Journal of Innovative Applied Mathematics and Computational Sciences

JIAMCS

ISSN: 2773-4196

Volume 1, Issue 1
July -December 2021

ARK: 49935/v1.i1

Institute of Sciences and Technology,
University Center Abdelhafid
Boussouf, Mila, Algeria.

PURL: 49935/v1.i1

http://jiamcs.centre-univ-mila.dz/

About the Journal

The Journal of Innovative Applied Mathematics and Computational Sciences (JIAMCS) is an online open access, peer-reviewed semiannual international journal published by the Institute of Sciences and Technology, University Center Abdelhafid Boussouf, MILA, ALGERIA.

The journal publishes high-quality original research papers from various fields related to applied mathematics, scientific computing, and computer science.

In particular, it publishes original papers in the following areas:

- Differential equations, (ODE's, PDE's, integral equations, difference equations, fractional differential equations)
- Dynamical systems and bifurcation-chaos theory
- Fractional calculus
- Operator's theory
- Mathematical physics
- Probability and statistics
- Mathematical modelling and simulation
- Numerical analysis
- Fuzzy logic and systems
- Lattice theory, number theory and cryptography
- Modern control theory and practice
- Mathematical optimization
- Combinatorial optimisation
- Computer science
- Graph theory
- Bioinformatics
- Artificial intelligence
- Pattern recognition
- Computer vision and image processing
- Naturel language processing
- Signal processing
- Communication theory and network security

Published by:

Institute of Sciences and Technology, University Center Abdelhafid Boussouf, MILA, ALGERIA.

Periodicity:

Biannually.

Contact

J. Innov. Appl. Math. Comput. Sci
Institute of Sciences and Technology, University Center Abdelhafid Boussouf , MILA 43000, ALGERIA.

Principal Contact

J. Innov. Appl. Math. Comput. Sci
jiamcs@centre-univ-mila.dz

Support Contact

Pr. Mohammed Salah Abdelouahab
m.abdelouahab@centre-univ-mila.dz

Editorial Team

- Soheyb Milles, Departement of mathematics and informatics,

  University Center Si El-Haoues, Barika, Algeria
- Ali Moussaoui, Departement of mathematics, University of Tlemcen,

  Tlemcen, Algeria
- Nafaa Nacereddine, Research Center in Industrial Technologies CRTI,

  Algeria
- Adel Ouannes, Department of Mathematics and Computer Science,University Larbi Ben M'hidi, Oum el Bouaghi,

  Algeria
- Fathalla A Rihan, Department of Mathematical Sciences, United Arab

  Emirates University, Al Ain, United Arab Emirates
- Anurag Shukla, Department of Applied Science (Mathematics), Rajkiya Engineering College Kannauj, Kannauj-209732,

  India
- Driss Sbibih, Department of Mathematics and Computer Science,

  University Mohammed First, Oujda, Morocco
- Nouressadat Touafek, Department of Mathematics, University of Jijel,

  Jijel, Algeria
- Cemil Tunç, Department of Mathematics, Van Yuzuncu Yil University,

  Van, Turkey
- Christos Volos, Department of Mathematics, Aristotle University of

  Thessaloniki, Thessaloníki, Greece
- Junwei Wang, Department of Applied Mathematics, Guangdong

  University of Foreign Studies, Guangzhou, China

Journal of Innovative Applied Mathematics and Computational Sciences

J. Innov. Appl. Math. Comput. Sci. **1**(1) (2021) 1–15 n2t.net/ark:/49935/jiamcs.v1i1.8

http://jiamcs.centre-univ-mila.dz/

On a system of difference equations of third order solved in closed form

Youssouf Akrour [1, 4], Nouressadat Touafek [*2, 4] and Yacine Halim [** 3, 4]

[1]Département des Sciences Exactes et d'Informatique, École Normale Supérieure Assia Djebar, Constantine, Algeria
[2]Department of Mathematics, Mohamed Seddik Ben Yahia University, Jijel, Algeria
[3]Department of Mathematics and Computer Science, Abdelhafid Boussouf University Center, Mila, Algeria
[4]LMAM Laboratory, Mohamed Seddik Ben Yahia University, Jijel, Algeria

Received 18 November 2021, Accepted 06 December 2021, Published 22 December 2021

Abstract. In this work, we show that the system of difference equations

$$x_{n+1} = \frac{ay_{n-2}x_{n-1}y_n + bx_{n-1}y_{n-2} + cy_{n-2} + d}{y_{n-2}x_{n-1}y_n},$$

$$y_{n+1} = \frac{ax_{n-2}y_{n-1}x_n + by_{n-1}x_{n-2} + cx_{n-2} + d}{x_{n-2}y_{n-1}x_n},$$

where $n \in \mathbb{N}_0$, x_{-2}, x_{-1}, x_0, y_{-2}, y_{-1} and y_0 are arbitrary nonzero real numbers and a, b, c and d are arbitrary real numbers with $d \neq 0$, can be solved in a closed form.
We will see that when $a = b = c = d = 1$ the solutions are expressed using the famous Tetranacci numbers. In particular, the results obtained here extend those in our recent work.

Keywords: System of difference equations, general solution, Tetranacci numbers.

2020 Mathematics Subject Classification: 39A05, 39A06, 39A10.

1 Introduction

Nonlinear difference equations and their systems are hot topics that attract the attention of several researchers. A significant number of papers are devoted to this field of research. One can consult, for example, the papers [3, 5–18, 20–23, 26, 27, 30, 31, 36–44, 46], where one can find concrete models of such equations and systems, as well as understand the techniques used to solve them and investigate the behavior of their solutions. Recently, in [1] and as a

✉ Corresponding author: youssouf.akrour@gmail.com
*ntouafek@gmail.com
**halyacine@yahoo.fr

ISSN (electronic): 2773-4196

generalization of the equations and systems studied in [4,19,32,45], we have solved in a closed form the system of difference equations

$$\begin{cases} x_{n+1} = \dfrac{a y_n x_{n-1} + b x_{n-1} + c}{x_{n-1} y_n}, \\ y_{n+1} = \dfrac{a x_n y_{n-1} + b y_{n-1} + c}{y_{n-1} x_n}. \end{cases} \tag{1.1}$$

Here, and motivated by the above papers, one shows that one can express in closed form the well-defined solutions of the following system of difference equations

$$\begin{cases} x_{n+1} = \dfrac{a y_{n-2} x_{n-1} y_n + b x_{n-1} y_{n-2} + c y_{n-2} + d}{y_{n-2} x_{n-1} y_n}, \\ y_{n+1} = \dfrac{a x_{n-2} y_{n-1} x_n + b y_{n-1} x_{n-2} + c x_{n-2} + d}{x_{n-2} y_{n-1} x_n}, \end{cases} \tag{1.2}$$

where $n \in \mathbb{N}_0$, the initial values x_{-2}, x_{-1}, x_0, y_{-2}, y_{-1} and y_0 are arbitrary nonzero real numbers and the parameters a, b, c and d are arbitrary real numbers with $d \neq 0$.

Clearly if $d = 0$, then System (1.2) is nothing other than system (1.1). For the readers interested in the solutions of this system, one refers to [1], where the system (1.1) has been completely solved.

Noting also that the system (1.2) can be seen as a generalization of the equation

$$x_{n+1} = \frac{a x_{n-2} x_{n-1} x_n + b x_{n-1} x_{n-2} + c x_{n-2} + d}{x_{n-2} x_{n-1} x_n}, \; n \in \mathbb{N}_0. \tag{1.3}$$

In fact, the solutions of (1.3) can be obtained from the solutions of (1.2) by choosing $y_{-i} = x_{-i}$, $i = 0, 1, 2$. The equation (1.3) was the subject of a substantial part of the paper [4], which also motivated our present study. The same equation was studied in complex numbers by Stevic in [29].

We will see that the explicit formulas of the well defined solutions of system (1.2) are expressed using the terms of the sequence $(J_n)_{n=0}^{+\infty}$ which are the solutions of the fourth-order linear homogeneous difference equation defined by the relation

$$J_{n+4} = a J_{n+3} + b J_{n+2} + c J_{n+1} + d J_n, \quad n \in \mathbb{N}_0, \tag{1.4}$$

and the special initial values

$$J_0 = 0, \quad J_1 = 0, \quad J_2 = 1 \text{ and } J_3 = a. \tag{1.5}$$

In this article one solves in closed form the equation (3.3). This well-known equation (with the same or different initial values and parameters) was the subject of some papers in the literature, see for example [25,29,47].

The characteristic equation associated to (3.3) is

$$\lambda^4 - a\lambda^3 - b\lambda^2 - c\lambda - d = 0, \tag{1.6}$$

and let α, β, γ and δ its four roots, then

$$\begin{cases} \alpha + \beta + \gamma + \delta = a \\ \alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = -b \\ \alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = c \\ \alpha\beta\gamma\delta = -d \end{cases} \tag{1.7}$$

One has:

Case 1: If all roots are real and equal. In this case,

$$J_n = \left(c_1 + c_2 n + c_3 n^2 + c_4 n^3\right) \alpha^n.$$

Now using (1.7) and the fact that $J_0 = 0$, $J_1 = 0$, $J_2 = 1$ and $J_3 = a$, one obtains

$$J_n = \left(\frac{-n + n^3}{6\alpha^2}\right) \alpha^n. \tag{1.8}$$

Case 2: If three roots are real and equal, i.e. $\beta = \gamma = \delta$. In this case

$$J_n = c_1 \alpha^n + \left(c_2 + c_3 n + c_4 n^2\right) \beta^n.$$

Now using (1.7) and the fact that $J_0 = 0$, $J_1 = 0$, $J_2 = 1$ and $J_3 = a$, one obtains

$$J_n = \frac{-\alpha}{(\beta-\alpha)^3}\alpha^n + \left(\frac{\alpha}{(\beta-\alpha)^3} - \frac{n(\alpha+\beta)}{2\beta(\beta-\alpha)^2} + \frac{n^2}{2\beta(\beta-\alpha)}\right) \beta^n, \tag{1.9}$$

Case 3: If two real roots are equal, i.e. $\gamma = \delta$. In this case

$$J_n = c_1 \alpha^n + c_2 \beta^n + (c_3 + c_4 n)\, \gamma^n.$$

Now using (1.7) and the fact that $J_0 = 0$, $J_1 = 0$, $J_2 = 1$ and $J_3 = a$, one obtains

$$J_n = \frac{-\alpha}{(\gamma-\alpha)^2(\beta-\alpha)}\alpha^n + \frac{\beta}{(\gamma-\beta)^2(\beta-\alpha)}\beta^n + \left(\frac{\alpha\beta-\gamma^2}{(\gamma-\alpha)^2(\gamma-\beta)^2} + \frac{n}{(\gamma-\alpha)(\gamma-\beta)}\right)\gamma^n, \tag{1.10}$$

Case 4: If two double real roots are equal, i.e. $\alpha = \beta \neq \gamma = \delta$. In this case

$$J_n = (c_1 + c_2 n)\, \alpha^n + (c_3 + c_4 n)\, \gamma^n.$$

Now using (1.7) and the fact that $J_0 = 0$, $J_1 = 0$, $J_2 = 1$ and $J_3 = a$, one obtains

$$J_n = \left(\frac{\gamma+\alpha}{(\gamma-\alpha)^3} + \frac{n}{(\gamma-\alpha)^2}\right)\alpha^n + \left(-\frac{\gamma+\alpha}{(\gamma-\alpha)^3} + \frac{n}{(\gamma-\alpha)^2}\right)\gamma^n, \tag{1.11}$$

Case 5: If all the roots are real and different. In this case

$$J_n = c_1\alpha^n + c_2\beta^n + c_3\gamma^n + c_4\delta^n.$$

Again, using (1.7) and the fact that $J_0 = 0$, $J_1 = 0$, $J_2 = 1$ and $J_3 = a$, one obtains

$$\begin{aligned} J_n &= \frac{-\alpha}{(\delta-\alpha)(\gamma-\alpha)(\beta-\alpha)}\alpha^n + \frac{\beta}{(\delta-\beta)(\gamma-\beta)(\beta-\alpha)}\beta^n + \frac{-\gamma}{(\delta-\gamma)(\gamma-\beta)(\gamma-\alpha)}\gamma^n \\ &\quad + \frac{\delta}{(\delta-\gamma)(\delta-\beta)(\delta-\alpha)}\delta^n. \end{aligned} \tag{1.12}$$

Case 6: If two real roots are equal, i.e. $\alpha = \beta$ and two roots are complex conjugate, i.e. $\delta = \overline{\gamma}$. In this case

$$J_n = (c_1 + c_2 n)\alpha^n + c_3\gamma^n + c_4\overline{\gamma}^n.$$

Again, using (1.7) and the fact that $J_0 = 0$, $J_1 = 0$, $J_2 = 1$ and $J_3 = a$, one obtains

$$J_n = \left(\frac{\overline{\gamma}\gamma - \alpha^2}{(\overline{\gamma}-\alpha)^2(\gamma-\alpha)^2} + \frac{n}{(\overline{\gamma}-\alpha)(\gamma-\alpha)}\right)\alpha^n + \frac{-\gamma}{(\overline{\gamma}-\gamma)(\gamma-\alpha)^2}\gamma^n + \frac{\overline{\gamma}}{(\overline{\gamma}-\gamma)(\overline{\gamma}-\alpha)^2}\overline{\gamma}^n. \tag{1.13}$$

Case 7: If two real roots α, β are different and two roots are complex conjugate, i.e. $\delta = \overline{\gamma}$. In this case

$$J_n = c_1\alpha^n + c_2\beta^n + c_3\gamma^n + c_4\overline{\gamma}^n.$$

Again, using (1.7) and the fact that $J_0 = 0$, $J_1 = 0$, $J_2 = 1$ and $J_3 = a$, one obtains

$$J_n = \frac{-\alpha}{(\overline{\gamma}-\alpha)(\gamma-\alpha)(\beta-\alpha)}\alpha^n + \frac{\beta}{(\overline{\gamma}-\beta)(\gamma-\beta)(\beta-\alpha)}\beta^n + \frac{-\gamma}{(\overline{\gamma}-\gamma)(\gamma-\beta)(\gamma-\alpha)}\gamma^n + \frac{\overline{\gamma}}{(\overline{\gamma}-\gamma)(\overline{\gamma}-\beta)(\overline{\gamma}-\alpha)}\overline{\gamma}^n. \tag{1.14}$$

Case 8: If two complex roots are equal, i.e. $\alpha = \gamma$ and $\beta = \delta = \overline{\alpha}$. In this case

$$J_n = (c_1 + c_2 n)\alpha^n + (c_3 + c_4 n)\overline{\alpha}^n.$$

Again, using (1.7) and the fact that $J_0 = 0$, $J_1 = 0$, $J_2 = 1$ and $J_3 = a$, one obtains

$$J_n = \left(\frac{\overline{\alpha}+\alpha}{(\overline{\alpha}-\alpha)^3} + \frac{n}{(\overline{\alpha}-\alpha)^2}\right)\alpha^n + \left(\frac{-\overline{\alpha}-\alpha}{(\overline{\alpha}-\alpha)^3} + \frac{n}{(\overline{\alpha}-\alpha)^2}\right)\overline{\alpha}^n. \tag{1.15}$$

Case 9: If the roots are all complex and different, i.e. $\beta = \overline{\alpha}$ and $\delta = \overline{\gamma}$. In this case

$$J_n = c_1\alpha^n + c_2\overline{\alpha}^n + c_3\gamma^n + c_4\overline{\gamma}^n.$$

Again, using (1.7) and the fact that $J_0 = 0$, $J_1 = 0$, $J_2 = 1$ and $J_3 = a$, one obtains

$$J_n = \frac{-\alpha}{(\overline{\gamma}-\alpha)(\gamma-\alpha)(\overline{\alpha}-\alpha)}\alpha^n + \frac{\overline{\alpha}}{(\overline{\gamma}-\overline{\alpha})(\gamma-\overline{\alpha})(\overline{\alpha}-\alpha)}\overline{\alpha}^n + \frac{-\gamma}{(\overline{\gamma}-\gamma)(\gamma-\overline{\alpha})(\gamma-\alpha)}\gamma^n + \frac{\overline{\gamma}}{(\overline{\gamma}-\gamma)(\overline{\gamma}-\overline{\alpha})(\overline{\gamma}-\alpha)}\overline{\gamma}^n. \tag{1.16}$$

2 The main theorem and some particular cases

Here, one gives a closed form for the well defined solutions of the system (1.2) with $d \neq 0$. One will use the same change of variables as in [1] to transform the system (1.2) to a linear one and then follows the same procedure as in [1] to obtain the closed-form of the solutions. To get the solutions of the corresponding linear system, one needs to solve some fourth-order linear difference equations. In particular, one derives from the main result (Main Theorem), for which one leaves the proof to the next section, the solutions of some particular systems and equations where their solutions are related to the famous Tetranacci numbers.

One recalls that by a well defined solution of system (1.2), one means a solution that satisfies $x_n y_n \neq 0$, $n \geq -2$. The set of well defined solutions is not empty. In fact, it suffices to choose the initial values and the parameters a, b, c and d positive, to see that every solution of (1.2) will be well defined.

2.1 Closed form of well defined solutions of the system (1.2)

The following result gives an explicit formula for well-defined solutions of the system (1.2).

Theorem 2.1. ***(Main Theorem)*** *Let $\{x_n, y_n\}_{n\geq -2}$ be a well defined solution of* (1.2). *Then, for $n \in \mathbb{N}_0$, one has*

$$x_{2n+1} = \frac{dJ_{2n+2} + (cJ_{2n+2} + dJ_{2n+1})\, y_{-2} + (J_{2n+4} - aJ_{2n+3})\, x_{-1}y_{-2} + J_{2n+3}y_0x_{-1}y_{-2}}{dJ_{2n+1} + (cJ_{2n+1} + dJ_{2n})\, y_{-2} + (J_{2n+3} - aJ_{2n+2})\, x_{-1}y_{-2} + J_{2n+2}y_0x_{-1}y_{-2}},$$

$$x_{2n+2} = \frac{dJ_{2n+3} + (cJ_{2n+3} + dJ_{2n+2})\, x_{-2} + (J_{2n+5} - aJ_{2n+4})\, y_{-1}x_{-2} + J_{2n+4}x_0y_{-1}x_{-2}}{dJ_{2n+2} + (cJ_{2n+2} + dJ_{2n+1})\, x_{-2} + (J_{2n+4} - aJ_{2n+3})\, y_{-1}x_{-2} + J_{2n+3}x_0y_{-1}x_{-2}},$$

$$y_{2n+1} = \frac{dJ_{2n+2} + (cJ_{2n+2} + dJ_{2n+1})\, x_{-2} + (J_{2n+4} - aJ_{2n+3})\, y_{-1}x_{-2} + J_{2n+3}x_0y_{-1}x_{-2}}{dJ_{2n+1} + (cJ_{2n+1} + dJ_{2n})\, x_{-2} + (J_{2n+3} - aJ_{2n+2})\, y_{-1}x_{-2} + J_{2n+2}x_0y_{-1}x_{-2}},$$

$$y_{2n+2} = \frac{dJ_{2n+3} + (cJ_{2n+3} + dJ_{2n+2})\, y_{-2} + (J_{2n+5} - aJ_{2n+4})\, x_{-1}y_{-2} + J_{2n+4}y_0x_{-1}y_{-2}}{dJ_{2n+2} + (cJ_{2n+2} + dJ_{2n+1})\, y_{-2} + (J_{2n+4} - aJ_{2n+3})\, x_{-1}y_{-2} + J_{2n+3}y_0x_{-1}y_{-2}},$$

where the initial values $x_{-2}, x_{-1}, x_0, y_{-2}, y_{-1}$ and $y_0 \in (\mathbb{R} - \{0\}) - F$, with F is the Forbidden set of system (1.2) *given by*

$$F = \bigcup_{n=0}^{\infty} \{(x_{-2}, x_{-1}, x_0, y_{-2}, y_{-1}, y_0) \in (\mathbb{R} - \{0\}) : A_n = 0 \, or \, B_n = 0\},$$

where

$$A_n = dJ_{n+1} + (cJ_{n+1} + dJ_n)\, y_{-2} + (J_{n+3} - aJ_{n+2})\, x_{-1}y_{-2} + J_{n+2}y_0x_{-1}y_{-2},$$
$$B_n = dJ_{n+1} + (cJ_{n+1} + dJ_n)\, x_{-2} + (J_{n+3} - aJ_{n+2})\, y_{-1}x_{-2} + J_{n+2}x_0y_{-1}x_{-2}.$$

2.2 Particular cases

Now, we focus our study on some particular cases of system (1.2).

2.2.1 The solutions of the equation $x_{n+1} = (ax_{n-2}x_{n-1}x_n + bx_{n-1}x_{n-2} + cx_{n-2} + d) \,/\, (x_{n-2}x_{n-1}x_n)$

If one chooses $y_{-2} = x_{-2}$, $y_{-1} = x_{-1}$ and $y_0 = x_0$, then system (1.2) is reduced to the equation

$$x_{n+1} = \frac{ax_{n-2}x_{n-1}x_n + bx_{n-1}x_{n-2} + cx_{n-2} + d}{x_{n-2}x_{n-1}x_n}, \quad n \in \mathbb{N}_0. \tag{2.1}$$

So, it follows from the **Main Theorem**

Corollary 2.2. *Let $\{x_n\}_{n\geq -2}$ be a well defined solution of the equation* (2.1). *Then for $n \in \mathbb{N}_0$, one has*

$$x_{2n+1} = \frac{dJ_{2n+2} + (cJ_{2n+2} + dJ_{2n+1})\, x_{-2} + (J_{2n+4} - aJ_{2n+3})\, x_{-1}x_{-2} + J_{2n+3}x_0x_{-1}x_{-2}}{dJ_{2n+1} + (cJ_{2n+1} + dJ_{2n})\, x_{-2} + (J_{2n+3} - aJ_{2n+2})\, x_{-1}x_{-2} + J_{2n+2}x_0x_{-1}x_{-2}},$$

$$x_{2n+2} = \frac{dJ_{2n+3} + (cJ_{2n+3} + dJ_{2n+2})\, x_{-2} + (J_{2n+5} - aJ_{2n+4})\, x_{-1}x_{-2} + J_{2n+4}x_0x_{-1}x_{-2}}{dJ_{2n+2} + (cJ_{2n+2} + dJ_{2n+1})\, x_{-2} + (J_{2n+4} - aJ_{2n+3})\, x_{-1}x_{-2} + J_{2n+3}x_0x_{-1}x_{-2}}.$$

It is worth noting that this equation was studied in [4, 29].

2.3 The solutions of the system (1.2) with $a = b = c = d = 1$

Consider the system

$$\begin{cases} x_{n+1} = \dfrac{y_{n-2}x_{n-1}y_n + x_{n-1}y_{n-2} + y_{n-2} + 1}{y_{n-2}x_{n-1}y_n}, \\ y_{n+1} = \dfrac{x_{n-2}y_{n-1}x_n + y_{n-1}x_{n-2} + x_{n-2} + 1}{x_{n-2}y_{n-1}x_n}, \ n \in \mathbb{N}_0, \end{cases} \tag{2.2}$$

which is a particular case of the system (1.2) with $a = b = c = d = 1$. In this case the sequence $\{J_n\}$ is nothing other than the sequence of Tetranacci numbers $\{T_n\}$, that is

$$T_{n+4} = T_{n+3} + T_{n+2} + T_{n+1} + T_n, \ n \in \mathbb{N}_0, \ \text{ where } \ T_0 = T_1 = 0, \ T_2 = 1 \text{ and } T_3 = 1,$$

and one has

$$\begin{aligned} T_n \ = \ & \frac{-\alpha}{(\overline{\gamma}-\alpha)(\gamma-\alpha)(\beta-\alpha)}\alpha^n + \frac{\beta}{(\overline{\gamma}-\beta)(\gamma-\beta)(\beta-\alpha)}\beta^n + \frac{-\gamma}{(\overline{\gamma}-\gamma)(\gamma-\beta)(\gamma-\alpha)}\gamma^n \\ & + \frac{\overline{\gamma}}{(\overline{\gamma}-\gamma)(\overline{\gamma}-\beta)(\overline{\gamma}-\alpha)}\overline{\gamma}^n, \qquad n \in \mathbb{N}_0, \end{aligned}$$

with

$$\alpha = \frac{1}{4} + \frac{1}{2}\omega + \frac{1}{2}\sqrt{\frac{11}{4} - \omega^2 + \frac{13}{4}\omega^{-1}}, \ \beta = \frac{1}{4} + \frac{1}{2}\omega - \frac{1}{2}\sqrt{\frac{11}{4} - \omega^2 + \frac{13}{4}\omega^{-1}},$$

$$\gamma = \frac{1}{4} - \frac{1}{2}\omega + \frac{1}{2}\sqrt{\frac{11}{4} - \omega^2 - \frac{13}{4}\omega^{-1}}, \ \delta = \frac{1}{4} - \frac{1}{2}\omega - \frac{1}{2}\sqrt{\frac{11}{4} - \omega^2 - \frac{13}{4}\omega^{-1}},$$

$$\omega = \sqrt{\frac{11}{12} + \left(\frac{-65}{54} + \sqrt{\frac{563}{108}}\right)^{\frac{1}{3}} + \left(\frac{-65}{54} - \sqrt{\frac{563}{108}}\right)^{\frac{1}{3}}}.$$

The 1-dimensional version of the system (2.2), is the equation

$$x_{n+1} = \frac{x_{n-2}x_{n-1}x_n + x_{n-1}x_{n-2} + x_{n-2} + 1}{x_{n-2}x_{n-1}x_n}, \quad n \in \mathbb{N}_0. \tag{2.3}$$

From the main theorem it follows respectively.

Corollary 2.3. *Let $\{x_n, y_n\}_{n \geq -2}$ be a well defined solution of (2.2). Then, for $n \in \mathbb{N}_0$, one has*

$$x_{2n+1} = \frac{T_{2n+2} + (T_{2n+2} + T_{2n+1})\,y_{-2} + (T_{2n+4} - T_{2n+3})\,x_{-1}y_{-2} + T_{2n+3}y_0x_{-1}y_{-2}}{T_{2n+1} + (T_{2n+1} + T_{2n})\,y_{-2} + (T_{2n+3} - T_{2n+2})\,x_{-1}y_{-2} + T_{2n+2}y_0x_{-1}y_{-2}},$$

$$x_{2n+2} = \frac{T_{2n+3} + (T_{2n+3} + T_{2n+2})\,x_{-2} + (T_{2n+5} - T_{2n+4})\,y_{-1}x_{-2} + T_{2n+4}x_0y_{-1}x_{-2}}{T_{2n+2} + (T_{2n+2} + T_{2n+1})\,x_{-2} + (T_{2n+4} - T_{2n+3})\,y_{-1}x_{-2} + T_{2n+3}x_0y_{-1}x_{-2}},$$

$$y_{2n+1} = \frac{T_{2n+2} + (T_{2n+2} + T_{2n+1})\,x_{-2} + (T_{2n+4} - T_{2n+3})\,y_{-1}x_{-2} + T_{2n+3}x_0y_{-1}x_{-2}}{T_{2n+1} + (T_{2n+1} + T_{2n})\,x_{-2} + (T_{2n+3} - T_{2n+2})\,y_{-1}x_{-2} + T_{2n+2}x_0y_{-1}x_{-2}},$$

$$y_{2n+2} = \frac{T_{2n+3} + (T_{2n+3} + T_{2n+2})\,y_{-2} + (T_{2n+5} - T_{2n+4})\,x_{-1}y_{-2} + T_{2n+4}y_0x_{-1}y_{-2}}{T_{2n+2} + (T_{2n+2} + T_{2n+1})\,y_{-2} + (T_{2n+4} - T_{2n+3})\,x_{-1}y_{-2} + T_{2n+3}y_0x_{-1}y_{-2}}.$$

Corollary 2.4. *Let $\{x_n\}_{n\geq -2}$ be a well defined solution of the equation (2.3). Then for $n \in \mathbb{N}_0$, one has*

$$x_{2n+1} = \frac{T_{2n+2} + (T_{2n+2} + T_{2n+1})\, x_{-2} + (T_{2n+4} - T_{2n+3})\, x_{-1}x_{-2} + T_{2n+3}x_0x_{-1}x_{-2}}{T_{2n+1} + (T_{2n+1} + T_{2n})\, x_{-2} + (T_{2n+3} - T_{2n+2})\, x_{-1}x_{-2} + T_{2n+2}x_0x_{-1}x_{-2}},$$

$$x_{2n+2} = \frac{T_{2n+3} + (T_{2n+3} + T_{2n+2})\, x_{-2} + (T_{2n+5} - T_{2n+4})\, x_{-1}x_{-2} + T_{2n+4}x_0x_{-1}x_{-2}}{T_{2n+2} + (T_{2n+2} + T_{2n+1})\, x_{-2} + (T_{2n+4} - T_{2n+3})\, x_{-1}x_{-2} + T_{2n+3}x_0x_{-1}x_{-2}}.$$

Remark 2.5. When $a = d = 0$, the system (1.2) takes the form

$$x_{n+1} = \frac{bx_{n-1} + c}{y_n x_{n-1}},\ y_{n+1} = \frac{by_{n-1} + c}{x_n y_{n-1}} \quad n \in \mathbb{N}_0. \tag{2.4}$$

As it is noted in [1], the solutions are expressed using Padovan numbers. This system, and some particular cases of it, were the subject of the papers [19,45].

If $d = c = 0$, the system (1.2) becomes

$$x_{n+1} = \frac{ay_n + b}{y_n},\ y_{n+1} = \frac{ax_n + b}{x_n},\ n \in \mathbb{N}_0. \tag{2.5}$$

Again, it is noted in [1] that:

- The system (2.5) is a particular case of the more general system

$$x_{n+1} = \frac{ay_n + b}{cy_n + d},\ y_{n+1} = \frac{\alpha x_n + \beta}{\gamma x_n + \lambda},\ n \in \mathbb{N}_0, \tag{2.6}$$

which was completely solved by Stevic in [33] and the solutions are expressed using a generalized Fibonacci sequence.

- Also, particular cases of System (2.6) were studied in [24,28,34,35].

- If also $b = 0$, then the solutions of the system (2.5) are given by

$$\{(x_0, y_0), (a, a), (a, a), ..., \}.$$

3 Proof of the Main Theorem

In order to solve the system (1.2), one needs first to solve the following two homogeneous fourth-order linear difference equations

$$R_{n+1} = aR_n + bR_{n-1} + cR_{n-2} + dR_{n-3},\ n \in \mathbb{N}_0, \tag{3.1}$$

$$S_{n+1} = -aS_n + bS_{n-1} - cS_{n-2} + dS_{n-3},\ n \in \mathbb{N}_0, \tag{3.2}$$

where the initial values $R_0, R_{-1}, R_{-2}, R_{-3}, S_0, S_{-1}, S_{-2}$ and S_{-3} and the constant coefficients a, b, c and d are real numbers with $d \neq 0$. In fact, one will express the terms of the sequences $(R_n)_{n=-3}^{+\infty}$ and $(S_n)_{n=-3}^{+\infty}$ using the sequence $(J_n)_{n=0}^{+\infty}$.

The difference equation (3.1) has the same characteristic equation as $(J_n)_{n=0}^{+\infty}$, that is the equation (1.6).

To solve the difference equation (3.2) using terms of (3.3), one needs the following fourth-order linear difference equation defined by

$$j_{n+4} = -aj_{n+3} + bj_{n+2} - cj_{n+1} + dj_n, \quad n \in \mathbb{N}_0, \tag{3.3}$$

and the special initial values

$$j_0 = 0, \quad j_1 = 0, \quad j_2 = 1 \text{ and } j_3 = -a. \tag{3.4}$$

The characteristic equation of (3.2) and (3.3) is

$$\lambda^4 + a\lambda^3 - b\lambda^2 + c\lambda - d = 0. \tag{3.5}$$

Clearly the roots of (3.5) are $-\alpha$, $-\beta$, $-\gamma$ and $-\delta$.
Now following the same procedure in solving $\{J_n\}$, it is not hard to see that

$$j_n = (-1)^n J_n.$$

Now, it is possible to prove the following result.

Lemma 3.1. *One has for all $n \in \mathbb{N}_0$,*

$$R_n = dJ_{n+1}R_{-3} + (cJ_{n+1} + dJ_n)\, R_{-2} + (J_{n+3} - aJ_{n+2})\, R_{-1} + J_{n+2}R_0, \tag{3.6}$$

$$S_n = (-1)^{n+1} \left[dJ_{n+1}S_{-3} - (cJ_{n+1} + dJ_n)\, S_{-2} + (J_{n+3} - aJ_{n+2})\, S_{-1} - J_{n+2}S_0 \right]. \tag{3.7}$$

Proof. Assume that α, β, γ and δ are the distinct roots of the characteristic equation (1.6), so

$$R_n = c'_1\alpha^n + c'_2\beta^n + c'_3\gamma^n + c'_4\delta^n, \; n \geq -3.$$

Using the initial values R_0, R_{-1}, R_{-2} and R_{-3}, one get

$$\begin{cases} \frac{1}{\alpha^3}c'_1 + \frac{1}{\beta^3}c'_2 + \frac{1}{\gamma^3}c'_3 + \frac{1}{\delta^3}c'_4 & = R_{-3} \\ \frac{1}{\alpha^2}c'_1 + \frac{1}{\beta^2}c'_2 + \frac{1}{\gamma^2}c'_3 + \frac{1}{\delta^2}c'_4 & = R_{-2} \\ \frac{1}{\alpha}c'_1 + \frac{1}{\beta}c'_2 + \frac{1}{\gamma}c'_3 + \frac{1}{\delta}c'_4 & = R_{-1} \\ c'_1 + c'_2 + c'_3 + c'_4 & = R_0, \end{cases} \tag{3.8}$$

after some calculations using the Cramer method one get

$$\begin{aligned}
c_1' &= \frac{\beta\gamma\delta\alpha^3}{(\delta-\alpha)(\gamma-\alpha)(\beta-\alpha)}R_{-3} - \frac{(\gamma\beta+\gamma\delta+\beta\delta)\alpha^3}{(\delta-\alpha)(\gamma-\alpha)(\beta-\alpha)}R_{-2} \\
&\quad + \frac{(\beta+\gamma+\delta)\alpha^3}{(\delta-\alpha)(\gamma-\alpha)(\beta-\alpha)}R_{-1} - \frac{\alpha^3}{(\delta-\alpha)(\gamma-\alpha)(\beta-\alpha)}R_0 \\
c_2' &= -\frac{\alpha\gamma\delta\beta^3}{(\delta-\beta)(\gamma-\beta)(\beta-\alpha)}R_{-3} + \frac{(\gamma\alpha+\gamma\delta+\alpha\delta)\beta^3}{(\delta-\beta)(\gamma-\beta)(\beta-\alpha)}R_{-2} \\
&\quad - \frac{(\alpha+\gamma+\delta)\beta^3}{(\delta-\beta)(\gamma-\beta)(\beta-\alpha)}R_{-1} + \frac{\beta^3}{(\delta-\beta)(\gamma-\beta)(\beta-\alpha)}R_0 \\
c_3' &= \frac{\alpha\beta\delta\gamma^3}{(\delta-\gamma)(\gamma-\beta)(\gamma-\alpha)}R_{-3} - \frac{(\alpha\beta+\alpha\delta+\beta\delta)\gamma^3}{(\delta-\gamma)(\gamma-\beta)(\gamma-\alpha)}R_{-2} \\
&\quad + \frac{(\alpha+\beta+\delta)\gamma^3}{(\delta-\gamma)(\gamma-\beta)(\gamma-\alpha)}R_{-1} - \frac{\gamma^3}{(\delta-\gamma)(\gamma-\beta)(\gamma-\alpha)}R_0 \\
c_4' &= -\frac{\alpha\beta\gamma\delta^3}{(\delta-\gamma)(\delta-\beta)(\delta-\alpha)}R_{-3} + \frac{(\gamma\alpha+\gamma\beta+\alpha\beta)\delta^3}{(\delta-\gamma)(\delta-\beta)(\delta-\alpha)}R_{-2} \\
&\quad - \frac{(\alpha+\beta+\gamma)\delta^3}{(\delta-\gamma)(\delta-\beta)(\delta-\alpha)}R_{-1} + \frac{\delta^3}{(\delta-\gamma)(\delta-\beta)(\delta-\alpha)}R_0
\end{aligned}$$

that is,

$$\begin{aligned}
R_n &= \Bigg(\frac{\beta\gamma\delta\alpha^3}{(\delta-\alpha)(\gamma-\alpha)(\beta-\alpha)}\alpha^n - \frac{\alpha\gamma\delta\beta^3}{(\delta-\beta)(\gamma-\beta)(\beta-\alpha)}\beta^n + \frac{\alpha\beta\delta\gamma^3}{(\delta-\gamma)(\gamma-\beta)(\gamma-\alpha)}\gamma^n \\
&\quad - \frac{\alpha\beta\gamma\delta^3}{(\delta-\gamma)(\delta-\beta)(\delta-\alpha)}\delta^n\Bigg) R_{-3} \\
&\quad + \Bigg(-\frac{(\gamma\beta+\gamma\delta+\beta\delta)\alpha^3}{(\delta-\alpha)(\gamma-\alpha)(\beta-\alpha)}\alpha^n + \frac{(\gamma\alpha+\gamma\delta+\alpha\delta)\beta^3}{(\delta-\beta)(\gamma-\beta)(\beta-\alpha)}\beta^n - \frac{(\alpha\beta+\alpha\delta+\beta\delta)\gamma^3}{(\delta-\gamma)(\gamma-\beta)(\gamma-\alpha)}\gamma^n \\
&\quad + \frac{(\gamma\alpha+\gamma\beta+\alpha\beta)\delta^3}{(\delta-\gamma)(\delta-\beta)(\delta-\alpha)}\delta^n\Bigg) R_{-2} \\
&\quad + \Bigg(\frac{(\beta+\gamma+\delta)\alpha^3}{(\delta-\alpha)(\gamma-\alpha)(\beta-\alpha)}\alpha^n - \frac{(\alpha+\gamma+\delta)\beta^3}{(\delta-\beta)(\gamma-\beta)(\beta-\alpha)}\beta^n + \frac{(\alpha+\beta+\delta)\gamma^3}{(\delta-\gamma)(\gamma-\beta)(\gamma-\alpha)}\gamma^n \\
&\quad - \frac{(\alpha+\beta+\gamma)\delta^3}{(\delta-\gamma)(\delta-\beta)(\delta-\alpha)}\delta^n\Bigg) R_{-1} \\
&\quad + \Bigg(-\frac{\alpha^3}{(\delta-\alpha)(\gamma-\alpha)(\beta-\alpha)}\alpha^n + \frac{\beta^3}{(\delta-\beta)(\gamma-\beta)(\beta-\alpha)}\beta^n - \frac{\gamma^3}{(\delta-\gamma)(\gamma-\beta)(\gamma-\alpha)}\gamma^n \\
&\quad + \frac{\delta^3}{(\delta-\gamma)(\delta-\beta)(\delta-\alpha)}\delta^n\Bigg) R_0.
\end{aligned}$$

$$R_n = dJ_{n+1}R_{-3} + (cJ_{n+1}+dJ_n)\,R_{-2} + (J_{n+3}-aJ_{n+2})\,R_{-1} + J_{n+2}R_0.$$

The proof of the other cases is similar and will be omitted.

Let $A := -a$, $B := b$, $C := -c$ and $D := d$ then, equation (3.2) takes the form (3.1) and the equation (3.3) takes the form (3.3). Then analogous to the formula of (3.1) one obtains

$$S_n = Dj_{n+1}S_{-3} + (Cj_{n+1}+Dj_n)\,S_{-2} + (j_{n+3}-Aj_{n+2})\,S_{-1} + j_{n+2}S_0.$$

Using the fact that $j_n = (-1)^n J_n$, $A = -a$ and $C := -c$ one get

$$S_n = (-1)^{n+1}\left[dJ_{n+1}S_{-3} - (cJ_{n+1}+dJ_n)\,S_{-2} + (J_{n+3}-aJ_{n+2})\,S_{-1} - J_{n+2}S_0\right].$$

□

Proof of the Main Theorem.
Replacing

$$x_n = \frac{u_n}{v_{n-1}}, \quad y_n = \frac{v_n}{u_{n-1}}, \, n \geq -2, \tag{3.9}$$

in system (1.2) one get the following linear fourth-order system of difference equations

$$u_{n+1} = av_n + bu_{n-1} + cv_{n-2} + du_{n-3}, \quad v_{n+1} = au_n + bv_{n-1} + cu_{n-2} + dv_{n-3}, \quad n \in \mathbb{N}_0, \tag{3.10}$$

where the initial values $u_{-3}, u_{-2}, u_{-1}, u_0, v_{-3}, v_{-2}, v_{-1}, v_0$ are nonzero real numbers.
From (3.10) one has for $n \in \mathbb{N}_0$,

$$\begin{cases} u_{n+1} + v_{n+1} = a(v_n + u_n) + b(u_{n-1} + v_{n-1}) + c(v_{n-2} + u_{n-2}) + d(u_{n-3} + v_{n-3}), \\ u_{n+1} - v_{n+1} = a(v_n - u_n) + b(u_{n-1} - v_{n-1}) + c(v_{n-2} - u_{n-2}) + d(u_{n-3} - v_{n-3}). \end{cases}$$

Putting again

$$R_n = u_n + v_n, \quad S_n = u_n - v_n, \, n \geq -3, \tag{3.11}$$

one obtains two fourth-order homogeneous linear difference equations:

$$R_{n+1} = aR_n + bR_{n-1} + cR_{n-2} + dR_{n-3}, \, n \in \mathbb{N}_0,$$

and

$$S_{n+1} = -aS_n + bS_{n-1} - cS_{n-2} + dS_{n-3}, \, n \in \mathbb{N}_0. \tag{3.12}$$

Using (3.11), one get for $n \geq -3$,

$$u_n = \frac{1}{2}(R_n + S_n), \; v_n = \frac{1}{2}(R_n - S_n).$$

From Lemma 3.1 one obtains,

$$\begin{cases} u_{2n-1} = & \frac{1}{2}\left[dJ_{2n}(R_{-3} + S_{-3}) + (cJ_{2n} + dJ_{2n-1})(R_{-2} - S_{-2}) + (J_{2n+2} - aJ_{2n+1})(R_{-1} + S_{-1}) \right. \\ & \left. + J_{2n+1}(R_0 - S_0)\right], n \in \mathbb{N}, \\ u_{2n} = & \frac{1}{2}\left[dJ_{2n+1}(R_{-3} - S_{-3}) + (cJ_{2n+1} + dJ_{2n})(R_{-2} + S_{-2}) + (J_{2n+3} - aJ_{2n+2})(R_{-1} - S_{-1}) \right. \\ & \left. + J_{2n+2}(R_0 + S_0)\right], n \in \mathbb{N}_0, \end{cases} \tag{3.13}$$

$$\begin{cases} v_{2n-1} = & \frac{1}{2}\left[dJ_{2n}(R_{-3} - S_{-3}) + (cJ_{2n} + dJ_{2n-1})(R_{-2} + S_{-2}) + (J_{2n+2} - aJ_{2n+1})(R_{-1} - S_{-1}) \right. \\ & \left. + J_{2n+1}(R_0 + S_0)\right], n \in \mathbb{N}, \\ v_{2n} = & \frac{1}{2}\left[dJ_{2n+1}(R_{-3} + S_{-3}) + (cJ_{2n+1} + dJ_{2n})(R_{-2} - S_{-2}) + (J_{2n+3} - aJ_{2n+2})(R_{-1} + S_{-1}) \right. \\ & \left. + J_{2n+2}(R_0 - S_0)\right], n \in \mathbb{N}_0. \end{cases} \tag{3.14}$$

Substituting (3.13) and (3.14) in (3.9), one get for $n \in \mathbb{N}_0$,

$$x_{2n+1} = \frac{dJ_{2n+2} + (cJ_{2n+2} + dJ_{2n+1})\frac{R_{-2} - S_{-2}}{R_{-3} + S_{-3}} + (J_{2n+4} - aJ_{2n+3})\frac{R_{-1} + S_{-1}}{R_{-3} + S_{-3}} + J_{2n+3}\frac{R_0 - S_0}{R_{-3} + S_{-3}}}{dJ_{2n+1} + (cJ_{2n+1} + dJ_{2n})\frac{R_{-2} - S_{-2}}{R_{-3} + S_{-3}} + (J_{2n+3} - aJ_{2n+2})\frac{R_{-1} + S_{-1}}{R_{-3} + S_{-3}} + J_{2n+2}\frac{R_0 - S_0}{R_{-3} + S_{-3}}}, \tag{3.15}$$

$$x_{2n+2}=\frac{dJ_{2n+3}+(cJ_{2n+3}+dJ_{2n+2})\dfrac{R_{-2}+S_{-2}}{R_{-3}-S_{-3}}+(J_{2n+5}-aJ_{2n+4})\dfrac{R_{-1}-S_{-1}}{R_{-3}-S_{-3}}+J_{2n+4}\dfrac{R_{0}+S_{0}}{R_{-3}-S_{-3}}}{dJ_{2n+2}+(cJ_{2n+2}+dJ_{2n+1})\dfrac{R_{-2}+S_{-2}}{R_{-3}-S_{-3}}+(J_{2n+4}-aJ_{2n+3})\dfrac{R_{-1}-S_{-1}}{R_{-3}-S_{-3}}+J_{2n+3}\dfrac{R_{0}+S_{0}}{R_{-3}-S_{-3}}}, \tag{3.16}$$

$$y_{2n+1}=\frac{dJ_{2n+2}+(cJ_{2n+2}+dJ_{2n+1})\dfrac{R_{-2}+S_{-2}}{R_{-3}-S_{-3}}+(J_{2n+4}-aJ_{2n+3})\dfrac{R_{-1}-S_{-1}}{R_{-3}-S_{-3}}+J_{2n+3}\dfrac{R_{0}+S_{0}}{R_{-3}-S_{-3}}}{dJ_{2n+1}+(cJ_{2n+1}+dJ_{2n})\dfrac{R_{-2}+S_{-2}}{R_{-3}-S_{-3}}+(J_{2n+3}-aJ_{2n+2})\dfrac{R_{-1}-S_{-1}}{R_{-3}-S_{-3}}+J_{2n+2}\dfrac{R_{0}+S_{0}}{R_{-3}-S_{-3}}}, \tag{3.17}$$

and

$$y_{2n+2}=\frac{dJ_{2n+3}+(cJ_{2n+3}+dJ_{2n+2})\dfrac{R_{-2}-S_{-2}}{R_{-3}+S_{-3}}+(J_{2n+5}-aJ_{2n+4})\dfrac{R_{-1}+S_{-1}}{R_{-3}+S_{-3}}+J_{2n+4}\dfrac{R_{0}-S_{0}}{R_{-3}+S_{-3}}}{dJ_{2n+2}+(cJ_{2n+2}+dJ_{2n+1})\dfrac{R_{-2}-S_{-2}}{R_{-3}+S_{-3}}+(J_{2n+4}-aJ_{2n+3})\dfrac{R_{-1}+S_{-1}}{R_{-3}+S_{-3}}+J_{2n+3}\dfrac{R_{0}-S_{0}}{R_{-3}+S_{-3}}}. \tag{3.18}$$

One has

$$x_{-2}=\frac{u_{-2}}{v_{-3}}=\frac{R_{-2}+S_{-2}}{R_{-3}-S_{-3}},\ x_{-1}=\frac{u_{-1}}{v_{-2}}=\frac{R_{-1}+S_{-1}}{R_{-2}-S_{-2}},\ x_{0}=\frac{u_{0}}{v_{-1}}=\frac{R_{0}+S_{0}}{R_{-1}-S_{-1}}, \tag{3.19}$$

$$y_{-2}=\frac{v_{-2}}{u_{-3}}=\frac{R_{-2}-S_{-2}}{R_{-3}+S_{-3}},\ y_{-1}=\frac{v_{-1}}{u_{-2}}=\frac{R_{-1}-S_{-1}}{R_{-2}+S_{-2}},\ y_{0}=\frac{v_{0}}{u_{-1}}=\frac{R_{0}-S_{0}}{R_{-1}+S_{-1}}. \tag{3.20}$$

From (3.19), (3.20) one get,

$$\begin{cases}\dfrac{R_{-1}+S_{-1}}{R_{-3}+S_{-3}}=\dfrac{R_{-1}+S_{-1}}{R_{-2}-S_{-2}}\times\dfrac{R_{-2}-S_{-2}}{R_{-3}+S_{-3}}=x_{-1}y_{-2},\\ \dfrac{R_{0}-S_{0}}{R_{-3}+S_{-3}}=\dfrac{R_{0}-S_{0}}{R_{-1}+S_{-1}}\times\dfrac{R_{-1}+S_{-1}}{R_{-2}-S_{-2}}\times\dfrac{R_{-2}-S_{-2}}{R_{-3}+S_{-3}}=y_0x_{-1}y_{-2},\end{cases} \tag{3.21}$$

$$\begin{cases}\dfrac{R_{-1}-S_{-1}}{R_{-3}-S_{-3}}=\dfrac{R_{-1}-S_{-1}}{R_{-2}+S_{-2}}\times\dfrac{R_{-2}+S_{-2}}{R_{-3}-S_{-3}}=y_{-1}x_{-2},\\ \dfrac{R_{0}+S_{0}}{R_{-3}-S_{-3}}=\dfrac{R_{0}+S_{0}}{R_{-1}-S_{-1}}\times\dfrac{R_{-1}-S_{-1}}{R_{-2}+S_{-2}}\times\dfrac{R_{-2}+S_{-2}}{R_{-3}-S_{-3}}=x_0y_{-1}x_{-2}.\end{cases} \tag{3.22}$$

Using (3.15), (3.16), (3.17), (3.18), (3.21) and (3.22), one obtains the closed form of the solutions of the system (1.2), that is for $n\in\mathbb{N}_0$, one has

$$\begin{cases}x_{2n+1}=\dfrac{dJ_{2n+2}+(cJ_{2n+2}+dJ_{2n+1})\,y_{-2}+(J_{2n+4}-aJ_{2n+3})\,x_{-1}y_{-2}+J_{2n+3}y_0x_{-1}y_{-2}}{dJ_{2n+1}+(cJ_{2n+1}+dJ_{2n})\,y_{-2}+(J_{2n+3}-aJ_{2n+2})\,x_{-1}y_{-2}+J_{2n+2}y_0x_{-1}y_{-2}},\\ x_{2n+2}=\dfrac{dJ_{2n+3}+(cJ_{2n+3}+dJ_{2n+2})\,x_{-2}+(J_{2n+5}-aJ_{2n+4})\,y_{-1}x_{-2}+J_{2n+4}x_0y_{-1}x_{-2}}{dJ_{2n+2}+(cJ_{2n+2}+dJ_{2n+1})\,x_{-2}+(J_{2n+4}-aJ_{2n+3})\,y_{-1}x_{-2}+J_{2n+3}x_0y_{-1}x_{-2}},\end{cases}$$

$$\begin{cases}y_{2n+1}=\dfrac{dJ_{2n+2}+(cJ_{2n+2}+dJ_{2n+1})\,x_{-2}+(J_{2n+4}-aJ_{2n+3})\,y_{-1}x_{-2}+J_{2n+3}x_0y_{-1}x_{-2}}{dJ_{2n+1}+(cJ_{2n+1}+dJ_{2n})\,x_{-2}+(J_{2n+3}-aJ_{2n+2})\,y_{-1}x_{-2}+J_{2n+2}x_0y_{-1}x_{-2}},\\ y_{2n+2}=\dfrac{dJ_{2n+3}+(cJ_{2n+3}+dJ_{2n+2})\,y_{-2}+(J_{2n+5}-aJ_{2n+4})\,x_{-1}y_{-2}+J_{2n+4}y_0x_{-1}y_{-2}}{dJ_{2n+2}+(cJ_{2n+2}+dJ_{2n+1})\,y_{-2}+(J_{2n+4}-aJ_{2n+3})\,x_{-1}y_{-2}+J_{2n+3}y_0x_{-1}y_{-2}}.\end{cases}$$

Remark 3.2. - The content of the present paper was posted on arXiv on 31.10.2019, ref. arXiv:1910.14365.

- Some parts of the results of this paper were used in the reference [2] in which the authors have generalized the system (1.2).

Acknowledgements

The authors would like to thank the two referees for their comments and suggestions. Also, the authors acknowledge the support of the Algerian General Directorate for Scientific Research and Technological Development DGRSDT.

Conflict of Interest

The authors have no conflicts of interest to declare.

References

[1] Y. Akrour, N. Touafek and Y. Halim, *On a system of difference equations of second order solved in closed form*, Miskolc Math. Notes **20**(2) (2019), 701–717.

[2] Y. Akrour, M. Kara, N. Touafek and Y. Yazlik, *Solutions formulas for some general systems of difference equations*, Miskolc Math. Notes in press (2021).

[3] A. Asiri, M. M. El-Dessoky and E. M. Elsayed, *Solution of a third order fractional system of difference equations*, J. Comput. Anal. Appl. **24**(3) (2018), 444–453.

[4] R. Azizi, *Global behaviour of the rational Riccati difference equation of order two: the general case*, J. Difference Equ. Appl. **18**(6) (2012), 947–961.

[5] F. Belhannache, N. Touafek, and R. Abo-Zeid, *Dynamics of a third-order rational difference equation*, Bull. Math. Soc. Sci. Math. Roumanie (N.S.) **59**(107) (2016), 13–22.

[6] M. M. El-Dessoky, *On a solvable for some systems of rational difference equations*, J. Nonlinear Sci. Appl. **9**(6) (2016), 3744–3759.

[7] M. M. El-Dessoky, E. M. Elsayed, E. M. Elabbasy and Asim Asiri, *Expressions of the solutions of some systems of difference equations*, J. Comput. Anal. Appl. **27**(7) (2019), 1161–1172.

[8] E. M. Elsayed, *On a system of two nonlinear difference equations of order two*, Proc. Jangjeon Math. Soc. **18**(1) (2015), 353–368.

[9] E. M. Elsayed and T. F. Ibrahim, *Periodicity and solutions for some systems of nonlinear rational difference equations*, Hacet. J. Math. Stat. **44**(1) (2015), 1361–1390.

[10] E. M. Elsayed, *Solution for systems of difference equations of rational form of order two*, Comput. Appl. Math. **33**(1) (2015), 751–765.

[11] M. Gumus, *The global asymptotic stability of a system of difference equations*, J. Difference Equ. Appl. **24**(1) (2018), 976–991.

[12] M. Gumus and O. Ocalan, *The qualitative analysis of a rational system of difference equations*, J. Fract. Calc. Appl. **9**(2) (2018), 113–126.

[13] M. Gumus and R. Abo-Zeid, *On the solutions of a (2k+2)th order difference equation*, Dyn. Contin. Discrete Impuls. Syst. Ser. B Appl. Algorithms **25**(1) (2018), 129–143.

[14] M. Gumus, *Analysis of periodicity for a new class of nonlinear difference equations by using a new method*, Electron. J. Math. Anal. Appl. **8**(1) (2020), 109–116.

[15] M. Gumus, *The periodic character in a higher order difference equation with delays*, Math. Methods Appl. Sci. **43**(3) (2020), 1112–1123.

[16] N. Haddad, N. Touafek and J. F. T. Rabago, *Solution form of a higher-order system of difference equations and dynamical behavior of its special case*, Math. Methods Appl. Sci. **40**(1) (2017), 3599–3607.

[17] N. Haddad, N. Touafek and E. M. Elsayed, *A note on a system of difference equations*, An. Ştiinţ. Univ. Al. I. Cuza Iaşi, Ser. Nouă, Mat.(N.S.) **63**(3) (2017), 599–606.

[18] N. Haddad, N. Touafek and J. F. T. Rabago, *Well-defined solutions of a system of difference equations*, J. Appl. Math. Comput. **56**(1) (2018), 439–458.

[19] Y. Halim and J. F. T. Rabago, *On the solutions of a second-order difference equation in terms of generalized Padovan sequences*, Math. Slovaca. **68**(3) (2018), 625–638.

[20] Y. Halim and J. F. T. Rabago, *On some solvable systems of difference equations with solutions associated to Fibonacci numbers*, Electron. J. Math. Anal. Appl. **5**(1) (2017), 166–178.

[21] Y. Halim, *A system of difference equations with solutions associated to Fibonacci numbers*, Int. J. Differ. Equ. **11**(1) (2016), 65–77.

[22] Y. Halim and M. Bayram, *On the solutions of a higher-order difference equation in terms of generalized Fibonacci sequences*, Math. Methods Appl. Sci. **39**(1) (2016), 2974–2982.

[23] Y. Halim, N. Touafek and E. M. Elsayed, *Closed form solution of some systems of rational difference equations in terms of Fibonacci numbers*, Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal. **21**(6) (2014), 473–486.

[24] Y. Halim, *Global character of systems of rational difference equations*, Electron. J. Math. Anal. Appl. **3**(1) (2016), 204–214.

[25] G. S. Hathiwala and D. V. Shah, *Binet-type formula for the sequence of Tetranacci numbers by alternate methods*, J. Interdiscip. Math. **6**(1) (2017), 37–48.

[26] M. Kara and Y. Yazlik, *Solvability of a system of nonlinear difference equations of higher order*, Turkish J. Math. **43**(3) (2019), 1533–1565.

[27] M. Kara and Y. Yazlik, *Solvability of a nonlinear three-dimensional system of difference equations with constant coefficients*, Math. Slovaca. **71**(5)(2021), 1133–1148.

[28] H. Matsunaga and R. Suzuki, *Classification of global behavior of a system of rational difference equations*, Appl. Math. Lett. **85**(1) (2018), 57–63.

[29] S. Stevic, *Some representations of the general solution to a difference equation of additive type*, Adv. Differ. Equ. **2019**(431) (2019), 1–19.

[30] S. Stevic, B. Iricanin and W. Kosmala, *Representations of general solutions to some classes of nonlinear difference equations*, Adv. Differ. Equ. **2019** (73) (2019), 1–21.

[31] S. Stevic, B. Iricanin, W. Kosmala and Z. Smarda, *Note on the bilinear difference equation with a delay*, Math. Methods Appl. Sci. **41**(18) (2018), 9349–9360.

[32] S. Stevic, B. Iricanin, W. Kosmala and Z. Smarda, *Representation of solutions of a solvable nonlinear difference equation of second order*, Electron. J. Qual. theory Differ. Equ. **2015** (95) (2018), 1–18.

[33] S. Stevic, *Representation of solutions of bilinear difference equations in terms of generalized Fibonacci sequences*, Electron. J. Qual. theory Differ. Equ. **2014**(67) (2014), 1–15.

[34] D. T. Tollu, Y. Yazlik and N. Taskara, *On the solutions of two special types of Riccati difference equation via Fibonacci numbers*, Adv. Differ. Equ. **2013**(174) (2013), 1–7.

[35] D. T. Tollu, Y. Yazlik and N. Taskara, *The solutions of four Riccati difference equations associated with Fibonacci numbers*, Balkan J. Math. **2**(1) (2014), 163–172.

[36] D. T. Tollu, Y. Yazlik and N. Taskara, *On fourteen solvable systems of difference equations*, Appl. Math. Comput. **233**(1) (2014), 310–319.

[37] N. Touafek, *On some fractional systems of difference equations*, Iran. J. Math. Sci. Inform. **9**(2) (2014), 303–305.

[38] N. Touafek, Y. Halim, *On max type difference equations: expressions of solutions*, Int. j. appl. nonlinear sci. **11**(4) (2011), 396–402.

[39] N. Touafek, E.M. Elsayed, *On the periodicity of some systems of nonlinear difference equations*, Bull. math. Soc. Sci. Math. Roum. **55**(1) (2012), 217–224.

[40] N. Touafek, E.M. Elsayed, *On the solutions of systems of rational difference equations*, Math. Comput. Model. **55**(1) (2012), 1987–1997.

[41] C. Wang, X. Fang and R. Li, *On the solution for a system of two rational difference equations*, J. Comput. Anal. Appl. **20**(1) (2016), 175–186.

[42] C. Wang, X. Fang and R. Li, *On the dynamics of a certain four-order fractional difference equations*, J. Comput. Anal. Appl. **22**(5) (2017), 968–976.

[43] C. Wang, X. Fang and R. Li, *On the periodicity of a max-type rational difference equation*, J. Nonlinear Sci. Appl. **10**(9) (2017), 4648–4661.

[44] C. Wang, Y. Zhou, S. Pan, R. Li, *On a system of three max-type nonlinear difference equations*, J. Comput. Anal. Appl. **25**(8) (2018), 1463–1479.

[45] Y. Yazlik, D. T. Tollu, N. Taskara, *On the solutions of difference equation systems with Padovan numbers*, Appl. Math. **4** (2013), 15–20.

[46] Y. Yazlik, M. Kara, *On a solvable system of difference equations of higher-order with period two coefficients*, Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat. **68**(1) (2019), 1675–1693.

[47] M. E. Waddill, *The Tetranacci sequence and generalizations*, Fibonacci Quart. **30**(1) (1992), 9–20.

Journal of Innovative Applied Mathematics and Computational Sciences

J. Innov. Appl. Math. Comput. Sci. **1**(1) (2021) 16–30. n2t.net/ark:/49935/jiamcs.v1i1.7

JIAMCS MILA UNIVERSITY CENTER

http://jiamcs.centre-univ-mila.dz/

A study on the sum of the squares of generalized balancing numbers: the sum formula $\sum_{k=0}^{n} x^k W_{mk+j}^2$

Yüksel Soykan [1], Erkan Taşdemir ✉ [2] and Can Murat Dikmen [1]

[1]Zonguldak Bülent Ecevit University, Department of Mathematics, Art and Science Faculty, 67100, Zonguldak, Turkey
[2]Kırklareli University, Pınarhisar Vocational School, 39300, Kırklareli, Turkey

Received 16 November 2021, Accepted 24 December 2021, Published 27 December 2021

Abstract. In this paper, closed forms of the sum formulas $\sum_{k=0}^{n} x^k W_{mk+j}^2$ for generalized balancing numbers are presented. As special cases, we give sum formulas of balancing, modified Lucas-balancing and Lucas-balancing numbers.

Keywords: Balancing numbers, modified Lucas-balancing numbers, Lucas-balancing numbers, sum formulas.

2020 Mathematics Subject Classification: 11B37, 11B39, 11B83.

1 Introduction

A generalized balancing sequence $\{W_n\}_{n\geq 0} = \{W_n(W_0, W_1)\}_{n\geq 0}$ is defined by the second-order recurrence relation

$$W_n = 6W_{n-1} - W_{n-2} \tag{1.1}$$

with the initial values $W_0 = c_0, W_1 = c_1$ not all being zero.

The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = 6W_{-(n-1)} - W_{-(n-2)}$$

for $n = 1, 2, 3, ...$ Therefore, recurrence (1.1) holds for all integer n.

The Binet formula of generalized balancing numbers can be written as

$$W_n = \frac{W_1 - \beta W_0}{\alpha - \beta}\alpha^n - \frac{W_1 - \alpha W_0}{\alpha - \beta}\beta^n,$$

where α and β are the roots of the quadratic equation $x^2 - 6x + 1 = 0$. Moreover

$$\alpha = 3 + 2\sqrt{2},$$
$$\beta = 3 - 2\sqrt{2}.$$

✉ Corresponding author. Email: erkantasdemir@hotmail.com

ISSN (electronic): 2773-4196

Note that

$$\begin{aligned} \alpha+\beta &= 6, \\ \alpha\beta &= 1, \\ \alpha-\beta &= 4\sqrt{2}. \end{aligned}$$

Now, one defines three special cases of the sequence $\{W_n\}$. Balancing sequence $\{B_n\}_{n\geq 0}$, modified Lucas-balancing sequence $\{H_n\}_{n\geq 0}$ and Lucas-balancing sequence $\{C_n\}_{n\geq 0}$ are defined, respectively, by the second-order recurrence relations

$$\begin{aligned} B_n &= 6B_{n-1}-B_{n-2}, \quad B_0=0, B_1=1, \qquad (1.2)\\ H_n &= 6H_{n-1}-H_{n-2}, \quad H_0=2, H_1=6, \qquad (1.3)\\ C_n &= 6C_{n-1}-C_{n-2}, \quad C_0=1, C_1=3. \qquad (1.4) \end{aligned}$$

The sequences $\{B_n\}_{n\geq 0}$, $\{H_n\}_{n\geq 0}$ and $\{C_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\begin{aligned} B_{-n} &= 6B_{-(n-1)}-B_{-(n-2)}, \\ H_{-n} &= 6H_{-(n-1)}-H_{-(n-2)}, \\ C_{-n} &= 6C_{-(n-1)}-C_{-(n-2)}, \end{aligned}$$

for $n=1,2,3,...$ respectively. Therefore, recurrences (1.2)-(1.4) hold for all integer n. For more information on generalized balancing numbers, see Soykan [29].

In [1], Behera and Panda defined balancing numbers n as solutions of the diophantine equation

$$1+2+\cdots+(n-1)=(n+1)+(n+2)+\cdots+(n+r)$$

for some natural number r, called the balancer corresponding to n. The nth balancing number is denoted by B_n. Moreover, $C_n=\sqrt{8B_n^2+1}$ is called the nth Lucas-balancing number (see [16]). In fact, B_n and C_n satisfy the second order linear recurrence relations (1.2) and (1.4) respectively. $(B_n)_{n\geq 0}$ is the sequence $A001109$ in the OEIS [27], whereas $(B_n)_{n\geq 0}$ is the idnumber $A001541$ in OEIS. Balancing and Lucas-balancing sequences have been studied by many authors and more detail can be found in the extensive literature dedicated to these sequences, see for example, [1–4,9–26].

2 The Sum Formula $\sum_{k=0}^{n} x^k W_{mk+j}^2$

The following theorem presents sum formulas of generalized balancing numbers.

Theorem 2.1. *Let x be a real (or complex) number. For all integers m and j, for generalized balancing numbers (the case $r=6,\ s=-1$), the following sum formulas hold:*

(a) *If $(1+(-s)^{2m}x^2-xH_{2m})((-s)^m x-1)\neq 0$, then*

$$\sum_{k=0}^{n} x^k W_{mk+j}^2=\frac{\Omega_1}{32(1+x^2-xH_{2m})(x-1)}, \qquad (2.1)$$

where

$\Omega_1 = 32(x-1)(x-H_{2m})x^{n+1}W_{mn+j}^2+32(x-1)x^{n+1}W_{mn-m+j}^2+32(x-1)W_j^2-32(x-1)xW_{j-m}^2+2(W_1^2+W_0^2-6W_1W_0)(x^n-1)(H_{2m}-2)x.$

(b) *If* $(1+(-s)^{2m}x^2-xH_{2m})((-s)^m x-1)=u(x-a)(x-b)(x-c)=0$ *for some* $u,a,b,c\in\mathbb{C}$ *with* $u\neq 0$ *and* $a\neq b\neq c$*, i.e.,* $x=a$ *or* $x=b$ *or* $x=c$*, then*

$$\sum_{k=0}^{n} x^k W_{mk+j}^2=\frac{\Omega_2}{32(3x^2-2(H_{2m}+1)x+H_{2m}+1)},$$

where

$\Omega_2=32((x-H_{2m})x^{n+1}+(x-1)((n+2)x-(n+1)H_{2m})x^nW_{mn+j}^2+32((n+2)x-(n+1))x^n)W_{mn-m+j}^2+32W_j^2-32(2x-1)W_{j-m}^2+2(W_1^2+W_0^2-6W_1W_0)(x^n(n+1)-1)(H_{2m}-2).$

(c) *If* $(1+(-s)^{2m}x^2-xH_{2m})((-s)^m x-1)=u(x-a)^2(x-c)=0$ *for some* $u,a,c\in\mathbb{C}$ *with* $u\neq 0$*,* $a\neq c$*, when* $x=c$*, then*

$$\sum_{k=0}^{n} x^k W_{mk+j}^2=\frac{\Omega_3}{32(3x^2-2(H_{2m}+1)x+H_{2m}+1)},$$

where

$\Omega_3=32((x-H_{2m})x^{n+1}+(x-1)((n+2)x-(n+1)H_{2m})x^nW_{mn+j}^2+32((n+2)x-(n+1))x^n)W_{mn-m+j}^2+32W_j^2-32(2x-1)W_{j-m}^2+2(W_1^2+W_0^2-6W_1W_0)(x^n(n+1)-1)(H_{2m}-2),$

and when $x=a$*, then*

$$\sum_{k=0}^{n} x^k W_{mk+j}^2=\frac{\Omega_4}{64(3x-1-H_{2m})},$$

where

$\Omega_4=32((n+3)(n+2)x^2-x(n+2)(n+1)(H_{2m}+1)+n(n+1)H_{2m})x^{n-1}W_{mn+j}^2+32(n+1)((2+n)x^n-nx^{n-1})W_{mn-m+j}^2-64W_{j-m}^2+2n(n+1)(W_1^2+W_0^2-6W_1W_0)(H_{2m}-2)x^{n-1}.$

(d) *If* $(1+(-s)^{2m}x^2-xH_{2m})((-s)^m x-1)=u(x-a)^3=0$ *for some* $u,a\in\mathbb{C}$ *with* $u\neq 0$*, i.e.,* $x=a$*, then*

$$\sum_{k=0}^{n} x^k W_{mk+j}^2=\frac{\Omega_5}{192},$$

where

$\Omega_5=32(n+1)((n+3)(n+2)x^2-n(n+2)(H_{2m}+1)x+n(n-1)H_{2m})x^{n-2}W_{mn+j}^2+32n(n+1)((n+2)x+1-n)x^{n-2}W_{mn-m+j}^2+2(n-1)n(n+1)(H_{2m}-2)(W_1^2+W_0^2-6W_1W_0)x^{n-2}.$

Proof. Take $r=6,\ s=-1$ in Soykan [28], Theorem 2.1.. □

Note that (2.1) can be written in the following form

$$\sum_{k=1}^{n} x^k W_{mk+j}^2=\frac{\Omega_6}{32(1+x^2-xH_{2m})(x-1)},$$

where

$\Omega_6=32(x-1)(x-H_{2m})x^{n+1}W_{mn+j}^2+32((-s)^m x-1)x^{n+1}W_{mn-m+j}^2-32(x-1)(x-H_{2m})x W_j^2-32(x-1)W_{j-m}^2x+2(W_1^2+W_0^2-6W_1W_0)(x^n-1)(H_{2m}-2)x.$

As special cases of m and j in the last theorem, one obtains the following proposition.

Proposition 2.2. *For generalized balancing numbers (the case $r = 6, s = -1$) one has the following sum formulas for $n \geq 0$:*

(a) $(m = 1, j = 0)$

If $(x-1)\left(x^2 - 34x + 1\right) \neq 0$, *i.e.,* $x \neq 1$, $x \neq 17 - 12\sqrt{2}$, $x \neq 17 + 12\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k W_k^2 = \frac{\Delta}{(x-1)\left(x^2 - 34x + 1\right)},$$

where

$\Delta = (x-1)(x-34)x^{n+1}W_n^2 + (x-1)x^{n+1}W_{n-1}^2 + (x-1)W_0^2 - (x-1)x(W_1 - 6W_0)^2 + 2(W_1^2 + W_0^2 - 6W_1W_0)(x^n - 1)x,$

and

if $(x-1)\left(x^2 - 34x + 1\right) = 0$, *i.e.,* $x = 1$ *or* $x = 17 - 12\sqrt{2}$ *or* $x = 17 + 12\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k W_k^2 = \frac{\Psi}{(3x^2 - 70x + 35)},$$

where

$\Psi = ((x-34)x^{n+1} + (x-1)((n+2)x - 34(n+1))x^n W_n^2 + ((n+2)x - (n+1))x^n W_{n-1}^2 + W_0^2 - (2x-1)(W_1 - 6W_0)^2 + 2(W_1^2 + W_0^2 - 6W_1W_0)(x^n(n+1) - 1)$

(b) $(m = 2, j = 0)$

If $(x-1)\left(x^2 - 1154x + 1\right) \neq 0$, *i.e.,* $x \neq 1$, $x \neq 577 - 408\sqrt{2}$, $x \neq 577 + 408\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k W_{2k}^2 = \frac{\Delta}{(x-1)\left(x^2 - 1154x + 1\right)},$$

where

$\Delta = (x-1)(x-1154)x^{n+1}W_{2n}^2 + (x-1)x^{n+1}W_{2n-2}^2 + (x-1)W_0^2 - (x-1)x(6W_1 - 35W_0)^2 + 72(W_1^2 + W_0^2 - 6W_1W_0)(x^n - 1)x,$

and

if $(x-1)\left(x^2 - 1154x + 1\right) = 0$, *i.e.,* $x = 1$ *or* $x = 577 - 408\sqrt{2}$ *or* $x = 577 + 408\sqrt{2}$ *then*

$$\sum_{k=0}^{n} x^k W_{2k}^2 = \frac{\Psi}{3\left(x^2 - 770x + 385\right)},$$

where

$\Psi = ((x-1154)x^{n+1} + (x-1)((n+2)x - 1154(n+1))x^n)W_{2n}^2 + ((n+2)x - (n+1))x^n W_{2n-2}^2 + W_0^2 - (2x-1)(6W_1 - 35W_0)^2 + 72(W_1^2 + W_0^2 - 6W_1W_0)(x^n(n+1) - 1)$

(c) $(m = 2, j = 1)$

If $(x-1)\left(x^2 - 1154x + 1\right) \neq 0$, *i.e.,* $x \neq 1$, $x \neq 577 - 408\sqrt{2}$, $x \neq 577 + 408\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k W_{2k+1}^2 = \frac{\Delta}{(x-1)\left(x^2 - 1154x + 1\right)},$$

where

$\Delta = (x-1)(x-1154)x^{n+1}W_{2n+1}^2 + (x-1)x^{n+1}W_{2n-1}^2 + (x-1)W_1^2 - (x-1)x\,(W_1 - 6W_0)^2 + 72(W_1^2 + W_0^2 - 6W_1W_0)(x^n - 1)x,$

and

if $(x-1)\left(x^2 - 1154x + 1\right) = 0$, *i.e.,* $x = 1$ *or* $x = 577 - 408\sqrt{2}$ *or* $x = 577 + 408\sqrt{2}$ *then*

$$\sum_{k=0}^{n} x^k W_{2k+1}^2 = \frac{\Psi}{3\,(x^2 - 770x + 385)},$$

where

$\Psi = ((x-1154)x^{n+1} + (x-1)((n+2)\,x - 1154(n+1))x^n)W_{2n+1}^2 + ((n+2)x - (n+1))x^n W_{2n-1}^2 + W_1^2 - (2x-1)(W_1 - 6W_0)^2 + 72(W_1^2 + W_0^2 - 6W_1W_0)(x^n(n+1) - 1)$

(d) $(m = -1, j = 0)$

If $(x-1)\left(x^2 - 34x + 1\right) \neq 0$, *i.e.,* $x \neq 1$, $x \neq 17 - 12\sqrt{2}$, $x \neq 17 + 12\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k W_{-k}^2 = \frac{\Delta}{(x-1)\,(x^2 - 34x + 1)},$$

where

$\Delta = (x-1)x^{n+1}W_{-n+1}^2 + (x-1)(x-34)x^{n+1}W_{-n}^2 + (x-1)W_0^2 - (x-1)xW_1^2 + 2(W_1^2 + W_0^2 - 6W_1W_0)(x^n - 1)x,$

and

if $(x-1)\left(x^2 - 34x + 1\right) = 0$, *i.e.,* $x = 1$ *or* $x = 17 - 12\sqrt{2}$ *or* $x = 17 + 12\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k W_{-k}^2 = \frac{\Psi}{(3x^2 - 70x + 35)},$$

where

$\Psi = ((n+2)x - (n+1))x^n W_{-n+1}^2 + ((x-34)x^{n+1} + (x-1)((n+2)\,x - 34(n+1))x^n)W_{-n}^2 + W_0^2 - (2x-1)W_1^2 + 2(W_1^2 + W_0^2 - 6W_1W_0)(x^n(n+1) - 1)$

(e) $(m = -2, j = 0)$

If $(x-1)\left(x^2 - 1154x + 1\right) \neq 0$, *i.e.,* $x \neq 1$, $x \neq 577 - 408\sqrt{2}$, $x \neq 577 + 408\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k W_{-2k}^2 = \frac{\Delta}{(x-1)\,(x^2 - 1154x + 1)},$$

where

$\Delta = (x-1)x^{n+1}W_{-2n+2}^2 + (x-1)(x-1154)x^{n+1}W_{-2n}^2 + (x-1)W_0^2 - (x-1)x(W_0 - 6W_1)^2 + 72(W_1^2 + W_0^2 - 6W_1W_0)(x^n - 1)x,$

and

if $(x-1)\left(x^2 - 1154x + 1\right) = 0$, *i.e.,* $x = 1$ *or* $x = 577 - 408\sqrt{2}$ *or* $x = 577 + 408\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k W_{-2k}^2 = \frac{\Psi}{3\,(x^2 - 770x + 385)},$$

where

$\Psi = ((n+2)x - (n+1))x^n W_{-2n+2}^2 + ((x-1154)x^{n+1} + (x-1)((n+2)\,x - 1154(n+1))x^n)W_{-2n}^2 + W_0^2 - (2x-1)(W_0 - 6W_1)^2 + 72(W_1^2 + W_0^2 - 6W_1W_0)(x^n(n+1) - 1)$

(f) $(m = -2, j = 1)$

If $(x-1)\left(x^2 - 1154x + 1\right) \neq 0$, *i.e.,* $x \neq 1$, $x \neq 577 - 408\sqrt{2}$, $x \neq 577 + 408\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k W_{-2k+1}^2 = \frac{\Delta}{(x-1)\left(x^2 - 1154x + 1\right)},$$

where

$\Delta = (x-1)x^{n+1}W_{-2n+3}^2 + (x-1)(x-1154)x^{n+1}W_{-2n+1}^2 + (x-1)W_1^2 - (x-1)x(6W_0 - 35W_1)^2 + 72(W_1^2 + W_0^2 - 6W_1W_0)(x^n - 1)x,$

and

if $(x-1)\left(x^2 - 1154x + 1\right) = 0$, *i.e.,* $x = 1$ *or* $x = 577 - 408\sqrt{2}$ *or* $x = 577 + 408\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k W_{-2k+1}^2 = \frac{\Psi}{3\left(x^2 - 770x + 385\right)},$$

where

$\Psi = ((n+2)x - (n+1))x^n W_{-2n+3}^2 + ((x-1154)x^{n+1} + (x-1)((n+2)x - 1154(n+1))x^n)W_{-2n+1}^2 + W_1^2 - (2x-1)(6W_0 - 35W_1)^2 + 72(W_1^2 + W_0^2 - 6W_1W_0)(x^n(n+1) - 1)$

From the above proposition, one has the following corollary, which gives sum formulas of balancing numbers (take $W_n = B_n$ with $B_0 = 0$, $B_1 = 1$).

Corollary 2.3. *For* $n \geq 0$, *balancing numbers have the following properties:*

(a) $(m = 1, j = 0)$

If $(x-1)\left(x^2 - 34x + 1\right) \neq 0$, *i.e.,* $x \neq 1$, $x \neq 17 - 12\sqrt{2}$, $x \neq 17 + 12\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k B_k^2 = \frac{(x-1)(x-34)x^{n+1}B_n^2 + (x-1)x^{n+1}B_{n-1}^2 + x(2x^n - x - 1)}{(x-1)\left(x^2 - 34x + 1\right)},$$

and

if $(x-1)\left(x^2 - 34x + 1\right) = 0$, *i.e.,* $x = 1$ *or* $x = 17 - 12\sqrt{2}$ *or* $x = 17 + 12\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k B_k^2 = \frac{\Theta_1}{\left(3x^2 - 70x + 35\right)},$$

where

$\Theta_1 = ((x-34)x^{n+1} + (x-1)((n+2)x - 34(n+1))x^n)B_n^2 + ((n+2)x - (n+1))x^n B_{n-1}^2 + 2(n+1)x^n - 2x - 1.$

(b) $(m = 2, j = 0)$

If $(x-1)\left(x^2 - 1154x + 1\right) \neq 0$, *i.e.,* $x \neq 1$, $x \neq 577 - 408\sqrt{2}$, $x \neq 577 + 408\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k B_{2k}^2 = \frac{(x-1)(x-1154)x^{n+1}B_{2n}^2 + (x-1)x^{n+1}B_{2n-2}^2 - 36x(-2x^n + x + 1)}{(x-1)\left(x^2 - 1154x + 1\right)}$$

and

if $(x-1)\left(x^2-1154x+1\right)=0$, *i.e.,* $x=1$ *or* $x=577-408\sqrt{2}$ *or* $x=577+408\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k B_{2k}^2 = \frac{\Theta_2}{3\left(x^2-770x+385\right)},$$

where

$\Theta_2 = ((x-1154)x^{n+1}+(x-1)((n+2)x-1154(n+1))x^n)B_{2n}^2+((n+2)x-(n+1))x^nB_{2n-2}^2+36(2(n+1)x^n-2x-1)$.

(c) $(m=2, j=1)$

If $(x-1)\left(x^2-1154x+1\right)\neq 0$, *i.e.,* $x\neq 1$, $x\neq 577-408\sqrt{2}$, $x\neq 577+408\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k B_{2k+1}^2 = \frac{(x-1)(x-1154)x^{n+1}B_{2n+1}^2+(x-1)x^{n+1}B_{2n-1}^2-(-72x^{n+1}+x^2+70x+1)}{(x-1)\left(x^2-1154x+1\right)},$$

and

if $(x-1)\left(x^2-1154x+1\right)=0$, *i.e.,* $x=1$ *or* $x=577-408\sqrt{2}$ *or* $x=577+408\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k B_{2k+1}^2 = \frac{\Theta_3}{3\left(x^2-770x+385\right)},$$

where

$\Theta_3 = ((x-1154)x^{n+1}+(x-1)((n+2)x-1154(n+1))x^n)B_{2n+1}^2+((n+2)x-(n+1))x^nB_{2n-1}^2+2(36(n+1)x^n-x-35)$.

(d) $(m=-1, j=0)$

If $(x-1)\left(x^2-34x+1\right)\neq 0$, *i.e.,* $x\neq 1$, $x\neq 17-12\sqrt{2}$, $x\neq 17+12\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k B_{-k}^2 = \frac{(x-1)x^{n+1}B_{-n+1}^2+(x-1)(x-34)x^{n+1}B_{-n}^2+x(2x^n-x-1)}{(x-1)\left(x^2-34x+1\right)},$$

and

if $(x-1)\left(x^2-34x+1\right)=0$, *i.e.,* $x=1$ *or* $x=17-12\sqrt{2}$ *or* $x=17+12\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k B_{-k}^2 = \frac{\Theta_4}{(3x^2-70x+35)}$$

where

$\Theta_4 = ((n+2)x-(n+1))x^nB_{-n+1}^2+((x-34)x^{n+1}+(x-1)((n+2)x-34(n+1))x^n)B_{-n}^2+2(n+1)x^n-2x-1$.

(e) $(m=-2, j=0)$

If $(x-1)\left(x^2-1154x+1\right)\neq 0$, *i.e.,* $x\neq 1$, $x\neq 577-408\sqrt{2}$, $x\neq 577+408\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k B_{-2k}^2 = \frac{(x-1)x^{n+1}B_{-2n+2}^2+(x-1)(x-1154)x^{n+1}B_{-2n}^2-36x(-2x^n+x+1)}{(x-1)\left(x^2-1154x+1\right)}$$

and

if $(x-1)\left(x^2-1154x+1\right)=0$, *i.e.*, $x=1$ *or* $x=577-408\sqrt{2}$ *or* $x=577+408\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k B_{-2k}^2 = \frac{\Theta_5}{3\left(x^2-770x+385\right)},$$

where

$\Theta_5 = ((n+2)x-(n+1))x^n B_{-2n+2}^2 + ((x-1154)x^{n+1} + (x-1)((n+2)x-1154(n+1))x^n)B_{-2n}^2 + 36(2(n+1)x^n - 2x - 1).$

(f) $(m=-2, j=1)$

If $(x-1)\left(x^2-1154x+1\right)\neq 0$, *i.e.*, $x\neq 1$, $x\neq 577-408\sqrt{2}$, $x\neq 577+408\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k B_{-2k+1}^2 = \frac{(x-1)x^{n+1}B_{-2n+3}^2 + (x-1)(x-1154)x^{n+1}B_{-2n+1}^2 + (72x^{n+1}-1225x^2+1154x-1)}{(x-1)\left(x^2-1154x+1\right)},$$

and

if $(x-1)\left(x^2-1154x+1\right)=0$, *i.e.*, $x=1$ *or* $x=577-408\sqrt{2}$ *or* $x=577+408\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k B_{-2k+1}^2 = \frac{\Theta_6}{3\left(x^2-770x+385\right)},$$

where

$\Theta_6 = ((n+2)x-(n+1))x^n B_{-2n+3}^2 + ((x-1154)x^{n+1} + (x-1)((n+2)x-1154(n+1))x^n)B_{-2n+1}^2 + 2(36(n+1)x^n - 1225x + 577).$

Taking $W_n = H_n$ with $H_0 = 2, H_1 = 6$ in the last proposition, one has the following corollary, which presents sum formulas of modified Lucas-balancing numbers.

Corollary 2.4. *For* $n\geq 0$, *modified Lucas-balancing numbers have the following properties:*

(a) $(m=1, j=0)$

If $(x-1)\left(x^2-34x+1\right)\neq 0$, *i.e.*, $x\neq 1$, $x\neq 17-12\sqrt{2}$, $x\neq 17+12\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k H_k^2 = \frac{(x-1)(x-34)x^{n+1}H_n^2 + (x-1)x^{n+1}H_{n-1}^2 - 4(16x^{n+1}+9x^2-26x+1)}{(x-1)\left(x^2-34x+1\right)},$$

and

if $(x-1)\left(x^2-34x+1\right)=0$, *i.e.*, $x=1$ *or* $x=17-12\sqrt{2}$ *or* $x=17+12\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k H_k^2 = \frac{\Theta_7}{\left(3x^2-70x+35\right)},$$

where

$\Theta_7 = ((x-34)x^{n+1} + (x-1)((n+2)x-34(n+1))x^n)H_n^2 + ((n+2)x-(n+1))x^n)H_{n-1}^2 - 8(8(n+1)x^n + 9x - 13).$

(b) $(m=2, j=0)$

If $(x-1)\left(x^2-1154x+1\right)\neq 0$, *i.e.*, $x\neq 1$, $x\neq 577-408\sqrt{2}$, $x\neq 577+408\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k H_{2k}^2 = \frac{(x-1)(x-1154)x^{n+1}H_{2n}^2+(x-1)x^{n+1}H_{2n-2}^2-4(576x^{n+1}+289x^2-866x+1)}{(x-1)\left(x^2-1154x+1\right)},$$

and

if $(x-1)\left(x^2-1154x+1\right)=0$, *i.e.*, $x=1$ *or* $x=577-408\sqrt{2}$ *or* $x=577+408\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k H_{2k}^2 = \frac{\Theta_8}{3\left(x^2-770x+385\right)},$$

where

$\Theta_8 = ((x-1154)x^{n+1}+(x-1)((n+2)x-1154(n+1))x^n)H_{2n}^2+((n+2)x-(n+1))x^nH_{2n-2}^2-8\left(288(n+1)x^n+289x-433\right))$.

(c) $(m=2, j=1)$

If $(x-1)\left(x^2-1154x+1\right)\neq 0$, *i.e.*, $x\neq 1$, $x\neq 577-408\sqrt{2}$, $x\neq 577+408\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k H_{2k+1}^2 = \frac{(x-1)(x-1154)x^{n+1}H_{2n+1}^2+(x-1)x^{n+1}H_{2n-1}^2-36(64x^{n+1}+x^2-66x+1)}{(x-1)\left(x^2-1154x+1\right)},$$

and

if $(x-1)\left(x^2-1154x+1\right)=0$, *i.e.*, $x=1$ *or* $x=577-408\sqrt{2}$ *or* $x=577+408\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k H_{2k+1}^2 = \frac{\Theta_9}{3\left(x^2-770x+385\right)},$$

where

$\Theta_9 = ((x-1154)x^{n+1}+(x-1)((n+2)x-1154(n+1))x^n)H_{2n+1}^2+((n+2)x-(n+1))x^nH_{2n-1}^2-72(32(n+1)x^n+x-33)$.

(d) $(m=-1, j=0)$

If $(x-1)\left(x^2-34x+1\right)\neq 0$, *i.e.*, $x\neq 1$, $x\neq 17-12\sqrt{2}$, $x\neq 17+12\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k H_{-k}^2 = \frac{(x-1)x^{n+1}H_{-n+1}^2+(x-1)(x-34)x^{n+1}H_{-n}^2-4(16x^{n+1}+9x^2-26x+1)}{(x-1)\left(x^2-34x+1\right)},$$

and

if $(x-1)\left(x^2-34x+1\right)=0$, *i.e.*, $x=1$ *or* $x=17-12\sqrt{2}$ *or* $x=17+12\sqrt{2}$ *then*

$$\sum_{k=0}^{n} x^k H_{-k}^2 = \frac{\Theta_{10}}{(3x^2-70x+35)},$$

where

$\Theta_{10} = ((n+2)x-(n+1))x^nH_{-n+1}^2+((x-34)x^{n+1}+(x-1)((n+2)x-34(n+1))x^n)H_{-n}^2-8(8(n+1)x^n+9x-13)$.

(e) $(m=-2, j=0)$

If $(x-1)\left(x^2-1154x+1\right)\neq 0$, *i.e.,* $x\neq 1$, $x\neq 577-408\sqrt{2}$, $x\neq 577+408\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k H_{-2k}^2 = \frac{(x-1)x^{n+1}H_{-2n+2}^2+(x-1)(x-1154)x^{n+1}H_{-2n}^2-4(576x^{n+1}+289x^2-866x+1)}{(x-1)\left(x^2-1154x+1\right)},$$

and

if $(x-1)\left(x^2-1154x+1\right)=0$, *i.e.,* $x=1$ *or* $x=577-408\sqrt{2}$ *or* $x=577+408\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k H_{-2k}^2 = \frac{\Theta_{11}}{3\left(x^2-770x+385\right)},$$

where

$\Theta_{11}=((n+2)x-(n+1))x^nH_{-2n+2}^2+((x-1154)x^{n+1}+(x-1)((n+2)x-1154(n+1))x^n)H_{-2n}^2-8(288(n+1)x^n+289x-433)$.

(f) $(m=-2, j=1)$

If $(x-1)\left(x^2-1154x+1\right)\neq 0$, *i.e.,* $x\neq 1$, $x\neq 577-408\sqrt{2}$, $x\neq 577+408\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k H_{-2k+1}^2 = \frac{(x-1)x^{n+1}H_{-2n+3}^2+(x-1)(x-1154)x^{n+1}H_{-2n+1}^2-36(64x^{n+1}+1089x^2-1154x+1)}{(x-1)\left(x^2-1154x+1\right)}$$

and

if $(x-1)\left(x^2-1154x+1\right)=0$, *i.e.,* $x=1$ *or* $x=577-408\sqrt{2}$ *or* $x=577+408\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k H_{-2k+1}^2 = \frac{\Theta_{12}}{3\left(x^2-770x+385\right)},$$

where

$\Theta_{12}=((n+2)x-(n+1))x^nH_{-2n+3}^2+((x-1154)x^{n+1}+(x-1)((n+2)x-1154(n+1))x^n)H_{-2n+1}^2-72(32(n+1)x^n+1089x-577)$.

From the above proposition, one has the following corollary, which gives sum formulas of Lucas-balancing numbers (take $W_n=C_n$ with $C_0=1, C_1=3$).

Corollary 2.5. *For* $n\geq 0$, *Lucas-balancing numbers have the following properties:*

(a) $(m=1, j=0)$

If $(x-1)\left(x^2-34x+1\right)\neq 0$, *i.e.,* $x\neq 1$, $x\neq 17-12\sqrt{2}$, $x\neq 17+12\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k C_k^2 = \frac{(x-1)(x-34)x^{n+1}C_n^2+(x-1)x^{n+1}C_{n-1}^2-(16x^{n+1}+9x^2-26x+1)}{(x-1)\left(x^2-34x+1\right)},$$

and

if $(x-1)\left(x^2-34x+1\right)=0$, *i.e.,* $x=1$ *or* $x=17-12\sqrt{2}$ *or* $x=17+12\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k C_k^2 = \frac{\Theta_{13}}{(3x^2-70x+35)},$$

where

$\Theta_{13}=((x-34)x^{n+1}+(x-1)((n+2)x-34(n+1))x^n)C_n^2+((n+2)x-(n+1))x^n)C_{n-1}^2-2(8(n+1)x^n+9x-13)$.

(b) $(m=2, j=0)$

If $(x-1)\left(x^2-1154x+1\right)\neq 0$, *i.e.,* $x\neq 1$, $x\neq 577-408\sqrt{2}$, $x\neq 577+408\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k C_{2k}^2 = \frac{(x-1)(x-1154)x^{n+1}C_{2n}^2+(x-1)x^{n+1}C_{2n-2}^2-(576x^{n+1}+289x^2-866x+1)}{(x-1)\left(x^2-1154x+1\right)},$$

and

if $(x-1)\left(x^2-1154x+1\right)=0$, *i.e.,* $x=1$ *or* $x=577-408\sqrt{2}$ *or* $x=577+408\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k C_{2k}^2 = \frac{\Theta_{14}}{3\left(x^2-770x+385\right)},$$

where

$\Theta_{14}=((x-1154)x^{n+1}+(x-1)((n+2)x-1154(n+1))x^n)C_{2n}^2+((n+2)x-(n+1))x^nC_{2n-2}^2-2(288(n+1)x^n+289x-433)$.

(c) $(m=2, j=1)$

If $(x-1)\left(x^2-1154x+1\right)\neq 0$, *i.e.,* $x\neq 1$, $x\neq 577-408\sqrt{2}$, $x\neq 577+408\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k C_{2k+1}^2 = \frac{(x-1)(x-1154)x^{n+1}C_{2n+1}^2+(x-1)x^{n+1}C_{2n-1}^2-9(64x^{n+1}+x^2-66x+1)}{(x-1)\left(x^2-1154x+1\right)},$$

and

if $(x-1)\left(x^2-1154x+1\right)=0$, *i.e.,* $x=1$ *or* $x=577-408\sqrt{2}$ *or* $x=577+408\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k C_{2k+1}^2 = \frac{\Theta_{15}}{3\left(x^2-770x+385\right)},$$

where

$\Theta_{15}=((x-1154)x^{n+1}+(x-1)((n+2)x-1154(n+1))x^n)C_{2n+1}^2+((n+2)x-(n+1))x^nC_{2n-1}^2-18(32(n+1)x^n+x-33)$.

(d) $(m=-1, j=0)$

If $(x-1)\left(x^2-34x+1\right)\neq 0$, *i.e.,* $x\neq 1$, $x\neq 17-12\sqrt{2}$, $x\neq 17+12\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k C_{-k}^2 = \frac{(x-1)x^{n+1}C_{-n+1}^2+(x-1)(x-34)x^{n+1}C_{-n}^2-(16x^{n+1}+9x^2-26x+1)}{(x-1)\left(x^2-34x+1\right)},$$

and

if $(x-1)\left(x^2-34x+1\right)=0$, *i.e.,* $x=1$ *or* $x=17-12\sqrt{2}$ *or* $x=17+12\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k C_{-k}^2 = \frac{\Theta_{16}}{(3x^2-70x+35)},$$

where

$\Theta_{16}=((n+2)x-(n+1))x^nC_{-n+1}^2+((x-34)x^{n+1}+(x-1)((n+2)x-34(n+1))x^n)C_{-n}^2-2(8(n+1)x^n+9x-13)$.

(e) $(m=-2, j=0)$

If $(x-1)\left(x^2-1154x+1\right) \neq 0$, *i.e.,* $x \neq 1$, $x \neq 577-408\sqrt{2}$, $x \neq 577+408\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k C_{-2k}^2 = \frac{(x-1)x^{n+1}C_{-2n+2}^2 + (x-1)(x-1154)x^{n+1}C_{-2n}^2 - (576x^{n+1}+289x^2-866x+1)}{(x-1)\left(x^2-1154x+1\right)},$$

and

if $(x-1)\left(x^2-1154x+1\right) = 0$, *i.e.,* $x=1$ *or* $x=577-408\sqrt{2}$ *or* $x=577+408\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k C_{-2k}^2 = \frac{\Theta_{17}}{3\left(x^2-770x+385\right)},$$

where

$\Theta_{17} = ((n+2)x-(n+1))x^n C_{-2n+2}^2 + ((x-1154)x^{n+1} + (x-1)((n+2)x-1154(n+1))x^n)C_{-2n}^2 - 2(288(n+1)x^n + 289x - 433).$

(f) $(m=-2, j=1)$

If $(x-1)\left(x^2-1154x+1\right) \neq 0$, *i.e.,* $x \neq 1$, $x \neq 577-408\sqrt{2}$, $x \neq 577+408\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k C_{-2k+1}^2 = \frac{(x-1)x^{n+1}C_{-2n+3}^2 + (x-1)(x-1154)x^{n+1}C_{-2n+1}^2 - 9(64x^{n+1}+1089x^2-1154x+1)}{(x-1)\left(x^2-1154x+1\right)},$$

and

if $(x-1)\left(x^2-1154x+1\right) = 0$, *i.e.,* $x=1$ *or* $x=577-408\sqrt{2}$ *or* $x=577+408\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k C_{-2k+1}^2 = \frac{\Theta_{18}}{3\left(x^2-770x+385\right)},$$

where

$\Theta_{18} = ((n+2)x-(n+1))x^n C_{-2n+3}^2 + ((x-1154)x^{n+1} + (x-1)((n+2)x-1154(n+1))x^n)C_{-2n+1}^2 - 18(32(n+1)x^n + 1089x - 577).$

Taking $x=1$ in the last two corollaries, one gets the following corollary.

Corollary 2.6. *For $n \geq 0$, balancing numbers, modified Lucas-balancing and Lucas-balancing numbers have the following properties:*

1.

(a) $\sum_{k=0}^{n} B_k^2 = \frac{1}{32}(33B_n^2 - B_{n-1}^2 - 2n + 1)$.

(b) $\sum_{k=0}^{n} B_{2k}^2 = \frac{1}{1152}(1153B_{2n}^2 - B_{2n-2}^2 - 72n + 36)$.

(c) $\sum_{k=0}^{n} B_{2k+1}^2 = \frac{1}{1152}(1153B_{2n+1}^2 - B_{2n-1}^2 - 72n)$.

(d) $\sum_{k=0}^{n} B_{-k}^2 = \frac{1}{32}(-B_{-n+1}^2 + 33B_{-n}^2 - 2n + 1)$.

(e) $\sum_{k=0}^{n} B_{-2k}^2 = \frac{1}{1152}(-B_{-2n+2}^2 + 1153B_{-2n}^2 - 72n + 36)$.

(f) $\sum_{k=0}^{n} B_{-2k+1}^2 = \frac{1}{1152}(-B_{-2n+3}^2 + 1153B_{-2n+1}^2 - 72n + 1224)$.

2.

(a) $\sum_{k=0}^{n} H_k^2 = \frac{1}{32}(33H_n^2 - H_{n-1}^2 + 64n + 32)$.

(b) $\sum_{k=0}^{n} H_{2k}^2 = \frac{1}{1152}(1153H_{2n}^2 - H_{2n-2}^2 + 2304n + 1152)$.

(c) $\sum_{k=0}^{n} H_{2k+1}^2 = \frac{1}{1152}(1153H_{2n+1}^2 - H_{2n-1}^2 + 2304n)$.

(d) $\sum_{k=0}^{n} H_{-k}^2 = \frac{1}{32}(-H_{-n+1}^2 + 33H_{-n}^2 + 64n + 32)$.

(e) $\sum_{k=0}^{n} H_{-2k}^2 = \frac{1}{1152}(-H_{-2n+2}^2 + 1153H_{-2n}^2 + 2304n + 1152)$.

(f) $\sum_{k=0}^{n} H_{-2k+1}^2 = \frac{1}{1152}(-H_{-2n+3}^2 + 1153H_{-2n+1}^2 + 2304n + 39168)$.

3.

(a) $\sum_{k=0}^{n} C_k^2 = \frac{1}{32}(33C_n^2 - C_{n-1}^2 + 16n + 8)$.

(b) $\sum_{k=0}^{n} C_{2k}^2 = \frac{1}{1152}(1153C_{2n}^2 - C_{2n-2}^2 + 576n + 288)$.

(c) $\sum_{k=0}^{n} C_{2k+1}^2 = \frac{1}{1152}(1153C_{2n+1}^2 - C_{2n-1}^2 + 576n)$.

(d) $\sum_{k=0}^{n} C_{-k}^2 = \frac{1}{32}(-C_{-n+1}^2 + 33C_{-n}^2 + 16n + 8)$.

(e) $\sum_{k=0}^{n} C_{-2k}^2 = \frac{1}{1152}(-C_{-2n+2}^2 + 1153C_{-2n}^2 + 576n + 288)$.

(f) $\sum_{k=0}^{n} C_{-2k+1}^2 = \frac{1}{1152}(-C_{-2n+3}^2 + 1153C_{-2n+1}^2 + 576n + 9792)$.

3 Conclusion

Recently, there have been so many studies of the sequences of numbers in the literature. The sequences of numbers were widely used in many research areas, such as architecture, nature, art, physics, and engineering. In this work, sum identities were proved. The method used in this paper can be used for the other linear recurrence sequences, too. We have written sum identities in terms of the generalized balancing sequence. Then, we have presented the formulas as special cases, the corresponding identity for the balancing, modified Lucas-balancing, and Lucas-balancing numbers. All the listed identities in the corollaries may be proved by induction, but that proof method gives no clue about their discovery. We have provided proofs to show how these identities were discovered in general.

Computations of the Frobenius norm, spectral norm, maximum column length norm, and maximum row length norm of circulant (r-circulant, geometric circulant, semicirculant) matrices with the generalized m-step Fibonacci sequences require the sum of the numbers of the sequences. So, our results can be used to study r-circulant matrices with m-order linear recurrence sequences.

Acknowledgements

No funding is available for this research.

Conflict of Interest

The authors have no conflicts of interest to declare.

References

[1] A. Behera and G. K. Panda, *On the Square Roots of Triangular Numbers*, Fibonacci Quart. **37**(2) (1999), 98–105.

[2] A. Berczes, K. Liptai and I. Pink, *On Generalized Balancing Sequences*, Fibonacci Quart. **48**(2) (2010), 121–128. .

[3] P. Catarino, H. Campos and P. Vasco, *On Some Identities for Balancing and Cobalancing Numbers*, Ann. Math. Inform. **45** (2015), 11–24.

[4] R.K. Davala and G.K. Panda, *On Sum and Ratio Formulas for Balancing Numbers*, J. Indian Math. Soc. (N.S.) **82**(1-2) (2015), 23–32.

[5] A.F. Horadam, *A Generalized Fibonacci Sequence*, Amer. Math. Monthly **68** (1961), 455–459.

[6] A.F. Horadam, *Basic Properties of a Certain Generalized Sequence of Numbers*, Fibonacci Quart. **3**(3) (1965), 161–176.

[7] A.F. Horadam, *Special Properties of The Sequence* $w_n(a,b;p,q)$, Fibonacci Quart. **5**(5) (1967), 424–434.

[8] A.F. Horadam, *Generating functions for powers of a certain generalized sequence of numbers*, Duke Math. J. **32** (1965), 437–446.

[9] R. Frontczak, *Identities for Generalized balancing Numbers*, Notes Number Theory Discrete Math. **25**(2) (2019) 169–180. DOI:10.7546/nntdm.2019.25.2.169-180

[10] R. Keskin and O. Karaatli, *Some New Properties of Balancing Numbers and Square Triangular Numbers*, J. Integer Seq. **15** (2012), 1–13.

[11] T. Kovacs, K. Liptai and P. Olajos, *On (a; b)-balancing Numbers*, Publ. Math. Debrecen **77**(3-4) (2010), 485–498.

[12] K. Liptai, *Fibonacci Balancing Numbers*, Fibonacci Quart. **42**(4) (2004), 330–340.

[13] K. Liptai, F. Luca, A. Pinter and L. Szalay, *Generalized Balancing Numbers*, Indag. Math. (N.S.) **20**(1) (2009), 87–100.

[14] P. Olajos, *Properties of Balancing, Cobalancing and Generalized Balancing Numbers*, Ann. Math. Inform. **37** (2010), 125–138.

[15] A. Özkoç, *Tridiagonal Matrices via k-Balancing Number*, British J. Math. Comput. Sci. **10**(4) (2015), 1–11.

[16] G.K. Panda, *Some Fascinating Properties of Balancing Numbers*, In Proc. of Eleventh Internat. Conference on Fibonacci Numbers and Their Applications, Congr. Numer. **194** (2009), 185–189.

[17] G.K. Panda, T. Komatsu and R.K. Davala, *Reciprocal Sums of Sequences Involving Balancing and Lucas-Balancing Numbers*, Math. Rep. **20**(70) (2018), 201–214.

[18] P.K. Ray, *On the Properties of Lucas-balancing Numbers by Matrix Method*, Sigmae **3**(1) (2014), 1–6.

[19] P.K. Ray, *Balancing Sequences of Matrices with Application to Algebra of balancing Numbers*, Notes Number Theory Discrete Math. **20**(1) (2014), 49–58.

[20] P.K. Ray, *Some Congruences for Balancing and Lucas-Balancing Numbers and Their Applications*, Integers **14** (2014), 1–8.

[21] P.K. Ray, *Balancing and Lucas-Balancing Sums by Matrix Methods*, Math. Rep. **17**(2) (2015), 225–233.

[22] P.K. Ray, S. Patel and M.K. Mandal, *Identities for Balancing Numbers Using Generating Function and Some New Congruence Relations*, Notes Number Theory Discrete Math. **22**(4) (2016), 41–48.

[23] P.K. Ray and J. Sahu, *Generating Functions for Certain Balancing and Lucas-Balancing Numbers*, Palestine J. Math. **5**(2) (2016), 122–129.

[24] P.K. Ray, *Balancing Polynomials and Their Derivatives*, Ukr. Math. J. **69** (2017), 646–663.

[25] S.G. Rayaguru and G.K. Panda, *Sum Formulas Involving Powers of Balancing and Lucas-balancing Numbers*, J. Ind. Math. Soc. **86**(1-2) (2019), 137–160.

[26] S.G. Rayaguru and G.K. Panda, *Sum Formulas Involving Powers of Balancing and Lucas-balancing Numbers-II*, Notes Number Theory Discrete Math. **25**(3) (2019), 102–110. DOI: 10.7546/nntdm.2019.25.3.102-110

[27] N.J.A. Sloane, *The On-Line Encyclopedia of Integer Sequences*, Published electronically at http://oeis.org.

[28] Y. Soykan, *Generalized Fibonacci Numbers with Indices in Arithmetic Progression and Sum of Their Squares: the Sum Formula* $\sum_{k=0}^{n} x^k W_{mk+j}^2$, J. Adv. Math. Comput. Sci. **36**(6) (2021), 30–62. DOI: 10.9734/JAMCS/2021/v36i630371

[29] Y. Soykan, *A Study on Generalized Balancing Numbers*, Asian J. Adv. Res. Rep. **15**(5) (2021), 78–100. DOI: 10.9734/AJARR/2021/v15i530401

Journal of Innovative Applied Mathematics and Computational Sciences

J. Innov. Appl. Math. Comput. Sci. **1**(1) (2021) 31–39. n2t.net/ark:/49935/jiamcs.v1i1.5

JIAMCS MILA UNIVERSITY CENTER

http://jiamcs.centre-univ-mila.dz/

Existence and asymptotic stability of continuous solutions for integral equations of product type

Mahmoud Bousselsal [1] and Azzeddine Bellour ✉ [2]

[1]Laboratoire d'EDP non linéaires, Ecole Normale Superieure de Kouba, Algiers-Algeria
[2]Laboratoire de Mathématiques Appliquées et Didactique, Ecole Normale Supérieure de Constantine, Algeria

Received 14 November 2021, Accepted 27 December 2021, Published 29 December 2021

Abstract. In this paper, we study the existence of a continuous solution for a nonlinear integral equation of a product type. The analysis uses the techniques of measures of noncompactness and Darbo's fixed point Theorem. Our results are obtained under rather general assumptions. Moreover, the method used in the proof allows us to obtain the asymptotic stability of the solutions.

Keywords: Integral equation of a product type, measure of weak noncompactness, fixed point theorem, continuous solutions.

2020 Mathematics Subject Classification: 45D05, 47H30.

1 Introduction

In this paper, we consider the following nonlinear integral equation of product type

$$x(t) = f(t, x(t)) + \left[p(t) + \int_0^t u(t, s, x(s)) ds \right] \times \left[q(t) + \int_0^t v(t, s, x(s)) ds \right], t \in \mathbb{R}^+, \quad (1.1)$$

where f, p, q, u, v are continuous functions and $x(t) \in C(\mathbb{R}^+, \mathbb{R})$ is an unknown function.
A variety of problems in physics and biology have their mathematical setting as integral equations of product type. In particular, in the study of the spread of an infectious disease that does not induce permanent immunity (see, for example [3,10,11,16]).
Recently, there has been a growing interest in integral equations of product type. In [12] Gripenberg studied the qualitative behavior of solutions of the following integral equation of product type

$$x(t) = k \left[p(t) + \int_0^t A(t-s)x(s) ds \right] \times \left[q(t) + \int_0^t B(t-s)x(s) ds \right]. \quad (1.2)$$

More exactly, the author studied the existence and uniqueness of a bounded continuous and nonnegative solution of (1.2). Moreover, Pachpatte [15], Abdeldaim [1] and Li et al. [13] studied the boundedness, the asymptotic behavior and continuous solutions of (1.2).

✉ Corresponding author. Email: bellourazze123@yahoo.com

ISSN (electronic): 2773-4196

Bellour et al. [8] studied the existence of an integrable solution of (1.1) on the interval $[0,1]$. On the other hand, Ardjouni and Djoudi [2] studied the existence and approximation of solutions of the initial value problems of nonlinear hybrid Caputo fractional integro-differential equations, which can be transformed to the following integral equation of product type

$$x(t) = \left[p(t) + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} g(s,x(s))ds\right] \times \left[\theta + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s,x(s))ds\right],$$

on a bounded interval $[0,a]$.
In the paper [14], Olaru studied the existence and the uniqueness of the continuous solution of the following integral equation

$$x(t) = \prod_{i=1}^{m} \left(g_i(t) + \int_a^t K_i(t,s,x(s))ds\right), \tag{1.3}$$

on a bounded interval $[a,b]$, where $K_i, i = 1, ..., n$ are continuous functions satisfying Lipschitz conditions with respect to the last variable.
Later, Boulfoul et al. [9] studied the existence of an integrable solution of a generalization of (1.3) on $\mathbb{R}^+$.

The purpose of the present work is to study the existence of a continuous solution and bounded solution to (1.1) under fairly simple conditions. Moreover, the method used in the proof allows us to obtain the asymptotic stability of the solutions. An example is presented to show the importance and the applicability of our results.

2 Auxiliary facts and results

In this section, we provide some notations, definitions and auxiliary facts which will be needed for stating our results. Denote by $BC(\mathbb{R}^+, \mathbb{R})$ the Banach space of all real functions defined, continuous and bounded on $\mathbb{R}^+$. It is equipped with the standard norm

$$\|x\| = \sup_{t\in\mathbb{R}^+} |x(t)|.$$

For later use, we assume that X be a Banach space. Let $\mathcal{B}(X)$ denote the family of all nonempty bounded subsets of X and $\mathcal{W}(X)$ the subset of $\mathcal{B}(X)$ consisting of all relatively compact subsets of X. Finally, let B_r denote the closed ball centered at 0 with radius r.
Recall the following definition of the concept of the axiomatic measure of noncompactness.

Definition 2.1. [6]. A function $\mu : \mathcal{B}(X) \longrightarrow \mathbb{R}^+$ is said to be a measure of noncompactness if it satisfies the following conditions:

(1) The family $ker(\mu) = \{M \in \mathcal{B}(X) : \mu(M) = 0\}$ is nonempty and $ker(\mu) \subset \mathcal{W}(X)$.

(2) $M_1 \subset M_2 \Rightarrow \mu(M_1) \leq \mu(M_2)$.

(3) $\mu(co(M)) = \mu(M)$, where $co(M)$ is the convex hull of M.

(4) $\mu(\lambda M_1 + (1-\lambda)M_2) \leq \lambda\mu(M_1) + (1-\lambda)\mu(M_2)$ for $\lambda \in [0,1]$.

(5) If $(M_n)_{n\geq 1}$ is a sequence of nonempty, weakly closed subsets of X with M_1 bounded and $M_1 \supseteq M_2 \supseteq ... \supseteq M_n \supseteq ...$ such that $\lim_{n\to\infty} \mu(M_n) = 0$, then $M_\infty := \bigcap_{n=1}^{\infty} M_n$ is nonempty.
A measure μ is said to be sublinear if it satisfies the following two conditions:

(6) $\mu(\lambda M) = |\lambda|\mu(M)$ for $\lambda \in \mathbb{R}$.

(7) $\mu(M_1 + M_2) \leq \mu(M_1) + \mu(M_2)$.

The family $ker(\mu)$ described in (1) is called the kernel of the measure of noncompactness μ. More information about measures of noncompactness and their properties can be found in [5]. For our purposes, we will only need the following fixed point theorem [5].

In what follows, we will use a measure of noncompactness in the space $BC(\mathbb{R}^+, \mathbb{R})$ which was introduced in [5]. In order to recall the definition of this measure let us fix a nonempty bounded subset $X \in BC(\mathbb{R}^+, \mathbb{R})$ and a positive number $T > 0$. For $x \in X$ and $\varepsilon > 0$, let us define the following quantities (cf. [5]):

$$\omega^T(x, \varepsilon) = \sup\left\{|x(s) - x(t)| : t, s \in [0, T], |t - s| \leq \varepsilon\right\}.$$

Further, let us put

$$\omega^T(X, \varepsilon) = \sup\left\{\omega^T(x, \varepsilon) : x \in X\right\},$$

$$\omega_0^T(X) = \lim_{\varepsilon \longrightarrow 0} \omega^T(X, \varepsilon), \quad \omega_0(X) = \lim_{T \longrightarrow \infty} \omega_0^T(X).$$

For a fixed number $t \geq 0$, we denote

$$d(X(t)) = \sup\left\{|x(t) - y(t)| : x, y \in X\right\}.$$

and

$$d(X) = \limsup_{t \longrightarrow \infty} d(X(t)).$$

Finally, the function μ is defined by putting

$$\mu(X) = \omega_0(X) + d(X).$$

It can be shown [5] that the function μ is a measure of noncompactness in the space $BC(\mathbb{R}^+, \mathbb{R})$ with the kernel $ker(\mu)$ consisting of all nonempty and bounded sets X such that functions from X are equicontinuous and nondecreasing on $\mathbb{R}^+$. For other properties of μ, see [5].

3 Main result

We will use the following fixed point theorem.

Theorem 3.1. *[4] Let $\mathcal{Q}$ be nonempty bounded closed convex subset of the space E and let $F : \mathcal{Q} \longrightarrow \mathcal{Q}$ be a continuous operator such that $\mu(FX) \leq k\mu(X)$ for any nonempty subset X of $\mathcal{Q}$, where $k \in [0, 1)$ is a constant. Then F has a fixed point in the set $\mathcal{Q}$.*

Equation (1.1) will be studied under the following assumptions:

(i) The functions $p, q : \mathbb{R}^+ \to \mathbb{R}$ are continuous and bounded functions on $\mathbb{R}^+$. Let $\|p\|$ be the norm of p in $BC(\mathbb{R}^+, \mathbb{R})$ and $\|q\|$ be the norm of q in $BC(\mathbb{R}^+, \mathbb{R})$.

(ii) The function $f : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ is Lipschizian with respect to the second variable with a Lipschitz constant α, that is, $|f(t, x) - f(t, y)| \leq \alpha|x - y|$ for all $t \in \mathbb{R}^+$ and all $x, y \in \mathbb{R}$. Let $\beta(t) = |f(t, 0)| \in BC(\mathbb{R}^+, \mathbb{R})$.

(iii) The function $u : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ is continuous and there exist a positive constant b_1 and a function $a_1 \in BC(\mathbb{R}^+, \mathbb{R})$ such that $|u(t,s,x)| \leq k_1(t,s)\,[a_1(s) + b_1\,|x|]$ for $(t,s,x) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}$, where $k_1 : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is measurable function and the linear Volterra operator K_1 generated by k_1,

$$(K_1 x)(t) = \int_0^t k_1(t,s)x(s)ds, \tag{3.1}$$

transforms the space $BC(\mathbb{R}^+, \mathbb{R})$ into itself. Let $\|K_1\|$ be the norm of this operator.

(iv) The function $v : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ is continuous and there exists a function $a_2 \in BC(\mathbb{R}^+, \mathbb{R})$ such that $|v(t,s,x)| \leq k_2(t,s)a_2(s)$ for $(t,s,x) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}$, where $k_2 : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is measurable function and the linear Volterra operator K_2 generated by k_2,

$$(K_2 x)(t) = \int_0^t k_2(t,s)x(s)ds, \tag{3.2}$$

transforms the space $BC(\mathbb{R}^+, \mathbb{R})$ into itself. Let $\|K_2\|$ be the norm of this operator.

(v) $\lim\limits_{t \longrightarrow +\infty} (K_i 1)(t) = \lim\limits_{t \longrightarrow +\infty} \int_0^t k_i(t,s)ds = 0$, for $i = 1, 2$.

(vi) $\alpha + b_1\|K_1\|(\|q\| + \|K_2\|\|a_2\|) < 1$.

To prove our main result, we need the following lemma.

Lemma 3.2. *Under the assumptions (i)-(v) the operators*

$$(Ux)(t) = p(t) + \int_0^t u(t,s,x(s))ds,$$
$$(Vx)(t) = q(t) + \int_0^t v(t,s,x(s))ds.$$

map $BC(\mathbb{R}^+, \mathbb{R})$ continuously into itself.

Proof. We prove only that U maps $BC(\mathbb{R}^+, \mathbb{R})$ continuously into itself and the proof of V is similarly.
It is clear that the operator U maps $BC(\mathbb{R}^+, \mathbb{R})$ into $C(\mathbb{R}^+, \mathbb{R})$. Moreover, let $x \in BC(\mathbb{R}^+, \mathbb{R})$, since

$$|(K_1 x)(t)| \leq \|x\|\,(K_1 1)(t).$$

On the other hand, from the assumption (v), there exists $T > 0$ such for all $t \geq T$

$$(K_1 1)(t) \leq 1.$$

Hence, from the assumption (iii), we have for all $t \geq T$

$$|(Ux)(t)| \leq (\|a_1\| + b_1\|x\|)\,(K_1 1)(t) \leq \|a_1\| + b_1\|x\|.$$

On the other hand, (Ux) is bounded on $[0, T]$, we deduce that U maps $BC(\mathbb{R}^+, \mathbb{R})$ into itself.
Now, to prove that U is continuous, let $\{x_n\}$ be an arbitrary sequence in $BC(\mathbb{R}^+, \mathbb{R})$ which converges to $x \in BC(\mathbb{R}^+, \mathbb{R})$.
Then, for $\varepsilon > 0$ there exist $n_1 \in \mathbb{N}$ and $T > 0$, such that for all $n \geq n_1$ and $t \geq T$, we have

$$\|x_n\| \leq 1 + \|x\|, (K_1 1)(t) \leq \frac{\varepsilon}{2\|a_1\| + b_1(2\|x\| + 1)}.$$

It follows that, for $n \geq n_1$ and $t \geq T$, we have

$$|(Ux_n - Ux)(t)| \leq (2\|a_1\| + b_1(2\|x\| + 1))\,(K_1 1)\,(t) \leq \varepsilon. \tag{3.3}$$

On the other hand, since u is uniformly bounded on the compact set $[0,T] \times [0,T] \times [-1 - \|x\|, 1 + \|x\|]$, hence there exists $n_2 \in \mathbb{N}$ such that for all $n \geq n_2$, we have

$$\sup\{|u(t,s,x_n(s)) - u(t,s,x(s))|, (t,s) \in [0,T] \times [0,T], n \geq n_2\} \leq \frac{\varepsilon}{T},$$

which implies that, for all $n \geq n_2$ and $t \in [0,T]$

$$|(Ux_n - Ux)(t)| \leq \varepsilon. \tag{3.4}$$

Then, from (3.3) and (3.4), we deduce that, for all $n \geq n_0 = \max(n_1, n_2)$

$$\|Ux_n - Ux\| \leq \varepsilon.$$

Thus, U maps $BC(\mathbb{R}^+, \mathbb{R})$ continuously into itself. □

Remark 3.3. [7] The concept of the asymptotic stability of a solution $x = x(t)$ of Eq. (1.1) is understood in the following sense.
For any $\varepsilon > 0$ there exist $T > 0$ and $r > 0$ such that if $x = x(t), y = y(t)$ are solutions of (1.1) then $|x(t) - y(t)| \leq \varepsilon$ for $t \geq T$.

Now we are able to state our main result.

Theorem 3.4. *Under the assumptions above the nonlinear integral equation (1.1) has at least an asymptotically stable solution $x \in BC(\mathbb{R}^+, \mathbb{R})$.*

Proof. Solving Eq. (1.1) is equivalent to finding a fixed point of the operator A, where $Ax(t) = f(t, x(t)) + (Ux)(t) \times (Vx)(t)$. We will show that A satisfies the conditions of Theorem 3.1. The proof is split into four steps.
Step 1. We first show that there exists B_{r_0} from $BC(\mathbb{R}^+, \mathbb{R})$ such that $A(B_{r_0}) \subset B_{r_0}$. To see this, let $x \in B_r$. Then

$$\begin{aligned}
\|Ax\| \leq& \|f(t,x(t))\| + \|(Ux)(t) \times (Vx)(t)\| \\
\leq& \alpha\|x\| + \|\beta\| + (\|p\| + \|K_1(a_1 + b_1 x)\|) \times (\|q\| + \|K_2(a_2)\|) \\
\leq& \alpha\|x\| + \|\beta\| + (\|p\| + \|K_1\|(\|a_1\| + b_1\|x\|)) \times (\|q\| + \|K_2\|\|a_2\|) \\
\leq& \alpha r + \|\beta\| + (\|p\| + \|K_1\|\|a_1\| + b_1\|K_1\|r) \times (\|q\| + \|K_2\|\|a_2\|) \\
\leq& (\alpha + b_1\|K_1\|(\|q\| + \|K_2\|\|a_2\|))r + \|\beta\| + (\|p\| + \|K_1\|\|a_1\|)(\|q\| + \|K_2\|\|a_2\|).
\end{aligned}$$

Since $\alpha + b_1\|K_1\|(\|q\| + \|K_2\|\|a_2\|) < 1$, we deduce that the operator A transforms the ball B_{r_0} into itself for $r_0 = \frac{\|\beta\| + (\|p\| + \|K_1\|\|a_1\|)(\|q\| + \|K_2\|\|a_2\|)}{1 - (\alpha + b_1\|K_1\|(\|q\| + \|K_2\|\|a_2\|))}$.
Step 2. The operator A maps B_{r_0} continuously into itself. To see this, take an arbitrary number $\epsilon > 0$ and a convergent sequence (x_n) to (x) in B_{r_0}.
Hence, by Lemma 3.2, there exists n_0 such that for all $n \geq n_0$, we have

$$\|x_n - x\| \leq \frac{\epsilon}{3\alpha}, \|Ux_n - Ux\| \leq \frac{\epsilon}{3(\|q\| + \|K_2\|\|a_2\|)},$$

$$\|Vx_n - Vx\| \leq \frac{\epsilon}{3(\|p\| + \|K_1\|(\|a_1\| + b_1 r_0))}.$$

Which implies, for all $n \geq n_0$,

$$\begin{aligned}\|Ax_n - Ax\| \leq& \alpha\|x_n - x\| + \|(Ux_n) \times (Vx_n) - (Ux) \times (Vx)\|\\ \leq& \alpha\|x_n - x\| + \|Ux_n\|\|Vx_n - Vx\| + \|Vx\|\|Ux_n - Ux\|\\ \leq& \alpha\|x_n - x\| + (\|p\| + \|K_1\|(\|a_1\| + b_1\|x_n\|))\|Vx_n - Vx\|\\ &+ (\|q\| + \|K_2\|\|a_2\|)\|Ux_n - Ux\|\\ \leq& \alpha\|x_n - x\| + (\|p\| + \|K_1\|(\|a_1\| + b_1 r_0))\|Vx_n - Vx\|\\ &+ (\|q\| + \|K_2\|\|a_2\|)\|Ux_n - Ux\|\\ \leq& \epsilon.\end{aligned}$$

We deduce that the operator A maps B_{r_0} continuously into itself.

Step 3. We illustrate that there exists $\gamma \in [0, 1)$ such that $\mu(AX) \leq \gamma\mu(X)$ for all subset X of B_{r_0}. To see this, take an arbitrary number $t \geq 0$. Then for any $x, y \in X$, we have

$$\begin{aligned}|Ax(t) - Ay(t)| \leq& \alpha|x(t) - y(t)| + |Ux(t)||Vx(t) - Vy(t)| + |Vy(t)||Ux(t) - Uy(t)|\\ \leq& \alpha\|x(t) - y(t)\| + (\|p\| + \|K_1\|(\|a_1\| + b_1 r_0))|Vx(t) - Vy(t)|\\ &+ (\|q\| + \|K_2\|\|a_2\|)|Ux(t) - Uy(t)|\\ \leq& \alpha\|x(t) - y(t)\| + 2(\|p\| + \|K_1\|(\|a_1\| + b_1 r_0))\|a_2\|K_2 1(t)\\ &+ 2(\|q\| + \|K_2\|\|a_2\|)(\|a_1\| + b_1 r_0))K_1 1(t).\end{aligned}$$

Which implies that

$$\begin{aligned}d(AX(t)) \leq& \alpha d(X(t)) + 2(\|p\| + \|K_1\|(\|a_1\| + b_1 r_0))\|a_2\|K_2 1(t)\\ &+ 2(\|q\| + \|K_2\|\|a_2\|)(\|a_1\| + b_1 r_0))K_1 1(t).\end{aligned}$$

Now, taking into account the assumption (v) we obtain the following estimate:

$$d(AX) \leq \alpha d(X). \tag{3.5}$$

Further, let us fix arbitrarily numbers $T > 0$, $\varepsilon > 0$, let $x \in X$ and take $t_1, t_2 \in [0, T]$ such that $|t_2 - t_1| \leq \varepsilon$. Without loss of generality we may assume that $t_1 < t_2$.

Then, in view of our assumptions, we have

$$\begin{aligned}|Ax(t_2) - Ax(t_1)| \leq& \alpha|x(t_2) - x(t_1)| + |Ux(t_2)||Vx(t_2) - Vx(t_1)|\\ &+ |Vx(t_1)||Ux(t_2) - Ux(t_1)|\\ \leq& \alpha|x(t_2) - x(t_1)| + (\|p\| + \|K_1\|(\|a_1\| + b_1 r_0))|Vx(t_2) - Vx(t_1)|\\ &+ (\|q\| + \|K_2\|\|a_2\|)|Ux(t_2) - Ux(t_1)|.\end{aligned} \tag{3.6}$$

Now, from the assumption (iii), we have

$$\begin{aligned}|Ux(t_2) - Ux(t_1)| \leq& \int_0^{t_2} |u(t_2, s, x(s)) - u(t_1, s, x(s))|ds\\ &+ \int_{t_1}^{t_2} |u(t_1, s, x(s))|ds\\ \leq& T\overline{\omega}^T(u, \varepsilon) + |t_2 - t_1|\overline{u}\\ \leq& T\overline{\omega}^T(u, \varepsilon) + \varepsilon\overline{u},\end{aligned} \tag{3.7}$$

where,

$$\overline{\omega}^T(u,\varepsilon) = \sup\{|u(t_2,s,x) - u(t_1,s,x)|, t_1,t_2,s \in [0,T], |t_2 - t_1| \le \varepsilon, |x| \le r_0\},$$
$$\overline{u} = \sup\{|u(t,s,x)|, t,s \in [0,T], |x| \le r_0\}.$$

Similarly, from the assumption (iv), we obtain

$$|Vx(t_2) - Vx(t_1)| \le T\overline{\omega}^T(v,\varepsilon) + \varepsilon\overline{v}, \tag{3.8}$$

where,

$$\overline{\omega}^T(v,\varepsilon) = \sup\{|v(t_2,s,x) - v(t_1,s,x)|, t_1,t_2,s \in [0,T], |t_2 - t_1| \le \varepsilon, |x| \le r_0\},$$
$$\overline{v}^T = \sup\{|v(t,s,x)|, t,s \in [0,T], |x| \le r_0\}.$$

Hence, from (3.6), (3.7) and (3.8), we obtain

$$\begin{aligned}\omega^T(Ax,\varepsilon) \le & \alpha\omega^T(x,\varepsilon) + (\|p\| + \|K_1\|(\|a_1\| + b_1 r_0))(T\overline{\omega}^T(v,\varepsilon) + \varepsilon\overline{v}) \\ & + (\|q\| + \|K_2\|\|a_2\|)(T\overline{\omega}^T(u,\varepsilon) + \varepsilon\overline{u}).\end{aligned}$$

Since $\lim\limits_{\varepsilon \longrightarrow 0} \overline{\omega}^T(u,\varepsilon) = \lim\limits_{\varepsilon \longrightarrow 0} \overline{\omega}^T(v,\varepsilon) = 0$, then

$$\omega_0(AX) \le \alpha\omega_0(X). \tag{3.9}$$

We deduce, from (3.5) and (3.9), that

$$\mu(AX) \le \alpha\mu(X).$$

Hence the third step is completed by taking $\gamma = \alpha < 1$.
Finally, using Theorem 3.1, we can see that (1.1) has at least one solution $x \in BC(\mathbb{R}^+,\mathbb{R})$.
Step 4. The solution x is asymptotically stable on $\mathbb{R}^+$.
Let $\varepsilon > 0$, and taking $r = r_0$, then, for any other solution $y \in B_{r_0}$, we have from Step 3

$$\begin{aligned}|Ax(t) - Ay(t)| \le & \alpha\|x(t) - y(t)\| + 2(\|p\| + \|K_1\|(\|a_1\| + b_1 r_0))\|a_2\|K_2 1(t) \\ & + 2(\|q\| + \|K_2\|\|a_2\|)(\|a_1\| + b_1 r_0))K_1 1(t).\end{aligned}$$

Since $\alpha < 1$, we obtain

$$\begin{aligned}|Ax(t) - Ay(t)| \le & \frac{2(\|p\| + \|K_1\|(\|a_1\| + b_1 r_0))\|a_2\|}{1-\alpha} K_2 1(t) \\ & + \frac{2(\|q\| + \|K_2\|\|a_2\|)(\|a_1\| + b_1 r_0)}{1-\alpha} K_1 1(t).\end{aligned}$$

By using Assumption (v), we deduce that there exists $T > 0$ such that for all $t \ge T$

$$|Ax(t) - Ay(t)| \le \varepsilon.$$

Which implies that the solution is asymptotically stable on $\mathbb{R}^+$. □

4 Example

Consider the following integral equation

$$x(t) = t\exp(-t) + 1 + \frac{1}{2}x(t) + \left(\frac{1}{5+t} + \int_0^t \frac{\cos(s+t)}{t+\lambda}\ln(1+x^2(s))ds\right) \times \left(\exp(-t) + \int_0^t \frac{\sin(t)}{(1+2t-s+|x(s)|)^2}ds\right), \tag{4.1}$$

where $t \in \mathbb{R}^+$ and λ is a positive number.
Set

$$f(t,x) = t\exp(-t) + 1 + \frac{1}{2}x, p(t) = \frac{1}{5+t}, q(t) = \exp(-t), k_1(t,s) = \frac{|\cos(s+t)|}{t+\lambda},$$

and

$$k_2(t,s) = \frac{1}{(1+2t-s)^2}, a_1(s) = 0, b_1 = 1, a_2(s) = |\sin(t)|.$$

Using the notations of Theorem 3.4, we can easily show that

$$\alpha = \frac{1}{2}, \|p\| = \frac{1}{5}, \|q\| = \|a_2\| = 1, K_1 1(t) \leq \frac{2}{t+\lambda}, K_2 1(t) \leq \frac{1}{1+t},$$

and

$$\|K_1\| \leq \frac{2}{\lambda}, \|K_2\| \leq 1.$$

Then the assumption (v) is satisfied, therefore, the inequality (vi) takes the form

$$\frac{1}{2} + \frac{4}{\lambda} < 1 \Longleftrightarrow \lambda > 8.$$

Then by Theorem 3.4, we conclude that the integral equation (4.1) has an asymptotically stable solution $x \in BC(\mathbb{R}^+, \mathbb{R})$ whenever $\lambda > 8$.

5 Conclusion

In this paper, we have considered a general form of integral equations of product type on the half-axis. The existence of a continuous solution and its asymptotic stability have been investigated using the measures of non-compactness and Darbo's fixed point theorem. Finally, an example is provided to illustrate our main result.

Conflict of Interest

The authors have no conflicts of interest to disclose.

References

[1] A. Abdeldaim, *On some new Gronwall-Bellman-Ou-Iang type integral inequalities to study certain epidemic models*, J. Integral Equations Appl. **24** (2012), 149–166.

[2] A. Ardjouni and A. Djoudi, *Approximating solutions of nonlinear hybrid Caputo fractional integro-differential equations via Dhage iteration principle*, Ural Math. J. **5** (2019), 3–12.

[3] N. T. J. Bailey, *The mathematical theory of infectious diseases and its applications*, Hafner, New York, 1975.

[4] J. Banas and K. Goebel, *Measures of noncompactness in Banach spaces*, Lecture Notes in Pure and Applied Mathematics, Dekker, New York, 1980.

[5] J. Banas and L. Olszowy, *Measures of noncompactness related to monotonicity*, Comment. Math. **41** (2001), 13–23.

[6] J. Banas and J. Rivero, *On measures of weak noncompactness*, Ann. Mat. Pura Appl. **151** (1988), 213–224.

[7] J. Banas and B. Rzepka, *On existence and asymptotic stability of solutions of a nonlinear integral equation*, J. Math. Anal. Appl. **284**(1) (2003), 165–173.

[8] A. Bellour, M. Bousselsal and M. A. Taoudi, *Integrable solutions of a nonlinear integral equation related to some epidemic models*, Glas. Mat. **49**(2) (2014), 395–406.

[9] B. Boulfoul, A. Bellour and S. Djebali, *Solvability of nonlinear integral equations of product type*, Electron. J. Differ. Equ. **19** (2018), 1–20.

[10] O. Diekmann, *A note on the asymptotic speed of propagation of an epidemic*, J. Differ. Equations **33**(1) (1979), 58–73.

[11] G. Gripenberg, *Periodic solutions of an epidemic model*, J. Math. Biol. **10** (1980), 271–280.

[12] G. Gripenberg, *On some epidemic models*, Quart. Appl. Math. **39**(3) (1981), 317–327.

[13] L. Li, F. Meng and P. Ju, *Some new integral inequalities and their applications in studying the stability of nonlinear integro-differential equations with time delay*, J. Math. Anal. Appl. **377** (2011), 853–862.

[14] I. M. Olaru, *Generalization of an integral equation related to some epidemic models*, Carpathain J. Math. **26**(1) (2010), 92–96.

[15] B. G. Pachpatte, *On a new inequality suggested by the study of certain epidemic models*, J. Math. Anal. Appl. **195**(1) (1995), 638–644.

[16] P. Waltman, *Deterministic threshold models in the theory of epidemics*, Lecture Notes in Biomathematics, vol. 1, Springer-Verlag, New York, 1974.

Journal of Innovative Applied Mathematics and Computational Sciences

J. Innov. Appl. Math. Comput. Sci. **1**(1) (2021) 40–47. n2t.net/ark:/49935/jiamcs.v1i1.11

http://jiamcs.centre-univ-mila.dz/

More on standard single valued neutrosophic metric spaces

Soheyb Milles [1], Abdelkrim Latreche [2] and Omar Barkat [3]

[1,3]Department of Mathematics and Computer Science, Barika University Center, Algeria
[2]Department of Technology, Faculty of Technology, University of Skikda, Algeria

Received 6 December 2021, Accepted 22 December 2021, Published 29 December 2021

Abstract. Recently, we have introduced the notion of standard single valued neutrosophic (SSVN) metric space as a generalization of the notion of standard fuzzy metric spaces given by J.R. Kider and Z.A. Hussain. In this paper, we study the fundamental properties of standard single valued neutrosophic metric spaces. Furthermore, we introduce the notion of continuous mapping and uniformly continuous mapping in standard single-valued neutrosophic metric spaces. To that end, we give a number of properties and characterizations of these notions.

Keywords: Neutrosophic set, single valued neutrosophic set, neutrosophic metric space.

2020 Mathematics Subject Classification: 03B52, 54E35, 47H10.

1 Introduction

In 1995, F. Smarandache [11] has generalized the concepts of fuzzy and intuitionistic fuzzy sets to the notion of neutrosophic sets to know the correct way of dealing with imprecise and indeterminate data. Neutrosophic sets are characterized by three independent components that truth membership function (T), indeterminacy membership function (I), and falsity membership function (F), and they have been useful in many real applications in several branches (see for e.g., [3,7,8,12,14]). Recently, we have introduced the notion of standard single valued neutrosophic metric space [2,6] and studied some of their fundamental properties.

Many authors have taken great care in studying the critical properties of various types of topological spaces. For instance, Latreche et al [5] have established the property of continuity in single valued neutrosophic topological space and investigated relationships among various types of single valued neutrosophic continuous mapping. Later on, Milles et al [6] have introduced other topological properties, such as the completeness and compactness in standard single valued neutrosophic metric spaces, where they have investigated their most interesting properties and characterizations. In particular, J Kider and Z Hussain [4] introduced a continuous mapping and uniformly continuous mapping from standard fuzzy metric space $(X, M, *)$ into a standard fuzzy metric space $(Y, M, *)$. In this paper, we will focus on

✉ Corresponding author. Email: soheyb.milles@cu-barika.dz

ISSN (electronic): 2773-4196

studying these properties, especially in standard single valued neutrosophic metric spaces. Furthermore, we discuss some characterizations of these notions. This paper is structured as follows. In Section 2, we recall the necessary basic notions and properties of standard fuzzy metric space and single valued neutrosophic sets with some related concepts that will be needed throughout this paper. In Section 3, the notion of standard fuzzy metric space is introduced, and some fundamental properties related to this concept are studied. By introducing the notions of continuous mapping and uniformly continuous mapping in a standard single-valued neutrosophic metric space, we discuss the interesting continuity properties in these spaces in Section 4. Finally, we present some conclusions and discuss future research in Section 5.

2 Preliminaries

This section contains the basic definitions and properties of single valued neutrosophic sets and some related notions that will be needed throughout this paper.

Definition 2.1. [16] Let X be a nonempty set. A fuzzy set $A = \{\langle x, \mu_A(x)\rangle \mid x \in X\}$ is characterized by a membership function $\mu_A : X \to [0,1]$, where $\mu_A(x)$ is interpreted as the degree of membership of the element x in the fuzzy subset A for any $x \in X$.

In 1983, Atanassov [1] proposed a generalization of Zadeh membership degree and introduced the notion of the intuitionistic fuzzy set.

Definition 2.2. [1] Let X be a nonempty set. An intuitionistic fuzzy set (IFS, for short) A on X is an object of the form $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in X\}$ characterized by a membership function $\mu_A : X \to [0,1]$ and a non-membership function $\nu_A : X \to [0,1]$ which satisfy the condition:

$$0 \le \mu_A(x) + \nu_A(x) \le 1, \text{ for any } x \in X.$$

In 1998, Smarandache [11] defined the concept of a neutrosophic set as a generalization of Atanassov's intuitionistic fuzzy set. Also, he introduced neutrosophic logic, neutrosophic set, and its applications in [9, 10]. In particular, Wang et al. [15] introduced the notion of a single valued neutrosophic set.

Definition 2.3. [9] Let X be a nonempty set. A neutrosophic set (NS, for short) A on X is an object of the form $A = \{\langle x, \mu_A(x), \sigma_A(x), \nu_A(x)\rangle \mid x \in X\}$ characterized by a membership function $\mu_A : X \to]^-0, 1^+[$ and an indeterminacy function $\sigma_A : X \to]^-0, 1^+[$ and a non-membership function $\nu_A : X \to]^-0, 1^+[$ which satisfy the condition:

$$^-0 \le \mu_A(x) + \sigma_A(x) + \nu_A(x) \le 3^+, \text{ for any } x \in X.$$

Certainly, intuitionistic fuzzy sets are neutrosophic sets by setting $\sigma_A(x) = 1 - \mu_A(x) - \nu_A(x)$.

Next, one shows the notion of single valued neutrosophic set as an instance of the neutrosophic set, which can be used in real scientific and engineering applications.

Definition 2.4. [15] Let X be a nonempty set. A single valued neutrosophic set (SVNS, for short) A on X is an object of the form $A = \{\langle x, \mu_A(x), \sigma_A(x), \nu_A(x)\rangle \mid x \in X\}$ characterized by a truth-membership function $\mu_A : X \to [0,1]$, an indeterminacy-membership function $\sigma_A : X \to [0,1]$ and a falsity-membership function $\nu_A : X \to [0,1]$.

The class of single valued neutrosophic sets on X is denoted by $SVN(X)$.

For any two SVNSs A and B on a set X, several operations are defined (see, e.g., [13,15]); mainly, we will present those related to the present paper.

(i) $A \subseteq B$ if $\mu_A(x) \leq \mu_B(x)$ and $\sigma_A(x) \leq \sigma_B(x)$ and $\nu_A(x) \geq \nu_B(x)$, for all $x \in X$,

(ii) $A = B$ if $\mu_A(x) = \mu_B(x)$ and $\sigma_A(x) = \sigma_B(x)$ and $\nu_A(x) = \nu_B(x)$, for all $x \in X$,

(iii) $A \cap B = \{\langle x, \mu_A(x) \wedge \mu_B(x), \sigma_A(x) \wedge \sigma_B(x), \nu_A(x) \vee \nu_B(x)\rangle \mid x \in X\}$,

(iv) $A \cup B = \{\langle x, \mu_A(x) \vee \mu_B(x), \sigma_A(x) \vee \sigma_B(x), \nu_A(x) \wedge \nu_B(x)\rangle \mid x \in X\}$,

(v) $\overline{A} = \{\langle x, 1 - \nu_A(x), 1 - \sigma_A(x), 1 - \mu_A(x)\rangle \mid x \in X\}$.

3 Standard single valued neutrosophic metric space

In this section, one generalizes the notion of standard fuzzy metric space introduced by J.R. Kider and Z.A. Hussain [4] to the setting of single valued neutrosophic sets.

Definition 3.1. [2] A quintuple $(X, M, *, \triangleleft, \diamond)$ is said to be a standard single valued neutrosophic metric space if X is an arbitrary set, $*$, $\triangleleft$ are a continuous $t-$norms, $\diamond$ is a $t-$conorm and M is a continuous single valued neutrosophic set on X^2 satisfying the following conditions:

(i) $\mu_M(x,y) > 0$, $\sigma_M(x,y) > 0$ and $\nu_M(x,y) < 1$ for all $x, y \in X$,

(ii) $\mu_M(x,y) = 1$, $\sigma_M(x,y) = 1$ and $\nu_M(x,y) = 0$ if and only if $x = y$,

(iii) $\mu_M(x,y) = \mu_M(y,x)$, $\sigma_M(x,y) = \sigma_M(y,x)$ and $\nu_M(x,y) = \nu_M(y,x)$ for all $x, y \in X$,

(iv) $\mu_M(x,z) \geq \mu_M(x,y) * \mu_M(y,z)$, $\sigma_M(x,z) \geq \sigma_M(x,y) \triangleleft \sigma_M(y,z)$ and $\nu_M(x,z) \leq \nu_M(x,y) \diamond \nu_M(y,z)$ for all $x, y, z \in X$.

The functions $\mu_M(x,y)$, $\sigma_M(x,y)$ and $\nu_M(x,y)$ denote the degree of nearness, the degree of neutralness and the degree of non-nearness between x and y, respectively.

Example 3.2. Let (X, d) be an ordinary metric space. Define the t-norms $x * y = min\{x, y\}$, $x \triangleleft y = min\{x, y\}$ and the t-conorm $x \diamond y = max\{x, y\}$, for all x, $y \in [0,1]$. Define the single valued neutrosophic set M on X^2 as:
$\mu_M(x,y) = \frac{1}{1+d(x,y)}$, $\sigma_M(x,y) = 1 + d(x,y)$, $\nu_M(x,y) = \frac{d(x,y)}{1+d(x,y)}$.
Then, $(X, M, *, \triangleleft, \diamond)$ is a standard single valued neutrosophic metric space.

Next, one introduces the standard single valued neutrosophic distance between an element and a subset of X and the standard single valued neutrosophic distance between two subsets of X.

Definition 3.3. Let $(X, M, *, \triangleleft, \diamond)$ be a standard single valued neutrosophic metric space. For $x \in X$ and A, B are a subsets of X. Then

(i) the standard single valued neutrosophic distance between x and A is defined as

$$\mu_M(x, A) = inf\{\mu_M(x,y) \mid y \in A\},\ \sigma_M(x, A) = inf\{\sigma_M(x,y) \mid y \in A\},$$

$$\text{and } \nu_M(x, A) = sup\{\nu_M(x,y) \mid y \in A\},$$

(ii) the standard single valued neutrosophic distance between A and B is defined as

$$\mu_M(A,B) = inf\{\mu_M(x,y) \mid x \in A, y \in B\},\ \sigma_M(A,B) = inf\{\sigma_M(x,y) \mid x \in A, y \in B\},$$

$$\text{and } \nu_M(A,B) = sup\{\nu_M(x,y) \mid x \in A, y \in B\}.$$

Definition 3.4. Let $(X, M, *, \triangleleft, \diamond)$ be a standard single valued neutrosophic metric space. For $x \in X$ and $r \in]0,1[$, the open ball $\mathcal{B}(x,r)$ with radius r and center x is defined by

$$\mathcal{B}(x,r) = \{y \in X \mid \mu_M(x,y) > 1-r,\ \sigma_M(x,y) > 1-r \text{ and } \nu_M(x,y) < r\}.$$

Definition 3.5. Let $(X, M, *, \triangleleft, \diamond)$ be a standard single valued neutrosophic metric space, a subset A of X is said to be an open set (OS, for short) if for any $x \in A$ there exists $r \in]0,1[$ such that $\mathcal{B}(x,r) \subseteq A$. The complement of an open set is called a closed set (CS, for short) in X.

Definition 3.6. Let $(X, M, *, \triangleleft, \diamond)$ be a standard single valued neutrosophic metric space, and $A \subseteq X$ a subset. One defines the interior of A to be the set $int(A) = \{a \in A \mid \mathcal{B}(x,r) \subseteq A \mid r \in]0,1[\}$.

Theorem 3.7. *Let $(X, M, *, \triangleleft, \diamond)$ be a standard single valued neutrosophic metric space, and $A \subseteq X$ a subset. Then $int(A)$ is open and is the largest open set of X inside of A.*

Proof. Firstly, one shows that $int(A)$ is open. By its definition if $x \in int(A)$ then $\mathcal{B}(x, r_x) \subseteq A$, $r_x \in]0,1[$. But since $\mathcal{B}(x, r_x)$ is itself an open set, one sees that any $y \in \mathcal{B}(x, r_x)$ has some $\mathcal{B}(y, r_y) \subseteq \mathcal{B}(x, r_x) \subseteq A$, $r_x \in]0,1[$, which forces $y \in int(A)$. That is, one has shown $\mathcal{B}(x, r_x) \subseteq int(A)$, whence $int(A)$ is open. If $U \subseteq A$ is an open set in X, then for each $u \in U$ there is $r_u \in]0,1[$ such that $\mathcal{B}(u, r_u) \subseteq U$, whence $\mathcal{B}(u, r_u) \subseteq A$, so $u \in int(A)$. This is true for all $u \in U$, so $U \subseteq int(A)$. □

Corollary 3.8. *A subset A in a standard single valued neutrosophic metric space X is open if and only if $A = int(A)$.*

Definition 3.9. Let $(X, M, *, \triangleleft, \diamond)$ be a standard single valued neutrosophic metric space. Then,

(i) a sequence (x_n) in X is said to be convergent to a point $x \in X$ (i.e., $\lim_{n\to\infty} x_n = x$) if,

$$\lim_{n\to\infty} \mu_M(x_n, x) = 1,\ \lim_{n\to\infty} \sigma_M(x_n, x) = 1 \text{ and } \lim_{n\to\infty} \nu_M(x_n, x) = 0,$$

(ii) a sequence (x_n) in X is said to be Cauchy sequence if for each $k > 0$,

$$\lim_{n\to\infty} \mu_M(x_{n+k}, x_n) = 1,\ \lim_{n\to\infty} \sigma_M(x_{n+k}, x_n) = 1 \text{ and } \lim_{n\to\infty} \nu_M(x_{n+k}, x_n) = 0.$$

Definition 3.10. Let $(X, M, *, \triangleleft, \diamond)$ be a standard single valued neutrosophic metric space. Then

(i) if every Cauchy sequence is convergent, then X is said to be complete.

(ii) X is said to be compact if every sequence contains a convergent subsequence.

3.1 Properties of standard single valued neutrosophic metric space

In this section, one investigates some properties of standard single valued neutrosophic metric space.

Proposition 3.11. *Every open ball in a standard single valued neutrosophic metric space* $(X, M, *, \triangleleft, \diamond)$ *is an open set.*

Proof. Let $\mathcal{B}(x,r)$ be an open ball with radius r and center x, where $r \in]0,1[$ and $x \in X$. Suppose that $y \in \mathcal{B}(x,r)$, this implies that

$$\mu_M(x,y) > 1-r,\ \sigma_M(x,y) > 1-r \text{ and } \nu_M(x,y) < r.$$

Let $r_0 = \mu_M(x,y)$. Then, there exist $s \in]0,1[$ such that $r_0 > 1-s > 1-r$. Now, for a given r_0 and s such that $r_0 > 1-s$. Then, there exist $r_1, r_2, r_3 \in]0,1[$ such that

$$r_0 * r_1 \geq 1-s, r_0 \triangleleft r_2 \geq 1-s \text{ and } (1-r_0) \diamond (1-r_3) \leq s.$$

Next, if one puts $r_4 = \max\{r_1, r_2, r_3\}$ and considers the open ball $\mathcal{B}(y, 1-r_4)$, then from the above, one can show that $\mathcal{B}(y, 1-r_4) \subset \mathcal{B}(x,r)$ as follows:
Let $z \in \mathcal{B}(y, 1-r_4)$. Then, $\mu_M(y,z) > r_4$, $\sigma_M(y,z) > r_4$ and $\nu_M(y,z) < 1-r_4$. Furthermore, one obtains

$$\mu_M(x,z) \geq \mu_M(x,y) * \mu_M(y,z) \geq r_0 * r_4 \geq r_0 * r_1 \geq 1-s > 1-r,$$

$$\sigma_M(x,z) \geq \sigma_M(x,y) \triangleleft \sigma_M(y,z) \geq r_0 \triangleleft r_4 \geq r_0 \triangleleft r_2 \geq 1-s > 1-r$$

$$\text{and } \nu_M(x,z) \leq \nu_M(x,y) \diamond \nu_M(y,z) \leq (1-r_0) \diamond (1-r_4) \leq (1-r_0) \diamond (1-r_3) \leq s < r.$$

It follows that $z \in \mathcal{B}(x,r)$, and hence $\mathcal{B}(y, 1-r_4) \subset \mathcal{B}(x,r)$. According to Definition 3.5, it holds that $\mathcal{B}(x,r)$ is an open set. □

Proposition 3.12. *Let* $\mathcal{B}(x,r_1)$ *and* $\mathcal{B}(x,r_2)$ *be two open balls with the same center* x *in a standard fuzzy metric space* $(X, M, *, \triangleleft, \diamond)$*. Then, either* $\mathcal{B}(x,r_1) \subseteq \mathcal{B}(x,r_2)$ *or* $\mathcal{B}(x,r_2) \subseteq \mathcal{B}(x,r_1)$ *where* $r_1, r_2 \in]0,1[$.

Proof. Let $x \in X$ and $r_1, r_2 \in]0,1[$. If $r_1 = r_2$, then $\mathcal{B}(x,r_1) = \mathcal{B}(x,r_2)$, hence the result trivially holds. Next, one assumes that $r_1 \neq r_2$. Then, one can distinguish two cases: $r_1 < r_2$ and $r_2 > r_1$.

(i) If $r_1 < r_2$ and suppose that $y \in \mathcal{B}(x,r_1)$, then $\mu_M(x,y) > 1-r_1$, $\sigma_M(x,y) > 1-r_1$ and $\nu_M(x,y) < r_1$, which implies that $\mu_M(x,y) > 1-r_2$, $\sigma_M(x,y) > 1-r_2$ and $\nu_M(x,y) < r_2$. Therefore, $y \in \mathcal{B}(x,r_2)$, and hence $\mathcal{B}(x,r_1) \subseteq \mathcal{B}(x,r_2)$.

(ii) If $r_1 > r_2$, then by applying a similar reasoning, one gets $\mathcal{B}(x,r_2) \subseteq \mathcal{B}(x,r_1)$.

□

Theorem 3.13. *Let* $(X, M, *, \triangleleft, \diamond)$ *be a standard single valued neutrosophic metric space. Then, it holds that the set*

$$\tau_M = \{A \subseteq X \mid x \in A \textit{ if and only if there exists } r \in]0,1[\textit{ such that } \mathcal{B}(x,r) \subseteq A\}$$

is a topology on X *called the topology induced by the single valued neutrosophic set* M.

4 Standard single valued neutrosophic continuous mappings

In this section, one will study some interesting properties of continuity in standard single valued neutrosophic metric spaces. First, one introduces the notion of continuous mapping and uniformly continuous mapping in a standard single valued neutrosophic metric space.

Definition 4.1. Let $(X, M, *, \triangleleft, \diamond)$ and $(Y, M', *, \triangleleft, \diamond)$ be two SSVN-metric spaces. A function $f : X \longrightarrow Y$ is said to be single valued neutrosophic continuous at $a \in X$, if for every $r \in]0,1[$, there exists $\delta \in]0,1[$ such that

$$\mu_{M'}(f(x), f(a)) > 1-r,\ \sigma_{M'}(f(x), f(a)) > 1-r \text{ and } \nu_{M'}(f(x), f(a)) < r,$$
$$\text{whenever } \mu_M(x,a) > 1-\delta,\ \sigma_M(x,a) > 1-\delta \text{ and } \nu_M(x,a) < \delta.$$

There is another approach to define the continuous mapping in single valued neutrosophic metric space.

Definition 4.2. Let $(X, M, *, \triangleleft, \diamond)$ and $(Y, M', *, \triangleleft, \diamond)$ be two SSVN-metric spaces. A function $f : X \longrightarrow Y$ is said to be single valued neutrosophic continuous at $a \in X$, if and only if whenever a sequence (x_n) in X converge to a, the sequence $(f(x_n))$ converges to $f(a)$.

Proposition 4.3. *Let $(X, M, *, \triangleleft, \diamond)$ and $(Y, M', *, \triangleleft, \diamond)$ be two SSVN-metric spaces. A function $f : X \longrightarrow Y$ is said to be single valued neutrosophic continuous at $a \in X$, if and only if for every $0 < \epsilon < 1$, there exists $0 < \delta < 1$ such that $B(a, \delta) \subseteq f^{-1}(B(f(a), \epsilon))$, where $B(a, \delta)$ denotes the open ball of radius δ with center a.*

Proof. The mapping $f : X \longrightarrow Y$ is continuous at $a \in X$ if and only if for every $\epsilon \in]0,1[$, there exists $\delta \in]0,1[$ such that

$$\mu_{M'}(f(x), f(a)) > 1-\epsilon,\ \sigma_{M'}(f(x), f(a)) > 1-\epsilon \text{ and } \nu_{M'}(f(x), f(a)) < \epsilon,$$
$$\text{whenever } \mu_M(x,a) > 1-\delta,\ \sigma_M(x,a) > 1-\delta \text{ and } \nu_M(x,a) < \delta.$$

i.e $x \in B(a, \delta)$ implies $f(x) \in B(f(a), \epsilon)$ or $f(B(a, \delta)) \subseteq B(f(a), \epsilon)$
This is equivalent to the condition

$$B(a, \delta) \subseteq f^{-1}(B(f(a), \epsilon)).$$

□

Theorem 4.4. *Let $(X, M, *, \triangleleft, \diamond)$ and $(Y, M', *, \triangleleft, \diamond)$ be two SSVN-metric spaces. A function $f : X \longrightarrow Y$ is said to be single valued neutrosophic continuous on X, if and only if $f^{-1}(G)$ is open in X for all open subset G of Y.*

Proof. Suppose f is a single valued neutrosophic continuous on X and let G be an open subset of Y.
One has to show that $f^{-1}(G)$ is open in X. Since $\emptyset$ and X are open, one may suppose that $f^{-1}(G) \neq \emptyset$ and $f^{-1}(G) \neq X$. Let $x \in f^{-1}(G)$. Then, $f(x) \in G$. Since G is open, there exists $0 < \varepsilon < 1$ such that $B(f(x), \epsilon) \subseteq G$. Since f is a single valued neutrosophic continuous at x, by Proposition 4.3 for this ϵ there exists $\delta \in]0,1[$ such that $B(x, \delta) \subseteq f^{-1}(B(f(x), \epsilon)) \subseteq f^{-1}(G)$ Thus, every point x of $f^{-1}(G)$ is an interior point, and so $f^{-1}(G)$ is open in X. Suppose, conversely, that $f^{-1}(G)$ is open in X for all open subsets G of Y. Let $x \in X$ for each $0 < \epsilon < 1$, the set $B(f(x), \epsilon)$ is open and so $f^{-1}(B(f(x), \epsilon))$ is open in X. Since $x \in f^{-1}(B(f(x), \epsilon))$ it follows that there exists $0 < \delta < 1$ such that $B(x, \delta) \subseteq f^{-1}(B(f(x), \epsilon))$. By Proposition 4.3 it follows that f is continuous of x. □

Corollary 4.5. *Let $(X, M, *, \triangleleft, \diamond)$ and $(Y, M', *, \triangleleft, \diamond)$ be two SSVN-metric spaces. A function $f : X \longrightarrow Y$ is said to be single valued neutrosophic continuous on X, if and only if $f^{-1}(G)$ is closed in X for all closed subset F of Y.*

Theorem 4.6. *Let $(X, M, *, \triangleleft, \diamond)$, $(Y, M', *, \triangleleft, \diamond)$, $(Z, M'', *, \triangleleft, \diamond)$ be three SSVN-metric spaces and let $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ be a continuous mappings, then the composition $g \circ f$ is a continuous mapping of X into Z.*

Proof. Let G be open subset of Z. By Theorem 4.4, $g^{-1}(G)$ is an open subset of Y and another application of the same theorem shows that $f^{-1}(g^{-1}(G))$ is an open subset of X. Since $(g \circ f)^{-1}(G) = f^{-1}(g^{-1}(G))$, it follows from the same theorem again that $g \circ f$ is continuous . □

Definition 4.7. Let $(X, M, *, \triangleleft, \diamond)$ and $(Y, M', *, \triangleleft, \diamond)$ be two SSVN-metric spaces. A function $f : X \longrightarrow Y$ is said to be single valued neutrosophic uniformly continuous on X, if for every $r \in]0, 1[$, there exists $\delta \in]0, 1[$ such that

$$\mu_{M'}(f(x_1), f(x_2)) > 1 - r,\ \sigma_{M'}(f(x_1), f(x_2)) > 1 - r \text{ and } \nu_{M'}(f(x_1), f(x_2)) < r\,,$$

$$\text{whenever } \mu_M(x_1, x_2) > 1 - \delta,\ \sigma_M(x_1, x_2) > 1 - \delta \text{ and } \nu_M(x_1, x_2) < \delta\,.$$

Theorem 4.8. *Let $f : (X, M, *, \triangleleft, \diamond) \longrightarrow (Y, M', *, \triangleleft, \diamond)$ to be a one-to-one and uniformly continuous. If f^{-1} is a single valued neutrosophic continuous and Y is complete, then X is complete.*

Proof. Suppose that (x_n) is a Cauchy sequence and let the sequence $y_n = f(x_n)$. One shows that (y_n) is a Cauchy sequence. Since (x_n) is a Cauchy sequence, it follows that

$$\mu_M(x_1, x_2) > 1 - \delta,\ \sigma_M(x_1, x_2) > 1 - \delta \text{ and } \nu_M(x_1, x_2) < \delta\,,$$

for any $\delta \in]0, 1[$. This implies that

$$\mu_{M'}(f(x_1), f(x_2)) > 1 - r,\ \sigma_{M'}(f(x_1), f(x_2)) > 1 - r \text{ and } \nu_{M'}(f(x_1), f(x_2)) < r\,,$$

for any $r \in]0, 1[$ and, there exists $k \in \mathbb{N}$ such that $m, n > k$ imply that

$$\mu_M(x_n, x_m) > 1 - \delta,\ \sigma_M(x_n, x_m) > 1 - \delta \text{ and } \nu_M(x_n, x_m) < \delta\,.$$

It follows that for $m, n > k$

$$\mu_{M'}(y_n, y_m) > 1 - r,\ \sigma_{M'}(y_n, y_m) > 1 - r \text{ and } \nu_{M'}(y_n, y_m) < r\,.$$

Hence, (y_n) is Cauchy sequence which implies that there exists a subsequence (y_{n_k}) such that y_{n_k} converge to y, where $y \in Y$. Since f^{-1} is a single valued neutrosophic continuous mapping, it follows that $x_{n_k} = f^{-1}(y_{n_k})$ converges to $f^{-1}(y) = x$. One concludes that X is complete. □

5 Conclusion

In this paper, we have studied the notions of continuous mapping and uniformly continuous mapping on standard single valued neutrosophic metric spaces, with their characterizations as interesting topological properties. Due to the usefulness of these notions, we think it makes sense to study these notions for other types of structure. Future efforts will be directed to the type of metric spaces with respect to SVN-sets.

Conflict of Interest

The authors have no conflicts of interest to disclose.

References

[1] K. Atanassov, *Intuitionistic fuzzy sets*, VII ITKRs Scientific Session, Sofia, 1983.

[2] O. Barkat, S. Milles and A. Latreche, *Standard Single Valued Neutrosophic Metric Spaces with application*, TWMS J. Apl. & Eng. Math. Accepted.

[3] Y. Guo and H.D. Cheng, *New neutrosophic approach to image segmentation*, Pattern Recognit. **42**(5) (2009), 587–595.

[4] J.R. Kider and Z.A. Hussain, *Continuous and Uniform Continuous Mappings on a Standard Fuzzy Metric Spaces*, Eng.& Tech. **32**(6) (2014), 1111–1119.

[5] A. Latreche, O. Barkat, S. Milles and F. Ismail, *Single valued neutrosophic mappings defined by single valued neutrosophic relations with applications*, Neutrosophic Sets Syst. **32**(1) (2020), 202–220.

[6] S. Milles, A. Latreche and O. Barkat, *Completeness and Compactness in Standard Single Valued Neutrosophic Metric Spaces*, Int. j. neutrosophic sci. **12**(2) (2020), 96-104.

[7] K. Mondal and S. Pramanik, *A study on problems of Hijras in West Bengal based on neutrosophic cognitive maps*, Neutrosophic Sets Syst. **5** (2014), 21–26.

[8] S. Pramanik and K. Mondal, *Weighted fuzzy similarity measure based on tangent function and its application to medical diagnosis*, Int. J. Innov. Res. Sci. Eng. Technol. **4** (2015), 158–164.

[9] F. Smarandache, *Neutrosophic Probability and Statistics*, In: A Unifying Field in Logics: Neutrosophic Logic. Neutrosophy, Neutrosophic Set, InfoLearnQuest, USA, 2007.

[10] F. Smarandache, *n-valued refined neutrosophic logic and its applications to Physics*, Prog. Phys. **8** (2013), 143–146.

[11] F. Smarandache, *In: Neutrosophy. Neutrisophic Property, Sets, and Logic*, American Research Press. Rehoboth. USA, 1998.

[12] F. Smarandache and S. Pramanik, *New Trends in Neutrosophic Theory and Applications*, Pons Editions, Brussels, 2016.

[13] H.L. Yang, Z.L.Guo and X. Liao, *On single valued neutrosophic relations*, J. Intell. Fuzzy. Syst. **30** (2016), 1045–1056.

[14] J. Ye, *Improved correlation coefficients of single-valued neutrosophic sets and interval neutrosophic sets for multiple attribute decision making*, J. Intell. Fuzzy Syst. **27** (2014), 2453–2462.

[15] H. Wang, F. Smarandache, Y.Q. Zhang and R. Sunderraman, *Single valued neutrosophic sets*, Multispace & Multistructure. Neutrosophic Transdisciplinarity **4** (2010), 410–413.

[16] L.A. Zadeh, *Fuzzy sets*, Inform. Control. **8** (1965), 338–353.

Journal of Innovative Applied Mathematics and Computational Sciences

J. Innov. Appl. Math. Comput. Sci. **1**(1) (2021) 48–63. n2t.net/ark:/49935/jiamcs.v1i1.4

http://jiamcs.centre-univ-mila.dz/

Recent progress in the conductivity reconstruction in Calderón's problem

Manal Aoudj [1]

[1]School of Mathematics and Statistics, Central China Normal University, Wuhan, 430079, China

Received 15 November 2021, Accepted 28 December 2021, Published 30 December 2021

Abstract. In this work, we study a nonlinear inverse problem for an elliptic partial differential equation known as the Calderón problem or the inverse conductivity problem. We give a quick survey on the reconstruction question of conductivity from measurements on the boundary, by covering the main currently known results regarding the isotropic problem with full data in two and higher dimensions. We present Nachman's reconstruction procedure and summarize the theoretical progress of the technique to more recent results in the field. An open problem of significant interest is proposed to check whether extending the method for Lipschitz conductivities is possible.

Keywords: Calderón problem, inverse conductivity problem, Dirichlet-to-Neumann map, complex geometrical optics solutions, $\bar{\partial}$-method, boundary integral equation.

2020 Mathematics Subject Classification: 35R30.

1 Introduction

The present paper aims to summarize some reconstruction results from boundary measurements for less regular conductivities in the inverse conductivity problem, which has been developed for over 30 years and provides references for further research and practical applications on the topic. The Calderón problem [15] asks to recover a conductivity of a domain from measurements that are taken on the boundary. For a formal definition, let $\Omega \subset \mathbb{R}^n, n \geq 2$ be a bounded domain with sufficiently smooth boundary $\partial\Omega$, and let γ be a positive real-valued function representing the electrical conductivity of Ω such that for almost every $x \in \Omega$ and for a constant $c_0 > 0$, the condition

$$\gamma(x) \geq c_0, \tag{1.1}$$

is satisfied. The application of a voltage $\psi \in H^{1/2}(\partial\Omega)$ on the boundary induces an electrical potential $w \in H^1(\Omega)$ in the interior of Ω, where w is the unique weak solution of the following elliptic boundary value problem

$$\begin{cases} \nabla \cdot \gamma \nabla w = 0 & \text{in } \Omega, \\ w = \psi & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

✉ Corresponding author. Email: 3370797235@qq.com

ISSN (electronic): 2773-4196

In this case, the Dirichlet-to-Neumann map (DN map) relating the boundary voltage ψ (Dirichlet data) to the flux at the boundary $\gamma\frac{\partial w}{\partial \nu}$ (Neumann data) is defined as follows

$$\Lambda_\gamma : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega),$$

$$\psi \mapsto \Lambda_\gamma(\psi) = \gamma \frac{\partial w}{\partial \nu}\bigg|_{\partial\Omega},$$

where $\frac{\partial}{\partial \nu}$ is the outward normal derivative at $\partial\Omega$.

In this paper, we consider the Calderón problem of reconstructing a conductivity from measurements on the boundary. Since the motivation to reconstruct a conductivity comes from its uniqueness, we should first ask if it is possible to determine γ from the knowledge of Λ_γ, i.e., whether the map $\gamma \mapsto \Lambda_\gamma$ is injective? In 1980, Alberto Calderón, who proposed the problem, gave a positive answer. He proved in his pioneer paper [15] that for γ a perturbation of the identity, the injectivity of the linearized inverse problem holds. For $n \geq 3$, Sylvester and Uhlmann [42] were the first to show uniqueness for C^2 conductivities. They reduced the problem to a similar one for a Schrödinger equation. This reduction is based on the well-known Liouville transformation: if z is a weak solution of the conductivity equation $\nabla \cdot \gamma \nabla z = 0$, then $w = \gamma^{1/2} z$ is a solution to the Schrödinger equation $(-\Delta + q)w = 0$, where the potential $q = \gamma^{-1/2}\Delta\gamma^{1/2}$. Under the standard assumption that 0 is not a Dirichlet eigenvalue for the Schrödinger equation, and for $q \in L^\infty(\Omega), \psi \in H^{1/2}(\partial\Omega)$, they considered the following Dirichlet problem

$$\begin{cases} -\Delta w + qw = 0 & \text{in } \Omega, \\ w = \psi & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

The DN map associated with q is well-defined from $H^{1/2}(\partial\Omega)$ into $H^{-1/2}(\partial\Omega)$ by $\psi \mapsto \Lambda_q(\psi) = \frac{\partial w}{\partial \nu}\Big|_{\partial\Omega}$. The idea of Sylvester and Uhlmann was to look for special solutions $w(x,\zeta), \zeta \in \mathbb{C}^n, \zeta \cdot \zeta = 0$ satisfying $(-\Delta + q)w = 0$, which are asymptotically exponential, i.e., $w \sim e^{i\zeta . x}$ when $|\zeta| \to \infty$. The functions $w(x,\zeta) = e^{i\zeta . x}(1 + y_\zeta(x))$ are called complex geometrical optics solutions (CGOs), where $y_\zeta(x) \in H^1(\Omega)$ is a correction term that is needed to transit from an approximate solution to the exact one by taking $|\zeta| \to \infty$. Their result inspired many authors to find the lowest regularity condition on the conductivity under which uniqueness holds. More recent uniqueness results, and the used techniques are listed in table 1.1.

Table 1.1: Recent uniqueness results for $n \geq 3$.

n	γ	Techniques	Ref
≥ 3	$W^{3/2,2n+}$	Approximation argument	[14]
≥ 3	$C^1, W^{1,\infty}$ with $\|\|\nabla \log \gamma\|\|_{L^\infty}$ small	Bs, averaging argument	[23]
≥ 3	$W^{1,\infty}$	Bourgain's spaces (Bs)	[16]
3	$H^{3/2+}$	Standard Sobolev spaces	[36]
3,4	$W^{1,n}$	Bs, L^p harmonic analysis	[22]
5,6	$W^{1+(1-\theta)(1/2-2/n),n/(1-\theta)}, \theta \in [0,1)$	Bs, L^p harmonic analysis	[22]
5	$W^{41/40+,5}$	Bilinear estimate	[24]
6	$W^{11/10+,6}$	Bilinear estimate	[24]
≥ 5	$W^{1+\frac{n-5}{2p}+,p}, p \in [n,\infty)$	Bilinear estimate	[39]

The observation of the table makes us wonder how much it would be interesting to check whether it is possible to prove Brown's conjecture [11], which affirms that in three and higher dimensions $\gamma \in W^{1,n}$ is the minimum possible regularity for which uniqueness holds. Notice that the approaches used in [22,24,36] are not useful for reconstructing γ, because the proofs there are not constructive, meaning that they did not give a procedure to recover γ from Λ_γ.

The two-dimensional problem is also of significant interest but differs mainly from the higher dimensional one so that different techniques are used to address this case. Nachman [33] was the first who proved uniqueness for $\gamma \in W^{2,d}, d > 1$ in the plane. This last regularity assumption was relaxed by Brown-Uhlmann [12] to $\gamma \in W^{1,2+}$, and by Astala-Päivärinta [8] to $\gamma \in L^\infty$.

Once uniqueness holds, one can be interested in the reconstruction problem. In practice, Nachman's reconstruction procedure was widely applied in the implementation of algorithms [40]. For example, in medical imaging technology, the electrical impedance tomography (EIT) with several applications, including the detection of breast cancer and pulmonary imaging. See the review papers [11,25] for more detailed arguments on this technique.

While the current paper deals mainly with the entire data problem, we note that the partial data problem is subject to huge advances. The partial data type problem aims to reduce as much as possible the part of the boundary, where measurements are taken, and excitations on the studied body are imposed because, from a realistic view, it is not practical to consider measurements on the whole boundary of some domain. We refer the reader to the excellent survey paper [26] by Kenig and Salo on the recent progress in this problem. For the reconstruction results with partial data, we give further references [3,5,35]. When γ depends on direction, we are in the presence of the anisotropic Calderón problem. In the plane, uniqueness was shown for L^∞ anisotropic conductivities in [7]. For $n \geq 3$, this problem is also called Calderón's inverse problem on Riemannian manifolds, and as Lassas and Uhlmann pointed out in [30], this is a geometrical problem that has up to now remained open.

We aim to offer the interested reader a short introduction to the reconstruction problem. We hope that this work could inspire a different way of proposing a method of reconstructing the conductivity. We have not attempted to be exhaustive in this introduction. In particular, we have neglected stability and numerical results and closely related inverse problems. As the research field on the Calderón problem is too broad, we refer the reader to the review works [4,9,17,25,46] on the general problem.

The rest of this article is organized in the following way: the applied notation and background knowledge are summarized in Section 2. In Section 3, we give the precise statements of the known reconstruction results. Section 4 discusses the proof strategy. Section 5 contains an open problem.

2 Preliminaries

Throughout this article

- Ω denotes a bounded open set of $\mathbb{R}^n$ with smooth boundary $\partial\Omega$.
- $n \geq 2$ denotes the space dimension.

- $q : \Omega \to \mathbb{R}$ denotes an electrical potential.
- dS denotes the surface on $\partial\Omega$.
- $\mathcal{S}(\mathbb{R}^n)$ denotes Schwartz space.
- $\mathcal{S}'(\mathbb{R}^n)$ denotes the space of tempered distributions.
- $\langle , \rangle$ denotes the dual pairing between $H^{1/2}(\partial\Omega)$ and $H^{-1/2}(\partial\Omega)$.
- $\mathbb{D}$ denotes the unit disc in $\mathbb{C}$.
- $B_R(0)$ denotes the closed ball with center 0 and radius $R > 0$.
- $a \lesssim b$ denotes that it exists a constant $c > 0$ such that $a \leq cb$.

2.1 Fourier transform and function spaces

For $\xi \in \mathbb{R}^n$, the applied notation for the Fourier transform is

$$\hat{w}(\xi) = \int_{\mathbb{R}^n} e^{-i\xi.x} w(x) dx.$$

The inverse Fourier transform is noted by

$$\check{w}(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i\xi.x} w(\xi) d\xi.$$

For $s \in \mathbb{R}$, we define Sobolev spaces $H^s(\mathbb{R}^n)$ via Fourier transform as follows:

$$H^s(\mathbb{R}^n) = \{ w \in \ \mathcal{S}'(\mathbb{R}^n) : \ \ \langle \xi \rangle^s \widehat{w} \in L^2(\mathbb{R}^n) \},$$

where $\langle \xi \rangle = (|\xi|^2 + 1)^{1/2}$.
The associated norm is
$\|w\|_{H^s(\mathbb{R}^n)} = \|\langle \xi \rangle^s \widehat{w}\|_{L^2(\mathbb{R}^n)}$.
Recalling the Schrödinger equation from the problem (1.3), substituting with $w(x,\zeta) = e^{i\zeta.x}(1 + y_\zeta(x))$, we deduce an equivalent equation for y_ζ, precisely

$$\triangle_\zeta y_\zeta = (\Delta + 2i\zeta \cdot \nabla) y_\zeta = q(1 + y_\zeta) \ \text{ in } \Omega.$$

The right inverse of the differential operator $\triangle_\zeta$ is defined by

$$\widehat{\triangle_\zeta^{-1} f}(\xi) = p_\zeta(\xi)^{-1} \hat{f}(\xi). \tag{2.1}$$

with symbol

$$p_\zeta(\xi) = -|\xi|^2 + 2i\zeta \cdot \xi.$$

Using this symbol, we can define the space $\dot{X}^b_\zeta$ with the associated norm

$$||w||_{\dot{X}^b_\zeta} = |||p_\zeta(\xi)|^b \hat{w}(\xi)||_{L^2},$$

and the inhomogeneous spaces X^b_ζ with the associated norm

$$||w||_{X^b_\zeta} = ||(|\zeta| + |p_\zeta(\xi)|)^b \hat{w}(\xi)||_{L^2}.$$

In Section 5, we will only need to use the exponent $b = \pm 1/2$. Notice that those two spaces were firstly considered by Haberman and Tataru [23] in the spirit of Bourgain's spaces, see [10, 45].

2.2 DN map and integral identity

From the variational formulation of the problem (1.2), it follows the following Alessandrini identity [2].

$$\langle \Lambda_\gamma \psi, \phi \rangle = \left\langle \gamma \frac{\partial w}{\partial \nu}, \phi \right\rangle = \int_\Omega \gamma \nabla w \nabla z dx \ \ \forall \psi, \phi \in H^{1/2}(\partial\Omega),$$

where $z \in H^1(\Omega), z\big|_{\partial\Omega} = \phi$.

By recalling from the introduction the DN map Λ_q associated with (1.3), we can give another useful identity when $q = \gamma^{-1/2}\Delta\gamma^{1/2}$. It is easy to check that the DN map Λ_q can be obtained from the DN map Λ_γ, where the explicit expression relating those two maps is given by

$$\Lambda_q f = \gamma^{-1/2}\Lambda_\gamma(\gamma^{-1/2} f) + \frac{1}{2}\gamma^{-1}\frac{\partial \gamma}{\partial \nu} f\Big|_{\partial\Omega}. \tag{2.2}$$

One other important relation is the following integral identity that relates boundary measurements with interior potentials.

$$\left\langle (\Lambda_{q_1} - \Lambda_{q_2}) w_1\big|_{\partial\Omega}, w_2\big|_{\partial\Omega} \right\rangle = \int_\Omega (q_1 - q_2) w_1 w_2 \, dx, \tag{2.3}$$

for $q \in L^\infty(\Omega)$ and $w_j \in H^1$ uniquely solve $-\Delta w_j + q_j w_j = 0$, for $j = 1, 2$.

2.3 Faddeev's Green's function and layer operator

While the equation (2.1) implicitly gives the right inverse G_ζ of $\triangle_\zeta$, the following explicit functions

$$g_\zeta(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \frac{e^{i\xi.x}}{p_\zeta(\xi)} d\xi, \qquad G_\zeta(x) = e^{i\zeta.x} g_\zeta(x), \tag{2.4}$$

are the Faddeev's Green's functions for $(\Delta + 2i\zeta.\nabla)$ and the Laplacian, respectively.

Now, we introduce some useful operators, which will be needed later in Section 4. Using the family G_ζ of Green's functions for $x \in \mathbb{R}^n \backslash \partial\Omega$, we define the following layer potentials.
Single layer potential:

$$S_\zeta f(x) = \int_{\partial\Omega} G_\zeta(x, y) f(y) dS(y). \tag{2.5}$$

Double layer potential:

$$D_\zeta f(x) = \int_{\partial\Omega} \frac{\partial G_\zeta(x, y)}{\partial \nu(y)} f(y) dS(y).$$

We define also for $x \in \partial\Omega$, the boundary double layer potential:

$$B_\zeta f(x) = p.v. \int_{\partial\Omega} \frac{\partial G_\zeta(x, y)}{\partial \nu(y)} f(y) dS(y). \tag{2.6}$$

3 Reconstruction results

Throughout this section, we try to give precise statements of the known reconstruction results. We will split the section into two subsections, depending on the study domain dimension. Notice that the used approach for the two-dimensional problem, which is essentially based on complex analysis, is quite different from the higher-dimensional problem. Thus, we first present the known reconstruction results in the plane.

3.1 Reconstruction in two dimensions

For the two-dimensional problem, Novikov and Nachman were the first to answer the reconstruction question in [38] and [33]. Nachman's result is presented as follows:

Theorem 3.1. *[33] Let $\Omega \subset \mathbb{R}^2$ be an open, bounded, smooth domain, and let $\gamma \in W^{2,p}(\Omega), p > 1$. Then there is a procedure to reconstruct γ uniquely from Λ_γ.*

Inspired by the uniqueness proof of Brown and Uhlmann [12], Knudsen and Tamasan [28] applied the $\bar{\partial}$-method to produce a reconstruction algorithm for $\gamma \in W^{1,p}, p > 2$. Their result is considered as a sharp improvement over the last one due to Nachman.

Theorem 3.2. *[28] Let $\Omega \subset \mathbb{R}^2$ be a bounded, smooth domain, and let $0 < \varsigma < 1$ with $\gamma \in W^{1+\varsigma,p}(\Omega), p > 2$ satisfying (1.1). Then γ can be reconstructed on Ω from the knowledge of Λ_γ.*

In 2018 Lytle, Perry, and Siltanen [31] proved that Nachman's reconstruction method still holds for L^∞ conductivities, which are 1 in a neighborhood of the boundary. Here we present their main Theorem, and further details on their work are given in Section 4.

Theorem 3.3. *[31] Let $\gamma \in L^\infty(\mathbb{D})$ satisfying (1.1), and suppose that the condition*

$$\text{there is a } x_0 \in (0,1) \text{such that } \gamma = 1 \text{ for } |x| \geq x_0, \tag{3.1}$$

holds. Then, for each $\zeta \in \mathbb{C}$, there exists a unique $w|_{\partial\mathbb{D}} \in H^{1/2}(\partial\mathbb{D})$ such that

$$w|_{\partial\mathbb{D}} = e^{i\zeta.x}|_{\partial\mathbb{D}} - S_\zeta(\Lambda_\gamma - \Lambda_1)w|_{\partial\mathbb{D}}. \tag{3.2}$$

By abuse of notation, the map $\Lambda_1 = \Lambda_0$ is the DN map for harmonic functions on $\mathbb{D}$ that correspond to $q = 0$ and $\gamma = 1$.

3.2 Reconstruction in higher dimensions

In 1988 for higher dimensions, Nachman [34] and Novikov [37] were also the first who provided a constructive procedure to recover $\gamma \in C^{1,1}$ from the knowledge of Λ_γ.

Theorem 3.4. *[34] Let $\Omega \subset \mathbb{R}^n, n \geq 3$ be a bounded domain with a $C^{1,1}$ boundary, and let $\gamma \in C^{1,1}(\bar{\Omega})$ satisfying (1.1). Then there is a procedure to reconstruct γ uniquely from Λ_γ.*

Novikov [37] has independently shown a similar result to the previous one given by Nachman. He was the first who introduced the key ingredient of the boundary integral equation, which will be explained later in the next section.

Based on the uniqueness result of Haberman and Tataru [23], Nachman's procedure was followed by García and Zhang in [20] to reconstruct C^1, or Lipschitz conductivities with $|\nabla \log \gamma|$ sufficiently small.

Theorem 3.5. *[20] Let Ω be a bounded Lipschitz domain on $\mathbb{R}^n, n \geq 3$, and let γ be a strictly positive real-valued function on Ω satisfying (1.1).*

1. *$\gamma \in C^1(\bar{\Omega})$.*
2. *$\gamma \in Lip(\Omega)$, such that $|\nabla \log \gamma(x)| < \delta_{\Omega,n}$ with $\delta_{\Omega,n}$ a constant.*

If 1 or 2 is satisfied, then γ can be reconstructed on Ω from the knowledge of Λ_γ.

In 2020, Tarikere extended the uniqueness result of Brown and Torres [14] to prove the validity of Nachman's method for $W^{3/2,2n}$ conductivities.

Theorem 3.6. *[44] Let Ω be a bounded Lipschitz domain on $\mathbb{R}^n, n \geq 3$, and let $\gamma \in W^{3/2,2n}(\Omega)$ be a strictly positive real-valued function on Ω satisfying (1.1) with $\gamma \equiv 1$ in a neighborhood of $\partial\Omega$. Then γ can be reconstructed from Λ_γ.*

While all the previous results concern the full data problem, Nachman was also interested in the reconstruction of the partial data type problem. Based on the well-known Carleman estimate approach in [27], Nachman and Street obtained a reconstruction proof with partial data measurements on a slightly overlapping partition of the boundary $\partial\Omega$. The reader is referred to ([35], Theorem 1.3) for the precise result. Their result was recently approved by Garde [21] to piecewise constant layered conductivities. Grade's reconstruction method only relies on the monotonicity principles of the local DN map, and therefore lends well to efficient numerical implementation models.

4 Proof strategy

In the present section, we briefly review the proof of the reconstruction results described above and the main theoretical tools used therein. The two-dimensional problem is quite different from the higher dimensional case. For example, it is no longer over-determined. To show that, we propose the following explanation. Since it is a linear operator from $H^{1/2}(\partial\Omega)$ to $H^{-1/2}(\partial\Omega)$, the DN map Λ_γ can be expressed in terms of the Schwartz kernel $K : \partial\Omega \times \partial\Omega \longrightarrow \mathbb{R}$ by

$$\Lambda_\gamma f(x) = \int_{\partial\Omega} K(x,y) f(y) dS(y). \tag{4.1}$$

From one side, it is known that the dimension of $\partial\Omega$ is $n-1$. Then, the kernel K is a function of $2(n-1)$ variables. On the other side, the conductivity γ, which we wish to recover, is defined in an n-dimensional domain. Thus, for $n = 2$, the Calderón problem in the plane is formally well-determined and fairly well-understood.
From (4.1), it is clear that for $n \geq 3$, the inverse problem is formally over-determined since the known data has more degree of freedom than the quantity γ, which we are trying to recover. That means that sometimes (but certainly not always) the problem may be easier to manipulate in higher dimensions.
The precedent motivates in some way that, to deal with the two-dimensional problem, we need to invoke a different technique than the one used when $n \geq 3$.

4.1 Preliminary reductions

To simplify the problem, we use the following two types of reductions. On the one hand, Nachman ([33], Section 6) proceeds to a reduction of γ in a neighborhood of $\partial\Omega$. His idea was to reduce the Calderón problem to a problem having a constant $\gamma \equiv 1$ near $\partial\Omega$, then to extend γ outside the study domain Ω such that the initial regularity assumption is conserved. Thus, solving the extended problem on the large domain means that the original problem on Ω is implicitly solved.

The main idea behind this reduction is based on the following step of reconstructing the boundary value of the unknown conductivity and its derivative from the DN map.

4.1.1 Reconstruction at the boundary

From identity (2.2), it is clear that to find the value of Λ_q, we need a procedure to recover the values of γ and $\frac{\partial\gamma}{\partial\nu}$ on the boundary $\partial\Omega$ from Λ_γ. Thus, we deduce the importance of boundary determination, which depends on the regularity of both the domain boundary and the conductivity itself. For the case of smooth conductivities in smooth domains, Kohn and Vogelius [29] proved that Λ_γ determines γ and all its normal derivatives on the boundary. More results and approaches to boundary determination of the conductivity were shown in [1, 43]. In particular, Brown [13] proved that we could recover the boundary values of a $W^{1,1}$, or a C^0 conductivity from the knowledge of Λ_γ.

In the appendix of [20], the gradient at the boundary of a C^1 conductivity in a Lipschitz domain was recovered by Brown in collaboration with García and Zhang. In all ways, this boundary determination is based on testing the DN map against highly oscillatory functions at the domain boundary.

On the other hand, we saw in the introduction that the conductivity problem (1.2) could be reduced to the Schrödinger problem (1.3) by a well-known transformation under the condition that the conductivities are sufficiently regular (which is the case here). The desired conclusion behind those reductions is to possess a potential q having a compact support in Ω.

4.2 Nachman's method

After reducing the inverse conductivity problem to the inverse problem for a Schrödinger equation, the reconstruction method of Nachman could be decomposed into three steps. First, we extend q to be 0 in $\mathbb{R}^2$ outside the study domain. The second step consists of computing the scattering transform t of the Schrödinger equation associated with the extended potential q from the given DN map. Finally, the $\bar{\partial}$-method permits solving the scattering problem, which is used to calculate the value of γ.

Below, we will give a discerption of the reconstruction process in the plane [33].

We identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$. For $q = \gamma^{-1/2}\Delta\gamma^{1/2}$, Nachman used Faddeev's [18] CGOs in the problem (1.3) to get

$$\begin{cases} -\Delta w + qw = 0, \\ \lim_{|x|\to\infty} e^{-i\zeta.x} w(x,\zeta) - 1 = 0. \end{cases} \tag{4.2}$$

We define the useful complex derivative operators $\bar{\partial}$ and ∂ as follows:

$$\bar{\partial} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\right), \qquad \partial = \frac{1}{2}\left(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\right).$$

By substituting with $a(x,\zeta) = e^{-i\zeta.x}w(x,\zeta)$ in (4.2), we get

$$\begin{cases} \bar{\partial}(\partial + ix)a = \frac{1}{4}qa, \\ \lim_{|x|\to\infty} a = 1. \end{cases} \tag{4.3}$$

Then, one can use (4.3) to define the scattering transform t

$$t(\zeta) = \int_{\mathbb{R}^2} b_\zeta(x)q(x)a(x,\zeta)dx, \tag{4.4}$$

where $b_\zeta(x) = e^{i(\zeta.x+\overline{\zeta.x})}$.
Nachman showed that the solutions $a(x,\zeta)$ solve

$$\begin{cases} \bar{\partial}_\zeta a = \frac{t(\zeta)}{4\pi\zeta}b_{-\zeta}(x)\bar{a}, \\ \lim_{|\zeta|\to\infty} a = 1. \end{cases} \tag{4.5}$$

Since we know from the preceding subsection that the used reduction guarantees that q has a compact support in Ω, then (4.3) and (4.4) can be reduced to the following boundary integral equations, respectively.

$$w|_{\partial\Omega} = e^{i\zeta.x}|_{\partial\Omega} - S_\zeta(\Lambda_q - \Lambda_0)w|_{\partial\Omega}. \tag{4.6}$$

$$t(\zeta) = \int_{\partial\Omega} e^{i\bar{\zeta}.\bar{x}}(\Lambda_q - \Lambda_0)w|_{\partial\Omega}\, dS. \tag{4.7}$$

Where S_ζ is defined in (2.5). As was mentioned in Section 3, the boundary integral identity (4.6) was developed for the first time by Novikov [37].
Finally, by giving the value of t from (4.7), we can solve (4.5) to recover the conductivity from the identity

$$\gamma(x) = a(x,0)^2. \tag{4.8}$$

In the plane, we recapitulate Nachman's reconstruction method for $\gamma \in W^{2,p}$ in the following four steps.

1. Solve (4.6) for $w|_{\partial\Omega}$.
2. Calculate the value of t from (4.7).
3. Solve the $\bar{\partial}_\zeta$-equation (4.5).
4. Recover γ from (4.8).

Remark 4.1.
- The Knudsen-Tamasan result in Theorem 3.2 for a less regular γ was proposed by following the uniqueness proof of Brown and Uhlmann [12], and by making every step in their proof constructive.
- The reconstruction algorithm of Knudsen-Tamasan [28] is a generalization of the above-summarized one, and the proof steps are almost the same. For other kinds of algorithms based on a linearized or iterative schema, see [9,17].

4.3 Beltrami equation

The construction of CGOs viewed before relies on the available regularity assumption on γ. Another construction that requires no smoothness on γ was introduced in [8] for $\gamma \in L^\infty$ strictly positive, using the Beltrami equation approach.
Next, we describe the analysis steps of the reconstruction process proposed by Lytle, Perry, and Siltanen in Theorem 3.3.
Without loss of generality, we assume that the domain Ω is the unit disc $\mathbb{D}$ and $\gamma = 1$ in a neighborhood of $\mathbb{D}$. More precisely, we consider that condition (3.1) holds. The Beltrami coefficient μ used by Astala and Päivärinta [8] is defined by

$$\mu = \frac{1-\gamma}{1+\gamma},$$

satisfying $|\mu(x)| < 1$, and having a compact support since that the conductivity is set to be equal to one outside a compact set. Then, for any function $w \in H^1(\mathbb{D})$ that solves the conductivity equation given in (1.2), there exists $\tilde{w} \in H^1(\mathbb{D})$ a real-valued function named the conjugate harmonic function of w such that the Beltrami equation

$$\bar{\partial}\dot{w} = \mu\partial\dot{w} \tag{4.9}$$

has a solution $\dot{w} = w + i\tilde{w}$.
The key ingredient in the analysis in [31] is this last Beltrami equation (4.9), which admits CGOs. Those CGOs can be used to define an associated scattering transform, which is identified as a natural analog of Nachman's one (4.7). This transform remains well-defined under the weaker regularity assumption $\mu \in L^\infty(\Omega)$ by Theorem 4.2 from [8]. Theorem 3.3 combined with Corollary 18.1.2 from [6] about the uniqueness of CGOs for the conductivity equation, establish the unique solvability of the integral equation (3.2).
Notice that the followed strategy to prove Theorem 3.3 is to show the compactness of the integral operator $T_\zeta = S_\zeta(\Lambda_\gamma - \Lambda_1)$ from $H^{1/2}(\partial\mathbb{D})$ to $H^{-1/2}(\partial\mathbb{D})$. Then, to prove that the integral equation (3.2) is uniquely solvable, it suffices by Fredholm theory, to show that the only vector $v \in H^{1/2}(\partial\mathbb{D})$ with $T_\zeta v = -v$ is the zero vector.
For more efficient algorithms for the computation of CGOs $\dot{w}$, and numerical examples, see ([32], Chapter 14, page: 215-221). Interested readers are referred to ([32], Chapter 15), and the references therein for readings on the D-bar method, which is based on Nachman's result [33].

4.4 Boundary integral equation

In the present subsection, we will describe more carefully each step in the reconstruction procedure in higher dimensions. For $n \geq 3$, the valuable tool of CGOs, which was presented in the introduction to show the uniqueness in Calderón problem in the work of Sylvester and Uhlmann [42], was used later by Nachman in Theorem 3.4 and by Novikov in [37] independently to reconstruct the conductivity γ. We will describe Nachman's idea [34] as follows. As it was already seen in subsection 4.1, we can give the boundary reconstruction of γ and $\frac{\partial\gamma}{\partial\nu}$ from the DN map. Then, if Λ_γ is knew, Λ_q is calculated from identity (2.2). Hence, the problem is reduced to the reconstruction of q from Λ_q. Once we have the value of $q = \gamma^{-1/2}\Delta\gamma^{1/2}$, we can solve the following problem to deduce γ.

$$\begin{cases} -\Delta w + qw = 0 & \text{in } \Omega, \\ \qquad w = \gamma^{1/2} & \text{on } \partial\Omega. \end{cases}$$

Now, let $q_1 = q$, $q_2 = 0$ in the integral identity (2.3). Then we get

$$\int_{\Omega} q w_1 w_2 \, dx = \int_{\partial\Omega} (\Lambda_q - \Lambda_0)(w_1|_{\partial\Omega}) w_2|_{\partial\Omega} \, dS, \tag{4.10}$$

where $w_1, w_2 \in H^1(\Omega)$ solves $-\Delta w_1 + q w_1 = 0$, and $-\Delta w_2 = 0$, respectively.
In the following, we use expression (4.10) and appropriate CGOs to reconstruct the Fourier transform of q. We consider $\xi \in \mathbb{R}^n, \xi \neq 0$, and we define the set B by $B = \{\zeta_j \in \mathbb{C}^n : \zeta_j \cdot \zeta_j = 0,\ |\zeta_1| = |\zeta_2| = h,\ \zeta_1 + \zeta_2 = \xi, j = 1,2\}$. The application of the argument from [42] ensures the existence of CGOs $w_1 = e^{i\zeta_1 . x}(1 + y_{\zeta_1})$ for $-\Delta w_1 + q w_1 = 0$, with the correction term y_{ζ_1} decaying to zero when $|\zeta_1| \to \infty$. Furthermore, the appropriate choice of $\zeta_2.\zeta_2 = 0$ implies that $\Delta e^{i\zeta_2 . x} = 0$.
By substituting in (4.10) and by using the decay property of y_{ζ_1}, we have

$$\hat{q}(\xi) = \lim_{h\to\infty} \int_{\partial\Omega} (\Lambda_q - \Lambda_0)(w_1|_{\partial\Omega}) e^{i\zeta_2 . x}|_{\partial\Omega} \, dS. \tag{4.11}$$

From (4.11), we deduce that the Fourier transform of q for $\xi \neq 0$ can be recovered from the DN map if $w_1|_{\partial\Omega}$ is knew. We know that q is compactly supported, then $\hat{q}(\xi)$ is continuous so that $\hat{q}(0)$ can be determined by continuity [41]. Hence, $\hat{q}(\xi)$ is known as a tempered distribution, and the potential q can be recovered in $\mathbb{R}^n$ by simply inverting the Fourier transform. Therefore, it is a question to get the value of $w_1|_{\partial\Omega}$ to recover $\hat{q}(\xi)$.
The aim now is to find a method to calculate $w_1\big|_{\partial\Omega}$. The idea is to look at the exterior problem, which means that we extend q to $\mathbb{R}^n$ to be $q = 0$ outside the study domain Ω. Since $q = 0$ in $\mathbb{R}^n \backslash \bar{\Omega}$, the equation $(-\Delta + q)w_1 = 0$ in $\mathbb{R}^n$ becomes $-\Delta w_1 = 0$ in $\mathbb{R}^n \backslash \bar{\Omega}$. Therefore, the function w_1 is a solution to the following exterior problem.

$$\begin{cases} -\Delta w_1 = 0 \text{ in } \mathbb{R}^n \backslash \bar{\Omega}, \\ w_1|_{\partial\Omega} = f_\zeta, \\ \frac{\partial w_1}{\partial \nu}|_{\partial\Omega} = \Lambda_q f_\zeta. \end{cases} \tag{4.12}$$

For a fixed $R > R_0$ such that $\Omega \subset B_R(0)$, it is known from [34] that if w_1 satisfies the following analog of Sommerfeld radiation condition

$$\lim_{R\to\infty} \int_{|y|=R} \left(G_\zeta(x,y) \frac{\partial (w_1 - e^{i\zeta . x})}{\partial \nu(y)} - (w_1 - e^{i\zeta . x}) \frac{\partial G_\zeta(x,y)}{\partial \nu(y)} \right) dS(y) = 0, \tag{4.13}$$

then, by using Green's formula in (4.12), we can show that the boundary value $w_1|_{\partial\Omega}$ can be characterized as the unique solution f_ζ of the following boundary integral equation of Fredholm type.

$$e^{i\zeta . x} - S_\zeta(\Lambda_q - \Lambda_0) f_\zeta = f_\zeta \quad \text{on } \partial\Omega. \tag{4.14}$$

As we notice that the operator on the left-hand side of the boundary integral equation (4.14), depends on the DN map and other known quantities, we can recover the value of $w_1|_{\partial\Omega}$ by solving (4.14). Moreover, (4.14) is an inhomogeneous integral equation for f_ζ having a unique solution $f_\zeta \in H^{3/2}(\partial\Omega)$. By Fredholm alternative, the uniqueness of the solution follows from the fact that the homogeneous equation

$$-S_\zeta(\Lambda_q - \Lambda_0) f_\zeta = f_\zeta \quad \text{on } \partial\Omega,$$

only has the zero solution, which follows by its turn from the uniqueness of the CGOs.

Remark 4.2.

- Nachman derived the slight different type of boundary integral equation:

$$e^{i\zeta.x} - (S_\zeta\Lambda_q - B_\zeta - \frac{1}{2}I)f_\zeta = f_\zeta \quad \text{on } \partial\Omega, \tag{4.15}$$

  where the operator B_ζ is defined in (2.6). Since we can easily show that $S_\zeta\Lambda_0 = B_\zeta + \frac{1}{2}I$, it is clear that the expressions (4.15) and (4.14) are equivalent.

- Because it is complicated to check that the condition (4.13) is satisfied by w_1, Nachman's idea was to construct from (4.15) CGOs to the Schrödinger equation $(-\Delta + q)w = 0$ in $\mathbb{R}^n$, that automatically satisfy condition (4.13), then to prove that those CGOs coincide with the ones constructed by Sylvester and Uhlmann [42].

Now, we turn to give the sketch of the proof of theorems 3.5 and 3.6. Mainly, the strategy used there was to follow the discussed Nachman's method for Theorem 3.4.

Due to the weak assumption regularity on γ in Theorem 3.5 ($\gamma \in C^1$ or γ Lipschitz with $|\nabla \log \gamma(x)| < \delta_{\Omega,n}$) and Theorem 3.6 ($\gamma \in W^{3/2,2n}$), some changes are made in the above steps. The proof outline consists of constructing CGOs to the conductivity equation or the Schrödinger equation in $\mathbb{R}^n$, respectively, from the boundary integral-equation on the boundary. Then, to show that these solutions coincide with the ones constructed by Haberman-Tataru [23] and Brown-Torres [14], respectively. Note that the reconstruction presentation in [44] follows mainly the analysis and notations from ([19], Chapter 4.7), which focuses on reconstructing $\gamma \in C^2(\Omega)$.
We know that by plugging $w(x,\zeta) = e^{i\zeta.x}(1 + y_\zeta(x))$ in the Schrödinger equation, we get

$$(-\Delta - 2i\zeta\cdot\nabla)y_\zeta(x) + q(x)y_\zeta(x) = -q(x) \quad \text{in } \mathbb{R}^n. \tag{4.16}$$

By convolving (4.16) with g_ζ which is defined in (2.4), we obtain the Lippmann-Schwinger-Faddeev integral equation

$$(I + g_\zeta * q)y_\zeta(x) = g_\zeta * q. \tag{4.17}$$

The last equation (4.17) is equivalent to the following integral equation

$$w(x,\zeta) + \int_{\mathbb{R}^n} G_\zeta(x,y)q(y)w(y,\zeta)dy = e^{i\zeta.x}, \tag{4.18}$$

where G_ζ is defined in (2.4). It is clear that the combination of (2.3) and (4.18) gives (4.14) for w. Moreover, the homogenous version of (4.18) is

$$w(x,\zeta) = \int_{\mathbb{R}^n} G_\zeta(x,y)q(y)w(y,\zeta)dy. \tag{4.19}$$

The analysis in [20] and [44] showed that the operator at the right-hand side of (4.19) is a contraction, provided the corresponding CGOs are constructed for sufficiently large $|\zeta|$. Finally, the problem is reduced to a fixed point problem.

5 Open problem, conjecture, and discussion

In the precedent sections, some methods for conductivity reconstruction were reviewed. Those methods were analyzed, compared, and their steps were summarized. The results show that all the cited methods are in some way a generalization of Nachman's (or Novikov's) method. Besides, those results can provide a reference to the reconstruction subject of the problem.

Under the broad research field of Calderón's problem, we wrote this note to motivate and draw more attention to the reconstruction topic. Therefore, we hope that something might lie beyond this paper. In this final section, we propose the following open question and discuss plausibly research extensions that can be subject to new results in the reconstruction direction of the problem.

Question. (Reconstruction of Lipschitz conductivities) *If Ω is a bounded Lipschitz domain on $\mathbb{R}^n, n \geq 3$, $\gamma \in Lip(\Omega)$ a strictly positive real-valued function on Ω satisfying (1.1), with $\gamma \equiv 1$ in a neighborhood of $\partial\Omega$, show that γ can be reconstructed on Ω from the knowledge of Λ_γ.*

Recently, Caro and Rogers [16] used Bourgain's spaces to prove the uniqueness of Lipschitz conductivities in three and higher dimensions. Their result makes us wonder how much it would be interesting to check whether it is possible to use this uniqueness proof to generalize Nachman's method to Lipschitz conductivities by taking off the smallness condition on $|\nabla \log \gamma|$ to improve the results of Theorem 3.5. The key ingredient in the uniqueness proof in [16] for Lipschitz conductivities without a smallness condition is the following a priori estimate:

$$||w||_{X_\zeta^{1/2}} \lesssim ||(-\Delta + 2\zeta \cdot \nabla + q)w||_{X_\zeta^{-1/2}},$$

for a function $w \in \mathcal{S}(\mathbb{R}^n)$ with support in Ω, and the function spaces $X_\zeta^{\pm 1/2}$ were defined in Section 2. From the last estimate and a standard functional analysis argument, it follows a key bound on the potential q

$$||y_\zeta||_{X_\zeta^{1/2}(\Omega)} \lesssim ||q||_{X_\zeta^{-1/2}},$$

for some corrector function y_ζ. The occurring complication is that the solutions here are local, but in our case, we need to extend them in some way to $\mathbb{R}^n$. Therefore, we conjecture that the techniques used until now, which have been reviewed in this survey, have reached some sort of limit. Thus, we can not follow the contraction mapping approach to apply the fixed point argument used in the above methods. However, it is straightforward that this problem seems more complicated and may require new ideas beyond the known techniques to overcome its difficulties.

Acknowledgements

The author is very grateful to Professor Guo Zhang for his introduction to the field of inverse problems, precious discussions, and timely guidance. Meanwhile, the author expressed his sincere gratitude to the editors and the anonymous reviewers who provided helpful comments and suggestions on the earlier drafts of the manuscript.

Conflict of Interest

The author has no conflicts of interest to disclose.

References

[1] G. Alessandrini, *Singular solutions of elliptic equations and the determination of conductivity by boundary measurements*, J. Differential Equations. **84**(2) (1990), 252–272.

[2] G. Alessandrini, *Stable determination of conductivity by boundary measurements*, Appl. Anal. **27**(1-3) (1988), 153–172.

[3] H. Ammari and G. Uhlmann, *Reconstruction of the potential from partial Cauchy data for the Schrödinger equation,* Indiana Univ. Math. J. **53**(1) (2004), 169–183.

[4] M. Aoudj, *On the main aspects of the inverse conductivity problem*, Journal of Nonlinear Modeling and Analysis. **4**(1) (2022), 1–17.

[5] Y. M. Assylbekov, *Reconstruction in the partial data Calderón problem on admissible manifolds*, Inverse Probl. Imaging. **1**1(3) (2017), 455–476.

[6] K. Astala, T. Iwaniec, and G. Martin, *Elliptic partial differential equations and quasiconformal mappings in the plane*, Princeton Mathematical Series, Princeton, 2009.

[7] K. Astala, M. Lassas and L. Päivärinta, *Calderón's inverse problem for anisotropic conductivity in the plane,* Comm. Partial Differential Equations. **30**(1-2) (2005), 207–224.

[8] K. Astala and L. Päivärinta, *Calderón's inverse conductivity problem in the plane*, Ann. of Math. **163**(1) (2006), 265–299.

[9] L. Borcea, *Electrical impedance tomography,* Inverse Problems. **18**(6) (2002), R99–R136.

[10] J. Bourgain, *Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations,* Geom. Funct. Anal. **3**(2) (1993), 107–156.

[11] B. H. Brown, *Electrical impedance tomography (eit): a review,* J. Med. Eng. Technol. **27**(3) (2003), 97–108.

[12] R. Brown and G. Uhlmann, *Uniqueness in the inverse conductivity problem for nonsmooth conductivities in two dimensions,* Comm. Partial Differential Equations. **22**(5-6) (1997), 1009–1027.

[13] R. Brown, *Recovering the conductivity at the boundary from the Dirichlet to Neumann map: a pointwise result,* J. Inverse Ill-Posed Probl. **9**(6) (2001), 567–574.

[14] R. M. Brown and R. H. Torres, *Uniqueness in the inverse conductivity problem for conductivities with* $3/2$ *derivatives in* $L^p, p > 2n$, J. Fourier Anal. Appl. **9**(6) (2003), 563–574.

[15] A. P. Calderón, On an inverse boundary value problem, in:*Seminar on Numerical Analysis and its Applications to Continuum Physics*, 1980, pp. 65–73.

[16] P. Caro and K. Rogers, *Global uniqueness for the Calderón problem with Lipschitz conductivities,* Forum Math. Pi. **4**(e2) 28 (2016).

[17] M. Cheney, D. Isaacson and J. C. Newell, *Electrical impedance tomography,* SIAM Review. **41**(1) (1999), 85–101.

[18] L. D. Faddeev, *Growing solutions to Schrödinger's equation,* Doklady Akademii nauk SSSR. **165**(3) (1965), 514–517.

[19] J. Feldman, M. Salo and G. Uhlmann, *The Calderón Problem – An Introduction to Inverse Problems*, Spring-Verlag, New York, 2010.

[20] A. García and G. Zhang, *Reconstruction from boundary measurements for less regular conductivities*, Inverse Problems. **32** (11) (2016), Article ID 115015, 20 pages.

[21] H. Garde, *Reconstruction of piecewise constant layered conductivities in electrical impedance tomography,* Comm. Partial Differential Equations. **45**(9) (2020), 1118–1133.

[22] B. Haberman, *Uniqueness in Calderón's problem for conductivities with unbounded gradient,* Comm. Math. Phys. **340**(2) (2015), 639–659.

[23] B. Haberman and D. Tataru, *Uniqueness in Calderón problem with Lipschitz conductivities*, Duke Math. **162**(3) (2013), 497–516.

[24] S. Ham, Y. Kwon and S. Lee, *Uniqueness in the Calderón problem and bilinear restriction estimates*, J. Funct. Anal. **281**(8) (2021), Article ID 109119, 58 pages.

[25] M. Hanke and M. Bröhl, *Recent progress in electrical impedance tomography,* Inverse Problems. **19**(6) (2003), S65–S90.

[26] C. Kenig and M. Salo, *Recent progress in the Calderón problem with partial data,* Inverse Problems and Applications. **165**(2) (2017), 567–591.

[27] C. Kenig, J. Sjöstrand and G. Uhlmann, *The Calderón problem with partial data.* Ann. of Math. **165**(2) (2007), 567–591.

[28] K. Knudsen and A. Tamasan, *Reconstruction of less regular conductivities in the plane*, Comm. Partial Differential Equations. **29**(3-4) (2005), 361–381.

[29] R. Kohn and M. Vogelius, *Determining conductivity by boundary measurements,* Comm. Pure Appl. Math. **37**(3)(1984), 289–298.

[30] M. Lassas and G. Uhlmann, *On determining a Riemannian manifold from the Dirichlet-to-Neumann map*, Ann. Sci. Ec. Norm. Supér. **34**(5) (2001), 771–787.

[31] G. Lytle, P. Perry and S. Siltanen, *Nachman's reconstruction method for the Calderón problem with discontinuous conductivities*, Inverse problems. **36**(3) (2019), 035018. 15 pages. DOI:10.1088/1361-6420/ab5a12

[32] J. L. Mueller and S. Siltanen, *Linear and Nonlinear Inverse Problems with Practical Applications*, Computational Science & Engineering, SIAM, Philadelphia, PA, 2012. DOI:10.1137/1.9781611972344

[33] A. I. Nachman, *Global uniqueness for a two-dimensional inverse boundary value problem,* Ann. of Math. **143**(2) (1996), 71–96.

[34] A. I. Nachman, *Reconstructions from boundary measurements*, Ann. of Math. **128**(3) (1988), 531–576.

[35] A. Nachman and B. Street, *Reconstruction in the Calderón problem with partial data,* Comm. Partial Differential Equations. **35**(2) (2010), 375–390.

[36] H. M. Nguyen and D. Spirn, *Recovering a potential from cauchy data via complex geometrical optics solutions*, (2014) preprint arXiv:1403.2255

[37] R. G. Novikov, *A multidimensional inverse spectral problem for the equation* $-\Delta\psi + (v(x) - Eu(x))\psi = 0$. Funct. Anal. Appl. **22**(4) (1988), 263–272.

[38] R. G. Novikov and G. M. Khenkin, *The ∂-equation in the multi-dimensional inverse scattering problem*, Russian Math. Surveys. **42**(3) (1987), 109–180.

[39] F. Ponc-Vanegas, *The bilinear strategy for Calderón's problem*, Rev. Mat. Iberoam. **37**(6) (2021), 2119–2160.

[40] S. Siltanen, J. L. Mueller and D. Isaacson, *An implementation of the reconstruction algorithm of A. Nachman for the 2D inverse conductivity problem*, Inverse Problems. **16**(3) (2000), 681–699.

[41] E. M. Stein and R. Shakarchi, *Fourier Analysis an Introduction*, Princeton University Press, Princeton, Oxford, 2003.

[42] J. Sylvester and G. Uhlmann, *A global uniqueness theorem for an inverse boundary value problem*, Ann. of Math. **125**(1) (1987), 153–169.

[43] J. Sylvester and G. Uhlmann, *Inverse boundary value problems at the boundary-continuous dependence,* Comm. Pure Appl. Math. **41**(2) (1988), 197–219.

[44] A. Tarikere, *Reconstruction of rough conductivities from boundary meuserments*, (2020), preprint arXiv:2001.05155v1

[45] D. Tataru, *The* X^s_θ *spaces and unique continuation for solutions to the semilinear wave equation*, Comm. Partial Differential Equations. **21**(5-6) (1996), 841–887.

[46] G. Uhlmann, *Electrical impedance tomography and Calderón's problem,* Inverse Problems. **25**(12) (2009), 123011, 42 pages.

Journal of Innovative Applied Mathematics and Computational Sciences

J. Innov. Appl. Math. Comput. Sci. **1**(1) (2021) 64–78. n2t.net/ark:/49935/jiamcs.v1i1.6

http://jiamcs.centre-univ-mila.dz/

On periodic solutions of fractional-order differential systems with a fixed length of sliding memory

Safa Bourafa [1], Mohammed Salah Abdelouahab [1] and René Lozi [2]

[1] Laboratory of Mathematics and their interactions, University Center of Mila, Algeria
[2] Université Côte d'Azur, CNRS, LJAD, Nice, France

Received 16 November 2021, Accepted 25 December 2021, Published 30 December 2021

Abstract. The fractional-order derivative of a non-constant periodic function is not periodic with the same period. Consequently, any time-invariant fractional-order systems do not have a non-constant periodic solution. This property limits the applicability of fractional derivatives and makes it unfavorable to model periodic real phenomena. This article introduces a modification to the Caputo and Rieman-Liouville fractional-order operators by fixing their memory length and varying the lower terminal. It is shown that this modified definition of fractional derivative preserves the periodicity. Therefore, periodic solutions can be expected in fractional-order systems in terms of the new fractional derivative operator. To confirm this assertion, one investigates two examples, one linear system for which one gives an exact periodic solution by its analytical expression and another nonlinear system for which one provides exact periodic solutions using qualitative and numerical methods.

Keywords: Fractional-order derivative; sliding fixed memory length; periodic solution.

2020 Mathematics Subject Classification: 34A35, 34K28, 34K40, 45L05.

1 Introduction

The history of fractional calculus goes back to the end of the 17th century when L'Hopital asked Leibniz what meaning could be ascribed to $D^n f$ if n were a fraction? Since that, time-fractional calculus has drawn the attention of many famous mathematicians, such as Euler, Laplace, Fourier, Abel, Liouville, Riemann, and Laurent [18]. The advantages of fractional calculus have been described and pointed out in the last few decades by many authors [8, 15–19]. It has been shown that the fractional-order models of realistic systems are regularly more adequate than usually used integer-order models. Applications of these fractional-order models spread in many fields, such as viscoelastic systems, dielectric polarization, electrode-electrolyte polarization, electromagnetic waves, quantum evolution of complex systems, and so on [6, 10, 11, 14, 20]. There are three definitions most frequently used

✉ Corresponding author. Email: s.bourafa@centre-univ-mila.dz

ISSN (electronic): 2773-4196

for the general fractional differential operators. The first one is the Grünwald-Letnikov (GL) fractional differential operator defined by the limit of a fractional-order backward difference and has an advantage for numerical simulations. The second type is the Riemann-Liouville (RL) definition; this operator played a pivotal role in developing the fractional calculus theory. Using these two fractional differential operators in modeling real phenomena leads to mathematical models with initial conditions expressed in terms of fractional derivatives that do not have known physical interpretation. The third type is the Caputo derivative having the advantage of dealing models with initial conditions expressed in terms of the field variables and their integer-order derivatives, having clear physical interpretations [9]. Recently it has been demonstrated that the fractional-order derivative of a non-constant periodic function is not a periodic function with the same period [13,22,23] and in [5] the authors studied quasi-periodic properties of fractional order integrals and derivatives of periodic functions. As a consequence of the non-periodicity of the fractional derivative of a $T-$periodic function, the time-invariant fractional-order systems do not have any non-constant exact periodic solution unless the lower terminal of the derivative is $\pm\infty$ [12,13,23], which is not realistic. This property limits the applicability of the fractional derivative and makes it unfavorable for periodic real phenomena. In [1], the authors have proposed a modification of the Grünwald-Letnikov fractional differential operator, which consists of fixing the memory length and varying the lower terminal of the derivative. They have demonstrated that the modified definition of fractional derivative preserves the periodicity. The present paper extends this modification to the Caputo and Rieman-Liouville fractional-order operators. Tow examples are investigated to confirm that periodic solutions arise in fractional-order systems when the new fractional derivative operator is used. One linear system for which one gives an exact periodic solution defined by its analytical expression and another nonlinear system for which one provides an exact periodic solution using both qualitative and numerical methods.

2 Fractional-Order Derivatives

As said above, the most usual definitions of fractional-order derivative are the Grünwald-Letnikov, the Riemann-Liouville and the Caputo definitions [17]. For $0 < \alpha \notin \mathbb{N}$, the $\alpha-$th order derivative of a function $f(t)$ with respect to t and a terminal value a is given in the sense of

- Grünwald-Letnikov by

$$ {}_{a}^{GL}D_t^{\alpha} f(x) = \lim_{\substack{h \to 0 \\ nh = x-a}} h^{-\alpha} \sum_{k=0}^{n} (-1)^k \begin{pmatrix} \alpha \\ k \end{pmatrix} f(x-kh), \tag{2.1} $$

where $\begin{pmatrix} \alpha \\ k \end{pmatrix} = \frac{\Gamma(\alpha+1)}{k!\Gamma(\alpha-k+1)}$.

- Riemann-Liouville by

$$ {}_{a}^{RL}D_t^{\alpha} f(t) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} (\int_a^t (t-\tau)^{m-\alpha-1} f(\tau) d\tau). \tag{2.2} $$

- Caputo by

$$ {}_{a}^{C}D_{t}^{\alpha}f(t) = \frac{1}{\Gamma(m-\alpha)} \int_{a}^{t} (t-\tau)^{m-\alpha-1} f^{(m)}(\tau) d\tau. \tag{2.3} $$

In (2.2) and (2.3), m is the first integer greater than α, and $\Gamma(.)$ is the Gamma function. The following theorems reveal a remarkable property for the fractional derivatives based on Caputo definition, Grünwald-Letnikov definition, Riemann-Liouville definition [22].

Theorem 2.1. *Suppose that $f(t)$ is a non constant periodic function with period T. If $f(t)$ is m-times differentiable, then the functions ${}_{a}^{C}D_{t}^{\alpha}f(t)$, where $0 < \alpha \notin \mathbb{N}$ and m is the first integer greater than α, cannot be a periodic functions with period T.*

Theorem 2.2. *Suppose that $f(t)$ is (m-1)-times continuously differentiable and $f^{(m)}(t)$ is bounded. If $f(t)$ is a non-constant periodic function with period T, then the functions ${}_{a}^{GL}D_{t}^{\alpha}f(t)$ and ${}_{a}^{RL}D_{t}^{\alpha}f(t)$, where $0 < \alpha \notin \mathbb{N}$ and m is the first integer greater than α, cannot be periodic functions with period T.*

Example 2.3. Let $f(t) = \sin(t)$. One has

$$ \sin(t) = \sum_{p=0}^{\infty} (-1)^{p} \frac{t^{2p+1}}{(2p+1)!}. $$

Hence

$$ {}_{a}^{RL}D_{t}^{\alpha} \sin(t) = t^{1-\alpha} E_{2,2-\alpha}(-t^{2}), $$

where $0 < \alpha < 1$ and $E_{\alpha,\beta}(t)$ is the generalized Mittag-Leffler function defined by

$$ E_{\alpha,\beta}(t) = \sum_{k=0}^{\infty} \frac{t^{k}}{\Gamma(\alpha k + \beta)}. $$

Numerical simulations showed that $t^{1-\alpha}E_{2,2-\alpha}(-t^{2})$ is not a periodic function where $0 < \alpha < 1$, even if $\alpha = 1$ this function is the periodic function $\cos(t)$.

As a consequence of the above theorems, periodic solution cannot be expected in fractional-order systems, under any circumstances [22,23].

Corollary 2.4. *A differential equation of fractional-order in the form*

$$ {}_{a}^{\cdot}D_{t}^{\alpha}x(t) = f(x(t)), $$

where $0 < \alpha \notin \mathbb{N}$, cannot have any non-constant smooth periodic solution.

This property highlights one of the basic differences between fractional-order derivative and integer-order one, and it makes fractional-order systems unfavourable for a wide range of real periodic phenomena. Therefore in this paper one overcomes this problem by imposing a simple modification to both Riemann-Liouville and Caputo definitions.

3 The Fractional-Order Derivative with Sliding Fixed Memory Length

one first recalls the Grünwald-Letnikov fractional-order derivative with fixed memory length introduced in [1].

Definition 3.1. (The Grünwald-Letnikov fractional derivative with fixed memory length) Let $\alpha \geq 0$, $L > 0$, m an integer such that $m-1 \leq \alpha < m$ and f an integrable function in the interval $[a-L, b]$. The operator ${}^{MG}_{\ L}D^{\alpha}_{t}$ defined by :

$$ {}^{MG}_{\ L}D^{\alpha}_{t} f(t) = \lim_{h\to 0} \frac{1}{h^{\alpha}} \sum_{k=0}^{\frac{L}{h}} (-1)^k \frac{\Gamma(\alpha+1)}{k!\Gamma(\alpha-k+1)} f(t-kh), \quad t \in [a,b], \tag{3.1} $$

is called the Grünwald-Letnikov fractional derivative with sliding fixed memory length.

The following proposition gives an evaluation of the limit in the definition of Grünwald-Letnikov fractional derivative with sliding fixed memory length.

Proposition 3.2. *Under the assumptions of definition (3.1), if the function f is m-differentiable with $f^{(m)} \in L^1[a-L,b]$, then*

$$ {}^{MG}_{\ L}D^{\alpha}_{t} f(t) = \sum_{k=0}^{m-1} \frac{f^{(k)}(t-L)L^{k-\alpha}}{\Gamma(k-\alpha+1)} + \frac{1}{\Gamma(m-\alpha)} \int_{t-L}^{t} (t-\tau)^{m-\alpha-1} f^{(m)}(\tau) d\tau. \tag{3.2} $$

It has been demonstrated that this modified fractional-order derivative possesses two important properties: the preservation of periodicity and the short memory, which considerably reduces the cost of numerical computations. Furthermore, it has been proven that contrarily to fractional autonomous systems defined using classical fractional derivative, the fractional autonomous systems in terms of the modified fractional derivative can generate exact periodic solutions.
In order to generalize this work, one introduces in this section a similar modification to both Caputo fractional-order derivative and Riemann-Liouville fractional-order derivative as follows.

Definition 3.3. (The Caputo fractional derivative with sliding fixed memory length) Let $\alpha > 0$, $L > 0$, m an integer such that $m = [\alpha]+1$ and $f \in C^m[a-L,b]$. The Caputo fractional derivative with sliding fixed memory length is defined by

$$ {}^{MC}_{\ L}D^{\alpha}_{t} f(t) = \frac{1}{\Gamma(m-\alpha)} \int_{t-L}^{t} (t-\tau)^{m-\alpha-1} f^{(m)}(\tau) d\tau. \tag{3.3} $$

Definition 3.4. (The Riemann-Liouville fractional derivative with sliding fixed memory length) Let $\alpha \geq 0$, $L > 0$, m an integer such that $m-1 \leq \alpha < m$ and f is a continuous function in $[a-L,b]$,the Riemann-Liouville fractional derivative with sliding fixed memory length is defined by

$$ {}^{MRL}_{\ \ L}D^{\alpha}_{t} f(t) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \int_{t-L}^{t} (t-\tau)^{m-\alpha-1} f(\tau) d\tau, \tag{3.4} $$

Remark 3.5. From (3.2) and (3.3) one gets

$$ {}^{MC}_{\ L}D^{\alpha}_{t} f(t) = {}^{MG}_{\ L}D^{\alpha}_{t} f(t) - \sum_{k=0}^{m-1} \frac{f^{(k)}(t-L)L^{k-\alpha}}{\Gamma(k-\alpha+1)}. \tag{3.5} $$

Proposition 3.6. *Under the assumption that the function $f(t)$ is m-times continuously differentiable*

$$ {}^{MRL}_{\ \ L}D_t^\alpha f(t) = {}^{MG}_{\ \ L}D_t^\alpha f(t) - \sum_{k=0}^{m-1} \frac{f^{(k)}(t-L)L^{k-\alpha}}{\Gamma(k-\alpha+1)}. \tag{3.6} $$

Proof. By differentiation and performing repeatedly integration by parts, one has

$$ \begin{aligned} {}^{MRL}_{\ \ L}D_t^\alpha f(t) &= \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \int_{t-L}^{t} (t-\tau)^{m-\alpha-1} f(\tau)d\tau, \\ &= -\frac{f^{(m-1)}L^{m-\alpha-1}(t-L)}{\Gamma(m-\alpha)} + \frac{1}{\Gamma(m-\alpha-1)} \frac{d^{m-1}}{dt^{m-1}} \int_{t-L}^{t} (t-\tau)^{m-\alpha-2} f(\tau)d\tau, \\ &\ \vdots \\ &= -\sum_{k=0}^{m-1} \frac{f^{(k)}(t-L)L^{k-\alpha}}{\Gamma(k-\alpha+1)} + \frac{1}{\Gamma(-\alpha)} \int_{t-L}^{t} (t-\tau)^{-\alpha-1} f(\tau)d\tau, \end{aligned} $$

setting $I = \frac{1}{\Gamma(-\alpha)} \int_{t-L}^{t} (t-\tau)^{-\alpha-1} f(\tau)d\tau$, and performing successive integrations by parts one obtains

$$ \begin{aligned} I &= \frac{f(t-L)L^{-\alpha}}{\Gamma(1-\alpha)} + \frac{1}{\Gamma(1-\alpha)} \int_{t-L}^{t} (t-\tau)^{-\alpha} f'(\tau)d\tau, \\ &= \frac{f(t-L)L^{-\alpha}}{\Gamma(1-\alpha)} + \frac{f'(t-L)L^{1-\alpha}}{\Gamma(2-\alpha)} + \frac{1}{\Gamma(2-\alpha)} \int_{t-L}^{t} (t-\tau)^{-\alpha+1} f^{(2)}(\tau)d\tau, \\ &\ \vdots \\ &= \sum_{k=0}^{m-1} \frac{f^{(k)}(t-L)L^{k-\alpha}}{\Gamma(k-\alpha+1)} + \frac{1}{\Gamma(m-\alpha)} \int_{t-L}^{t} (t-\tau)^{-\alpha+m-1} f^{(m)}(\tau)d\tau, \\ &= {}^{MG}_{\ \ L}D_t^\alpha f(t). \end{aligned} $$

Therefore

$$ {}^{MRL}_{\ \ L}D_t^\alpha f(t) = {}^{MG}_{\ \ L}D_t^\alpha f(t) - \sum_{k=0}^{m-1} \frac{f^{(k)}(t-L)L^{k-\alpha}}{\Gamma(k-\alpha+1)}. $$

□

Remark 3.7. From (3.5) and (3.6) one has

$$ {}^{MRL}_{\ \ L}D_t^\alpha f(t) = {}^{MC}_{\ \ L}D_t^\alpha f(t) = {}^{MG}_{\ \ L}D_t^\alpha f(t) - \sum_{k=0}^{m-1} \frac{f^{(k)}(t-L)L^{k-\alpha}}{\Gamma(k-\alpha+1)}. \tag{3.7} $$

In the following parts, one denotes the operators of Caputo and Riemann-Liouville fractional derivative with sliding fixed memory length by ${}^{M}_{L}D_t^\alpha$.

3.1 Fractional derivative of some elementary functions

In order to highlight the amazing properties of the fractional derivative with sliding fixed memory length one consider two elementary functions (the power and exponential functions), for which one computes their new derivatives.

3.1.1 New fractional derivative of the power function

Let $f(t) = t^n$, $n \in \mathbb{N}^*$, $\alpha > 0$, $L > 0$ and m is an integer such that $m - 1 < \alpha < m$.

If $n < m$, then $f^{(m)}(t) = 0$, substituting in (3.3) yields ${}^M_L D^\alpha_t (t^n) = 0$.
If $n \geq m$ then by repeated integrations by parts of the relation (3.3) one obtains

$$ {}^M_L D^\alpha_t (t^n) = \sum_{k=0}^{n-m} \frac{n! L^{-\alpha+m+k} (t-L)^{n-m-k}}{(n-m-k)!\Gamma(-\alpha+m+k+1)}. \tag{3.8} $$

Remark 3.8. (Fractional derivative of a constant function)
If f is a constant function (i.e.$f(t) = C$ for all $t \in [a-L, b]$, and C any constant including zero) then one has

$$ {}^M_L D^\alpha_t C = 0. $$

3.1.2 Fractional derivative of the exponential function

Let $f(t) = e^t = \sum\limits_{p=0}^{\infty} \frac{t^p}{p!}$, $\alpha > 0$, $L > 0$ and m is an integer such that $m - 1 < \alpha < m$.

One has

$$ {}^M_L D^\alpha_t e^t = {}^M_L D^\alpha_t \sum_{p=0}^{\infty} \frac{t^p}{p!} = \sum_{p=0}^{\infty} \frac{1}{p!} \, {}^M_L D^\alpha_t t^p. $$

From (3.8), one obtains that

$$ \begin{aligned} {}^M_L D^\alpha_t (e^t) &= \sum_{p=0}^{\infty} \sum_{k=0}^{p-m} \frac{L^{-\alpha+m+k}(t-L)^{p-m-k}}{(p-m-k)!\Gamma(-\alpha+m+1+k)}, \\ &= \sum_{p=0}^{\infty} \sum_{k=0}^{p-m} \frac{L^{-\alpha+m+k}(t-L)^{p-m-k}}{(p-m-k)!\Gamma(k-\alpha+m+1)}, \\ &= \sum_{p=0}^{\infty} \sum_{k=0}^{p} \frac{L^{-\alpha+m+k}(t-L)^{p-k}}{(p-k)!\Gamma(k-\alpha+m+1)}, \\ &= \sum_{p=0}^{\infty} \frac{L^{-\alpha+m}(t-L)^p}{p!\Gamma(-\alpha+m+1)} + \sum_{p=0}^{\infty} \frac{L^{-\alpha+m+1}(t-L)^p}{p!\Gamma(-\alpha+m+2)} + \ldots, \\ &= \left(\sum_{p=0}^{\infty} \frac{(t-L)^p}{p!\Gamma(-\alpha+m+1)} \right) \left(\sum_{k=0}^{\infty} \frac{L^{-\alpha+m+k}}{\Gamma(-\alpha+m+1+k)} \right), \\ &= e^{t-L} L^{-\alpha+m} \sum_{k=0}^{\infty} \frac{L^k}{\Gamma(-\alpha+m+1+k)}, \\ &= e^{t-L} L^{m-\alpha} E_{1,m+1-\alpha}(L). \end{aligned} $$

3.2 Derivative of a periodic function

The main result of this paper is stated in the following theorem.

Theorem 3.9. *Let $\alpha > 0$, $L > 0$ and m an integer such that $m - 1 < \alpha < m$ and $f \in C^m[a-L, b]$. If f is a periodic function with period T, Then ${}^M_L D^\alpha_t f$ is a periodic function with the same period T.*

Proof. Suppose that f is a periodic function with a period T. The aim of this proof is to demonstrate that the function $g(t) = {}^M_L D^\alpha_t f$ is a periodic function with the same period T (i.e.

$g(t+T) = g(t))$.
One has

$$g(t+T) = {}^{M}_{L}D^{\alpha}_{t+T} f(t+T) = \frac{1}{\Gamma(m-\alpha)} \int_{t+T-L}^{t+T} (t+T-\tau)^{m-\alpha-1} f^{(m)}(\tau+T) d\tau,$$
$$= \frac{1}{\Gamma(m-\alpha)} \int_{t-L}^{t} (t-s)^{m-\alpha-1} f^{(m)}(s+2T) ds,$$
$$= \frac{1}{\Gamma(m-\alpha)} \int_{t-L}^{t} (t-s)^{m-\alpha-1} f^{(m)}(s) ds,$$
$$= {}^{M}_{L}D^{\alpha}_{t} f(t) = g(t).$$

Thus, ${}^{M}_{L}D^{\alpha}_{t} f$ is a periodic function with the same period T.

□

3.2.1 Fractional derivative of some fundamental periodic functions

Note first that the functions ${}^{MG}_{L}D^{\alpha}_{t} \sin(t)$ and ${}^{MG}_{L}D^{\alpha}_{t} \cos(t)$ have been calculated in [1].

Example 3.10. (Fractional derivative with sliding fixed memory length of the sine function)
By definition

$${}^{M}_{L}D^{\alpha}_{t} f(t) = {}^{MG}_{L}D^{\alpha}_{t} f(t) - \sum_{k=0}^{m-1} \frac{f^{(k)}(t-L)L^{k-\alpha}}{\Gamma(k-\alpha+1)}.$$

Therefore

$${}^{M}_{L}D^{\alpha}_{t} \sin(t) = {}^{MG}_{L}D^{\alpha}_{t} \sin(t) - \sum_{k=0}^{m-1} \frac{\frac{d^k}{dt^k}(\sin(t-L))L^{k-\alpha}}{\Gamma(k-\alpha+1)},$$
$$= L^{-\alpha} \sin(t-L) E_{2,1-\alpha}(-L^2) + L^{1-\alpha} \cos(t-L) E_{2,2-\alpha}(-L^2)$$
$$- L^{-\alpha} \sum_{k=0}^{[\frac{m-1}{2}]} \frac{(-L^2)^k}{\Gamma(2k+1-\alpha)} \sin(t-L) - L^{1-\alpha} \sum_{k=0}^{[\frac{m-2}{2}]} \frac{(-L^2)^k}{\Gamma(2k+2-\alpha)} \cos(t-L),$$
$$= L^{-\alpha} \sin(t-L)(E_{2,1-\alpha}(-L^2) - \sum_{k=0}^{[\frac{m-1}{2}]} \frac{(-L^2)^k}{\Gamma(2k+1-\alpha)})$$
$$+ L^{1-\alpha} \cos(t-L)(E_{2,2-\alpha}(-L^2) - \sum_{k=0}^{[\frac{m-2}{2}]} \frac{(-L^2)^k}{\Gamma(2k+2-\alpha)}),$$
$$= a \sin(t-L) + b \cos(t-L), \tag{3.9}$$

where, $a = L^{-\alpha}(E_{2,1-\alpha}(-L^2) - \sum_{k=0}^{[\frac{m-1}{2}]} \frac{(-L^2)^k}{\Gamma(2k+1-\alpha)})$, $b = L^{1-\alpha}(E_{2,2-\alpha}(-L^2) - \sum_{k=0}^{[\frac{m-2}{2}]} \frac{(-L^2)^k}{\Gamma(2k+2-\alpha)})$.
One notices that, ${}^{M}_{L}D^{\alpha}_{t} \sin(t)$ is a periodic function with the period $T = 2\pi$. This analytical result is displayed in figure (3.1), for some values of α and $L = 32.1$.

Example 3.11. (Fractional derivative of cosine function)

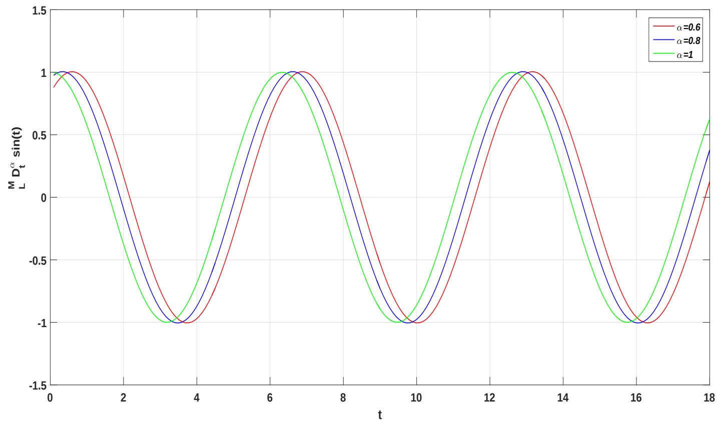

Figure 3.1: Fractional derivative of the Sine function for $L = 32.1$ and some values of α.

By definition

$$\begin{aligned}
{}^{M}_{L}D^{\alpha}_{t}\cos(t) &= {}^{MG}_{\;\;L}D^{\alpha}_{t}\cos(t) - \sum_{k=0}^{m-1}\frac{\frac{d^k}{dt^k}(\cos(t-L))L^{k-\alpha}}{\Gamma(k-\alpha+1)},\\
&= L^{-\alpha}\cos(t-L)E_{2,1-\alpha}(-L^2) - L^{1-\alpha}\sin(t-L)E_{2,2-\alpha}(-L^2)\\
&- L^{-\alpha}\sum_{k=0}^{[\frac{m-1}{2}]}\frac{(-L^2)^k}{\Gamma(2k+1-\alpha)}\cos(t-L) + L^{1-\alpha}\sum_{k=0}^{[\frac{m-2}{2}]}\frac{(-L^2)^k}{\Gamma(2k+2-\alpha)}\sin(t-L),\\
&= L^{-\alpha}\cos(t-L)(E_{2,1-\alpha}(-L^2) - \sum_{k=0}^{[\frac{m-1}{2}]}\frac{(-L^2)^k}{\Gamma(2k+1-\alpha)})\\
&- L^{1-\alpha}\sin(t-L)(E_{2,2-\alpha}(-L^2) - \sum_{k=0}^{[\frac{m-2}{2}]}\frac{(-L^2)^k}{\Gamma(2k+2-\alpha)}),\\
&= a\cos(t-L) - b\sin(t-L), \qquad (3.10)
\end{aligned}$$

where,

$$a = L^{-\alpha}(E_{2,1-\alpha}(-L^2) - \sum_{k=0}^{[\frac{m-1}{2}]}\frac{(-L^2)^k}{\Gamma(2k+1-\alpha)}),$$

and

$$b = L^{1-\alpha}(E_{2,2-\alpha}(-L^2) - \sum_{k=0}^{[\frac{m-2}{2}]}\frac{(-L^2)^k}{\Gamma(2k+2-\alpha)}).$$

Obviously ${}^{M}_{L}D^{\alpha}_{t}\cos(t)$ is a periodic function with period $T = 2\pi$.

3.3 An interpolation property

It is known that the operator of Grünwald-Letnikov fractional derivative with sliding fixed memory length is an extension of the integer-order operator $\frac{d^m}{t^m}$, (see [1]).
The following proposition proves that the Caputo and Riemann-Liouville operators of the

fractional derivative with sliding fixed memory length verifies this property for $\alpha \to m$, but not for $\alpha \to m-1$.

Proposition 3.12. *Let $L > 0$ and $0 \leq m-1 < \alpha < m$ such that m is an integer number, and let $f(t)$ having $(m+1)$ continuous bounded derivatives in $[a-L, b]$. Then , for all $t \in [a,b]$, one has*

$$\lim_{\alpha \to m} \ {}^{M}_{L}D^{\alpha}_{t} f(t) = f^{(m)}(t),$$

and

$$\lim_{\alpha \to m-1} \ {}^{M}_{L}D^{\alpha}_{t} f(t) = f^{(m-1)}(t) - f^{(m-1)}(t-L).$$

Proof. One has

$$\begin{aligned}
\lim_{\alpha \to m} \ {}^{M}_{L}D^{\alpha}_{t} f(t) &= \lim_{\alpha \to m} \frac{1}{\Gamma(m-\alpha)} \int_{t-L}^{t} (t-\tau)^{m-\alpha-1} f^{(m)}(\tau) d\tau, \\
&= \lim_{\alpha \to m} \frac{L^{m-\alpha} f^{(m)}(t-L)}{\Gamma(m-\alpha+1)} + \lim_{\alpha \to m} \frac{1}{\Gamma(m-\alpha+1)} \\
&\int_{t-L}^{t} (t-\tau)^{m-\alpha} f^{(m+1)}(\tau) d\tau, \\
&= f^{(m)}(t-L) + \int_{t-L}^{t} f^{(m+1)}(\tau) d\tau, \\
&= f^{(m)}(t).
\end{aligned}$$

For $\alpha \to m-1$, one has

$$\begin{aligned}
\lim_{\alpha \to m-1} \ {}^{MC}_{L}D^{\alpha}_{t} f(t) &= \lim_{\alpha \to m-1} \frac{1}{\Gamma(m-\alpha)} \int_{t-L}^{t} (t-\tau)^{m-\alpha-1} f^{(m)}(\tau) d\tau, \\
&= \int_{t-L}^{t} f^{(m)}(\tau) d\tau, \\
&= f^{(m-1)}(t) - f^{(m-1)}(t-L).
\end{aligned}$$

□

Example 3.13.
Let $f(t) = e^t$, then

$${}^{M}_{L}D^{\alpha}_{t} e^t = e^{t-L} L^{m-\alpha} E_{1,m+1-\alpha}(L),$$

Therefore,

$$\lim_{\alpha \to m} \ {}^{M}_{L}D^{\alpha}_{t} e^t = e^{t-L} E_{1,1}(L) = e^t = f^{(m)}(t).$$

However,

$$\begin{aligned}
\lim_{\alpha \to m-1} \ {}^{M}_{L}D^{\alpha}_{t} e^t &= e^{t-L} L E_{1,2}(L) = e^{t-L}(e^L - 1), \\
&= e^t - e^{t-L} = f^{(m)}(t) - f^{(m-1)}(t-L).
\end{aligned}$$

Example 3.14.
Let $f(t) = t^n$, one has

$${}^{M}_{L}D^{\alpha}_{t}(t^n) = \sum_{k=0}^{n-m} \frac{n! L^{-\alpha+m+k} (t-L)^{n-m-k}}{(n-m-k)! \Gamma(-\alpha+m+k+1)}.$$

Putting $N = n - m$ and $t - L = a$, then

$$\begin{aligned}
\lim_{\alpha\to m} {}_L^M D_t^\alpha(t^n) &= \sum_{k=0}^{N} \frac{n! L^k a^{N-k}}{(N-k)!k!}, \\
&= \frac{n!}{N!}\sum_{k=0}^{N} \frac{N! L^k a^{N-k}}{(N-k)!k!}, \\
&= \frac{n!}{N!}(a+L)^N = \frac{n!}{(n-m)!} t^{n-m}, \\
&= \frac{d^m}{dt} t^n = f^{(m)}(t).
\end{aligned}$$

However,

$$\begin{aligned}
\lim_{\alpha\to m-1} {}_L^M D_t^\alpha(t^n) &= \sum_{k=0}^{N} \frac{n! L^{k+1} a^{N-k}}{(N-k)!(k+1)!}, \\
&= \frac{n!}{(N+1)!}\sum_{k=0}^{N+1} \frac{(N+1)! L^k a^{N+1-k}}{(N+1-k)!k!} - \frac{n!}{(n-m+1)!}(t-L)^{n-m+1}, \\
&= \frac{n!}{(N+1)!} t^{N+1} - \frac{n!}{(n-(m-1))!}(t-L)^{n-(m-1)}, \\
&= \frac{n!}{(n-(m-1))!} t^{n-(m-1)} - \frac{n!}{(n-(m-1))!}(t-L)^{n-(m-1)}, \\
&= \frac{d^{m-1}}{dt} t^n - \frac{d^{m-1}}{dt}(t-L)^n = f^{(m-1)}(t) - f^{(m-1)}(t-L).
\end{aligned}$$

3.4 Comparison between some results of classical fractional-order derivatives and fractional order derivatives with sliding fixed memory length

The previous results are summarized in the table (3.1), in order to highlight the differences between classical fractional-order derivative and fractional-order derivative with sliding fixed memory length.

3.5 Fractional-order autonomous system with exact periodic solution

As previously mentioned, any autonomous fractional-order system expressed in terms of classical fractional derivatives cannot have any exact periodic solutions [13,22,23].
Conversely to these results, one presents some examples showing that fractional-order autonomous systems (linear and nonlinear) expressed in terms of fractional derivatives with sliding fixed memory length can have exact periodic solutions.

Example 3.15. (Linear fractional-order system)
Let consider the following linear fractional-order autonomous system

$$ {}_L^M D_t^\alpha X(t) = AX(t), \tag{3.11}$$

where $X(t) \in \mathbb{R}$ and $A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$, with $a = L^{-\alpha}(E_{2,1-\alpha}(-L^2) - \sum_{p=0}^{[\frac{m-1}{2}]} \frac{(-L^2)^p}{\Gamma(2p+1-\alpha)})$,
$b = L^{1-\alpha}(E_{2,2-\alpha}(-L^2) - \sum_{p=0}^{[\frac{m-2}{2}]} \frac{(-L^2)^p}{\Gamma(2p+2-\alpha)})$.

Classical fractional derivative ${}^{C}_{a}D^{\alpha}_{t}$ or ${}^{RL}_{a}D^{\alpha}_{t}$	**Fractional derivative with sliding fixed memory length** ${}^{M}_{L}D^{\alpha}_{t}$
${}^{C}_{a}D^{\alpha}_{t}f(t) = {}^{RL}_{a}D^{\alpha}_{t}f(t) - \sum_{k=0}^{m-1}\frac{f^{(k)}(a)(t-a)^{k-\alpha}}{\Gamma(k-\alpha+1)}$	${}^{MC}_{L}D^{\alpha}_{t}f(t) = {}^{MR}_{L}D^{\alpha}_{t}f(t)$
$\lim_{\alpha\to m} {}^{RL}_{a}D^{\alpha}_{t}f(t) = \lim_{\alpha\to m} {}^{C}_{a}D^{\alpha}_{t}f(t) = f^{(m)}(t)$	$\lim_{\alpha\to m} {}^{M}_{L}D^{\alpha}_{t}f(t) = f^{(m)}(t)$
$\lim_{\alpha\to m-1} {}^{RL}_{a}D^{\alpha}_{t}f(t) = f^{(m-1)}(t)$, $\lim_{\alpha\to m-1} {}^{C}_{a}D^{\alpha}_{t}f(t) = f^{(m-1)}(t) - f^{(m-1)}(a)$	$\lim_{\alpha\to m-1} {}^{M}_{L}D^{\alpha}_{t}f(t) = f^{(m-1)}(t) - f^{(m-1)}(t-L)$
${}^{RL}_{0}D^{\alpha}_{t}(t^n) = {}^{C}_{0}D^{\alpha}_{t}(t^n) = \frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)}t^{n-\alpha}$	${}^{M}_{L}D^{\alpha}_{t}(t^n) = \sum_{k=0}^{n-m}\frac{n!L^{-\alpha+m+k}(t-L)^{n-m-k}}{(n-m-k)!\Gamma(-\alpha+m+k+1)}$
${}^{RL}_{a}D^{\alpha}_{t}C = \frac{C}{\Gamma(1-\alpha)}(t-a)^{\alpha} \neq 0$, ${}^{C}_{a}D^{\alpha}_{t}C = 0$	${}^{M}_{L}D^{\alpha}_{t}C = 0$
${}^{RL}_{a}D^{\alpha}_{t}\sin t = t^{1-\alpha}E_{2,2-\alpha}(-t^2)$	${}^{M}_{a}D^{\alpha}_{t}\sin t = a\sin(t-L) + b\cos(t-L)$.

Table 3.1: Comparison between some results of classical fractional-order derivatives and fractional order derivatives with sliding fixed memory length.

- For $L = 2k\pi$, where k is a non-zero integer. The vector function $X(t) = c\begin{pmatrix}\cos(t)\\ \sin(t)\end{pmatrix}$, $c \in \mathbb{R}$ is an exact $2\pi-$periodic solution for the system (3.11).
By definition,

$$ {}^{M}_{2k\pi}D^{\alpha}_{t}X(t) = c\begin{pmatrix} {}^{M}_{2k\pi}D^{\alpha}_{t}\cos(t) \\ {}^{M}_{2k\pi}D^{\alpha}_{t}\sin(t)\end{pmatrix}. $$

Then, from (3.9) and (3.10) one obtains

$$ \begin{aligned} {}^{M}_{2k\pi}D^{\alpha}_{t}X(t) &= c\begin{pmatrix} a\cos(t-2k\pi) - b\sin(t-2k\pi) \\ a\sin(t-2k\pi) + b\cos(t-2k\pi)\end{pmatrix}, \\ &= c\begin{pmatrix} a & -b \\ b & a\end{pmatrix}\begin{pmatrix}\cos(t-2k\pi)\\ \sin(t-2k\pi)\end{pmatrix}, \\ &= cA\begin{pmatrix}\cos(t-2k\pi)\\ \sin(t-2k\pi)\end{pmatrix}, \\ &= AX(t). \end{aligned} $$

Therefore, $X(t) = c\begin{pmatrix}\cos(t)\\ \sin(t)\end{pmatrix}$ is an exact $2\pi-$periodic solution of (3.11) with $L = 2k\pi$.
- For $L = \frac{\pi}{2}$, one has

$$ \begin{aligned} {}^{M}_{\frac{\pi}{2}}D^{\alpha}_{t}X(t) &= c\begin{pmatrix} a\cos(t-\frac{\pi}{2}) - b\sin(t-\frac{\pi}{2}) \\ a\sin(t-\frac{\pi}{2}) + b\cos(t-\frac{\pi}{2})\end{pmatrix}, \\ &= c\begin{pmatrix} a\sin(t) + b\cos(t) \\ -a\cos(t) + b\sin(t)\end{pmatrix}, \\ &= c\begin{pmatrix} b & a \\ -a & b\end{pmatrix}\begin{pmatrix}\cos(t)\\ \sin(t)\end{pmatrix}, \\ &= cB\begin{pmatrix}\cos(t)\\ \sin(t)\end{pmatrix}, \\ &= BX(t), \end{aligned} $$

with $B = \begin{pmatrix} b & a \\ -a & b \end{pmatrix} \neq A$. Thus, $X(t) = c \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix}$ is not solution of (3.11), but it is an exact $2\pi-$periodic solution of the system ${}^{M}_{\frac{\pi}{2}}D^{\alpha}_{t}X(t) = BX(t)$.

Example 3.16. (The predator-prey model with Holling type II response function)
All population species possess the property of heredity, which means the passing on traits from parents to their offspring, either through asexual reproduction or sexual reproduction. The offspring cells or organisms acquire the genetic information of their parents through heredity. This property makes fractional differential systems models more efficiently regarding some specific problems than ordinary differential ones.
Motivated by this fact, we introduce the fractional version of the Holling-Tanner model [21] as follows

$$\begin{cases} D^{\alpha}x = r_1 x(1 - \frac{x}{K}) - \frac{qxy}{m+x}, \\ D^{\alpha}y = r_2 y(1 - \frac{y}{\gamma x}). \end{cases} \tag{3.12}$$

Where $D^{\cdot}$ denotes a standard fractional-order derivative operator and $\alpha \in [0,1]$ is the fractional-order related to the hereditary property of the population (a value of α close to an integer number means that the population has a weak hereditary property), $x(t) \geq 0$ and $y(t) \geq 0$ are the density of prey and predator populations at time t respectively. The parameters r_1 and r_2 are the intrinsic growth rates, K represents the carrying capacity of the prey, q is the maximum number of preys that can be eaten per predator per unit of time, m is the saturation value (it corresponds to the number of preys necessary to achieve one half the maximum rate q), γ is a measure of the quality of the prey as a portion of food for the predator.

Since exact analytical resolution of this nonlinear system is unavailable, one resorts to qualitative and numerical study. For this purpose the parameters are set to $r_1 = 1$, $r_2 = 0.2$, $K = 25$, $q = \frac{6}{7}$, $m = 1$ and $\gamma = 0.95$, the system (3.12) has two equilibrium points $E_0 = (25, 0)$ and $E_1 \approx (7.1429, 6.7857)$.

- The characteristic polynomial of the Jacobian matrix evaluated at E_0 is given by

  $$P(\lambda) = \lambda^2 + a_1\lambda + a_2 = \lambda^2 + 0.8\lambda - 0.2.$$

  So $a_2 = -0.2 < 0$, then according to Proposition 1 in [7] E_0 is unstable for all $\alpha \in [0, 2)$.

- The characteristic polynomial of the Jacobian matrix evaluated at E_1 is given by

  $$P(\lambda) = \lambda^2 - 0.1409\lambda + 0.0747.$$

  So $a_1 \approx -0.1409$ and $a_2 \approx 0.0747 > 0$.

  Applying Hopf-Like Bifurcation theory [2–4] and using Proposition 1 in [7], one obtains the Hopf-Like bifurcation value

  $$\alpha^* = \frac{2}{\pi} cos^{-1}\left(\frac{-a_1}{2\sqrt{a_2}}\right) \approx 0.8341.$$

  The fixed point E_1 losses its stability, and a periodic motion ($S-$asymptotically periodic for the classical fractional derivative and exact periodic for fractional derivative with sliding fixed memory length) appears.

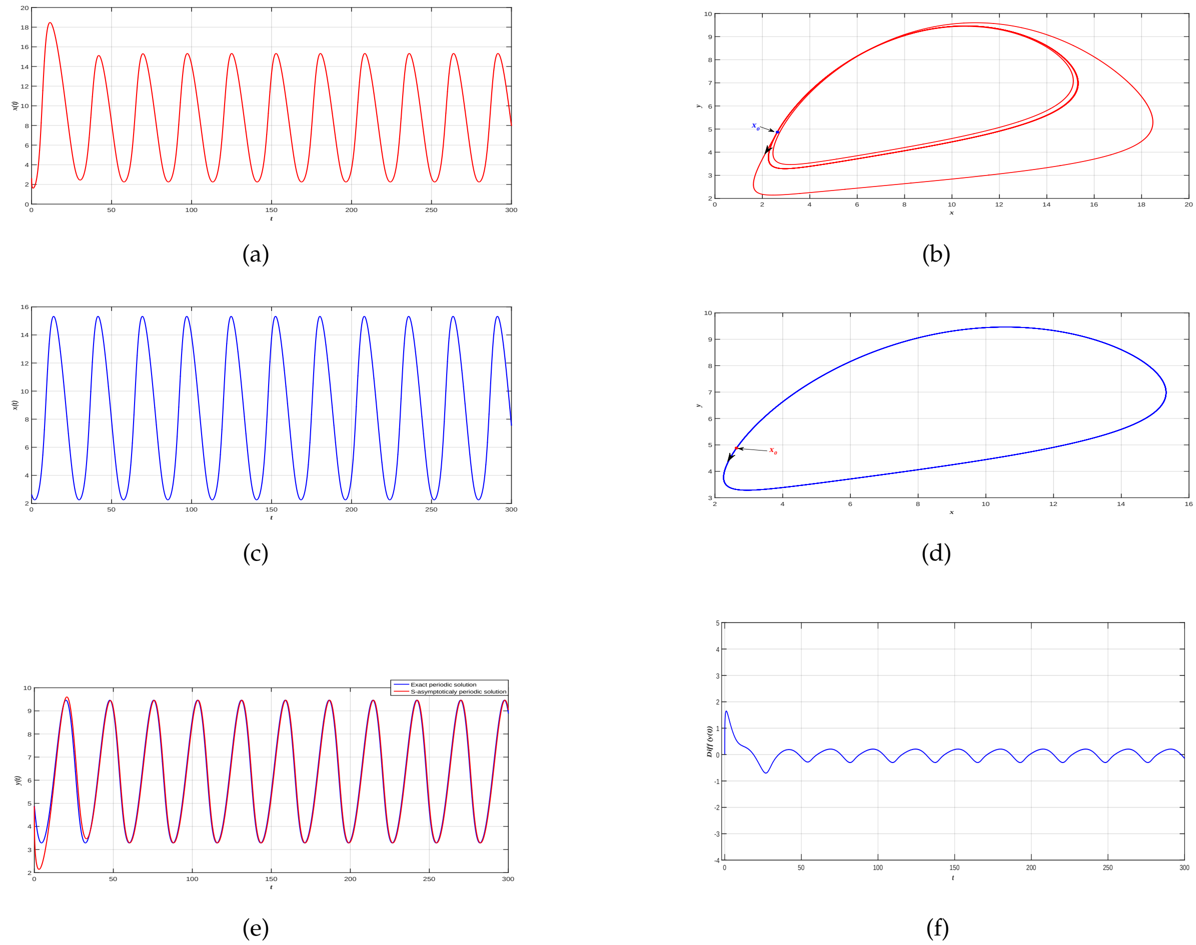

Figure 3.2: Time evolution and phase portrait of system (3.12) for $\alpha = 0.9$ (a,b) $S-$asymptotically $T-$periodic solution with $T \approx 27.2$ for classical fractional operator. (c,d) Exact $T-$periodic solution for the fractional derivative operator with sliding fixed memory length. (e,f) Comparison between the two solutions.

To illustrate these results, one solves the system (3.12) numerically by developing a Matlab code using a discretization technique based on the formula (3.7).
Choosing a value for α greater than α^*, for example, $\alpha = 0.9$, one compares the solution of (3.12) in terms of classical fractional operator and its solution in terms of the fractional operator with sliding fixed memory length $L = 30$. The two trajectories are start from the same initial point $X_0 = (2.64, 4.88)$, belonging to the attracting limit cycle. The results are shown in Fig. 3.2.
An $S-$asymptotically $T-$periodic solution with $T \approx 27.2$ is obtained for classical fractional operator as shown in Fig. 3.2(a,b), and an exact $T-$periodic solution is obtained for the fractional derivative operator with sliding fixed memory length as shown in Fig. 3.2(c,d).

4 Conclusion

In this article, one modifies the Caputo and Rieman-Liouville fractional-order derivatives by fixing the memory length and varying the lower terminal of the derivative. It is shown that the modified fractional derivative operator preserves the periodicity. Consequently, periodic solutions can be obtained in fractional-order systems expressed in terms of the new operator. Two examples are investigated to highlight this property for a linear system provides an analytic expression of an exact periodic solution is computed and for another nonlinear system for which exact periodic solutions are obtained using qualitative and numerical methods.

Acknowledgments

This research was supported by the Algerian General Directorate for Scientific Research and Technological Development (DGRSDT).

Conflict of Interest

The authors have no conflicts of interest to disclose.

References

[1] M-S. Abdelouahab and N. Hamri, *The Grünwald-Letnikov fractional-order derivative with fixed memory length*, Mediterr. J. Math. **13** (2016), 557–572.

[2] M-S. Abdelouahab, N. Hamri and J. W. Wang, *Hopf bifurcation and chaos in fractional-order modified hybrid optical system*, Nonlinear Dyn. **69** (2012), 275–284.

[3] M-S. Abdelouahab and R. Lozi, *Hopf Bifurcation and Chaos in Simplest Fractional-Order Memristor-based Electrical Circuit*, Indian. J. Ind. Appl. Math. **6**(2) (2015), 105–119.

[4] M-S. Abdelouahab, R. Lozi and G. Chen, *Complex Canard Explosion in a Fractional-Order FitzHugh–Nagumo Model*, Int. J. Bifurcation Chaos. **29**(8) (2019), 1950111–1950133.

[5] I. Area, J. Losada and J. J. Nieto, *On quasi-periodicity properties of fractional integrals and fractional derivatives of periodic functions*, Integral Transforms Spec. Funct. **27** (2016), 1–16.

[6] R. L. Bagley and R. A. Calico, *Fractional order state equations for the control of viscoelastically damped structures*, J. Guid. Control Dyn. **14** (1991), 304–311.

[7] S. Bourafa, M-S. Abdelouahab and A. Moussaoui, *On some extended Routh-Hurwitz conditions for fractional-order autonomous systems of order $\alpha \in [0, 2)$ and their applications to some population dynamic models*, Chaos Solitons Fractals. **133** (2020), 109623.

[8] P. L. Butzer and U. Westphal, *An introduction to fractional calculus*, In: Hilfer, R. Applications of Fractional Calculus in Physics, World Scientific, Singapore, 2000.

[9] M. Caputo, *Linear models of dissipation whose Q is almost frequency independent-II*, Geophys. J. R. Astron. Soc. **13** (1967), 529–539.

[10] O. Heaviside, H. J. Josephs and B. A. Behrend, *Electromagnetic Theory*, Chelsea Publishing Company, New York, 1971.

[11] M. Ichise, Y. Nagayanagi and T. Kojima, *An analog simulation of noninteger order transfer functions for analysis of electrode process*, J. Electroanal. Chem. **33** (1971), 253–265.

[12] Y-M. Kang, Y. Xie, J-C. Lu and J. Jiang, *On the nonexistence of non-constant exact periodic solutions in a class of the Caputo fractional-order dynamical systems*, Nonlinear Dyn. **82** (2015), 1259–1267.

[13] E. Kaslik and S. Sivasundaram, *Non-existence of periodic solutions in fractional-order dynamical systems and a remarkable difference between integer and fractional-order derivatives of periodic functions*, Nonlinear Analysis: Real World Applications. **13** (2012), 1489–1497.

[14] D. Kusnezov, A. Bulgac and G. D. Dang, *Quantum levy processes and fractional kinetics*, Phys. Rev. Lett. **82** (1999), 1136–1139.

[15] C. A. Monje, Y. Q. Chen, B. M. Vinagre, D. Xue, and V. Feliu, *Fractional-Order Systems and Controls Fundamentals and Applications*, Springer-Verlag London Limited, London, 2010.

[16] K. B. Oldham and J. Spanier, *The Fractional Calculus Theory and Applications of Differentiation and Integration to Arbitrary Order*, Academic press, inc, USA, 1974.

[17] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, 1999.

[18] B. Ross, *The development of fractional calculus 1695-1900*, Hist. Math. **4** (1977), 75–89.

[19] S. G. Samko, A. A. Kilbas and O. I. Marichev, *Fractional Integrals and Derivatives: Theory and Applications*, Gordon and Breach, Amsterdam, 1993.

[20] H. H. Sun, A. A. Abdelwahab and B. Onaral, *Linear approximation of transfer function with a pole of fractional order*, IEEE Trans. Autom. Control. **29** (1984), 441–444.

[21] J. T. Tanner, *The stability and the intrinsic growth rates of prey and predator populations*, Ecology. **56** (1975), 855–867.

[22] M. S. Tavazoei, *A note on fractional-order derivatives of periodic functions*. Automatica J. IFAC. **46** (2010), 945–948.

[23] M. S. Tavazoei and M. Haeri, *A proof for non existence of periodic solutions in time invariant fractional order systems*, Automatica J. IFAC. **45** (2009), 1886–1890.

Journal of Innovative Applied Mathematics and Computational Sciences

J. Innov. Appl. Math. Comput. Sci. **1**(1) (2021) 79–92. n2t.net/ark:/49935/jiamcs.v1i1.2

http://jiamcs.centre-univ-mila.dz/

Generalization of fractional Laplace transform for higher order and its application

Ahmed Bouchenak ✉

Department of Mathematics, University of Jordan, Amman, Jordan

Received 31 October 2021, Accepted 28 December 2021, Published 30 December 2021

Abstract. In this paper, we first introduce the conformable fractional Laplace transform. Then, we give its generalization for higher-order. Finally, as an application, we solve a non-homogeneous conformable fractional differential equation with variable coefficients and a system of fractional differential equations.

Keywords: Conformable fractional derivative, Conformable fractional Laplace transform, System of fractional differential equations.

2020 Mathematics Subject Classification: 39A10, 67B89.

1 Introduction

Fractional derivative emergence date back to the time of calculus. In 1695, L'Hospital wondered about the meaning of $\frac{d^n f}{dx^n}$ if $n = \frac{1}{2}$, since then, researchers have been attempting to define a fractional derivative. Some of which are : Riemann-Liouville fractional definitions [15], Caputo fractional definitions [9, 15], Grünwald-Letnikov fractional derivative [16], Atangana-Baleanu fractional definitions [5], Hadamard fractional integral [14], Caputo-Fabrizio fractional derivative [9] and conformable fractional definitions [12]. Most of the definitions give numerical solution to the problems. However, the conformable fractional derivative is a natural definition which gives us simple and easy solutions for the problems. For more different applications on conformable fractional derivative, the reader can refer to [1, 2, 4, 6–8, 10, 11].

In 2015, Abdeljawad Thabet defined the conformable fractional Laplace transform [1] which will help to solve many fractional differential equations. In order to study the solution of the most challenging problems, like a non-homogeneous fractional differential equation with variables coefficients for higher-order, we generalize the conformable fractional Laplace transform for higher-order. Finally, we use this generalization to solve fractional differential equations and a system as an application.

For more details on conformable fractional Laplace transform, we refer the reader to [1, 3, 6, 13, 18].

✉ Corresponding author. Email: ahm9170471@ju.edu.jo

ISSN (electronic): 2773-4196

2 Basics of conformable fractional Laplace transform

Definition 2.1. Let $f : [0, \infty) \to \mathbb{R}$ be a real valued function and $0 < \alpha \leq 1$. Then the conformable fractional Laplace transform of f is defined as:

$$\mathfrak{L}_\alpha \{f(x)\} = \mathcal{F}_\alpha(\xi) = \int_0^\infty e^{-\xi \frac{x^\alpha}{\alpha}} f(x)\, d_\alpha x = \int_0^\infty e^{-\xi \frac{x^\alpha}{\alpha}} f(x)\, x^{\alpha-1} dx.$$

provided the integral exists.

Let us have as an example for the conformable fractional Laplace transform of the usual functions in the theorem bellow.

Theorem 2.2. *Let $a, p, c \in \mathbb{R}$ and $0 < \alpha \leq 1$. Then*
(1) $\mathfrak{L}_\alpha \{c\}(\xi) = \frac{c}{\xi}$, $\xi > 0$.
(2) $\mathfrak{L}_\alpha \{x^p\}(\xi) = \alpha^{\frac{p}{\alpha}} \frac{\Gamma\left(1+\frac{p}{\alpha}\right)}{\xi^{1+\frac{p}{\alpha}}}$, $\xi > 0$.
(3) $\mathfrak{L}_\alpha \left\{e^{a\frac{x^\alpha}{\alpha}}\right\}(\xi) = \frac{1}{\xi - a}$, $\xi > 0$.
(4) $\mathfrak{L}_\alpha \left\{\sin a\frac{x^\alpha}{\alpha}\right\}(\xi) = \frac{a}{\xi^2+a^2}$, $\xi > 0$.
(5) $\mathfrak{L}_\alpha \left\{\cos a\frac{x^\alpha}{\alpha}\right\}(\xi) = \frac{\xi}{\xi^2+a^2}$, $\xi > 0$.
(6) $\mathfrak{L}_\alpha \left\{\sinh a\frac{x^\alpha}{\alpha}\right\}(\xi) = \frac{a}{\xi^2-a^2}$, $\xi > |a|$.
(7) $\mathfrak{L}_\alpha \left\{\cosh a\frac{x^\alpha}{\alpha}\right\}(\xi) = \frac{\xi}{\xi^2-a^2}$, $\xi > |a|$.

Proof. Follows by applying Definition 2.1 □

One of the excellent results is the relation between the usual, and the conformable fractional Laplace transforms, given in the theorem below.

Theorem 2.3. *Let $f : [0, \infty) \to \mathbb{R}$ be a function such that $\mathfrak{L}_\alpha \{f(x)\}(\xi) = \mathcal{F}_\alpha(\xi)$ exists. Then*

$$\mathfrak{L}_\alpha \{f(x)\}(\xi) = \mathcal{F}_\alpha(\xi) = \mathfrak{L}\left\{f\left((\alpha x)^{\frac{1}{\alpha}}\right)\right\}(\xi), \quad 0 < \alpha \leq 1.$$

Proof. See [1,3]. □

Theorem 2.4. *Let $f : [0, \infty) \to \mathbb{R}$, $g : [0, \infty) \to \mathbb{R}$ and let $\lambda, \mu, a \in \mathbb{R}$ and $0 < \alpha \leq 1$. Then*
(1) $\mathfrak{L}_\alpha \{\lambda f(x) + \mu g(x)\} = \lambda \mathcal{F}_\alpha(\xi) + \mu G_\alpha(\xi)$, $\xi > 0$.
(2) $\mathfrak{L}_\alpha \left\{e^{-a\frac{x^\alpha}{\alpha}} f(x)\right\}(\xi) = \mathcal{F}_\alpha(\xi + a)$, $\xi > |a|$.
(3) $\mathfrak{L}_\alpha \{I^\alpha f(x)\}(\xi) = \frac{\mathcal{F}_\alpha(\xi)}{\xi}$, $\xi > 0$.
(4) $\mathfrak{L}_\alpha \left\{\frac{x^{n\alpha}}{\alpha^n} f(x)\right\}(\xi) = (-1)^n \frac{d^n}{d\xi^n} \mathcal{F}_\alpha(\xi)$, $\xi > 0$.
(5) $\mathfrak{L}_\alpha \{(f * g)(x)\} = \mathcal{F}_\alpha(\xi) G_\alpha(\xi)$, $\xi > 0$.
where $\mathcal{F}_\alpha$ and G_α are the conformable fractional Laplace transform of the functions f and g respectively, $f * g$ is the convolution product of f and g and $I^\alpha f(x)$ is the conformable fractional integral.

Proof. See [1,3]. □

2.1 Generalization of fractional Laplace transform

Theorem 2.5. *Let $f : [0, \infty) \to \mathbb{R}$ be a continuous real valued differentiable function and $0 < \alpha \leq 1$. Then*

$$\mathfrak{L}_\alpha \{D^\alpha f(x)\} = \xi \mathcal{F}_\alpha(\xi) - f(0), \ \xi > 0.$$

Proof. See [1,3]. □

Theorem 2.6. *Let $f : [0, \infty) \to \mathbb{R}$ be a continuous real valued differentiable function and $0 < \alpha \leq 1$. Then*

$$\mathfrak{L}_\alpha \left\{D^{2\alpha} f(x)\right\} = \xi^2 \mathcal{F}_\alpha(\xi) - f^\alpha(0) - \xi f(0), \quad \xi > 0.$$

Proof. By using Definition 2.1 and integration by parts, we find:

$$\begin{aligned}
\mathfrak{L}_\alpha \left\{D^{2\alpha} f(x)\right\} &= \int_0^\infty e^{-\xi \frac{x^\alpha}{\alpha}} D^{2\alpha} f(x)\, d_\alpha x \\
&= \int_0^\infty D^\alpha D^\alpha f(x)\, e^{-\xi \frac{x^\alpha}{\alpha}} x^{\alpha-1} dx \\
&= \int_0^\infty e^{-\xi \frac{x^\alpha}{\alpha}} x^{1-\alpha} \frac{d}{dx} D^\alpha f(x)\, x^{\alpha-1} dx \\
&= \int_0^\infty e^{-\xi \frac{x^\alpha}{\alpha}} \frac{d}{dx} D^\alpha f(x)\, dx \\
&= \lim_{b\to\infty} \left[e^{-\xi \frac{x^\alpha}{\alpha}} D^\alpha f(x)\right]_0^b + \int_0^\infty D^\alpha f(x) \left(\frac{\xi}{\alpha} \alpha x^{\alpha-1}\right) e^{-\xi \frac{x^\alpha}{\alpha}} dx \\
&= -f^\alpha(0) + \xi \int_0^\infty e^{-\xi \frac{x^\alpha}{\alpha}} D^\alpha f(x)\, d_\alpha x \\
&= -f^\alpha(0) + \xi \mathfrak{L}_\alpha \{D^\alpha f(x)\}.
\end{aligned}$$

By the previous theorem we get the result.

$$\mathfrak{L}_\alpha \left\{D^{2\alpha} f(x)\right\} = \xi^2 \mathcal{F}_\alpha(\xi) - f^\alpha(0) - \xi f(0).$$

□

Theorem 2.7. *Let $f : [0, \infty) \to \mathbb{R}$ be a continuous real valued differentiable function and $0 < \alpha \leq 1$. Then*

$$\mathfrak{L}_\alpha \left\{D^{3\alpha} f(x)\right\} = \xi^3 \mathcal{F}_\alpha(\xi) - f^{2\alpha}(0) - \xi f^\alpha(0) - \xi^2 f(0), \quad \xi > 0.$$

Proof. By using Definition 2.1 and integration by parts, we have:

$$\begin{aligned}
\mathfrak{L}_\alpha \left\{D^{3\alpha} f(x)\right\} &= \int_0^\infty e^{-\xi \frac{x^\alpha}{\alpha}} D^{3\alpha} f(x)\, d_\alpha x \\
&= \int_0^\infty D^\alpha D^{2\alpha} f(x)\, e^{-\xi \frac{x^\alpha}{\alpha}} x^{\alpha-1} dx \\
&= \int_0^\infty e^{-\xi \frac{x^\alpha}{\alpha}} \frac{d}{dx} D^{2\alpha} f(x)\, dx \\
&= \lim_{b\to\infty} \left[e^{-\xi \frac{x^\alpha}{\alpha}} D^{2\alpha} f(x)\right]_0^b + \int_0^\infty D^{2\alpha} f(x)\, \xi x^{\alpha-1} e^{-\xi \frac{x^\alpha}{\alpha}} dx \\
&= -f^{2\alpha}(0) + \xi \int_0^\infty e^{-\xi \frac{x^\alpha}{\alpha}} \left(D^{2\alpha} f(x)\right) d_\alpha x \\
&= -f^{2\alpha}(0) + \xi \mathfrak{L}_\alpha \left\{D^{2\alpha} f(x)\right\}.
\end{aligned}$$

By the previous theorem we get the result.

$$\mathfrak{L}_\alpha\left\{D^{3\alpha}f(x)\right\} = \xi^3\mathcal{F}_\alpha(\xi) - f^{2\alpha}(0) - \xi f^{\alpha}(0) - \xi^2 f(0).$$

□

Theorem 2.8. *Generalization of (C.F.L.T)*
Let $f : [0,\infty) \to \mathbb{R}$ be a continuous real valued differentiable function and $0 < \alpha \leq 1$, then for any integer number n we have :

$$\begin{aligned}
\mathfrak{L}_\alpha\left\{D^{(n\alpha)}f(x)\right\} &= \xi^n\mathcal{F}_\alpha(\xi) - f^{(n-1)\alpha}(0) - \xi f^{(n-2)\alpha}(0) - \xi^{(2)} f^{(n-3)\alpha}(0) \\
&- \ldots\ldots - \xi^{(n-2)} f^{\alpha}(0) - \xi^{(n-1)} f(0) \\
&= \xi^n\mathcal{F}_\alpha(\xi) - \xi^{(0)} f^{(n-1-0)\alpha}(0) - \xi^{(1)} f^{(n-1-1)\alpha}(0) - \xi^{(2)} f^{(n-1-2)\alpha}(0) \\
&- \ldots\ldots - \xi^{(n-1-1)} f^{(1)\alpha}(0) - \xi^{(n-1)} f(0).
\end{aligned}$$

Hence

$$\mathfrak{L}_\alpha\left\{D^{(n\alpha)}f(x)\right\} = \xi^n\mathcal{F}_\alpha(\xi) - \sum_{j=0}^{n-1}\xi^j f^{(n-j-1)\alpha}(0), \quad \xi > 0.$$

Proof. We are going to prove this theorem by induction.
For $n = 1,2,3$ the formula is true (see the previous theorems).
Now, suppose that the formula is true for n and prove it for $n+1$.
that is $\mathfrak{L}_\alpha\left\{D^{(n\alpha)}f(x)\right\} = \xi^n\mathcal{F}_\alpha(\xi) - \sum_{j=0}^{n-1}\xi^j f^{(n-j-1)\alpha}(0), \quad \xi > 0$ is true.
By using Definition 2.1 and integration by parts, we have:

$$\begin{aligned}
\mathfrak{L}_\alpha\left\{D^{(n+1)\alpha}f(x)\right\} &= \int_0^\infty e^{-\xi\frac{x^\alpha}{\alpha}} D^{(n+1)\alpha} f(x)\, d_\alpha x \\
&= \int_0^\infty e^{-\xi\frac{x^\alpha}{\alpha}} D^{(n+1)\alpha} f(x)\, x^{\alpha-1} dx \\
&= \int_0^\infty D^\alpha D^{n\alpha} f(x)\, e^{-\xi\frac{x^\alpha}{\alpha}} x^{\alpha-1} dx \\
&= \int_0^\infty e^{-\xi\frac{x^\alpha}{\alpha}} x^{1-\alpha}\frac{d}{dx} D^{n\alpha} f(x)\, x^{\alpha-1} dx \\
&= \int_0^\infty e^{-\xi\frac{x^\alpha}{\alpha}} \frac{d}{dx} D^{n\alpha} f(x)\, dx \\
&= \left[e^{-\xi\frac{x^\alpha}{\alpha}} D^{n\alpha} f(x)\right]_0^\infty + \int_0^\infty D^{n\alpha} f(x)\, \xi x^{\alpha-1} e^{-\xi\frac{x^\alpha}{\alpha}} dx \\
&= -f^{n\alpha}(0) + \xi\int_0^\infty e^{-\xi\frac{x^\alpha}{\alpha}} \left(D^{n\alpha} f(x)\right) d_\alpha x \\
&= -f^{n\alpha}(0) + \xi\int_0^\infty e^{-\xi\frac{x^\alpha}{\alpha}} \left(D^{n\alpha} f(x)\right) d_\alpha x \\
&= -f^{n\alpha}(0) + \xi\mathfrak{L}_\alpha\left\{D^{n\alpha} f(x)\right\} \text{ (since the formula is true)} \\
&= -f^{n\alpha}(0) + \xi\left(\xi^n\mathcal{F}_\alpha(\xi) - \sum_{j=0}^{n-1}\xi^j f^{(n-j-1)\alpha}(0)\right).
\end{aligned}$$

Therefore

$$\begin{aligned}\mathfrak{L}_\alpha\left\{D^{(n+1)\alpha}f(x)\right\} &= -f^{n\alpha}(0)+\xi^{(n+1)}\mathcal{F}_\alpha(\xi)-\xi\sum_{j=0}^{n-1}\xi^j f^{(n-j-1)\alpha}(0)\\ &= \xi^{(n+1)}\mathcal{F}_\alpha(\xi)-f^{n\alpha}(0)-\sum_{j=0}^{n-1}\xi^{(j+1)}f^{(n-j-1)\alpha}(0)\\ &= \xi^{(n+1)}\mathcal{F}_\alpha(\xi)-\sum_{j=0}^{n}\xi^j f^{(n-j-1)\alpha}(0).\end{aligned}$$

Which complete the proof of the theorem. □

3 Applications

We use the conformable fractional Laplace transform as an application to solve some problems. In the first one, we solve a system of fractional differential equations with constant coefficients of three unknowns. In the second, we apply the generalization of **(C.F.L.T)** to solve a non-homogeneous fractional differential equation with variables coefficients.

Problem 1 :

$$\begin{cases} Y_1^{(\alpha)} &= Y_1 - Y_2 + Y_3,\\ Y_2^{(\alpha)} &= -2Y_1 + Y_2 - Y_3,\\ Y_3^{(\alpha)} &= -Y_2 + Y_3.\end{cases}$$

Conditions 1 :

$$Y_1(0) = Y_2(0) = Y_3(0) = 1, \quad 0 < \alpha \le 1.$$

Solution :
Let $\mathfrak{L}_\alpha\{Y_1\} = F_\alpha(\xi)$, $\mathfrak{L}_\alpha\{y_2\} = G_\alpha(\xi)$ and $\mathfrak{L}_\alpha\{Y_3\} = H_\alpha(\xi)$.
When applying the conformable fractional Laplace transform on all the system of fractional differential equation and using the giving conditions, we get:

$$\begin{cases} \xi F_\alpha(\xi) - 1 &= F_\alpha(\xi) - G_\alpha(\xi) + H_\alpha(\xi),\\ \xi G_\alpha(\xi) - 1 &= -2F_\alpha(\xi) + G_\alpha(\xi) - H_\alpha(\xi),\\ \xi H_\alpha(\xi) - 1 &= -G_\alpha(\xi) + H_\alpha(\xi).\end{cases}$$

Which implies

$$\begin{cases} (\xi-1)F_\alpha(\xi) + G_\alpha(\xi) - H_\alpha(\xi) &= 1,\\ 2F_\alpha(\xi) + (\xi-1)G_\alpha(\xi) + H_\alpha(\xi) &= 1,\\ G_\alpha(\xi) + (\xi-1)H_\alpha(\xi) &= 1.\end{cases}$$

Now, we can use Cramers rule to obtain solutions for $F_\alpha(\xi)$, $G_\alpha(\xi)$ and $H_\alpha(\xi)$.
First

$$\Delta = \begin{vmatrix} (\xi-1) & 1 & -1\\ 2 & (\xi-1) & 1\\ 0 & 1 & (\xi-1)\end{vmatrix} = (\xi^3 - 3\xi^2).$$

Hence

$$F_\alpha\{\xi\} = \frac{1}{\Delta}\begin{vmatrix} 1 & 1 & -1 \\ 1 & (\xi-1) & 1 \\ 1 & 1 & (\xi-1) \end{vmatrix} = \frac{(\xi^2-2\xi)}{(\xi^3-3\xi^2)}.$$

We are going to find $G_\alpha\{\xi\}$ again using Cramer's Rule.

$$G_\alpha\{\xi\} = \frac{1}{\Delta}\begin{vmatrix} (\xi-1) & 1 & -1 \\ 2 & 1 & 1 \\ 0 & 1 & (\xi-1) \end{vmatrix} = \frac{(\xi^2-5\xi+2)}{(\xi^3-3\xi^2)}.$$

In the similar way, we get $H_\alpha(\xi)$

$$H_\alpha\{\xi\} = \frac{1}{\Delta}\begin{vmatrix} (\xi-1) & 1 & 1 \\ 2 & (\xi-1) & 1 \\ 0 & 1 & 1 \end{vmatrix} = \frac{(\xi^2-3\xi+2)}{(\xi^3-3\xi^2)}$$

Using partial fraction to rewrite F_α, G_α and H_α in this way

$$c_1\frac{1}{\xi} + c_2\frac{1}{\xi^2} + c_3\frac{1}{(\xi-3)},$$

for some constants c_1, c_2 and c_3 to make the calculation easy.
Therefore, we get

$$\begin{cases} F_\alpha(\xi) &= \frac{6}{9}\frac{1}{\xi} + \frac{3}{9}\frac{1}{(\xi-3)}, \\ G_\alpha(\xi) &= \frac{13}{9}\frac{1}{\xi} - \frac{6}{9}\frac{1}{\xi^2} - \frac{4}{9}\frac{1}{(\xi-3)}, \\ H_\alpha(\xi) &= \frac{7}{9}\frac{1}{\xi} - \frac{6}{9}\frac{1}{\xi^2} + \frac{2}{9}\frac{1}{(\xi-3)}. \end{cases} \tag{3.1}$$

Applying the conformable fractional Laplace inverse transform on all the system (3.1) using the properties in Theorem 2.2 and Theorem 2.4 we obtain the solution of our problem.

$$\begin{cases} \mathfrak{L}_\alpha^{-1}\{F_\alpha(\xi)\} &= \frac{6}{9}\mathfrak{L}_\alpha^{-1}\left\{\frac{1}{\xi}\right\} + \frac{3}{9}\mathfrak{L}_\alpha^{-1}\left\{\frac{1}{(\xi-3)}\right\}, \\ \mathfrak{L}_\alpha^{-1}\{G_\alpha(\xi)\} &= \frac{13}{9}\mathfrak{L}_\alpha^{-1}\left\{\frac{1}{\xi}\right\} - \frac{6}{9}\mathfrak{L}_\alpha^{-1}\left\{\frac{1}{\xi^2}\right\} - \frac{4}{9}\mathfrak{L}_\alpha^{-1}\left\{\frac{1}{(\xi-3)}\right\}, \\ \mathfrak{L}_\alpha^{-1}\{H_\alpha(\xi)\} &= \frac{7}{9}\mathfrak{L}_\alpha^{-1}\left\{\frac{1}{\xi}\right\} - \frac{6}{9}\mathfrak{L}_\alpha^{-1}\left\{\frac{1}{\xi^2}\right\} + \frac{2}{9}\mathfrak{L}_\alpha^{-1}\left\{\frac{1}{(\xi-3)}\right\}. \end{cases}$$

Then

$$\begin{cases} Y_1(x) &= \frac{6}{9} + \frac{3}{9}e^{3\frac{x^\alpha}{\alpha}}, \\ Y_2(x) &= \frac{13}{9} - \frac{6}{9}\frac{x^\alpha}{\alpha} - \frac{4}{9}e^{3\frac{x^\alpha}{\alpha}}, \\ Y_3(x) &= \frac{7}{9} - \frac{6}{9}\frac{x^\alpha}{\alpha} + \frac{2}{9}e^{3\frac{x^\alpha}{\alpha}}. \end{cases}$$

Hence a result as required.

Problem 2 :

$$Y^{3\alpha}(x) + Y^{2\alpha}(x) - \frac{x^\alpha}{\alpha}Y(x) + 2Y(x) = \cos\frac{x^\alpha}{\alpha}. \tag{3.2}$$

Conditions 2 :

$$Y^{2\alpha}(0) = Y^\alpha(0) = Y(0) = 0. \tag{3.3}$$

Solution :
Let us take the conformable fractional Laplace transform of both sides and using the given conditions, we get equation 3.4

$$\xi^3\Psi(\xi)+\xi^2\Psi(\xi)-(-1)\frac{d}{d\xi}\left(\xi\Psi(\xi)\right)+2\Psi(\xi)=\frac{\xi}{\xi^2+1}. \tag{3.4}$$

Where $\Psi(\xi)=\mathfrak{L}_\alpha\{Y\}$ and $\xi>0$.
This follows from the properties of **(C.F.L.T)** in Theorem 2.2 and Theorem 2.4,
$\left(\mathfrak{L}_\alpha\left\{\frac{x^{n\alpha}}{\alpha^n}f\left(x\right)\right\}(\xi)=(-1)^n\frac{d^n}{d\xi^n}\mathcal{F}_\alpha(\xi),\ \ \xi>0\right)$.
Then :

$$(\xi^3+\xi^2+2)\Psi(\xi)+\Psi(\xi)+\xi\Psi^{'}(\xi)=\frac{\xi}{\xi^2+1}. \tag{3.5}$$

This equation can be simplified to:

$$\xi\Psi^{'}(\xi)+(\xi^3+\xi^2+3)\Psi(\xi)+=\frac{\xi}{\xi^2+1}.$$

Hence, we find :

$$\Psi^{'}(\xi)+\left(\frac{\xi^3+\xi^2+3}{\xi}\right)\Psi(\xi)+=\frac{1}{\xi^2+1}. \tag{3.6}$$

Which is a first order ordinary non-homogeneous linear differential equation with variable coefficients.
Applying theory of linear differential equations we obtain:

$$\begin{aligned}\Psi\left(\xi\right) &= e^{-\int\left(\frac{\xi^3+\xi^2+3}{\xi}\right)d\xi}\left[\int e^{\int\left(\frac{\xi^3+\xi^2+3}{\xi}\right)d\xi}\frac{1}{\xi^2+1}d\xi+k\right]\\ &= e^{-\left(\frac{\xi^3}{3}+\frac{\xi^2}{2}+3\ln\xi\right)}\left[\int e^{\left(\frac{\xi^3}{3}+\frac{\xi^2}{2}+3\ln\xi\right)}\frac{1}{\xi^2+1}d\xi+k\right].\end{aligned}$$

for some constant k.
Therefore

$$\Psi\left(\xi\right)=\frac{\int e^{\left(\frac{\xi^3}{3}+\frac{\xi^2}{2}+3\ln\xi\right)}\frac{1}{\xi^2+1}d\xi+k}{e^{\left(\frac{\xi^3}{3}+\frac{\xi^2}{2}+3\ln\xi\right)}}\quad\text{for some constant k.}$$

Claim :
The conformable fractional Laplace inverse transform exists $(\Psi(\xi)\in Dom(\mathfrak{L}_\alpha^{-1}))$.

Proof.
1. $\lim\limits_{\xi\to\infty}\Psi\left(\xi\right)=\frac{\infty}{\infty}$ indeterminate.
Then we have to use $\text{L}^{'}\text{Hopital}^{'}\text{s}$ rule to get :

$$\lim_{\xi\to\infty}\Psi\left(\xi\right)\;=\;\lim_{\xi\to\infty}\frac{e^{\left(\frac{\xi^3}{3}+\frac{\xi^2}{2}+3\ln\xi\right)}\left(\frac{1}{\xi^2+1}\right)}{\left(\xi^2+\xi+\frac{3}{\xi}\right)e^{\left(\frac{\xi^3}{3}+\frac{\xi^2}{2}+3\ln\xi\right)}}=\lim_{\xi\to\infty}\frac{1}{\left(\xi^2+\xi+\frac{3}{\xi}\right)(\xi^2+1)}=0.$$

2. $\lim\limits_{\xi\to\infty}\xi\Psi\left(\xi\right)=\lim\limits_{\xi\to\infty}\xi\dfrac{\int e^{\left(\frac{\xi^3}{3}+\frac{\xi^2}{2}+3\ln\xi\right)}\frac{1}{\xi^2+1}d\xi+k}{e^{\left(\frac{\xi^3}{3}+\frac{\xi^2}{2}+3\ln\xi\right)}}=\dfrac{\infty}{\infty}$ indeterminate.

Thus we have to use L$^{'}$Hopital$^{'}$s rule to find :

$$\lim_{\xi\to\infty}\xi\Psi\left(\xi\right)=\lim_{\xi\to\infty}\left[\frac{e^{\left(\frac{\xi^3}{3}+\frac{\xi^2}{2}+3\ln\xi\right)}\left(\frac{\xi}{\xi^2+1}\right)+\int e^{\left(\frac{\xi^3}{3}+\frac{\xi^2}{2}+3\ln\xi\right)}\frac{1}{\xi^2+1}d\xi+k}{\left(\xi^2+\xi+\frac{3}{\xi}\right)e^{\left(\frac{\xi^3}{3}+\frac{\xi^2}{2}+3\ln\xi\right)}}\right]=\frac{\infty}{\infty}.$$

Which is also indeterminate, so we reuse L$^{'}$Hopital$^{'}$s rule again :

$$\lim_{\xi\to\infty}\xi\Psi\left(\xi\right)=\lim_{\xi\to\infty}\left[\frac{\left(\frac{1}{\xi^2+1}\right)+\left(\xi^2+\xi+\frac{3}{\xi}\right)\left(\frac{\xi}{\xi^2+1}\right)+\left(\frac{1-\xi^2}{(\xi^2+1)^2}\right)}{\left(2\xi+1-\frac{3}{\xi^2}\right)+\left(\xi^2+\xi+\frac{3}{\xi}\right)^2}\right].$$

After simplifying and using the properties of limits calculations, we get :

$$\lim_{\xi\to\infty}\xi\Psi\left(\xi\right)=\lim_{\xi\to\infty}\frac{\xi^5}{\xi^6}=0.$$

Hence a result as required. □

Now, we can reformulate $\Psi(\xi)$ to become :

$$\Psi\left(\xi\right)=\frac{\int\xi^3e^{\left(\frac{\xi^3}{3}+\frac{\xi^2}{2}\right)}\frac{1}{\xi^2+1}d\xi+k}{\xi^3e^{\left(\frac{\xi^3}{3}+\frac{\xi^2}{2}\right)}}\quad\text{for some constant k.}$$

Let us approximate the Exponential by the first 2-terms of the series expansion.
ie : $\left(e^{\left(\frac{\xi^3}{3}+\frac{\xi^2}{2}\right)}\approx\left(1+\frac{\xi^3}{3}+\frac{\xi^2}{2}\right)\right)$.
Therefore

$$\Psi\left(\xi\right)\approx\frac{\int\xi^3\left(1+\frac{\xi^3}{3}+\frac{\xi^2}{2}\right)\frac{1}{\xi^2+1}d\xi+k}{\xi^3\left(1+\frac{\xi^3}{3}+\frac{\xi^2}{2}\right)}=\frac{\frac{1}{6}\left[\int\left(\xi^6+3\xi^5+6\xi^3\right)\frac{1}{\xi^2+1}d\xi+6k\right]}{\frac{1}{6}\left(2\xi^6+3\xi^5+6\xi^3\right)}.$$

Hence

$$\Psi\left(\xi\right)\approx\frac{\int\left(2\xi^6+3\xi^5+6\xi^3\right)\frac{1}{\xi^2+1}d\xi+6k}{\left(2\xi^6+3\xi^5+6\xi^3\right)}.$$

Choose

$$I=\frac{\left(2\xi^6+3\xi^5+6\xi^3\right)}{\left(\xi^2+1\right)}.$$

By division algorithm we obtain :

$$\begin{aligned}I&=2\xi^4+3\xi^3-2\xi^2+3\xi+2+\frac{-2-3\xi}{\xi^2+1}\\&=2\xi^4+3\xi^3-2\xi^2+3\xi+2-\frac{2}{\xi^2+1}-\frac{3}{2}\frac{2\xi}{\xi^2+1}.\end{aligned}$$

Then

$$\begin{aligned}\int I\, d\xi + 6k &= \int 2\xi^4 + 3\xi^3 - 2\xi^2 + 3\xi + 2 - \frac{2}{\xi^2+1} - \frac{3}{2}\frac{2\xi}{\xi^2+1} d\xi + 6k \\ &= 2\frac{\xi^5}{5} + 3\frac{\xi^4}{4} - 2\frac{\xi^3}{3} + 3\frac{\xi^2}{2} + 2\xi - 2\tan^{-1}\xi - \frac{3}{2}\ln|\xi^2+1| + 6k.\end{aligned}$$

So $\Psi(\xi)$ after simplification becomes :

$$\Psi(\xi) \approx \frac{2\frac{\xi^5}{5} + 3\frac{\xi^4}{4} - 2\frac{\xi^3}{3} + 3\frac{\xi^2}{2} - 2\xi + 2\tan^{-1}\xi - \frac{3}{2}\ln|\xi^2+1| + 6k}{(2\xi^6 + 3\xi^5 + 6\xi^3)}.$$

For some constant k.
Now, we approach $\tan^{-1}\xi + 6k$ and $\ln|\xi^2+1|$ using the series expansion (1-term).
Starting by $\tan^{-1}\xi + 6k$:

$$\begin{aligned}\tan^{-1}\xi + 6k &= \int \frac{1}{1+\xi^2} d\xi = \int \frac{1}{1-(-\xi^2)} d\xi = \int \sum_{n=0}^{\infty} (-\xi^2)^n d\xi,\ |\xi| < 1 \\ &= \int 1 - \xi^2 + \xi^4 - \xi^6 + ...d\xi = \int \sum_{n=0}^{\infty} (-1)^n \xi^{2n} d\xi \\ &= c + \xi - \frac{\xi^3}{3} + \frac{\xi^5}{5} - \frac{\xi^7}{7} + ... \text{ for some constant } c.\end{aligned}$$

Then

$$\tan^{-1}\xi = (c - 6k) + \sum_{n=0}^{\infty} (-1)^n \frac{\xi^{2n+1}}{2n+1}.$$

Letting $\xi = 0$ then we obtain $(c - 6k) = 0$, so

$$\tan^{-1}\xi = \sum_{n=0}^{\infty} (-1)^n \frac{\xi^{2n+1}}{2n+1} = \xi - \frac{\xi^3}{3} + \frac{\xi^5}{5} - \frac{\xi^7}{7} + ...$$

So the approach can be taken as :

$$\tan^{-1}\xi \approx \xi.$$

Secondly $\ln\left|\xi^2+1\right|$:

$$\begin{aligned}\ln\left|\xi^2+1\right| &= \sum_{n=0}^{\infty} (-1)^n \frac{\xi^{2n+2}}{n+1} \\ &= \xi^2 - \frac{\xi^4}{2} + \frac{\xi^6}{3}...\end{aligned}$$

Hence

$$\ln\left|\xi^2+1\right| \approx \xi^2.$$

Finally, after these estimations $\Psi(\xi)$ becomes :

$$\begin{aligned}\Psi(\xi) &\approx \frac{2\frac{\xi^5}{5}+3\frac{\xi^4}{4}-2\frac{\xi^3}{3}+3\frac{\xi^2}{2}+2\xi-2\xi-\frac{3}{2}\xi^2}{(2\xi^6+3\xi^5+6\xi^3)}\\ &= \frac{2\frac{\xi^5}{5}+3\frac{\xi^4}{4}-2\frac{\xi^3}{3}}{(2\xi^6+3\xi^5+6\xi^3)}\\ &= \frac{2\frac{\xi^2}{5}+3\frac{\xi}{4}-\frac{2}{3}}{(2\xi^3+3\xi^2+6)}\\ &= \frac{1}{6}\left[\frac{(2.4\xi^2+4.5\xi-4)}{(2\xi^3+3\xi^2+6)}\right].\end{aligned}$$

Now, we have to reformulate $\Psi(\xi)$ to take the conformable fractional Laplace inverse transform easier.
Let us start by the denominator.

$$2\xi^3+3\xi^2+6=0.$$

Rewrite the equation as,

$$\xi^3+\frac{3}{2}\xi^2+3=0. \tag{3.7}$$

It is important to mention a formula called the cubic formula for finding the roots of (2.6) .
The cubic formula for finding roots of (2.6) as contained is given by,
let $P=b-\frac{a^2}{3}=-\frac{3}{4}$ and $q=\frac{2a^3}{27}-\frac{ab}{3}+c=\frac{13}{4}$, where, $a=\frac{3}{2}$, $b=0$ and $c=3$.

Discriminant

$$(\triangle)=\frac{q^2}{4}+\frac{p^3}{27}=\frac{168}{64}>0.$$

As noted earlier, the nature of the roots of a cubic equation depends on whether the associated discriminant is positive, negative or zero.
Roots of a cubic equation when $\triangle>0$ there is only one real solution.

$$\begin{aligned}\xi &= \left(-\frac{q}{2}+\sqrt{\triangle}\right)^{\frac{1}{3}}+\left(-\frac{q}{2}-\sqrt{\triangle}\right)^{\frac{1}{3}}-\frac{a}{3}\\ &= -2.14937...\end{aligned}$$

By division algorithm we conclude

$$\frac{2\xi^3+3\xi^2+6}{\xi+2.14937...}=2\xi^2-1.29875...\xi+2.79150...$$

$$2\xi^3+3\xi^2+6=(\xi+2.14937...)(2\xi^2-1.29875...\xi+2.79150...)$$

$$\Psi(\xi)\approx\frac{1}{6}\left[\frac{(2.4\xi^2+4.5\xi-4)}{2\xi^3+3\xi^2+6}\right]=\frac{1}{6}\left[\frac{(2.4\xi^2+4.5\xi-4)}{(\xi+2.14937...)(2\xi^2-1.29875...\xi+2.79150...)}\right].$$

Now, we have to use partial fraction decomposition where the degree of the polynomial in the numerator is less than the degree of the polynomial in the denominator to make the conformable fractional Laplace inverse ($\mathfrak{L}^{-1}$) transform exist.
Hence

$$\Psi(\xi)\approx\frac{1}{6}\left[\frac{c_1}{(\xi+2.14937...)}+\frac{c_2\xi+c_3}{(2\xi^2-1.29875...\xi.+2,79150...)}\right].$$

By identification we get

$$\begin{cases} c_1 & = -\ 0.17437... \\ c_2 & = +\ 2.74874... \\ c_3 & = -\ 1.63455... \end{cases}$$

Therefore

$$\Psi(\xi) \approx \frac{1}{6}\left[\frac{-0.17437...}{(\xi + 2.14937...)} + \frac{2.74874...\xi - 1.63455...}{(2\xi^2 - 1.29875...\xi + 2,79150...)}\right]. \tag{3.8}$$

Applying the conformable fractional Laplace inverse transform to the both sides of equation (2.7) we obtain

$$\mathfrak{L}_\alpha^{-1}\{\Psi(\xi)\} \approx \frac{1}{6}\mathfrak{L}_\alpha^{-1}\left\{\frac{-0.17437...}{(\xi + 2.14937...)}\right\} + \frac{1}{6}\mathfrak{L}_\alpha^{-1}\left\{\frac{2.74874...\xi - 1.63455...}{(2\xi^2 - 1.29875...\xi + 2,79150...)}\right\}.$$

By linearity of $\mathfrak{L}_\alpha^{-1}$ we get

$$\begin{aligned} \mathfrak{L}_\alpha^{-1}\{\Psi(\xi)\} &= -\frac{1}{6}\mathfrak{L}_\alpha^{-1}\left\{\frac{0.17437...}{(\xi + 2.14937...)}\right\} + \frac{1}{6}\mathfrak{L}_\alpha^{-1}\left\{\frac{2.74874...\xi}{(2\xi^2 - 1.29875...\xi + 2,79150...)}\right\} \\ &- \frac{1}{6}\mathfrak{L}_\alpha^{-1}\left\{\frac{1.63455...}{(2\xi^2 - 1.29875...\xi + 2,79150...)}\right\}. \end{aligned}$$

To facilitate and simplify our calculation we must rewrite the second denominator as we apply the property of Theorem 2.4 $\left(\mathfrak{L}_\alpha\left\{e^{-a\frac{x^\alpha}{\alpha}}f(x)\right\}(\xi) = \mathcal{F}_\alpha(\xi + a), \quad \xi > |a|\right)$.

$$\begin{aligned} Y(x) &\approx -\frac{1}{6}\mathfrak{L}_\alpha^{-1}\left\{\frac{0.17437...}{(\xi + 2.14937...)}\right\} + \frac{1}{12}\mathfrak{L}_\alpha^{-1}\left\{\frac{2.74874...\xi}{(\xi - 0.325...)^2 + 1.29012...}\right\} \\ &- \frac{1}{12}\mathfrak{L}_\alpha^{-1}\left\{\frac{1.63455...}{(\xi - 0.325...)^2 + 1.29012...}\right\}. \end{aligned}$$

Finally we conclude that

$$\begin{aligned} Y(x) &\approx -0.029061...\, e^{-\left(2.14937...\frac{x^\alpha}{\alpha}\right)} + 0.22906...\, e^{\left(0.325...\frac{x^\alpha}{\alpha}\right)} \cos\left(\sqrt{1.29012...}\,\frac{x^\alpha}{\alpha}\right) \\ &- \frac{1.63455...}{12\sqrt{1.29012...}}\, e^{\left(0.325...\frac{x^\alpha}{\alpha}\right)} \sin\left(\sqrt{1.29012...}\,\frac{x^\alpha}{\alpha}\right). \end{aligned}$$

Hence a result as required.

Problem 3 :

Now, we will use the conformable fractional Laplace transform method to find the induced deflection function $Y(x)$ of a cantilever beam subjected to a uniform distributed load with intensity W_0 on half of the beam span, as illustrated in the figure bellow.

$$Y^{(4\alpha)}(x) = \frac{W(x)}{EI}. \tag{3.9}$$

Where E and I are respectively the Young's modulus of the beam material and the section moment of inertia of the beam and

$$W(x) = \begin{cases} W_0 & \text{if } 0 \le x \le \frac{L}{2}, \\ 0 & \text{if } \frac{L}{2} \le x \le L. \end{cases}$$

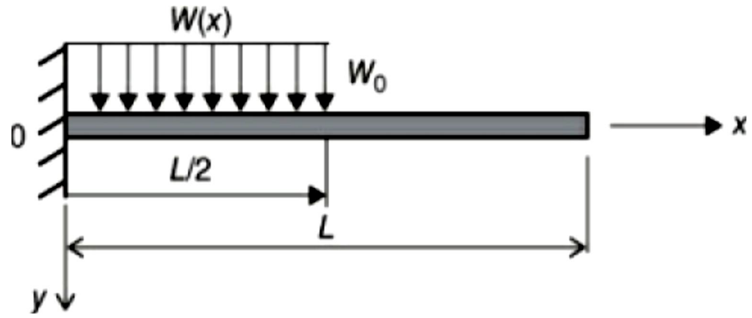

Figure 3.1: A cantilever beam subjected to uniform distributed load.

Conditions 3 :

$$Y^{(3\alpha)}(L) = Y^{(2\alpha)}(L) = 0, \qquad Y^{(\alpha)}(0) = Y(0) = 0. \tag{3.10}$$

Solution :

Take the conformable fractional Laplace transform of both sides of equation (2.8) and using the conditions (2.9) we get :

$$\xi^4 \Psi(\xi) - Y^{3\alpha}(0) - \xi Y^{2\alpha}(0) = \frac{W_0}{EI} \frac{(1-e^{-\xi k})}{\xi}, \quad k = \frac{L^\alpha}{\alpha 2^\alpha}.$$

Since

$$\begin{aligned} \mathfrak{L}_\alpha\{W(x)\} &= \int_0^\infty e^{-\xi\frac{x^\alpha}{\alpha}} W(x) d_\alpha x = \int_0^{\frac{L}{2}} e^{-\xi\frac{x^\alpha}{\alpha}} W_0 d_\alpha x \\ &= -\frac{W_0}{\xi}\int_0^{\frac{L}{2}} -\xi x^{\alpha-1} e^{-\xi\frac{x^\alpha}{\alpha}} dx = -\frac{W_0}{\xi}\left[e^{-\xi\frac{x^\alpha}{\alpha}}\right]_0^{\frac{L}{2}} \\ &= \frac{W_0}{\xi}\left[1-e^{-\xi k}\right], \quad k = \frac{L^\alpha}{\alpha 2^\alpha}. \end{aligned}$$

Thus

$$\Psi(\xi) = \frac{Y^{(3\alpha)}(0)}{\xi^4} + \frac{Y^{(2\alpha)}(0)}{\xi^3} + \frac{W_0}{EI}\frac{(1-e^{-\xi k})}{\xi^5}, \quad k = \frac{L^\alpha}{\alpha 2^\alpha}. \tag{3.11}$$

Clearly $\begin{cases} 1. \lim\limits_{\xi\to\infty} \Psi(\xi) = 0, \\ 2. \lim\limits_{\xi\to\infty} \xi\Psi(\xi) = 0 \quad (bounded). \end{cases}$

Then we apply the conformable fractional Laplace inverse transform on all equation (2.10) to get the solution.

$$\mathfrak{L}_\alpha^{-1}\{\Psi(\xi)\} = \mathfrak{L}_\alpha^{-1}\left\{\frac{Y^{(3\alpha)}(0)}{\xi^4}\right\} + \mathfrak{L}_\alpha^{-1}\left\{\frac{Y^{(2\alpha)}(0)}{\xi^3}\right\} + \mathfrak{L}_\alpha^{-1}\left\{\frac{W_0}{EI}\frac{(1-e^{-\xi k})}{\xi^5}\right\}, \quad k = \frac{L^\alpha}{\alpha 2^\alpha}.$$

From the property $\left(\mathfrak{L}_\alpha\left\{x^p\right\}(\xi)=\alpha^{\frac{p}{\alpha}}\frac{\Gamma\left(1+\frac{p}{\alpha}\right)}{\xi^{1+\frac{p}{\alpha}}},\ \ \xi>0,\ \ \frac{p}{\alpha}>-1\right)$, we get

$$\begin{aligned}\mathfrak{L}_\alpha^{-1}\left\{\frac{Y^{(3\alpha)}(0)}{\xi^4}\right\} &= Y^{(3\alpha)}(0)\mathfrak{L}_\alpha^{-1}\left\{\xi^4\right\}\\ &= \frac{Y^{(3\alpha)}(0)}{\alpha^3\Gamma(1+3)}\frac{\alpha^3\Gamma(1+3)}{\xi^{(1+3)}},\quad p=3\alpha\\ &= \frac{Y^{(3\alpha)}(0)}{24\alpha^3}x^{3\alpha}.\end{aligned}$$

similarly

$$\mathfrak{L}_\alpha^{-1}\left\{\frac{Y^{(2\alpha)}(0)}{\xi^3}\right\}=\frac{Y^{(2\alpha)}(0)}{6\alpha^2}x^{2\alpha}.$$

Let approach the exponential by two first terms of expansion $\left(e^{-\xi k}\approx 1-\xi k\right)$, then

$$\begin{aligned}\mathfrak{L}_\alpha^{-1}\left\{\frac{W_0}{EI}\frac{(1-e^{-\xi k})}{\xi^5}\right\} &\approx \frac{W_0}{EI}\mathfrak{L}_\alpha^{-1}\left\{\frac{1}{\xi^5}+\frac{1}{\xi^5}-\frac{(\xi k)}{\xi^5}\right\}\\ &= \frac{W_0}{EI}\left(\frac{2}{120\alpha^4}x^{4\alpha}-\frac{k}{24\alpha^3}x^{3\alpha}\right).\end{aligned}$$

Hence, the solution is given as

$$Y(x)\ \approx\ \frac{Y^{(3\alpha)}(0)}{24\alpha^3}x^{3\alpha}+\frac{Y^{(2\alpha)}(0)}{6\alpha^2}x^{2\alpha}+\frac{W_0}{EI}\left(\frac{2}{120\alpha^4}x^{4\alpha}-\frac{k}{24\alpha^3}x^{3\alpha}\right).$$

It is easy to use the conditions to calculate $Y^{(3\alpha)}(0)$ and $Y^{(2\alpha)}(0)$.
Finally, the solution of equation (2.8) is given as:

$$Y(x)\approx\begin{cases}\frac{Y^{(3\alpha)}(0)}{24\alpha^3}x^{3\alpha}+\frac{Y^{(2\alpha)}(0)}{6\alpha^2}x^{2\alpha}+\frac{W_0}{EI}\left(\frac{2}{120\alpha^4}x^{4\alpha}-\frac{k}{24\alpha^3}x^{3\alpha}\right) & \text{if} 0\le x\le\frac{L}{2},\\ 0 & \text{if}\frac{L}{2}\le x\le L.\end{cases}$$

4 Conclusion

We conclude that the conformable fractional Laplace transform can be used in solving the most difficult fractional differential equations and systems, as we provide in the solution of **Problem 1** and **Problem 2**. Also, this fractional transform has many applications in physics and engineering, as mentioned in **Problem 3**.

Conflict of Interest

The authors have no conflicts of interest to disclose.

References

[1] T. Abdeljawad, *On conformable fractional calculus*, J. Comput. Appl. Math. **279**(1) (2015), 57–66.

[2] M. Abu Hammad and R. Khalil, *Fractional Fourier Series with Applications*, Amer. J. Comput. Appl. Math. **4**(6) (2014), 187–191.

[3] Z. Al-zhour, F. Alrawajeh, N. Al-mutairi and R. Alkhasawneh, *New results on the conformable fractional Sumudu transforms: Theories and applications*, International Journal of Analysis and Applications. **17**(6) (2019), 1019–1033.

[4] D. R. Anderson, E. Camrud and D. J. Ulness, *On the nature of the conformable derivative and its applications to physics*, JJ. Fract. Calc. Appl. **10**(2) (2019), 92–135.

[5] A. Atangana and D. Baleanu, *New fractional derivatives with nonlocal and non-singular kernel: Theory and application to heat transfer model*, Therm. Sci. **20**(2) (2016), 763–769.

[6] M. Ayata and O. Ozkan, *A new application of conformable Laplace decomposition method for fractional Newell-Whitehead-Segel equation*, AIMS Math. **5**(6) (2020), 7402–7412.

[7] A. Bouchenak, R. Khalil, and M. AlHorani, *Fractional Fourier Series with Separation of Variables Technique and its Application on Fractional Differential Equations*, Wseas Transactions on Mathematics. **20** (2021), 461-469. DOI: 10.37394/23206.2021.20.48

[8] A. Bushnaque, M. AlHorani and R. Khalil, *Tensor product technique and atomic solution of fractional Bate Man Burger equation*, J. math. Comput. Sci. **11**(1) (2021), 330–336.

[9] M. Caputo and M. Fabrizio, *A new definition of fractional derivative without singular kernel*, Progr. Fract. Differ. Appl. **1**(2) (2015), 1–13.

[10] A. Gokdogan, E. Unal, and E. Celik, *Existence and Uniqueness Theorems for Sequential Linear Conformable Fractional Differential Equations*, Miskolc Math. Notes **17** (1) (2016), 267–279. DOI: 10.18514/MMN.2016.1635

[11] R. Khalil, M. Al Horani and D. Anderson, *Undetermined coefficients for local fractional differential equations*, J. Math. Computer Sci. **16** (2016), 140–146.

[12] R. Khalil, M. Al Horani, A. Yousef and M. Sababheh, *A new definition of fractional derivative*, J. Comput. Appl. Math. **264**(2014), 65–70.

[13] N. A. Khan, O. A. Razzaq, and M. Ayaz, *Some properties and applications of conformable fractional Laplace transform (CFLT)*, J. Fract. Calc. Appl. **9**(1) (2018), 72–81.

[14] A. Kilbas, *Hadamard-type fractional calculus*. J. Korean Math. Soc. **38**(6) (2001), 1191–1204.

[15] K. S. Miller, *An introduction to fractional calculus and fractional differential equations*, J. Wiley Sons, New York, 1993.

[16] D. A. Murio, *Stable numerical evaluation of GrünwaldâĂŞLetnikov fractional derivatives applied to a fractional IHCP*, Inverse Prob. Sci. Eng. **17**(2) (2009), 229–243.

[17] K. Oldham, J. Spanier, *The Fractional Calculus, Theory and Applications of Differentiation and Integration of Arbitrary Order*, Academic Press, USA, 1974.

[18] F. S. Silva, D. M. Moreira, M. A. Moret, *Conformable Laplace Transform of Fractional Differential Equations*. J. axioms, **7**(3) (2018), 50. DOI: 10.20944/preprints201807.0025.v1