alpha)}\delta^n \right) R_{-1} \\
&\quad + \left(-\frac{\alpha^3}{(\delta-\alpha)(\gamma-\alpha)(\beta-\alpha)}\alpha^n + \frac{\beta^3}{(\delta-\beta)(\gamma-\beta)(\beta-\alpha)}\beta^n - \frac{\gamma^3}{(\delta-\gamma)(\gamma-\beta)(\gamma-\alpha)}\gamma^n \right. \\
&\quad \left. + \frac{\delta^3}{(\delta-\gamma)(\delta-\beta)(\delta-\alpha)}\delta^n \right) R_0.
\end{aligned}
$$

$$
R_n = dJ_{n+1}R_{-3} + (cJ_{n+1} + dJ_n)R_{-2} + (J_{n+3} - aJ_{n+2})R_{-1} + J_{n+2}R_0.
$$

The proof of the other cases is similar and will be omitted.

Let $A := -a$, $B := b$, $C := -c$ and $D := d$ then, equation (3.2) takes the form (3.1) and the equation (3.3) takes the form (3.3). Then analogous to the formula of (3.1) one obtains

$$
S_n = Dj_{n+1}S_{-3} + (Cj_{n+1} + Dj_n)S_{-2} + (j_{n+3} - Aj_{n+2})S_{-1} + j_{n+2}S_0.
$$

Using the fact that $j_n = (-1)^n J_n$, $A = -a$ and $C := -c$ one get

$$
S_n = (-1)^{n+1}\left[dJ_{n+1}S_{-3} - (cJ_{n+1} + dJ_n)S_{-2} + (J_{n+3} - aJ_{n+2})S_{-1} - J_{n+2}S_0 \right].
$$

□

Proof of the Main Theorem.
Replacing

$$x_n = \frac{u_n}{v_{n-1}}, \quad y_n = \frac{v_n}{u_{n-1}}, \, n \geq -2, \tag{3.9}$$

in system (1.2) one get the following linear fourth-order system of difference equations

$$u_{n+1} = av_n + bu_{n-1} + cv_{n-2} + du_{n-3}, \quad v_{n+1} = au_n + bv_{n-1} + cu_{n-2} + dv_{n-3}, \quad n \in \mathbb{N}_0, \tag{3.10}$$

where the initial values $u_{-3}, u_{-2}, u_{-1}, u_0, v_{-3}, v_{-2}, v_{-1}, v_0$ are nonzero real numbers.
From (3.10) one has for $n \in \mathbb{N}_0$,

$$\begin{cases} u_{n+1} + v_{n+1} = a(v_n + u_n) + b(u_{n-1} + v_{n-1}) + c(v_{n-2} + u_{n-2}) + d(u_{n-3} + v_{n-3}), \\ u_{n+1} - v_{n+1} = a(v_n - u_n) + b(u_{n-1} - v_{n-1}) + c(v_{n-2} - u_{n-2}) + d(u_{n-3} - v_{n-3}). \end{cases}$$

Putting again

$$R_n = u_n + v_n, \quad S_n = u_n - v_n, \, n \geq -3, \tag{3.11}$$

one obtains two fourth-order homogeneous linear difference equations:

$$R_{n+1} = aR_n + bR_{n-1} + cR_{n-2} + dR_{n-3}, \, n \in \mathbb{N}_0,$$

and

$$S_{n+1} = -aS_n + bS_{n-1} - cS_{n-2} + dS_{n-3}, \, n \in \mathbb{N}_0. \tag{3.12}$$

Using (3.11), one get for $n \geq -3$,

$$u_n = \frac{1}{2}(R_n + S_n), \, v_n = \frac{1}{2}(R_n - S_n).$$

From Lemma 3.1 one obtains,

$$\begin{cases} u_{2n-1} = & \frac{1}{2} \left[dJ_{2n}(R_{-3} + S_{-3}) + (cJ_{2n} + dJ_{2n-1})(R_{-2} - S_{-2}) + (J_{2n+2} - aJ_{2n+1})(R_{-1} + S_{-1}) \right. \\ & \left. + J_{2n+1}(R_0 - S_0) \right], n \in \mathbb{N}, \\ u_{2n} = & \frac{1}{2} \left[dJ_{2n+1}(R_{-3} - S_{-3}) + (cJ_{2n+1} + dJ_{2n})(R_{-2} + S_{-2}) + (J_{2n+3} - aJ_{2n+2})(R_{-1} - S_{-1}) \right. \\ & \left. + J_{2n+2}(R_0 + S_0) \right], n \in \mathbb{N}_0, \end{cases} \tag{3.13}$$

$$\begin{cases} v_{2n-1} = & \frac{1}{2} \left[dJ_{2n}(R_{-3} - S_{-3}) + (cJ_{2n} + dJ_{2n-1})(R_{-2} + S_{-2}) + (J_{2n+2} - aJ_{2n+1})(R_{-1} - S_{-1}) \right. \\ & \left. + J_{2n+1}(R_0 + S_0) \right], n \in \mathbb{N}, \\ v_{2n} = & \frac{1}{2} \left[dJ_{2n+1}(R_{-3} + S_{-3}) + (cJ_{2n+1} + dJ_{2n})(R_{-2} - S_{-2}) + (J_{2n+3} - aJ_{2n+2})(R_{-1} + S_{-1}) \right. \\ & \left. + J_{2n+2}(R_0 - S_0) \right], n \in \mathbb{N}_0. \end{cases} \tag{3.14}$$

Substituting (3.13) and (3.14) in (3.9), one get for $n \in \mathbb{N}_0$,

$$x_{2n+1} = \frac{dJ_{2n+2} + (cJ_{2n+2} + dJ_{2n+1})\dfrac{R_{-2} - S_{-2}}{R_{-3} + S_{-3}} + (J_{2n+4} - aJ_{2n+3})\dfrac{R_{-1} + S_{-1}}{R_{-3} + S_{-3}} + J_{2n+3}\dfrac{R_0 - S_0}{R_{-3} + S_{-3}}}{dJ_{2n+1} + (cJ_{2n+1} + dJ_{2n})\dfrac{R_{-2} - S_{-2}}{R_{-3} + S_{-3}} + (J_{2n+3} - aJ_{2n+2})\dfrac{R_{-1} + S_{-1}}{R_{-3} + S_{-3}} + J_{2n+2}\dfrac{R_0 - S_0}{R_{-3} + S_{-3}}},$$

$$\tag{3.15}$$

$$x_{2n+2} = \frac{dJ_{2n+3} + (cJ_{2n+3} + dJ_{2n+2})\dfrac{R_{-2}+S_{-2}}{R_{-3}-S_{-3}} + (J_{2n+5} - aJ_{2n+4})\dfrac{R_{-1}-S_{-1}}{R_{-3}-S_{-3}} + J_{2n+4}\dfrac{R_0+S_0}{R_{-3}-S_{-3}}}{dJ_{2n+2} + (cJ_{2n+2} + dJ_{2n+1})\dfrac{R_{-2}+S_{-2}}{R_{-3}-S_{-3}} + (J_{2n+4} - aJ_{2n+3})\dfrac{R_{-1}-S_{-1}}{R_{-3}-S_{-3}} + J_{2n+3}\dfrac{R_0+S_0}{R_{-3}-S_{-3}}},$$
(3.16)

$$y_{2n+1} = \frac{dJ_{2n+2} + (cJ_{2n+2} + dJ_{2n+1})\dfrac{R_{-2}+S_{-2}}{R_{-3}-S_{-3}} + (J_{2n+4} - aJ_{2n+3})\dfrac{R_{-1}-S_{-1}}{R_{-3}-S_{-3}} + J_{2n+3}\dfrac{R_0+S_0}{R_{-3}-S_{-3}}}{dJ_{2n+1} + (cJ_{2n+1} + dJ_{2n})\dfrac{R_{-2}+S_{-2}}{R_{-3}-S_{-3}} + (J_{2n+3} - aJ_{2n+2})\dfrac{R_{-1}-S_{-1}}{R_{-3}-S_{-3}} + J_{2n+2}\dfrac{R_0+S_0}{R_{-3}-S_{-3}}},$$
(3.17)

and

$$y_{2n+2} = \frac{dJ_{2n+3} + (cJ_{2n+3} + dJ_{2n+2})\dfrac{R_{-2}-S_{-2}}{R_{-3}+S_{-3}} + (J_{2n+5} - aJ_{2n+4})\dfrac{R_{-1}+S_{-1}}{R_{-3}+S_{-3}} + J_{2n+4}\dfrac{R_0-S_0}{R_{-3}+S_{-3}}}{dJ_{2n+2} + (cJ_{2n+2} + dJ_{2n+1})\dfrac{R_{-2}-S_{-2}}{R_{-3}+S_{-3}} + (J_{2n+4} - aJ_{2n+3})\dfrac{R_{-1}+S_{-1}}{R_{-3}+S_{-3}} + J_{2n+3}\dfrac{R_0-S_0}{R_{-3}+S_{-3}}}.$$
(3.18)

One has

$$x_{-2} = \frac{u_{-2}}{v_{-3}} = \frac{R_{-2}+S_{-2}}{R_{-3}-S_{-3}}, \ x_{-1} = \frac{u_{-1}}{v_{-2}} = \frac{R_{-1}+S_{-1}}{R_{-2}-S_{-2}}, \ x_0 = \frac{u_0}{v_{-1}} = \frac{R_0+S_0}{R_{-1}-S_{-1}}, \qquad (3.19)$$

$$y_{-2} = \frac{v_{-2}}{u_{-3}} = \frac{R_{-2}-S_{-2}}{R_{-3}+S_{-3}}, \ y_{-1} = \frac{v_{-1}}{u_{-2}} = \frac{R_{-1}-S_{-1}}{R_{-2}+S_{-2}}, \ y_0 = \frac{v_0}{u_{-1}} = \frac{R_0-S_0}{R_{-1}+S_{-1}}. \qquad (3.20)$$

From (3.19), (3.20) one get,

$$\begin{cases} \dfrac{R_{-1}+S_{-1}}{R_{-3}+S_{-3}} = \dfrac{R_{-1}+S_{-1}}{R_{-2}-S_{-2}} \times \dfrac{R_{-2}-S_{-2}}{R_{-3}+S_{-3}} = x_{-1}y_{-2}, \\[2ex] \dfrac{R_0-S_0}{R_{-3}+S_{-3}} = \dfrac{R_0-S_0}{R_{-1}+S_{-1}} \times \dfrac{R_{-1}+S_{-1}}{R_{-2}-S_{-2}} \times \dfrac{R_{-2}-S_{-2}}{R_{-3}+S_{-3}} = y_0x_{-1}y_{-2}, \end{cases} \qquad (3.21)$$

$$\begin{cases} \dfrac{R_{-1}-S_{-1}}{R_{-3}-S_{-3}} = \dfrac{R_{-1}-S_{-1}}{R_{-2}+S_{-2}} \times \dfrac{R_{-2}+S_{-2}}{R_{-3}-S_{-3}} = y_{-1}x_{-2}, \\[2ex] \dfrac{R_0+S_0}{R_{-3}-S_{-3}} = \dfrac{R_0+S_0}{R_{-1}-S_{-1}} \times \dfrac{R_{-1}-S_{-1}}{R_{-2}+S_{-2}} \times \dfrac{R_{-2}+S_{-2}}{R_{-3}-S_{-3}} = x_0y_{-1}x_{-2}. \end{cases} \qquad (3.22)$$

Using (3.15), (3.16), (3.17), (3.18), (3.21) and (3.22), one obtains the closed form of the solutions of the system (1.2), that is for $n \in \mathbb{N}_0$, one has

$$\begin{cases} x_{2n+1} = \dfrac{dJ_{2n+2} + (cJ_{2n+2} + dJ_{2n+1})y_{-2} + (J_{2n+4} - aJ_{2n+3})x_{-1}y_{-2} + J_{2n+3}y_0x_{-1}y_{-2}}{dJ_{2n+1} + (cJ_{2n+1} + dJ_{2n})y_{-2} + (J_{2n+3} - aJ_{2n+2})x_{-1}y_{-2} + J_{2n+2}y_0x_{-1}y_{-2}}, \\[2ex] x_{2n+2} = \dfrac{dJ_{2n+3} + (cJ_{2n+3} + dJ_{2n+2})x_{-2} + (J_{2n+5} - aJ_{2n+4})y_{-1}x_{-2} + J_{2n+4}x_0y_{-1}x_{-2}}{dJ_{2n+2} + (cJ_{2n+2} + dJ_{2n+1})x_{-2} + (J_{2n+4} - aJ_{2n+3})y_{-1}x_{-2} + J_{2n+3}x_0y_{-1}x_{-2}}, \end{cases}$$

$$\begin{cases} y_{2n+1} = \dfrac{dJ_{2n+2} + (cJ_{2n+2} + dJ_{2n+1})x_{-2} + (J_{2n+4} - aJ_{2n+3})y_{-1}x_{-2} + J_{2n+3}x_0y_{-1}x_{-2}}{dJ_{2n+1} + (cJ_{2n+1} + dJ_{2n})x_{-2} + (J_{2n+3} - aJ_{2n+2})y_{-1}x_{-2} + J_{2n+2}x_0y_{-1}x_{-2}}, \\[2ex] y_{2n+2} = \dfrac{dJ_{2n+3} + (cJ_{2n+3} + dJ_{2n+2})y_{-2} + (J_{2n+5} - aJ_{2n+4})x_{-1}y_{-2} + J_{2n+4}y_0x_{-1}y_{-2}}{dJ_{2n+2} + (cJ_{2n+2} + dJ_{2n+1})y_{-2} + (J_{2n+4} - aJ_{2n+3})x_{-1}y_{-2} + J_{2n+3}y_0x_{-1}y_{-2}}. \end{cases}$$

Remark 3.2. - The content of the present paper was posted on arXiv on 31.10.2019, ref. arXiv:1910.14365.

- Some parts of the results of this paper were used in the reference [2] in which the authors have generalized the system (1.2).

Acknowledgements

Conflict of Interest

• • •

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