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On some extended Routh–Hurwitz conditions for fractional-order autonomous systems of order $\alpha \in (0, 2)$ and their applications to some population dynamic models

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A B S T R A C T

The Routh–Hurwitz stability criterion is a useful tool for investigating the stability property of linear and nonlinear dynamical systems by analyzing the coefficients of the corresponding characteristic polynomial without calculating the eigenvalues of its Jacobian matrix. Recently some of these conditions have been generalized to fractional systems of order $\alpha \in [0, 1)$. In this paper we extend these results to fractional systems of order $\alpha \in [0, 2)$. Stability diagram and phase portraits classification in the (τ, Δ)-plane for planer fractional-order system are reported. Finally some numerical examples from population dynamics are employed to illustrate our theoretical results.

1. Introduction

In the past few decades, fractional calculus theory has been improved significantly and has been successfully applied to various research fields. Compared with integer calculus, fractional calculus is more suitable in describing the memory and genetic characteristics. We can find numerous applications of fractional order derivatives in the mathematical modeling of physical and biological phenomena in various fields of science and engineering, (see for example [1–17]). The fractional order derivatives have many definitions such as the Riemann–Liouville definition, the Gürwald–Letnikov definition, the Caputo definition and so on. In this paper, we consider the standard fractional differential equation:

$$D^{\alpha}x(t) = f(x(t)), \qquad \alpha \in [0, 2), \tag{1}$$

where $x(t) \in \mathbb{R}^n$ and D^{α} is the Caputo derivative operator defined as follows:

$$D^{\alpha}f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} f^{(m)}(\tau)d\tau. \tag{2}$$

Where, m is the first integer greater than α, and $\Gamma(.)$ is the Gamma function. For convention, we put: $D^0 f(t) = f(t)$. The stability of a hyperbolic equilibrium point of any dynamical system with integer-order derivative is determined by the signs of the real parts of the eigenvalues of its Jacobian matrix. If all the eigenvalues of the Jacobian matrix have negative real parts then this hyperbolic equilibrium point is asymptotically stable. This result is equivalent to the algebraic procedure *Routh–Hurwitz criterion*. The *Routh–Hurwitz criterion* is well known for determining the stability of linear systems of the form

$$\dot{x}(t) = Ax(t), \quad x(t) \in \mathbb{R}^n \text{ and } \quad A \quad \text{is} \quad n \times n \text{ real matrix}, \tag{3}$$

without involving root solving. So this criterion provides also an answer to the question of stability by considering the characteristic equation of the system, which can be written as

$$P(\lambda) = \lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots + a_n = 0, \tag{4}$$

where all the coefficients a_i are real constants. The n Hurwitz matrices are given by

$$H_1 = (a_1), H_2 = \begin{pmatrix} a_1 & 1 \\ a_3 & a_2 \end{pmatrix}, \; H_3 = \begin{pmatrix} a_1 & 1 & 0 \\ a_3 & a_2 & a_1 \\ a_5 & a_4 & a_3 \end{pmatrix}, \ldots$$

$$H_n = \begin{pmatrix} a_1 & 1 & 0 & \cdots & 0 \\ a_3 & a_2 & a_1 & \cdots & 0 \\ a_5 & a_4 & a_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & a_n \end{pmatrix},$$

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where $a_j = 0$ if $j > n$. All of the roots of the polynomial $P(\lambda)$ have negative real part if and only if the determinants of all Hurwitz matrices are positive, that is:

$$Det(H_j) > 0,\ j = 1, ..., n. \tag{5}$$

As in integer calculus, stability analysis is a central task in the study of fractional differential system and fractional control. Stability analysis of fractional differential equations was investigated by Matignon who produced the following theorem when the order of derivative is between 0 and 1.

Theorem 1. *[18] The autonomous system:*

$$D^{\alpha}x(t) = Ax(t) \text{ with } x(t_0) = x_0, \tag{6}$$

is asymptotically stable if and only if

$$|\arg(spec(A))| > \frac{\alpha\pi}{2}, \tag{7}$$

where $\alpha \in [0, 1)$, $\arg(.)$ *is the principal argument of a given complex number and* $spec(A)$ *is the spectrum (set of all eigenvalues) of* A.

This work is in fact the starting point of several results in the field. In recent papers in [19–22], the authors derived some optimal *Routh–Hurwitz conditions* of the dynamical systems involving the Caputo fractional derivative of orders between 0 and 1. These new optimal *Routh–Hurwitz conditions* serve as necessary and sufficient conditions to guarantee that all roots of the characteristic polynomial obtained from the linearization process are located inside the Matignon stability sector when the order of the derivative is between 0 and 1.

If $0 < \alpha < 2$, an extension of Matignon's theorem is given in [23]. The given result permits to check the stability of any system of the form given by (6) with $\alpha \in [0, 2)$ can be analyzed in a unified way by the location of the eigenvalues of matrix A in the complex plane. System described by (6) is hence asymptotically stable if and only if $|\arg(spec(A))| > \frac{\alpha\pi}{2}$, where $0 < \alpha < 2$. In this paper we extend the *Routh–Hurwitz conditions* to fractional order systems of order $\alpha \in [0, 2)$, and for the first time we report the stability diagram and phase portraits classification in the (τ, Δ)-plane for planer fractional-order systems. We use these results to investigate the stability properties of some population models. Numerical simulations which support our theoretical analysis are also given.

2. The Routh–Hurwitz conditions for fractional-order systems of order $\alpha \in (0, 2)$

Since most biological systems are 1, 2, 3 or 4– dimensional, we will consider only fractional-order system with dimension $n = 2, 3$ and 4.

Remark 1.

- For $\alpha \in [0, 1[$, the Routh–Hurwitz conditions (5) are sufficient but not necessary to have (7) satisfied.
- For $\alpha \in]1, 2)$, the Routh–Hurwitz conditions (5) are necessary but not sufficient in general case to have (7) satisfied.

2.1. Fractional-order two dimensional systems

Proposition 1. *Consider the fractional linear system (6) with its corresponding characteristic Eq. (4). For* $n = 2$, *the necessary and sufficient conditions for every* $\alpha \in [0, 2[$ *to have (7) satisfied are*

$$a_2 > 0 \text{ and } a_1 > -2\sqrt{a_2}\cos(\alpha\frac{\pi}{2}). \tag{8}$$

Proof. For $n = 2$ the characteristic polynomial is

$$P(\lambda) = \lambda^2 + a_1\lambda + a_2.$$

Its discriminant is $D(P) = a_1^2 - 4a_2$.

1. If $D(P) \geq 0$ $\left(\text{i.e } a_2 \in \left]-\infty, \frac{a_1^2}{4}\right]\right)$, then $P(\lambda)$ have two real roots given by
$$\lambda_{\pm} = -\frac{1}{2}\left(a_1 \mp \sqrt{a_1^2 - 4a_2}\right).$$
For $\alpha \in [0, 2[$, we have
   - **a)** $(a_2 < 0 \text{ or } (a_2 \in [0, \frac{a_1^2}{4}] \text{ and } a_1 \leq -2\sqrt{a_2})) \Rightarrow \lambda_+ > 0$, then $\arg(\lambda_+) = 0 \leq \alpha\frac{\pi}{2}$, thus (7) is not satisfied.
   - **b)** $\left(a_2 \in [0, \frac{a_1^2}{4}] \text{ and } a_1 \geq 2\sqrt{a_2}\right) \Rightarrow \lambda_{\pm} < 0$, then $\arg(\lambda_+) = \pi > \alpha\frac{\pi}{2}$, thus (7) is satisfied.
2. If $D(P) < 0$ $\left(\text{i.e } a_2 \in]\frac{a_1^2}{4}, \infty[\right)$, then $P(\lambda)$ have two complex conjugate roots given by
$$\lambda_{\pm} = -\frac{1}{2}\left(a_1 \mp i\sqrt{4a_2 - a_1^2}\right),$$
then $\tan(\theta) = -\frac{\sqrt{4a_2 - a_1^2}}{a_1}$, where $\theta = |\arg(\lambda_{\pm})|$. One emphasis two possibility
   - **(a)** When $\alpha \in [0, 1[$ $\left(\text{i.e } \alpha\frac{\pi}{2} \in [0, \frac{\pi}{2}[\right)$ then if $\left(a_1 > 0 \text{ and } a_2 \in]\frac{a_1^2}{4}, \infty[\right)$, it follows that $\tan(\theta) < 0$, then $\theta \in]\frac{\pi}{2}, \pi[$. Therefore $\theta > \alpha\frac{\pi}{2}$. Thus, (7) is satisfied.
   But if $-2\sqrt{a_2}\cos(\alpha\frac{\pi}{2}) < a_1 < 0$ and $a_2 \in]\frac{a_1^2}{4}, \infty[$, then $\tan(\theta) > 0$ and $\tan^2(\theta) > \tan^2(\alpha\frac{\pi}{2})$, it follows that $\tan(\theta) > \tan(\alpha\frac{\pi}{2})$. Therefore $\theta > \alpha\frac{\pi}{2}$, and, (7) is satisfied. On the other hand if $(a_1 < -2\sqrt{a_2}\cos(\alpha\frac{\pi}{2}) < 0$ and $a_2 \in]\frac{a_1^2}{4}, \infty[)$, then $(\tan(\theta) > 0$ and $\tan^2(\theta) < \tan^2(\alpha\frac{\pi}{2}))$, it follows that $\tan(\theta) < \tan(\alpha\frac{\pi}{2})$. Therefore $\theta < \alpha\frac{\pi}{2}$, and, (7) is not satisfied.
   - **(b)** When $\alpha \in [1, 2[$ $\left(\text{i.e } \alpha\frac{\pi}{2} \in [\frac{\pi}{2}, \pi[\right)$, then if $\left(a_1 < 0 \text{ and } a_2 \in]\frac{a_1^2}{4}, \infty[\right)$, it follows that $\tan(\theta) > 0$, hence $\theta \in]0, \frac{\pi}{2}[$. Therefore $\theta < \alpha\frac{\pi}{2}$. Thus, (7) is not satisfied.
   But if $\left(0 < a_1 < -2\sqrt{a_2}\cos(\alpha\frac{\pi}{2}) \text{ and } a_2 \in]\frac{a_1^2}{4}, \infty[\right)$, then $\left(\tan(\theta) < 0 \text{ and } \tan^2(\theta) > \tan^2(\alpha\frac{\pi}{2})\right)$, it follows that $\tan(\theta) < \tan(\alpha\frac{\pi}{2})$. Therefore $\theta < \alpha\frac{\pi}{2}$. Thus, (7) is not satisfied.
   On the other hand if $(0 < -2\sqrt{a_2}\cos(\alpha\frac{\pi}{2}) < a_1$ and $a_2 \in]\frac{a_1^2}{4}, \infty[)$, then $(\tan(\theta) < 0$ and $\tan^2(\theta) < \tan^2(\alpha\frac{\pi}{2}))$, it follows that $\tan(\theta) > \tan(\alpha\frac{\pi}{2})$. Therefore $\theta > \alpha\frac{\pi}{2}$. Thus, (7) is satisfied. Finally we summarizes the proof as follows
     - From **(1-b), (2-a)** and **(2-b)** we have
       - if $(a_2 > 0$ and $a_1 > -2\sqrt{a_2}\cos(\alpha\frac{\pi}{2}))$, then $\theta = |\arg(\lambda_{\pm})| > \alpha\frac{\pi}{2}$. Thus, (7) is satisfied.
       - When $a_2 = 0$ then $\lambda_+ = 0$. Thus, $\arg(\lambda_+)$ is not defined and (7) is not satisfied.
     - From **(1-a), (2-a)** and **(2-b)** we have
       - if $\left(a_2 < 0 \text{ or } (a_2 > 0 \text{ and } a_1 \leq -2\sqrt{a_2}\cos(\alpha\frac{\pi}{2}))\right)$, then $|\arg(\lambda_+)| \leq \alpha\frac{\pi}{2}$. Thus, (7) is not satisfied.

□

2.2. Stability diagram and phase portraits classification for fractional-order planar systems

Consider the planar case of system (6), where $\alpha \in [0, 2)$. The characteristic equation of the matrix A can be written as

$$P(\lambda) = \lambda^2 - \tau\lambda + \Delta = 0.$$

where $\tau = Tr(A) = -a_1$ is the trace of the matrix A and $\Delta = Det(A) = a_2$ its determinant.

Remark 2. The conditions in (8) to have (7) satisfied are equivalent to:

$$\Delta > 0 \text{ and } \frac{\tau}{2} < \sqrt{\Delta}.cos(\frac{\alpha\pi}{2}). \tag{9}$$

- For $0 \le \alpha < 1$, (9) is equivalent to

$$\frac{\tau^2}{4}sec^2(\frac{\alpha\pi}{2}) < \Delta. \tag{10}$$

- For $1 < \alpha < 2$, (9) is equivalent to

$$\frac{\tau^2}{4}sec^2(\frac{\alpha\pi}{2}) > \Delta > 0. \tag{11}$$

Using the conditions (10), (11) and taking into account the following observations:

- For $\Delta < 0$, the two eigenvalues are real and have opposite signs; hence the equilibrium point is a saddle.
- For $\Delta > 0$, the eigenvalues are either real with the same sign (node point if $\tau^2 - 4\Delta > 0$), or complex conjugate (spiral point if $\tau^2 - 4\Delta < 0$.).
- The parabola $\tau^2 - 4\Delta = 0$. is the borderline between nodes and spirals.
- The curve of equation

  $$\tau = 2\sqrt{\Delta}\cos(\alpha\frac{\pi}{2})$$

  (i.e a branches of parabola of equation $\Delta = \frac{\tau^2}{4}sec^2(\frac{\alpha\pi}{2})$) is the borderline between stability and instability region of the equilibrium point in the half plane $\Delta > 0$.

We can draw the stability diagram and phase portraits classification in the (τ, Δ) plane as shown in Fig. 1, where the stability area is with green colour. From this figure we observe that:

- When $\alpha \to 1$ we find the same stability diagram and phase portrait classification as in the integer systems.
- The stability area for $\alpha < 1$ is wider than stability area for the integer case.
- The stability area for $\alpha > 1$ is narrower than stability area for the integer case.

2.3. Fractional-order three dimensional systems

Proposition 2. For $n = 3$

(1) *If $D(P) > 0$, then the Routh–Hurwitz conditions (5) are the necessary and sufficient conditions for every $\alpha \in [0, 2[$ to have (7) satisfied:*

$$a_1 > 0,\ a_3 > 0 \text{ and } a_1a_2 > a_3.$$

(2) *If $D(P) < 0$ and $\alpha \in [0, 2[$, then*
(i) *If $a_1 \ge 0$, $a_2 \ge 0$, $a_3 > 0$ then we have: If $\alpha < \frac{2}{3}$, then (7) is satisfied, but if $\alpha > \frac{4}{3}$, then (7) is not satisfied.*
(ii) *If $a_1 > 0$, $a_2 > 0$, $a_1a_2 = a_3$, then (7) is satisfied for all $\alpha \in [0, 1[$, and (7) is not satisfied for all $\alpha \in]1, 2[$.*

Proof. For $n = 3$ the characteristic polynomial is

$$P(\lambda) = \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3. \tag{12}$$

(1) If $D(P) > 0$, then $P(\lambda) = 0$ have three real roots hence *Routh–Hurwitz conditions* are necessary and sufficient for (7).
(2) If $D(P) < 0$, then $P(\lambda) = 0$ have one real root $\lambda_1 = -b$ and two complex conjugate roots $\lambda_{2,3} = \beta \pm i\gamma$. Thus,

$$P(\lambda) = (\lambda + b)(\lambda - \beta - i\gamma)(\lambda - \beta + i\gamma),$$

$$\text{it follow that} \begin{cases} a_1 = b - 2\beta, \\ a_2 = \beta^2 + \gamma^2 - 2b\beta, \\ a_3 = \left(\beta^2 + \gamma^2\right)b, b > 0. \end{cases}$$

(i)
* If $\begin{cases} a_1 \ge 0, \\ a_2 \ge 0, \end{cases}$ then $\begin{cases} b \ge 2\beta, \\ \beta^2 + \gamma^2 \ge 2b\beta \ge 4\beta^2, \end{cases}$ hence $\theta \in [\frac{\pi}{3}, \frac{2\pi}{3}]$, where $\theta = |arg(\lambda)|$.
* If $\alpha < \frac{2}{3}$, then $\theta > \frac{\alpha\pi}{2}$. But if $\alpha > \frac{4}{3}$, then $\theta < \frac{\alpha\pi}{2}$.

(ii) If $a_1a_2 = a_3$, then $\beta(\beta^2 + \gamma^2 + b^2 - 2b\beta) = 0$, hence $\beta = 0$ or $\beta^2 + \gamma^2 + b^2 - 2b\beta = 0$. The second equality is not valid, that is $\beta = 0$, then $|arg(\lambda_\pm)| = \frac{\pi}{2}$. Thus, (7) is satisfied for all $\alpha \in [0, 1[$, and (7) is not satisfied for all $\alpha \in [1, 2[$.

□

In the general case we use the following proposition.

Proposition 3. For $n = 3$ *and* $\alpha \in [0, 2)$. *If $D(P) < 0$, then, the necessary and sufficient conditions to have (7) satisfied are*

$$\begin{cases} a_3 > 0, \\ \frac{2}{\pi}\left|tan^{-1}(-3\sqrt{3}\frac{u-v}{3(u+v)+2a_1})\right| > \alpha, \end{cases}$$

where

$$u = \sqrt[3]{\frac{-q + \sqrt{\frac{4}{27}p^3 + q^2}}{2}} \text{ and } v = \sqrt[3]{\frac{-q - \sqrt{\frac{4}{27}p^3 + q^2}}{2}}, \tag{13}$$

with

$$p = a_2 - \frac{a_1^2}{3} \text{ and } q = \frac{a_1}{27}(2a_1^2 - 9a_2) + a_3. \tag{14}$$

Proof. If $D(P) < 0$. Then, $P(\lambda)$ has one real root λ_1 and two complex conjugate roots λ_i, $i = 2, 3$. Substituting λ in Eq. (12) by $x - \frac{a_1}{3}$ (the Tschirnhaus transformation) we get the equation

$$x^3 + px + q = 0, \tag{15}$$

where p and q are given by (14). The left hand side of Eq. (15) is a monic trinomial called a depressed cubic. Any formula for the roots of a depressed cubic may be transformed into a formula for the roots of Eq. (12) using (14) and the relation

$$\lambda = x - \frac{a_1}{3}. \tag{16}$$

following Cardano's method the real root of (15) is given by

$$x_1 = u + v, \tag{17}$$

where the two variables u and v are given by (13). The complex roots are given by

$$x_2 = ju + \bar{j}v \text{ and } x_3 = j^2u + \bar{j}^2v, \tag{18}$$

where $j = e^{i\frac{2\pi}{3}} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$. Using (14) and (16) we obtain the roots of $P(\lambda)$. Namely,

$$\lambda_1 = u + v - \frac{a_1}{3},$$

$$\lambda_2 = ju + \bar{j}v - \frac{a_1}{3} = -\frac{1}{6}(3(u+v) + 2a_1 - i3\sqrt{3}(u-v)),$$

$$\lambda_3 = j^2u + \bar{j}^2v - \frac{a_1}{3} = -\frac{1}{6}(3(u+v) + 2a_1 + i3\sqrt{3}(u-v)).$$

We have

$$P(\lambda) = (\lambda - \lambda_1)(\lambda + \frac{1}{6}(3(u+v) + 2a_1 - i3\sqrt{3}(u-v)))$$
$$\times(\lambda + \frac{1}{6}(3(u+v) + 2a_1 + i3\sqrt{3}(u-v))),$$

it follow that $a_3 = -\lambda_1((3\sqrt{3}(u-v))^2 + (\frac{1}{6}(3(u+v) + 2a_1)^2)$ then $a_3 > 0$ imply that $\lambda_1 < 0$. Thus, $|\arg(\lambda_1)| > \frac{\alpha\pi}{2}$.

On the other hand $|\arg(\lambda_i)| = \left|tan^{-1}(-3\sqrt{3}\frac{u-v}{3(u+v)+2a_1})\right|$, for all $i = 2, 3$.

Thus, $\frac{2}{\pi}\left|tan^{-1}(-3\sqrt{3}\frac{u-v}{3(u+v)+2a_1})\right| > \alpha$, imply that $|\arg(\lambda_{2,3})| > \frac{\alpha\pi}{2}$. □

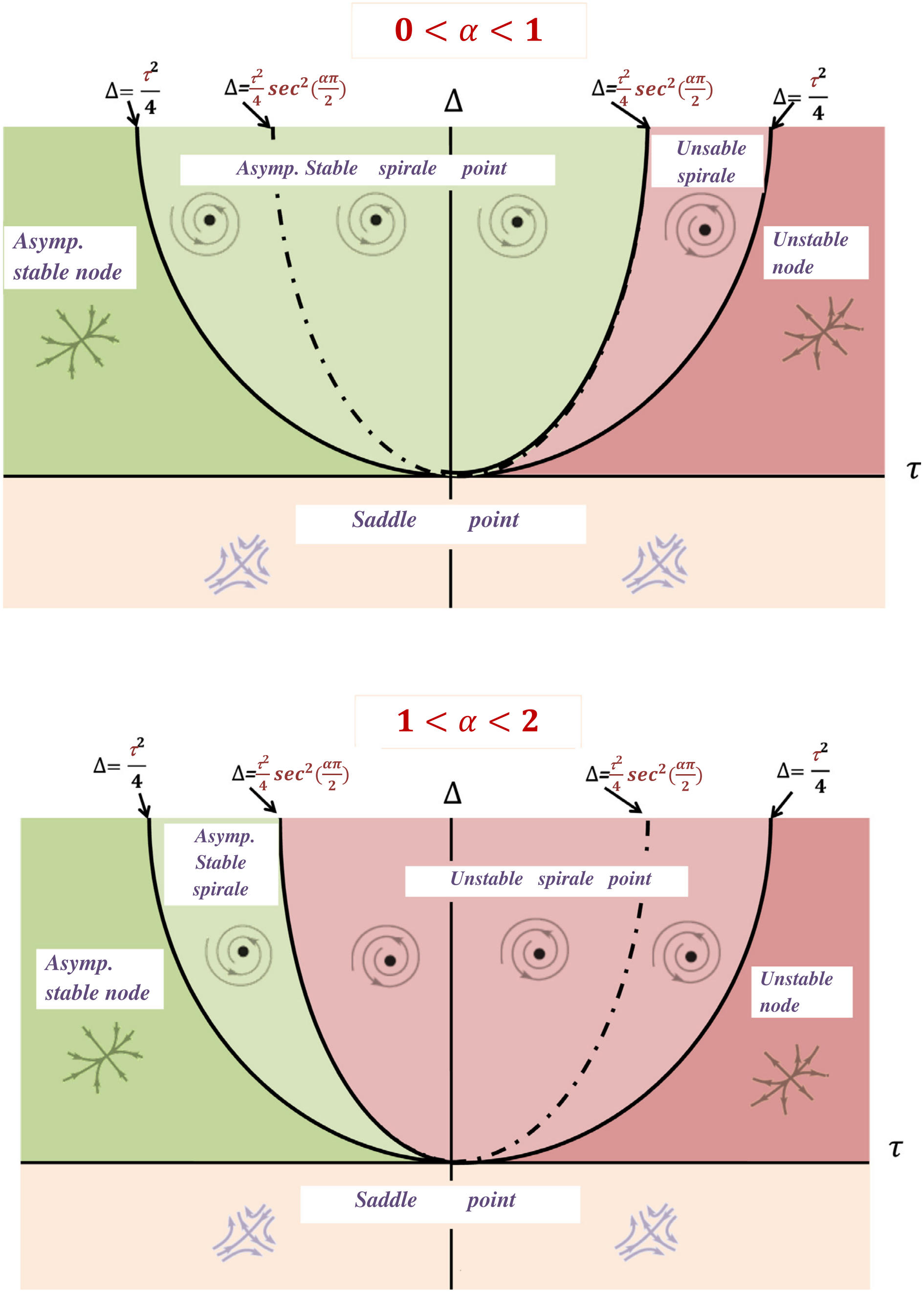

Fig. 1. Stability diagram and phase portraits classification in the (τ, Δ)-plane for planer fractional-order system. (For interpretation of the references to color in this figure, the reader is referred to the web version of this article.)

2.4. Fractional-order four dimensional systems

Proposition 4. For $n = 4$

1. *The conditions (5) are sufficient conditions for the equilibrium point x^* to be locally asymptotically stable for all $\alpha \in [0, 1)$, but they are necessary conditions for all $\alpha \in [1, 2)$.*
2. *If $D(P) > 0$, $a_1 > 0$, $a_2 < 0$ and $\alpha \in [\frac{2}{3}, 2]$ then the equilibrium point x^* is unstable.*
3. *If $D(P) < 0$, $a_1 > 0$, $a_2 > 0$, $a_3 > 0$, $a_4 > 0$, then the equilibrium point x^* is locally asymptotically stable for all $\alpha \in [0, \frac{1}{2}[$. Also, if $D(P) < 0$, $a_1 < 0$, $a_2 > 0$, $a_3 < 0$, $a_4 > 0$, then the equilibrium point x^* is unstable for all $\alpha \in [0, 2]$.*
4. *If $D(P) < 0$, $a_1 > 0$, $a_2 > 0$, $a_3 > 0$, $a_4 > 0$ and $a_2 = \frac{a_1 a_4}{a_3} + \frac{a_3}{a_1}$, then the equilibrium point x^* is locally asymptotically stable, for all $\alpha \in [0, 1[$ and unstable for all $\alpha \in]1, 2]$.*
5. *$a_4 > 0$ is the necessary condition for the equilibrium point x^* to be locally asymptotically stable.*

Proof.

1. We emphasis two cases:
   - For $\alpha \in [0, 1[$, assume that the conditions (5) are satisfied, then all real eigenvalues and all real parts of complex conjugate eigenvalues of Eq. (4) are negative, hence, conditions (5) implies that all the eigenvalues of (4) lie in the left-half complex plane then $|\arg(\lambda_i)| > \frac{\pi}{2}$. Thus, $|\arg(\lambda_i)| > \frac{\pi}{2} > \alpha\frac{\pi}{2}$. Therefore x^* is locally asymptotically stable.

- For $\alpha \in [1, 2]$, we have $\alpha\frac{\pi}{2} \geq \frac{\pi}{2}$. Assume that (7) is satisfied then $|\arg(\lambda_i)| > \alpha\frac{\pi}{2}$, implies that $|\arg(\lambda_i)| > \frac{\pi}{2}$. Therefore the asymptotic stability of x^* imply that the conditions (5) are satisfied.

2. Notice that if $D(P) > 0$ then there exists 4 distinct real roots r_1, r_2, r_3, r_4 or two pairs of complex eigenvalues $\lambda_{1,2} = \beta_1 \pm j\gamma_1$, and $\lambda_{3,4} = \beta_2 \pm j\gamma_2$.
In the case of real roots we have

$$\begin{aligned}
a_1 &= -(r_1 + r_2 + r_3 + r_4),\\
a_2 &= r_1r_2 + r_1r_3 + r_1r_4 + r_2r_3 + r_2r_4 + r_3r_4,\\
a_3 &= -[r_1r_2r_3 + r_1r_2r_4 + r_1r_3r_4 + r_2r_3r_4],\\
a_4 &= r_1r_2r_3r_4.
\end{aligned}$$

Clearly, $a_2 < 0$ implies that at last two real roots have opposite signs. Hence the equilibrium point x^* is unstable. In the other case:

$$\begin{aligned}
a_1 &= -2(\beta_1 + \beta_2),\\
a_2 &= \beta_1^2 + \gamma_1^2 + \beta_2^2 + \gamma_2^2 + 4\beta_1\beta_2,\\
a_3 &= -2[\beta_1(\beta_2^2 + \gamma_2^2) + \beta_2(\beta_1^2 + \gamma_1^2)],\\
a_4 &= (\beta_1^2 + \gamma_1^2)(\beta_2^2 + \gamma_2^2).
\end{aligned}$$

We have $a_2 < 0$ imply that $\beta_2^2 \sec^2\theta + \beta_1^2 + \gamma_1^2 + 4\beta_1\beta_2 < 0$, where $\theta = |\arg\lambda_{3,4}|$. Therefore $\beta_2^2 \sec^2\theta < -4\beta_1\beta_2$, which imply that $\beta_1\beta_2 < 0$ (i.e β_1 and β_2 are of opposite signs),
Without loss of generality, suppose that $\beta_1 < 0, \beta_2 > 0$, then using the condition $a_1 > 0$, we get

$$\beta_2^2 \sec^2\theta < -4\beta_1\beta_2 < 4\beta_2^2.$$

This implies that $\theta < \pi/3$. Hence, the equilibrium point x^* is unstable for all $\alpha \in [\frac{2}{3}, 2]$.

3. If $D(P) < 0$ then there exists two real roots $\lambda_1 = r_1, \lambda_2 = r_2$, and one pair of complex eigenvalues $\lambda_{3,4} = \beta \pm j\gamma$. Then we have

$$\begin{aligned}
a_1 &= -(r_1 + r_2 + 2\beta),\\
a_2 &= r_1r_2 + \beta^2 + \gamma^2 + 2\beta(r_1 + r_2),\\
a_3 &= -2\beta r_1r_2 - (r_1 + r_2)(\beta^2 + \gamma^2),\\
a_4 &= r_1r_2(\beta^2 + \gamma^2).
\end{aligned}$$

Assume that $a_1 > 0$, $a_2 > 0$, $a_3 > 0$, $a_4 > 0$ there are zero changes in sign of the coefficients of the characteristic polynomial $P(\lambda)$, then by Descartes' rule of signs, it follows that there is no positive real roots of $P(\lambda)$, this implies that $r_1 < 0$ and $r_2 < 0$. On the other hand $a_3 > 0$ implies that $2\beta r_1r_2 + (r_1 + r_2)\beta^2 sec^2\theta < 0$.
We distinguish two cases:
    1. If $\beta \leq 0$, then x^* is locally asymptotically stable for all $\alpha \in [0, 1[$, particularly for $\alpha \in [0, \frac{1}{2}[$.
    2. If $\beta > 0$, then $-\frac{(r_1+r_2)}{2}\beta \sec^2\theta > r_1r_2$ and $a_2 > 0$ implies that $r_1r_2 > -2\beta(r_1 + r_2) - \beta^2 \sec^2\theta$, where $\theta = |\arg\lambda_{3,4}|$. Therefore, $-\frac{(r_1+r_2)}{2}\beta \sec^2\theta > -2\beta(r_1 + r_2) - \beta^2 \sec^2\theta$, then $-(r_1 + r_2)[\frac{\beta}{2}\sec^2\theta - 2\beta] > -\beta^2 \sec^2\theta$, it follow that $\beta^2 \sec^2\theta > -(r_1 + r_2)[-\frac{\beta}{2}\sec^2\theta + 2\beta]$, then $a_1 > 0$ implies that $-(r_1 + r_2) > 2\beta$, therefore $\beta^2 \sec^2\theta > 2\beta[-\frac{\beta}{2}\sec^2\theta + 2\beta]$. Thus, $\beta^2 \sec^2\theta > -\beta^2 \sec^2\theta + 4\beta^2$ which implies that $\sec^2\theta > 2$, therefore $\frac{\pi}{4} < \theta < \frac{\pi}{2}$. Then x^* is locally asymptotically stable for all $\alpha \in [0, \frac{1}{2}[$.

If the conditions $a_1 < 0$, $a_2 > 0$, $a_3 < 0$, $a_4 > 0$ are satisfied, then there are zero changes in sign of the coefficients of the polynomial $P(-\lambda)$, then by Descartes' rule of signs, it follows that there is no positive real roots of $P(-\lambda)$, this mean that there is no negative real roots for the characteristic polynomial $P(\lambda)$, therefore $r_1 > 0$ and $r_2 > 0$. Thus, the equilibrium point x^* is unstable for all $\alpha \in [0, 2]$.

4. Notice that $D(P) < 0$, $a_1 > 0$, $a_2 > 0$, $a_3 > 0$, $a_4 > 0$ imply that there are two negative real eigenvalues, and the condition $a_2 = \frac{a_1a_4}{a_3} + \frac{a_3}{a_1}$ implies that the two other eigenvalues are $\lambda_{3,4} = \pm i\sqrt{\frac{a_3}{a_1}}$, which lie on the imaginary axis (i.e $|\arg\lambda_{3,4}| = \frac{\pi}{2}$).
Consequently, if $\alpha \in [0, 1[$, then all eigenvalues lie in the stable region, and if $\alpha \in]1, 2]$, then $\lambda_{3,4}$ lie on the unstable region.
5. The part (5) is proved in [20] for general n, which includes our current case.

□

Remark 3.

- Although the stability criteria given by inequality (7) with the fractional order α as the main variable, remain valid for both cases $\alpha \in [0, 1)$ and $\alpha \in (1, 2]$ the stability area in the parameter space does not remain the same as illustrated in Fig. 1, where the stability region (green colour) for $\alpha \in (1, 2]$ is restricted than the stability region for $\alpha \in [0, 1)$, indicating a high requirement on the parameter to have stability for $\alpha \in (1, 2]$ than for $\alpha \in [0, 1)$.
  We have reported a common form of stability conditions on parameters for both cases $\alpha \in [0, 1)$ and $\alpha \in (1, 2]$, for dimension $n = 2$ and $n = 3$ in Propositions 1, and 3, respectively, but for $n = 4$ no common form where fond.
- Although the results presented in proposition $1 - 4$, elaborate conditions on parameters for satisfying necessary and sufficient conditions for stability of equilibrium points, the proposed analysis is limited to restricted order of characteristic equation resulted from the Jacobian matrix.

Remark 4. The validity of Routh–Hurwitz conditions derived in [20], for fractional order differential systems, is limited to fractional order $\alpha \in [0, 1)$, but the validity of conditions proposed in the present paper is demonstrated for fractional order $\alpha \in [0, 2)$. Furthermore for the first time the stability diagram and phases portrait classification for fractional order planar differential systems in the (τ, Δ) plane are reported in the present paper.

3. Applications to population dynamics

The interactions between population models either prey and predator species or epidemiological models can be predicted by simple mathematical models [24–26]. All population species posses the property of heredity which means the passing on of traits from parents to their offspring, either through asexual reproduction or sexual reproduction, the offspring cells or organisms acquire the genetic information of their parents. Through heredity. Variations between individuals can accumulate and cause species to evolve by natural selection. This property makes fractional differential equations may model more efficiently certain problems than ordinary differential equations. In this work we apply our theoretical results to three population fractional-order models. We consider some classical models existing in the literature, but modeled by a system of fractional differential equations. The first one is the fractional-order Holling-Tanner model [27], the second one is the fractional-order super-predator, predator and prey community model [28] and the last one is a Heroin epidemic model [29].

Example 1: Let consider the fractional order Holling-Tanner model

$$\begin{cases} D^\alpha x = r_1x(1 - \frac{x}{K}) - \frac{qxy}{m+x}, \\ D^\alpha y = r_2y(1 - \frac{y}{\gamma x}). \end{cases} \tag{19}$$

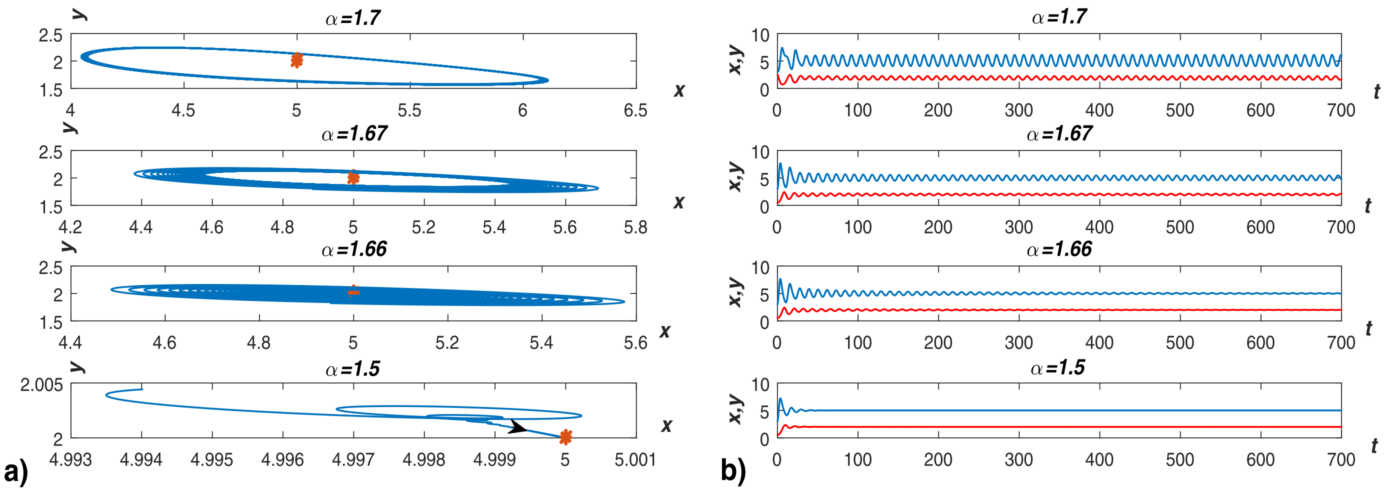

Fig. 2. (**a**) Phase portrait and (**b**) Time evolutions of system (19) for some values of α with the parameter values $r_1 = 1,\ r_2 = 0.2,\ K = 7,\ q = \frac{6}{7},\ m = 1$ and $\gamma = 0.4$.

Where $\alpha \in [0, 2)$, $x(t) \geq 0$ and $y(t) \geq 0$ are the density of prey and predator populations at time t respectively. The parameters r_1 and r_2 are the intrinsic growth rates, K represents the carrying capacity of the prey, q is the maximum number of prey that can be eaten per predator per unit of time, m is a saturation value, it corresponds to the number of prey necessary to achieve one half the maximum rate q, γ is a measure of the quality of the prey as a food for the predator. For example for $r_1 = 1,\ r_2 = 0.2,\ K = 7,\ q = \frac{6}{7},\ m = 1$ and $\gamma = 0.4$, the system (19) has two equilibrium points $E_0 = (7, 0)$ and $E_1 = (5, 2)$.

- The characteristic polynomial of the Jacobian matrix evaluated at E_0 is given by

  $$P(\lambda) = \lambda^2 + 0.8\lambda - 0.2.$$

  So $a_2 = -0.2 < 0$, then according to Proposition 1 E_0 is unstable for all $\alpha \in [0, 2)$.
- The characteristic polynomial of the Jacobian matrix evaluated at E_1 is given by

  $$P(\lambda) = \lambda^2 + \frac{71}{105}\lambda + \frac{16}{105}.$$

  So $a_1 = \frac{71}{105}$ and $a_2 = \frac{16}{105} > 0$, according to Proposition 1 the critical value of α is

  $$\alpha_c = \frac{2}{\pi}cos^{-1}(\frac{-a_1}{2\sqrt{a_2}}) \approx 1.6668.$$

  Then E_1 is locally asymptotically stable for all $\alpha < \alpha_c$, Fig. 2 illustrate these results. We observe that for $\alpha = 1.5$ and for $\alpha = 1.66$ the trajectory initiated near E_1 spiral toward E_1, which is locally asymptotically stable for all fractional order $\alpha < \alpha_c$, but for $\alpha = 1.67$ and $\alpha = 1.7$ the trajectories initiated near E_1 are repulsed by E_1 which is unstable for $\alpha > \alpha_c$. Particularly for α not too far from α_c the trajectories spiral toward an $S-$asymptotically periodic solution of (19) [30,31], giving rise to a periodic behavior of the model.

Example 2: The fractional-order super-predator, predator and prey community model introduced in [28] by

$$\begin{cases} D^\alpha x = x(\rho - \omega y), \\ D^\alpha y = y(-\mu + \beta x - \gamma z), \\ D^\alpha z = z(1 - z) + \delta yz. \end{cases} \tag{20}$$

Where $\alpha \in [0, 2)$, $x \geq 0, y \geq 0$ and $z \geq 0$ are the density of prey, predator and super-predator respectively. All parameters of the model are positive and constant values. The equilibrium point of (20) is $E^* = (x^*, y^*, z^*)$ such that:

$$x^* = \frac{\mu}{\beta} + \frac{\gamma}{\beta}(1 + \frac{\delta\rho}{\omega}),\ y^* = \frac{\rho}{\omega},\ z^* = 1 + \frac{\delta\rho}{\omega}.$$

The characteristic polynomial of the Jacobian matrix of (20) at E^* is

$$P(\lambda) = \lambda^3 + z^*\lambda^2 + (\gamma\delta z^* + \omega\beta x^*)y^*\lambda + \omega\beta x^* y^* z^*.$$

We have

$$\begin{cases} a_1 = z^* > 0, \\ a_2 = (\gamma\delta z^* + \omega\beta x^*)y^* > 0, \\ a_3 = \omega\beta x^* y^* z^* > 0. \end{cases}$$

If $D(P) > 0$, we have $a_1 a_2 > a_3$, by Proposition 2, we have the local asymptotic stability of E^* for all $\alpha \in [0, 2[$. If $D(P) < 0$, then according to the Proposition 2, E^* is locally asymptotically stable for all $\alpha < \frac{2}{3}$ and it is unstable for all $\alpha > \frac{4}{3}$, as shown in Fig. 3, where for $\alpha = 0.66 < \frac{2}{3}$ the trajectory starting near E^* is attracted by it indicating local asymptotic stability, but for $\alpha = 1.34 > \frac{4}{3}$ the trajectory starting near E^* is repulsed by it indicating its instability. Two values of the fractional order α. For $\alpha \in [\frac{2}{3}, \frac{4}{3}]$, we use the Proposition 3, for example for $\omega = 1,\ \beta = 2,\ \mu = 1,\ \gamma = 1,\ \rho = 4$ and $\delta = 3$. We have $D(P) = -15109584 < 0$, $u = 7.2936$ and $v = -7.1142$. The critical value of α is

$$\alpha_c = \frac{2}{\pi}\left|tan^{-1}\left(\frac{-3\sqrt{3}(u - v)}{3(u + v) + 2a_1}\right)\right| \approx 1.2169.$$

Thus the equilibrium point E^* is locally asymptotically stable for all $\alpha < \alpha_c$, as illustrated in Fig. 3, where for $\alpha = 1.21 < \alpha_c$ the trajectory starting in the vicinity of E^* is attracted by it which confirm that E^* is locally asymptotically stable, but for $\alpha = 1.22 > \alpha_c$ the trajectory starting in the vicinity of E^* is repulsed by it indicating its instability.

Example 3: Let consider the following fractional order Heroin epidemic model of four subpopulation [29], with susceptibles x, heroin users not in treatment y, heroin users undergoing treatment z and heroin users who have been successfully treated from heroin use w:

$$\begin{cases} D^\alpha x = \Lambda - \beta yx - \mu x, \\ D^\alpha y = \beta yx + \rho y - (\mu + \delta_1 + \xi)y - \dfrac{\kappa y}{1 + \omega y}, \\ D^\alpha z = \dfrac{\kappa y}{1 + \omega y} - (\rho + \sigma + \delta_2 + \mu)z, \\ D^\alpha w = \sigma z + \xi y - \mu w, \end{cases} \tag{21}$$

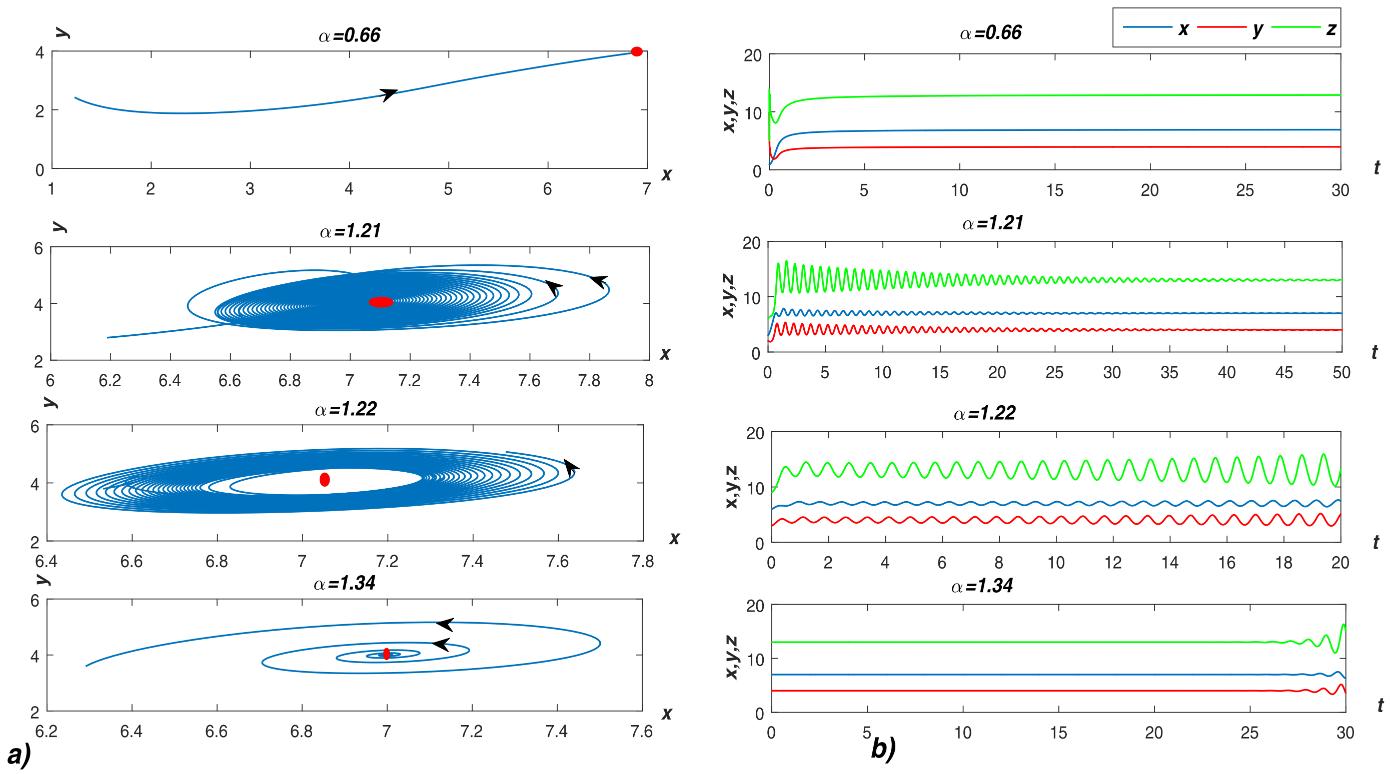

Fig. 3. (**a**) Phase portrait and (**b**) Time evolutions of system (20) for some values of α, with the parameter values $\omega = 1$, $\beta = 2$, $\mu = 1$, $\gamma = 1$, $\rho = 4$ and $\delta = 3$

where $\alpha \in [0, 2)$, and all parameters of the model are positive. The system (21) has two equilibrium points $E = (\frac{\Lambda}{\mu}, 0, 0, 0)$ and $E^* = (x^*, y^*, z^*, w^*)$, such that

$$\begin{cases} x^* = \dfrac{\Lambda}{\beta y^* + \mu}, \\ z^* = \dfrac{\kappa y^*}{Q_0 Q_2}, \\ w^* = \dfrac{\sigma \kappa y^* + \xi Q_0 Q_2 y^*}{\mu Q_0 Q_2}. \end{cases}$$

and y^* is solution of the second order equation

$$Ay^{*2} + By^* + C = 0,$$

where

$$\begin{cases} A = \beta\omega Q_1 Q_2, \\ B = (\beta + \mu\omega) Q_1 Q_2 + \kappa\beta(Q_2 - \rho) - \beta\Lambda, \\ C = [\mu Q_1 Q_2 + \kappa\mu(Q_2 - \rho)](1 - R_0). \end{cases}$$

And

$$\begin{cases} Q_0 = 1 + \omega y^*, \\ Q_1 = \mu + \delta_1 + \xi, \\ Q_2 = \rho + \sigma + \delta_2 + \mu, \\ R_0 = \dfrac{\kappa\beta Q_2}{\mu Q_1 Q_2 + \kappa\mu(Q_2 - \rho)}. \end{cases}$$

The characteristic polynomial of the Jacobian matrix of (21) at E is

$$P_E(\lambda) = (\lambda + \mu)^2 (\lambda^2 + p_1 \lambda + p_2),$$

where

$$\begin{cases} p_1 = Q_1 + Q_2 + \kappa - \frac{\beta\Lambda}{\mu}, \\ p_2 = Q_2 (Q_1 + \kappa - \frac{\beta\Lambda}{\mu}) - \kappa\rho. \end{cases}$$

The characteristic polynomial of the jacobian matrix of (21) at E^* is

$$P_{E^*}(\lambda) = \lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4,$$

where

$$\begin{cases} a_1 = \mu + I_1 + I_2, \\ a_2 = I_1 I_2 + I_3 - I_4 + \mu(I_1 + I_2), \\ a_3 = I_1 I_3 - Q_2 I_4 + \mu(I_1 I_2 + I_3 - I_4), \\ a_4 = \mu(I_1 I_3 - Q_2 I_4), \end{cases}$$

and

$$\begin{cases} I_1 = \beta y^* + \mu, \\ I_2 = Q_1 + Q_2 + \dfrac{\kappa}{Q_0^2} - \beta x^*, \\ I_3 = Q_2\left(Q_1 + \dfrac{\kappa}{Q_0^2} - \beta x^*\right) - \dfrac{\kappa\rho}{Q_0^2}, \\ I_4 = \beta^2 x^* y^*. \end{cases}$$

For example we use the following parameter values $\Lambda = 4.434486182758694$, $\kappa = 0.5$, $\beta = 0.001185$, $\omega = 0.1654$, $\sigma = 20$, $\mu = 0.0099909$, $\rho = 0.001$, $\delta_1 = 0.001$, $\delta_2 = 0.002$, $\xi = 0.014999324798155$. We have: $D(P_E) > 0$, $p_1 > 0$ and $p_2 > 0$, it means that all the roots of $P_E(\lambda) = 0$ are real negative, then E is locally asymptotically stable for all $\alpha \in [0, 2)$. $D(P_{E^*}) < 0$, $a_i > 0$ for all $i = 1, 2, 3, 4$ and $a_2 = \frac{a_1 a_4}{a_3} - \frac{a_3}{a_1}$, then according to Proposition 4 E^* is locally asymptotically stable for all $\alpha \in [0, 1[$ and unstable for all $\alpha \in]1, 2]$, Fig. 4 illustrates these results, where we observe that for $\alpha \in [0, 1[$ all trajectory initiated near E^* converge to it but for $\alpha \in [1, 2]$ all trajectory initiated near E^* are repulsed by it and attracted by E which is locally asymptotically stable for all $\alpha \in [0, 2]$.

Remark 5. Assume that a $3-D$ integer-order system displays a chaotic attractor and suppose that Ω is the set of equilibrium

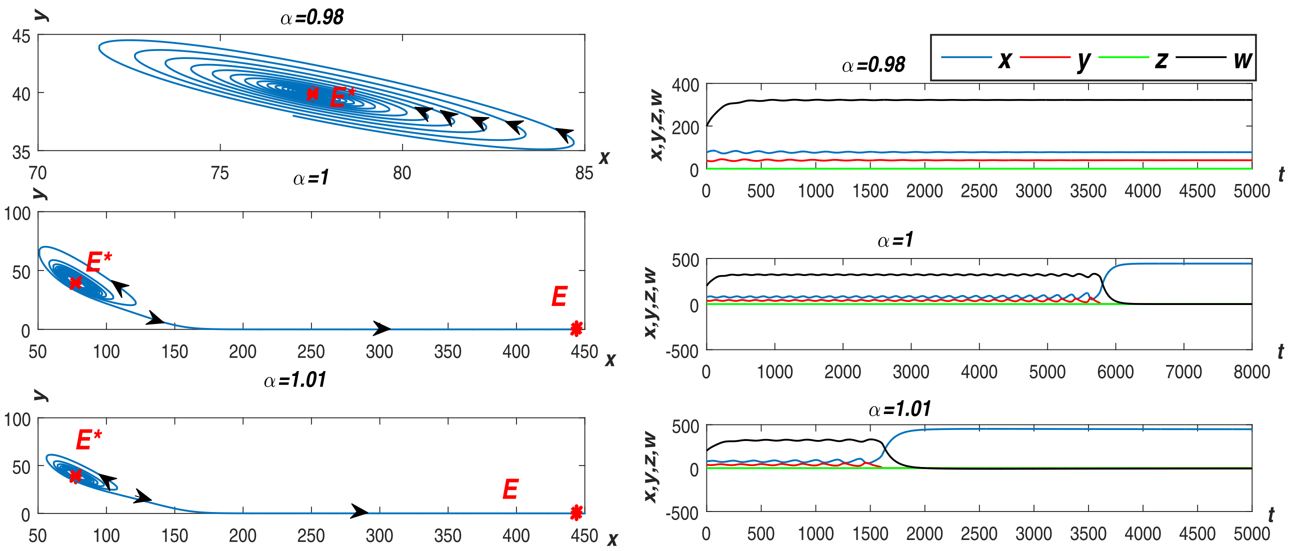

Fig. 4. (**a**) Phase portrait and (**b**) Time evolutions of system (21) for some values of α, with the parameter values $\Lambda = 4.434486182758694$, $\kappa = 0.5$, $\beta = 0.001185$, $\omega = 0.1654$, $\sigma = 20$, $\mu = 0.0099909$, $\rho = 0.001$, $\delta_1 = 0.001$, $\delta_2 = 0.002$, $\xi = 0.014999324798155$.

points surrounded by scrolls. A necessary condition for the corresponding fractional order system to exhibit a chaotic attractor similar to its integer order counterpart is instability of the equilibrium points in Ω. Otherwise, one of these equilibrium points becomes asymptotically stable and attracts the nearby trajectories [32,33]. The proposed stability conditions are a powerful tool for determining regions of possible chaos (instability region) in the parameters space (including fractional order) where chaotic phenomenon can be developed. Different figures of the presented examples show variation of state evolution (from stationary to periodic and divergent) as value of fractional order α changes indicating possibility of chaos. In the forthcoming papers we will investigate possible appearance of chaotic phenomena in such models.

4. Conclusion

In this paper we have derived some new optimal (non-improvable) *Routh-Hurwitz conditions* for fractional type models of orders between 0 and 2., i.e., some necessary and sufficient conditions guaranteeing that all zeros of the corresponding characteristic polynomial are located inside the Matignon stability sector. The effect of parameter α (i.e. the order of model (1)) on the model dynamics has been highlighted. These results can be regarded as a generalization of the classical Routh-Hurwitz stability conditions. As application, the stability properties of some fractional-order mathematical models in population dynamics and epidemiology have been explored. Numerical simulations are provided to exemplify the theoretical findings.

Declaration of Competing Interest

The authors declare that they have no conflict of interest.

CRediT authorship contribution statement

S. Bourafa: Data curation, Writing - original draft, Visualization, Investigation. **M-S. Abdelouahab:** Supervision, Conceptualization, Methodology, Software, Validation, Writing - review & editing. **A. Moussaoui:** Software, Methodology, Validation, Writing - review & editing.

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