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Generalization of fractional Laplace transform for higher order and its application

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Abstract. In this paper, we first introduce the conformable fractional Laplace transform. Then, we give its generalization for higher-order. Finally, as an application, we solve a non-homogeneous conformable fractional differential equation with variable coefficients and a system of fractional differential equations.

Keywords: Conformable fractional derivative, Conformable fractional Laplace transform, System of fractional differential equations.

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1 Introduction

Fractional derivative emergence date back to the time of calculus. In 1695, L'Hospital wondered about the meaning of $\frac{d^n f}{dx^n}$ if $n = \frac{1}{2}$, since then, researchers have been attempting to define a fractional derivative. Some of which are : Riemann-Liouville fractional definitions [15], Caputo fractional definitions [9, 15], Grünwald-Letnikov fractional derivative [16], Atangana-Baleanu fractional definitions [5], Hadamard fractional integral [14], Caputo-Fabrizio fractional derivative [9] and conformable fractional definitions [12]. Most of the definitions give numerical solution to the problems. However, the conformable fractional derivative is a natural definition which gives us simple and easy solutions for the problems. For more different applications on conformable fractional derivative, the reader can refer to [1, 2, 4, 6–8, 10, 11].

In 2015, Abdeljawad Thabet defined the conformable fractional Laplace transform [1] which will help to solve many fractional differential equations. In order to study the solution of the most challenging problems, like a non-homogeneous fractional differential equation with variables coefficients for higher-order, we generalize the conformable fractional Laplace transform for higher-order. Finally, we use this generalization to solve fractional differential equations and a system as an application.

For more details on conformable fractional Laplace transform, we refer the reader to [1, 3, 6, 13, 18].

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2 Basics of conformable fractional Laplace transform

Definition 2.1. Let $f : [0, \infty) \to \mathbb{R}$ be a real valued function and $0 < \alpha \leq 1$. Then the conformable fractional Laplace transform of f is defined as:

$$\mathfrak{L}_\alpha \{f(x)\} = \mathcal{F}_\alpha(\xi) = \int_0^\infty e^{-\xi \frac{x^\alpha}{\alpha}} f(x)\, d_\alpha x = \int_0^\infty e^{-\xi \frac{x^\alpha}{\alpha}} f(x)\, x^{\alpha-1} dx.$$

provided the integral exists.

Let us have as an example for the conformable fractional Laplace transform of the usual functions in the theorem bellow.

Theorem 2.2. *Let $a, p, c \in \mathbb{R}$ and $0 < \alpha \leq 1$. Then*
(1) $\mathfrak{L}_\alpha \{c\}(\xi) = \frac{c}{\xi}$, $\xi > 0$.
(2) $\mathfrak{L}_\alpha \{x^p\}(\xi) = \alpha^{\frac{p}{\alpha}} \frac{\Gamma\left(1+\frac{p}{\alpha}\right)}{\xi^{1+\frac{p}{\alpha}}}$, $\xi > 0$.
(3) $\mathfrak{L}_\alpha \left\{e^{a\frac{x^\alpha}{\alpha}}\right\}(\xi) = \frac{1}{\xi - a}$, $\xi > 0$.
(4) $\mathfrak{L}_\alpha \left\{\sin a\frac{x^\alpha}{\alpha}\right\}(\xi) = \frac{a}{\xi^2 + a^2}$, $\xi > 0$.
(5) $\mathfrak{L}_\alpha \left\{\cos a\frac{x^\alpha}{\alpha}\right\}(\xi) = \frac{\xi}{\xi^2 + a^2}$, $\xi > 0$.
(6) $\mathfrak{L}_\alpha \left\{\sinh a\frac{x^\alpha}{\alpha}\right\}(\xi) = \frac{a}{\xi^2 - a^2}$, $\xi > |a|$.
(7) $\mathfrak{L}_\alpha \left\{\cosh a\frac{x^\alpha}{\alpha}\right\}(\xi) = \frac{\xi}{\xi^2 - a^2}$, $\xi > |a|$.

Proof. Follows by applying Definition 2.1 □

One of the excellent results is the relation between the usual, and the conformable fractional Laplace transforms, given in the theorem below.

Theorem 2.3. *Let $f : [0, \infty) \to \mathbb{R}$ be a function such that $\mathfrak{L}_\alpha \{f(x)\}(\xi) = \mathcal{F}_\alpha(\xi)$ exists. Then*

$$\mathfrak{L}_\alpha \{f(x)\}(\xi) = \mathcal{F}_\alpha(\xi) = \mathfrak{L}\left\{f\left((\alpha x)^{\frac{1}{\alpha}}\right)\right\}(\xi), \quad 0 < \alpha \leq 1.$$

Proof. See [1,3]. □

Theorem 2.4. *Let $f : [0, \infty) \to \mathbb{R}$, $g : [0, \infty) \to \mathbb{R}$ and let λ, μ, $a \in \mathbb{R}$ and $0 < \alpha \leq 1$. Then*
(1) $\mathfrak{L}_\alpha \{\lambda f(x) + \mu g(x)\} = \lambda \mathcal{F}_\alpha(\xi) + \mu G_\alpha(\xi)$, $\xi > 0$.
(2) $\mathfrak{L}_\alpha \left\{e^{-a\frac{x^\alpha}{\alpha}} f(x)\right\}(\xi) = \mathcal{F}_\alpha(\xi + a)$, $\xi > |a|$.
(3) $\mathfrak{L}_\alpha \{I^\alpha f(x)\}(\xi) = \frac{\mathcal{F}_\alpha(\xi)}{\xi}$, $\xi > 0$.
(4) $\mathfrak{L}_\alpha \left\{\frac{x^{n\alpha}}{\alpha^n} f(x)\right\}(\xi) = (-1)^n \frac{d^n}{d\xi^n} \mathcal{F}_\alpha(\xi)$, $\xi > 0$.
(5) $\mathfrak{L}_\alpha \{(f * g)(x)\} = \mathcal{F}_\alpha(\xi) G_\alpha(\xi)$, $\xi > 0$.
where $\mathcal{F}_\alpha$ and G_α are the conformable fractional Laplace transform of the functions f and g respectively, $f * g$ is the convolution product of f and g and $I^\alpha f(x)$ is the conformable fractional integral.

Proof. See [1,3]. □

2.1 Generalization of fractional Laplace transform

Theorem 2.5. *Let $f : [0, \infty) \to \mathbb{R}$ be a continuous real valued differentiable function and $0 < \alpha \leq 1$. Then*

$$\mathfrak{L}_\alpha \left\{ D^\alpha f(x) \right\} = \xi \mathcal{F}_\alpha(\xi) - f(0), \ \xi > 0.$$

Proof. See [1,3]. □

Theorem 2.6. *Let $f : [0, \infty) \to \mathbb{R}$ be a continuous real valued differentiable function and $0 < \alpha \leq 1$. Then*

$$\mathfrak{L}_\alpha \left\{ D^{2\alpha} f(x) \right\} = \xi^2 \mathcal{F}_\alpha(\xi) - f^\alpha(0) - \xi f(0), \quad \xi > 0.$$

Proof. By using Definition 2.1 and integration by parts, we find:

$$\begin{aligned}
\mathfrak{L}_\alpha \left\{ D^{2\alpha} f(x) \right\} &= \int_0^\infty e^{-\xi \frac{x^\alpha}{\alpha}} D^{2\alpha} f(x) \, d_\alpha x \\
&= \int_0^\infty D^\alpha D^\alpha f(x) \, e^{-\xi \frac{x^\alpha}{\alpha}} x^{\alpha-1} dx \\
&= \int_0^\infty e^{-\xi \frac{x^\alpha}{\alpha}} x^{1-\alpha} \frac{d}{dx} D^\alpha f(x) \, x^{\alpha-1} dx \\
&= \int_0^\infty e^{-\xi \frac{x^\alpha}{\alpha}} \frac{d}{dx} D^\alpha f(x) \, dx \\
&= \lim_{b \to \infty} \left[e^{-\xi \frac{x^\alpha}{\alpha}} D^\alpha f(x) \right]_0^b + \int_0^\infty D^\alpha f(x) \left(\frac{\xi}{\alpha} \alpha x^{\alpha-1} \right) e^{-\xi \frac{x^\alpha}{\alpha}} dx \\
&= -f^\alpha(0) + \xi \int_0^\infty e^{-\xi \frac{x^\alpha}{\alpha}} D^\alpha f(x) \, d_\alpha x \\
&= -f^\alpha(0) + \xi \mathfrak{L}_\alpha \left\{ D^\alpha f(x) \right\}.
\end{aligned}$$

By the previous theorem we get the result.

$$\mathfrak{L}_\alpha \left\{ D^{2\alpha} f(x) \right\} = \xi^2 \mathcal{F}_\alpha(\xi) - f^\alpha(0) - \xi f(0).$$

□

Theorem 2.7. *Let $f : [0, \infty) \to \mathbb{R}$ be a continuous real valued differentiable function and $0 < \alpha \leq 1$. Then*

$$\mathfrak{L}_\alpha \left\{ D^{3\alpha} f(x) \right\} = \xi^3 \mathcal{F}_\alpha(\xi) - f^{2\alpha}(0) - \xi f^\alpha(0) - \xi^2 f(0), \quad \xi > 0.$$

Proof. By using Definition 2.1 and integration by parts, we have:

$$\begin{aligned}
\mathfrak{L}_\alpha \left\{ D^{3\alpha} f(x) \right\} &= \int_0^\infty e^{-\xi \frac{x^\alpha}{\alpha}} D^{3\alpha} f(x) \, d_\alpha x \\
&= \int_0^\infty D^\alpha D^{2\alpha} f(x) \, e^{-\xi \frac{x^\alpha}{\alpha}} x^{\alpha-1} dx \\
&= \int_0^\infty e^{-\xi \frac{x^\alpha}{\alpha}} \frac{d}{dx} D^{2\alpha} f(x) \, dx \\
&= \lim_{b \to \infty} \left[e^{-\xi \frac{x^\alpha}{\alpha}} D^{2\alpha} f(x) \right]_0^b + \int_0^\infty D^{2\alpha} f(x) \, \xi x^{\alpha-1} e^{-\xi \frac{x^\alpha}{\alpha}} dx \\
&= -f^{2\alpha}(0) + \xi \int_0^\infty e^{-\xi \frac{x^\alpha}{\alpha}} \left(D^{2\alpha} f(x) \right) d_\alpha x \\
&= -f^{2\alpha}(0) + \xi \mathfrak{L}_\alpha \left\{ D^{2\alpha} f(x) \right\}.
\end{aligned}$$

By the previous theorem we get the result.

$$\mathfrak{L}_\alpha\left\{D^{3\alpha}f(x)\right\} = \xi^3\mathcal{F}_\alpha(\xi) - f^{2\alpha}(0) - \xi f^{\alpha}(0) - \xi^2 f(0).$$

□

Theorem 2.8. *Generalization of (C.F.L.T)*
Let $f:[0,\infty)\to\mathbb{R}$ be a continuous real valued differentiable function and $0<\alpha\le 1$, then for any integer number n we have :

$$\begin{aligned}\mathfrak{L}_\alpha\left\{D^{(n\alpha)}f(x)\right\} &= \xi^n\mathcal{F}_\alpha(\xi) - f^{(n-1)\alpha}(0) - \xi f^{(n-2)\alpha}(0) - \xi^{(2)}f^{(n-3)\alpha}(0)\\ &- \ldots\ldots - \xi^{(n-2)}f^{\alpha}(0) - \xi^{(n-1)}f(0)\\ &= \xi^n\mathcal{F}_\alpha(\xi) - \xi^{(0)}f^{(n-1-0)\alpha}(0) - \xi^{(1)}f^{(n-1-1)\alpha}(0) - \xi^{(2)}f^{(n-1-2)\alpha}(0)\\ &- \ldots\ldots - \xi^{(n-1-1)}f^{(1)\alpha}(0) - \xi^{(n-1)}f(0).\end{aligned}$$

Hence

$$\mathfrak{L}_\alpha\left\{D^{(n\alpha)}f(x)\right\} = \xi^n\mathcal{F}_\alpha(\xi) - \sum_{j=0}^{n-1}\xi^j f^{(n-j-1)\alpha}(0), \quad \xi>0.$$

Proof. We are going to prove this theorem by induction.
For $n=1,2,3$ the formula is true (see the previous theorems).
Now, suppose that the formula is true for n and prove it for $n+1$.
that is $\mathfrak{L}_\alpha\left\{D^{(n\alpha)}f(x)\right\} = \xi^n\mathcal{F}_\alpha(\xi) - \sum_{j=0}^{n-1}\xi^j f^{(n-j-1)\alpha}(0), \quad \xi>0$ is true.
By using Definition 2.1 and integration by parts, we have:

$$\begin{aligned}\mathfrak{L}_\alpha\left\{D^{(n+1)\alpha}f(x)\right\} &= \int_0^\infty e^{-\xi\frac{x^\alpha}{\alpha}}D^{(n+1)\alpha}f(x)\,d_\alpha x\\ &= \int_0^\infty e^{-\xi\frac{x^\alpha}{\alpha}}D^{(n+1)\alpha}f(x)\,x^{\alpha-1}dx\\ &= \int_0^\infty D^{\alpha}D^{n\alpha}f(x)\,e^{-\xi\frac{x^\alpha}{\alpha}}x^{\alpha-1}dx\\ &= \int_0^\infty e^{-\xi\frac{x^\alpha}{\alpha}}x^{1-\alpha}\frac{d}{dx}D^{n\alpha}f(x)\,x^{\alpha-1}dx\\ &= \int_0^\infty e^{-\xi\frac{x^\alpha}{\alpha}}\frac{d}{dx}D^{n\alpha}f(x)\,dx\\ &= \left[e^{-\xi\frac{x^\alpha}{\alpha}}D^{n\alpha}f(x)\right]_0^\infty + \int_0^\infty D^{n\alpha}f(x)\,\xi x^{\alpha-1}e^{-\xi\frac{x^\alpha}{\alpha}}dx\\ &= -f^{n\alpha}(0) + \xi\int_0^\infty e^{-\xi\frac{x^\alpha}{\alpha}}\left(D^{n\alpha}f(x)\right)d_\alpha x\\ &= -f^{n\alpha}(0) + \xi\int_0^\infty e^{-\xi\frac{x^\alpha}{\alpha}}\left(D^{n\alpha}f(x)\right)d_\alpha x\\ &= -f^{n\alpha}(0) + \xi\mathfrak{L}_\alpha\left\{D^{n\alpha}f(x)\right\}\ \text{(since the formula is true)}\\ &= -f^{n\alpha}(0) + \xi\left(\xi^n\mathcal{F}_\alpha(\xi) - \sum_{j=0}^{n-1}\xi^j f^{(n-j-1)\alpha}(0)\right).\end{aligned}$$

Therefore

$$\begin{aligned}
\mathfrak{L}_\alpha\left\{D^{(n+1)\alpha}f(x)\right\} &= -f^{n\alpha}(0)+\xi^{(n+1)}\mathcal{F}_\alpha(\xi)-\xi\sum_{j=0}^{n-1}\xi^j f^{(n-j-1)\alpha}(0)\\
&= \xi^{(n+1)}\mathcal{F}_\alpha(\xi)-f^{n\alpha}(0)-\sum_{j=0}^{n-1}\xi^{(j+1)}f^{(n-j-1)\alpha}(0)\\
&= \xi^{(n+1)}\mathcal{F}_\alpha(\xi)-\sum_{j=0}^{n}\xi^j f^{(n-j-1)\alpha}(0).
\end{aligned}$$

Which complete the proof of the theorem. □

3 Applications

We use the conformable fractional Laplace transform as an application to solve some problems. In the first one, we solve a system of fractional differential equations with constant coefficients of three unknowns. In the second, we apply the generalization of **(C.F.L.T)** to solve a non-homogeneous fractional differential equation with variables coefficients.

Problem 1 :

$$\begin{cases}
Y_1^{(\alpha)} &= Y_1 - Y_2 + Y_3,\\
Y_2^{(\alpha)} &= -2Y_1 + Y_2 - Y_3,\\
Y_3^{(\alpha)} &= -Y_2 + Y_3.
\end{cases}$$

Conditions 1 :

$$Y_1(0) = Y_2(0) = Y_3(0) = 1, \quad 0 < \alpha \leq 1.$$

Solution :
Let $\mathfrak{L}_\alpha\{Y_1\} = F_\alpha(\xi)$, $\mathfrak{L}_\alpha\{y_2\} = G_\alpha(\xi)$ and $\mathfrak{L}_\alpha\{Y_3\} = H_\alpha(\xi)$.
When applying the conformable fractional Laplace transform on all the system of fractional differential equation and using the giving conditions, we get:

$$\begin{cases}
\xi F_\alpha(\xi)-1 &= F_\alpha(\xi)-G_\alpha(\xi)+H_\alpha(\xi),\\
\xi G_\alpha(\xi)-1 &= -2F_\alpha(\xi)+G_\alpha(\xi)-H_\alpha(\xi),\\
\xi H_\alpha(\xi)-1 &= -G_\alpha(\xi)+H_\alpha(\xi).
\end{cases}$$

Which implies

$$\begin{cases}
(\xi-1)F_\alpha(\xi)+G_\alpha(\xi)-H_\alpha(\xi) &= 1,\\
2F_\alpha(\xi)+(\xi-1)G_\alpha(\xi)+H_\alpha(\xi) &= 1,\\
G_\alpha(\xi)+(\xi-1)H_\alpha(\xi) &= 1.
\end{cases}$$

Now, we can use Cramers rule to obtain solutions for $F_\alpha(\xi)$, $G_\alpha(\xi)$ and $H_\alpha(\xi)$.
First

$$\Delta = \begin{vmatrix} (\xi-1) & 1 & -1\\ 2 & (\xi-1) & 1\\ 0 & 1 & (\xi-1)\end{vmatrix} = (\xi^3-3\xi^2).$$

Hence

$$F_\alpha\{\xi\} = \frac{1}{\Delta}\begin{vmatrix} 1 & 1 & -1 \\ 1 & (\xi-1) & 1 \\ 1 & 1 & (\xi-1) \end{vmatrix} = \frac{(\xi^2-2\xi)}{(\xi^3-3\xi^2)}.$$

We are going to find $G_\alpha\{\xi\}$ again using Cramer's Rule.

$$G_\alpha\{\xi\} = \frac{1}{\Delta}\begin{vmatrix} (\xi-1) & 1 & -1 \\ 2 & 1 & 1 \\ 0 & 1 & (\xi-1) \end{vmatrix} = \frac{(\xi^2-5\xi+2)}{(\xi^3-3\xi^2)}.$$

In the similar way, we get $H_\alpha(\xi)$

$$H_\alpha\{\xi\} = \frac{1}{\Delta}\begin{vmatrix} (\xi-1) & 1 & 1 \\ 2 & (\xi-1) & 1 \\ 0 & 1 & 1 \end{vmatrix} = \frac{(\xi^2-3\xi+2)}{(\xi^3-3\xi^2)}$$

Using partial fraction to rewrite F_α, G_α and H_α in this way

$$c_1\frac{1}{\xi} + c_2\frac{1}{\xi^2} + c_3\frac{1}{(\xi-3)},$$

for some constants c_1, c_2 and c_3 to make the calculation easy.
Therefore, we get

$$\begin{cases} F_\alpha(\xi) &= \frac{6}{9}\frac{1}{\xi} + \frac{3}{9}\frac{1}{(\xi-3)}, \\ G_\alpha(\xi) &= \frac{13}{9}\frac{1}{\xi} - \frac{6}{9}\frac{1}{\xi^2} - \frac{4}{9}\frac{1}{(\xi-3)}, \\ H_\alpha(\xi) &= \frac{7}{9}\frac{1}{\xi} - \frac{6}{9}\frac{1}{\xi^2} + \frac{2}{9}\frac{1}{(\xi-3)}. \end{cases} \tag{3.1}$$

Applying the conformable fractional Laplace inverse transform on all the system (3.1) using the properties in Theorem 2.2 and Theorem 2.4 we obtain the solution of our problem.

$$\begin{cases} \mathfrak{L}_\alpha^{-1}\{F_\alpha(\xi)\} &= \frac{6}{9}\mathfrak{L}_\alpha^{-1}\left\{\frac{1}{\xi}\right\} + \frac{3}{9}\mathfrak{L}_\alpha^{-1}\left\{\frac{1}{(\xi-3)}\right\}, \\ \mathfrak{L}_\alpha^{-1}\{G_\alpha(\xi)\} &= \frac{13}{9}\mathfrak{L}_\alpha^{-1}\left\{\frac{1}{\xi}\right\} - \frac{6}{9}\mathfrak{L}_\alpha^{-1}\left\{\frac{1}{\xi^2}\right\} - \frac{4}{9}\mathfrak{L}_\alpha^{-1}\left\{\frac{1}{(\xi-3)}\right\}, \\ \mathfrak{L}_\alpha^{-1}\{H_\alpha(\xi)\} &= \frac{7}{9}\mathfrak{L}_\alpha^{-1}\left\{\frac{1}{\xi}\right\} - \frac{6}{9}\mathfrak{L}_\alpha^{-1}\left\{\frac{1}{\xi^2}\right\} + \frac{2}{9}\mathfrak{L}_\alpha^{-1}\left\{\frac{1}{(\xi-3)}\right\}. \end{cases}$$

Then

$$\begin{cases} Y_1(x) &= \frac{6}{9} + \frac{3}{9}e^{3\frac{x^\alpha}{\alpha}}, \\ Y_2(x) &= \frac{13}{9} - \frac{6}{9}\frac{x^\alpha}{\alpha} - \frac{4}{9}e^{3\frac{x^\alpha}{\alpha}}, \\ Y_3(x) &= \frac{7}{9} - \frac{6}{9}\frac{x^\alpha}{\alpha} + \frac{2}{9}e^{3\frac{x^\alpha}{\alpha}}. \end{cases}$$

Hence a result as required.

Problem 2 :

$$Y^{3\alpha}(x) + Y^{2\alpha}(x) - \frac{x^\alpha}{\alpha}Y(x) + 2Y(x) = \cos\frac{x^\alpha}{\alpha}. \tag{3.2}$$

Conditions 2 :

$$Y^{2\alpha}(0) = Y^\alpha(0) = Y(0) = 0. \tag{3.3}$$

Solution :
Let us take the conformable fractional Laplace transform of both sides and using the given conditions, we get equation 3.4

$$\xi^3\Psi(\xi)+\xi^2\Psi(\xi)-(-1)\frac{d}{d\xi}(\xi\Psi(\xi))+2\Psi(\xi)=\frac{\xi}{\xi^2+1}. \tag{3.4}$$

Where $\Psi(\xi)=\mathfrak{L}_\alpha\{Y\}$ and $\xi>0$.
This follows from the properties of **(C.F.L.T)** in Theorem 2.2 and Theorem 2.4,
$\left(\mathfrak{L}_\alpha\left\{\frac{x^{n\alpha}}{\alpha^n}f(x)\right\}(\xi)=(-1)^n\frac{d^n}{d\xi^n}\mathcal{F}_\alpha(\xi),\ \ \xi>0\right)$.
Then :

$$(\xi^3+\xi^2+2)\Psi(\xi)+\Psi(\xi)+\xi\Psi'(\xi)=\frac{\xi}{\xi^2+1}. \tag{3.5}$$

This equation can be simplified to:

$$\xi\Psi'(\xi)+(\xi^3+\xi^2+3)\Psi(\xi)+=\frac{\xi}{\xi^2+1}.$$

Hence, we find :

$$\Psi'(\xi)+\left(\frac{\xi^3+\xi^2+3}{\xi}\right)\Psi(\xi)+=\frac{1}{\xi^2+1}. \tag{3.6}$$

Which is a first order ordinary non-homogeneous linear differential equation with variable coefficients.
Applying theory of linear differential equations we obtain:

$$\begin{aligned}\Psi(\xi) &= e^{-\int\left(\frac{\xi^3+\xi^2+3}{\xi}\right)d\xi}\left[\int e^{\int\left(\frac{\xi^3+\xi^2+3}{\xi}\right)d\xi}\frac{1}{\xi^2+1}d\xi+k\right]\\ &= e^{-\left(\frac{\xi^3}{3}+\frac{\xi^2}{2}+3\ln\xi\right)}\left[\int e^{\left(\frac{\xi^3}{3}+\frac{\xi^2}{2}+3\ln\xi\right)}\frac{1}{\xi^2+1}d\xi+k\right].\end{aligned}$$

for some constant k.
Therefore

$$\Psi(\xi)=\frac{\int e^{\left(\frac{\xi^3}{3}+\frac{\xi^2}{2}+3\ln\xi\right)}\frac{1}{\xi^2+1}d\xi+k}{e^{\left(\frac{\xi^3}{3}+\frac{\xi^2}{2}+3\ln\xi\right)}}\quad\text{for some constant k.}$$

Claim :
The conformable fractional Laplace inverse transform exists $(\Psi(\xi)\in Dom(\mathfrak{L}_\alpha^{-1}))$.

Proof.
1. $\lim\limits_{\xi\to\infty}\Psi(\xi)=\frac{\infty}{\infty}$ indeterminate.
Then we have to use $L'Hopital's$ rule to get :

$$\lim_{\xi\to\infty}\Psi(\xi)=\lim_{\xi\to\infty}\frac{e^{\left(\frac{\xi^3}{3}+\frac{\xi^2}{2}+3\ln\xi\right)}\left(\frac{1}{\xi^2+1}\right)}{\left(\xi^2+\xi+\frac{3}{\xi}\right)e^{\left(\frac{\xi^3}{3}+\frac{\xi^2}{2}+3\ln\xi\right)}}=\lim_{\xi\to\infty}\frac{1}{\left(\xi^2+\xi+\frac{3}{\xi}\right)(\xi^2+1)}=0.$$

2. $\lim_{\xi\to\infty} \xi\Psi(\xi) = \lim_{\xi\to\infty} \xi \dfrac{\int e^{\left(\frac{\xi^3}{3}+\frac{\xi^2}{2}+3\ln\xi\right)} \frac{1}{\xi^2+1} d\xi + k}{e^{\left(\frac{\xi^3}{3}+\frac{\xi^2}{2}+3\ln\xi\right)}} = \dfrac{\infty}{\infty}$ indeterminate.

Thus we have to use L'Hopital's rule to find :

$$\lim_{\xi\to\infty} \xi\Psi(\xi) = \lim_{\xi\to\infty} \left[\frac{e^{\left(\frac{\xi^3}{3}+\frac{\xi^2}{2}+3\ln\xi\right)} \left(\frac{\xi}{\xi^2+1}\right) + \int e^{\left(\frac{\xi^3}{3}+\frac{\xi^2}{2}+3\ln\xi\right)} \frac{1}{\xi^2+1} d\xi + k}{\left(\xi^2+\xi+\frac{3}{\xi}\right) e^{\left(\frac{\xi^3}{3}+\frac{\xi^2}{2}+3\ln\xi\right)}} \right] = \frac{\infty}{\infty}.$$

Which is also indeterminate, so we reuse L'Hopital's rule again :

$$\lim_{\xi\to\infty} \xi\Psi(\xi) = \lim_{\xi\to\infty} \left[\frac{\left(\frac{1}{\xi^2+1}\right) + \left(\xi^2+\xi+\frac{3}{\xi}\right)\left(\frac{\xi}{\xi^2+1}\right) + \left(\frac{1-\xi^2}{(\xi^2+1)^2}\right)}{\left(2\xi+1-\frac{3}{\xi^2}\right) + \left(\xi^2+\xi+\frac{3}{\xi}\right)^2} \right].$$

After simplifying and using the properties of limits calculations, we get :

$$\lim_{\xi\to\infty} \xi\Psi(\xi) = \lim_{\xi\to\infty} \frac{\xi^5}{\xi^6} = 0.$$

Hence a result as required. □

Now, we can reformulate $\Psi(\xi)$ to become :

$$\Psi(\xi) = \frac{\int \xi^3 e^{\left(\frac{\xi^3}{3}+\frac{\xi^2}{2}\right)} \frac{1}{\xi^2+1} d\xi + k}{\xi^3 e^{\left(\frac{\xi^3}{3}+\frac{\xi^2}{2}\right)}} \quad \text{for some constant k.}$$

Let us approximate the Exponential by the first 2-terms of the series expansion.

ie : $\left(e^{\left(\frac{\xi^3}{3}+\frac{\xi^2}{2}\right)} \approx \left(1+\frac{\xi^3}{3}+\frac{\xi^2}{2}\right)\right)$.

Therefore

$$\Psi(\xi) \approx \frac{\int \xi^3 \left(1+\frac{\xi^3}{3}+\frac{\xi^2}{2}\right) \frac{1}{\xi^2+1} d\xi + k}{\xi^3\left(1+\frac{\xi^3}{3}+\frac{\xi^2}{2}\right)} = \frac{\frac{1}{6}\left[\int \left(\xi^6+3\xi^5+6\xi^3\right) \frac{1}{\xi^2+1} d\xi + 6k\right]}{\frac{1}{6}\left(2\xi^6+3\xi^5+6\xi^3\right)}.$$

Hence

$$\Psi(\xi) \approx \frac{\int \left(2\xi^6+3\xi^5+6\xi^3\right) \frac{1}{\xi^2+1} d\xi + 6k}{\left(2\xi^6+3\xi^5+6\xi^3\right)}.$$

Choose

$$I = \frac{\left(2\xi^6+3\xi^5+6\xi^3\right)}{\left(\xi^2+1\right)}.$$

By division algorithm we obtain :

$$\begin{aligned} I &= 2\xi^4+3\xi^3-2\xi^2+3\xi+2+\frac{-2-3\xi}{\xi^2+1} \\ &= 2\xi^4+3\xi^3-2\xi^2+3\xi+2-\frac{2}{\xi^2+1}-\frac{3}{2}\frac{2\xi}{\xi^2+1}. \end{aligned}$$

Then

$$\begin{aligned}\int I\,d\xi+6k &= \int 2\xi^4+3\xi^3-2\xi^2+3\xi+2-\frac{2}{\xi^2+1}-\frac{3}{2}\frac{2\xi}{\xi^2+1}d\xi+6k\\ &= 2\frac{\xi^5}{5}+3\frac{\xi^4}{4}-2\frac{\xi^3}{3}+3\frac{\xi^2}{2}+2\xi-2\tan^{-1}\xi-\frac{3}{2}\ln|\xi^2+1|+6k.\end{aligned}$$

So $\Psi(\xi)$ after simplification becomes :

$$\Psi(\xi)\approx\frac{2\frac{\xi^5}{5}+3\frac{\xi^4}{4}-2\frac{\xi^3}{3}+3\frac{\xi^2}{2}-2\xi+2\tan^{-1}\xi-\frac{3}{2}\ln|\xi^2+1|+6k}{(2\xi^6+3\xi^5+6\xi^3)}.$$

For some constant k.
Now, we approach $\tan^{-1}\xi+6k$ and $\ln|\xi^2+1|$ using the series expansion (1-term).
Starting by $\tan^{-1}\xi+6k$:

$$\begin{aligned}\tan^{-1}\xi+6k &= \int\frac{1}{1+\xi^2}d\xi=\int\frac{1}{1-(-\xi^2)}d\xi=\int\sum_{n=0}^{\infty}(-\xi^2)^n d\xi,\ |\xi|<1\\ &= \int 1-\xi^2+\xi^4-\xi^6+....d\xi=\int\sum_{n=0}^{\infty}(-1)^n\,\xi^{2n}d\xi\\ &= c+\xi-\frac{\xi^3}{3}+\frac{\xi^5}{5}-\frac{\xi^7}{7}+...\ \text{ for some constant } c.\end{aligned}$$

Then

$$\tan^{-1}\xi = (c-6k)+\sum_{n=0}^{\infty}(-1)^n\frac{\xi^{2n+1}}{2n+1}.$$

Letting $\xi=0$ then we obtain $(c-6k)=0$, so

$$\tan^{-1}\xi = \sum_{n=0}^{\infty}(-1)^n\frac{\xi^{2n+1}}{2n+1}=\xi-\frac{\xi^3}{3}+\frac{\xi^5}{5}-\frac{\xi^7}{7}+...$$

So the approach can be taken as :

$$\tan^{-1}\xi\approx\xi.$$

Secondly $\ln\left|\xi^2+1\right|$:

$$\begin{aligned}\ln\left|\xi^2+1\right| &= \sum_{n=0}^{\infty}(-1)^n\frac{\xi^{2n+2}}{n+1}\\ &= \xi^2-\frac{\xi^4}{2}+\frac{\xi^6}{3}...\end{aligned}$$

Hence

$$\ln\left|\xi^2+1\right|\approx\xi^2.$$

Finally, after these estimations $\Psi(\xi)$ becomes :

$$\begin{aligned}\Psi(\xi) &\approx \frac{2\frac{\xi^5}{5}+3\frac{\xi^4}{4}-2\frac{\xi^3}{3}+3\frac{\xi^2}{2}+2\xi-2\xi-\frac{3}{2}\xi^2}{(2\xi^6+3\xi^5+6\xi^3)}\\ &= \frac{2\frac{\xi^5}{5}+3\frac{\xi^4}{4}-2\frac{\xi^3}{3}}{(2\xi^6+3\xi^5+6\xi^3)}\\ &= \frac{2\frac{\xi^2}{5}+3\frac{\xi}{4}-\frac{2}{3}}{(2\xi^3+3\xi^2+6)}\\ &= \frac{1}{6}\left[\frac{(2.4\xi^2+4.5\xi-4)}{(2\xi^3+3\xi^2+6)}\right].\end{aligned}$$

Now, we have to reformulate $\Psi(\xi)$ to take the conformable fractional Laplace inverse transform easier.
Let us start by the denominator.

$$2\xi^3+3\xi^2+6=0.$$

Rewrite the equation as,

$$\xi^3+\frac{3}{2}\xi^2+3=0. \tag{3.7}$$

It is important to mention a formula called the cubic formula for finding the roots of (2.6).
The cubic formula for finding roots of (2.6) as contained is given by,
let $P=b-\frac{a^2}{3}=-\frac{3}{4}$ and $q=\frac{2a^3}{27}-\frac{ab}{3}+c=\frac{13}{4}$, where, $a=\frac{3}{2}$, $b=0$ and $c=3$.

Discriminant

$$(\triangle)=\frac{q^2}{4}+\frac{p^3}{27}=\frac{168}{64}>0.$$

As noted earlier, the nature of the roots of a cubic equation depends on whether the associated discriminant is positive, negative or zero.
Roots of a cubic equation when $\triangle>0$ there is only one real solution.

$$\begin{aligned}\xi &= \left(-\frac{q}{2}+\sqrt{\triangle}\right)^{\frac{1}{3}}+\left(-\frac{q}{2}-\sqrt{\triangle}\right)^{\frac{1}{3}}-\frac{a}{3}\\ &= -2.14937...\end{aligned}$$

By division algorithm we conclude

$$\frac{2\xi^3+3\xi^2+6}{\xi+2.14937...}=2\xi^2-1.29875...\xi+2.79150...$$

$$2\xi^3+3\xi^2+6=(\xi+2.14937...)(2\xi^2-1.29875...\xi+2.79150...)$$

$$\Psi(\xi)\approx\frac{1}{6}\left[\frac{(2.4\xi^2+4.5\xi-4)}{2\xi^3+3\xi^2+6}\right]=\frac{1}{6}\left[\frac{(2.4\xi^2+4.5\xi-4)}{(\xi+2.14937...)(2\xi^2-1.29875...\xi+2.79150...)}\right].$$

Now, we have to use partial fraction decomposition where the degree of the polynomial in the numerator is less than the degree of the polynomial in the denominator to make the conformable fractional Laplace inverse ($\mathfrak{L}^{-1}$) transform exist.
Hence

$$\Psi(\xi)\approx\frac{1}{6}\left[\frac{c_1}{(\xi+2.14937...)}+\frac{c_2\xi+c_3}{(2\xi^2-1.29875...\xi.+2,79150...)}\right].$$

By identification we get

$$\begin{cases} c_1 & = -\,0.17437... \\ c_2 & = +\,2.74874... \\ c_3 & = -\,1.63455... \end{cases}$$

Therefore

$$\Psi\left(\xi\right) \approx \frac{1}{6}\left[\frac{-0.17437...}{(\xi + 2.14937...)} + \frac{2.74874...\xi - 1.63455...}{(2\xi^2 - 1.29875...\xi + 2,79150...)}\right]. \tag{3.8}$$

Applying the conformable fractional Laplace inverse transform to the both sides of equation (2.7) we obtain

$$\mathfrak{L}_\alpha^{-1}\left\{\Psi\left(\xi\right)\right\} \approx \frac{1}{6}\mathfrak{L}_\alpha^{-1}\left\{\frac{-0.17437...}{(\xi + 2.14937...)}\right\} + \frac{1}{6}\mathfrak{L}_\alpha^{-1}\left\{\frac{2.74874...\xi - 1.63455...}{(2\xi^2 - 1.29875...\xi + 2,79150...)}\right\}.$$

By linearity of $\mathfrak{L}_\alpha^{-1}$ we get

$$\begin{aligned} \mathfrak{L}_\alpha^{-1}\left\{\Psi\left(\xi\right)\right\} &= -\frac{1}{6}\mathfrak{L}_\alpha^{-1}\left\{\frac{0.17437...}{(\xi + 2.14937...)}\right\} + \frac{1}{6}\mathfrak{L}_\alpha^{-1}\left\{\frac{2.74874...\xi}{(2\xi^2 - 1.29875...\xi + 2,79150...)}\right\} \\ &- \frac{1}{6}\mathfrak{L}_\alpha^{-1}\left\{\frac{1.63455...}{(2\xi^2 - 1.29875...\xi + 2,79150...)}\right\}. \end{aligned}$$

To facilitate and simplify our calculation we must rewrite the second denominator as we apply the property of Theorem 2.4 $\left(\mathfrak{L}_\alpha\left\{e^{-a\frac{x^\alpha}{\alpha}}f\left(x\right)\right\}\left(\xi\right) = \mathcal{F}_\alpha(\xi + a), \quad \xi > |a|\right)$.

$$\begin{aligned} Y\left(x\right) &\approx -\frac{1}{6}\mathfrak{L}_\alpha^{-1}\left\{\frac{0.17437...}{(\xi + 2.14937...)}\right\} + \frac{1}{12}\mathfrak{L}_\alpha^{-1}\left\{\frac{2.74874...\xi}{(\xi - 0.325...)^2 + 1.29012...}\right\} \\ &- \frac{1}{12}\mathfrak{L}_\alpha^{-1}\left\{\frac{1.63455...}{(\xi - 0.325...)^2 + 1.29012...}\right\}. \end{aligned}$$

Finally we conclude that

$$\begin{aligned} Y(x) &\approx -0.029061...\, e^{-\left(2.14937...\frac{x^\alpha}{\alpha}\right)} + 0.22906...\, e^{\left(0.325...\frac{x^\alpha}{\alpha}\right)} \cos\left(\sqrt{1.29012...}\,\frac{x^\alpha}{\alpha}\right) \\ &- \frac{1.63455...}{12\sqrt{1.29012...}}\, e^{\left(0.325...\frac{x^\alpha}{\alpha}\right)} \sin\left(\sqrt{1.29012...}\,\frac{x^\alpha}{\alpha}\right). \end{aligned}$$

Hence a result as required.

Problem 3 :

Now, we will use the conformable fractional Laplace transform method to find the induced deflection function $Y(x)$ of a cantilever beam subjected to a uniform distributed load with intensity W_0 on half of the beam span, as illustrated in the figure bellow.

$$Y^{(4\alpha)}(x) = \frac{W(x)}{EI}. \tag{3.9}$$

Where E and I are respectively the Young's modulus of the beam material and the section moment of inertia of the beam and

$$W(x) = \begin{cases} W_0 & \text{if } 0 \leq x \leq \frac{L}{2}, \\ 0 & \text{if } \frac{L}{2} \leq x \leq L. \end{cases}$$

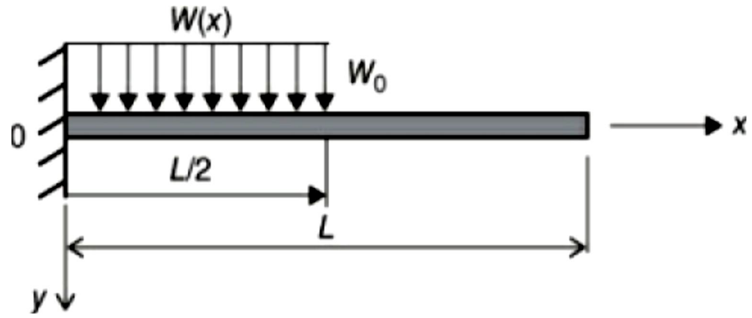

Figure 3.1: A cantilever beam subjected to uniform distributed load.

Conditions 3 :

$$Y^{(3\alpha)}(L) = Y^{(2\alpha)}(L) = 0, \qquad Y^{(\alpha)}(0) = Y(0) = 0. \tag{3.10}$$

Solution :

Take the conformable fractional Laplace transform of both sides of equation (2.8) and using the conditions (2.9) we get :

$$\xi^4\Psi(\xi) - Y^{3\alpha}(0) - \xi Y^{2\alpha}(0) = \frac{W_0}{EI}\frac{(1-e^{-\xi k})}{\xi}, \quad k = \frac{L^\alpha}{\alpha 2^\alpha}.$$

Since

$$\begin{aligned}
\mathfrak{L}_\alpha\{W(x)\} &= \int_0^\infty e^{-\xi\frac{x^\alpha}{\alpha}}W(x)d_\alpha x = \int_0^{\frac{L}{2}} e^{-\xi\frac{x^\alpha}{\alpha}}W_0 d_\alpha x \\
&= -\frac{W_0}{\xi}\int_0^{\frac{L}{2}} -\xi x^{\alpha-1}e^{-\xi\frac{x^\alpha}{\alpha}}dx = -\frac{W_0}{\xi}\left[e^{-\xi\frac{x^\alpha}{\alpha}}\right]_0^{\frac{L}{2}} \\
&= \frac{W_0}{\xi}\left[1-e^{-\xi k}\right], \quad k = \frac{L^\alpha}{\alpha 2^\alpha}.
\end{aligned}$$

Thus

$$\Psi(\xi) = \frac{Y^{(3\alpha)}(0)}{\xi^4} + \frac{Y^{(2\alpha)}(0)}{\xi^3} + \frac{W_0}{EI}\frac{(1-e^{-\xi k})}{\xi^5}, \quad k = \frac{L^\alpha}{\alpha 2^\alpha}. \tag{3.11}$$

Clearly $\begin{cases} 1. \lim_{\xi\to\infty}\Psi(\xi) = 0, \\ 2. \lim_{\xi\to\infty}\xi\Psi(\xi) = 0 \ \ (bounded). \end{cases}$

Then we apply the conformable fractional Laplace inverse transform on all equation (2.10) to get the solution.

$$\mathfrak{L}_\alpha^{-1}\{\Psi(\xi)\} = \mathfrak{L}_\alpha^{-1}\left\{\frac{Y^{(3\alpha)}(0)}{\xi^4}\right\} + \mathfrak{L}_\alpha^{-1}\left\{\frac{Y^{(2\alpha)}(0)}{\xi^3}\right\} + \mathfrak{L}_\alpha^{-1}\left\{\frac{W_0}{EI}\frac{(1-e^{-\xi k})}{\xi^5}\right\}, \quad k = \frac{L^\alpha}{\alpha 2^\alpha}.$$

From the property $\left(\mathfrak{L}_\alpha\left\{x^p\right\}(\xi) = \alpha^{\frac{p}{\alpha}}\frac{\Gamma\left(1+\frac{p}{\alpha}\right)}{\xi^{1+\frac{p}{\alpha}}},\ \ \xi > 0,\ \ \frac{p}{\alpha} > -1\right)$, we get

$$\begin{aligned}\mathfrak{L}_\alpha^{-1}\left\{\frac{Y^{(3\alpha)}(0)}{\xi^4}\right\} &= Y^{(3\alpha)}(0)\mathfrak{L}_\alpha^{-1}\left\{\xi^4\right\}\\ &= \frac{Y^{(3\alpha)}(0)}{\alpha^3\Gamma(1+3)}\frac{\alpha^3\Gamma(1+3)}{\xi^{(1+3)}},\quad p = 3\alpha\\ &= \frac{Y^{(3\alpha)}(0)}{24\alpha^3}x^{3\alpha}.\end{aligned}$$

similarly

$$\mathfrak{L}_\alpha^{-1}\left\{\frac{Y^{(2\alpha)}(0)}{\xi^3}\right\} = \frac{Y^{(2\alpha)}(0)}{6\alpha^2}x^{2\alpha}.$$

Let approach the exponential by two first terms of expansion $\left(e^{-\xi k} \approx 1 - \xi k\right)$, then

$$\begin{aligned}\mathfrak{L}_\alpha^{-1}\left\{\frac{W_0}{EI}\frac{(1-e^{-\xi k})}{\xi^5}\right\} &\approx \frac{W_0}{EI}\mathfrak{L}_\alpha^{-1}\left\{\frac{1}{\xi^5} + \frac{1}{\xi^5} - \frac{(\xi k)}{\xi^5}\right\}\\ &= \frac{W_0}{EI}\left(\frac{2}{120\alpha^4}x^{4\alpha} - \frac{k}{24\alpha^3}x^{3\alpha}\right).\end{aligned}$$

Hence, the solution is given as

$$Y(x) \approx \frac{Y^{(3\alpha)}(0)}{24\alpha^3}x^{3\alpha} + \frac{Y^{(2\alpha)}(0)}{6\alpha^2}x^{2\alpha} + \frac{W_0}{EI}\left(\frac{2}{120\alpha^4}x^{4\alpha} - \frac{k}{24\alpha^3}x^{3\alpha}\right).$$

It is easy to use the conditions to calculate $Y^{(3\alpha)}(0)$ and $Y^{(2\alpha)}(0)$.
Finally, the solution of equation (2.8) is given as:

$$Y(x) \approx \begin{cases}\frac{Y^{(3\alpha)}(0)}{24\alpha^3}x^{3\alpha} + \frac{Y^{(2\alpha)}(0)}{6\alpha^2}x^{2\alpha} + \frac{W_0}{EI}\left(\frac{2}{120\alpha^4}x^{4\alpha} - \frac{k}{24\alpha^3}x^{3\alpha}\right) & \text{if} 0 \le x \le \frac{L}{2},\\ 0 & \text{if} \frac{L}{2} \le x \le L.\end{cases}$$

4 Conclusion

We conclude that the conformable fractional Laplace transform can be used in solving the most difficult fractional differential equations and systems, as we provide in the solution of **Problem 1** and **Problem 2**. Also, this fractional transform has many applications in physics and engineering, as mentioned in **Problem 3**.

Conflict of Interest

The authors have no conflicts of interest to disclose.

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