Noname manuscript No.
(will be inserted by the editor)

Taylor collocation method for a system of nonlinear Volterra delay integro-differential equations with application to COVID-19 epidemic

Hafida Laib · Azzeddine Bellour · Aissa Boulmerka

Received: date / Accepted: date

Abstract The present paper deals with the numerical solution for a general form of a system of nonlinear Volterra delay integro-differential equations (VDIDEs). The main purpose of this work is to provide a current numerical method based on the use of continuous collocation Taylor polynomials for the numerical solution of nonlinear VDIDEs systems. It is shown that this method is convergent. Numerical results will be presented to prove the validity and effectiveness of this convergent algorithm. We apply two models to the COVID-19 epidemic in China and one for the Predator-Prey model in mathematical ecology.

Keywords Nonlinear Volterra delay integro-differential equations · Collocation method · Taylor polynomials · Epidemic mathematical model · Corona virus

Mathematics Subject Classification (2010) 45G15 · 34K28 · 45D05 · 45L05 · 65L60

H. Laib
Laboratory of Mathematics and their interactions, University Center Abdelhafid Boussouf, Mila, Algeria.
E-mail: hafida.laib@gmail.com

A. Bellour
Laboratory of Applied Mathematics and Didactics Ecole Normale Supérieure de Constantine, Constantine, Algeria.
E-mail: bellourazze123@yahoo.com

A. Boulmerka
Laboratory of Mathematics and their interactions, University Center Abdelhafid Boussouf, Mila, Algeria.
E-mail: aissaboulmerka@gmail.com

1 Introduction

In this paper, a numerical method is presented to obtain an approximate solution for the following system of nonlinear delay integro-differential equations

$$y'(t) = f(t, y(t), y(t-\tau)) + \int_{t-\tau}^{t} k(t, s, y(t), y(s))ds, \tag{1}$$

for $t \in [0, T]$ and $y(t) = \Phi(t)$ for $t \in [-\tau, 0]$ with $\Phi : \mathbb{R} \to \mathbb{R}^d$, where the functions $f : \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ and $k : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ are sufficiently smooth.
The existence and the uniqueness of the solution of (1) can be found, for example, in [10,11]. The delay integro-differential equations and their systems have become important in the mathematical modeling of many fields of sciences and engineering (see, e.g., [9,12,18,21,25,29,30]). As a particular case in traditional population biology, The system of 'predator–prey' dynamics, which was first modeled by Volterra [27]. Moreover, Liu et al. [22] presents a COVID-19 epidemic model, which can be described by a particular form of the system of nonlinear delay integro-differential (1). We will present some applications of this system in Section 4.
Recently, many numerical methods have been proposed to approximate the solution of system of nonlinear integro-differential equations. For example, operational Tau method [1], linear barycentric rational method [2], Adomian decomposition method [3], differential transform method [5], He's homotopy perturbation method [8], variational iteration method ([20,24,25]), collocation method [13–15,17,26].
The Taylor polynomial method for approximating the solution of integral equations and integro-differential equations has been proposed. Bellour and Bousselsal [6,7] used Taylor collocation method for solving delay integral equations and integro-differential equations, Taylor collocation method for the Volterra Fredholm integral equations is used in [28], Gülsu and Sezer [16] applied a Taylor collocation method for the solution of systems of high-order Fredholm Volterra integro-differential equations.
The aim of this paper is to generalize the Taylor collocation method in [6] and [7] to construct an approximate solution for a general form of the system of nonlinear Volterra delay integro-differential equations (1). In our method, the approximate solution is explicit, direct, high order of convergence and obtained by using simple iterative formulas.
The paper is organized as follows: In section 2, we divide the interval $[0, T]$ into subintervals, and we approximate the solution of (1) in each interval by a Taylor polynomial. The convergence analysis is established in section 3, and the numerical illustrations are provided in section 4.

2 Description of the Method

We suppose that $T = r\tau$, where $r \in \{1, 2, 3, ...\}$. Let Π_N be a uniform partition of the interval $I = [0, T]$ defined by $t_n^i = i\tau + nh,\ n = 0, 1, .., N,\ i = 0, 1, .., r-1,$

where the stepsize is given by $h = \frac{\tau}{N}$. Define the subintervals $\sigma_n^i = [t_n^i; t_{n+1}^i), n = 0, 1, .., N-1,\ i = 0, 1, .., r-1$ and $\sigma_{N-1}^{r-1} = [t_{N-1}^{r-1}, t_N^{r-1}]$. Moreover, denote by π_m the set of all real polynomials of degree not exceeding m. We define the real polynomial spline space of degree m as follows:

$$S_m^{(0)}(\Pi_N) = \{u \in C(I, \mathbb{R}^d) : u_n^i = u|_{\sigma_n^i} \in \pi_m, n = 0, ..., N-1,\ i = 0, 1, .., r-1\}.$$

This is the space of piecewise polynomials of degree (at most) m. Its dimension is $rNm+1$, i.e., the same as the total number of the coefficients of the polynomials $u_n^i, n = 0, ..., N-1, i = 0, 1, .., r-1$. To find these coefficients, we use Taylor polynomial on each subinterval.

First, we approximate the exact solution y in the interval σ_0^0 by the polynomial

$$u_0^0(t) = \sum_{j=0}^{m} \frac{y^{(j)}(0)}{j!} t^j \ ; \quad t \in \sigma_0^0. \tag{2}$$

To find $y^{(j)}(0)$, we differentiate equation (1) j-times, we obtain

$$y^{(j+1)}(0) = f^{(j)}(0, \Phi(0), \Phi(-\tau)) + \left(\int_{t-\tau}^{0} k(t, s, \Phi(t), \Phi(s)) ds \right)^{(j)} (0) + \sum_{i=0}^{j-1} \left[\partial_t^{(i)} k(t, t, y(t), y(t)) \right]^{(j-1-i)} (0),$$

for $j = 0, 1, ..., m$, where $y(0) = \Phi(0)$.

Second, we approximate y by the polynomial u_n^0 $(n \in \{1, 2, ..., N-1\})$ on the interval σ_n^0 such that

$$u_n^0(t) = \sum_{j=0}^{m} \frac{\hat{u}_{n,0}^{(j)}(t_n^0)}{j!} (t - t_n^0)^j; \quad t \in \sigma_n^0, \tag{3}$$

where $\hat{u}_{n,0}$ is the exact solution of the integro-differential equation

$$\begin{aligned} \hat{u}'_{n,0}(t) =& f(t, u_{n-1}^0(t), \Phi(t-\tau)) + \int_{t-\tau}^{0} k(t, s, \Phi(t), \Phi(s)) ds \\ &+ \sum_{i=0}^{n-1} \int_{t_i^0}^{t_{i+1}^0} k(t, s, u_i^0(t), u_i^0(s)) ds + \int_{t_n^0}^{t} k(t, s, \hat{u}_{n,0}(t), \hat{u}_{n,0}(s)) ds, \end{aligned} \tag{4}$$

for $t \in \sigma_n^0$ such that $\hat{u}_{n,0}(t_n^0) = u_{n-1}^0(t_n^0)$.
The coefficients $\hat{u}_{n,0}^{(j)}(t_n^0)$ are given by the following formula

$$\begin{aligned}\hat{u}_{n,0}^{(j+1)}(t_n^0) =& f^{(j)}(t_n^0, u_{n-1}^0(t_n^0), \Phi((t_n^0-\tau)) + \left(\int_{t-\tau}^{0} k(t,s,\Phi(t),\Phi(s))ds\right)^{(j)}(t_n^0)\\ &+ \sum_{i=0}^{j-1}\left[\partial_t^{(i)}k(t,t,\hat{u}_{n,0}(t),\hat{u}_{n,0}(t))\right]^{(j-1-i)}(t_n^0)\\ &+ \sum_{i=0}^{n-1}\int_{t_i^0}^{t_{i+1}^0}\partial_t^{(j)}k(t_n^0,s,u_i^0(t_n^0),u_i^0(s))ds,\end{aligned} \tag{5}$$

for $j \in \{0,1,...,m\}$ and $\hat{u}_{n,0}(t_n^0) = u_{n-1}^0(t_n^0)$.
Third, for y to be approximated by u_0^p ($p \in \{1,2,...,r-1\}$) on the interval σ_0^p, y is to be approximated by u_k^j ($0 \le k \le N-1$ and $0 \le j < p$) on each interval σ_k^j such that

$$u_0^p(t) = \sum_{j=0}^{m}\frac{\hat{u}_{0,p}^{(j)}(t_0^p)}{j!}(t-t_0^p)^j; \quad t \in \sigma_0^p, \tag{6}$$

where $\hat{u}_{0,p}$ is the exact solution of the integro-differential equation

$$\begin{aligned}\hat{u}_{0,p}'(t) = f(t,u_{N-1}^{p-1}(t),u_0^{p-1}(t-\tau)) + \int_{t-\tau}^{t_0^p}k(t,s,u^{p-1}(t),u^{p-1}(s))ds\\ + \int_{t_0^p}^{t}k(t,s,\hat{u}_{0,p}(t),\hat{u}_{0,p}(s))ds,\end{aligned} \tag{7}$$

for $t \in \sigma_0^p$ such that $\hat{u}_{0,p}(t_0^p) = u_{N-1}^{p-1}(t_0^p)$, where $u^{p-1} = u$ on the interval $\sigma^{p-1} = [t_0^{p-1}, t_0^p]$ for $p \in \{0,...,r\}$.
The coefficients $\hat{u}_{0,p}^{(j)}(t_0^p)$ are given by the following formula

$$\begin{aligned}\hat{u}_{0,p}^{(j+1)}(t_0^p) =& f^{(j)}(t_0^p, u_{N-1}^{p-1}(t_0^p), u_0^{p-1}(t_0^p-\tau))\\ &- \sum_{i=0}^{j-1}\left[\partial_t^{(i)}k(t,t-\tau,u_0^{p-1}(t),u_0^{p-1}(t-\tau))\right]^{(j-1-i)}(t_0^p)\\ &+ \sum_{i=0}^{N-1}\int_{t_i^{p-1}}^{t_{i+1}^{p-1}}\partial_t^{(j)}k(t_0^p,s,u_i^{p-1}(t_0^p),u_i^{p-1}(s))ds\\ &+ \sum_{i=0}^{j-1}\left[\partial_t^{(i)}k(t,t,\hat{u}_{0,p}(t),\hat{u}_{0,p}(t))\right]^{(j-1-i)}(t_0^p),\end{aligned} \tag{8}$$

for $j \in \{0,1,...,m\}$ and $\hat{u}_{0,p}(t_0^p) = u_{N-1}^{p-1}(t_0^p)$.
Finally, on the interval σ_n^p ($n \in \{1,2,...,N-1\}$), the polynomial u_n^p is defined by the following formula

$$u_n^p(t) = \sum_{j=0}^{m}\frac{\hat{u}_{n,p}^{(j)}(t_n^p)}{j!}(t-t_n^p)^j; \quad t \in \sigma_n^p, \tag{9}$$

where $\hat{u}_{n,p}$ is the exact solution of the integro-differential equation

$$\begin{aligned}\hat{u}'_{n,p}(t) =& f(t, u^p_{n-1}(t), u^{p-1}_n(t-\tau)) + \int_{t-\tau}^{t_0^p} k(t,s,u^{p-1}(t),u^{p-1}(s))ds \\ &+ \sum_{i=0}^{n-1} \int_{t_i^p}^{t_{i+1}^p} k(t,s,u_i^p(t),u_i^p(s))ds + \int_{t_n^p}^{t} k(t,s,\hat{u}_{n,p}(t),\hat{u}_{n,p}(s))ds,\end{aligned} \tag{10}$$

for $t \in \sigma_n^p$ such that $\hat{u}_{n,p}(t_n^p) = u^p_{n-1}(t_n^p)$.
The coefficients $\hat{u}^{(j)}_{n,p}(t_n^p)$ are given by the following formula

$$\begin{aligned}\hat{u}^{(j+1)}_{n,p}(t_n^p) =& f^{(j)}(t_n^p, u^p_{n-1}(t_n^p), u^{p-1}_n(t_n^p-\tau)) \\ &- \sum_{i=0}^{j-1} \left[\partial_t^{(i)} k(t,t-\tau,u_n^{p-1}(t),u_n^{p-1}(t-\tau))\right]^{(j-1-i)} (t_n^p) \\ &+ \sum_{i=0}^{N-1} \int_{t_i^{p-1}}^{t_{i+1}^{p-1}} \partial_t^{(j)} k(t_n^p,s,u_i^{p-1}(t_n^p),u_i^{p-1}(s))ds \\ &+ \sum_{i=0}^{j-1} \left[\partial_t^{(i)} k(t,t,\hat{u}_{n,p}(t),\hat{u}_{n,p}(t))\right]^{(j-1-i)} (t_n^p) \\ &+ \sum_{i=0}^{n-1} \int_{t_i^p}^{t_{i+1}^p} \partial_t^{(j)} k(t_n^p,s,u_i^p(t_n^p),u_i^p(s))ds,\end{aligned} \tag{11}$$

for $j \in \{0,1,...,m\}$ and $\hat{u}_{n,p}(t_n^p) = u^p_{n-1}(t_n^p)$.

3 Analysis of Convergence

For ease of exposition, we will consider a feasible linear form of (1), namely

$$y'(t) = g(t) + a(t)y(t) + b(t)y(t-\tau) + \int_{t-\tau}^{t} k(t,s)y(s)ds, \tag{12}$$

More precisely, the equations (4), (5), (7), (8), (10) and (11) may be written in the following linear forms, respectively,

$$\begin{aligned}\hat{u}'_{n,0}(t) = g(t) + a(t)\hat{u}_{n,0}(t) + b(t)\Phi(t-\tau) + \int_{t-\tau}^{0} k(t,s)\Phi(s)ds \\ + \sum_{i=0}^{n-1} \int_{t_i^0}^{t_{i+1}^0} k(t,s)u_i^0(s)ds + \int_{t_n^0}^{t} k(t,s)\hat{u}_{n,0}(s)ds,\end{aligned} \tag{13}$$

$$
\begin{aligned}
\hat{u}_{n,0}^{(j+1)}(t_n^0) &= f^{(j)}(t_n^0) + \sum_{l=0}^{j} \binom{j}{l} \left(a^{(j-l)}(t_n^0)\hat{u}_{n,0}^{(l)}(t_n^0) + b^{(j-l)}(t_n^0)\Phi^{(l)}(t_n^0-\tau) \right) \\
&\quad + \left(\int_{t-\tau}^{0} k(t,s)\Phi(s)ds \right)^{(j)} (t_n^0) + \sum_{i=0}^{j-1} [\partial_t^{(i)} k(t,t)\hat{u}_{n,0}(t)]^{(j-1-i)}(t_n^0) \\
&\quad + \sum_{i=0}^{n-1} \int_{t_i^0}^{t_{i+1}^0} \partial_t^{(j)} k(t_n^0,s) u_i^0(s)ds \\
&= f^{(j)}(t_n^0) + \sum_{l=0}^{j} \binom{j}{l} \left(a^{(j-l)}(t_n^0)\hat{u}_{n,0}^{(l)}(t_n^0) + b^{(j-l)}(t_n^0)\Phi^{(l)}(t_n^0-\tau) \right) \\
&\quad + \left(\int_{t-\tau}^{0} k(t,s)\Phi(s)ds \right)^{(j)} (t_n^0) \\
&\quad + \sum_{i=0}^{j-1} \sum_{l=0}^{j-1-i} \binom{j-1-i}{l} [\partial_t^{(i)} k(t,t)]^{(j-1-i-l)}(t_n^0)\hat{u}_{n,0}^{(l)}(t_n^0) \\
&\quad + \sum_{i=0}^{n-1} \sum_{l=0}^{m} \frac{\hat{u}_{i,0}^{(l)}(t_i^0)}{l!} \int_{t_i^0}^{t_{i+1}^0} \partial_t^{(j)} k(t_n^0,s)(s-t_i^0)^l ds,
\end{aligned}
\tag{14}
$$

$$
\begin{aligned}
\hat{u}_{0,p}'(t) &= g(t) + a(t)\hat{u}_{0,p}(t) + b(t)\hat{u}_{0,p-1}(t-\tau) + \int_{t-\tau}^{t_0^p} k(t,s)u^{p-1}(s)ds \\
&\quad + \int_{t_0^p}^{t} k(t,s)\hat{u}_{0,p}(s)ds,
\end{aligned}
\tag{15}
$$

$$
\begin{aligned}
\hat{u}_{0,p}^{(j+1)}(t_0^p) &= f^{(j)}(t_0^p) + \sum_{l=0}^{j} \binom{j}{l} \left(a^{(j-l)}(t_0^p)\hat{u}_{0,p}^{(l)}(t_0^p) + b^{(j-l)}(t_0^p))\hat{u}_{0,p-1}^{(l)}(t_0^{p-1}) \right) \\
&\quad - \sum_{i=0}^{j-1} \sum_{l=0}^{j-1-i} \binom{j-1-i}{l} [\partial_t^{(i)} k(t,t-\tau)]^{(j-1-i-l)}(t_0^p)\hat{u}_{0,p-1}^{(l)}(t_0^{p-1}) \\
&\quad + \sum_{i=0}^{N-1} \sum_{l=0}^{m} \frac{\hat{u}_{i,p-1}^{(l)}(t_i^{p-1})}{l!} \int_{t_i^{p-1}}^{t_{i+1}^{p-1}} \partial_t^{(j)} k(t_0^p,s)(s-t_i^{p-1})^l ds \\
&\quad + \sum_{i=0}^{j-1} \sum_{l=0}^{j-1-i} \binom{j-1-i}{l} [\partial_t^{(i)} k(t,t)]^{(j-1-i-l)}(t_0^p)\hat{u}_{0,p}^{(l)}(t_0^p),
\end{aligned}
\tag{16}
$$

$$
\begin{aligned}
\hat{u}_{n,p}'(t) &= g(t) + a(t)\hat{u}_{n,p}(t) + b(t)\hat{u}_{n,p-1}(t-\tau) + \int_{t-\tau}^{t_0^p} k(t,s)u^{p-1}(s)ds \\
&\quad + \sum_{i=0}^{n-1} \int_{t_i^p}^{t_{i+1}^p} k(t,s)u_i^p(s)ds + \int_{t_n^p}^{t} k(t,s)\hat{u}_{n,p}(s)ds,
\end{aligned}
\tag{17}
$$

and

$$\begin{aligned}\hat{u}_{n,p}^{(j+1)}(t_n^p) = f^{(j)}(t_n^p) &+ \sum_{l=0}^{j}\binom{j}{l}\left(a^{(j-l)}(t_n^p)\hat{u}_{n,p}^{(l)}(t_n^p) + b^{(j-l)}(t_n^p))\hat{u}_{n,p-1}^{(l)}(t_n^{p-1})\right)\\
&- \sum_{i=0}^{j-1}\sum_{l=0}^{j-1-i}\binom{j-1-i}{l}[\partial_t^{(i)}k(t,t-\tau)]^{(j-1-i-l)}(t_n^p)\hat{u}_{n,p-1}^{(l)}(t_n^{p-1})\\
&+ \sum_{i=n}^{N-1}\sum_{l=0}^{m}\frac{\hat{u}_{i,p-1}^{(l)}(t_i^{p-1})}{l!}\int_{t_i^{p-1}}^{t_{i+1}^{p-1}}\partial_t^{(j)}k(t_n^p,s)(s-t_i^{p-1})^l ds\\
&+ \sum_{i=0}^{j-1}\sum_{l=0}^{j-1-i}\binom{j-1-i}{l}[\partial_t^{(i)}k(t,t)]^{(j-1-i-l)}(t_n^p)\hat{u}_{n,p}^{(l)}(t_n^p)\\
&+ \sum_{i=0}^{n-1}\sum_{l=0}^{m}\frac{\hat{u}_{i,p}^{(l)}(t_i^p)}{l!}\int_{t_i^p}^{t_{i+1}^p}\partial_t^{(j)}k(t_n^p,s)(s-t_i^p)^l ds,\end{aligned} \tag{18}$$

The following three lemmas will be used in this section.

Lemma 1 (Discrete Gronwall-type inequality [10]) Let $\{k_j\}_{j=0}^n$ be a given non-negative sequence and the sequence $\{\varepsilon_n\}$ satisfies $\varepsilon_0 \le p_0$ and

$$\varepsilon_n \le p_0 + \sum_{i=0}^{n-1} k_i\varepsilon_i, \quad n \ge 1,$$

with $p_0 \ge 0$.Then ε_n can be bounded by

$$\varepsilon_n \le p_0 \exp\left(\sum_{j=0}^{n-1} k_j\right), \quad n \ge 1.$$

Lemma 2 (Discrete Gronwall-type inequality [4]) If $\{f_n\}_{n\ge0}$, $\{g_n\}_{n\ge0}$ and $\{\varepsilon_n\}_{n\ge0}$ are nonnegative sequences and

$$\varepsilon_n \le f_n + \sum_{i=0}^{n-1} g_i\varepsilon_i, \quad n \ge 0,$$

Then,

$$\varepsilon_n \le f_n + \sum_{i=0}^{n-1} f_i g_i \exp\left(\sum_{k=0}^{n-1} g_k\right), \quad n \ge 0.$$

Lemma 3 [19] Assume that the sequence $\{\varepsilon_n\}_{n\ge0}$ of nonnegative numbers satisfies

$$\varepsilon_n \le A\varepsilon_{n-1} + B\sum_{i=0}^{n-1}\varepsilon_i + K, \quad n \ge 1,$$

where A, B and K are nonnegative constants, then

$$\varepsilon_n \le \frac{\varepsilon_0}{R_2 - R_1}[(R_2-1)R_2^n + (1-R_1)R_1^n] + \frac{K}{R_2 - R_1}[R_2^n - R_1^n],$$

where

$$R_1 = \left(1+A+B-\sqrt{(1-A)^2+B^2+2AB+2B}\right)/2,$$
$$R_2 = \left(1+A+B+\sqrt{(1-A)^2+B^2+2AB+2B}\right)/2,$$

therefore, $0 \leq R_1 \leq 1 \leq R_2$.

The next lemma will be crucial for establishing the convergence of the approximate solution.

In the following, for a given fonction $\psi \in C(I,\mathbb{R}^d)$, we define the norm $\|\psi\|$ by

$$\|\psi\| = \{\max|\psi_i(t)|, \quad t \in I, \quad i = 1,...,d\}$$

Lemma 4 Let g,k,a,b and Φ be m times continuously differentiable on their respective domains. Then, there exists a positive number $\alpha(m)$ such that for all $n = 0,1,...,N-1$, $p = 0,1,...,r-1$ and $j = 0,1,...,m+1$, we have,

$$\|\hat{u}_{n,p}^{(j)}\| \leq \alpha(m)$$

provided that h is sufficiently small, where $\hat{u}_{0,0}(t) = y(t)$ for $t \in \sigma_0^0$.

Proof. The proof is split into two steps.
Claim 1. There exists a positive constant $\alpha_1(m)$ such that $\|\hat{u}_{n,0}^{(j)}\| \leq \alpha_1(m)$.
Let $a_n^j = \|\hat{u}_{n,0}^{(j)}\|$, we have for all $j = 0,1,...,m+1$,

$$a_0^j \leq \max\{\|y^{(j)}\|, j = 0,1,...,m+1\} = \alpha_1^1(m). \tag{19}$$

On the other hand, for $n \geq 1$, by differentiating equation (13) j-times, we obtain, for all $j = 0,1,...,m$,

$$a_n^{j+1} \leq c_1 + A\sum_{k=0}^{j} a_n^k + (mb_1^1 + b_1^2)\sum_{k=0}^{j-1} a_n^k + hd_1 \sum_{i=0}^{n-1}\sum_{k=0}^{m} a_i^k,$$

where

$$A = \max\{\binom{j}{l}\left\|a^{(j-l)}\right\|, l = 0,...,j; \qquad j = 0,...,m\},$$
$$c_1 = \max\{\left\|g^{(j)} + (\int_{t-\tau}^{0} k(t,s)\Phi(s)ds)^{(j)} + \Big(b(t)\Phi(t-\tau)\Big)^{(j)}\right\|, \; j = 0,...,m\},$$
$$b_1^1 = \max\{\binom{j-1-i}{l}\|[\partial_t^{(i)}k(t,t)]^{(j-1-i-l)}\|; \; j = 1,...,m;$$
$$i = 0,...,j-1; \; l = 0,...,j-1-i\},$$
$$b_1^2 = \max\{\|\int_0^t \partial_t^{(j)} k(t,s)ds\|, j = 0,...,m\},$$

and

$$d_1 = \max\{\frac{1}{l!}\;\|\partial_t^{(j)}k(t,s)(s-v)^l\|; \quad j = 0,...,m; \; l = 0,...,m\},$$

which implies that, for all $j \geq 1$,

$$\begin{aligned} a_n^j &\leq c_1 + A\sum_{k=0}^{j-1} a_n^k + (mb_1 + b_1^2)\sum_{k=0}^{j-2} a_n^k + hd_1 \sum_{i=0}^{n-1}\sum_{k=0}^{m} a_i^k \\ &\leq c_1 + \underbrace{(A + mb_1^1 + b_1^2)}_{b_1}\sum_{k=0}^{j-1} a_n^k + hd_1 \sum_{i=0}^{n-1}\sum_{k=0}^{m} a_i^k \\ &\leq c_1 + b_1 a_n^0 + b_1 \sum_{k=1}^{j-1} a_n^k + hd_1 \sum_{i=0}^{n-1}\sum_{k=0}^{m} a_i^k. \end{aligned} \tag{20}$$

Now, for each fixed $n \geq 1$, we consider the sequence $y_j = a_n^j$ for $j = 1, 2, ..., m+1$. Then by (20), the sequence (y_j) satisfies for all $j = 1, 2, ..., m+1$

$$y_j \leq c_1 + b_1 a_n^0 + b_1 \sum_{k=1}^{j-1} y_k + hd_1 \sum_{i=0}^{n-1}\sum_{k=0}^{m} a_i^k,$$

hence, by Lemma 1, for all $j = 1, 2, ..., m+1$

$$\begin{aligned} y_j &\leq \underbrace{c_1 \exp(b_1 m)}_{c_2} + \underbrace{b_1 \exp(b_1 m)}_{b_2} a_n^0 + h\underbrace{d_1 \exp(b_1 m)}_{d_2} \sum_{i=0}^{n-1}\sum_{k=0}^{m} a_i^k \\ &\leq c_2 + b_2 a_n^0 + hd_2 \sum_{i=0}^{n-1}\sum_{k=0}^{m+1} a_i^k. \end{aligned} \tag{21}$$

Consider the sequence $z_n = \sum_{j=1}^{m+1} y_j = \sum_{j=1}^{m+1} a_n^j$ for $n \geq 0$.
Then by (21), the sequence (z_n) satisfies

$$\begin{aligned} z_n &\leq \underbrace{(m+1)c_2}_{c_3^1} + \underbrace{(m+1)b_2}_{b_3} a_n^0 + h\underbrace{(m+1)d_2}_{d_3} \sum_{i=0}^{n-1}\sum_{k=0}^{m+1} a_i^k \\ &\leq c_3^1 + b_3 a_n^0 + hd_3 \sum_{i=0}^{n-1} a_i^0 + hd_3 \sum_{i=0}^{n-1}\sum_{k=1}^{m+1} a_i^k \\ &\leq c_3^1 + b_3 a_n^0 + hd_3 \sum_{i=0}^{n} a_i^0 + hd_3 \sum_{i=0}^{n-1} z_i. \end{aligned} \tag{22}$$

Moreover, from (19), we obtain,

$$z_0 \leq (m+2)\alpha_1^1(m) = c_3^2. \tag{23}$$

Let $c_3 = \max(c_3^1, c_3^2)$, then from (22) and (23), we get for all $n \geq 0$

$$z_n \leq c_3 + b_3 a_n^0 + hd_3 \sum_{i=0}^{n} a_i^0 + hd_3 \sum_{i=0}^{n-1} z_i.$$

Then, by Lemma 2, we obtain

$$\begin{aligned} z_n &\le c_3 + b_3 a_n^0 + hd_3 \sum_{i=0}^{n} a_i^0 + \sum_{j=0}^{n-1} hd_3 (c_3 + b_3 a_j^0 + hd_3 \sum_{i=0}^{j} a_i^0) \exp(\tau d_3) \\ &\le \underbrace{c_3(1+\tau d_3 \exp(\tau d_3))}_{c_4} + (b_3 + hd_3) a_n^0 \\ &+ h \underbrace{(d_3 + d_3 b_3 \exp(\tau d_3) + \tau d_3^2 \exp(\tau d_3))}_{d_4} \sum_{i=0}^{n-1} a_i^0 \\ &\le c_4 + \underbrace{(b_3 + \tau d_3)}_{b_4} a_n^0 + hd_4 \sum_{i=0}^{n-1} a_i^0. \end{aligned} \tag{24}$$

On the other hand, by integrating (13) from t_n^0 to $t \in \sigma_n^0$, we get,

$$\begin{aligned} a_n^0 &\le |u_{n-1}^0(t_n^0)| + hc_1 + hAa_n^0 + hb_1^3 a_n^0 + h^2 d_1 \sum_{i=0}^{n-1} \sum_{k=0}^{m} a_i^k \\ &\le |u_{n-1}^0(t_n^0)| + hc_1 + hb_1 a_n^0 + h^2 d_1 \sum_{i=0}^{n-1} a_i^0 + h^2 d_1 \sum_{i=0}^{n-1} z_i. \end{aligned}$$

Moreover, from (3), we obtain,

$$\begin{aligned} |u_{n-1}^0(t_n^0)| &\le a_{n-1}^0 + h \sum_{j=1}^{m} a_{n-1}^j \\ &\le a_{n-1}^0 + hz_{n-1}, \end{aligned}$$

hence,

$$a_n^0 \le a_{n-1}^0 + hz_{n-1} + hc_1 + hb_1 a_n^0 + h^2 d_1 \sum_{i=0}^{n-1} a_i^0 + h^2 d_1 \sum_{i=0}^{n-1} z_i,$$

using (24), we deduce that

$$\begin{aligned} a_n^0 &\le a_{n-1}^0 + h(c_4 + b_4 a_{n-1}^0 + hd_4 \sum_{i=0}^{n-1} a_i^0) + hc_1 + hb_1 a_n^0 \\ &+ h^2 d_1 \sum_{i=0}^{n-1} a_i^0 + h^2 d_1 \sum_{i=0}^{n-1} (c_4 + b_4 a_i^0 + hd_4 \sum_{k=0}^{n-1} a_k^0) \\ &\le (1 + hb_4) a_{n-1}^0 + h \underbrace{(c_4 + c_1 + \tau d_1 c_4)}_{c_5} + hb_1 a_n^0 \\ &+ h^2 \underbrace{(d_4 + d_1 + d_1 b_4 + \tau d_1 d_4)}_{d_5} \sum_{i=0}^{n-1} a_i^0, \end{aligned}$$

this implies that

$$(1-hb_1)a_n^0 \leq (1+hb_4)a_{n-1}^0 + hc_5 + h^2 d_5 \sum_{i=0}^{n-1} a_i^0,$$

hence, for all $h \in (0, \frac{1}{b_1})$, we have

$$a_n^0 \leq \frac{1+hb_4}{1-hb_1} a_{n-1}^0 + \frac{h^2 d_5}{1-hb_1} \sum_{i=0}^{n-1} a_i^0 + \frac{hc_5}{1-hb_1}.$$

Then, by Lemma 3, we get for all $n \in \{0,1,...,N-1\}$

$$\begin{aligned} a_n^0 &\leq \frac{a_0^0}{R_2-R_1}[(R_2-1)R_2^n + (1-R_1)R_1^n] + \frac{hc_5}{(R_2-R_1)(1-hb_1)}[R_2^n - R_1^n] \\ &\leq \frac{\alpha_1^1(m)}{R_2-R_1}[(R_2-1)R_2^n + (1-R_1)R_1^n] + \frac{hc_5}{(R_2-R_1)(1-hb_1)}[R_2^n - R_1^n], \end{aligned} \tag{25}$$

where

$$R_1 = \left(1 + \frac{1+hb_4+h^2 d_5}{1-hb_1} - h\sqrt{\frac{(b_1+b_4)^2 + h^2 d_5^2 + 2d_5(1+hb_4)}{(1-hb_1)^2} + 2\frac{d_5}{1-hb_1}}\right)/2,$$

$$R_2 = \left(1 + \frac{1+hb_4+h^2 d_5}{1-hb_1} + h\sqrt{\frac{(b_1+b_4)^2 + h^2 d_5^2 + 2d_5(1+hb_4)}{(1-hb_1)^2} + 2\frac{d_5}{1-hb_1}}\right)/2.$$

Since $0 < R_1 \leq 1 \leq R_2$, then for all $h \in (0, \frac{1}{b_1})$, we have

$$R_1^n \leq 1 \leq R_2^n \leq R_2^N = R_2^{\frac{\tau}{h}}, n = 0,1,...,N-1,$$

which implies, from (25), that

$$a_n^0 \leq \alpha_1^1(m) \frac{(R_2-1)R_2^{\frac{\tau}{h}} + (1-R_1)}{R_2-R_1} + c_5 \frac{hR_2^{\frac{\tau}{h}}}{(R_2-R_1)(1-hb_1)}.$$

Then, there exist $h_1 \in (0, \frac{1}{b_1})$ and $\alpha_1^2(m) \geq 0$ such that for all $h \in (0, h_1]$

$$a_n^0 \leq \alpha_1^2(m), n = 0,1,...,N-1,$$

which implies, from (24), that for all $j = 1,2,...,m+1$ and $n = 0,1,...,N-1$

$$a_n^j \leq z_n \leq c_4 + b_4 \alpha_1^2(m) + d_4 \alpha_1^2(m)\tau = \alpha_1^3(m).$$

Hence, the first step is completed by setting

$$\alpha_1(m) = \max(\alpha_1^2(m), \alpha_1^3(m)).$$

Claim 2. There exists a positive constant $\alpha(m)$ such that $\|\hat{u}_{n,p}^{(j)}\| \leq \alpha(m)$ for all $n = 0,1,...,N-1,\ j = 0,1,...,m+1$ and $p = 1,...,r-1$.

Let $a_{n,p}^{j}=\|\hat{u}_{n,p}^{(j)}\|$ and $\xi_p=\max\{a_{i,p}^{j}, j=0,...,m+1, i=0,...,N-1\}$ for $p=0,...,r-1$.
Similarly to Claim 1, by differentiating equation (15) j-times, we obtain for all $j=1,...,m+1$,

$$a_{0,p}^{j}\leq c_1+b_1\xi_{p-1}+d_1\sum_{l=0}^{j-1}a_{0,p}^{l},$$

where c_1,b_1,d_1 are positive numbers.
On the other hand, by integrating (15) from t_0^p to $t\in\sigma_0^p$, we get,

$$a_{0,p}^{0}\leq c_2+b_2\xi_{p-1}+hd_2a_{0,p}^{0},$$

where c_2,b_2,d_2 are positive numbers.
hence, there exists $h_2\in(0,h_1]$ and positive numbers c_3,b_3,d_3 such that for all $h\in(0,h_2]$ and $j\in\{0,1,...,m+1\}$, we have

$$a_{0,p}^{j}\leq c_3+b_3\xi_{p-1}+d_3\sum_{l=0}^{j-1}a_{0,p}^{l}.$$

Then, by Lemma 1, for all $j\in\{0,1,...,m+1\}$

$$a_{0,p}^{j}\leq\underbrace{c_3\exp(d_3(m+1))}_{c_4^1}+\underbrace{b_3\exp(d_3(m+1))}_{b_4}\xi_{p-1},$$

hence, for $c_4=\max(\alpha_1(m),c_4^1)$, we get for all $p=0,1,...,r-1$, $j\in\{0,1,...,m+1\}$

$$a_{0,p}^{j}\leq c_4+b_4\xi_{p-1}. \tag{26}$$

Next, by differentiating equation (17) j-times, we obtain for all $n=1,...,N-1$ and $j=1,...,m+1$,

$$a_{n,p}^{j}\leq c_5+b_5\xi_{p-1}+e_5\sum_{l=0}^{j-1}a_{n,p}^{l}+d_5h\sum_{i=0}^{n-1}\sum_{l=0}^{m+1}a_{i,p}^{l},$$

where c_5,b_5,e_5,d_5 are positive numbers.
Then, by Lemma 1, for all $j\in\{1,...,m+1\}$

$$a_{n,p}^{j}\leq\underbrace{c_5\exp(e_5(m+1))}_{c_6}+\underbrace{b_5\exp(e_5(m+1))}_{b_6}\xi_{p-1}+\underbrace{e_5\exp(e_5(m+1))}_{e_6}a_{n,p}^{0}$$
$$+\underbrace{d_5\exp(e_5(m+1))}_{d_6}h\sum_{i=0}^{n-1}\sum_{l=0}^{m+1}a_{i,p}^{l}.$$

Consider the sequence $y_n = \sum_{j=1}^{m+1} a_{n,p}^j$, $n = 0, 1, ..., N-1$, hence, by the above inequality, the sequence (y_n) satisfies for all $n = 1, ..., N-1$,

$$
\begin{aligned}
y_n \le & \underbrace{(m+1)c_6}_{c_7^1} + \underbrace{(m+1)b_6}_{b_7^1} \xi_{p-1} + \underbrace{(m+1)e_6}_{e_7} a_{n,p}^0 + \underbrace{(m+1)d_6}_{d_7} h \sum_{i=0}^{n} a_{i,p}^0 \\
& + \underbrace{(m+1)d_6}_{d_7} h \sum_{i=0}^{n-1} y_i.
\end{aligned} \tag{27}
$$

Moreover, from (26), we obtain,

$$
y_0 \le \underbrace{(m+1)c_4}_{c_7^2} + \underbrace{(m+1)b_4}_{b_7^2} \xi_{p-1}. \tag{28}
$$

Let $c_7 = \max\{c_7^1, c_7^2\}$ and $b_7 = \max\{b_7^1, b_7^2\}$.
Then, from (27) and (28), we get for all $n = 0, 1, ..., N-1$,

$$
y_n \le c_7 + b_7 \xi_{p-1} + e_7 a_{n,p}^0 + d_7 h \sum_{i=0}^{n} a_{i,p}^0 + d_7 h \sum_{i=0}^{n-1} y_i,
$$

hence, by Lemma 2, we obtain

$$
\begin{aligned}
y_n \le & \underbrace{c_7(1+\tau d_7 \exp(\tau d_7))}_{c_8} + \underbrace{b_7(1+\tau d_7 \exp(\tau d_7))}_{b_8} \xi_{p-1} + \underbrace{(e_7 + \tau d_7)}_{e_8} a_{n,p}^0 \\
& + h \underbrace{\left(d_7 + d_7 e_7 \exp(\tau d_7) + \tau d_7^2 \exp(\tau d_7)\right)}_{d_8} \sum_{i=0}^{n-1} a_{i,p}^0.
\end{aligned} \tag{29}
$$

On the other hand, by integrating (17) from t_n^p to $t \in \sigma_n^p$, we get

$$
\begin{aligned}
a_{n,p}^0 & \le |u_{n-1}^p(t_n^p)| + hc_9 + hb_9 \xi_{p-1} + he_9 a_{n,p}^0 + d_9 h^2 \sum_{i=0}^{n-1} \sum_{l=0}^{m} a_{i,p}^l \\
& \le |u_{n-1}^p(t_n^p)| + hc_9 + hb_9 \xi_{p-1} + he_9 a_{n,p}^0 + d_9 h^2 \sum_{i=0}^{n-1} a_{i,p}^0 + d_9 h^2 \sum_{i=0}^{n-1} y_i,
\end{aligned}
$$

for all $n = 1, ..., N-1$, where c_9, b_9, e_9, d_9 are positive numbers.
Moreover, from (9), we obtain,

$$
\begin{aligned}
|u_{n-1}^p(t_n^p)| & \le a_{n-1,p}^0 + h \sum_{j=1}^{m} a_{n-1,p}^j \\
& \le a_{n-1,p}^0 + h y_{n-1},
\end{aligned}
$$

then,

$$a_{n,p}^0 \leq a_{n-1,p}^0 + hy_{n-1} + hc_9 + hb_9\xi_{p-1} + he_9 a_{n,p}^0 + d_9 h^2 \sum_{i=0}^{n-1} a_{i,p}^0 + d_9 h^2 \sum_{i=0}^{n-1} y_i,$$

which implies, by using (29), that

$$\begin{aligned}(1-he_9)a_{n,p}^0 \leq & a_{n-1,p}^0 + h(c_8 + b_8\xi_{p-1} + e_8 a_{n-1,p}^0 + hd_8 \sum_{i=0}^{n-1} a_{i,p}^0) + hc_9 + hb_9\xi_{p-1} \\ & + d_9 h^2 \sum_{i=0}^{n-1} a_{i,p}^0 + d_9 h^2 \sum_{i=0}^{n-1} (c_8 + b_8\xi_{p-1} + e_8 a_{i,p}^0 + hd_8 \sum_{k=0}^{n-1} a_{k,p}^0) \\ \leq & (1+he_8)a_{n-1,p}^0 + h\underbrace{(c_8 + c_9 + \tau d_9 c_8)}_{c_{10}} + h\underbrace{(b_8 + b_9 + \tau d_9 b_8)}_{b_{10}}\xi_{p-1} \\ & + h^2 \underbrace{(d_8 + d_9 + d_9 e_8 + \tau d_9 d_8)}_{d_{10}} \sum_{i=0}^{n-1} a_{i,p}^0,\end{aligned}$$

hence, there exists $h_3 \in (0, h_2]$ such that for all $h \in (0, h_3]$, we have

$$a_{n,p}^0 \leq \frac{1+he_8}{1-he_9} a_{n-1,p}^0 + \frac{h(c_{10} + b_{10}\xi_{p-1})}{1-he_9} + \frac{h^2 d_{10}}{1-he_9} \sum_{i=0}^{n-1} a_{i,p}^0.$$

Then, by Lemma 3, we get for all $n \in \{0, 1, ..., N-1\}$

$$a_{n,p}^0 \leq \frac{a_{0,p}^0}{R_2 - R_1} [(R_2 - 1)R_2^n + (1 - R_1)R_1^n] + \frac{h(c_{10} + b_{10}\xi_{p-1})}{(R_2 - R_1)(1 - he_9)} [R_2^n - R_1^n],$$

where

$$R_1 = \left(1 + \frac{1 + he_8 + h^2 d_{10}}{1 - he_9} - h\sqrt{\frac{(e_8 + e_9)^2 + h^2 d_{10}^2 + 2d_{10}(1 + he_8)}{(1 - he_9)^2} + 2\frac{d_{10}}{1 - hb_9}}\right)/2,$$

$$R_2 = \left(1 + \frac{1 + he_8 + h^2 d_{10}}{1 - he_9} + h\sqrt{\frac{(e_8 + e_9)^2 + h^2 d_{10}^2 + 2d_{10}(1 + he_8)}{(1 - he_9)^2} + 2\frac{d_{10}}{1 - hb_9}}\right)/2.$$

Hence, similar as in (25), there exist $h_4 \in (0, h_3]$ and $\overline{R} > 0$ such that for all $h \in (0, h_4]$, we have

$$a_{n,p}^0 \leq a_{0,p}^0 \overline{R} + (c_{10} + b_{10}\xi_{p-1})\overline{R},$$

which implies, by using(26), that for all $n \in \{0, 1, ..., N-1\}$ and $p \in \{0, 1, ..., r-1\}$

$$a_{n,p}^0 \leq \underbrace{(c_4 + c_{10})\overline{R}}_{c_{11}} + \underbrace{(b_4 + b_{10})\overline{R}}_{b_{11}} \xi_{p-1}.$$

Then, from (29), we get for all $n \in \{0,1,...,N-1\}$, $j \in \{1,...,m+1\}$ and $p \in \{0,1,...,r-1\}$,

$$a_{n,p}^{j} \leq y_n \leq \underbrace{(c_8 + e_8 c_{11} + \tau d_8 c_{11})}_{c_{12}^1} + \underbrace{(b_8 + e_8 b_{11} + \tau d_8 b_{11})}_{b_{12}^1} \xi_{p-1},$$

Let $c_{12} = \max(c_{11}, c_{12}^1)$ and $b_{12} = \max(b_{11}, b_{12}^1)$.
We deduce that, for all $p \in \{0,1,...,r-1\}$,

$$\begin{aligned}\xi_p &\leq c_{12} + b_{12}\xi_{p-1}\\ &\leq c_{12} + b_{12}\sum_{i=0}^{p-1}\xi_i.\end{aligned}$$

Then, by Lemma 1, we get for all $p \in \{0,1,...,r-1\}$, $n \in \{0,1,...,N-1\}$ and $j \in \{0,1,...,m+1\}$,

$$a_{n,p}^{j} \leq \xi_p \leq c_{12}\exp(rb_{12}) = \alpha(m).$$

This completes the proof of Lemma 4.

The following theorem describes the order of convergence of the method.

Theorem 1 Let g,k,a,b and Φ be m times continuously differentiable on their respective domains. Then equations (2), (3), (6), (9) define a unique approximation $u \in S_m^{(0)}(\Pi_N)$, and the resulting error function $e := y - u$ satisfies:

$$\|e\| \leq Ch^m$$

provided that h is sufficiently small, where C is a finite constant independent of h.

Proof. The proof is split into two steps.
Claim 1. There exists a constant C_0 independent of h such that $\|e^0\| \leq C_0 h^m$, where the error $e^0 = e|_{\sigma^0}$ which is defined on σ_n^0, by $e^0(t) = e_n^0(t) = y(t) - u_n^0(t)$ for all $n \in \{0,1,...,N-1\}$.
Let $t \in \sigma_0^0$, we have from Lemma 4, for sufficient small h

$$\|e_0^0(t)\| = \|y(t) - u_0^0(t)\| \leq \frac{\left\|y^{(m+1)}\right\|}{(m+1)!}h^{m+1} \leq \frac{\alpha(m)}{(m+1)!}h^{m+1}.$$

In general for $n = 1,2,...,N-1$ and $t \in \sigma_n^0$, we have from (13),

$$\begin{aligned}y'(t) - \hat{u}_{n,0}'(t) = a(t)(y(t) - \hat{u}_{n,0}(t)) &+ \sum_{i=0}^{n-1}\int_{t_i^0}^{t_{i+1}^0} k(t,s)(y(s) - u_i^0(s))ds\\ &+ \int_{t_n^0}^{t} k(t,s)(y(s) - \hat{u}_{n,0}(s))ds,\end{aligned}$$

this implies that,

$$\|y' - \hat{u}'_{n,0}\| \leq hk_0 \sum_{i=0}^{n-1} \left\|e_i^0\right\| + \underbrace{(A + \tau k_0)}_{\widetilde{A}} \|y - \hat{u}_{n,0}\|, \tag{30}$$

where, $k_0 = \|k\|_{L^\infty}(\sigma^0 \times \sigma^0)$.
On the other hand, for $t \in \sigma_n^0$,

$$\begin{aligned} y(t) - \hat{u}_{n,0}(t) &= y(t_n^0) - \hat{u}_{n,0}(t_n^0) + \int_{t_n^0}^{t} (y'(s) - \hat{u}'_{n,0}(s))ds \\ &= e_{n-1}^0(t_n^0) + \int_{t_n^0}^{t} (y'(s) - \hat{u}'_{n,0}(s))ds, \end{aligned}$$

it follows that,

$$\|y - \hat{u}_{n,0}\| \leq \left\|e_{n-1}^0\right\| + h\|y' - \hat{u}'_{n,0}\|,$$

hence, by using (30), we get,

$$\|y - \hat{u}_{n,0}\| \leq \frac{1}{1 - h\widetilde{A}} \left\|e_{n-1}^0\right\| + \frac{h^2 k_0}{1 - h\widetilde{A}} \sum_{i=0}^{n-1} \left\|e_i^0\right\|,$$

Therefore, by Lemma 4,

$$\begin{aligned} \left\|e_n^0\right\| &\leq \|y - \widehat{u}_{n,0}\| + \left\|\widehat{u}_{n,0} - u_n^0\right\| \\ &\leq \|y - \widehat{u}_{n,0}\| + \frac{\alpha(m)}{(m+1)!} h^{m+1}. \end{aligned}$$

Then,

$$\left\|e_n^0\right\| \leq \frac{1}{1 - h\widetilde{A}} \left\|e_{n-1}^0\right\| + \frac{h^2 k_0}{1 - h\widetilde{A}} \sum_{i=0}^{n-1} \left\|e_i^0\right\| + \frac{\alpha(m)}{(m+1)!} h^{m+1},$$

hence by Lemma 3, for all $n \in \{0, 1, ..., N-1\}$

$$\begin{aligned} \left\|e_n^0\right\| &\leq \frac{\left\|e_0^0\right\|}{R_2 - R_1} [(R_2 - 1)R_2^n + (1 - R_1)R_1^n] + \frac{\frac{\alpha(m)}{(m+1)!} h^{m+1}}{R_2 - R_1} [R_2^n - R_1^n] \\ &\leq \frac{\alpha(m)}{(m+1)!} h^{m+1} \frac{[(R_2 - 1)R_2^n + (1 - R_1)R_1^n] + [R_2^n - R_1^n]}{R_2 - R_1}, \end{aligned} \tag{31}$$

where

$$\begin{aligned} R_1 &= \left(1 + \frac{1 + h^2 k_0}{1 - h\widetilde{A}} - \frac{h}{1 - h\widetilde{A}} \sqrt{\xi}\right) /2 = \frac{1 - h(\widetilde{A} + \sqrt{\xi})}{2(1 - h\widetilde{A})}, \\ R_2 &= \left(1 + \frac{1 + h^2 k_0}{1 - h\widetilde{A}} + \frac{h}{1 - h\widetilde{A}} \sqrt{\xi}\right) /2 = \frac{1 - h(\widetilde{A} - \sqrt{\xi})}{2(1 - h\widetilde{A})}, \\ \xi &= \widetilde{A}^2 + (hk_0)^2 + 2k_0(2 - h\widetilde{A}). \end{aligned} \tag{32}$$

Since $0 < R_1 \leq 1 \leq R_2$, then

$$R_1^n \leq 1 \leq R_2^n \leq R_2^N = R_2^{\frac{\tau}{h}}, n = 0,1,...,N-1,$$

which implies, from (31), that

$$\|e_n^0\| \leq \frac{\alpha(m)}{(m+1)!} h^m (1 - h\widetilde{A}) \frac{[(R_2 - 1)R_2^{\frac{\tau}{h}} + (1 - R_1)] + R_2^{\frac{\tau}{h}}}{\sqrt{\xi}},$$

we deduce that, there exist $\widetilde{R}_1$ and h_1 such that, for all $h \in (0, h_1]$

$$\|e_n^0\| \leq \widetilde{R}_1 h^m.$$

Thus,

$$\|e^0\| = \max_{n=0,\dots,N-1} \|e_n^0\| \leq \widetilde{R}_1 h^m. \tag{33}$$

Hence, the first step is completed by taking $C_0 = \widetilde{R}_1$.
Claim 2. There exists a constant C independent of h such that $\|e\| \leq Ch^m$. Define the error $e^p(t)$ on σ^p by $e^p(t) = y(t) - u^p(t)$ and on σ_n^p by $e^p(t) = e_n^p(t) = y(t) - u_n^p(t)$ for all $n \in \{0,1,...,N-1\}$ and $p \in \{0,1,...,r-1\}$.
First, let $t \in \sigma_0^p$, we have from (15),

$$\begin{aligned} y'(t) - \hat{u}'_{0,p}(t) &= a(t)(y(t) - \hat{u}_{0,p}(t)) + b(t)(e_0^{p-1}(t-\tau)) \\ &\quad + \int_{t-\tau}^{t_0^p} k(t,s) e^{p-1}(s) ds + \int_{t_0^p}^{t} k(t,s)(y(s) - \hat{u}_{0,p}(s)) ds, \end{aligned}$$

such that $y(t_0^p) - \hat{u}_{0,p}(t_0^p) = y(t_0^p) - u^{p-1}(t_0^p) = e^{p-1}(t_0^p)$,
hence,

$$\begin{aligned} \|y - \hat{u}_{0,p}\| &\leq \|e^{p-1}\| + h\|y' - \hat{u}'_{0,p}\| \\ &\leq \|e^{p-1}\| + h\left(\underbrace{(B + \tau k_0)}_{\widetilde{B}} \|e^{p-1}\| + \widetilde{A}\|y - \hat{u}_{0,p}\|\right), \end{aligned}$$

where $B = \|b\|$ and $\widetilde{A} = A + \tau k_0$, this implies that

$$\|y - \hat{u}_{0,p}\| \leq \frac{1 + h\widetilde{B}}{1 - h\widetilde{A}} \|e^{p-1}\|.$$

Therefore, by Lemma 4,

$$\begin{aligned} \|e_0^p\| &\leq \|y - \widehat{u}_{0,p}\| + \|\widehat{u}_{0,p} - u_0^p\| \\ &\leq \|y - \widehat{u}_{0,p}\| + \frac{\alpha(m)}{(m+1)!} h^{m+1}, \end{aligned}$$

then,

$$\left\|e_0^p\right\| \leq \frac{1+h\widetilde{B}}{1-h\widetilde{A}}\left\|e^{p-1}\right\| + \frac{\alpha(m)}{(m+1)!}h^{m+1}. \tag{34}$$

Next, let $t \in \sigma_n^p$ for $n \in \{1,2,...,n\}$, we have from (17)

$$\begin{aligned} y'(t)-\hat{u}'_{n,p}(t) &= a(t)(y(t)-\hat{u}_{n,p}(t)) + b(t)(e_n^{p-1}(t-\tau)) + \int_{t-\tau}^{t_0^p} k(t,s)e^{p-1}(s)ds \\ &\quad + \sum_{i=0}^{n-1}\int_{t_i^p}^{t_{i+1}^p} k(t,s)e_i^p(s)ds + \int_{t_n^p}^{t} k(t,s)(y(s)-\hat{u}_{n,p}(s))ds, \end{aligned}$$

such that $y(t_n^p)-\hat{u}_{n,p}(t_n^p) = y(t_n^p)-u_{n-1}^p(t_n^p) = e_{n-1}^p(t_n^p)$,
hence,

$$\begin{aligned} \left\|y-\hat{u}_{n,p}\right\| &\leq \left\|e_{n-1}^p\right\| + h\|y'-\hat{u}'_{n,p}\| \\ &\leq \left\|e_{n-1}^p\right\| + h\widetilde{B}\left\|e^{p-1}\right\| + \widetilde{A}\left\|y-\hat{u}_{n,p}\right\| + h^2k_0\sum_{i=0}^{n-1}\left\|e_i^p\right\|, \end{aligned}$$

this implies that

$$\left\|y-\hat{u}_{n,p}\right\| \leq \frac{1}{1-h\widetilde{A}}\left\|e_{n-1}^p\right\| + \frac{h\widetilde{B}}{1-h\widetilde{A}}\left\|e^{p-1}\right\| + \frac{h^2k_0}{1-h\widetilde{A}}\sum_{i=0}^{n-1}\left\|e_i^p\right\|.$$

Then, by Lemma 4,

$$\begin{aligned} \|e_n^p\| &\leq \left\|y-\widehat{u}_{n,p}\right\| + \left\|\widehat{u}_{n,p}-u_n^p\right\| \\ &\leq \frac{1}{1-h\widetilde{A}}\left\|e_{n-1}^p\right\| + \frac{h\widetilde{B}}{1-h\widetilde{A}}\left\|e^{p-1}\right\| + \frac{h^2k_0}{1-h\widetilde{A}}\sum_{i=0}^{n-1}\left\|e_i^p\right\| + \frac{\alpha(m)}{(m+1)!}h^{m+1}. \end{aligned}$$

It follows from Lemma 3, for all $n \in \{0,1,...,N-1\}$,

$$\begin{aligned} \|e_n^p\| &\leq \frac{\left\|e_0^p\right\|}{R_2-R_1}[(R_2-1)R_2^n+(1-R_1)R_1^n] + \frac{\frac{h\widetilde{B}}{1-h\widetilde{A}}\left\|e^{p-1}\right\| + \frac{\alpha(m)}{(m+1)!}h^{m+1}}{R_2-R_1}[R_2^n-R_1^n] \\ &\leq \left\|e_0^p\right\|\frac{(R_2-1)R_2^{\frac{\tau}{h}}+(1-R_1)}{R_2-R_1} + \frac{\widetilde{B}\left\|e^{p-1}\right\| + (1-h\widetilde{A})\frac{\alpha(m)}{(m+1)!}h^m}{\sqrt{\xi}}R_2^{\frac{\tau}{h}}, \end{aligned}$$

where R_1 and R_2 are defined by (32).
So, there exist $\widetilde{R}_2$ and h_2 such that, for all $h \in (0,h_2]$

$$\|e_n^p\| \leq (\left\|e_0^p\right\| + \left\|e^{p-1}\right\| + h^m)\widetilde{R}_2,$$

which implies, by (34), that for all $h \leq h_2$,

$$\begin{aligned} \|e_n^p\| &\leq ((\frac{1+h\widetilde{B}}{1-h\widetilde{A}}+1)\left\|e^{p-1}\right\| + \frac{\alpha(m)}{(m+1)!}h^{m+1} + h^m)\widetilde{R}_2 \\ &\leq ((\frac{1+h_2\widetilde{B}}{1-h_2\widetilde{A}}+1)\left\|e^{p-1}\right\| + (\frac{\alpha(m)}{(m+1)!}h_2+1)h^m)\widetilde{R}_2, \end{aligned}$$

hence, for $\widetilde{R}_3 = \max\{(\frac{1+h_2\widetilde{B}}{1-h_2\widetilde{A}}+1)\widetilde{R}_2, (\frac{\alpha(m)}{(m+1)!}h_2+1)\widetilde{R}_2\}$, we obtain,

$$\|e_n^p\| \leq \widetilde{R}_3 \left\|e^{p-1}\right\| + \widetilde{R}_3 h^m,$$

we deduce that,

$$\|e^p\| \leq \widetilde{R}_3 \sum_{i=0}^{p-1} \left\|e^i\right\| + \widetilde{R}_3 h^m \leq \widetilde{R}_3 \sum_{i=0}^{p-1} \left\|e^i\right\| + \widetilde{R}_4 h^m, \tag{35}$$

where $\widetilde{R}_4 = \max\{\widetilde{R}_1, \widetilde{R}_3\}$.
Then, from (33), (35) and by using Lemma 1, we get,

$$\|e^p\| \leq \widetilde{R}_4 h^m \exp(r\widetilde{R}_3). \tag{36}$$

Thus, the proof is completed by taking $C = \widetilde{R}_4 \exp(r\widetilde{R}_3)$.

4 Numerical Examples

In this section, we present several examples with analytical solutions to show the performance of the described method in Section 2 for solving the system of Eqs. (1). In each example, we present a different dimensional nonlinear system (one-dimensional in example (1), two-dimensional in examples (2), (3) , (4), (5), (6), four-dimensional in example (8) and five-dimensional in examples (7) and (9)). We calculate the error between y and u. Also, the results obtained in examples (3), (4), (5), and (6), which arise in mathematical modeling of "predator-prey" dynamics in Ecology, are compared with those obtained from the application of Variational Iteration Method (VIM) [20], Adomian Decomposition Method (ADM) [3] and Pseudospectral Legendre Method (PLM) [25]. Moreover, we apply two mathematical models that simulate the evolution of the COVID-19 epidemic in China, namely the SEIR [18] model in example (8) and the SEIRU [22] model in example (9).

Example 1 ([23]) Consider the Volterra delay integro-differential equation
$y'(t) = -(6+\sin(t))y(t) + y(t-\frac{\pi}{4}) - \int_{t-\frac{\pi}{4}}^{t} \sin(s)y(s)ds$,
for $t \in [0, \frac{\pi}{2}]$ and $y(t) = e^{\cos(t)}$ for $t \in [-\frac{\pi}{4}, 0]$ with $\Phi(t) = e^{\cos(t)}$.
The absolute errors for $(N,m) = \{(4,4),(6,6),(8,8),(10,10)\}$ are presented in Table 1.

Example 2 ([2]) Consider the nonlinear system of VDIDEs:

$$\begin{pmatrix} y_1'(t) \\ y_2'(t) \end{pmatrix} = -4\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} + \begin{pmatrix} 0 & \sin(t) \\ \cos(t) & 0 \end{pmatrix}\begin{pmatrix} y_1(t-\frac{\pi}{5}) \\ y_2(t-\frac{\pi}{5}) \end{pmatrix} + \int_{t-\frac{\pi}{5}}^{t} \begin{pmatrix} \frac{(1+\sin^2(s)y_1^2(s))}{\sqrt{2}(1+y_1^2(s))} \\ \frac{(1+\cos^2(s)y_2^2(s))}{\sqrt{2}(1+y_2^2(s))} \end{pmatrix} ds + \begin{pmatrix} g_1(t) \\ g_2(t) \end{pmatrix}$$

for $t \in [0, \frac{2\pi}{5}]$, $g(t) = (g_1(t), g_2(t))^t$ is chosen so that the exact solution is $y(t) = (\sin(t), \cos(t))$ and $\Phi(t) = y(t)$ for $t \in [-\frac{\pi}{5}, 0]$

The absolute errors for $(N,m) = \{(4,4),(8,8)\}$ are presented in Table 2.

Table 1: Absolute errors for $y(t)$ of Example 1

t	$(N,m)=(4,4)$	$(N,m)=(6,6)$	$(N,m)=(8,8)$	$(N,m)=(10,10)$
0.0	0.0	0.0	0.0	0.0
$\pi/10$	1.79×10^{-6}	2.58×10^{-10}	2.76×10^{-10}	7.68×10^{-11}
$\pi/9$	3.68×10^{-5}	3.21×10^{-8}	3.02×10^{-10}	1.33×10^{-10}
$\pi/8$	6.54×10^{-5}	1.16×10^{-7}	3.76×10^{-10}	1.99×10^{-10}
$\pi/7$	3.60×10^{-5}	6.21×10^{-8}	6.41×10^{-10}	4.02×10^{-10}
$\pi/6$	1.22×10^{-5}	1.61×10^{-8}	2.39×10^{-10}	3.35×10^{-11}
$\pi/5$	3.95×10^{-5}	2.17×10^{-8}	1.89×10^{-9}	2.50×10^{-10}
$\pi/4$	2.12×10^{-4}	1.26×10^{-6}	3.40×10^{-8}	9.93×10^{-10}
$\pi/3$	2.16×10^{-4}	5.35×10^{-6}	1.96×10^{-8}	1.52×10^{-7}
$\pi/2$	1.59×10^{-2}	2.44×10^{-4}	3.00×10^{-6}	4.31×10^{-7}

Table 2: Absolute errors for $y(t)$ of Example 2

t	$y_1(t)$		$y_2(t)$	
	$(N,m)=(4,4)$	$(N,m)=(8,8)$	$(N,m)=(4,4)$	$(N,m)=(8,8)$
0.0	0.0	0.0	0.0	0.0
$\pi/20$	7.96×10^{-7}	1.56×10^{-14}	2.08×10^{-8}	1.22×10^{-12}
$2\pi/20$	3.07×10^{-6}	1.48×10^{-11}	1.62×10^{-8}	1.66×10^{-10}
$3\pi/20$	3.46×10^{-6}	2.84×10^{-11}	2.04×10^{-7}	2.88×10^{-10}
$4\pi/20$	6.39×10^{-6}	1.72×10^{-11}	1.63×10^{-7}	3.55×10^{-10}
$5\pi/20$	7.17×10^{-6}	1.40×10^{-10}	8.12×10^{-7}	2.42×10^{-10}
$6\pi/20$	1.16×10^{-5}	3.07×10^{-12}	5.98×10^{-7}	1.79×10^{-10}
$7\pi/20$	1.32×10^{-5}	5.08×10^{-11}	1.38×10^{-6}	3.17×10^{-11}
$8\pi/20$	2.01×10^{-5}	1.49×10^{-10}	4.73×10^{-7}	2.37×10^{-10}

Example 3 ([25]) In this example, we solve the system (1) of two nonlinear delay integro-differential equations with
$\Phi_1(t)=-t^2$, $\Phi_2(t)=\frac{1}{2}te^{-t}$ for $t\in[-\frac{1}{3},0]$,
$k_1(t,s,y(t),y(s))=(3-2(t-s))y_1(t)y_2(s)$,
$k_2(t,s,y(t),y(s))=(t-s)y_1(s)y_2(t)$,
$f_1(t,y(t))=t^2\left(2-3te^{-t}-\frac{7}{2}e^{-t}+\frac{13}{6}te^{\frac{1}{3}-t}+\frac{22}{9}e^{\frac{1}{3}-t}\right)-2t+y_1(t)\,(2-y_2(t))$,
$f_2(t,y(t))=\frac{1}{648}e^{-t}(342t^3-8t^2+325t+324)+y_2(t)\,(-2+y_1(t))$.
The exact solution of this system is in the following form $y_1(t)=-t^2$, $y_2(t)=\frac{1}{2}te^{-t}$.
The absolute errors of Taylor Collocation Method (TCM) for $(N,m)=(8,8)$ are compared with the absolute error of the Variational Iteration Method (VIM), Adomian Decomposition Method (ADM) and Pseudospectral Legendre Method (PLM) given in [25] in Table 3 and Table 4.

Example 4 ([25]) Consider the system (1) of two nonlinear delay integro-differential equations
$y_1'(t)=\left(-\frac{5}{2}t^3+\frac{49}{12}t^2+\frac{17}{12}t-\frac{23}{6}\right)+y_1(t)\left(1-\frac{1}{3}y_2(t)\right)+\int_{t-\frac{1}{2}}^{t}-y_1(t)y_2(s)ds$
$y_2'(t)=\left(\frac{15}{8}t^3-\frac{1}{4}t^2+\frac{3}{8}t-1\right)+y_2(t)\,(-2+y_1(t))+\int_{t-\frac{1}{2}}^{t}(t-s-1)y_1(s)y_2(t)ds$,
for $t\in[0,1]$ and $y(t)=\Phi(t)$ for $t\in[-\frac{1}{2},0]$ with
$\Phi_1(t)=-3t+1$, $\Phi_2(t)=t^2-t$,

Table 3: Comparison of the absolute errors for $y_1(t)$ of Example 3

t	PLM	ADM	VIM	TCM
0.1	1.02×10^{-4}	1.68×10^{-6}	4.50×10^{-10}	1.40×10^{-13}
0.2	1.76×10^{-4}	2.56×10^{-6}	4.07×10^{-9}	2.75×10^{-12}
0.3	2.29×10^{-4}	3.70×10^{-5}	4.72×10^{-8}	1.21×10^{-11}
0.4	2.69×10^{-4}	1.88×10^{-4}	3.64×10^{-7}	6.08×10^{-12}
0.5	3.04×10^{-4}	6.92×10^{-4}	2.03×10^{-6}	6.66×10^{-15}
0.6	3.42×10^{-4}	2.07×10^{-3}	8.80×10^{-6}	3.39×10^{-11}
0.7	3.90×10^{-4}	5.27×10^{-3}	3.12×10^{-5}	5.61×10^{-12}
0.8	4.56×10^{-4}	1.17×10^{-2}	9.44×10^{-5}	1.02×10^{-10}
0.9	5.48×10^{-4}	2.39×10^{-2}	2.51×10^{-4}	9.29×10^{-11}
1.0	6.74×10^{-4}	4.51×10^{-2}	6.04×10^{-4}	1.49×10^{-10}

Table 4: Comparison of the absolute errors for $y_2(t)$ of Example 3

t	PLM	ADM	VIM	Present method
0.1	1.76×10^{-3}	2.53×10^{-6}	9.80×10^{-8}	4.82×10^{-15}
0.2	2.20×10^{-3}	2.12×10^{-5}	6.93×10^{-8}	2.88×10^{-13}
0.3	1.91×10^{-3}	1.50×10^{-4}	2.69×10^{-7}	1.43×10^{-12}
0.4	1.33×10^{-3}	6.29×10^{-4}	3.55×10^{-7}	2.82×10^{-12}
0.5	7.39×10^{-4}	1.89×10^{-3}	2.49×10^{-6}	9.11×10^{-12}
0.6	3.17×10^{-4}	4.69×10^{-3}	1.08×10^{-5}	1.13×10^{-11}
0.7	1.30×10^{-4}	1.00×10^{-2}	3.85×10^{-5}	9.67×10^{-12}
0.8	1.51×10^{-4}	1.95×10^{-2}	1.14×10^{-4}	1.64×10^{-11}
0.9	2.70×10^{-4}	3.48×10^{-2}	3.00×10^{-4}	2.63×10^{-11}
1.0	3.08×10^{-4}	5.84×10^{-2}	7.11×10^{-4}	3.76×10^{-11}

$k_1(t,s,y(t),y(s)) = -y_1(t)y_2(s)$,
$k_2(t,s,y(t),y(s)) = (t-s-1)y_1(s)y_2(t)$,
$f_1(t,y(t)) = -\frac{5}{2}t^3 + \frac{49}{12}t^2 + \frac{17}{12}t - \frac{23}{6} + y_1(t)\left(1-\frac{1}{3}y_2(t)\right)$,
$f_2(t,y(t)) = \frac{15}{8}t^3 - \frac{1}{4}t^2 + \frac{3}{8}t - 1 + y_2(t)\left(-2+y_1(t)\right)$.
The exact solution of this system is in the following form
$y_1(t) = -3t+1,\ y_2(t) = t^2 - t$.
The absolute errors of Taylor Collocation Method (TCM) for $m = 10$, $N = 10$ are compared with the absolute error of the Variational Iteration Method (VIM), Adomian Decomposition Method (ADM) and Pseudospectral Legendre Method (PLM) given in [25] in Table 5 and Table 6.

Example 5 ([25]) As the last example, consider the system (1) of two nonlinear delay integro-differential equations
for $t \in [0,3]$ and $y(t) = \Phi(t)$ for $t \in [-\frac{3}{10}, 0]$ with
$\Phi_1(t) = \frac{1}{4}\sin(t),\ \Phi_2(t) = -\frac{1}{4}\sin(t)$,
$k_1(t,s,y(t),y(s)) = -y_1(t)y_2(s)$,
$k_2(t,s,y(t),y(s)) = e^{s-t}y_1(s)y_2(t)$,
$f_1(t,y(t)) = g_1(t) + y_1(t)\left(1-\frac{1}{3}y_2(t)\right)$,
$f_2(t,y(t)) = g_2(t) + y_2(t)\left(-2+y_1(t)\right)$ where
$g_1(t) = \frac{1}{4}\cos(t) - \frac{1}{4}\sin(t)\left(\frac{1}{3} + \frac{1}{2}\sin(t) - \frac{1}{4}\cos(t) + \frac{1}{4}\cos(t-\frac{3}{10})\right)$ and

Table 5: Comparison of the absolute errors for $y_1(t)$ of Example 4

t	PLM	ADM	VIM	TCM
0.1	1.98×10^{-13}	1.09×10^{-4}	3.15×10^{-4}	1.66×10^{-11}
0.2	3.23×10^{-13}	1.78×10^{-4}	4.27×10^{-4}	6.49×10^{-11}
0.3	3.61×10^{-13}	1.08×10^{-4}	4.72×10^{-4}	1.14×10^{-10}
0.4	3.21×10^{-13}	3.09×10^{-4}	4.85×10^{-4}	1.92×10^{-10}
0.5	2.09×10^{-13}	1.35×10^{-3}	4.74×10^{-4}	3.03×10^{-10}
0.6	3.37×10^{-14}	1.35×10^{-3}	4.45×10^{-4}	4.17×10^{-10}
0.7	1.98×10^{-13}	6.37×10^{-3}	4.36×10^{-4}	4.62×10^{-10}
0.8	4.81×10^{-13}	1.04×10^{-2}	5.35×10^{-4}	5.82×10^{-10}
0.9	8.05×10^{-13}	1.52×10^{-2}	9.10×10^{-4}	6.53×10^{-10}
1.0	1.16×10^{-12}	1.99×10^{-2}	1.82×10^{-3}	7.02×10^{-10}

Table 6: Comparison of the absolute errors for $y_2(t)$ of Example 4

t	PLM	ADM	VIM	TCM
0.1	4.46×10^{-14}	7.58×10^{-6}	3.34×10^{-5}	0
0.2	3.74×10^{-14}	1.36×10^{-4}	8.54×10^{-5}	2.50×10^{-14}
0.3	1.10×10^{-14}	7.65×10^{-4}	1.33×10^{-4}	4.32×10^{-13}
0.4	9.03×10^{-14}	2.75×10^{-3}	1.79×10^{-4}	6.36×10^{-12}
0.5	1.89×10^{-13}	7.63×10^{-3}	2.22×10^{-4}	1.03×10^{-11}
0.6	2.98×10^{-13}	1.77×10^{-2}	2.37×10^{-4}	2.13×10^{-11}
0.7	4.06×10^{-13}	3.61×10^{-2}	1.62×10^{-4}	2.00×10^{-11}
0.8	5.02×10^{-13}	6.66×10^{-2}	1.07×10^{-4}	3.29×10^{-11}
0.9	5.76×10^{-13}	1.13×10^{-1}	7.31×10^{-4}	4.45×10^{-11}
1.0	6.18×10^{-13}	1.83×10^{-1}	1.90×10^{-3}	8.10×10^{-11}

$$g_2(t) = -\frac{1}{4}\cos(t) + \frac{1}{4}\sin(t)\left(-\frac{1}{2} + \frac{3}{8}\sin(t) - \frac{1}{8}\cos(t) + \frac{1}{8}e^{-\frac{3}{10}\left(\cos(t-\frac{3}{10})-\sin(t-\frac{3}{10})\right)}\right).$$

The exact solution of this system is in the following form
$y_1(t) = \frac{1}{4}\sin(t)$, $y_2(t) = -\frac{1}{4}\sin(t)$.
The absolute errors for $N = 3$ and $m = \{5, 10\}$ are presented in Table 7.

Table 7: Absolute errors for $y(t)$ of Example 5

t	$y_1(t)$		$y_2(t)$	
	$m=5$	$m=10$	$m=5$	$m=10$
0.0	0.0	0.0	0.0	0.0
0.5	7.16×10^{-10}	6.60×10^{-13}	2.63×10^{-10}	4.08×10^{-13}
1.0	5.04×10^{-9}	7.02×10^{-12}	1.51×10^{-9}	6.99×10^{-12}
1.5	1.64×10^{-8}	1.88×10^{-11}	4.71×10^{-9}	5.71×10^{-12}
2.0	3.74×10^{-8}	9.08×10^{-11}	1.01×10^{-8}	1.79×10^{-12}
2.5	6.53×10^{-8}	1.49×10^{-10}	1.55×10^{-8}	4.72×10^{-11}
3.0	9.18×10^{-8}	7.20×10^{-11}	1.72×10^{-8}	1.21×10^{-10}

Example 6 ([25]) Consider the system (1) of two nonlinear delay integro-differential equations with

$\Phi_1(t) = t^2 - t,\ \Phi_2(t) = -e^{-3t}$ for $t \in [-\frac{1}{4}, 0]$,
$k_1(t,s,y(t),y(s)) = (s-t)y_1(t)y_2(s)$,
$k_2(t,s,y(t),y(s)) = (t-s+1)y_1(s)y_2(t)$,
$f_1(t,y(t)) = 2t - 1 - (t^2 - t)\left(1 + \frac{11}{18}e^{-3t} - \frac{1}{36}e^{\frac{3}{4}-3t}\right) + y_1(t)\left(1 - \frac{1}{2}y_2(t)\right)$,
$f_2(t,y(t)) = \frac{1}{3072}e^{-3t}(10080t^2 - 10304t + 6275) + y_2(t)(-1 + 3y_1(t))$.
The exact solution of this system is in the following form $y_1(t) = t^2 - t$, $y_2(t) = -e^{-3t}$.
The absolute errors of Taylor Collocation Method (TCM) for $m = 8$, $N = 8$ are compared with the absolute error of the Variational Iteration Method (VIM), Adomian Decomposition Method (ADM) and Pseudospectral Legendre Method (PLM) given in [25] in Table 8 and Table 9.

Table 8: Comparison of the absolute errors for $y_1(t)$ of Example 6

t	PLM	ADM	VIM	TCM
0.1	7.99×10^{-4}	2.48×10^{-5}	3.59×10^{-7}	1.51×10^{-12}
0.2	1.48×10^{-3}	9.91×10^{-5}	2.66×10^{-7}	7.21×10^{-12}
0.3	2.05×10^{-3}	2.16×10^{-4}	4.66×10^{-7}	3.86×10^{-12}
0.4	2.52×10^{-3}	3.80×10^{-4}	1.64×10^{-5}	8.68×10^{-12}
0.5	2.91×10^{-3}	6.11×10^{-4}	6.93×10^{-5}	2.48×10^{-11}
0.6	3.22×10^{-3}	9.58×10^{-4}	1.73×10^{-4}	2.94×10^{-11}
0.7	3.47×10^{-3}	1.50×10^{-3}	3.23×10^{-4}	3.78×10^{-11}
0.8	3.67×10^{-3}	2.40×10^{-3}	4.92×10^{-4}	7.34×10^{-12}
0.9	3.82×10^{-3}	3.81×10^{-3}	6.41×10^{-4}	4.08×10^{-11}
1.0	3.95×10^{-3}	5.92×10^{-3}	7.37×10^{-4}	9.12×10^{-11}

Table 9: Comparison of the absolute errors for $y_2(t)$ of Example 6

t	PLM	ADM	VIM	TCM
0.1	3.61×10^{-2}	4.87×10^{-5}	1.09×10^{-5}	2.88×10^{-12}
0.2	4.53×10^{-2}	4.44×10^{-5}	1.55×10^{-5}	4.44×10^{-12}
0.3	4.06×10^{-2}	1.62×10^{-4}	8.22×10^{-6}	3.35×10^{-11}
0.4	3.06×10^{-2}	8.70×10^{-4}	9.26×10^{-5}	5.90×10^{-11}
0.5	2.04×10^{-2}	2.49×10^{-3}	3.95×10^{-4}	3.91×10^{-11}
0.6	1.30×10^{-2}	5.43×10^{-3}	9.37×10^{-4}	4.34×10^{-11}
0.7	9.06×10^{-3}	9.83×10^{-3}	1.62×10^{-3}	5.66×10^{-11}
0.8	8.19×10^{-3}	1.53×10^{-2}	2.26×10^{-3}	2.15×10^{-10}
0.9	8.96×10^{-3}	2.10×10^{-2}	2.63×10^{-3}	7.67×10^{-11}
1.0	9.14×10^{-3}	2.54×10^{-2}	2.57×10^{-3}	2.61×10^{-10}

Example 7 Consider the following five-dimensional nonlinear system:

$$\begin{cases} y_1'(t) = g_1(t) - y_1(t)y_3(t) - y_1(t) \\ y_2'(t) = y_1(t) - y_5(t) - y_2(t) \\ y_3'(t) = y_3(t)(2 - y_3(t)) - y_4(t)y_5(t) & ,t \in [0,2]. \\ y_4'(t) = g_2(t) + \int_{t-\frac{1}{4}}^{t} (s - y_3(t))ds - \frac{1}{4}y_4(t) \\ y_5'(t) = g_3(t) + \int_{t-\frac{1}{4}}^{t} sy_3(t)ds - \frac{1}{2}y_5(t) \end{cases} \tag{37}$$

$g_1(t), g_2(t), g_3(t)$ are chosen so that the exact solution is $y(t) = \left(-e^{-2t} + \cos(t), e^{-2t}, t^2 + 1, 1 - 2t - t^4 - \cos(t), \cos(t)\right)$ and $y(t) = \Phi(t)$ for $t \in [-\frac{1}{4}, 0]$. The maximum errors that have been obtained for system (37) for $(N,m) = \{(3,3),(4,4),(5,5),(6,6)\}$ are presented in Table 10.

Table 10: The maximum errors of system (37)

(N,m)	$(3,3)$	$(4,4)$	$(5,5)$	$(6,6)$
$y_1(t)$	4.83×10^{-5}	1.96×10^{-7}	1.25×10^{-9}	3.68×10^{-10}
$y_2(t)$	3.83×10^{-5}	2.15×10^{-7}	8.87×10^{-10}	7.26×10^{-10}
$y_3(t)$	2.61×10^{-4}	7.12×10^{-9}	1.19×10^{-9}	1.12×10^{-9}
$y_4(t)$	9.37×10^{-4}	1.32×10^{-7}	2.01×10^{-9}	7.41×10^{-10}
$y_5(t)$	7.84×10^{-5}	9.89×10^{-8}	4.01×10^{-10}	1.06×10^{-10}

Example 8 ([18]) In this example, we found a numerical solution for the SEIR model based on the four nonlinear ordinary differential equations describing the COVID-19 epidemic in China. It has four elements which are S (susceptible), E (exposed), I (infectious) and R (recovered) and can be represented as follows:

$$\begin{cases} \frac{dS}{dt}(t) = -\beta\frac{S(t)}{N}I(t) - \frac{Z}{N}S(t) + \rho_I + \rho_E - (\frac{\rho_I}{N} + \frac{\rho_E}{N})S(t) + \nu N(t) - \mu S(t) \\ \frac{dE}{dt}(t) = \beta\frac{S(t)}{N}I(t) + \frac{Z}{N}S(t) - \alpha E(t) - (\frac{\rho_I}{N} + \frac{\rho_E}{N})E(t) - \mu E(t) - \sigma E(t) \\ \frac{dI}{dt}(t) = \alpha E(t) - \gamma I(t) - (\frac{\rho_I}{N} + \frac{\rho_E}{N})I(t) - \mu I(t) \\ \frac{dR}{dt}(t) = \gamma I(t) - \mu R(t) + \sigma E(t). \end{cases} \tag{38}$$

For $t \in [0,20]$, with initial conditions $S(0) = 2500$, $E(0) = 1$, $I(0) = 1$, $R(0) = 0$, $N(t) = S(t)+E(t)+I(t)+R(t) = 2502$ and parameters $\beta = 0.8$, $\alpha = 0.75$, $\sigma = 0.1$, $\gamma = 0.05$, $\nu = 0.009/N$, $\mu = 0.01$, $Z = 0.001$, $\rho_I = 0.15$, $\rho_E = 0.15$, $\rho_I = 0.01$, $\rho_E = 0.03$.
We used Taylor Collocation Method (TCM) for $h = \frac{1}{4}$, $(N,m) = (80,6)$ and

compared the approximate solution obtained with the approximate solution of the variational iteration method (VIM) and the differential transformation method (DTM) given in [18] in table (11). For more information about the model refer to [9].

Table 11: Comparison of approximate solutions of the SEIR model (38) for compartment S, E, I and R using VIM, DTM and TCM.

t	DTM	VIM	TCM
0	2500	2500	2500
2	2448	2448	2448
4	2396	2396	2394
6	2345	2345	2334
8	2294	2293	2260
10	2244	2243	2157
12	2197	2195	2000
14	2152	2149	1754
16	2111	2105	1398
18	2075	2065	967
20	2043	2028	567

Compartment S

t	DTM	VIM	TCM
0	1	1	1
2	1	1	2
4	−1	−1	4
6	−10	−9	8
8	−31	−29	17
10	−69	−64	34
12	−128	−117	66
14	−214	−194	119
16	−331	−296	184
18	−484	−429	230
20	−677	−594	221

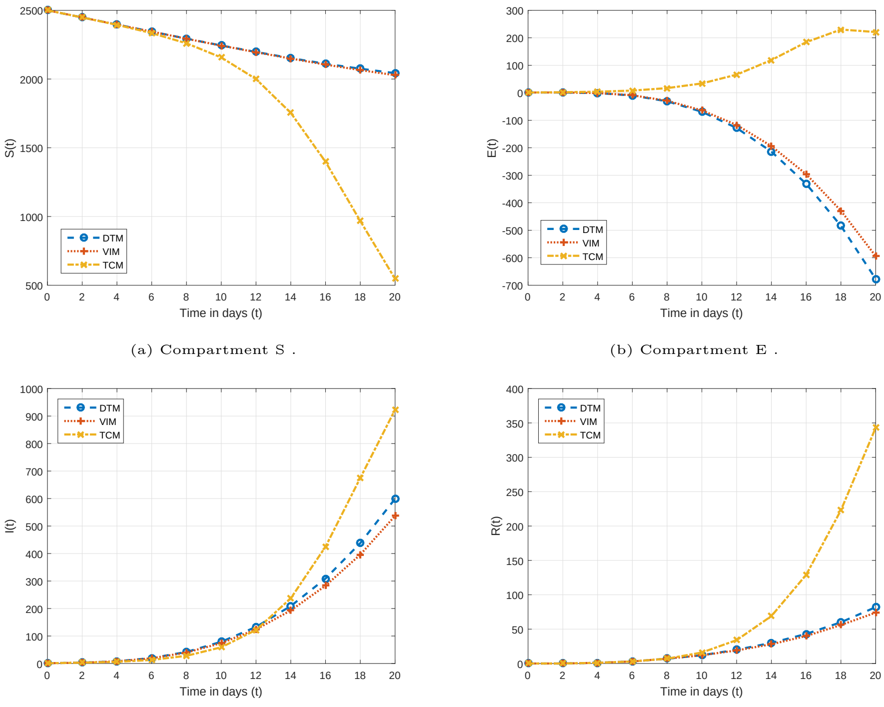
Compartment E

t	DTM	VIM	TCM
0	1	1	1
2	3	3	3
4	8	8	6
6	20	19	13
8	42	40	28
10	79	74	59
12	133	124	122
14	209	193	237
16	309	283	425
18	438	396	675
20	599	537	923

Compartment I

t	DTM	VIM	TCM
0	0	0	0
2	0	0	0
4	1	1	1
6	3	3	3
8	7	7	7
10	12	12	16
12	20	19	34
14	30	28	69
16	43	40	129
18	60	56	223
20	82	74	343

Compartment R



(a) Compartment S .
(b) Compartment E .
(c) Compartment I .
(d) Compartment R .

Fig. 1: Comparison of approximate solutions of the SEIR model (38) for compartment S, E, I and R using VIM, DTM and TCM.

Example 9 ([22]) In this example, we found a numerical solution for the *SEIRU* model based on a DDEs (delay differential equations), describing the COVID-19 epidemic in China. This system has five elements which are:
S is the number of individuals susceptible to infection, E is the number of asymptomatic noninfectious individuals, I is the number of asymptomatic but infectious individuals, R is the number of reported symptomatic infectious individuals, and U is the number of unreported symptomatic infectious individuals, can be represented in the folowing system:

$$\begin{cases} \frac{dS}{dt}(t) = -D(t)S(t)(I(t)+U(t)) \\ \frac{dE}{dt}(t) = D(t)S(t)(I(t)+U(t)) - D(t-\tau)S(t-\tau)(I(t-\tau)+U(t-\tau)) \\ \frac{dI}{dt}(t) = D(t-\tau)S(t-\tau)(I(t-\tau)+U(t-\tau)) - \nu I(t) \\ \frac{dR}{dt}(t) = \nu_1 I(t) - \eta R(t) \\ \frac{dU}{dt}(t) = \nu_2 I(t) - \eta U(t). \end{cases} \tag{39}$$

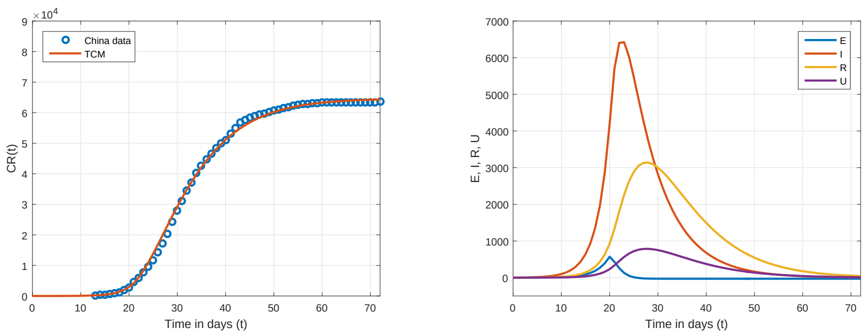
For $t \in [0,72] = [January \;\; 6, March \;\; 18]$. This system is supplemented by initial fonctions for $t \in [-\tau, 0]$

$$\begin{cases} S(t) = 1400050000 \\ E(t) = \tau(0.3762 + v)I(t) \\ I(t) = \dfrac{0.3762}{0.8v} exp(0.3762t) \\ R(t) = 0 \\ U(t) = \dfrac{0.2v}{\eta + 0.3762} I(t). \end{cases}$$

The time-dependent transmission rate parameter D is

$$D(t) = \begin{cases} D_0 = \left(\dfrac{0.3762 + v}{1400050000}\right)\left(\dfrac{\eta + 0.3762}{v_2 + \eta + 0.3762}\right) exp(0.3762\tau) & 0 \leq t \leq N_0 \\ D_0 \; exp(-\mu(t - N_0)) \quad N_0 < t \end{cases}$$

Where the day $N_0 = 19$ (January 25) corresponds to the day when the public measures take effect, $\mu = 0.62$ is the rate at which they take effect, it is chosen so that the simulations align with the cumulative reported case data and the parameters $\tau = -\frac{1}{4}$, $v = \frac{1}{7}$, $v_1 = \frac{0.8}{7}$, $v_2 = \frac{0.2}{7}$, $\eta = \frac{1}{7}$. For more information about the model refer to [22].



(a) Reported case data and $t \to CR(t)$.

(b) Compartment E, I, R and U.

Fig. 2: Approximate solutions of the SEIRU model (39) for compartment E, I,R and U using TCM.

5 Conclusion

In this paper, we have proposed a collocation method based on the use of Taylor polynomials to approximate the solution of the general system of nonlinear delay integro-differential equations (1) in the spline space $S_m^{(0)}(\Pi_N)$. We have shown that the numerical solution is convergent. This method is easy to implement and the coefficients of the approximation solution are determined by

Table 12: Comparison of cumulative daily reported case data [22] with approximate solutions of the SEIRU model (39) for the cumulative number of reported symptomatic infectious cases ($t \to CR(t)$) using TCM.

January								
Day	16	17	18	19	20	21	22	23
CR(China data)				198	291	440	571	830
CR(TCM)	63	94	139	205	302	442	647	945
Day	24	25	26	27	28	29	30	31
CR(China data)	1287	1975	2744	4515	5974	7711	9692	11791
CR(TCM)	1379	2011	2933	4266	6075	8331	10952	13838
February								
Day	1	2	3	4	5	6	7	8
CR(China data)	14380	17205	20438	24324	28018	31161	34546	37198
CR(TCM)	16889	20019	23158	26250	29253	32136	34878	37466
Day	9	10	11	12	13	14	15	16
CR(China data)	40171	42638	44653	46472	48467	49970	51091	53139
CR(TCM)	39891	42153	44251	46188	47971	49606	51101	52464
Day	17	18	19	20	21	22	23	24
CR(China data)	55027	56776	57593	58482	58879	59527	59741	60249
CR(TCM)	53704	54829	55849	56770	57602	58352	59026	59633
Day	25	26	27	28	29			
CR(China data)	60655	61088	61415	61806	62415			
CR(TCM)	60177	60665	61101	61492	61841			
March								
Day	1	2	3	4	5	6	7	8
CR(China data)	62617	62742	62861	63000	63143	63242	63286	63326
CR(TCM)	62153	62432	62680	62901	63097	63272	63428	63566
Day	9	10	11	12	13	14	15	16
CR(China data)	63345	63369	63384	63404	63415	63435	63451	63472
CR(TCM)	63688	63797	63894	63979	64055	64122	64182	64235
Day	17	18	19	20	21	22	23	24
CR(China data)	63485	63519						
CR(TCM)	64281	64322						

iterative formulas without the need to solve any system of algebraic equations. The numerical examples which were introduced have shown that the method is convergent with a good accuracy.

Acknowledgements This work is supported by the Directorate General for Scientific Research and Technological Development (DGRSDT), Ministry of Higher Education and Scientific Research, Algeria.

References

1. Abbasbandy, S., Taati, A.: Numerical solution of the system of nonlinear Volterra integro-differential equations with nonlinear differential part by the operational Tau method and error estimation. Journal of Computational and Applied Mathematics. 231(1), 106-113 (2009)
2. Abdi, A., Berrut, J.P, Hosseini, S.A.: The linear barycentric rational method for a class of delay Volterra integro-differential equations. Journal of Scientific Computing. 75(3), 1757-1775 (2018)
3. Adomian, G.: A review of the decomposition method and some recent results for nonlinear equations. Math. Comput. Model. 13(7), 17-43 (1990)
4. Agarwal, R.P.: Difference equations and inequalities: theory, methods, and applications. CRC Press. (2000)
5. Arikoglu, A., Ozkol, I.: Solutions of integral and integro-differential equation systems by using differential transform method. Computers and Mathematics with Applications. 56(9), 2411-2417 (2008)
6. Bellour, A., Bousselsal, M.: Numerical solution of delay integro-differential equations by using Taylor collocation method. Math. Meth. Appl. Sci. 37, 1491-1506 (2014)
7. Bellour, A., Bousselsal, M.: A Taylor collocation method for solving delay integral equations. Numerical Algorithms. 65, 843-857 (2014)
8. Biazar, J., Ghazvini, H., Eslami, M.: He's homotopy perturbation method for systems of integro-differential equations. Chaos, Solitons and Fractals. 39(3), 1253-1258 (2009)
9. Biswas, M.H.A., Paiva, L.T., De Pinho, M.D.R.: A SEIR model for control of infectious diseases with constraints. Mathematical Biosciences and Engineering. 11(4), 761 (2014)
10. Brunner, H.: Collocation methods for Volterra integral and related functional differential equations. Cambridge university press, Cambridge (2004)
11. Burton, T.A.: Volterra Integral and Differential Equations. Academic Press, New York (1983)
12. Cushing, J.M.: Integrodifferential equations and delay models in population dynamics (Vol. 20). Springer Science and Business Media (2013)
13. Darania, P.: Multistep collocation method for nonlinear delay integral equations. Sahand Commun. Math. Anal. (SCMA). 3, 47-65 (2016)
14. Darania, P., Pishbin, S.: High-order collocation methods for nonlinear delay integral equation. J. Comput. Appl. 326, 284-295 (2017)
15. Fazeli, S., Hojjati, G.: Numerical solution of Volterra integro-differential equations by superimplicit multistep collocation methods. Numer. Algorithms. 68, 741-768 (2015)
16. Gülsu, M., Sezer, M.: Taylor collocation method for solution of systems of high-order linear Fredholm Volterra integro-differential equations. Int. J. Comput. 83, 429-448 (2006)
17. Gu, Z.: Chebyshev spectral collocation method for system of nonlinear Volterra integral equations. Numerical Algorithms. 83(1), 243-263 (2020)
18. Harir, A., Melliani, S., El Harfi, H., Chadli, L.S.: Variational Iteration Method and Differential Transformation Method for Solving the SEIR Epidemic Model. Int. J. Differ. Equ. vol 2020 (2020)
19. Hairer, E., Lubich, C., Nϕrsett, S.P.: Order of convergence of one-step methods for Volterra integral equations of the second kind. SIAM J. Numer. Anal. 20, 569-579 (1983)
20. He, J.H.: Approximate solution of nonlinear differential equations with convolution product nonlinearities. Comput. Methods Appl. Mech. Eng. 167, 69-73 (1998)
21. Jerri, A.J.: Introduction to Integral Equations with Applications. John Wiley and Sons, INC (1999)
22. Liu, Z., Magal, P., Seydi, O., Webb, G.: A COVID-19 epidemic model with latency period. Infectious Disease Modelling. 5, 323-337 (2020)
23. Rihan, F. A., Doha, E.H., Hassan, M., Kamel, N.M.: Numerical treatments for volterra delay integro-differential equations. Comput. Methods Appl. Math. 9(3), 292-308 (2009)
24. Saberi-Nadjafi, J., Tamamgar, M.: The variational iteration method: a highly promising method for solving the system of integro-differential equations. Computers and Mathematics with Applications. 56(2), 346-351 (2008)
25. Shakeri, F., Dehghan, M.: Solution of a model describing biological species living together using the variational iteration method. Math. Comput. Model. 48, 685-699 (2008)

26. Shakourifar, M., Enright, W.: Superconvergent interpolants for collocation methods applied to Volterra integro-differential equations with delay. BIT Numerical Mathematics. 52(3), 725-740 (2012)
27. Volterra, V.: Variazioni e fluttuazioni del numero d'individui in specie animali conviventi (p. 142). C. Ferrari. (1927)
28. Wang, K., Wang, Q.: Taylor collocation method and convergence analysis for the Volterra Fredholm integral equations. J. Comput. Appl. 260, 294-300 (2014)
29. Wazwaz, A.M.: Linear and nonlinear integral equations (Vol. 639). Berlin, Springer. (2011)
30. Yüzbaşı, Ş., Sezer, M.: An exponential approach for the system of nonlinear delay integro-differential equations describing biological species living together. Neural Computing and Applications. 27(3), 769-779 (2016)