Well-defined solutions of a system of difference equations

Article *in* Journal of Applied Mathematics and Computing · January 2017

DOI: 10.1007/s12190-017-1081-8

J. Appl. Math. Comput. (2018) 56:439–458
https://doi.org/10.1007/s12190-017-1081-8

CrossMark

ORIGINAL RESEARCH

Well-defined solutions of a system of difference equations

Nabila Haddad[1] · **Nouressadat Touafek**[2] · **Julius Fergy T. Rabago**[3]

Received: 14 October 2016 / Published online: 11 January 2017

Abstract This note deals with the solution form of the system of difference equations

$$x_{n+1} = \frac{ax_n y_{n-1}}{y_n - \alpha} + \beta, \quad y_{n+1} = \frac{bx_{n-1} y_n}{x_n - \beta} + \alpha, \quad n \in \mathbb{N}_0,$$

where the parameters a, b, α, β and initial values x_{-i}, y_{-i}, $i = 0, 1$, are non-zero real numbers. The special case $a = b$ is treated separately, and the qualitative behavior of its solutions is examined. Also, conditions are determined so that the system admits periodic solutions. Finally, numerical examples are provided to support the theoretical results exhibited in the paper.

Keywords Difference equations · Periodic solutions · System of difference equations

Mathematics Subject Classification 39A10

✉ Nabila Haddad
nabilahaddadt@yahoo.com

Nouressadat Touafek
ntouafek@gmail.com

Julius Fergy T. Rabago
jfrabago@gmail.com

1 LMAM Laboratory, Mohamed Seddik Ben Yahia University, 18000 Jijel, Algeria

2 LMAM Laboratory, Department of Mathematics, Mohamed Seddik Ben Yahia University, 18000 Jijel, Algeria

3 Department of Mathematics and Computer Science, College of Science, University of the Philippines Baguio, 2600 Baguio City, Philippines

1 Introduction

Difference equations have been of great interest to mathematicians and scientists in recent decades. This interest continue to grow as more fascinating results and applications are obtained and discovered. Typically, this line of research is approach in two directions: the first one is the study of the qualitative behavior of solutions and the second one is to find for closed forms of the solutions whenever it is possible (see, e.g., [1–3,5–15,28–31]). In general, solving nonlinear difference equations is a very challenging task. The main reason behind this difficulty is due to the fact that there is no known systematic method to follow in dealing with the solutions of these type of problems. However, in some occasions, the form of solutions of some nonlinear difference equations are derive through reduction to equations with known explicit solutions. Several recent results can be found in the following papers [17–27].

In this work, we shall use appropriate substitutions on variables and reduction to first order linear difference equations to explicitly solve for the well-defined solutions of the following system of difference equations:

$$x_{n+1} = \frac{a x_n y_{n-1}}{y_n - \alpha} + \beta, \qquad y_{n+1} = \frac{b x_{n-1} y_n}{x_n - \beta} + \alpha, \qquad n \in \mathbb{N}_0, \tag{1.1}$$

where the parameters a, b, α, β and initial values $x_{-i}, y_{-i}, i = 0, 1$ are non-zero real numbers.

This study is actually motivated by a recent result of Elabassy et al. found in [4]. In particular, the authors of [4] obtain, among other things, the form of solutions of the difference equation

$$x_{n+1} = \frac{x_n x_{n-1}}{x_n - 1} + 1, \qquad n \in \mathbb{N}_0.$$

Remark 1 By a well-defined solution of system (1.1), we mean a solution such that

$$y_n - \alpha \neq 0, \quad x_n - \beta \neq 0, \qquad n \in \mathbb{N}_0.$$

For example, if we choose the parameters a, b, α, β and initial values positive such that $x_0 > \beta$ and $y_0 > \alpha$, then all solutions of system (1.1) are well-defined.

On the other hand, if at least one of the initial values $x_{-i}, y_{-i}, i = 0, 1$, is zero, then the solutions of system (1.1) are not defined. Assume for example that $x_{-1} = 0$, then we get $y_1 = \alpha$, and so x_2 is not defined. This explains why we have to choose the initial values to be nonzero.

For every well-defined solution of system (1.1) we have

$$x_n \neq 0, \quad y_n \neq 0, \qquad \forall\, n \geq 1.$$

In fact suppose, for example, that there is $n_0 \geq 1$ such that $x_{n_0} = 0$, then from system (1.1), we get

$$x_{n_0+1} = \frac{a y_{n_0-1} x_{n_0}}{y_{n_0} - \alpha} + \beta = \beta$$

and so, it follows that, y_{n_0+2} is not defined.

The following well known lemma shall be central to our investigation.

Lemma 1.1 ([16]). *Let $(a_n)_{n\in\mathbb{N}_0}$ and $(b_n)_{n\in\mathbb{N}_0}$ be two sequences of real numbers and consider the linear difference equation*

$$y_{n+1} = a_n y_n + b_n, \qquad n \in \mathbb{N}_0.$$

Then,

$$y_n = \left(\prod_{i=0}^{n-1} a_i\right) y_0 + \sum_{r=0}^{n-1}\left(\prod_{i=r+1}^{n-1} a_i\right) b_r.$$

Moreover, if $(a_n)_{n\in\mathbb{N}_0}$ and $(b_n)_{n\in\mathbb{N}_0}$ are constant (i.e. $a_n = a$ and $b_n = b$ for some real numbers a and b for all $n \in \mathbb{N}_0$), then

$$y_n = \begin{cases} y_0 + bn, & a = 1, \\ a^n y_0 + \left(\dfrac{a^n - 1}{a - 1}\right) b, & \text{otherwise}, \end{cases} \qquad n \in \mathbb{N}_0.$$

Throughout the rest of the discussion we assume, $\prod_{j=i}^{k} A_j = 1$ and $\sum_{j=i}^{k} A_j = 0$ for all $k < i$.

The rest of the paper is structured as follows: in Sect. 2, we derive the closed-form solution of system (1.1). In Sect. 3, we examine the asymptotic behavior and periodicity of solutions of system (1.1) when $a = b$. We present some numerical examples in Sect. 4, and finally, in Sect. 5 we provide a summary and conclusion of the present study.

2 Form of solutions of system (1.1)

We give in this section the following result describing the form of (well-defined) solutions of system (1.1).

Theorem 2.1 *Let $\{(x_n, y_n)\}_{n\geq -1}$ be a well-defined solution of system (1.1). Then, for $n \in \mathbb{N}_0$,*

$$x_{2n} = \left(\prod_{t=0}^{n-1}\left(\frac{a}{b}\right)^{2t+1} d\right) x_0 + \sum_{r=0}^{n-1}\left(\prod_{t=r+1}^{n-1}\left(\frac{a}{b}\right)^{2t+1} d\right)\left(\left(\frac{a}{b}\right)^{r+1}\frac{x_0 - \beta}{x_{-1}} + 1\right)\beta,$$

$$x_{2n-1} = \left(\prod_{t=0}^{n-1}\left(\frac{a}{b}\right)^{2t} d\right) x_{-1} + \sum_{r=0}^{n-1}\left(\prod_{t=r+1}^{n-1}\left(\frac{a}{b}\right)^{2t} d\right)\left(\left(\frac{a}{b}\right)^{r}\frac{a y_{-1}}{y_0 - \alpha} + 1\right)\beta,$$

$$y_{2n} = \left(\prod_{t=0}^{n-1}\left(\frac{b}{a}\right)^{2t+1}\frac{ab}{d}\right)y_0$$
$$+\sum_{r=0}^{n-1}\left(\prod_{t=r+1}^{n-1}\left(\frac{b}{a}\right)^{2t+1}\frac{ab}{d}\right)\left(\left(\frac{b}{a}\right)^{r+1}\frac{y_0-\alpha}{y_{-1}}+1\right)\beta,$$
$$y_{2n-1} = \left(\prod_{t=0}^{n-1}\left(\frac{b}{a}\right)^{2t}\frac{ab}{d}\right)y_{-1}+\sum_{r=0}^{n-1}\left(\prod_{t=r+1}^{n-1}\left(\frac{b}{a}\right)^{2t}\frac{ab}{d}\right)\left(\left(\frac{b}{a}\right)^{r}\frac{bx_{-1}}{x_0-\beta}+1\right)\beta,$$

where

$$d = \frac{ay_{-1}(x_0-\beta)}{x_{-1}(y_0-\alpha)}.$$

Proof We first rearrange system (1.1) as follows

$$\frac{x_{n+1}-\beta}{x_n} = \frac{a\ y_{n-1}}{y_n-\alpha},\quad \frac{y_{n+1}-\alpha}{y_n} = \frac{b\ x_{n-1}}{x_n-\beta}.$$

Putting

$$v_n := \frac{x_n-\beta}{x_{n-1}},\quad u_n := \frac{y_n-\alpha}{y_{n-1}},\qquad n\in\mathbb{N}_0, \tag{2.1}$$

we get

$$v_{n+1} = \frac{a}{u_n},\quad u_{n+1} = \frac{b}{v_n},\qquad n\in\mathbb{N}_0, \tag{2.2}$$

and so

$$v_{n+2} = \frac{a}{b}v_n,\quad u_{n+2} = \frac{b}{a}u_n,\qquad n\in\mathbb{N}_0.$$

Hence, for $n\in\mathbb{N}_0$, we have

$$v_{2n} = \left(\frac{a}{b}\right)^n v_0,\quad v_{2n+1} = \left(\frac{a}{b}\right)^n v_1,\quad u_{2n} = \left(\frac{b}{a}\right)^n u_0,\quad u_{2n+1} = \left(\frac{b}{a}\right)^n u_1. \tag{2.3}$$

Rearranging Eq. (2.1), we get

$$x_n = v_n x_{n-1}+\beta,\quad y_n = u_n y_{n-1}+\alpha. \tag{2.4}$$

Replacing n by $2n$ and respectively by $2n+1$ in Eq. (2.4), we get (using the relations in Eq. (2.3))

$$
\begin{aligned}
x_{2n} &= v_{2n}x_{2n-1} + \beta = \left(\frac{a}{b}\right)^n v_0 x_{2n-1} + \beta, \quad n \in \mathbb{N}_0,\\
x_{2n+1} &= v_{2n+1}x_{2n} + \beta = \left(\frac{a}{b}\right)^n v_1 x_{2n} + \beta, \quad n \in \mathbb{N}_0,\\
y_{2n} &= u_{2n}y_{2n-1} + \alpha = \left(\frac{b}{a}\right)^n u_0 y_{2n-1} + \alpha, \quad n \in \mathbb{N}_0,\\
y_{2n+1} &= u_{2n+1}y_{2n} + \alpha = \left(\frac{b}{a}\right)^n u_1 y_{2n} + \alpha, \quad n \in \mathbb{N}_0 \qquad .
\end{aligned}
$$

which implies that

$$
x_{2n+1} = \left(\frac{a}{b}\right)^{2n} v_0v_1x_{2n-1} + \left(\frac{a}{b}\right)^n v_1\beta + \beta, \quad n \in \mathbb{N}_0, \tag{2.5}
$$

$$
x_{2n+2} = \left(\frac{a}{b}\right)^{2n+1} v_0v_1x_{2n} + \left(\frac{a}{b}\right)^{n+1} v_0\beta + \beta, \quad n \in \mathbb{N}_0, \tag{2.6}
$$

$$
y_{2n+1} = \left(\frac{b}{a}\right)^{2n} u_0u_1y_{2n-1} + \left(\frac{b}{a}\right)^n u_1\alpha + \alpha, \quad n \in \mathbb{N}_0, \tag{2.7}
$$

$$
y_{2n+2} = \left(\frac{b}{a}\right)^{2n+1} u_0u_1y_{2n} + \left(\frac{b}{a}\right)^{n+1} u_0\alpha + \alpha, \quad n \in \mathbb{N}_0 \qquad . \tag{2.8}
$$

Let,

$$
K_n = x_{2n-1}, \quad L_n = x_{2n}, \quad R_n = y_{2n-1}, \quad S_n = y_{2n}, \qquad n \in \mathbb{N}_0. \tag{2.9}
$$

Then, from (2.5)–(2.8), we get the following non-homogeneous linear first order recursive sequence with variable coefficients

$$
\begin{aligned}
K_{n+1} &= \left(\frac{a}{b}\right)^{2n} v_0v_1\ K_n + \left(\left(\frac{a}{b}\right)^n v_1 + 1\right)\beta, \quad n \in \mathbb{N}_0,\\
L_{n+1} &= \left(\frac{a}{b}\right)^{2n+1} v_0v_1\ L_n + \left(\left(\frac{a}{b}\right)^{n+1} v_0 + 1\right)\beta, \quad n \in \mathbb{N}_0,\\
R_{n+1} &= \left(\frac{b}{a}\right)^{2n} u_0u_1\ R_n + \left(\left(\frac{b}{a}\right)^n u_1 + 1\right)\alpha, \quad n \in \mathbb{N}_0,\\
S_{n+1} &= \left(\frac{b}{a}\right)^{2n+1} u_0u_1\ S_n + \left(\left(\frac{b}{a}\right)^{n+1} u_0 + 1\right)\alpha, \quad n \in \mathbb{N}_0 \qquad .
\end{aligned}
$$

By virtue of Lemma 1.1 and from Eq. (2.9), it follows that for $n \in \mathbb{N}_0$, we have

$$
\begin{aligned}
x_{2n-1} &= \left(\prod_{t=0}^{n-1}\left(\frac{a}{b}\right)^{2t} v_0v_1\right)x_{-1} + \sum_{r=0}^{n-1}\left(\prod_{t=r+1}^{n-1}\left(\frac{a}{b}\right)^{2t} v_0v_1\right)\left(\left(\frac{a}{b}\right)^r v_1 + 1\right)\beta,\\
x_{2n} &= \left(\prod_{t=0}^{n-1}\left(\frac{a}{b}\right)^{2t+1} v_0v_1\right)x_0 + \sum_{r=0}^{n-1}\left(\prod_{t=r+1}^{n-1}\left(\frac{a}{b}\right)^{2t+1} v_0v_1\right)\left(\left(\frac{a}{b}\right)^{r+1} v_0 + 1\right)\beta,
\end{aligned}
$$

$$y_{2n-1} = \left(\prod_{t=0}^{n-1}\left(\frac{b}{a}\right)^{2t} u_0u_1\right) y_{-1} + \sum_{r=0}^{n-1}\left(\prod_{t=r+1}^{n-1}\left(\frac{b}{a}\right)^{2t} u_0u_1\right)\left(\left(\frac{b}{a}\right)^{r} u_1 + 1\right)\beta,$$

$$y_{2n} = \left(\prod_{t=0}^{n-1}\left(\frac{b}{a}\right)^{2t+1} u_0u_1\right) y_0 + \sum_{r=0}^{n-1}\left(\prod_{t=r+1}^{n-1}\left(\frac{b}{a}\right)^{2t+1} u_0u_1\right)\left(\left(\frac{b}{a}\right)^{r+1} u_0 + 1\right)\beta.$$

Now, let $d = v_0v_1$. Then, from (2.1) and (2.2), we have $v_0 = \frac{x_0-\beta}{x_{-1}}$, $u_0 = \frac{y_0-\alpha}{y_{-1}}$, $v_1 = \frac{ay_{-1}}{y_0-\alpha}$, $u_1 = \frac{bx_{-1}}{x_0-\beta}$, $d = \frac{ay_{-1}(x_0-\beta)}{x_{-1}(y_0-\alpha)}$, and $u_0u_1 = \frac{bx_{-1}(y_0-\alpha)}{y_{-1}(x_0-\beta)} = \frac{ab}{d}$. Upon substituting these values to the above formulas, we arrive at the conclusion. This ends the proof. □

3 Asymptotic behavior and periodicity of solutions of system (1.1) when $a = b$

In this section we study the case when $a = b$. Particularly, we examine the asymptotic behavior and periodicity of well-defined solutions of system (1.1). In this regard, the following corollary, which is a direct consequence of Theorem 2.1, is needed.

Corollary 3.1 *Let $\{(x_n, y_n)\}_{n\geq -1}$ be a well-defined solution of system (1.1) with $a = b$. Then, for $n \in \mathbb{N}_0$,*

$$x_{2n-1} = \begin{cases} x_{-1} + h_1\beta n, & d = 1, \\ d^n x_{-1} + \left(\dfrac{d^n-1}{d-1}\right) h_1\beta, & \text{otherwise}, \end{cases} \tag{3.1}$$

$$x_{2n} = \begin{cases} x_0 + h_0\beta n, & d = 1, \\ d^n x_0 + \left(\dfrac{d^n-1}{d-1}\right) h_0\beta, & \text{otherwise}, \end{cases} \tag{3.2}$$

$$y_{2n-1} = \begin{cases} y_{-1} + t_1\alpha n, & d = a^2, \\ \left(\dfrac{a^2}{d}\right)^n y_{-1} + \left[\dfrac{\left(\frac{a^2}{d}\right)^n - 1}{\frac{a^2}{d} - 1}\right] t_1\alpha, & \text{otherwise}, \end{cases} \tag{3.3}$$

$$y_{2n} = \begin{cases} y_0 + t_0\alpha n, & d = a^2, \\ \left(\dfrac{a^2}{d}\right)^n y_0 + \left[\dfrac{\left(\frac{a^2}{d}\right)^n - 1}{\frac{a^2}{d} - 1}\right] t_0\alpha, & \text{otherwise}, \end{cases} \tag{3.4}$$

where

$$h_0 = \frac{x_0-\beta}{x_{-1}} + 1,\ h_1 = \frac{ay_{-1}}{y_0-\alpha} + 1,\ t_0 = \frac{y_0-\alpha}{y_{-1}} + 1,\ t_1 = \frac{ax_{-1}}{x_0-\beta} + 1.$$

In the following theorem, we study the limiting properties of solutions of system (1.1).

Theorem 3.2 *Let $\{(x_n, y_n)\}_{n\geq -1}$ be a well-defined solution of system (1.1) with $a = b$. Then, the following statements are true.*

(a) If $(d-1)x_0 + h_0\beta \neq 0$, then we have

$$\lim_{n\to\infty} |x_{2n}| = \begin{cases} \left|\dfrac{h_0\beta}{d-1}\right|, & |d| < 1, \\ \infty, & |d| > 1. \end{cases}$$

Otherwise, if $(d-1)x_0 + h_0\beta = 0$ and $d \neq 1$, then $x_{2n} = x_0$ for all $n \in \mathbb{N}_0$.

(b) Suppose $d = 1$. If $x_0 + x_{-1} \neq \beta$ (i.e. $h_0 \neq 0$), then $|x_{2n}| \to \infty$, as $n \to \infty$. Otherwise, if $x_0 + x_{-1} = \beta$ (i.e. $h_0 = 0$), then $x_{2n} = x_0$ for all $n \in \mathbb{N}_0$.

(c) If $(d-1)x_{-1} + h_1\beta \neq 0$, then we have

$$\lim_{n\to\infty} |x_{2n-1}| = \begin{cases} \left|\dfrac{h_1\beta}{d-1}\right|, & |d| < 1, \\ \infty, & |d| > 1. \end{cases}$$

Otherwise, if $(d-1)x_1 + h_1\beta = 0$ and $d \neq 1$, then $x_{2n-1} = x_{-1}$ for all $n \in \mathbb{N}_0$.

(d) Suppose $d = 1$. If $ay_{-1} + y_0 \neq \alpha$ (i.e. $h_1 \neq 0$), then $|x_{2n-1}| \to \infty$, as $n \to \infty$. Otherwise, if $ay_{-1} + y_0 = \alpha$ (i.e. $h_1 = 0$), then $x_{2n-1} = x_{-1}$ for all $n \in \mathbb{N}_0$.

(e) If $(\frac{a^2}{d} - 1)y_0 + t_0\alpha \neq 0$, then we have

$$\lim_{n\to\infty} |y_{2n}| = \begin{cases} \infty, & |d| < a^2, \\ \left|\dfrac{t_0\alpha d}{d-a^2}\right|, & |d| > a^2. \end{cases}$$

Otherwise, if $(\frac{a^2}{d} - 1)y_0 + t_0\alpha = 0$ and $d \neq a^2$, then $y_{2n} = y_0$ for all $n \in \mathbb{N}_0$.

(f) Suppose $d = a^2$. If $y_0 + y_{-1} \neq \alpha$, then $|y_{2n}| \to \infty$, as $n \to \infty$. Otherwise, if $y_0 + y_{-1} = \alpha$, then $y_{2n} = y_0$ for all $n \in \mathbb{N}_0$.

(g) If $(\frac{a^2}{d} - 1)y_{-1} + t_1\alpha \neq 0$, then we have

$$\lim_{n\to\infty} |y_{2n-1}| = \begin{cases} \infty, & |d| < a^2, \\ \left|\dfrac{t_1\alpha d}{d-a^2}\right|, & |d| > a^2. \end{cases}$$

Otherwise, if $(\frac{a^2}{d} - 1)y_{-1} + t_1\alpha = 0$ and $d \neq a^2$, then $y_{2n-1} = y_{-1}$ for all $n \in \mathbb{N}_0$.

(h) Suppose $d = a^2$. If $ax_{-1} + x_0 \neq \beta$, then $|y_{2n-1}| \to \infty$, as $n \to \infty$. Otherwise, if $ax_{-1} + x_0 = \beta$, then $y_{2n-1} = y_{-1}$ for all $n \in \mathbb{N}_0$.

Proof We'll only prove properties (a) and (b). The rest follows the same inductive lines. First, suppose that $(d-1)x_0 + h_0\beta \neq 0$. Then, it follows that, $x_{2n} \neq 0$.

Evidently, if $|d| < 1$, then $|d|^n \to 0$ as $n \to \infty$. On the other hand, if $|d| > 1$, then $|d|^n \to \infty$ as $n \to \infty$. So, from (3.2), we have

$$\begin{aligned}\lim_{n\to\infty} |x_{2n}| &= \lim_{n\to\infty} \left| \frac{(d-1)x_0 + h_0\beta}{d-1} d^n + \frac{h_0\beta}{1-d} \right| \\ &= \left| \frac{(d-1)x_0 + h_0\beta}{d-1} \lim_{n\to\infty} d^n + \frac{h_0\beta}{1-d} \right| \\ &= \begin{cases} \left| \dfrac{h_0\beta}{d-1} \right|, & |d| < 1, \\ \infty, & |d| > 1. \end{cases}\end{aligned}$$

Now, on the other hand, if $(d-1)x_0 + h_0\beta = 0$ and $d \neq 1$. Then, we get

$$\begin{aligned}x_{2n} &= d^n x_0 + \left(\frac{d^n - 1}{d-1}\right) h_0\beta = d^n x_0 + \left(\frac{d^n - 1}{d-1}\right)(-(d-1)x_0) \\ &= d^n x_0 - (d^n - 1)x_0 = x_0, \qquad \forall n \in \mathbb{N}_0.\end{aligned}$$

This proves property (a). Now we proceed on proving (b). So we suppose that $d = 1$. If $x_0 + x_{-1} \neq \beta$ $(i.e.\, h_0 \neq 0)$, then from (3.2) we have

$$x_{2n} = x_0 + \left(\frac{x_0 + x_{-1} - \beta}{x_{-1}}\right) \beta n \neq 0.$$

Letting $n \to \infty$ in above equation implies that $|x_{2n}| \to \infty$. On the other hand, if $x_0 + x_{-1} = \beta$ $(i.e.\, h_0 = 0)$, then obviously,

$$x_{2n} = x_0 + \left(\frac{x_0 + x_{-1} - \beta}{x_{-1}}\right) \beta n = x_0 + 0 \cdot \beta n = x_0, \qquad \forall\, n \in \mathbb{N}_0.$$

This proves property (b). □

The following result is devoted to the periodicity of the solutions.

Corollary 3.3 *Let* $\{(x_n, y_n)\}_{n \geq -1}$ *be a well-defined solution of system* (1.1) *with* $a = b$. *Then the following statements are true.*

(a) If $d = -1$, *then for all* $n \in \mathbb{N}_0$, *we have*

$$\begin{cases} x_{4n-1} = x_{-1}, \\ x_{4n} \quad\;\; = x_0, \\ x_{4n+1} = -x_{-1} + h_1\beta, \\ x_{4n+2} = -x_0 + h_0\beta. \end{cases}$$

(b) If $d=-a^2$*, then for all* $n\in\mathbb{N}_0$*, we have*

$$\begin{cases} y_{4n-1} = y_{-1}, \\ y_{4n} \quad\; = y_0, \\ y_{4n+1} = -y_{-1} + t_1\alpha, \\ y_{4n+2} = -y_0 + t_0\alpha. \end{cases}$$

(c) If $a = 1, \alpha x_0 + \beta y_0 = \alpha x_{-1} + \beta y_{-1} = \alpha\beta$ *and* $x_{-1} + x_0 \neq \beta$*, then for all* $n\in\mathbb{N}_0$*, we have*

$$\begin{cases} x_{2n-1} = x_{-1}, \\ x_{2n} \quad\; = x_0, \\ y_{2n-1} = y_{-1}, \\ y_{2n} \quad\; = y_0. \end{cases}$$

Proof (a) When $d=-1$, then from (3.1) and (3.2), we have for $n\in\mathbb{N}_0$

$$x_{2n-1} = (-1)^n\, x_{-1} + \left(\frac{1-(-1)^n}{2}\right) h_1\beta,$$
$$x_{2n} = (-1)^n\, x_0 + \left(\frac{1-(-1)^n}{2}\right) h_0\beta.$$

Depending on the parity of n, we get for $n\in\mathbb{N}_0$

$$\begin{cases} x_{4n-1} = (-1)^{2n}\, x_{-1} + \left(\frac{1-(-1)^{2n}}{2}\right) h_1\beta = x_{-1}, \\ x_{4n+1} = (-1)^{2n+1}\, x_{-1} + \left(\frac{1-(-1)^{2n+1}}{2}\right) h_1\beta = -x_{-1} + h_1\beta, \\ x_{4n} \quad\; = (-1)^{2n}\, x_0 + \left(\frac{1-(-1)^{2n}}{2}\right) h_0\beta = x_0, \\ x_{4n+2} = (-1)^{2n+1}\, x_0 + \left(\frac{1-(-1)^{2n+1}}{2}\right) h_0\beta = -x_0 + h_0\beta. \end{cases}$$

(b) When $d=-a^2$, from (3.3) and (3.4), we get for $n\in\mathbb{N}_0$

$$y_{2n-1} = (-1)^n y_{-1} + \frac{(-1)^n - 1}{-2} t_1\alpha,$$
$$y_{2n} = (-1)^n y_0 + \frac{(-1)^n - 1}{-2} t_0\alpha.$$

Depending on the parity of n, we get for $n \in \mathbb{N}_0$

$$\begin{cases} y_{4n-1} = (-1)^{2n}\, y_{-1} + \left(\dfrac{1-(-1)^{2n}}{2}\right) t_1\alpha \qquad = y_{-1}, \\ y_{4n+1} = (-1)^{2n+1}\, y_{-1} + \left(\dfrac{1-(-1)^{2n+1}}{2}\right) t_1\alpha = -y_{-1} + t_1\alpha, \\ y_{4n} \quad = (-1)^{2n}\, y_0 + \left(\dfrac{1-(-1)^{2n}}{2}\right) t_0\alpha \qquad = y_0, \\ y_{4n+2} = (-1)^{2n+1}\, y_0 + \left(\dfrac{1-(-1)^{2n+1}}{2}\right) t_0\alpha \quad = -y_0 + t_0\alpha. \end{cases}$$

(c) When $a = 1$ and $\alpha x_0 + \beta y_0 = \alpha x_{-1} + \beta y_{-1} = \alpha\beta$, we get

$$d - 1 = \frac{\beta - x_{-1} - x_0}{x_{-1}x_0}\beta, \quad h_1 = \frac{x_{-1} - \beta}{x_0} + 1, \quad \text{and} \quad t_0 = \frac{x_0}{x_{-1} - \beta} + 1$$

from which it follows that $(d-1)x_0 + h_0\beta$, $(d-1)x_{-1} + h_1\beta$, $(1-d)y_0 + t_0\alpha d$ and $(1-d)y_{-1} + t_1\alpha d$ are zero. As $x_{-1} + x_0 \neq \beta$ $(i.e. d \neq 1)$, then from the results of Theorem 3.2, we obtain

$$x_{2n-1} = x_{-1}, \quad x_{2n} = x_0, \quad y_{2n-1} = y_{-1}, \quad y_{2n} = y_0, \qquad \forall n \in \mathbb{N}_0.$$

□

The following result follows from Theorem 3.2 and Corollary 3.3.

Corollary 3.4 *Let $\{(x_n, y_n)\}_{n\geq -1}$ be a well-defined solution of system (1.1) with $a = b$. Then, the following statements are true.*

(a) If $|a| = 1$ and $ay_{-1}(x_0 - \beta) = x_{-1}(\alpha - y_0)$ (i.e., $d = -1$), then the solution is periodic of period 4.
(b) If $a = 1$, $x_{-1} + x_0 = \beta$ and $y_{-1} + y_0 = \alpha$ (i.e., $d = 1$), then the solution is periodic of period 2.
(c) If $a = 1$, $x_{-1} + x_0 \neq \beta$, and $\alpha x_0 + \beta y_0 = \alpha x_{-1} + \beta y_{-1} = \alpha\beta$, then the solution is periodic of period 2.

The following remark provides an observation regarding the one-dimensional case of system (1.1).

Remark 2 Let $a = b$ and $\alpha = \beta$. If we choose initial conditions which satisfy the relation $x_{-i} = y_{-i}$, $i = 0, 1$, then system (1.1) will reduced to a one-dimensional case. Particularly, we shall obtain the nonlinear difference equation

$$x_{n+1} = \frac{a x_n x_{n-1}}{x_n - \alpha} + \alpha, \qquad n \in \mathbb{N}_0. \tag{3.5}$$

By Corollary 3.1, we get for $n \in \mathbb{N}_0$, the following form of solutions of Eq. (3.5)

$$x_{2n} = \begin{cases} x_0 + h_0\alpha n, & a = 1, \\ a^n x_0 + \left(\dfrac{a^n - 1}{a - 1}\right) h_0\alpha, & \text{otherwise.} \end{cases} \tag{3.6}$$

$$x_{2n-1} = \begin{cases} x_{-1} + h_1\alpha n, & a = 1, \\ a^n x_{-1} + \left(\dfrac{a^n - 1}{a - 1}\right) h_1\alpha, & \text{otherwise.} \end{cases} \tag{3.7}$$

In [4], the authors gave the form of the solutions of the equation

$$x_{n+1} = \frac{x_n x_{n-1}}{x_n - 1} + 1, \quad n \in \mathbb{N}_0,$$

which is a special case of Eq. (3.5) with $a = \alpha = 1$. Clearly the formulas of the solutions given in [4] follows from (3.6) and (3.7).

4 Numerical examples

In this section we provide some numerical examples which represent different types of the asymptotic behavior and periodicity of well-defined solutions of system (1.1) with $a = b$.

Example 4.1 (a) Consider the parameters

a	b	α	β	x_{-1}	x_0	y_{-1}	y_0
1	1	1	1/2	−5/12	1/4	5/4	1/4

(4.1)

We have $d = -1 = -a^2$, then in view of cases (a) and (b) of Corollary 3.3, the solution is periodic of period 4 and takes the form

$$\left\{\left(\frac{-5}{12}, \frac{5}{4}\right), \left(\frac{1}{4}, \frac{1}{4}\right), \left(\frac{1}{12}, \frac{17}{12}\right), \left(\frac{11}{20}, \frac{3}{20}\right), \left(\frac{-5}{12}, \frac{5}{4}\right), \left(\frac{1}{4}, \frac{1}{4}\right), \left(\frac{1}{12}, \frac{17}{12}\right), \left(\frac{11}{20}, \frac{3}{20}\right), \ldots\right\}.$$

See, Figs. 1 and 2.

(b) Consider the parameters

a	b	α	β	x_{-1}	x_0	y_{-1}	y_0
1	1	1	1/2	$-\frac{61}{150}$	1/4	5/4	1/4

(4.2)

We have choose the same parameters as in case (a) except for x_{-1} we have take the value $-\frac{61}{150} = \frac{-5}{12} + \frac{1}{100}$. From Figs. 3, 4 and 5, we can see that

$$\lim_{n\to\infty} |x_{2n}| = \lim_{n\to\infty} |x_{2n+1}| = \infty$$

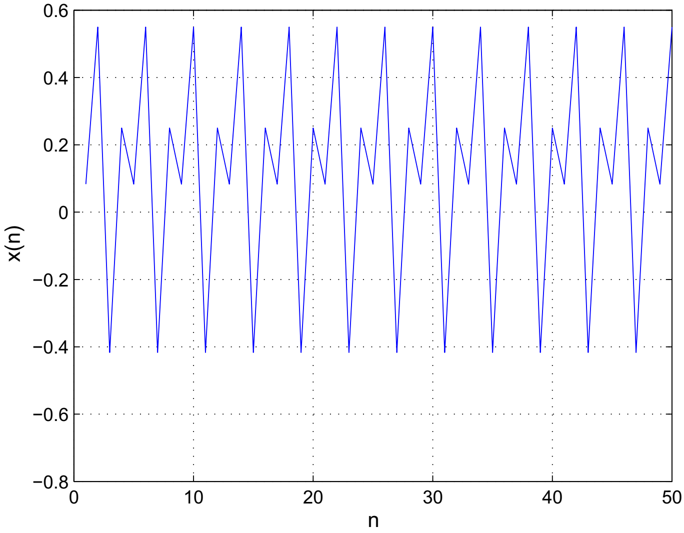

Fig. 1 Plots of x_n of Example 4.1.(a)

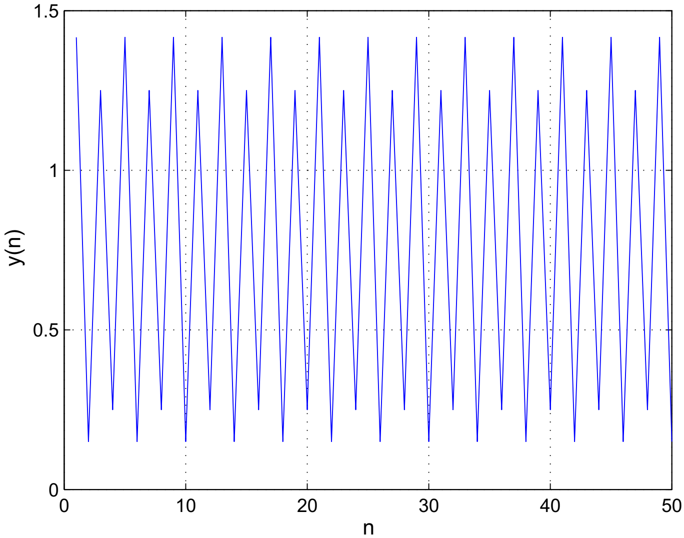

Fig. 2 Plots of y_n of Example 4.1.(a)

and

$$\lim_{n\to\infty}|y_{2n}| \simeq 0.2 \simeq \left|\frac{t_0\alpha d}{d-a^2}\right| = \frac{50}{247}, \lim_{n\to\infty}|y_{2n+1}| \simeq 1.33 \simeq \left|\frac{t_1\alpha d}{d-a^2}\right| = \frac{985}{741}.$$

This is true because $|d| = \frac{125}{122} > 1 = a^2$, see Theorem 3.2.

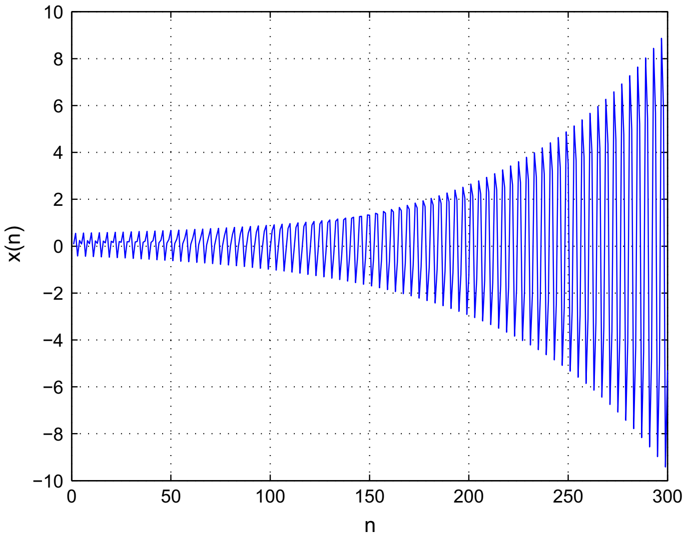

Fig. 3 Plots of x_n of Example 4.1.(b)

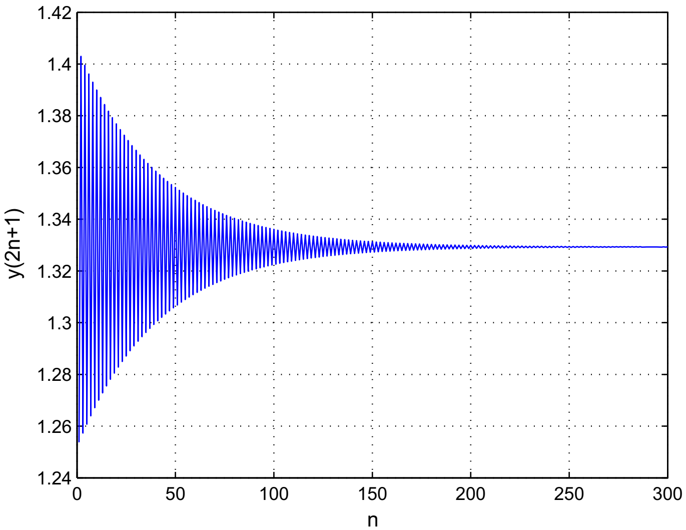

Fig. 4 Plots of y_{2n+1} of Example 4.1.(b)

(c) Consider the parameters

a	b	α	β	x_{-1}	x_0	y_{-1}	y_0
1	1	1	$1/2$	$-\frac{32}{75}$	$1/4$	$5/4$	$1/4$

(4.3)

We have choose the same parameters as in case (a) except for x_{-1} we have take the value $-\frac{32}{75} = \frac{-5}{12} - \frac{1}{100}$. From Figs. 6, 7 and 8, we can see that

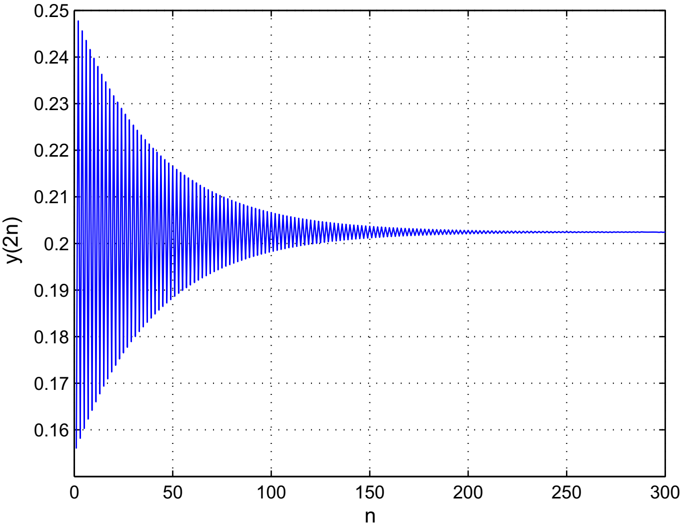

Fig. 5 Plots of y_{2n} of Example 4.1.(b)

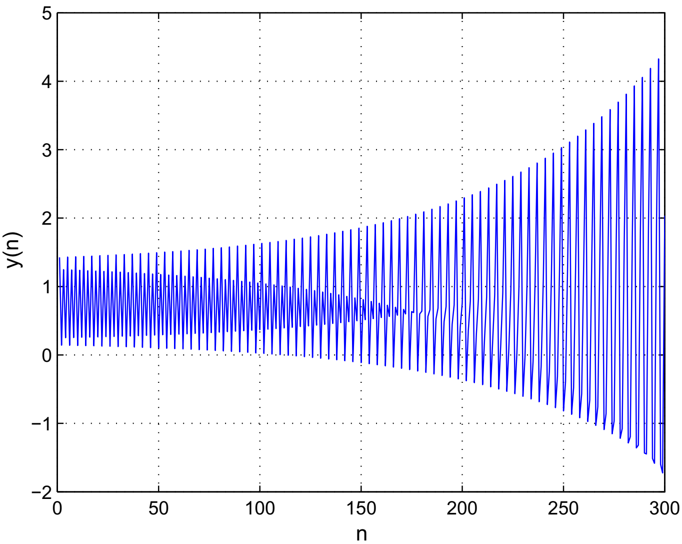

Fig. 6 Plots of y_n of Example 4.1.(c)

$$\lim_{n\to\infty} |y_{2n}| = \lim_{n\to\infty} |y_{2n+1}| = \infty$$

and

$$\lim_{n\to\infty} |x_{2n}| \simeq 0.4 \simeq \left|\frac{h_0\beta}{d-1}\right| = \frac{203}{506}, \ \lim_{n\to\infty} |x_{2n+1}| \simeq 0.17 \simeq \left|\frac{h_1\beta}{d-1}\right| = \frac{128}{759}.$$

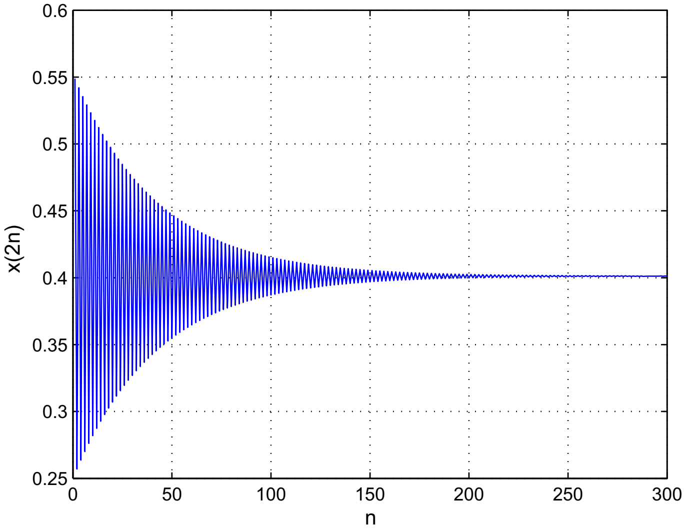

Fig. 7 Plots of x_{2n} of Example 4.1.(c)

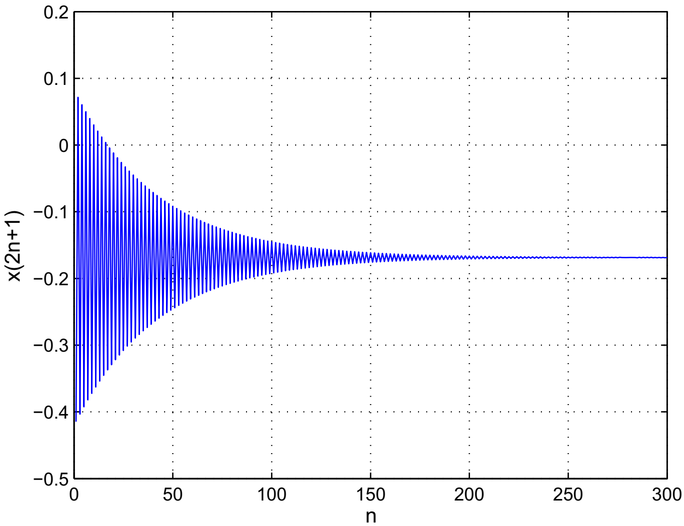

Fig. 8 Plots of x_{2n+1} of Example 4.1.(c)

This is true because $|d| = \frac{125}{128} < 1 = a^2$, see Theorem 3.2.

Example 4.2 Consider the parameters

a	b	α	β	x_{-1}	x_0	y_{-1}	y_0
1	1	3	5	−10	10	9	−3

(4.4)

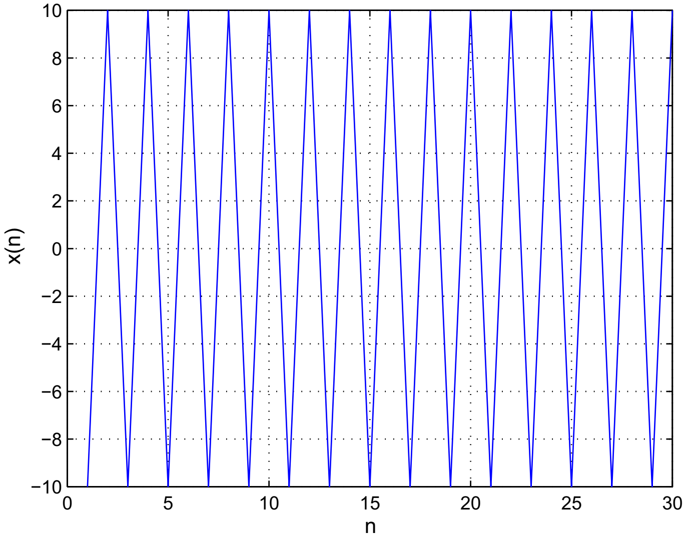

Fig. 9 Plots of x_n of Example 4.2

We have

$$x_{-1} + x_0 \neq \beta, \ \alpha x_0 + \beta y_0 = \alpha x_{-1} + \beta y_{-1} = \alpha\beta.$$

So, in view of case (c) of Corollary 3.4, the solution is periodic of period 2 and takes the form

$$\{(-10, 9), (10, -3), (-10, 9), (10, -3), \ldots\}.$$

See, Figs. 9 and 10.

Example 4.3 Consider the parameters

a	b	α	β	x_{-1}	x_0	y_{-1}	y_0
1/2	1/2	2	1	1	2	1	3

. (4.5)

We have $\frac{1}{4} = a^2 < |d| = \frac{1}{2} < 1$ and we can see from Figs. 11, 12, 13 and 14 that the sub-sequences (x_{2n}), (x_{2n+1}), (y_{2n}) and (y_{2n+1}) are convergent. This, agree with our result stated in Theorem 3.2, that is

$$\lim_{n\to\infty} x_{2n} = \left|\frac{h_0\beta}{d-1}\right| = 4, \ \lim_{n\to\infty} x_{2n+1} = \left|\frac{h_1\beta}{d-1}\right| = 3,$$
$$\lim_{n\to\infty} y_{2n} = \left|\frac{t_0\alpha d}{d-a^2}\right| = 8, \ \lim_{n\to\infty} y_{2n+1} = \left|\frac{t_1\alpha d}{d-a^2}\right| = 6.$$

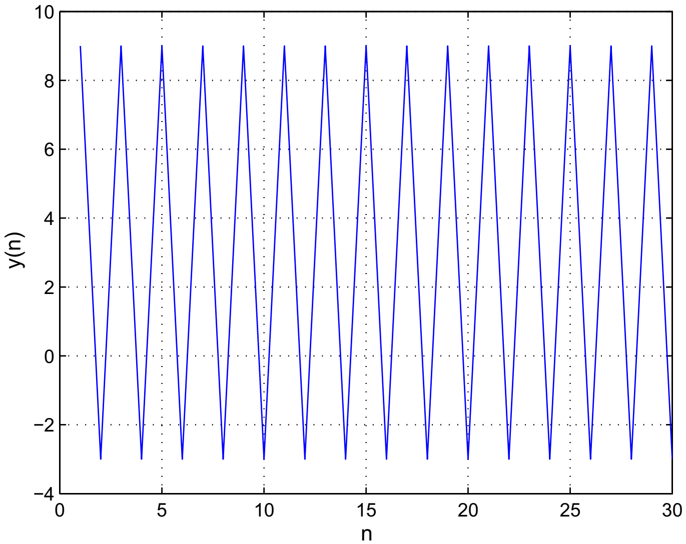

Fig. 10 Plots of y_n of Example 4.2

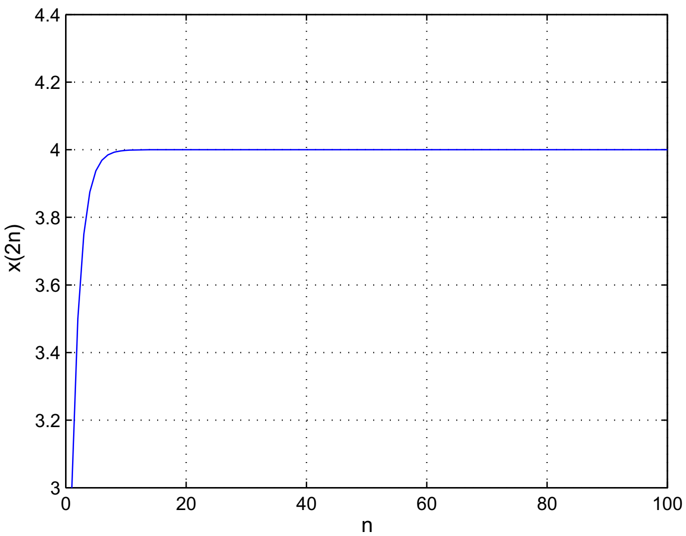

Fig. 11 Plots of x_{2n} of Example 4.3

5 Summary and conclusion

In this paper, we are able to derive analytically the form of well-defined solutions of system (1.1). Also, we have obtained conditions on when the system (with $a = b$) admits periodic solutions. In fact, we have shown that, under appropriate conditions imposed on the parameters a, α and β, and the initial values x_{-1}, x_0, y_{-1} and y_0,

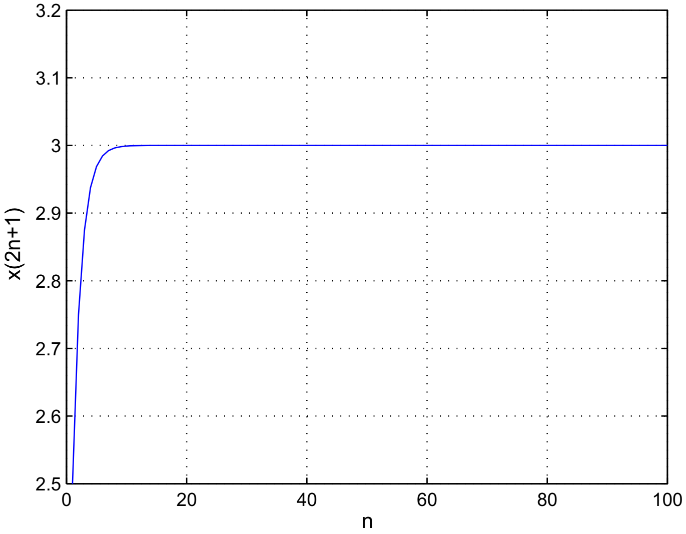

Fig. 12 Plots of x_{2n+1} of Example 4.3

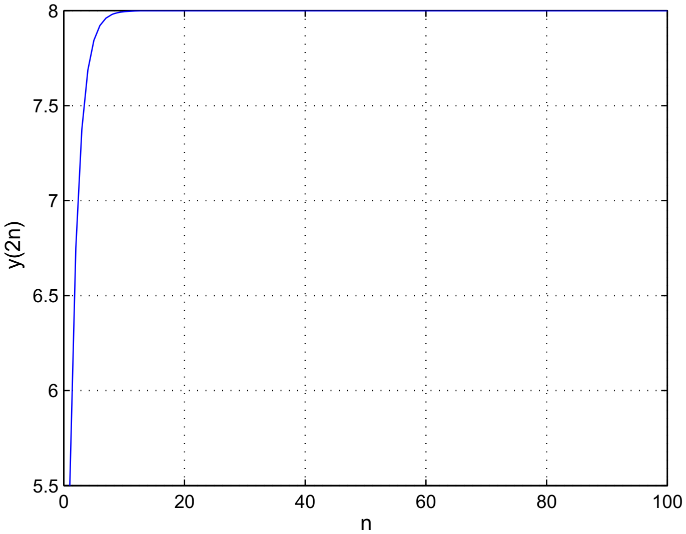

Fig. 13 Plots of y_{2n} of Example 4.3

system (1.1) may admits periodic solutions with periodicity two or four. Moreover, we have illustrate (through numerical examples) the asymptotic behavior and periodicity character of the solutions. Consequently, the results presented here were analytically justified, and verified through numerical examples. In addition, the results delivered here contributed to the understanding of the complex behavior of solutions of the class of nonlinear system of difference equations considered in this paper. We expect that more interesting results will be obtained when $a \neq b$.

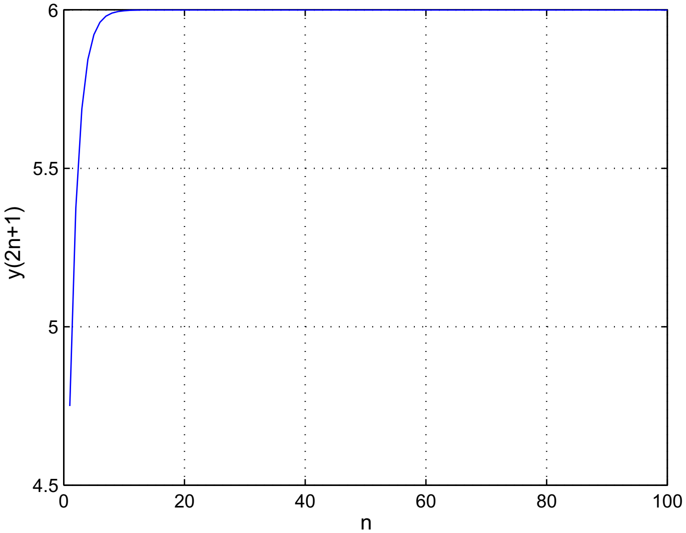

Fig. 14 Plots of y_{2n+1} of Example 4.3

Acknowledgements The authors are grateful to the four reviewers for their constructive comments and valuable suggestions.

References

1. Belhannache, F., Touafek, N., Abo-Zeid, R.: Dynamics of a third-order rational difference equation. Bull. Math. Soc. Sci. Math. Roum. Nouv. Sr., **59(107)(1)**, 13–22 (2016)
2. Din, Q.: On a system of fourth-order rational difference equations. Acta Univ. Apulensis, Math. Inform. **39**, 137–150 (2014)
3. Din, Q.: Asymptotic behavior of an anti-competitive system of second-order difference equations. J. Egypt. Math. Soc. **24**(1), 37–43 (2016)
4. Elabbasy, E.M., El-Metwally, H., Elsayed, E.M.: Qualitative behavior of higher order difference equation. Soochow J. Math. **33**, 861–873 (2007)
5. Elaydi, S.: An Introduction to Difference Equations. Springer, New York (1999)
6. El-Dessoky, M.M.: On the solutions and periodicity of some nonlinear systems of difference equations. J. Nonlinear Sci. Appl. **9**, 2190–2207 (2016)
7. Elsayed, E.M.: Qualitative behavior of difference equation of order two. Math. Comput. Model. **50**, 1130–1141 (2009)
8. Fotiades, N., Papaschinopoulos, G.: On a system of difference equations with maximum. Appl. Math. Comput. **221**, 684–690 (2013)
9. Grove, E.A., Ladas, G.: Periodicities in Nonlinear Difference Equations, Advances in Discrete Mathematics and Applications, vol. 4. Chapman & Hall/CRC, London (2005)
10. Halim, Y., Touafek, N., Yazlik, Y.: Dynamic behavior of a second-order nonlinear rational difference equation. Turk. J. Math. **39**, 1004–1018 (2015)
11. Ibrahim, T.F.: Solving a class of three-order max-type difference equations. Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal. **21**, 333–342 (2014)
12. Ibrahim, T.F., Touafek, N.: Max-type system of difference equations with positive two-periodic sequences. Math. Methods Appl. Sci. **37**, 2562–2569 (2014)
13. Kocic, V.L., Ladas, G.: Global Behavior of Nonlinear Difference Equations of Higher Order with Applications. Kluwer Academic Publishers, Dordrecht (1993)

14. Kulenovic, M.R.S., Ladas, G.: Dynamics of Second Order Rational Difference Equations with Open Problems and Conjectures. Chapman & Hall, CRC Press, London (2001)
15. Papaschinopoulos, G., Fotiades, N., Schinas, C.J.: On a system of difference equations including negative exponential terms. J. Differ. Equ. Appl. **20**, 717–732 (2014)
16. Elaydi, S.: An Introduction to Difference Equations. Springer, New York (1996)
17. Rabago, J.F.T.: Effective methods on determining the periodicity and form of solutions of some systems of nonlinear difference equations. Int. J. Dyn. Syst. Differ. Equ. (2016) (in press)
18. Stević, S.: More on a rational recurrence relation. Appl. Math. E-Notes **4**, 80–85 (2004)
19. Stević, S.: On a solvable rational system of difference equations. Appl. Math. Comput. **219**, 2896–2908 (2012)
20. Stević, S.: On a solvable system of difference equations of kth order. Appl. Math. Comput. **219**, 7765–7771 (2013)
21. Stević, S.: On a system of difference equations. Appl. Math. Comput. **218**, 3372–3378 (2011)
22. Stević, S.: On a system of difference equations with period two coefficients. Appl. Math. Comput. **218**, 4317–4324 (2011)
23. Stević, S.: On a third-order system of difference equations. Appl. Math. Comput. **218**, 7649–7654 (2012)
24. Stević, S.: On some solvable systems of difference equations. Appl. Math. Comput. **218**, 5010–5018 (2012)
25. Stević, S.: On the difference equation $x_n = \frac{x_{n-2}}{b_n + c_n x_n x_{n-2}}$. Appl. Math. Comput. **218**, 4507–4513 (2011)
26. Stević, S.: On the difference equation $x_n = \frac{x_{n-k}}{b + c x_{n-1} \cdots x_{n-k}}$. Appl. Math. Comput. **218**, 6291–6296 (2012)
27. Stević, S.: On the system of difference equations $x_n = \frac{c_n y_{n-3}}{a_n + b_n y_{n-1} x_{n-2} y_{n-3}}$, $y_n = \frac{\gamma_n x_{n-3}}{\alpha_n + \beta_n x_{n-1} y_{n-2} x_{n-3}}$. Appl. Math. Comput. **219**, 4755–4764 (2013)
28. Touafek, N.: On a second order rational difference equation. Hacet. J. Math. Stat. **41**, 867–874 (2012)
29. Yazlik, Y.: On the solutions and behavior of rational difference equations. J. Comput. Anal. Appl. **17**(3), 584–594 (2014)
30. Yazlik, Y., Elsayed, E.M., Taskara, N.: On the behaviour of the solutions the solutions of difference equation system. J. Comput. Anal. Appl. **16**(5), 932–941 (2014)
31. Yazlik, Y., Tollu, D.T., Taskara, N.: On the behaviour of solutions for some systems of difference equations. J. Comput. Anal. Appl. **18**(1), 166–178 (2015)