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Article *in* Electronic Research Archive · April 2021

DOI: 10.3934/era.2021029

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Manuscript submitted to
Electronic Research Archive
Volume **X**, Number **0X**, XX **200X**

doi:10.3934/xx.xxxxxxx

pp. **X–XX**

GLOBAL BEHAVIOR OF P-DIMENSIONAL DIFFERENCE EQUATIONS SYSTEM

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(Communicated by the associate editor name)

Abstract. The global asymptotic stability of the unique positive equilibrium point and the rate of convergence of positive solutions of the system of two recursive sequences has been studied recently. Here we generalize this study to the system of p recursive sequences $x_{n+1}^{(j)} = A + (x_{n-m}^{(j+1)mod(p)}/x_n^{(j+1)mod(p)})$, $n = 0, 1, \dots, m, p \in \mathbb{N}$, where $A \in (0, +\infty)$, $x_{-i}^{(j)}$ are arbitrary positive numbers for $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, p$. We also give some numerical examples to demonstrate the effectiveness of the results obtained.

1. **Introduction.** Difference equations are the essentials required to understand even the simplest epidemiological model: the SIR-susceptible, infected, recovered-model. This model is a compartmental model, which results in the basic difference equation used to measure the actual reproduction number. It is this basic model that helps us determine whether a pathogen is going to die out or whether we end up having an epidemic. This is also the basis for more complex models, including the SVIR, which requires a vaccinated state, which helps us to estimate the probability of herd immunity.

There has been some recent interest in studying the qualitative analysis of difference equations and system of difference equations. Since the beginning of nineties there has be considerable interest in studying systems of difference equations composed by two or three rational difference equations (see, e.g., [2, 3, 4, 6, 8, 9, 10, 11, 14, 15, 17, 19, 20] and the references therein). However, given the multiplicity of factors involved in any epidemic, it will be important to study systems of difference equations composed by many rational difference equations, which is what we will do in this paper.

In [6], Devault et al. studied the boundedness, global stability and periodic character of solutions of the difference equation

$$x_{n+1} = p + \frac{x_{n-m}}{x_n} \tag{1}$$

2020 *Mathematics Subject Classification.* Primary: 39A10; Secondary: 40A05.

Key words and phrases. System of difference equations, global asymptotic stability, boundedness, rate of convergence.

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where $m \in \{2,3,\ldots\}$, p is positive and the initial conditions are positive numbers.

In [20], Zhang et al. investigated the behavior of the following symmetrical system of difference equations

$$x_{n+1} = A + \frac{y_{n-m}}{y_n}, \quad y_{n+1} = A + \frac{x_{n-m}}{x_n} \tag{2}$$

where the parameter A is positive, the initial conditions x_i, y_i are arbitrary positive numbers for $i = -m, -m+1, \ldots, 0$ and $m \in \mathbb{N}$. While this study is good, we note that the authors did not investigate various device properties, such as the stability nature, the rate of convergence and the asymptotic behavior.

Complement of the work above, in [8], Gümüş studied the global asymptotic stability of positive equilibrium, the rate of convergence of positive solutions and he presented some results about the general behavior of solutions of system (2). Our aim in this paper is to generalize the results concerning equation (1) and system (2) to the system of p nonlinear difference equations

$$x_{n+1}^{(1)} = A + \frac{x_{n-m}^{(2)}}{x_n^{(2)}}, \quad x_{n+1}^{(2)} = A + \frac{x_{n-m}^{(3)}}{x_n^{(3)}}, \ldots, \quad x_{n+1}^{(p)} = A + \frac{x_{n-m}^{(1)}}{x_n^{(1)}}, \quad n, m, p \in \mathbb{N}_0 \tag{3}$$

where A is a nonnegative constant and $x_{-m}^{(j)}, x_{-m+1}^{(j)}, \ldots, x_{-1}^{(j)}, x_0^{(j)}, j = 1, 2, \ldots, p$ are positive real numbers.

The remainder of the paper is organized as follows. In Section (2), we introduce some definitions and notations that will be needed in the sequel. Moreover, we present, in Theorem (2.4), a result concerning the linearized stability that will be useful in the main part of the paper. Section (3) discuses the behavior of positive solutions of system (3) via semi-cycle analysis method. Furthermore, Section (4) is devoted to study the local stability of the equilibrium points and the asymptotic behavior of the solutions when $0 \leq A < 1, A = 1$ and $A > 1$. In Section (5), we turn our attention to estimate the rate of convergence of a solution that converges to the equilibrium point of the system (3) in the region of parameters described by $A > 1$. Some numerical examples are carried out to support the analysis results in Section (6). Section (7) summarizes the results of this work, draws conclusions and give some interesting open problems for difference equations theory researchers.

2. **Preliminaries.** In this section we recall some definitions and results that will be useful in our investigation, for more details see [1, 7, 14, 13].

Definition 2.1. (see, [14]) A 'string' of sequential terms $\{x_\mu^{(j)}, \ldots, x_\nu^{(j)}\}$, $\mu \geq -1$, $\nu \leq +\infty$ is said to be a positive semi-cycle if $x_i^{(j)} \geq \overline{x^{(j)}}$, $i \in \{\mu, \ldots, \nu\}$, $x_{\mu-1}^{(j)} < \overline{x^{(j)}}$ and $x_{\nu+1}^{(j)} < \overline{x^{(j)}}$, $j \in \{1, 2, \ldots, p\}$.

A 'string' of sequential terms $\{x_\mu^{(j)}, \ldots, x_\nu^{(j)}\}$, $\mu \geq -1$, $\nu \leq +\infty$ is said to be a negative semi-cycle if $x_i^{(j)} < \overline{x^{(j)}}$, $i \in \{\mu, \ldots, \nu\}$, $x_{\mu-1}^{(j)} \geq \overline{x^{(j)}}$ and $x_{\nu+1}^{(j)} \geq \overline{x^{(j)}}$, $j \in \{1, 2, \ldots, p\}$.

A 'string' of sequential terms $\left\{(x_\mu^{(1)}, x_\mu^{(2)}, \ldots, x_\mu^{(p)}), \ldots, (x_\nu^{(1)}, x_\nu^{(2)}, \ldots, x_\nu^{(p)})\right\}$, $\mu \geq -1$, $\nu \leq +\infty$ is said to be a positive semi-cycle (resp. negative semi-cycle) if if $\{x_\mu^{(1)}, \ldots, x_\nu^{(1)}\}$,..., $\{x_\mu^{(p)}, \ldots, x_\nu^{(p)}\}$ are positive semi-cycles (resp. negative semi-cycles).

A 'string' of sequential terms $\left\{(x_\mu^{(1)}, x_\mu^{(2)}, \ldots, x_\mu^{(p)}), \ldots, (x_\nu^{(1)}, x_\nu^{(2)}, \ldots, x_\nu^{(p)})\right\}$, $\mu \geq -1$, $\nu \leq +\infty$ is said to be a positive semi-cycle (resp. negative semi-cycle) with respect to $x_n^{(q)}$ and negative semi-cycle (resp. positive semi-cycle) with respect to $x_n^{(s)}$ if $\{x_\mu^{(q)}, \ldots, x_\nu^{(q)}\}$ is a positive semi-cycle (resp. negative semi-cycle) and $\{x_\mu^{(s)}, \ldots, x_\nu^{(s)}\}$ is a negative semi-cycle (resp. positive semi-cycle).

Definition 2.2. (see, [14]) A function $x_n^{(i)}$ oscillates about $\overline{x^{(i)}}$ if for every $\xi \in \mathbb{N}$ there exist $\mu, \nu \in \mathbb{N}$, $\mu \geq \xi$, $\nu \geq \xi$ such that

$$(x_\mu^{(i)} - \overline{x^{(i)}})(x_\mu^{(i)} - \overline{x^{(i)}}) \leq 0, \quad i = 1, 2, \ldots, p.$$

We say that a solution $\left\{x_n^{(1)}, x_n^{(2)}, \ldots, x_n^{(p)}\right\}_{n \geq -m}$ of system (3) oscillates about $(\overline{x^{(1)}}, \overline{x^{(2)}}, \ldots, \overline{x^{(p)}})$ if $x_n^{(q)}$ oscillates about $\overline{x^{(q)}}$, $q \in \{1, 2, \ldots, p\}$.

Let $f^{(1)}, f^{(2)}, \ldots, f^{(p)}$ be p continuously differentiable functions:

$$f^{(i)} : I_1^{k+1} \times I_2^{k+1} \times \ldots \times I_p^{k+1} \to I_i^{k+1}, \quad i = 1, 2, \ldots, p,$$

where $I_i, \quad i = 1, 2, \ldots, p$ are some intervals of real numbers. Consider the system of difference equations

$$\left\{\begin{array}{lcl} x_{n+1}^{(1)} & = & f^{(1)}\left(x_n^{(1)}, x_{n-1}^{(1)}, \ldots, x_{n-k}^{(1)}, x_n^{(2)}, x_{n-1}^{(2)}, \ldots, x_{n-k}^{(2)}, \ldots, x_n^{(p)}, x_{n-1}^{(p)}, \ldots, x_{n-k}^{(p)}\right) \\ x_{n+1}^{(2)} & = & f^{(2)}\left(x_n^{(1)}, x_{n-1}^{(1)}, \ldots, x_{n-k}^{(1)}, x_n^{(2)}, x_{n-1}^{(2)}, \ldots, x_{n-k}^{(2)}, \ldots, x_n^{(p)}, x_{n-1}^{(p)}, \ldots, x_{n-k}^{(p)}\right) \\ & \vdots & \\ x_{n+1}^{(p)} & = & f^{(p)}\left(x_n^{(1)}, x_{n-1}^{(1)}, \ldots, x_{n-k}^{(1)}, x_n^{(2)}, x_{n-1}^{(2)}, \ldots, x_{n-k}^{(2)}, \ldots, x_n^{(p)}, x_{n-1}^{(p)}, \ldots, x_{n-k}^{(p)}\right) \end{array}\right. \tag{4}$$

where $n, k \in \mathbb{N}_0$, $\left(x_{-k}^{(i)}, x_{-k+1}^{(i)}, \ldots, x_0^{(i)}\right) \in I_i^{k+1}, \quad i = 1, 2, \ldots, p.$

Define the map

$$F : I_1^{(k+1)} \times I_2^{(k+1)} \times \ldots \times I_p^{(k+1)} \longrightarrow I_1^{(k+1)} \times I_2^{(k+1)} \times \ldots \times I_p^{(k+1)}$$

by

$$F(W) = \big(f_0^{(1)}(W), f_1^{(1)}(W), \ldots, f_k^{(1)}(W), f_0^{(2)}(W), f_1^{(2)}(W), \ldots, \\ \ldots, f_k^{(2)}(W), \ldots, f_0^{(p)}(W), f_1^{(p)}(W), \ldots, f_k^{(p)}(W)\big),$$

where

$$W = \left(u_0^{(1)}, u_1^{(1)}, \ldots, u_k^{(1)}, u_0^{(2)}, u_1^{(2)}, \ldots, u_k^{(2)}, \ldots, u_0^{(p)}, u_1^{(p)}, \ldots, u_k^{(p)}\right)^T,$$

$$f_0^{(i)}(W) = f^{(i)}(W), \quad f_1^{(i)}(W) = u_0^{(i)}, \ldots, f_k^{(i)}(W) = u_{k-1}^{(i)}, \quad i = 1, 2, \ldots, p.$$

Let

$$W_n = \left(x_n^{(1)}, x_{n-1}^{(1)}, \ldots, x_{n-k}^{(1)}, x_n^{(2)}, x_{n-1}^{(2)}, \ldots, x_{n-k}^{(2)}, \ldots, x_n^{(p)}, x_{n-1}^{(p)}, \ldots, x_{n-k}^{(p)}\right)^T.$$

Then, we can easily see that system (4) is equivalent to the following system written in vector form

$$W_{n+1} = F(W_n), \quad n \in \mathbb{N}_0. \tag{5}$$

Definition 2.3. (see, [13]) Let $(\overline{x^{(1)}}, \overline{x^{(2)}}, \ldots, \overline{x^{(p)}})$ be an equilibrium point of the map F where $f^{(i)}$, $i = 1, 2, \ldots, p$ are continuously differentiable functions at $(\overline{x^{(1)}}, \overline{x^{(2)}}, \ldots, \overline{x^{(p)}})$. The linearized system of (3) about the equilibrium point $(\overline{x^{(1)}}, \overline{x^{(2)}}, \ldots, \overline{x^{(p)}})$ is

$$X_{n+1} = F(X_n) = BX_n$$

where $X_n = \left(x_n^{(1)}, x_{n-1}^{(1)}, \ldots, x_{n-k}^{(1)}, x_n^{(2)}, x_{n-1}^{(2)}, \ldots, x_{n-k}^{(2)}, \ldots, x_n^{(p)}, x_{n-1}^{(p)}, \ldots, x_{n-k}^{(p)}\right)^T$ and B is a Jacobian matrix of the system (3) about the equilibrium point $(\overline{x^{(1)}}, \overline{x^{(2)}}, \ldots, \overline{x^{(p)}})$.

Theorem 2.4. (see, [13])

1. *If all the eigenvalues of the Jacobian matrix B lie in the open unit disk $|\lambda| < 1$, then the equilibrium point $\overline{X}$ of system (3) is asymptotically stable.*
2. *If at least one eigenvalue of the Jacobian matrix B has absolute value greater than one, then the equilibrium point $\overline{X}$ of system (3) is unstable.*

3. **Semi-cycle analysis.** In this section, we discuss the behavior of positive solutions of system (3) via semi-cycle analysis method. It is easy to see that system (3) has a unique positive equilibrium point $\left(\overline{x^{(1)}}, \overline{x^{(2)}}, \ldots, \overline{x^{(p)}}\right) = (A+1, A+1, \ldots, A+1)$.

Lemma 3.1. *Let $\{(x_n^{(1)}, x_n^{(2)}, \ldots, x_n^{(p)})\}_{n\geq -m}$ be a solution to system (3). Then, either $\{(x_n^{(1)}, x_n^{(2)}, \ldots, x_n^{(p)})\}_{n\geq -m}$ consists of a single semi-cycle or $\{(x_n^{(1)}, x_n^{(2)}, \ldots, x_n^{(p)})\}_{n\geq -m}$ oscillates about the equilibrium $(\overline{x^{(1)}}, \overline{x^{(2)}}, \ldots, \overline{x^{(p)}}) = (A+1, A+1, \ldots, A+1)$ with semi-cycles having at most m terms.*

Proof. Suppose that $\{(x_n^{(1)}, x_n^{(2)}, \ldots, x_n^{(p)})\}_{n\geq -m}$ has at least two semi-cycles. Then, there exists $n_0 \geq -m$ such that either

$$x_{n_0}^{(j)} < A+1 \leq x_{n_0+1}^{(j)} \text{ or } x_{n_0+1}^{(j)} < A+1 \leq x_{n_0}^{(j)}, \quad j = 1, 2, \ldots, p.$$

We suppose the first case, that is, $x_{n_0}^{(j)} < A+1 \leq x_{n_0+1}^{(j)}$. The other case is similar and will be omitted. Assume that the positive semi-cycle beginning with the term $(x_{n_0+1}^{(1)}, x_{n_0+1}^{(2)}, \ldots, x_{n_0+1}^{(p)})$ have m terms. In this case we have

$$x_{n_0}^{(j)} < A+1 \leq x_{n_0+m}^{(j)}, \quad j = 1, 2, \ldots, p.$$

So, we get from system (3)

$$x_{n_0+m+1}^{(j)} = A + \frac{x_{n_0}^{(j+1)mod(p)}}{x_{n_0+m}^{(j+1)mod(p)}} < A+1, \quad j = 1, 2, \ldots, p.$$

The Lemma is proved. □

Lemma 3.2. *Let $\{(x_n^{(1)}, x_n^{(2)}, \ldots, x_n^{(p)})\}_{n\geq -m}$ be a solution to system (3) which has $m-1$ sequential semi-cycles of length one. Then, every semi-cycle after this point is of length one.*

Proof. Assume that there exists $n_0 \geq -m$ such that either

$$x_{n_0}^{(j)}, x_{n_0+2}^{(j)}, \ldots, x_{n_0+m-1}^{(j)} < A+1 \leq x_{n_0+1}^{(j)}, x_{n_0+3}^{(j)}, \ldots, x_{n_0+m}^{(j)}, \quad j = 1, 2, \ldots, p, \quad (6)$$

or

$$x_{n_0+1}^{(j)}, x_{n_0+3}^{(j)}, \ldots, x_{n_0+m}^{(j)} < A+1 \leq x_{n_0}^{(j)}, x_{n_0+2}^{(j)}, \ldots, x_{n_0+m-1}^{(j)}, \quad j = 1, 2, \ldots, p. \quad (7)$$

We will prove the case (6). The case (7) Is identical and will not be included. According to system (3) we obtain

$$x_{n_0+m+1}^{(j)} = A + \frac{x_{n_0}^{(j+1)mod(p)}}{x_{n_0+m}^{(j+1)mod(p)}} < A+1, \quad j = 1, 2, \ldots, p,$$

and

$$x^{(j)}_{n_0+m+2} = A + \frac{x^{(j+1)mod(p)}_{n_0+1}}{x^{(j+1)mod(p)}_{n_0+m+1}} > A+1, \quad j = 1, 2, \ldots, p,$$

The result proceeds by induction. Thus, the proof is completed. □

Lemma 3.3. *System (3) has no nontrivial periodic solutions of (not necessarily prime) period m.*

Proof. Suppose that

$(\alpha^{(1)}_1, \alpha^{(2)}_1, \ldots, \alpha^{(p)}_1), (\alpha^{(1)}_2, \alpha^{(2)}_2, \ldots, \alpha^{(p)}_2), \ldots, (\alpha^{(1)}_m, \alpha^{(2)}_m, \ldots, \alpha^{(p)}_m), (\alpha^{(1)}_1, \alpha^{(2)}_1, \ldots, \alpha^{(p)}_1), \ldots$

is a m-periodic solution of system (3). It is obvious then that for this solution,

$$(x^{(1)}_{n-m}, x^{(2)}_{n-m}, \ldots, x^{(p)}_{n-m}) = (x^{(1)}_n, x^{(2)}_n, \ldots, x^{(p)}_n), \quad n \geq 0.$$

So, the equilibrium solution $\left(\overline{x^{(1)}}, \overline{x^{(2)}}, \ldots, \overline{x^{(p)}}\right) = (A+1, A+1, \ldots, A+1)$ must be this solution. Thus, the proof is completed. □

Lemma 3.4. *All non-oscillatory solutions of system (3) converge to the equilibrium $\left(\overline{x^{(1)}}, \overline{x^{(2)}}, \ldots, \overline{x^{(p)}}\right) = (A+1, A+1, \ldots, A+1)$.*

Proof. We assume there exists non-oscillatory solutions of system (3). We will prove this lemma for the case of a single positive semi-cycle, the situation is identical for a single negative semi-cycle, so it will be omitted. Assume that $(x^{(1)}_n, x^{(2)}_n, \ldots, x^{(p)}_n) \geq (\overline{x^{(1)}}, \overline{x^{(2)}}, \ldots, \overline{x^{(p)}})$ for all $n \geq -m$. From system (3) we have

$$x^{(j)}_{n+1} = A + \frac{x^{(j+1)mod(p)}_{n-m}}{x^{(j+1)mod(p)}_{n}} \geq A+1, \quad j = 1, 2, \ldots, p,$$

So, we get

$$A+1 \leq x^{(j)}_n \leq x^{(j)}_{n-m}, \quad n \geq 0, \quad j = 1, 2, \ldots, p \tag{8}$$

From (8), there exists $\delta^{(j)}_i$ for$i = 0, 1, \ldots, m-1$ such that

$$\lim_{n\to+\infty} x^{(j)}_{nm+i} = \delta^{(j)}_i.$$

Hence,

$$(\delta^{(1)}_0, \delta^{(2)}_0, \ldots, \delta^{(p)}_0), (\delta^{(1)}_1, \delta^{(2)}_1, \ldots, \delta^{(p)}_1), \ldots, (\delta^{(1)}_{m-1}, \delta^{(2)}_{m-1}, \ldots, \delta^{(p)}_{m-1})$$

is a periodic solution of (not necessarily prime period) period m. But, from Lemma (3.3), we saw system (3) has no nontrivial periodic solutions of (not necessarily prime period) period m. Thus, the solution must be the equilibrium solution. So, the proof is over. □

4. The asymptotic behavior.

4.1. The Case $0 < A < 1$.

Theorem 4.1. *Suppose* $0 < A < 1$ *and* $\{(x_n^{(1)}, x_n^{(2)}, \dots, x_n^{(p)})\}_{n\geq -m}$ *be a positive solution to system* (3). *Then the following statements hold.*

i): *If* m *is odd, and* $0 < x_{2k-1}^{(j)} < 1$, $x_{2k}^{(j)} > \frac{1}{1-A}$ *for* $k = \frac{1-m}{2}, \frac{3-m}{2}, \dots, 0$, *then*

$$\lim_{n\to+\infty} x_{2n}^{(j)} = +\infty, \qquad \lim_{n\to+\infty} x_{2n+1}^{(j)} = A.$$

ii): *If* m *is odd, and* $0 < x_{2k}^{(j)} < 1$, $x_{2k-1}^{(j)} > \frac{1}{1-A}$ *for* $k = \frac{1-m}{2}, \frac{3-m}{2}, \dots, 0$, *then*

$$\lim_{n\to+\infty} x_{2n}^{(j)} = A, \qquad \lim_{n\to+\infty} x_{2n+1}^{(j)} = +\infty.$$

Proof. **(i):** From (3), for $i = 1, 2, \dots, p$, we get

$$\begin{aligned} x_1^{(i)} &= A + \frac{x_{-m}^{(i+1)mod(p)}}{x_0^{(i+1)mod(p)}} < A + \frac{1}{x_0^{(i+1)mod(p)}} < A + (1-A) = 1, \\ x_2^{(i)} &= A + \frac{x_{1-m}^{(i+1)mod(p)}}{x_1^{(i+1)mod(p)}} > A + x_{1-m}^{(i+1)mod(p)} > x_{1-m}^{(i+1)mod(p)} > \frac{1}{1-A}. \end{aligned}$$

By induction, for n $= 0, 1, 2, \dots$ and $i = 1, 2, \dots, p$, we obtain

$$x_{2n-1}^{(i)} < 1, \quad x_{2n}^{(i)} > \frac{1}{1-A}. \tag{9}$$

So, from (3) and (9), we have

$$x_{2n}^{(i)} = A + \frac{x_{2n-1-m}^{(i+1)mod(p)}}{x_{2n-1}^{(i+1)mod(p)}} > A + x_{2n-1-m}^{(i+1)mod(p)} > 2A + x_{2n-3-m}^{(i+1)mod(p)} > 3A + x_{2n-5-m}^{(i+1)mod(p)} > \cdots$$

So

$$x_{2n}^{(i)} > nA + x_0^{(i+1)mod(p)}. \tag{10}$$

By limiting the inequality (10), we get

$$\lim_{n\to\infty} x_{2n}^{(i)} = \infty. \tag{11}$$

On the other hand, from(3), (9) and (11), we get

$$\lim_{n\to\infty} x_{2n+1}^{(i)} = \lim_{n\to\infty} \left(A + \frac{x_{2n-m}^{(i+1)mod(p)}}{x_{2n}^{(i+1)mod(p)}} \right) = A.$$

(ii): The proof is similar to the proof of (i).

□

Open Problem: Investigate the asymptotic behavior of the system (3) when m is even.

4.2. **The Case $A = 1$.**

Lemma 4.2. *Suppose $A = 1$. Then every positive solution of the system (3) is bounded and persists.*

Proof. Let $\{(x_n^{(1)}, x_n^{(2)}, \dots, x_n^{(p)})\}_{n \geq -m}$ be a positive solution to system (3). Then, it is clear that for $n \geq 1$, $x_n^{(j)} > A = 1, \quad j = 1, 2, \dots, p$. So, we get

$$x_i^{(j)} \in \left[L, \frac{L}{L-1}\right], \quad i = 1, 2, \dots, m+1, \quad j = 1, 2, \dots, p,$$

where

$$L = \min\left\{\alpha, \frac{\beta}{\beta - 1}\right\} > 1, \quad \alpha = \min_{1 \leq j \leq m+1} \{x_j^{(1)}, x_j^{(2)}, \dots, x_j^{(p)}\},$$

$$\beta = \max_{1 \leq j \leq m+1} \{x_j^{(1)}, x_j^{(2)}, \dots, x_{ji}^{(p)}\}.$$

So, we get

$$L = 1 + \frac{L}{L/(L-1)} \leq x_{m+2}^{(j)} = 1 + \frac{x_1^{(j+1) mod(p)}}{x_{m+1}^{(j+1) mod(p)}} \leq \frac{L}{L-1},$$

thus, the following is obtained

$$L \leq x_m^{(j)} \leq \frac{L}{L-1}.$$

By induction, we get

$$x_i^{(j)} \in \left[L, \frac{L}{L-1}\right], \quad j = 1, 2, \dots, p, \quad i = 1, 2, \dots \tag{12}$$

□

Theorem 4.3. *Suppose $A = 1$ and $\{(x_n^{(1)}, x_n^{(2)}, \dots, x_n^{(p)})\}_{n \geq -m}$ be a positive solution to system (3). Then*

$$\begin{aligned} \liminf_{n \to +\infty} x_n^{(i)} &= \liminf_{n \to +\infty} x_n^{(j)}, \quad i, j = 1, 2, \dots, p, \\ \limsup_{n \to +\infty} x_n^{(i)} &= \limsup_{n \to +\infty} x_n^{(j)}, \quad i, j = 1, 2, \dots, p. \end{aligned}$$

Proof. From (17), we can set

$$L_i = \lim_{n \to \infty} \sup x_n^{(i)}, \quad m_i = \lim_{n \to \infty} \inf x_n^{(i)}, \quad i = 1, 2, \dots, p. \tag{13}$$

We first prove the theorem for $p = 2$. From system (3), we have

$$L_1 \leq 1 + \frac{L_2}{m_2}, L_2 \leq 1 + \frac{L_1}{m_1}, m_1 \geq 1 + \frac{m_2}{L_2}, m_2 \geq 1 + \frac{m_2}{L_2},$$

which implies

$$L_1 m_2 \leq m_2 + L_2 \leq m_1 L_2 \leq m_1 + L_1 \leq m_2 L_1$$

thus, the following equalities are obtained

$$m_2 + L_2 = m_1 + L_1, \quad L_1 m_2 = m_1 L_2.$$

So, we get that $m_1 = m_2$ and $L_1 = L_2$. Now we suppose that

$$L_i = L_j, \quad m_i = m_j, \quad i, j = 1, 2, \dots, p-1,$$

From system (3), we have

$$L_{p-1} \leq 1 + \frac{L_p}{m_p}, L_p \leq 1 + \frac{L_{p-1}}{m_{p-1}}, m_{p-1} \geq 1 + \frac{m_p}{L_p}, m_p \geq 1 + \frac{m_p}{L_p},$$

hence, we get

$$L_{p-1}m_p \leq m_p + L_p \leq m_{p-1}L_p \leq m_{p-1} + L_{p-1} \leq m_p L_{p-1},$$

consequently , the following equalities are obtained

$$m_p + L_p = m_{p-1} + L_{p-1}, \quad L_{p-1}m_p = m_{p-1}L_p.$$

So, we get that $m_p = m_{p-1}$ and $L_p = L_{p-1}$. Thus, the proof completes. □

4.3. **The Case** $A > 1$.

Theorem 4.4. *Assume that $A > 1$. Then, the unique positive equilibrium $\left(\overline{x^{(1)}}, \overline{x^{(2)}}, \dots, \overline{x^{(p)}}\right) = (A+1, A+1, \dots, A+1)$ of system (3) is locally asymptotically stable.*

Proof. The linearized equation of system (3) about the equilibrium point $(\overline{x^{(1)}}, \overline{x^{(2)}}, \dots, \overline{x^{(p)}})$ is

$$X_{n+1} = BX_n$$

where $X_n = (x_n^{(1)}, x_{n-1}^{(1)}, \dots, x_{n-m}^{(1)}, x_n^{(2)}, x_{n-1}^{(2)}, \dots, x_{n-m}^{(2)}, \dots, x_n^{(p)}, x_{n-1}^{(p)}, , \dots, x_{n-m}^{(p)})^t$, and $B = (b_{ij})$, $1 \leq i, j \leq pm + p$ is an $(pm + p) \times (pm + p)$ matrix such that

$$B = \begin{pmatrix} \mathcal{J} & \mathcal{A} & \mathcal{O} & \mathcal{O} & \dots & \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{J} & \mathcal{A} & \mathcal{O} & \dots & \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} & \mathcal{J} & \mathcal{A} & \dots & \mathcal{O} & \mathcal{O} \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ \mathcal{O} & \mathcal{O} & \mathcal{O} & \mathcal{O} & \dots & \mathcal{J} & \mathcal{A} \\ \mathcal{A} & \mathcal{O} & \mathcal{O} & \mathcal{O} & \dots & \mathcal{O} & \mathcal{J} \end{pmatrix}$$

where $\mathcal{A}, \mathcal{J}$ and $\mathcal{O}$ are $(m + 1) \times (m + 1)$ matrix defined as follows

$$\mathcal{J} = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix}, \quad \mathcal{O} = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 \end{pmatrix}, \tag{14}$$

$$\mathcal{A} = \begin{pmatrix} -\frac{1}{A+1} & 0 & \dots & 0 & \frac{1}{A+1} \\ 0 & 0 & \dots & 0 & 0 \\ \vdots & \ddots & \dots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 \end{pmatrix}. \tag{15}$$

Let $\lambda_1, \lambda_2, \dots, \lambda_{pm+p}$ denote the eigenvalues of matrix B and let

$$D = diag(d_1, d_2, \dots, d_{pm+p})$$

be a diagonal matrix where $d_1 = d_{m+2} = d_{2m+3} = \dots = d_{(p-1)m+p} = 1$, $d_k = d_{m+1+k} = 1 - k\varepsilon$ for $k \in \{1, 2, \dots, \frac{p}{2}(m + 1)\}$. Since $A > 1$, we can take a positive number ε such that

$$0 < \varepsilon < \frac{A - 1}{(m + 1)(A + 1)}. \tag{16}$$

It is obvious that D is an invertible matrix. Computing matrix DBD^{-1}, we get

$$DBD^{-1} = \begin{pmatrix} \mathcal{J}^{(1)} & \mathcal{A}^{(1)} & \mathcal{O} & \mathcal{O} & \dots & \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{J}^{(2)} & \mathcal{A}^{(2)} & \mathcal{O} & \dots & \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} & \mathcal{J}^{(3)} & \mathcal{A}^{(3)} & \dots & \mathcal{O} & \mathcal{O} \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ \mathcal{O} & \mathcal{O} & \mathcal{O} & \mathcal{O} & \dots & \mathcal{J}^{(p-1)} & \mathcal{A}^{(p-1)} \\ \mathcal{A}^{(p)} & \mathcal{O} & \mathcal{O} & \mathcal{O} & \dots & \mathcal{O} & \mathcal{J}^{(p)} \end{pmatrix},$$

where

$$\mathcal{J}^{(j)} = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ \frac{d_{(j-1)m+j+1}}{d_{(j-1)m+j}} & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \frac{d_{(j-1)m+m+j}}{d_{(j-1)m+m+j-1}} & 0 \end{pmatrix}, \quad j = 0, 1, \dots, p,$$

$$\mathcal{A}^{(j)} = \begin{pmatrix} -\frac{1}{A+1}\frac{d_j}{d_{jm+j+1}} & 0 & \dots & 0 & \frac{1}{A+1}\frac{d_j}{d_{jm+j+1}} \\ 0 & 0 & \dots & 0 & 0 \\ \vdots & \ddots & \dots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 \end{pmatrix}, \quad j = 0, 1, \dots, p-1,$$

and

$$\mathcal{A}^{(p)} = \begin{pmatrix} -\frac{1}{A+1}\frac{d_{(p-1)m+p}}{d_1} & 0 & \dots & 0 & \frac{1}{A+1}\frac{d_{(p-1)m+p}}{d_{m+1}} \\ 0 & 0 & \dots & 0 & 0 \\ \vdots & \ddots & \dots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 \end{pmatrix}.$$

From $d_1 > d_2 > \ldots > d_{\frac{p}{2}(m+1)}$ and $d_{\frac{p}{2}(m+1)} + 1 > d_{\frac{p}{2}(m+1)} + 2 > \ldots > d_{pm+p}$ we can get that

$$\begin{aligned} d_2 d_1^{-1} &< 1 \\ d_3 d_2^{-1} &< 1 \\ &\vdots \\ d_{m+1} d_m^{-1} &< 1 \\ d_{m+3} d_{m+2}^{-1} &< 1 \\ &\vdots \\ d_{pm+p} d_{pm+p-1} &< 1 \end{aligned}$$

Moreover, from $A > 1$ and (16) we have

$$\begin{aligned} \frac{1}{A+1} + \frac{1}{(1-(m+1)\varepsilon)(A+1)} &< \frac{1}{(1-(m+1)\varepsilon)(A+1)} + \frac{1}{(1-(m+1)\varepsilon)(A+1)} \\ &< \frac{2}{(1-(m+1)\varepsilon)(A+1)} \\ &< 1. \end{aligned}$$

It is common knowledge that B has the same eigenvalues as DBD^{-1}, we have that

$$\begin{aligned}
\max|\lambda_i| &\leq \|DBD^{-1}\|_\infty \\
&= \max\left\{\begin{array}{c} d_2d_1^{-1},\dots,d_{m+1}d_m^{-1},d_{m+3}d_{m+2}^{-1},\dots,d_{pm+p}d_{pm+p-1}, \\ \frac{1}{A+1}+\frac{1}{(1-(m+1)\varepsilon)(A+1)} \end{array}\right\} \\
&< 1.
\end{aligned}$$

We have that all eigenvalues of B lie inside the unit disk. According to Theorem (2.4) we obtain that the unique positive equilibrium $\left(\overline{x^{(1)}},\overline{x^{(2)}},\dots,\overline{x^{(p)}}\right)=(A+1,A+1,\dots,A+1)$ is locally asymptotically stable. Thus, the proof is completed. □

To prove the global stability of the positive equilibrium, we need the following lemma.

Lemma 4.5. *Suppose $A>1$. Then every positive solution of the system (3) is bounded and persists.*

Proof. Let $\{(x_n^{(1)},x_n^{(2)},\dots,x_n^{(p)})\}_{n\geq -m}$ be a positive solution to system (3). Then, it is clear that for $n\geq 1$, $x_n^{(j)}>A>1,\quad j=1,2,\dots,p$. So, we get

$$x_i^{(j)}\in\left[L,\frac{L}{L-A}\right],\quad i=1,2,\dots,m+1,\quad j=1,2,\dots,p,$$

where

$$L=\min\left\{\alpha,\frac{\beta}{\beta-1}\right\}>1,\quad \alpha=\min_{1\leq j\leq m+1}\{x_j^{(1)},x_j^{(2)},\dots,x_j^{(p)}\},$$
$$\beta=\max_{1\leq j\leq m+1}\{x_j^{(1)},x_j^{(2)},\dots,x_{ji}^{(p)}\}.$$

So, we get

$$L=A+\frac{L}{L/(L-A)}\leq x_{m+2}^{(j)}=A+\frac{x_1^{(j+1)mod(p)}}{x_{m+1}^{(j+1)mod(p)}}\leq\frac{L}{L-1},$$

thus, the following is obtained

$$L\leq x_m^{(j)}\leq\frac{L}{L-1}.$$

By induction, we get

$$x_i^{(j)}\in\left[L,\frac{L}{L-1}\right],\quad j=1,2,\dots,p,\quad i=1,2,\dots \tag{17}$$

□

Theorem 4.6. *Assume that $A>1$. Then the positive equilibrium of system (3) is globally asymptotically stable.*

Proof. Let $\left\{\left(x_n^{(1)},x_n^{(2)},\dots,x_n^{(p)}\right)\right\}_{n\geq -m}$ be a solution of system (3). By Theorem (4.4) we need only to prove that the equilibrium point $(A+1,A+1,\dots,A+1)$ is global attractor, that is

$$\lim_{n\to\infty}\left(x_n^{(1)},x_n^{(2)},\dots,x_n^{(p)}\right)=(A+1,A+1,\dots,A+1).$$

To do this, we prove that for $i=1,2,\dots,p$, we have

$$\lim_{n\to\infty}x_n^{(i)}=A+1.$$

From Lemma (4.5), we can set

$$L_i = \lim_{n\to\infty} \sup x_n^{(i)}, \quad m_i = \lim_{n\to\infty} \inf x_n^{(i)}, \quad i = 1, 2, \ldots, p. \tag{18}$$

So, from (3) and (18), we have

$$L_i \leq A + \frac{L_{(i+1)mod(p)}}{m_{(i+1)mod(p)}}, \quad m_i \geq A + \frac{m_{(i+1)mod(p)}}{L_{(i+1)mod(p)}}. \tag{19}$$

We first prove the theorem for $p = 2$. From (19), we get

$$AL_1 + m_1 \leq L_1 m_2 \leq Am_2 + L_2, \quad AL_2 + m_2 \leq L_2 m_1 \leq Am_1 + L_1.$$

So,

$$AL_1 + m_1 - (Am_1 + L_1) \leq Am_2 + L_2 - (AL_2 + m_2),$$

hence

$$(A - 1)(L_1 - m_1 + L_2 - m_2) \leq 0,$$

since $A > 1$, It follows that

$$L_1 - m_1 + L_2 - m_2 = 0,$$

we know that $L_1 - m_1 \geq 0$ and $L_2 - m_2 \geq 0$, so we obtain $L_1 = m_1$ and $L_2 = m_2$. Now we assume that the theorem holds for $p-1$, that is $L_i = m_i, \quad i = 1, 2, \ldots, p-1$ and prove the theorem for p. From (19), we have

$$AL_p + m_p \leq L_p m_1 \leq Am_1 + L_1, \quad AL_1 + m_1 \leq L_1 m_p \leq Am_p + L_p.$$

So,

$$AL_p + m_p - (Am_p + L_p) \leq Am_1 + L_1 - (AL_1 + m_1),$$

Thus, the following inequality is obtained

$$(A - 1)(L_p - m_p + L_1 - m_1) \leq 0,$$

since $A > 1$, $L_1 - m_1 \geq 0$ and $L_p - m_p \geq 0$, we obtain $L_p = m_p$, it signify that

$$L_i = m_i, \quad = 1, 2, \ldots, p.$$

Therefore every positive solution $\left\{\left(x_n^{(1)}, x_n^{(2)}, \ldots, x_n^{(p)}\right)\right\}_{n\geq -1}$ of system (3) tends to $(A + 1, A + 1, \ldots, A + 1)$ as $n \to +\infty$. □

5. **Rate of convergence.** In this section, we estimate the rate of convergence of a solution that converges to the equilibrium point $\left(\overline{x^{(1)}}, \overline{x^{(2)}}, \ldots, \overline{x^{(p)}}\right) = (A+1, A+1, \ldots, A+1)$ of the system (3) in the region of parameters described by $A > 1$. We give precise results about the rate of convergence of the solutions that converge to the equilibrium point by using Perron's theorems. The following result gives the rate of convergence of solutions of a system of difference equations

$$X_{n+1} = (A + B_n)X_n \tag{20}$$

where X_n is a $(pm + p)$-dimensional vector, $A \in C^{(pm+p)\times(pm+p)}$ is a constant matrix and $B : \mathbb{Z}^+ \to C^{(pm+p)\times(pm+p)}$ is a matrix function satisfying

$$\|B_n\| \to 0, \ when\ n \to \infty \tag{21}$$

where $\|.\|$ indicates any matrix norm which is associated with the vector norm $\|.\|$.

Theorem 5.1. (Perron's first Theorem, see [16]) *Suppose that condition (21) holds. If Xn is a solution of (20), then either $X_n = 0$ for all largen or*

$$\rho = \lim_{n\to+\infty} \frac{\|X_{n+1}\|}{\|X_n\|}$$

exists and is equal to the modulus of one of the eigenvalues of matrix A.

Theorem 5.2. (Perron's second Theorem, see [16]) *Suppose that condition (21) holds. If Xn is a solution of (20), then either $X_n = 0$ for all largen or*

$$\rho = \lim_{n\to+\infty} (\|X_n\|)^{\frac{1}{n}}$$

exists and is equal to the modulus of one of the eigenvalues of matrix A.

Theorem 5.3. *Assume that a solution $\{(x_n^{(1)}, x_n^{(2)}, \ldots, x_n^{(p)})\}_{n\geq -m}$ of system (3) converges to the equilibrium $(\overline{x^{(1)}}, \overline{x^{(2)}}, \ldots, \overline{x^{(p)}})$ which is globally asymptotically stable. Then, the error vector*

$$e_n = \begin{pmatrix} e_n^{(1)} \\ e_{n-1}^{(1)} \\ \vdots \\ e_{n-m}^{(1)} \\ \vdots \\ e_n^{(p)} \\ e_{n-1}^{(p)} \\ \vdots \\ e_{n-m}^{(p)} \end{pmatrix} = \begin{pmatrix} x_n^{(1)} - \overline{x^{(1)}} \\ x_{n-1}^{(1)} - \overline{x^{(1)}} \\ \vdots \\ x_{n-m}^{(1)} - \overline{x^{(1)}} \\ \vdots \\ x_n^{(p)} - \overline{x^{(p)}} \\ x_{n-1}^{(p)} - \overline{x^{(p)}} \\ \vdots \\ x_{n-m}^{(p)} - \overline{x^{(p)}} \end{pmatrix}$$

of every solution of system (3) satisfies both of the following asymptotic relations:

$$\lim_{n\to+\infty} \frac{\|e_{n+1}\|}{\|e_n\|} = |\lambda_i J_F((\overline{x^{(1)}}, \overline{x^{(2)}}, \ldots, \overline{x^{(p)}}))|, \quad i = 1, 2, \ldots, m$$

$$\lim_{n\to+\infty} (\|e_n\|)^{\frac{1}{n}} = |\lambda_i J_F((\overline{x^{(1)}}, \overline{x^{(2)}}, \ldots, \overline{x^{(p)}}))|, \quad i = 1, 2, \ldots, m$$

where $|\lambda_i J_F((\overline{x^{(1)}}, \overline{x^{(2)}}, \ldots, \overline{x^{(p)}}))|$ is equal to the modulus of one the eigenvalues of the Jacobian matrix evaluated at the equilibrium point $(\overline{x^{(1)}}, \overline{x^{(2)}}, \ldots, \overline{x^{(p)}})$.

Proof. First, we will find a system that satisfies the error terms. The error terms are given as

$$x_{n+1}^{(j)} - \overline{x^{(j)}} = \sum_{i=0}^{m} {}^{(j)}A_i^{(1)}(x_{n-i}^{(1)} - \overline{x^{(1)}}) + \sum_{i=0}^{m} {}^{(j)}A_i^{(2)}(x_{n-i}^{(2)} - \overline{x^{(2)}}) + \cdots + \sum_{i=0}^{m} {}^{(j)}A_i^{(1)}(x_{n-i}^{(p)} - \overline{x^{(p)}}), \tag{22}$$

for $i = 1, 2, \ldots, m, j = 1, 2, \ldots, p$. □

Set

$$e_n^{(j)} = x_n^{(j)} - \overline{x^{(j)}}, \quad j = 1, 2, \ldots, p$$

Then, system (22) can be written as

$$e_{n+1}^{(j)} = \sum_{i=0}^{m} {}^{(j)}A_i^{(1)} e_{n-i}^{(1)} + \sum_{i=0}^{m} {}^{(j)}A_i^{(2)} e_{n-i}^{(2)} + \cdots + \sum_{i=0}^{m} {}^{(j)}A_i^{(1)} e_{n-i}^{(p)}$$

where

$$^{(i+1)mod(p)}A_0^{(i)} = -\frac{x_{n-m}^{(i+1)mod(p)}}{\left(x_n^{(i+1)mod(p)}\right)^2}, \quad ^{(i+1)mod(p)}A_m^{(i)} = \frac{1}{x_n^{(i+1)mod(p)}}, \quad i = 1, 2, \dots, m$$

and the others parameters $^{(k)}A_i^{(j)}$ are equal zero.

If we consider the limiting case, It is obvious then that

$$\begin{aligned} \lim_{n\to\infty} {}^{(i+1)mod(p)}A_0^{(i)} &= -\frac{1}{\overline{x_n^{(i+1)mod(p)}}}, \\ \lim_{n\to\infty} {}^{(i+1)mod(p)}A_m^{(i)} &= \frac{1}{\overline{x_n^{(i+1)mod(p)}}}, \quad i = 1, 2, \dots, m. \end{aligned}$$

That is

$$^{(i+1)mod(p)}A_0^{(i)} = -\frac{1}{\overline{x_n^{(i+1)mod(p)}}} + \alpha_n^{(i)}, \quad ^{(i+1)mod(p)}A_m^{(i)} = \frac{1}{\overline{x_n^{(i+1)mod(p)}}} + \beta_n^{(i)}$$

where $\alpha_n^{(i)}, \beta_n^{(i)} \to 0$ when $n \to \infty$. Now we have the following system of the form (20)

$$e_{n+1} = (A + B_n)e_n$$

where $e_n = \left(e_n^{(1)}, e_{n-1}^{(1)}, \dots, e_{n-m}^{(1)}, e_n^{(2)}, e_{n-1}^{(2)}, \dots, e_{n-m}^{(2)}, \dots, e_n^{(p)}, e_{n-1}^{(p)}, \dots, e_{n-m}^{(p)}\right)^t$ and

$$A = J_F((\overline{x^{(1)}}, \overline{x^{(2)}}, \dots, \overline{x^{(p)}})) = \begin{pmatrix} \mathcal{J} & \mathcal{A}_n^{(1)} & \mathcal{O} & \mathcal{O} & \dots & \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{J} & \mathcal{A}_n^{(2)} & \mathcal{O} & \dots & \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} & \mathcal{J} & \mathcal{A}_n^{(3)} & \dots & \mathcal{O} & \mathcal{O} \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ \mathcal{O} & \mathcal{O} & \mathcal{O} & \mathcal{O} & \dots & \mathcal{J} & \mathcal{A}_n^{(p-1)} \\ \mathcal{A}_n^{(p)} & \mathcal{O} & \mathcal{O} & \mathcal{O} & \dots & \mathcal{O} & \mathcal{J} \end{pmatrix}$$

$$B_n = \begin{pmatrix} \mathcal{J} & \mathcal{A} & \mathcal{O} & \mathcal{O} & \dots & \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{J} & \mathcal{A} & \mathcal{O} & \dots & \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} & \mathcal{J} & \mathcal{A} & \dots & \mathcal{O} & \mathcal{O} \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ \mathcal{O} & \mathcal{O} & \mathcal{O} & \mathcal{O} & \dots & \mathcal{J} & \mathcal{A} \\ \mathcal{A} & \mathcal{O} & \mathcal{O} & \mathcal{O} & \dots & \mathcal{O} & \mathcal{J} \end{pmatrix}$$

where

$$\mathcal{A}_n^{(j)} = \begin{pmatrix} \alpha_n^{(j)} & 0 & \dots & 0 & \beta_n^{(j)} \\ 0 & 0 & \dots & 0 & 0 \\ \vdots & \ddots & \dots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 \end{pmatrix}, \quad j = 1, 2, \dots, p.$$

and $\mathcal{A}, \mathcal{J}$ and $\mathcal{O}$ are the $(m+1) \times (m+1)$ matrix defined in (14) and (15).

$\|B_n\| \to 0$ when $n \to \infty$. Therefore, the limiting system of error terms can be written as

$$e_{n+1}=\begin{pmatrix}\mathcal{J}&\mathcal{A}&\mathcal{O}&\mathcal{O}&\dots&\mathcal{O}&\mathcal{O}\\ \mathcal{O}&\mathcal{J}&\mathcal{A}&\mathcal{O}&\dots&\mathcal{O}&\mathcal{O}\\ \mathcal{O}&\mathcal{O}&\mathcal{J}&\mathcal{A}&\dots&\mathcal{O}&\mathcal{O}\\ \vdots&\vdots&\vdots&\vdots&&\vdots&\vdots\\ \mathcal{O}&\mathcal{O}&\mathcal{O}&\mathcal{O}&\dots&\mathcal{J}&\mathcal{A}\\ \mathcal{A}&\mathcal{O}&\mathcal{O}&\mathcal{O}&\dots&\mathcal{O}&\mathcal{J}\end{pmatrix}\begin{pmatrix}e_n^{(1)}\\ e_{n-1}^{(1)}\\ \vdots\\ e_{n-m}^{(1)}\\ \vdots\\ e_n^{(p)}\\ e_{n-1}^{(p)}\\ \vdots\\ e_{n-m}^{(p)}\end{pmatrix}$$

and $\|B_n\|\to 0$ when $n\to\infty$. This system is exactly the linearized system of (3) evaluated at the equilibrium point $\left(\overline{x^{(1)}},\overline{x^{(2)}},\dots,\overline{x^{(p)}}\right)$. From Theorems (5.1) and (5.2), the result follows.

6. **Numerical examples.** In this section we will consider several interesting numerical examples to verify our theoretical results. These examples shows different types of qualitative behavior of solutions of the system (3). All plots in this section are drawn with Matlab.

Exemple 6.1. *Let $m=1$ and $p=10$ in system (3), then we obtain the system*

$$x_{n+1}^{(1)}=1.2+\frac{x_{n-1}^{(2)}}{x_n^{(2)}},\quad x_{n+1}^{(2)}=A+\frac{x_{n-1}^{(3)}}{x_n^{(3)}},\dots,\quad x_{n+1}^{(10)}=1.2+\frac{x_{n-1}^{(1)}}{x_n^{(1)}},\quad n\in\mathbb{N}_0 \tag{23}$$

with $A=1.2>1$ and the initial values $x_{-1}^{(1)}=3.3, x_0^{(1)}=2, x_{-1}^{(2)}=1.1, x_0^{(2)}=0.3, x_{-1}^{(3)}=2.3, x_0^{(3)}=1.5, x_{-1}^{(4)}=0.5, x_0^{(4)}=2, x_{-1}^{(5)}=1.9, x_0^{(5)}=0.8, x_{-1}^{(6)}=4, x_0^{(6)}=1.3, x_{-1}^{(7)}=1.2, x_0^{(7)}=1.3, x_{-1}^{(8)}=2.1, x_0^{(8)}=2.3, x_{-1}^{(9)}=3.6, x_0^{(9)}=0.2, x_{-1}^{(10)}=2.3, x_0^{(10)}=1.1$. Then the positive equilibrium point $(\overline{x^{(1)}},\overline{x^{(2)}},\dots,\overline{x^{(10)}})=(2.2,2.2,\dots,2.2)$ of system (23)) is globally asymptotically stable (see Figure (1), Theorem (4.4)).

Exemple 6.2. *Consider the system (23) with $A=1$ and the initial values $x_{-1}^{(1)}=0.3, x_0^{(1)}=1.1, x_{-1}^{(2)}=1.3, x_0^{(2)}=0.3, x_{-1}^{(3)}=1.4, x_0^{(3)}=1.5, x_{-1}^{(4)}=0.5, x_0^{(4)}=2, x_{-1}^{(5)}=1.9, x_0^{(5)}=0.8, x_{-1}^{(6)}=4, x_0^{(6)}=1.3, x_{-1}^{(7)}=1.4, x_0^{(7)}=1.3, x_{-1}^{(8)}=0.1, x_0^{(8)}=1.1, x_{-1}^{(9)}=1.6, x_0^{(9)}=1.7, x_{-1}^{(10)}=1.9, x_0^{(10)}=1.1$. Then the solution oscillates about the positive equilibrium point $(\overline{x^{(1)}},\overline{x^{(2)}},\dots,\overline{x^{(10)}})=(2,2,\dots,2)$ of system (23) with semi-cycles having at most five terms. Also, the equilibrium is not globally asymptotically stable (see Figure(2), Theorem 4.2).*

Exemple 6.3. *Consider the system (23) with $A=0.9$ and the initial values $x_{-1}^{(1)}=1.2, x_0^{(1)}=0.7, x_{-1}^{(2)}=1.2, x_0^{(2)}=2.3, x_{-1}^{(3)}=0.4, x_0^{(3)}=1.1, x_{-1}^{(4)}=0.8, x_0^{(4)}=8, x_{-1}^{(5)}=1.3, x_0^{(5)}=1.8, x_{-1}^{(6)}=2.6, x_0^{(6)}=0.9, x_{-1}^{(7)}=1.4, x_0^{(7)}=1.1, x_{-1}^{(8)}=0.1, x_0^{(8)}=1.4, x_{-1}^{(9)}=0.9, x_0^{(9)}=1.3, x_{-1}^{(10)}=1.2, x_0^{(10)}=2.1$. Then the positive equilibrium point $(\overline{x^{(1)}},\overline{x^{(2)}},\overline{x^{(3)}},\overline{x^{(4)}})=(1.9,1.9,\dots,1.9)$ of system (23) is not*

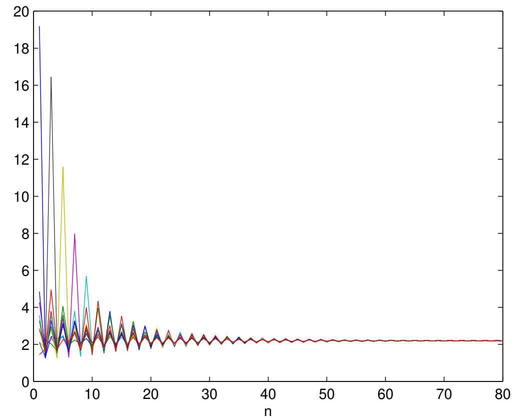

FIGURE 1. The plot of system (23) with $A = 1.2 > 1$

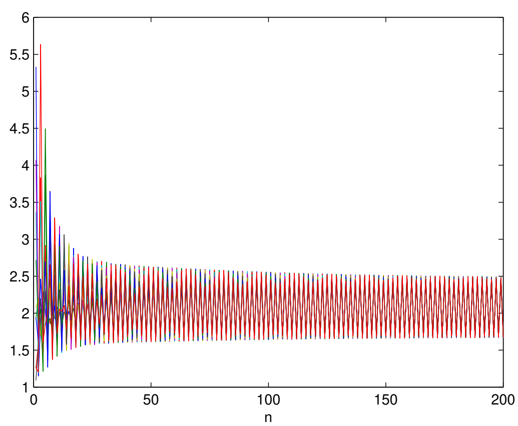

FIGURE 2. The plot of system (23) with $A = 1$

globally asymptotically stable. Also, this solution is unbounded solution see Figure (3), Theorem 4.2).

Exemple 6.4. *Let $m = 5$ and $p = 4$ in system (3), then we obtain the system*

$$x^{(1)}_{n+1} = A+\frac{x^{(2)}_{n-5}}{x^{(2)}_{n}},\quad x^{(2)}_{n+1} = A+\frac{x^{(3)}_{n-5}}{x^{(3)}_{n}},\quad x^{(3)}_{n+1} = A+\frac{x^{(4)}_{n-5}}{x^{(4)}_{n}},\quad x^{(4)}_{n+1} = A+\frac{x^{(1)}_{n-5}}{x^{(1)}_{n}},\quad n\in\mathbb{N}_0 \tag{24}$$

with $A = 1.4 > 1$ and the initial values $x^{(1)}_{-5} = 1.2, x^{(1)}_{-4} = 0.8, x^{(1)}_{-3} = 1.9, x^{(1)}_{-2} = 2.2, x^{(1)}_{-1} = 0.3, x^{(1)}_{0} = 1.7, x^{(2)}_{-5} = 1.3, x^{(2)}_{-4} = 2.4, x^{(2)}_{-3} = 1.2, x^{(2)}_{-2} = 0.5, x^{(2)}_{-1} = 1.6, x^{(2)}_{0} = 2.3, x^{(3)}_{-5} = 0.4, x^{(3)}_{-4} = 1.1, x^{(3)}_{-3} = 1.4, x^{(3)}_{-2} = 2.1, x^{(3)}_{-1} = 0.3, x^{(3)}_{0} = 1.1, x^{(4)}_{-5} = 0.8, x^{(4)}_{-4} = 1.2, x^{(4)}_{-3} = 1.8, x^{(4)}_{-2} = 3.1, x^{(4)}_{-1} = 0.7, x^{(4)}_{0} = 1.8$,. Then the

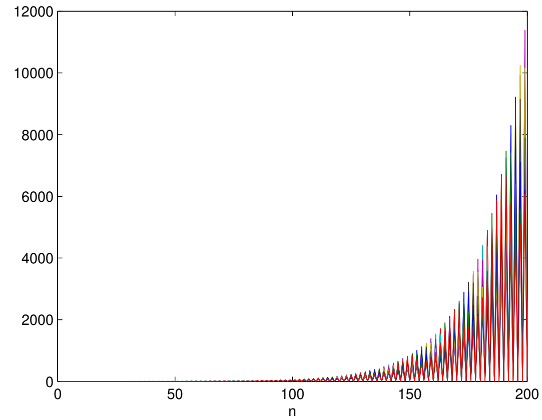

FIGURE 3. The plot of system (23) with $A = 0.9 < 1$

positive equilibrium point $(\overline{x^{(1)}}, \overline{x^{(2)}}, \overline{x^{(3)}}, \overline{x^{(4)}}) = (2.4, 2.4, 2.4, 2.4)$ *of system* (24)) *is globally asymptotically stable (see Figure* (4)*, Theorem* (4.4)*).*

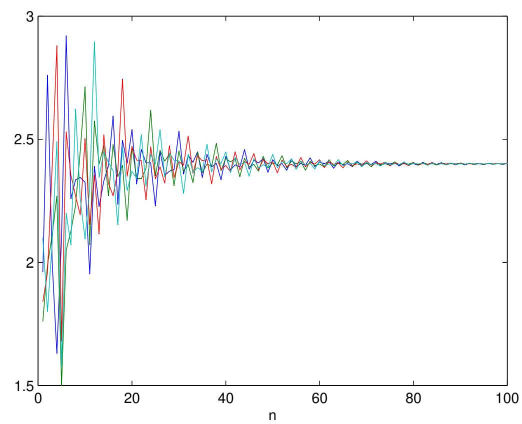

FIGURE 4. The plot of system (24) with $A = 1.4 > 1$

Exemple 6.5. *Consider the system* (24) *with* $A = 1$ *and the initial values* $x^{(1)}_{-5} = 0.4, x^{(1)}_{-4} = 1.3, x^{(1)}_{-3} = 2.9, x^{(1)}_{-2} = 1.2, x^{(1)}_{-1} = 0.8, x^{(1)}_{0} = 1.2, x^{(2)}_{-5} = 0.3, x^{(2)}_{-4} = 1.4, x^{(2)}_{-3} = 1.3x^{(2)}_{-2} = 0.5, x^{(2)}_{-1} = 1.6, x^{(2)}_{0} = 2.1, x^{(3)}_{-5} = 1.3, x^{(3)}_{-4} = 2.1, x^{(3)}_{-3} = 1.4, x^{(3)}_{-2} = 2.1, x^{(3)}_{-1} = 0.3, x^{(3)}_{0} = 1.5, x^{(4)}_{-5} = 0.6, x^{(4)}_{-4} = 1.2, x^{(4)}_{-3} = 1.3, x^{(4)}_{-2} = 0.8, x^{(4)}_{-1} = 1.7, x^{(4)}_{0} = 0.1,$. *Then the solution oscillates about the positive equilibrium point* $(\overline{x^{(1)}}, \overline{x^{(2)}}, \overline{x^{(3)}}, \overline{x^{(4)}}) = (2.4, 2.4, 2.4, 2.4)$ *of system* (24) *with semi-cycles having at most five terms. Also, the equilibrium is not globally asymptotically stable (see Figure*(5)*, Theorem 4.2).*

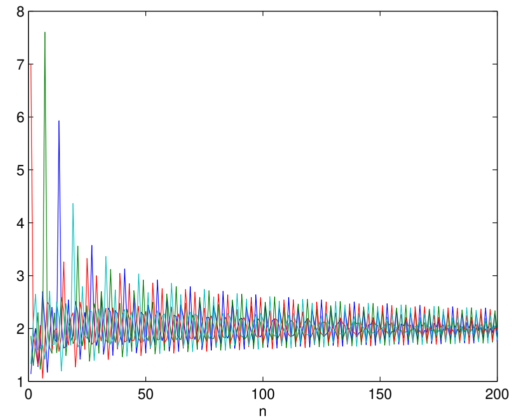

FIGURE 5. The plot of system (24) with $A = 1$

Exemple 6.6. *Consider the system* (24) *with* $A = 0.7$ *and the initial values* $x^{(1)}_{-5} = 1.3, x^{(1)}_{-4} = 0.9, x^{(1)}_{-3} = 2.1, x^{(1)}_{-2} = 0.9, x^{(1)}_{-1} = 0.7, x^{(1)}_{0} = 2.2, x^{(2)}_{-5} = 1.3, x^{(2)}_{-4} = 0.4, x^{(2)}_{-3} = 1.3x^{(2)}_{-2} = 1.5, x^{(2)}_{-1} = 1.2, x^{(2)}_{0} = 1.1, x^{(3)}_{-5} = 1.7, x^{(3)}_{-4} = 1.6, x^{(3)}_{-3} = 1.5, x^{(3)}_{-2} = 2.3, x^{(3)}_{-1} = 0.9, x^{(3)}_{0} = 1.5, x^{(4)}_{-5} = 0.6, x^{(4)}_{-4} = 1.4, x^{(4)}_{-3} = 2.3, x^{(4)}_{-2} = 3.1, x^{(4)}_{-1} = 2.7, x^{(4)}_{0} = 1.9$. *Then the positive equilibrium point* $(\overline{x^{(1)}}, \overline{x^{(2)}}, \overline{x^{(3)}}, \overline{x^{(4)}}) = (1.7, 1.7, 1.7, 1.7)$ *of system* (23) *is not globally asymptotically stable. Also, this solution is unbounded solution see Figure* (6)*, Theorem 4.2).*

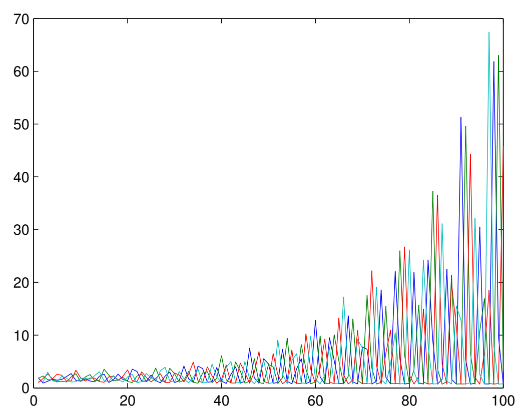

FIGURE 6. The plot of system (24) with $A = 0.7 < 1$

7. **Conclusions and some open problems.** In the paper, we studied the global behavior of solutions of system (3) composed by p rational difference equations. More exactly, we here study the global asymptotic stability of equilibrium, the rate of convergence of positive solutions. Also, we present some results about the general

behavior of solutions of system (3) and some numerical examples are carried out to support the analysis results. Our system generalized the equations and systems studied in [6, 8] and [20].

The findings suggest that this approach could also be useful for extended to a system with arbitrary constant different parameters, or to a system with a nonautonomous parameter, or to a system with different parameters and arbitrary powers. So, we will give the following some important open problems for difference equations theory researchers.

Open Problem 1: study the dynamical behaviors of the system of difference equations

$$x_{n+1}^{(1)} = A_1 + \frac{x_{n-m}^{(2)}}{x_n^{(2)}}, \quad x_{n+1}^{(2)} = A_2 + \frac{x_{n-m}^{(3)}}{x_n^{(3)}}, \dots, \quad x_{n+1}^{(p)} = A_p + \frac{x_{n-m}^{(1)}}{x_n^{(1)}}, \quad n, m, p \in \mathbb{N}_0$$

where A_i, $i = 1, 2, \dots, p$ are nonnegative constants and $x_{-m}^{(j)}, x_{-m+1}^{(j)}, \dots, x_{-1}^{(j)}, x_0^{(j)}, j = 1, 2, \dots, p$ are positive real numbers.

Open Problem 2: study the dynamical behaviors of the system of difference equations

$$x_{n+1}^{(1)} = \alpha_n + \frac{x_{n-m}^{(2)}}{x_n^{(2)}}, \quad x_{n+1}^{(2)} = \alpha_n + \frac{x_{n-m}^{(3)}}{x_n^{(3)}}, \dots, \quad x_{n+1}^{(p)} = \alpha_n + \frac{x_{n-m}^{(1)}}{x_n^{(1)}}, \quad n, m, p \in \mathbb{N}_0$$

where α_n is a sequence (this sequence can be chosen as convergent, periodic or bounded), and $x_{-m}^{(j)}, x_{-m+1}^{(j)}, \dots, x_{-1}^{(j)}, x_0^{(j)}, j = 1, 2, \dots, p$ are positive real numbers.

Open Problem 3: study the dynamical behaviors of the system of difference equations

$$x_{n+1}^{(1)} = A_1 + \frac{\left(x_{n-m}^{(2)}\right)^{p_1}}{\left(x_n^{(2)}\right)^{q_1}}, \quad x_{n+1}^{(2)} = A_2 + \frac{\left(x_{n-m}^{(3)}\right)^{p_2}}{\left(x_n^{(3)}\right)^{q_2}}, \dots, \quad x_{n+1}^{(p)} = A_p + \frac{\left(x_{n-m}^{(1)}\right)^{p_p}}{\left(x_n^{(1)}\right)^{q_p}},$$

where$n, m, p \in \mathbb{N}_0$, A_i, $i = 1, 2, \dots, p$ are nonnegative constants, the parameters p_i, q_i, $i = 1, 2, \dots, p$ are non-negative and $x_{-m}^{(j)}, x_{-m+1}^{(j)}, \dots, x_{-1}^{(j)}, x_0^{(j)}, j = 1, 2, \dots, p$ are positive real numbers.

Acknowledgement(s). This work was supported by Directorate General for Scientific Research and Technological Development (DGRSDT), Algeria.

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Received xxxx 20xx; revised xxxx 20xx.

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