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Bifurcations et chaos dans un système fractionnaire

Préparée par : Bourafa Safa **Directeur de thèse:** Mohammed-Salah Abdelouahab

Filière : Mathématiques **Spécialité:** Mathématiques et Applications

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Bifurcations and Chaos in a Fractional-Order System

Presented by: Bourafa Safa

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Faculty : Mathematics

Specialty: Mathematics and Applications

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General Introduction

"*Can the meaning of a derivative of integer order* $d^n y/dx^n$ *be extended to have meaning when* n *is a fractional?*"

" *Can* n *be any number fractional, irrational or complex?*"

These questions are the original questions leading to the name of ***Fractional Calculus***.

The history of fractional calculus goes back to the end of the 17th century, when *Leibniz* invented the notation $d^n y/dx^n$. this notation prompted *L'Hospital* in 1695 to ask *Leibniz* "*What if* n *be* 1/2?", this question was answered affirmatively by *Leibniz* in his letter [1]. After this Letter, *Leibnitz* wrote two letters, the first one in *28 December 1695* to *Johann Bernoulli* and the second one in *28 May 1697* to *John Wallis.* since that time fractional calculus has drawn the attention of many famous mathematicians, such as *L. Euler* (1730), *J.L. Lagrange* (1772), *P.S Laplace* (1812), *S.F. Lacroix* (1819), *J.B.J. Fourier*(1822) [3]. But the Fractional operations were used firstly in (1823) by *N.H. Abel* in the solution of an integral equation which arises in the formulation of the tautochrone problem [4].

After almost a decade and specifically in 1832, *J. Liouville* published his three large memoirs. The first one is considered as the first major study of fractional calculus and he was successful in applying his definition to problems in potential theory his results were presented in several more publications between 1834 and 1873. Over and after this period many mathematicians made very important work in fractional calculus such as *G.F.B. Riemann*(1847), *A.K. Grunwald*(1867), *A.V. Letnikove*(1868), *H. Laurent*(1884) *M. Caputo*(1967), *K.S. Miller*, *B.Ross*(1993) and many others [1, 3, 4, 2]. The interaction of all the previous works has produced many different types of fractional-order derivatives, but the three famous and most commonly used in theory of fractional calculus are *the Riemann-Liouville (RL), Grünwald-Letnikov* and *Caputo* types.

The idea of fractional calculus is a generalization of integration and differentiation to non-integer order fundamental operator ${}_aD_t^\alpha$, where a and t are the terminals of the operator. The continuous integro-differential operator is defined by

$$
{}_aD_t^\alpha = \begin{cases} \frac{d^\alpha}{dt^\alpha}; & \alpha > 0 \\ I; & \alpha = 0 \\ \int\limits_a^t (d\tau)^{-\alpha}; & \alpha < 0 \end{cases};
$$

where $\alpha \in \mathbb{R}$ is the order of the operation.

As major causes the fractional derivative provide an excellent instrument for the description of memory and hereditary properties of various materials and processes. there is no doubt that fractional calculus has become an exciting new mathematical method to solve many problems in diverse fields of our life. For example: in physics, chemistry, biology, viscoelasticity, engineering, medicine and many others [5, 6, 7, 8, 9, 10].

Bifurcations and chaos theories are among the important topics that have attracted more attention of many researchers, they have been greatly influenced in several fields of natural sciences such as in physics [11], biology [12], chemistry [13], economics [14] and the others [15]. With the development of the fractional calculus, the bifurcations and chaos in fractional-order systems also have received much attention in a number of areas through the works of *Hartley et al* [16], *Li et al* [17, 18, 19], *Abdelouahab et al* [71, 21, 22]. The content of this thesis is divided into four chapters:

Chapter 1: Entitled *Fractional Calculus.*

In this chapter, we begin presenting the basic notations of fractional calculus that will be recurrently used during all this thesis, we give definitions of the special functions and some of their properties such as *Gamma, Beta* and *Mittag-Leffler* functions. In the second part we present the three definitions most frequently used for the general fractional differential operators with their most important properties (Linearity, Leibniz rule and Laplace transform). In the last one section we discuss the question of the existence and uniqueness of solutions of fractional-order differential equations and analytical and numerical resolution of these equation are also considered.

Chapter 2: *Fractional Order Systems.*

According to the basic rule of " the generalization and extension of meaning to fractional calculus", This chapter is a review of the most important tools and theory of fractional order dynamical systems. The memory dependence property of the solution, the question of stability for the fractional-order linear and nonlinear systems, bifurcation and chaos of the dynamical system have been presented.

Chapter 3: *Routh-Hurwitz Conditions for Fractional Order Systems.*

In this chapter we extend the *Routh-Hurwitz conditions* to fractional systems of order $\alpha \in [0, 2)$. We use these results to investigate the stability properties of some population models. Numerical simulations which support our theoretical analysis are given [23].

Chapter 4: Entitled *Periodic Solutions of Fractional Order Systems.*

This chapter focuses on the issues of periodicity property in fractional-order derivative. We display the absence of periodicity property in fractional-order derivatives unless the lower terminal of the derivative is $\pm\infty$ [24, 25, 26], and since this property limits the applicability areas of fractional-order systems for a wide range of periodic real phenomena we have to find practical solutions for this problem. The proposed solution consists of fixing the memory length and varying the lower terminal of the derivative [27, 28].

Chapter 1

Fractional Calculus

In this chapter we commence with some basic theory of the special functions, we give here some informations on the Gamma function, the Beta function and the Mittag-Liffler function. The second section contains the definitions and some properties of fractional integrals and fractional derivatives of different types, and the next section is devoted to proving the existence and uniqueness, the analytical and numerical solutions of fractional differential equations.

1.1 Basic functions

In this section we present some basic tools. Which play the most important role in the theory of fractional calculus, more details may be found in [29, 30].

1.1.1 Gamma function

One of the basic functions of the fractional calculus is the Euler's gamma function (or Euler's integral of the second kind), which generalizes the factorial to non-integer values. The gamma function $\Gamma(z)$ is defined by the following integral:

$$\Gamma(z) = \int_0^\infty \mathrm{e}^{-t} t^{z-1} dt; \ \Re(z) > 0. \tag{1.1}$$

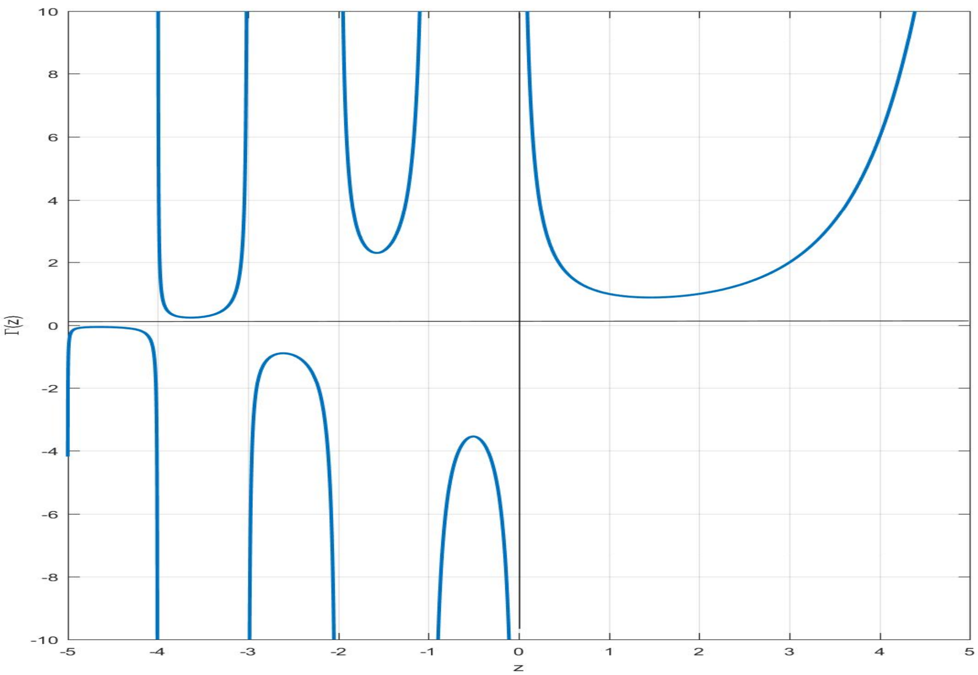


Figure 1.1: Graphical representation of Euler gamma function.

The fundamental property of the gamma function is the recurrence relationship which can be easily proved by integration by parts

$$\Gamma(z) = \int_0^{\infty} \mathrm{e}^{-t} t^{z-1} dt = \frac{1}{z} t^z \mathrm{e}^{-t}|_0^{\infty} + \int_0^{\infty} \mathrm{e}^{-t} \frac{t^z}{z} dt = \frac{1}{z} \int_0^{\infty} \mathrm{e}^{-t} t^z dt = \frac{1}{z} \Gamma(z+1).$$

So

$$\Gamma(z+1) = z\Gamma(z) \tag{1.2}$$

If $z = n$, where n is a positive integer, then

$$\Gamma(n+1) = n(n-1)(n-2)...1 = n!.$$

The figure (1.1) represents the graph of the gamma function.

1.1.2 Beta function

The beta function is defined by the Euler integral of the first kind:

$$\beta(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt;\ (\Re(x) > 0;\ \Re(y) > 0). \tag{1.3}$$

This function is connected with the gamma function by the relation

$$\beta(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. \tag{1.4}$$

The relationship between the gamma function and the beta function (1.4) can be also provided by using the Laplace transform (1.30).

1.1.3 Mittag-Liffler function

The Mittag-Liffler function $E_\alpha(z)$ plays a very important role in the theory of fractional calculus, its role is similar to that played by the exponenetial function in the theory of integer-order differential equations.

The Mittag-Liffler function with one parameter was introduced by G.M.Mittag-Liffler and the basic properties of this function were studied by Mittag-Liffler and Wiman. The Mittag-Liffler function is defined by:

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)};\ \alpha > 0. \tag{1.5}$$

When $\alpha = 1$, we have

$$E_1(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+1)} = \sum_{k=0}^{\infty} \frac{z^k}{k!} = e^z.$$

The Mittag-Liffler function with two parameters α and β was introduced by Wiman at 1905. This function is defined by:

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)};\ \alpha > 0,\ \beta > 0. \tag{1.6}$$

When $\beta = 1$, $E_{\alpha,\beta}(z)$ coincides with the Mittag-Leffler function (1.5).

The figure (1.2) represents the graph of the Mittag-Liffler function

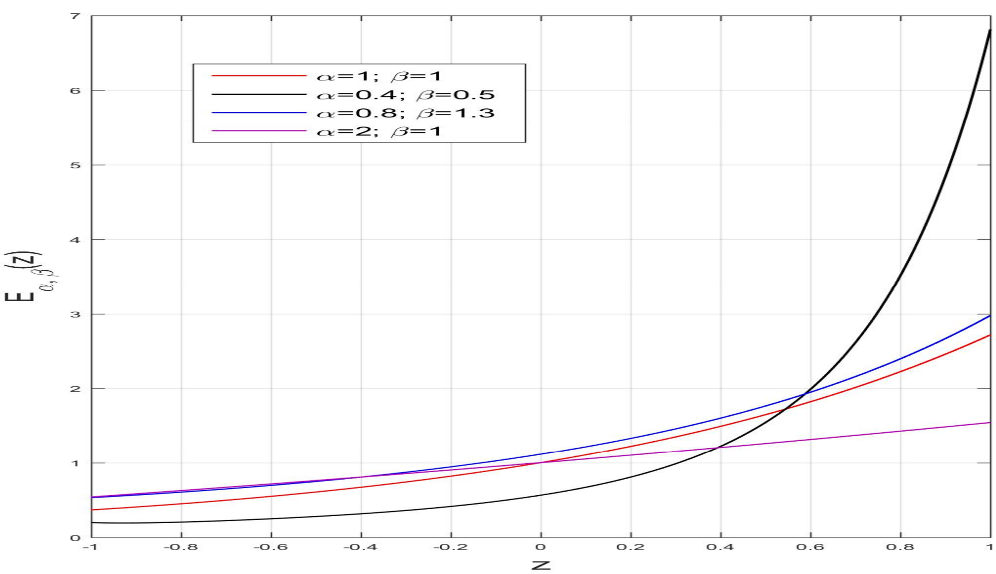


Figure 1.2: Graphical representation of Mittag-Liffler function with two parameters α and β.

1.2 Definitions

The most famous definitions for fractional-order derivative are the Riemann-Liouville definition, the Caputo definition and the Grünwald-Letnikov definition.

1.2.1 Riemann-Liouville fractional derivatives

In this section we give a definition for fractional integral and differential operators ${}_a^{RL}D_t^{-\alpha}$ and ${}_a^{RL}D_t^{\alpha}$, of order $\alpha \notin \mathbb{N}$. we begin with the integral operator.

Integral of arbitrary order

The Riemann-Liouville integral operator ${}_a^{RL}D_t^{-\alpha}$ is an extension of Cauchy's integral:

$$f^{(-n)}(t) = \frac{1}{\Gamma(n)} \int_a^t (t-\tau)^{n-1} f(\tau) d\tau, \tag{1.7}$$

and replace the integer n by a real $\alpha > 0$:

$$ {}_aD_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau) d\tau. \tag{1.8}$$

In (1.7) the integer n must satisfy the condition $n \geq 1$, the corresponding condition for α is weaker, for the existence of the integral (1.8) we must have $\alpha > 0$.

Example 1.1 *Let consider the power function $f(t) = (t-a)^{\beta}$, where β is a real number. We have*

$$ {}_aD_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} (\tau - a)^{\beta} d\tau.$$

By using the variable change $\tau = a + x(t-a)$, and then using the definition of the beta function, we obtain

$$\begin{aligned} \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} (\tau - a)^{\beta} d\tau &= \tfrac{(t-a)^{\alpha+\beta}}{\Gamma(\alpha)} \int_0^1 (1-x)^{\alpha-1} x^{\beta} dx \\ &= \tfrac{(t-a)^{\alpha+\beta}}{\Gamma(\alpha)} \beta(\alpha, \beta+1) \\ &= (t-a)^{\alpha+\beta} \tfrac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}. \end{aligned} \tag{1.9}$$

If $f(t) = K$, then ${}_aD_t^{-\alpha} K = (t-a)^{\alpha} \frac{K}{\Gamma(\alpha+1)}$.

Some properties of the Riemann-Liouville integral

1. Let $f \in C^0([a,b])$, $\alpha > 0$. Then we have

$$\lim_{\alpha \longrightarrow 0} ({}_aD_t^{-\alpha} f(t)) = f(t). \tag{1.10}$$

So we can put ${}_aD_t^0 = I$, the identity operator.

2. Let $f \in C^0([a,b])$, $\alpha > 0$, and $\beta > 0$. Then

$$ {}_aD_t^{-\alpha} ({}_aD_t^{-\beta} f(t)) = {}_a D_t^{-(\alpha+\beta)} f(t) = {}_a D_t^{-\beta} ({}_aD_t^{-\alpha} f(t)). \tag{1.11}$$

Derivative of arbitrary order

Let $\alpha \in \mathbb{R}_+$ and $n = \lceil \alpha \rceil$ ($\lceil \alpha \rceil = \min\{z \in Z : z \geq \alpha\}$). The Riemann-Liouville fractional derivative of a function $f(t)$ is defined by

$$\begin{aligned} {}_a^{RL}D_t^{\alpha} f(t) &= \tfrac{1}{\Gamma(n-\alpha)} \tfrac{d^n}{dt^n} \int_a^t (t-\tau)^{n-\alpha-1} f(\tau) d\tau, \\ &= \tfrac{d^n}{dt^n} ({}_aD_t^{-(n-\alpha)} f(t)). \end{aligned} \tag{1.12}$$

If $\alpha = n-1$, then we obtain a conventional integer order derivative of order $n-1$:

$$ {}_aD_t^{n-1} f(t) = \frac{d^n}{dt^n} ({}_aD_t^{-1} f(t)) = f^{(n-1)}(t).$$

Example 1.2 *Let consider the function $f(t) = (t-a)^{\beta}$, for $n-1 \leq \alpha < n$ we have*

$$ {}_a^{RL}D_t^{\alpha} f(t) = \tfrac{d^n}{dt^n} ({}_aD_t^{-(n-\alpha)} f(t)). \tag{1.13}$$

Substituting the integral of order $(n-\alpha)$ of this function (1.9) into the formula (1.13), then

$$\begin{aligned} {}_a^{RL}D_t^{\alpha} f(t) &= \tfrac{d^n}{dt^n} ((t-a)^{n-\alpha+\beta} \tfrac{\Gamma(\beta+1)}{\Gamma(n+\beta-\alpha+1)}). \\ &= \tfrac{\Gamma(\beta+1)}{\Gamma(-\alpha+\beta+1)} (t-a)^{\beta-\alpha}. \end{aligned} \tag{1.14}$$

If $f(t) = K$. So ${}_a^{RL}D_t^{\alpha} K = (t-a)^{-\alpha} \frac{K}{\Gamma(1-\alpha)}$.

Some properties of the Riemann-Liouville derivative

1. $$ {}_a^{RL}D_t^{\alpha} ({}_aD_t^{-\alpha} f(t)) = f(t). \tag{1.15}$$

   This property means that the Riemann-Liouville differentiation operator is a left inverse to the Riemann-Liouville integration operator of the same order α.

2. $$ {}_aD_t^{-\alpha} ({}_a^{RL}D_t^{\alpha} f(t)) = f(t) - \sum_{k=1}^{n} [{}_a^{RL}D_t^{\alpha-k} f(t)]_{t=a} \frac{(t-a)^{\alpha-k}}{\Gamma(\alpha-k+1)};\ \alpha > 0; t > a. \tag{1.16}$$

3. The following properties are the generalisation of (1.15) and (1.16), for $\alpha \geq \beta \geq 0$:

$$ {}_{a}^{RL}D_t^{\alpha}({}_aD_t^{-\beta}f(t)) = {}_{a}^{RL}D_t^{\alpha-\beta}f(t). \tag{1.17}$$

and

$$ {}_aD_t^{-\alpha}({}_{a}^{RL}D_t^{\beta}f(t)) = {}_{a}^{RL}D_t^{\alpha-\beta}f(t) - \sum_{k=1}^{n}[{}_{a}^{RL}D_t^{\alpha-k}f(t)]_{t=a}\frac{(t-a)^{\alpha-k}}{\Gamma(\beta-k+1)}. \tag{1.18}$$

4. Let $(m-1 \leq \alpha < m)$ and $(n-1 \leq \beta < n)$, we have

$$ {}_{a}^{RL}D_t^{\alpha}({}_{a}^{RL}D_t^{\beta}f(t)) = ({}_{a}^{RL}D_t^{\alpha+\beta}f(t)) - \sum_{k=1}^{n}[{}_{a}^{RL}D_t^{\beta-k}f(t)]_{t=a}\frac{(t-a)^{-\alpha-k}}{\Gamma(1-\alpha-k)}. \tag{1.19}$$

and

$$ {}_{a}^{RL}D_t^{\beta}({}_{a}^{RL}D_t^{\alpha}f(t)) = ({}_{a}^{RL}D_t^{\alpha+\beta}f(t)) - \sum_{k=1}^{m}[{}_{a}^{RL}D_t^{\alpha-k}f(t)]_{t=a}\frac{(t-a)^{-\beta-k}}{\Gamma(1-\beta-k)}. \tag{1.20}$$

From the properties (1.19) and (1.20), we deduce that the Riemann-Liouville differentiation operators ${}_{a}^{RL}D_t^{\alpha}$ and ${}_{a}^{RL}D_t^{\beta}$ commute only if $\alpha = \beta$, or $f^{(k)}(a) = 0$ for all $k = 1, 2, ..., r-1$ such that $r = \max(n, m)$.

1.2.2 Caputo's fractional derivative

Since Riemann-Liouville derivatives failed in the description and modeling of some complex phenomena. Caputo derivative is considered as a solution of this problem it is proposed by M. Caputo in 1967.

Definition 1.1 *Let $f \in C^n([a,b])$, $\alpha > 0$. The Caputo's fractional derivative of the function $f(t)$ is defined by*

$$ {}_{a}^{C}D_t^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)}\int_a^t \frac{f^{(n)}(\tau)d\tau}{(t-\tau)^{\alpha-n+1}} = {}_aD_t^{-(n-\alpha)}(\frac{d^n}{dt^n}f(t)) \tag{1.21}$$

where $n-1 < \alpha < n$ and $t > a$.

Lemma 1.1 *Let $\alpha \geq 0$ and $n = \lceil \alpha \rceil$. Assume that f is such that both ${}_{a}^{C}D_t^{\alpha}$ and ${}_{a}^{RL}D_t^{\alpha}$ exist. Then*

$$ {}_{a}^{C}D_t^{\alpha}f(t) = {}_{a}^{RL}D_t^{\alpha}f(t) - \sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{\Gamma(-\alpha+k+1)}(t-a)^{-\alpha+k} \tag{1.22}$$

Example 1.3 *Let consider the function* $f(t) = (t-a)^{\beta}$*, such that* $\beta > n$ *and* $n = \lceil \alpha \rceil$*. We have*

$$ {}_{a}^{C}D_{t}^{\alpha} f(t) = {}_{a}^{RL}D_{t}^{\alpha} f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{\Gamma(k-\alpha+1)} (t-a)^{k-\alpha}, $$

and

$$ f^{(k)}(a) = 0; \ \forall k = 0, 1, \ldots, n-1, $$

then

$$ \begin{aligned} {}_{a}^{C}D_{t}^{\alpha} (t-a)^{\beta} &= {}_{a}^{RL}D_{t}^{\alpha} (t-a)^{\beta} \\ &= \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} (t-a)^{\beta-\alpha}. \end{aligned} $$

If $\beta = 0, 1, \ldots, n-1$*, then*

$$ {}_{a}^{C}D_{t}^{\alpha} (t-a)^{\beta} = 0. $$

Lemma 1.2 *Let* $\alpha > 0$*. We have*

1. *If* $f \in C([a,b])$*, then*

$$ {}_{a}^{C}D_{t}^{\alpha} ({}_{a}D_{t}^{-\alpha} f(t)) = f(t). $$

2. *If* $f \in C^{n}([a,b])$*, then*

$$ {}_{a}D_{t}^{-\alpha} ({}_{a}^{C}D_{t}^{\alpha} f(t)) = f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} (t-a)^{k}. $$

In particular, if $0 < \alpha \leq 1$ *and* $f \in C([a,b])$ *then*

$$ {}_{a}D_{t}^{-\alpha} ({}_{a}^{C}D_{t}^{\alpha} f(t)) = f(t) - f(a). $$

Some properties of the Caputo's derivative

Let $n-1 \leq \alpha < n$ and $f \in C^{n+1}([a,b])$

1. The Caputo derivative of a constant function is 0, but its Riemann-Liouville fractional derivative is not equal to 0.

2. For all $t \in [a, b]$ we have

$$\lim_{\alpha \to n} ({}_a^C D_t^\alpha f(t)) = f^{(n)}(t),$$

and

$$\lim_{\alpha \to n-1} ({}_a^C D_t^\alpha f(t)) = f^{(n-1)}(t) - f^{(n-1)}(a).$$

3. Let $m \in \mathbb{N}^*$, we have

$${}_a^C D_t^\alpha ({}_a^C D_t^m f(t)) = {}_a^C D_t^{\alpha+m} f(t),$$

but

$${}_a^C D_t^\alpha ({}_a^C D_t^m f(t)) ={}_a^C D_t^m ({}_a^C D_t^\alpha f(t)) = {}_a^C D_t^{\alpha+m} f(t).$$

The interchange of the differentiation operators is allowed under the following conditions

$$f^{(k)}(a) = 0 \text{ for } k = n, n+1, \ldots, m.$$

1.2.3 Grünwald-Letnikov fractional derivative

Grünwald-Letnikov Derivative introduced by *Anton Karl Grünwald* in 1867, and then by *Aleksey vasilievich Letnikov* in 1868. As well known the classical derivatives can be expressed as differential quotients, i.e. as limits of difference quotients. For example, the n-th order derivative of a function $f(t) \in C^n([a, b])$ is defined by

$$f^{(n)}(t) = \frac{d^n f}{dt^n} = \lim_{h \to 0} \frac{1}{h^n} \sum_{k=0}^{n} (-1)^k \binom{n}{k} f(t - kh), \tag{1.23}$$

where

$$\binom{n}{k} = \frac{n(n-1)(n-2)...(n-k+1)}{k!},$$

is the usual notation for the binomial coefficients.

The equality (1.23) may be used to define a fractional derivative of Grünwald-Letnikov by direct replacing n by $\alpha \in \mathbb{R}_+$

$${}_a^{GL} D_t^\alpha f(t) = \lim_{\substack{h \to 0 \\ nh = t-a}} \frac{1}{h^\alpha} \sum_{k=0}^{n} (-1)^k \binom{\alpha}{k} f(t - kh),$$

Since $\alpha \in \mathbb{R}_+$, so the binomial coefficient is given by

$$\binom{\alpha}{k} = \frac{\Gamma(\alpha+1)}{k!\Gamma(\alpha-k+1)}.$$

Hence,

$$ {}_{a}^{GL}D_t^{\alpha} f(t) = \lim_{\substack{h \to 0 \\ nh = t-a}} \frac{1}{h^{\alpha}} \sum_{k=0}^{\frac{t-a}{h}} (-1)^k \frac{\Gamma(\alpha+1)}{k!\Gamma(\alpha-k+1)} f(t-kh). \tag{1.24}$$

Integrals of arbitrary order

Let us consider the case of $\alpha < 0$. For convenience let us replace α by $-\alpha$ in the expression (1.24). Then (1.24) takes the form

$$ {}_{a}^{GL}D_t^{-\alpha} f(t) = \lim_{\substack{h \to 0 \\ nh = t-a}} h^{\alpha} \sum_{k=0}^{\frac{t-a}{h}} \frac{\Gamma(\alpha+k)}{k!\Gamma(\alpha)} f(t-kh). \tag{1.25}$$

The expression (1.25) is called the Grünwald-Letnikov integral of order α of the function $f(t)$.

Link to the Riemann-Liouville and the Caputo derivatives

Under the assumption that the derivatives $f^{(k)}(t)$, $(k = 1, 2, \ldots, n)$ are continuous in the closed interval $[a, T]$ and n is an integer number such that $\alpha < n$, then the Grünwald-Letnikov fractional derivative (1.24) can be written as follows:

$$ {}_{a}^{GL}D_t^{\alpha} f(t) = \sum_{k=0}^{n-1} \frac{f^{(k)}(a)(t-a)^{-\alpha+k}}{\Gamma(-\alpha+k+1)} + \frac{1}{\Gamma(-\alpha+n)} \int_a^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau. \tag{1.26}$$

The right hand side of the formula (1.26) can be written as

$$\frac{d^n}{dt^n}\Big(\sum_{k=0}^{n-1} \frac{f^{(k)}(a)(t-a)^{-\alpha+n+k}}{\Gamma(-\alpha+n+k+1)} + \frac{1}{\Gamma(-\alpha+2n)} \int_a^t (t-\tau)^{2n-\alpha-1} f^{(n)}(\tau) d\tau\Big), \tag{1.27}$$

after n integrations by parts it takes the form of the Riemann-Liouville derivative

$$\frac{d^n}{dt^n}\Big(\frac{1}{\Gamma(-\alpha+n)} \int_a^t (t-\tau)^{n-\alpha-1} f(\tau) d\tau\Big) = \frac{d^n}{dt^n}\big({}_aD_t^{-(n-\alpha)} f(t)\big) = {}_{a}^{RL}D_t^{\alpha} f(t). \tag{1.28}$$

Finally, under the above assumptions and according to the relationship (1.22) we get:

$$ {}_{a}^{GL}D_t^{\alpha} f(t) = {}_{a}^{RL}D_t^{\alpha} f(t) = {}_{a}^{C}D_t^{\alpha} f(t) + \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{\Gamma(-\alpha+k+1)}(t-a)^{-\alpha+k} \tag{1.29} $$

1.3 Properties of fractional-order operators

1.3.1 Linearity

Similarly to integer order differentiation, fractional differentiation is a linear operation:

$$ D^{\alpha}(\lambda f(t) + \beta g(t)) = \lambda D^{\alpha} f(t) + \beta D^{\alpha} g(t), \text{ for } \alpha > 0, \ \lambda, \beta \in \mathbb{R}, $$

where D^{α} denotes any mutation of the fractional differentiation considered in this work.

1.3.2 The Leibniz rule for fractional derivatives

The Leibniz rule for Fractional differentiation can be formulated as follow. If $f(t)$ is continuous in $[a,t]$ and $\varphi(t)$ has $(n+1)$ continuous derivatives in $[a,t]$, then the fractional derivative of the product $\varphi(t)f(t)$ is given by

$$ {}_{a}D_t^{\alpha}(\varphi(t)f(t)) = \sum_{k=0}^{n} \binom{\alpha}{k} \varphi^{(k)}(t)\,{}_{a}D_t^{\alpha-k} f(t) - R_n^{\alpha}(t); $$

where $n \geq \alpha + 1$ and

$$ R_n^{\alpha}(t) = \frac{1}{n!\Gamma(-\alpha)} \int_a^t (t-\tau)^{-\alpha-1} f(\tau) d\tau \int_{\tau}^{t} \varphi^{(n+1)}(\xi)(\tau-\xi)^n d\xi, $$

with

$$ \lim_{n\to\infty} R_n^{\alpha}(t) = 0. $$

If f and φ along with all its derivatives are continuous in $[a,t]$ then the Liebniz rule for fractional differentiation takes the form:

$$ {}_{a}D_t^{\alpha}(\varphi(t)f(t)) = \sum_{k=0}^{\infty} \binom{\alpha}{k} \varphi^{(k)}(t)\,{}_{a}D_t^{\alpha-k} f(t). $$

1.3.3 Laplace transforms of fractional derivatives

The Laplace transform $F(s)$ of a function $f(t)$ of a real variable $t \in \mathbb{R}^+$ is defined by

$$F(s) = L\{f(t), s\} = \int_0^\infty e^{-st} f(t) dt; \; s \in \mathbb{C} \tag{1.30}$$

The original $f(t)$ can be restored from the Laplace transform $F(s)$ with the help of the inverse Laplace transform

$$f(t) = L^{-1}(F(s)) = \int_{c-i\infty}^{c+i\infty} e^{st} F(s) ds, \;\; c = \Re(s) > c_0, \tag{1.31}$$

where c_0 is called the abscissa of convergence of the Laplace integral (1.30).

For the existence of the integral (1.30) the function $f(t)$ must be of exponential order α, which means that there exist positive constants M and T such that

$$e^{-\alpha t} \mid f(t) \mid \leq M \text{ for all } t > T$$

The Laplace transform of the convolution

$$\begin{aligned} f(t) * g(t) &= \int_0^t f(t-\tau) g(\tau) d\tau, \\ &= \int_0^t f(\tau) g(t-\tau) d\tau, \end{aligned} \tag{1.32}$$

of two functions $f(t)$ and $g(t)$, which are equal to zero for all $t < 0$, is given by

$$L\{f(t) * g(t), s\} = F(s)G(s), \tag{1.33}$$

under the assumption that both $F(s)$ and $G(s)$ exist.

The Laplace transform of the integer-order derivative $f^{(n)}(t)$ is given by:

$$\begin{aligned} L\{f^{(n)}(t), s\} &= s^n F(s) - \sum_{k=0}^{n-1} s^k f^{(n-k-1)}(0) \\ &= s^n F(s) - \sum_{k=0}^{n-1} s^{n-k-1} f^{(k)}(0). \end{aligned} \tag{1.34}$$

In the following section on the Laplace transforms of fractional derivatives we consider the lower terminal $a = 0$.

Laplace transform of the Riemann-Liouville fractional integral

The Riemann-Liouville fractional integral (1.8) can be written as a convolution of the functions $g(t) = t^{\alpha-1}$ and $f(t)$

$$ {}_0D_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau = t^{\alpha-1} * f(t). \tag{1.35} $$

The Laplace transform of the power function $t^{\alpha-1}$ is given by

$$ G(s) = L\{t^{\alpha-1}, s\} = \Gamma(\alpha) s^{-\alpha}. \tag{1.36} $$

Therefore, using the formula (1.33) we obtain the Laplace transform of the Riemann-Liouville fractional integral:

$$ L\{{}_0D_t^{-\alpha} f(t), s\} = s^{-\alpha} F(s). \tag{1.37} $$

Laplace transform of the Riemann-Liouville fractional derivative

In order to evaluate the Laplace transform of the Riemann-Liouville fractional derivative, we write (1.12) in the form

$$ {}_a^{RL}D_t^{\alpha} f(t) = g^{(n)}(t), \tag{1.38} $$

where

$$ g(t) = {}_0^{RL}D_t^{-(n-\alpha)} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f(\tau) d\tau; \ n-1 \leq \alpha < n. \tag{1.39} $$

The use of the formula (1.34) give

$$ L\{{}_0^{RL}D_t^{\alpha} f(t), s\} = s^n G(s) - \sum_{k=0}^{n-1} s^k g^{(n-k-1)}(0), \tag{1.40} $$

such that

$$ G(s) = s^{-(n-p)} F(s). \tag{1.41} $$

From the definition of the Riemann-Liouville fractional derivative it follows that

$$ g^{(n-k-1)}(t) = \frac{d^{n-k-1}}{dt^{n-k-1}} {}_0D_t^{-(n-\alpha)} f(t) = {}_0^{RL}D_t^{\alpha-k-1} f(t) \tag{1.42} $$

Substituting (1.41) and (1.42) into (1.40) we obtain the final expression for the Laplace transform of the Riemann-Liouville fractional derivative

$$ L\{{}_0^{RL}D_t^{\alpha} f(t), s\} = s^{\alpha} F(s) - \sum_{k=0}^{n-1} s^k [{}_0^{RL}D_t^{\alpha-k-1} f(t)]_{t=0}; \ n-1 \leq \alpha < n \tag{1.43} $$

Laplace transform of the Caputo's fractional derivative

To establish the Laplace transform of the Caputo derivative let us write the Caputo derivative (1.21) in the form:

$$ {}_a^C D_t^{\alpha} f(t) = {}_a D_t^{-(n-\alpha)} g(t), \; g(t) = f^{(n)}(t). \quad n-1 < \alpha \leq n. \tag{1.44} $$

Using the formula (1.37) and (1.34) we obtain the Laplace transform of the Caputo fractional derivative:

$$ L\{{}_0^C D_t^{\alpha} f(t), s\} = s^{\alpha} F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0); \; n-1 < \alpha \leq n. \tag{1.45} $$

Laplace transform of the Grünwald-Letnikov fractional derivative

First case $0 \leq \alpha < 1$: The Grünwald-Letnikov fractional derivative (1.26) can be written as follows

$$ {}_0^{GL} D_t^{\alpha} f(t) = \frac{f(0) t^{-\alpha}}{\Gamma(1-\alpha)} + \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} f'(\tau) d\tau. \tag{1.46} $$

Using the Laplace transforms (1.36), (1.33) and (1.34) we obtain the Laplace transform of the Grünwald-Letnikov fractional derivative of order $0 \leq \alpha < 1$

$$ L\{{}_0^{GL} D_t^{\alpha} f(t), s\} = s^{\alpha} F(s). \tag{1.47} $$

If $\alpha > 1$: In this case the Laplace transform of the Grünwald-Letnikov fractional derivative does not exist in the classical sense, because in such a case we have non integrable function in the sum in the formula (1.26).

1.4 Fractional differential equations

1.4.1 Existence and uniqueness results

In this part we will be focused on equations with Riemann-Liouville differential operators and Caputo derivatives. We assume in this result and in the ensuing developments that the fractional derivatives are developed at the point 0.

Existence and uniqueness results for Riemann–Liouville fractional differential equations

The initial value problem (Cauchy problem) with Riemann–Liouville differential is given by:

$$\begin{cases} {}^{RL}D^{\alpha}y(t) = f(t, y(t)), \\ {}^{RL}D^{\alpha-k}y(0) = b_k, \qquad k = 1, 2, \ldots, n-1, \\ \lim\limits_{z \to 0_+} D^{-(n-\alpha)}y(z) = b_n. \end{cases} \tag{1.48}$$

Such that $\alpha > 0$, $\alpha \notin \mathbb{N}$ and $n = \lceil \alpha \rceil$.

Theorem 1.1 *Let $\alpha > 0$, $\alpha \notin \mathbb{N}$ and $n = \lceil \alpha \rceil$. Moreover let $K > 0$, $h^* > 0$ and $b_1, \ldots, b_n \in \mathbb{R}$. Define*

$$G := \{(t, y) \in \mathbb{R}^2 : 0 \leq t \leq h^*, y \in \mathbb{R} \text{ for } t = 0 \text{ and } |t^{n-\alpha}y - \sum_{k=1}^{n} b_k t^{n-k}/\Gamma(\alpha - k + 1)| < K \text{ else,}\}$$

and assume that the function $f : G \to \mathbb{R}$ is continuous and bounded in G and that it fulfils a Lipschitz condition with respect to the second variable, i.e. there exist a constant $L > 0$ such that, for all (t, y_1) and $(t, y_2) \in G$, we have

$$|f(t, y_1) - f(t, y_2)| < L|y_1 - y_2|.$$

Then the problem (1.48) has a uniquely defined continuous solution $y \in C(0, h]$ where

$$h := \min\{h^*, \tilde{h}, (\frac{\Gamma(\alpha + 1)K}{M})^{\frac{1}{n}}\}$$

with $M := \sup_{(t,z)\in G} |f(t, z)|$ and $\tilde{h}$ begin an arbitrary positive number satisfying the constraint

$$\tilde{h} < \frac{\Gamma(2\alpha - n + 1)}{(\Gamma(\alpha - n + 1)L)^{\frac{1}{\alpha}}}.$$

This result is very similar to the known classical results for first-order equations.

Specifically, we shall first transform the initial value problem (1.48) into an equivalent Volterra integral equation (Lemma 1.3), and then we prove the existence and uniquenss of the solution of this integral equation by a Picard type iteration process (i,e, by using a variant of Banach's fixed point theorem in a suitably chosen complete metric space), (Lemma 1.4). Theorem 1.1 is thus an immediate consequence of these two lemmas.

Lemma 1.3 *Assume the hypotheses of Theorem 1.1 and let $h > 0$. The function $y \in C(0,h]$ is a solution of the problem (1.48), if and only if it is a solution of the Volterra integral equation*

$$y(t) = \sum_{k=1}^{n} \frac{b_k t^{\alpha-k}}{\Gamma(\alpha-k+1)} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau, y(\tau)) d\tau. \tag{1.49}$$

Lemma 1.4 *Under the assumptions of Theorem 1.1. The Volterra equation (1.49) possesses a uniquely determined solution $y \in C(0,h]$.*

For a detailed proof of Theorem 1.1, Lemma 1.3 and Lemma 1.4 one can refer to [31, 32].

Existence and uniqueness results for Caputo fractional differential equations

For the fractional differential equation of Caputo type we can obtain a similar results of Riemann-Liouville differential equation.

Let $\alpha \in \mathbb{R}^*$ and $n = \lceil \alpha \rceil$, we consider the Cauchy problem with Caputo's fractional derivatives :

$$\begin{cases} {}^C D^\alpha y(t) = f(t, y(t)), \\ D^k y(0) = y_0^{(k)}; \quad k = 0, 1, \dots, n-1. \end{cases} \tag{1.50}$$

Theorem 1.2 *Let $\alpha > 0$, $\alpha \notin \mathbb{N}$ and $n = \lceil \alpha \rceil$. Moreover let $y_0^{(0)}, \dots, y_0^{(n-1)} \in \mathbb{R}$, $K > 0$ and $h^* > 0$. Define*

$$G := [0, h^*] \times [y_0^{(0)} - K, y_0^{(0)} + K],$$

and let the function $f : G \longrightarrow \mathbb{R}$ be continuous. Then, there exists some $h > 0$ and a function $y \in C[0,h]$ solving the problem (1.50). For the case $\alpha \in (0,1)$ the parameter h is given by

$$h := \min\{h^*, (K\Gamma(\alpha+1)/M)^{\frac{1}{\alpha}}\}, \text{ with } M := \sup_{(t,z)\in G} |f(t,z)|.$$

If furthermore f fulfils a Lipschitz condition with respect to the second variable, i.e.

$$|f(t, y_1) - f(t, y_2)| \leq L|y_1 - y_2|.$$

With some constant $L > 0$ independent of t, y_1 and y_2, the function $y \in C[0,h]$ is unique.

This result is also similar to their counterparts in the classical case of first order equations, this means that we will not prove this theorem directly, but rather show that (1.50) can be formulated as Volterra integral equation (Lemma 1.5 and Lemma 1.6).

Lemma 1.5 *Under the assumptions of Theorem 1.2. The function $y \in C(0,h]$ is a solution of the problem (1.48), if and only if it is a solution of the Volterra integral equation*

$$y(t) = \sum_{k=0}^{n-1} \frac{t^k}{k!} y_0^{(k)} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau, y(\tau)) d\tau. \tag{1.51}$$

Lemma 1.6 *Under the assumptions of Theorem 1.2. The Volterra equation (1.51) possesses a uniquely determined solution $y \in C(0,h]$.*

For the proofs of Theorem 1.2, Lemma 1.5 and Lemma 1.6 one can refer to [32].

1.4.2 Analytical solution of linear fractional differential equations

One dimensional case

Theorem 1.3 *[31]*

Let $\alpha > 0$, $n = \lceil \alpha \rceil$ and $\lambda \in \mathbb{R}$. The solution of the initial value problem

$$\begin{cases} {}^C D^\alpha y(t) = \lambda y(t) + q(t), \\ y^{(k)}(0) = y_0^{(k)}; \qquad k = 0, 1, \ldots, n-1. \end{cases} \tag{1.52}$$

where $q \in C[0,h]$ is a given function, can be expressed in the form

$$y(t) = \sum_{k=0}^{n-1} y_0^{(k)} u_k(t) + \tilde{y}(t) \tag{1.53}$$

with

$$\tilde{y}(t) = \begin{cases} D^{-\alpha} q(t) & \text{if } \lambda = 0 \\ \frac{1}{\lambda} \int_0^t q(t-\tau) u_0'(\tau) d\tau & \text{if } \lambda \neq 0 \end{cases} ;$$

where $u_k(t) := D^{-k}(e_\alpha(t))$, $k = 0, 1, ..., n-1$ and $e_\alpha(t) := E_\alpha(\lambda t^\alpha)$.

Example 1.4 *Let consider the problem*

$$\begin{cases} D^\alpha y(t) = -y(t) + 1, \\ y(0) = 0; \quad y'(0) = 0. \end{cases} \tag{1.54}$$

we have: $\lambda = -1$ and $q(t) = 1$.

So;

$$y(t) = \sum_{k=0}^{1} y_0^{(k)} u_k(t) + \tilde{y}(t),$$

such that

$$\tilde{y}(t) = \frac{1}{\lambda}\int_0^t q(t-\tau)u_0^{'}(\tau)d\tau,$$
$$= -E_\alpha(-\tau^\alpha) + 1,$$

and

$$\sum_{k=0}^{1} y_0^{(k)} u_k(t) = y(0)E_\alpha(-t^\alpha) + y'(0)\int_0^t E_\alpha(-\tau^\alpha)d\tau = 0.$$

So, the general solution of (1.54) is

$$y(t) = 1 - E_\alpha(-t^\alpha).$$

Multidimensional case

Let us consider the fractional differential equation

$$D^\alpha y(t) = Ay(t) + q(t), \tag{1.55}$$

with $0 < \alpha < 1$, $A \in M_n(\mathbb{R})$, a given function $q : [0,h] \to \mathbb{R}^n$ and an unknown solution $y : [0,h] \to \mathbb{R}^n$.

As usual we start with the homogeneous problem corresponding to (1.55)

$$D^\alpha y(t) = Ay(t), \tag{1.56}$$

We know that in the classical situation $\alpha = 1$, the general solution of (1.56) is $y(t) = u\exp(At)$ with a suitable vector u. Since we found that the Mittag-Liffler function $E_\alpha(t^\alpha)$ takes the role of $\exp(t)$ in the one-dimensional case, it is natural to seek a solution that is a linear combination of expressions of the form

$$y(t) = uE_\alpha(\lambda t^\alpha), \tag{1.57}$$

with suitable vectors $u \in \mathbb{C}^n$ and scalars $\lambda \in \mathbb{C}$ that need to be determined. Inserting (1.57) into the homogeneous equation (1.56) we obtain

$$u\lambda E_\alpha(\lambda t^\alpha) = AuE_\alpha(\lambda t^\alpha). \tag{1.58}$$

Since $E_\alpha(\lambda t^\alpha) \neq 0$, this implies

$$\lambda u = Au,$$

it means that λ must be an eigenvalue of the matrix A, and u must be a corresponding eigenvector. Now, if all k-fold eigenvalues of A have k eigenvectors, then the set of all these eigenvectors is linearly independent and it forms a basis of $\mathbb{C}^n$. Hence, the following result holds

Theorem 1.4 *[33] Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of A and $u_1, u_2, \ldots, u_n$ be the corresponding eigenvectors. Then, the general solution of (1.56) has the form*

$$y(t) = \sum_{l=1}^{n} c_l u_l E_\alpha(\lambda_l t^\alpha), \tag{1.59}$$

with certain constants $c_l \in \mathbb{C}$. The unique solution of this differential equation subject to the initial condition $y(0) = y_0$ is characterized by the linear system

$$y_0 = (u_1, u_2, \ldots, u_n)(c_1, c_2, \ldots, c_n)^T.$$

If the matrix A has a repeated eigenvalue λ of multiplicity k, then we have two possibilities: either there are k linearly independent eigenvectors corresponding to the eigenvalue λ, in this case $y_1 = u_l E_\alpha(\lambda t^\alpha), \ldots, y_k = u_k E_\alpha(\lambda t^\alpha)$ are k linearly independent solutions of the system (1.56). However, if there are m linearly independent eigenvectors corresponding to an eigenvalue λ of multiplicity k, where $m < k$ then the following

$$\begin{cases} y_1 = u_1 E_\alpha(\lambda t^\alpha), \\ y_2 = u_1 t^\alpha E_\alpha^{(1)}(\lambda t^\alpha) + u_2 E_\alpha(\lambda t^\alpha), \\ \vdots \\ y_{k-m} = u_1 t^{\alpha(k-m-1)} E_\alpha^{(k-m-1)}(\lambda t^\alpha) + u_2 t^{\alpha(k-m-2)} E_\alpha^{(k-m-2)}(\lambda t^\alpha) + \cdots + u_{k-m} E_\alpha(\lambda t^\alpha), \end{cases}$$

are $k - m$ linearly independent solutions of the system (1.56), where $E_\alpha^{(1)}(t) = \frac{d}{dt} E_\alpha(t)$.

Remark 1.1 *Let $[y_1(t), y_2(t), \ldots, y_n(t)]^T$ be the solution of the initial value problem consisting of the fractional order linear system (1.56) and the initial condition $y(0) = y_0$. Then the initial value problem for the nonhomogeneous fractional order system (1.55) and the initial condition $y(0) = y_0$ has the solution $[Y_1(t), Y_2(t), \ldots, Y_n(t)]^T$, such that [33]*

$$Y_i(t) = y_i(t) + \int_0^t y_i(\tau - t) q_i(\tau) d\tau. \tag{1.60}$$

1.4.3 Numerical solution of fractional differential equations

For most fractional differential equations we cannot provide methods to compute the exact solutions analytically. Therefore it is necessary to revert to numerical methods. There are lot of methods used to solve fractional differential equations. An efficient method for solving fractional differential equations in term of Caputo type fractional derivative, is the predictor-corrector scheme or more precisely, PECE (Predict, Evaluate, Correct, Evaluate) [31, 34], which represents a generalization of Adams-Bashforth-Moulton algorithm.

Classical formulation

We first recall the idea behind the classical Adams-Bashforth-Moulton algorithm for the first-order equations:

$$\begin{cases} \dot{y}(t) = f(t, y(t)), \\ y(0) = y_0. \end{cases} \tag{1.61}$$

We assume the function f to be such that a unique solution exists on some interval $[0, T]$, and we are working on a uniform grid $t_j = jh : j = 0, 1, \ldots, n$ with some integer The basic idea is, assume that we have already calculated approximations $y_j \approx y(t_j)(j = 1, 2, \ldots, k)$, that we try to obtain the approximation y_{k+1} by means of the equation

$$y(t_{k+1}) = y(t_k) + \int_{t_k}^{t_{k+1}} f(\tau, y(\tau)) f\tau. \tag{1.62}$$

This equation follows upon integration of (1.61) on the interval $[t_k, t_{k+1}]$. The integral on the right-hand side of (1.62) is then replaced by two point trapezoidal quadrature formula

$$\int_a^b g(t)dt \approx \frac{b-a}{2}(g(a) + g(b)), \tag{1.63}$$

thus giving an equation for the unknown approximation y_{k+1}; it begin

$$y_{k+1} = y_k + \frac{t_{k+1} - t_k}{2}(f(t_k, y(t_k)) + f(t_{k+1}, y(t_{k+1}))), \tag{1.64}$$

where again we have to replace $y(t_k)$ and $y(t_{k+1})$ by their approximations y_k and y_{k+1} respectively. This yield the equation for the implicit one-step Adams-Moulton method, which is

$$y_{k+1} = y_k + \frac{t_{k+1} - t_k}{2}(f(t_k, y_k) + f(t_{k+1}, y_{k+1})). \tag{1.65}$$

The problem with this equation is that the unknown quantity y_{k+1} appears on both sides, and due to the nonlinear nature of the function f, we cannot solve it for y_{k+1} directly in general. Therefore, we may use (1.65) in an iterative process, inserting a preliminary approximation for y_{k+1} in the right-hand side in order to determine a better approximation that we can then use. The preliminary approximation y^P_{k+1}, the so-called predictor, is obtained in a very similar way, only replacing the trapezoidal quadrature formula by the rectangle rule

$$\int_a^b g(t)dt \approx (b-a)g(a), \tag{1.66}$$

giving the explicit(one-step Adams-Bashforth) method

$$y^P_{k+1} = y_k + hf(t_k, y_k). \tag{1.67}$$

It is well known that the process defined by(1.67) and

$$y^P_{k+1} = y_k + \frac{h}{2}(f(t_k, y_k) + f(t_{k+1}, y^P_{k+1})), \tag{1.68}$$

known as the one-step Adams-Bashforth-Moulton technique, it is said to be the PECE (Predict, Evaluate, Correct, Evaluate) type.

Fractional formulation

We now try to carry over the essential ideas to the fractional-order problem with unavoidable modifications, for that we need to derive an equation similar to (1.62). Fortunately, such an equation is available, namely (1.51). This equation looks somewhat different from (1.62), because the range of integration now starts at 0 instead of t_k. This is a consequence of the non-local structure of the fractional-order differential operators. This however does not cause major problems in our attempts to generalize the Adams method. What we do is simply use the product trapezoibal quadrature formula to replace the integral, ie. we use the nodes (j=0,1,. . . , k+1) and interpret function $(t_{k+1} - .)^{\alpha-1}$ as a weight function for the integral. In other words, we apply the approximation

$$\int_0^{t_{k+1}} (t_{k+1} - \tau)^{\alpha-1} g(\tau) d\tau \approx \int_0^{t_{k+1}} (t_{k+1} - \tau)^{\alpha-1} \tilde{g}_{k+1}(\tau) d\tau, \tag{1.69}$$

where $\tilde{g}_{k+1}$ is the piecewise linear interpolate for g with nodes and knots chosen at the t_j, $j = 0, 1, \ldots, k+1$. We can write the integral on the right-hand side of (1.69) as

$$\int_0^{t_{k+1}} (t_{k+1} - \tau)^{\alpha-1} \tilde{g}_{k+1}(\tau) d\tau = \sum_{j=0}^{k+1} a_{j,k+1} g(t_j), \tag{1.70}$$

where

$$a_{j,k+1} = \int_0^{t_{k+1}} (t_{k+1} - \tau)^{\alpha-1} \phi_{j,k+1}(\tau) d\tau, \tag{1.71}$$

and

$$\phi_{j,k+1}(\tau) = \begin{cases} \frac{\tau - t_{j-1}}{t_j - t_{j-1}} & \text{if } t_{j-1} < \tau \leq t_j \\ \frac{t_{j+1} - \tau}{t_{j+1} - t_j} & \text{if } t_j < \tau < t_{j+1} \\ 0 & \text{else} \end{cases}. \tag{1.72}$$

An easy explicit calculation yields that for an arbitrary choice of the t_j, (1.71) and (1.72) produce

$$a_{0,k+1} = \frac{(t_{k+1} - t_1)^{\alpha+1} + t_{k+1}^{\alpha}[\alpha t_1 + t_1 - t_{k+1}]}{\alpha(\alpha+1)t_1}, \tag{1.73}$$

if $1 \leq j \leq k$

$$\begin{aligned} a_{j,k+1} &= \frac{(t_{k+1} - t_{j-1})^{\alpha+1} + (t_{k+1} - t_j)^{\alpha}[\alpha(t_{j-1} + t_j) + t_{j-1} - t_{k+1}]}{\alpha(\alpha+1)(t_j - t_{j-1})} \\ &+ \frac{(t_{k+1} - t_{j+1})^{\alpha+1} - (t_{k+1} - t_j)^{\alpha}[\alpha(t_j + t_{j+1}) - t_{j+1} + t_{k+1}]}{\alpha(\alpha+1)(t_{j+1} - t_j)}, \end{aligned}$$

and

$$a_{k+1,k+1} = \frac{(t_{k+1} - t_k)^{\alpha}}{\alpha(\alpha+1)}. \tag{1.74}$$

In the case of equispaced nodes ($t_j = jh$), these relations reduce to

$$a_{j,k+1}(\tau) = \begin{cases} \frac{h^{\alpha}}{\alpha(\alpha+1)}(k^{\alpha+1} - (k-\alpha)(k+1)^{\alpha}) & \text{if } j = 0 \\ \frac{h^{\alpha}}{\alpha(\alpha+1)}(k-j+2)^{\alpha+1} + (k-j)^{\alpha+1} - 2(k-j+1)^{\alpha+1}) & \text{if } 1 \leq j \leq k \\ \frac{h^{\alpha}}{\alpha(\alpha+1)} & \text{if } j = k+1 \end{cases}. \tag{1.75}$$

This then gives us corrector formula (the fractional variant of the one-step Adams-Moulton mathod), which is

$$y_{k+1} = \sum_{j=0}^{m-1} \frac{t_{k+1}^{j}}{j!} y_0^{(j)} + \frac{1}{\Gamma(\alpha)} (\sum_{j=0}^{k} a_{j,k+1} f(t_j, y_j) + a_{k+1,k+1} f(t_{k+1}, y_{k+1}^{p})). \tag{1.76}$$

The remaining problem is the determination of the predictor formula that we require to calculate the value y_{k+1}^{p}. That idea we use to generalize the one-step Adams-Bashforth method is the same as the one described above for the Adams-Moulton technique: We replace the integral on the right-hand side of (1.51) by the product rectangle rule

$$\int_0^{t_{k+1}} (t_{k+1} - \tau)^{\alpha-1} g(\tau) d\tau \approx \sum_{j=0}^{k} b_{j,k+1} g(t_j), \tag{1.77}$$

where now

$$b_{j,k+1} = \int_{t_j}^{t_{j+1}} (t_{k+1} - \tau)^{\alpha-1} d\tau = \frac{(t_{k+1} - t_j)^{\alpha} - (t_{k+1} - t_{j+1})^{\alpha}}{\alpha}. \tag{1.78}$$

In the equispaced case, we have the simpler expression

$$b_{j,k+1} = \frac{h^{\alpha}}{\alpha} ((k + 1 - j)^{\alpha} - (k - j)^{\alpha}). \tag{1.79}$$

Thus, the predictor y_{k+1}^{p} is determined by the fractional Adams-Bashforth method

$$y_{k+1}^{p} = \sum_{j=0}^{m-1} \frac{t_{k+1}^{j}}{j!} y_0^{(j)} + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{k} b_{j,k+1} f(t_j, y_j). \tag{1.80}$$

In the basic algorithm, the fractional Adams-Bashforth-Moulton (ABM) method, is therefore completely described now by (1.80) and (1.76) with the weights $a_{j,k+1}$ and $b_{j,k+1}$ being defined according to (1.75) and (1.78), respectively.

Chapter 2

Fractional Order Systems

This part is a review of the most important subjects in dynamical systems. The memory dependence property of the solution in fractional order systems and the question of stability for the fractional-order linear and nonlinear systems have been discussed in the first section. In the second section we present some important concepts like bifurcation and chaos of the dynamical system.

2.1 Fractional order dynamical systems

2.1.1 Memory dependency of solutions

One of the basic differences between the integer-order systems and the fractional-order systems is the dependence of the solution at time t on its memory from the starting time to t, this result is stated in the following theorem [26].

Theorem 2.1 *Let $f(t)$ satisfy the Lipschitz condition, then the solutions of the following fractional-order system are memory dependent.*

$$\begin{cases} {}_a^cD_t^\alpha x = f(x) \\ x(a) = x_a \end{cases} ; \tag{2.1}$$

It means that solution of (2.1) which is denoted by $\phi(t, x_a)$*, and the solution of*

$$\begin{cases} {}_b^c D_t^\alpha y = f(y) \\ y(b) = y_b \triangleq \phi(b, x_a) \end{cases} ; b > a \tag{2.2}$$

which is denoted by $\psi(t, y_b)$*, do not coincide for* $t \geq b$*.*

According to the theorem (2.1), the solution of fractional-order system does not satisfy the semigroup property.

2.1.2 Stability of fractional order systems

In this section we study the question of stability of solutions of fractional differential equations. The stability theory of fractional differential equations is of main interest in physical systems. Moreover, some stability results have been found [33, 35, 36, 37, 38, 39, 40]. First we consider the stability results of linear fractional differential equations, then we give these results for a general fractional differential equations.

Let us consider the following differential equation:

$$D^\alpha y(t) = f(t, y(t)), \tag{2.3}$$

where $\alpha \in (0, 1)$, $y(t) \in \mathbb{R}^N$ with $N \in \mathbb{N}$ and f is defined on a suitable subset of $\mathbb{R}^{N+1}$.

For existing of the solutions y of (2.3) on $[0, \infty)$, we consider the following assumptions:

i) The first of these assumptions is that f is defined on a set $G := [0, \infty) \times w \in \mathbb{R}^N : \|w\| < W$ with some $0 < W \leqslant \infty$. The norm in this definition of G may be an arbitrary norm on $\mathbb{R}^N$.

ii) The second assumption is that f is continuous on its domain of definition and that it satisfies a Lipschitz condition there. This asserts that the initial value problem consisting of (2.3) and the initial condition $y(0) = y_0$ has a unique solution on the interval $[0, b)$ with some $b \leqslant \infty$ if $\|y_0\| \leqslant W$.

iii) And last one assumption is that

$$f(t, 0) = 0 \text{ for all } t \geqslant 0.$$

This condition implies that the function $y(t) = 0$ is a solution of (2.3).

Definition 2.1 **a)** *The zero solution $y(t) = 0$ of the differential equation (2.3), is called stable if, for any $\epsilon > 0$ there exists some $\delta > 0$ such that the solution of the initial value problem consisting of the differential equation (2.3) and the initial condition $y(0) = y_0$ satisfies $\|y(t)\| < \epsilon$ for all $t \geqslant 0$ whenever $\|y_0\| < \delta$.*

b) *The solution $y(t) = 0$ of the differential equation (2.3), is called asymptotically stable if it is stable and there exists some $\gamma > 0$ such that $\|y(t)\| \to 0$ as $t \to +\infty$ whenever $\|y_0\| < \gamma$.*

Stability of fractional order linear systems

A necessary and sufficient condition on asymptotic stability of linear fractional differential system with order $0 < \alpha \leq 1$ was first given in 1996 Matignon [36]. Then, some literatures on the stability of linear fractional differential systems with order $0 < \alpha < 1$ have been appeared [31, 37, 38, 39, 40]. In our work we begin the analysis of stability by a very simple special case, the homogeneous linear differential equation with constant coefficients.

Theorem 2.2 *[41]*
Autonomous system:

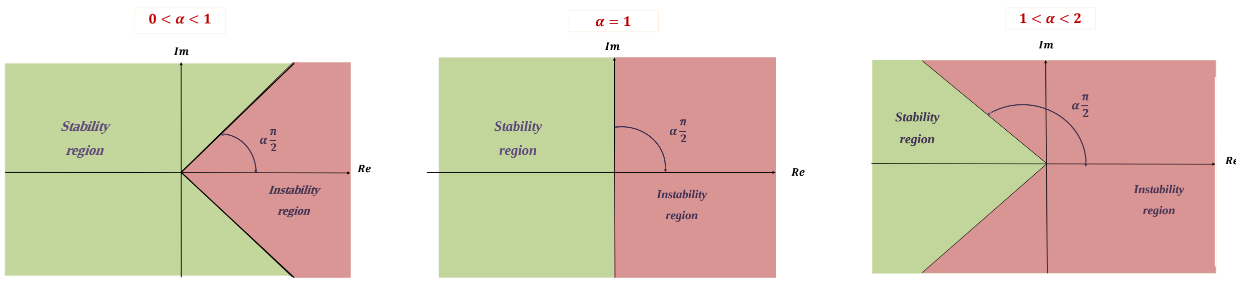
$$D^{\alpha}x(t) = Ax(t) \text{ with } x(t_0) = x_0, \tag{2.4}$$

is asymptotically stable if and only if

$$|\arg(spec(A))| > \frac{\alpha\pi}{2}, \tag{2.5}$$

where $\alpha \in [0, 1)$, $\arg(.)$ is the principal argument of a given complex number and $spec(A)$ is the spectrum (set of all eigenvalues) of A.

But not all the fractional differential systems have fractional orders in $(0, 1)$. There exist fractional models which have fractional orders lying in $(1, 2)$, for example, super-diffusion [42]. Hence, the stability of linear fractional differential systems with order $1 < \alpha < 2$ has also been studied in [35, 43].



(a) (b) (c)

Figure 2.1: Stability and instability regions for fractional-order systems .

Now, we consider the N-dimensional fractional differential equation system (2.4) such that $1 < \alpha < 2$, under the initial conditions

$$y^{(k)}(0) = y_k \ (k = 0, 1). \tag{2.6}$$

The stability result of this case is presented in the following theorem

Theorem 2.3 *[43]*
The autonomous fractional differential system (2.4) with the initial conditions (2.6) is asymptotically stable iff $|\arg(spec(A))| > \frac{\alpha\pi}{2}$. *Moreover, the system (2.4) is stable iff either it is asymptotically stable, or those critical eigenvalues which satisfy* $|\arg(spec(A))| = \frac{\alpha\pi}{2}$ *have the same algebraic and geometric multiplicities.*

Proof This theorem was proved in [43]. ■

The figure (2.1) represents stability and instability regions of the complex plane, for $\alpha \in (0, 2)$.

Stability of fractional order nonlinear systems

In this subsection we introduce some necessary definitions, before we give the linearization and the stability theorems of fractional dynamical system.
Let consider the autonomous nonlinear differential system given as follows:

$$D^{\alpha} y(t) = f(y(t)), \tag{2.7}$$

where $y(t) \in \mathbb{R}^N$, $y(0) = y_0$ and $f(y)$ is continuous.

Definition 2.2 *[44] Suppose that E is an equilibrium point of system (2.7), and that all the eigenvalues λ of the linearized matrix $Df(E)$ at the equilibrium point E satisfy: $|\lambda| \neq 0$ and $|\arg(\lambda)| \neq \frac{\alpha\pi}{2}$, then we call E an hyperbolic equilibrium point.*

Suppose $f(t)$ and $g(t)$ are continuous vector fields (defined on U, $V \subseteq \mathbb{R}^N$), and they generate flows $\phi_{t,f} : U \to U$, $\phi_{t,g} : V \to V$.

Definition 2.3 *[44] If there is an homeomorphism $h : U \to V$, satisfying: $h \circ \phi_{t,f}(y) = \phi_{t,g} \circ h(y)$, $y \in \delta(y_0, r) \subset U$, $y_0 \in U$ then $f(y)$ and $g(y)$ are locally topologically equivalent. If the above relation holds in the whole space U, then they are globally topologically equivalent.*

Let the equilibrium point E be the origin.

Theorem 2.4 *[44]*
If the origin O is an hyperbolic equilibrium point of (2.7), then vector field $f(y)$ is topologically equivalent with its linearization vector field $Df(O)y$ in the neighborhood $\delta(O)$ of the origin O.

It follows from theorem 2.3 and theorem 2.4 that the equilibrium point E of the system (2.7) is locally asymptotically stable if all eigenvalues λ of the Jacobian matrix $Df(E)$ evaluated at the equilibrium point satisfy [45]:

$$|\arg(\lambda)| > \frac{\alpha\pi}{2}.$$

As well known that in integer-order derivative, the stability of any hyperbolic equilibrium point of any dynamical system is determined by the signs of the real parts of the eigenvalues of its jacobian matrix. This result is equivalent to the algebraic procedure *Routh-Hurwitz criterion.* The Routh-Hurwitz criterion is well known for determining the stability of linear systems without involving root solving. In the chapter 3 we generalize this criterion to fractional-order systems of order $\alpha \in [0, 2)$.

2.2 Bifurcation and Chaos theories

In this section we present some important concepts like bifurcation and chaos of the dynamical system

$$\frac{dy}{dt} = f(y, \mu); \tag{2.8}$$

or the general form

$$\frac{d^\alpha y}{dt} = f(y, \mu); \quad \alpha \in \mathbb{R}^+, \tag{2.9}$$

where $y \in \Omega \subseteq \mathbb{R}^n$ and $\mu \in \mathbb{R}^r$, with the initial condition $y_0 = y(t_0)$.

2.2.1 Bifurcation theory

In the theory of dynamical systems a bifurcation is a change of the topological type of the system as its parameters pass through a bifurcation value (critical value), in this part we discuss the most important classes of bifurcations.

a) Saddle-node bifurcation

The saddle-node bifurcation is the basic mechanism by which fixed points are created and destroyed. As a parameter is varied, two fixed points move toward each other, collide, and mutually annihilate.

The prototypical example of a saddle-node bifurcation is given by the first order equation

$$\frac{dy}{dt} = \mu - y^2; \tag{2.10}$$

where μ is a real control parameter, and $y \in \mathbb{R}$.

When $\mu > 0$, there are two fixed points given by $y^*_\pm = \pm\sqrt{\mu}$ one stable and another one unstable, but for $\mu = 0$, the fixed points coalesce into a half-stable fixed point at $y^* = 0$, and there are no fixed points for $\mu < 0$, as illustrated in fig(2.2).

b) Transcritical bifurcation

The normal form of the transcritical bifurcation is

$$\frac{dy}{dt} = \mu y - y^2, \tag{2.11}$$

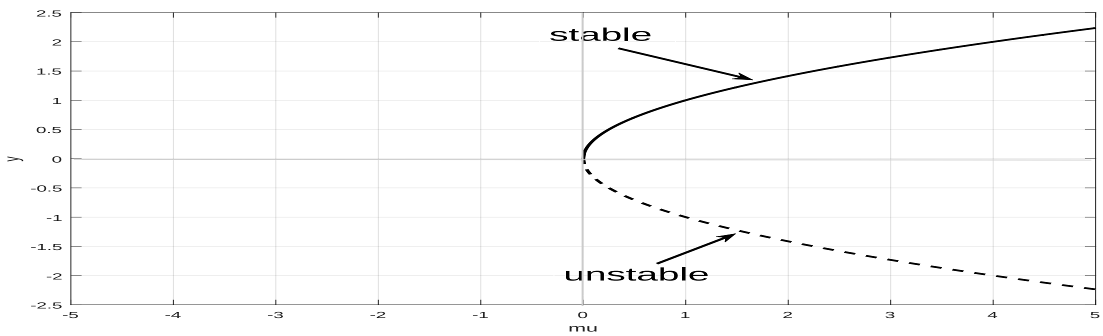


Figure 2.2: Saddle-node bifurcation diagram.

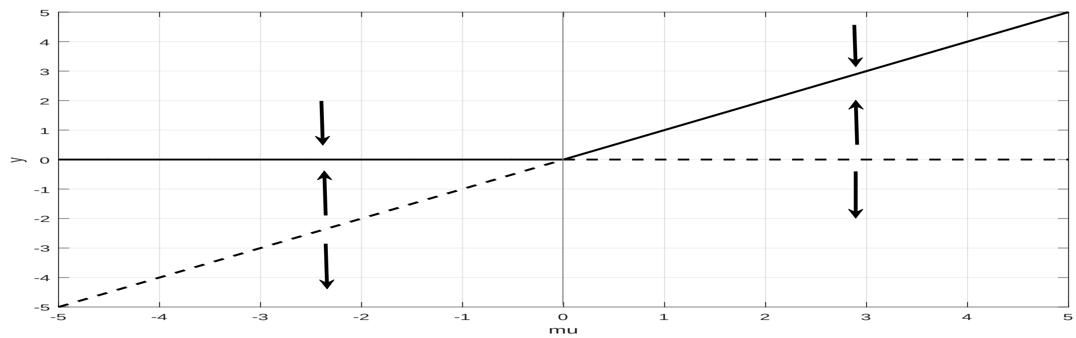


Figure 2.3: Transcritical bifurcation diagram.

where μ is a real bifurcation parameter, and $y \in \mathbb{R}$.

The two fixed points of (2.11) are $y_0^* = 0$ and $y_1^* = \mu$. For $\mu > 0$ there is a stable fixed point at $y_1^* = \mu$ and an unstable fixed point at $y_0^* = 0$, as μ increases, the unstable fixed point approaches the origin, and coalesces with it when $\mu = 0$, but for $\mu < 0$ the fixed points y_0^* and y_1^* switch their stability, as illustrated in fig(2.3).

c) Pitchfork bifurcation

There are two types of Pitchfork bifurcation, the supercritical and subcritical pitchfork bifurcations. The normal forms of the supercritical and the subcritical Pitchfork bifurcations are given respectively by:

$$\frac{dy}{dt} = \mu y - y^3, \tag{2.12}$$

and

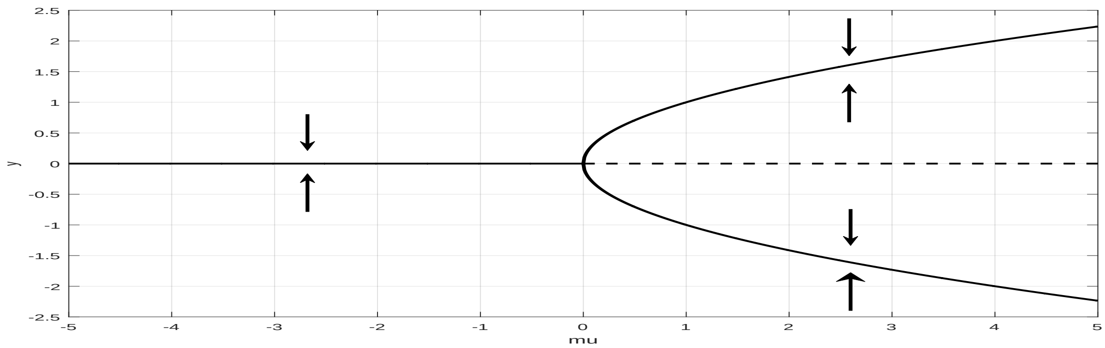


Figure 2.4: Supercritical Pitchfork Bifurcation Diagram

$$\frac{dy}{dt} = \mu y + y^3, \tag{2.13}$$

where $y \in \mathbb{R}$, and μ is a real bifurcation parameter.

The equilibrium points of (2.12) are $y_0^* = 0$ if $\mu \leqslant 0$, this equilibrium point is stable for $\mu < 0$, but if $\mu = 0$ the linearization vanishes. When $\mu > 0$, (2.12) has three equilibrium points $y_0^* = 0$ which is unstable, and $y_\pm^* = \pm\sqrt{\mu}$ which are stable, as illustrated in fig(2.4).

d) Hopf bifurcation

A Hopf bifurcation occurs when an equilibrium point of the system (2.8) changes its stability property and the system starts to expand oscillating.

Assume that $(y^*, \mu^*) \in \mathbb{R}^n \times \mathbb{R}$ is an equilibrium point of (2.8), the conditions of Hopf bifurcation are [46]:

C1) The Jacobian matrix $Df(y^*, \mu^*)$ has algebraically simple eigenvalues $\pm i\omega(\mu^*) \neq 0$ and no other eigenvalues are lying on the imaginary axis.

C2) $\theta'(\mu^*) \neq 0$,

where $\theta(\mu) \pm i\omega(\mu)$ are an eigenvalues of $Df(y^*, \mu^*)$. The two conditions C1 and C2 are satisfied meaning that there exists a unique branch of periodic orbits of the system (2.8) bifurcating from (y^*, μ^*).

2.2.2 Hopf bifurcation in fractional order systems

Let consider the system (2.8), such that $y \in \mathbb{R}^3$. As well known that the equilibrium point y^* of (2.8) is asymptotically stable if the real parts of all eigenvalues of the Jacobian matrix $Df(y^*)$ are negative, and it is unstable if there exist an eigenvalues such that it's real part positive. The conditions of system (2.8) to undergo a Hopf bifurcation at the equilibrium point y^* when $\mu = \mu^*$ are:

* $D(P_{y^*}(\mu^*)) < 0$ (it means that the Jacobian matrix of (2.8) have one real eigenvalues $\lambda_1(\mu)$ and two complex conjugate $\lambda_{2,3} = \theta(\mu) \pm i\omega(\mu)$).

* $\theta(\mu^*) = 0$ and $\lambda_1(\mu^*) \neq 0$.

* $\omega(\mu^*) \neq 0$.

* $\frac{d\theta}{d\mu}|_{\mu=\mu^*} \neq 0$.

But in the case of fractional differential system (2.9), the stability of y^* is determined by the sign of

$$m_i(\alpha, \mu) = \frac{\alpha\pi}{2} - |\arg(\lambda_i(\mu))|, \; i = 1, 2, 3.$$

If $m_i(\alpha, \mu) < 0$ for all $i = 1, 2, 3$, then y^* is locally asymptotically stable. If there exist i such that $m_i(\alpha, \mu) > 0$, then y^* is unstable. So, the function $m_i(\alpha, \mu)$ has a similar effect as the real part of eigenvalue in integer systems, therefore, the Hopf bifurcation conditions have been extended to the fractional systems by replacing $Re(\lambda_i(\mu))$ with $m_i(\alpha, \mu)$ as follows [21, 22]:

* $Df_{y^*}(\mu^*) < 0$

* $m_{2,3}(\alpha, \mu^*) = 0$ and $\lambda_1(\mu^*) \neq 0$.

* $\frac{dm}{d\mu}|_{\mu=\mu^*} \neq 0$.

2.2.3 Chaos theory

Chaos theory is one of the main themes of dynamical system theory, there are many possible definitions of chaos in dynamical system. We begin by presenting the following definitions [47]:

Definition 2.4 *$f : J \to J$ is said to be topologically transitive if for any pair of open sets $U, V \subset J$ there exists $k > 0$ such that $f^k(U) \cap V \neq \emptyset$.*

Definition 2.5 *$f : J \to J$ has sensitive dependence on initial conditions if there exists $\delta > 0$ such that, for any $x \in J$ and any neighborhood N of x, there exists $y \in N$ and $n \geq 0$ such that $|f^n(x) - f^n(y)| > \delta$.*

Definition 2.6 *A subset Y of X is called dense in X; if any point in X can be "well-approximated" by points in Y in the sense that any point in X is either an element or a limit point of Y. Equivalently, the closure of a subset Y in X is X itself.*

Definition 2.7 *Let V be a set. $f : V \to V$ is said to be chaotic on V if*

- *f is topologically transitive.*

- *f has sensitive dependence on initial conditions.*

- *Periodic points are dense in V.*

But in general there is no widely accepted definition of chaos, because this phenomenon is more a philosophical notion than a scientific notion. We can observe the phenomenon of chaos in several areas, but how to formalize it? The answer is negative because until now, there is no general theory that gives an explanation or a final characterization of this phenomenon. All that can be said is that there are several physical criteria by which to confirm that a system is chaotic.

Note that there are some definitions of chaos, but they remain restrictive, the most effective from a practical point of view is that given in [48]: Chaos can be defined as bounded steady-state behavior that is not an equilibrium point, not periodic, and not quasi-periodic.

2.2.4 Chaos quantification tools

It is not always easy to use the definition only to check for chaos. It is therefore essential to come up with other tests that are easier to use. In this section, we will consider two tests, the Lyapunov Exponent test and the $0-1$ test, which can both be used to determine the dynamics of a system.

Lyapunov exponents

Lyapunov exponents are of interest in the study of dynamical systems, which provide a qualitative and quantitative characterization of dynamical behavior, they are related to the exponentially fast divergence or convergence of nearby orbits in phase space. The signs of the Lyapunov exponents provide a qualitative picture of a system's dynamics. For example in a three-dimensional continuous dissipative dynamical system the only possible spectra, and the attractors they describe, are as follows: $(+,0,-)$ a strange attractor, $(0,0,-)$ a tow-torus, $(0,-,-)$ a limit cycle and $(-,-,-)$ a fixed point. So, a system with one or more positive Lyapunov exponents is defined to be chaotic. The Lyapunov exponents have been studied widely in many papers such as [49, 50, 51, 52, 53, 54].

Let consider a dynamical system with evolution equation

$$\dot{x}_i = f_i(x),$$

in an $N-$dimensional phase space.

The Lyapunov exponents describe the behavior of vectors in the tangent space of the phase space and are defined from the Jacobian matrix

$$J_{ij} = \frac{df_i(x(t))}{dx_j},$$

this Jacobian defines the evolution of the tangent vectors, given by the matrix Y, via the equation

$$\dot{Y} = JY,$$

The matrix Y describes how a small change at the point $x(0)$ propagates to the final point $x(t)$. The Lyapunov exponents are defined by the eigenvalues of the matrix $\frac{1}{2t}\log(Y^TY)$

$$\{\lambda_i(t)\} = \{\text{eigenvalues of } (\log(Y^T(t)Y(t))^{\frac{1}{2t}})\}. \tag{2.14}$$

The conditions for the convergence of $\log(Y^T(t)Y(t))^{\frac{1}{2t}}$ as $t \to \infty$ are given by the Oseledets theorem [49]. So, The Lyapunov exponents L_i are defined by

$$L_i = \lim_{t\to\infty} \lambda_i(t). \tag{2.15}$$

There are many algorithms for calculating Lyapunov exponents, one of the famous algorithms is the algorithm of Wolf [51], this algorithm allow the estimation of non-negative Lyapunov exponent from an experimental time series.

The steps of this algorithm are as follows:

1. Change of control parameter.
2. Random selection of an initial condition.
3. Creation of a new trajectory from the current trajectory to which we add a small disturbance.
4. Evolution in the attractor of these two neighboring trajectories and calculation of the average of the renormalized divergence between these two trajectories.
5. Readjustment of the deviation, thus allowing at each time step of the evolution of the previous point the calculation of an average of the divergence.
6. Return to step 4 performed according to a given number.
7. Return to step 1.
8. Representation of the largest Lyapunov exponent as a function of the given control parameter.

Lyapuov exponents of fractional-order systems

In this part we present the Benttin-Wolf algorithm to find all Lyapunov exponents for a class of fractional order systems. The existence of the variational equations which are necessary to determine those LEs which is ensures in the theorem (2.5), also this algorithm requires the numerical integration of differential equations of integer or fractional order, these numerical integrations are performed for example with the Adams-Bashforth-Moulton (ABM) method.

Let us consider the autonomous fractional-order system (2.7)

Theorem 2.5 *[55] System (2.7) has the following variational equations which define the LEs*

$$D^{\alpha}\phi(t) = D_y f(y)\phi(t), \ \phi(0) = I, \tag{2.16}$$

where ϕ is the matrix solution of the system (2.7), D_y is the Jacobian of f and I is the identity matrix.

The main steps of the algorithm to determine numerically all the Lyapunov exponents LEs are:

1. Numerical integration of the fractional-order system (2.7) together with the variational system (2.16), these numerical integrations are performed for example with the Adams-Bashforth-Moulton (ABM) method.

2. Gram-Schmidt procedure and picking up the exponents during the renormalization procedure.

3. The LEs begin determined as the average of the logarithm of the stretching factor of each perturbation.

These steps are presented in the following algorithm [56]

Input:

- ne (number of equations)

- y_start (initial conditions of (2.7))

- t_start, t_end (time span)

- h_norm(Normalisation step-size)

- $n\_it \leftarrow (t\_end - t\_start)/h\_norm$ (iterations number)

for $i \leftarrow ne + 1$ **to** $ne(ne + 1)$ **do**

$y(i) = 1$ (initial conditiond of (2.16))

end

$t \leftarrow t\_start$

for $i \leftarrow 1$ **to** n_it **do**

$y \leftarrow$ integration of fractional order systems (2.7)-(2.16))

$t \leftarrow t + h\_norm$

$zn(1), ..., zn(ne) \leftarrow$ Gram-Schmidt procedure

for $k \leftarrow 1$ **to** ne **do**

$s(k) \leftarrow 0$

$s(k) \leftarrow s(k) + log(zn(k))$ (vector magnitudes)

$LE(k) \leftarrow s(k)/(t - t\_start)$ (LEs)

end

end

Output: LE

The 0-1 test for validating chaos

Additional to the Lyapunov exponent test which determine if a given dynamical system is chaotic or non-chaotic, the $0 - 1$ test is an other useful test. This test proposed by Gottwald and Melbourne in [57] for integer order differential equations, the input of this test is the time series data and the output is 0 or 1, depending on whether the dynamics is chaotic or non-chaotic. The extend of the $0 - 1$ test to fractional order systems has been proposed by Cafagna and Grassi in

[58].

We first define the following real valued function

$$p(n) = \sum_{j=1}^{n} \phi(j) \cos(\theta(j)), \tag{2.17}$$

where ϕ is a one-dimensional observable data set obtained from the underlying dynamics and

$$\theta(j) = jc + \sum_{i=1}^{j} \phi(i), \quad j = 1, 2, \cdots, n \tag{2.18}$$

and $c \in \mathbb{R}^+$ is a constant chosen at random.

On the basis of the function $p(n)$, define the mean square displacement

$$M(n) = \lim_{N \to \infty} \frac{1}{N} \sum_{j=1}^{N} [p(j+n) - p(j)]^2, \quad n = 1, 2, 3, \cdots \tag{2.19}$$

The mean square displacement $M(n)$ grows linearly in time when the behaviour of $p(n)$ is Brownian (i.e. the underling dynamics is chaotic). On the other hand $M(n)$ proves to be bounded when the behaviour of $p(n)$ is regular (i.e. the underling dynamics is non-chaotic).

By defining the asymptotic growth rate

$$K = \lim_{n \to \infty} \frac{\log M(n)}{\log n} \tag{2.20}$$

The growth rate K takes either the value $K = 0$ or $K = 1$, whene $K = 0$ means that the system is regular, and $K = 1$ means that the system is chaotic.

Chapter 3

Routh-Hurwitz Conditions for Fractional-Order Systems

The stability of a hyperbolic equilibrium point of any dynamical system with integer-order derivative is determined by the signs of the real parts of the eigenvalues of its Jacobian matrix. If all the eigenvalues of the Jacobian matrix have negative real parts then this hyperbolic equilibrium point is asymptotically stable. This result is equivalent to the algebraic procedure *Routh-Hurwitz criterion.* The *Routh-Hurwitz criterion* is well known for determining the stability of linear systems of the form

$$\dot{x}(t) = Ax(t), \quad x(t) \in \mathbb{R}^n \text{ and } A \text{ is } n \times n \text{ real matrix }, \tag{3.1}$$

without involving root solving. So this criterion provides also an answer to the question of stability by considering the characteristic equation of the system, which can be written as

$$P(\lambda) = \lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + ... + a_n = 0, \tag{3.2}$$

where all the coefficients a_i are real constants.

The n Hurwitz matrices are given by

$$H_1 = (a_1), H_2 = \begin{pmatrix} a_1 & 1 \\ a_3 & a_2 \end{pmatrix}, \; H_3 = \begin{pmatrix} a_1 & 1 & 0 \\ a_3 & a_2 & a_1 \\ a_5 & a_4 & a_3 \end{pmatrix}, ...$$

$$H_n = \begin{pmatrix} a_1 & 1 & 0 & \cdots & 0 \\ a_3 & a_2 & a_1 & \cdots & 0 \\ a_5 & a_4 & a_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & a_n \end{pmatrix},$$

where $a_j = 0$ if $j > n$. All of the roots of the polynomial $P(\lambda)$ have negative real part if and only if the determinants of all Hurwitz matrices are positive, that is:

$$Det(H_j) > 0, j = 1, ..., n. \tag{3.3}$$

As in integer calculus, stability analysis is a central task in the study of fractional differential system and fractional control. Stability analysis of fractional differential equations was investigated by Matignon who produced the theorem(2.2) when the order of derivative is between 0 and 1.

This work is in fact the starting point of several results in the field. In recent papers in [59, 60, 61, 62], the authors derived some optimal *Routh-Hurwitz conditions* of the dynamical systems involving the Caputo fractional derivative of orders between 0 and 1. These new optimal *Routh-Hurwitz conditions* serve as necessary and sufficient conditions to guarantee that all roots of the characteristic polynomial obtained from the linearization process are located inside the Matignon stability sector when the order of the derivative is between 0 and 1.

If $0 < \alpha < 2$, an extension of Matignon's theorem is given in [63]. The given result permits to check the stability of any system of form given by (2.4) with $\alpha \in [0, 2)$ can be analyzed in a unified way by the location of the eigenvalues of matrix A in the complex plane. System described by (2.4) is hence asymptotically stable if and only if $|\arg(spec(A))| > \frac{\alpha\pi}{2}$, where $0 < \alpha < 2$.

In our work which referred by [23], we extend the *Routh-Hurwitz conditions* to fractional order systems of order $\alpha \in [0, 2)$. We use these results to investigate the stability properties of some population models. Numerical simulations which support our theoretical analysis are also given.

3.1 The Routh-Hurwitz conditions for fractional-order systems of order $\alpha \in [0, 1)$

First we recall that the equilibrium point of the system (2.4) for $\alpha \in (0, 1]$ is asymptotically stable if and only if the condition (2.5) is satisfied according to the Matignon's theorem (2.2). In [59, 62] some Routh-Hurwitz conditions have been generalized for $n = 1, 2, 3, 4$ to this case, the main results of these two papers are presented in the following proposition:

Proposition 3.1 *[59]*

1) *For* $n = 1$ *the condition for* (2.5) *is* $a_1 > 0$.

2) *For* $n = 2$ *the condition for* (2.5) *are either Routh-Hurwitz conditions or*

$$a_1 < 0, \quad 4a_2 > a_1^2 \text{ and } |\tan^{-1} \frac{\sqrt{4a_2 - a_1^2}}{a_1}| > \frac{\alpha\pi}{2}. \tag{3.4}$$

3) *For* $n = 3$

a) If $D(P) > 0$, *then the Routh-Hurwitz conditions are necessary and sufficient conditions for every* $\alpha \in [0, 1)$ *to have (2.5) satisfied:*

$$a_1 > 0, \; a_3 > 0 \text{ and } a_1 a_2 > a_3.$$

b) If $D(P) < 0$, *then*

i) If $a_1 \geq 0$, $a_2 \geq 0$, $a_3 > 0$ *and* $\alpha < \frac{2}{3}$, *then (2.5) is satisfied.*

ii) $a_1 < 0$, $a_2 < 0$, $a_3 > 0$ *and* $\alpha > \frac{2}{3}$, *then (2.5) is not satisfied.*

iii) If $a_1 > 0$, $a_2 > 0$, $a_1 a_2 = a_3$, *then (2.5) is satisfied for all* $\alpha \in [0, 1[$.

4) *For* $n = 4$

a) If $D(P) > 0$, $a_1 > 0$, $a_2 < 0$ *and* $\alpha > \dfrac{2}{3}$ *then (2.5) is not satisfied.*

b) If $D(P) < 0$, *then*

i) If $a_1 > 0$, $a_2 > 0$, $a_3 > 0$, $a_4 > 0$, *and* $\alpha < \frac{1}{3}$ *then (2.5) is satisfied .*

ii) If $a_1 < 0$, $a_2 > 0$, $a_3 < 0$, $a_4 > 0$, then (2.5) is not satisfied.

iii) If $a_1 > 0, a_2 > 0, a_3 > 0, a_4 > 0$ and $a_2 = \frac{a_1 a_4}{a_3} + \frac{a_3}{a_1}$, then(2.5) is satisfied, for all $\alpha \in [0, 1[$.

5) *For general n, $a_n > 0$ is a necessary conditions for (2.5).*

Proof This proposition was proved in the following section. ■

3.2 The Routh-Hurwitz conditions for fractional-order systems of order $\alpha \in [0, 2)$

Since most biological systems are $1, 2, 3$ or $4-$dimensional, we will consider only fractional-order system with dimension $n = 2, 3$ and 4.

Remark 3.1 - *For $\alpha \in [0, 1[$, the Routh-Hurwitz conditions (3.3) are sufficient but not necessary to have (2.5) satisfied.*

- *For $\alpha \in]1, 2)$, the Routh-Hurwitz conditions (3.3) are necessary but not sufficient in general case to have (2.5) satisfied.*

3.2.1 Routh-Hurwitz conditions for fractional-order two dimensional systems

Proposition 3.2

Consider the fractional linear system (2.4) with its corresponding characteristic equation (3.2). For $n = 2$, the necessary and sufficient conditions for every $\alpha \in [0, 2[$ to have (2.5) satisfied are

$$a_2 > 0 \text{ and } a_1 > -2\sqrt{a_2}\cos(\alpha\frac{\pi}{2}). \tag{3.5}$$

Proof For $n = 2$ the characteristic polynomial is

$$P(\lambda) = \lambda^2 + a_1\lambda + a_2.$$

Its discriminant is $D(P) = a_1^2 - 4a_2$.

1. If $D(P) \geq 0$ $\left(\text{i.e } a_2 \in \left] -\infty, \frac{a_1^2}{4}\right]\right)$, then $P(\lambda)$ have two real roots given by

$$\lambda_{\pm} = -\frac{1}{2}\left(a_1 \mp \sqrt{a_1^2 - 4a_2}\right).$$

For $\alpha \in [0, 2[$, we have

a) $\left(a_2 < 0 \text{ or } (a_2 \in [0, \frac{a_1^2}{4}] \text{ and } a_1 \leq -2\sqrt{a_2})\right) \Rightarrow \lambda_+ > 0$, then $\arg(\lambda_+) = 0 \leq \alpha\frac{\pi}{2}$, thus (2.5) is not satisfied.

b) $\left(a_2 \in [0, \frac{a_1^2}{4}] \text{ and } a_1 \geq 2\sqrt{a_2}\right) \Rightarrow \lambda_{\pm} < 0$, then $\arg(\lambda_+) = \pi > \alpha\frac{\pi}{2}$, thus (2.5) is satisfied.

2. If $D(P) < 0$ $\left(\text{i.e } a_2 \in]\frac{a_1^2}{4}, \infty[\right)$, then $P(\lambda)$ have two complex conjugate roots given by

$$\lambda_{\pm} = -\frac{1}{2}\left(a_1 \mp i\sqrt{4a_2 - a_1^2}\right),$$

then $\tan(\theta) = -\frac{\sqrt{4a_2 - a_1^2}}{a_1}$, where $\theta = |\arg(\lambda_{\pm})|$.

One emphasis two possibility

a) When $\alpha \in [0, 1[$ $\left(\text{i.e } \alpha\frac{\pi}{2} \in [0, \frac{\pi}{2}[\right)$ then if $\left(a_1 > 0 \text{ and } a_2 \in]\frac{a_1^2}{4}, \infty[\right)$, it follows that $\tan(\theta) < 0$, then $\theta \in]\frac{\pi}{2}, \pi[$. Therefore $\theta > \alpha\frac{\pi}{2}$. Thus, (2.5) is satisfied.

But if $-2\sqrt{a_2}\cos(\alpha\frac{\pi}{2}) < a_1 < 0$ and $a_2 \in]\frac{a_1^2}{4}, \infty[$, then $\tan(\theta) > 0$ and $\tan^2(\theta) > \tan^2(\alpha\frac{\pi}{2})$, it follows that $\tan(\theta) > \tan(\alpha\frac{\pi}{2})$. Therefore $\theta > \alpha\frac{\pi}{2}$.

Thus, (2.5) is satisfied.

On the other hand if ($a_1 < -2\sqrt{a_2}\cos(\alpha\frac{\pi}{2}) < 0$ and $a_2 \in]\frac{a_1^2}{4}, \infty[$), then ($\tan(\theta) > 0$ and $\tan^2(\theta) < \tan^2(\alpha\frac{\pi}{2})$), it follows that $\tan(\theta) < \tan(\alpha\frac{\pi}{2})$. Therefore $\theta < \alpha\frac{\pi}{2}$. Thus, (2.5) is not satisfied.

b) When $\alpha \in [1, 2[$ $\left(\text{i.e } \alpha\frac{\pi}{2} \in [\frac{\pi}{2}, \pi[\right)$, then if $\left(a_1 < 0 \text{ and } a_2 \in]\frac{a_1^2}{4}, \infty[\right)$, it follows that $\tan(\theta) > 0$, hence $\theta \in]0, \frac{\pi}{2}[$. Therefore $\theta < \alpha\frac{\pi}{2}$. Thus, (2.5) is not satisfied.

But if $\left(0 < a_1 < -2\sqrt{a_2}\cos(\alpha\frac{\pi}{2}) \text{ and } a_2 \in]\frac{a_1^2}{4}, \infty[\right)$,

then $\left(\tan(\theta) < 0 \text{ and } \tan^2(\theta) > \tan^2(\alpha\frac{\pi}{2})\right)$, it follows that $\tan(\theta) < \tan(\alpha\frac{\pi}{2})$. Therefore $\theta < \alpha\frac{\pi}{2}$. Thus, (2.5) is not satisfied.

On the other hand if ($0 < -2\sqrt{a_2}\cos(\alpha\frac{\pi}{2}) < a_1$ and $a_2 \in]\frac{a_1^2}{4}, \infty[$), then ($\tan(\theta) < 0$ and $\tan^2(\theta) < \tan^2(\alpha\frac{\pi}{2})$), it follows $\tan(\theta) > \tan(\alpha\frac{\pi}{2})$. Therefore $\theta > \alpha\frac{\pi}{2}$. Thus, (2.5) is satisfied.

Finally we summarizes the proof as follows

* From **(1-b)**, **(2-a)** and **(2-b)** we have

- if ($a_2 > 0$ and $a_1 > -2\sqrt{a_2}\cos(\alpha\frac{\pi}{2})$), then $\theta = |\arg(\lambda_\pm)| > \alpha\frac{\pi}{2}$. Thus, (2.5) is satisfied.

- When $a_2 = 0$ then $\lambda_+ = 0$. Thus, $\arg(\lambda_+)$ is not defined and (2.5) is not satisfied.

* From **(1-a)**, **(2-a)** and **(2-b)** we have

- if $\left(a_2 < 0 \text{ or } (a_2 > 0 \text{ and } a_1 \le -2\sqrt{a_2}\cos(\alpha\frac{\pi}{2}))\right)$, then $|\arg(\lambda_+)| \le \alpha\frac{\pi}{2}$. Thus, (2.5) is not satisfied.

■

3.2.2 Stability diagram and phase portraits classification for fractional-order planar systems

Consider the planar case of system (2.4), where $\alpha \in [0, 2)$. The characteristic equation of the matrix A can be written as

$$P(\lambda) = \lambda^2 - \tau\lambda + \Delta = 0.$$

where $\tau = Tr(A) = -a_1$ is the trace of the matrix A, and $\Delta = Det(A) = a_2$ its determinant.

Remark 3.2 *The conditions (3.5) to have (2.5) satisfied are equivalent to:*

$$\Delta > 0 \text{ } and \text{ } \frac{\tau}{2} < \sqrt{\Delta}.cos(\frac{\alpha\pi}{2}). \tag{3.6}$$

- *For* $0 \le \alpha < 1$, *(3.6) is equivalent to*

$$\frac{\tau^2}{4}sec^2(\frac{\alpha\pi}{2}) < \Delta. \tag{3.7}$$

- *For* $1 < \alpha < 2$, *(3.6) is equivalent to*

$$\frac{\tau^2}{4}sec^2(\frac{\alpha\pi}{2}) > \Delta > 0. \tag{3.8}$$

Using the conditions (3.7), (3.8) and taking into-account the following observations:

- For $\Delta < 0$, the two eigenvalues are real and have opposite signs; hence the fixed point is a saddle.
- For $\Delta > 0$, the eigenvalues are either real with the same sign (node point if, $\tau^2 - 4\Delta > 0$), or complex conjugate (spiral point, if $\tau^2 - 4\Delta < 0$.).
- The parabola $\tau^2 - 4\Delta = 0$. is the borderline between nodes and spirals.
- The curve of equation

$$\tau = 2\sqrt{\Delta}\cos(\alpha\frac{\pi}{2}).$$

  (i.e a branches of parabola of equation $\Delta = \frac{\tau^2}{4}sec^2(\frac{\alpha\pi}{2})$) is the borderline between stability and instability region of the fixed point in the half plane $\Delta > 0$.

We can draw the stability diagram and phase portraits classification in the (τ, Δ) plane as shown in Figure. (3.1), where the stability area is with green colour. From this figure we observe that:

- When $\alpha \to 1$ we find the same stability diagram and phase portrait classification as in the integer systems.
- The stability area for $\alpha < 1$ is wider than stability area for the integer case.
- The stability area for $\alpha > 1$ is narrower than stability area for the integer case.

3.2.3 Routh-Hurwitz conditions for fractional-order three dimensional systems

Proposition 3.3

For $n = 3$

1) *If* $D(P) > 0$*, then the Routh-Hurwitz conditions (3.3) are the necessary and sufficient conditions for every* $\alpha \in [0, 2[$ *to have (2.5) satisfied:*

$$a_1 > 0,\ a_3 > 0 \ and \ a_1a_2 > a_3.$$

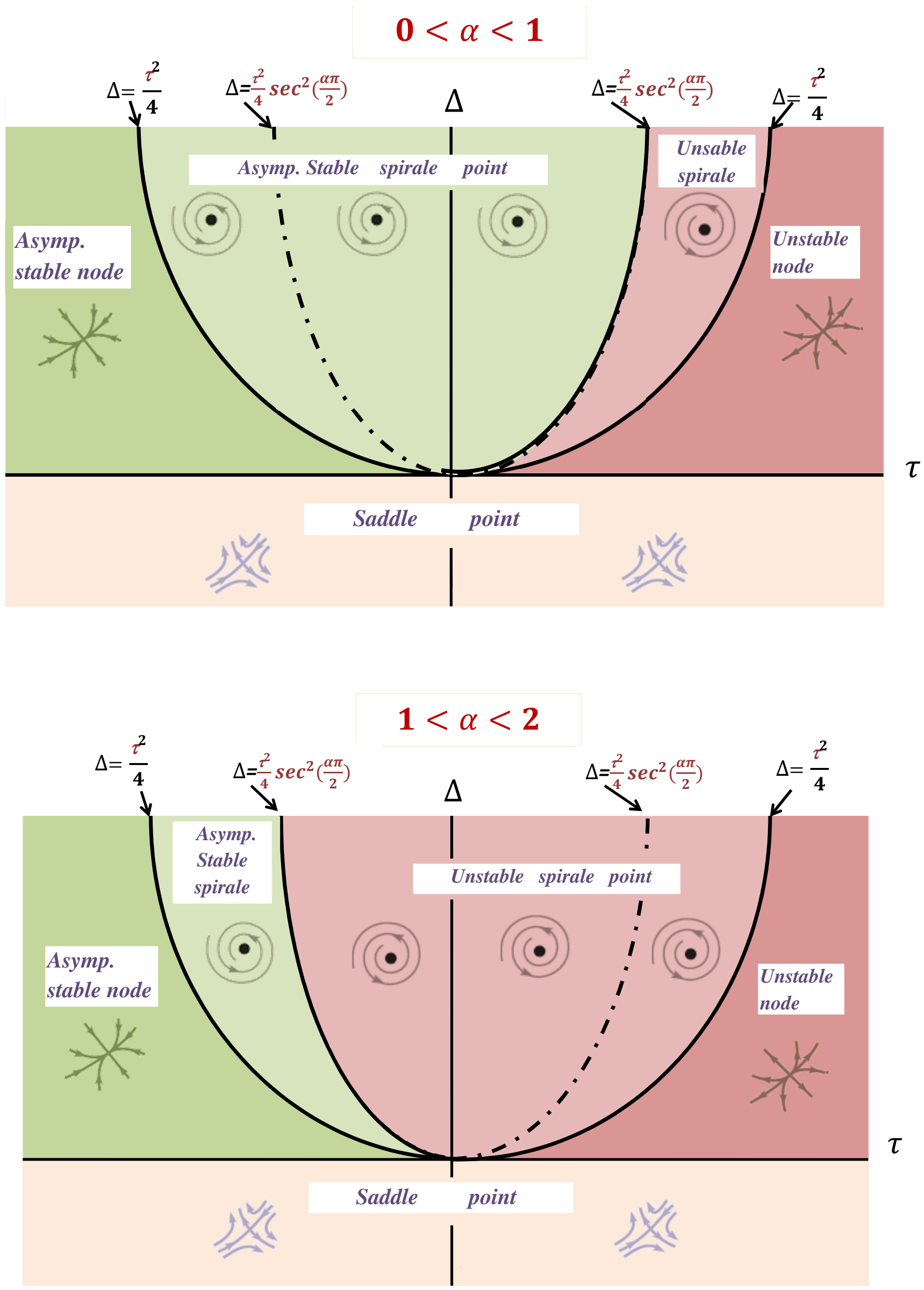


Figure 3.1: Stability diagram and phase portraits classification in the (τ, Δ)-plane for planer fractional-order system.

2) If $D(P) < 0$ and $\alpha \in [0, 2[$, then

i) If $a_1 \geq 0$, $a_2 \geq 0$, $a_3 > 0$ then we have:
If $\alpha < \frac{2}{3}$, then (2.5) is satisfied, but if $\alpha > \frac{4}{3}$, then (2.5) is not satisfied.

ii) If $a_1 > 0$, $a_2 > 0$, $a_1 a_2 = a_3$, then (2.5) is satisfied for all $\alpha \in [0, 1[$, and (2.5) is not satisfied for all $\alpha \in]1, 2[$.

Proof For $n = 3$ the characteristic polynomial is

$$P(\lambda) = \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3. \tag{3.9}$$

1) If $D(P) > 0$, then $P(\lambda) = 0$ have three real roots hence *Routh-Hurwitz conditions* are necessary and sufficient for (2.5).

2) If $D(P) < 0$, then $P(\lambda) = 0$ have one real root $\lambda_1 = -b$ and two complex conjugate roots $\lambda_{2,3} = \beta \pm i\gamma$. Thus,

$$P(\lambda) = (\lambda+b)(\lambda-\beta-i\gamma)(\lambda-\beta+i\gamma), \text{ it follow that } \begin{cases} a_1 = b - 2\beta, \\ a_2 = \beta^2 + \gamma^2 - 2b\beta, \\ a_3 = (\beta^2 + \gamma^2)b \ , \ b > 0. \end{cases}$$

i) * If $\begin{cases} a_1 \geq 0, \\ a_2 \geq 0, \end{cases}$ then $\begin{cases} b \geq 2\beta, \\ \beta^2 + \gamma^2 \geq 2b\beta \geq 4\beta^2, \end{cases}$

hence $\theta \in [\frac{\pi}{3}, \frac{2\pi}{3}]$, where $\theta = |arg(\lambda)|$.
* If $\alpha < \frac{2}{3}$, then $\theta > \frac{\alpha\pi}{2}$. But if $\alpha > \frac{4}{3}$, then $\theta < \frac{\alpha\pi}{2}$.

ii) If $a_1 a_2 = a_3$, then $\beta(\beta^2 + \gamma^2 + b^2 - 2b\beta) = 0$, hence $\beta = 0$ or $\beta^2 + \gamma^2 + b^2 - 2b\beta = 0$. The second equality is not valid, that is $\beta = 0$, then $|arg(\lambda_\pm)| = \frac{\pi}{2}$. Thus, (2.5) is satisfied for all $\alpha \in [0, 1[$, and (2.5) is not satisfied for all $\alpha \in [1, 2[$.

■

In the general case we use the following proposition.

Proposition 3.4

For $n = 3$ and $\alpha \in [0, 2)$. If $D(P) < 0$. Then, the necessary and sufficient conditions to have (2.5) satisfied are

$$\begin{cases} a_3 > 0, \\ (1 + sign(3(u+v) + 2a_1)) - sign(3(u+v) + 2a_1)\frac{2}{\pi}\left|tan^{-1}(-3\sqrt{3}\frac{u-v}{3(u+v)+2a_1})\right| > \alpha, \end{cases}$$

where

$$u = \sqrt[3]{\frac{-q + \sqrt{\frac{4}{27}p^3 + q^2}}{2}} \quad and \quad v = \sqrt[3]{\frac{-q - \sqrt{\frac{4}{27}p^3 + q^2}}{2}}, \tag{3.10}$$

with

$$p = a_2 - \frac{a_1^2}{3} \ and \ q = \frac{a_1}{27}(2a_1^2 - 9a_2) + a_3. \tag{3.11}$$

Proof If $D(P) < 0$. Then, $P(\lambda)$ has one real root λ_1 and two complex conjugate roots $\lambda_i, \ i = 2, 3$.

Substituting λ in equation (3.9) by $x - \frac{a_1}{3}$ (the Tschirnhaus transformation) we get the equation

$$x^3 + px + q = 0, \tag{3.12}$$

where p and q are given by (3.11).

The left hand side of equation (3.12) is a monic trinomial called a depressed cubic.

Any formula for the roots of a depressed cubic may be transformed into a formula for the roots of equation (3.9) using (3.11) and the relation

$$\lambda = x - \frac{a_1}{3}. \tag{3.13}$$

following Cardano's method the real root of (3.12) is given by

$$x_1 = u + v, \tag{3.14}$$

where the two variables u and v are given by (3.10). The complex roots are given by

$$x_2 = ju + \bar{j}v \text{ and } x_3 = j^2u + \bar{j}^2v, \tag{3.15}$$

where $j = e^{i\frac{2\pi}{3}} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$.

Using (3.11) and (3.13) we obtain the roots of $P(\lambda)$. Namely,

$$\lambda_1 = u + v - \frac{a_1}{3},$$
$$\lambda_2 = ju + \bar{j}v - \frac{a_1}{3} = -\frac{1}{6}(3(u+v) + 2a_1 - i3\sqrt{3}(u-v)),$$
$$\lambda_3 = j^2u + \bar{j}^2v - \frac{a_1}{3} = -\frac{1}{6}(3(u+v) + 2a_1 + i3\sqrt{3}(u-v)).$$

We have

$$P(\lambda) = (\lambda - \lambda_1)(\lambda + \frac{1}{6}(3(u+v) + 2a_1 - i3\sqrt{3}(u-v)))(\lambda + \frac{1}{6}(3(u+v) + 2a_1 + i3\sqrt{3}(u-v))),$$

it follow that $a_3 = -\lambda_1((3\sqrt{3}(u-v))^2 + (\frac{1}{6}(3(u+v)+2a_1)^2)$, then $a_3 > 0$ imply that $\lambda_1 < 0$. Thus, $|\arg(\lambda_1)| > \frac{\alpha\pi}{2}$.

On the other hand

- If $3(u+v) + 2a_1 < 0$ then $|\arg(\lambda_{2,3})| = \left|tan^{-1}(-3\sqrt{3}\frac{u-v}{3(u+v)+2a_1})\right|$, thus,
  $(1 + sign(3(u+v) + 2a_1)) - sign(3(u+v) + 2a_1)\frac{2}{\pi}\left|tan^{-1}(-3\sqrt{3}\frac{u-v}{3(u+v)+2a_1})\right| > \alpha$
  imply that $\frac{2}{\pi}\left|tan^{-1}(-3\sqrt{3}\frac{u-v}{3(u+v)+2a_1})\right| > \alpha$, then $|\arg(\lambda_{2,3})| > \frac{\alpha\pi}{2}$.
- If $3(u+v) + 2a_1 > 0$ then
  $$|\arg(\lambda_{2,3})| = \pi - \left|tan^{-1}(-3\sqrt{3}\frac{u-v}{3(u+v)+2a_1})\right|$$
  $$= \frac{(1 + sign(3(u+v) + 2a_1))}{2}\pi - sign(3(u+v) + 2a_1)\left|tan^{-1}(-3\sqrt{3}\frac{u-v}{3(u+v)+2a_1})\right|,$$
  thus, $(1 + sign(3(u+v) + 2a_1)) - sign(3(u+v) + 2a_1)\frac{2}{\pi}\left|tan^{-1}(-3\sqrt{3}\frac{u-v}{3(u+v)+2a_1})\right| > \alpha$
  imply that $|\arg(\lambda_{2,3})| > \frac{\alpha\pi}{2}$.

■

3.2.4 Routh-Hurwitz conditions for fractional-order four dimensional systems

Proposition 3.5

For $n = 4$

1. *The conditions (3.3) are sufficient conditions for the equilibrium point* x^* *to be locally asymptotically stable for all* $\alpha \in [0, 1)$, *but they are necessary conditions for all* $\alpha \in [1, 2)$.

2. *If* $D(P) > 0$, $a_1 > 0$, $a_2 < 0$ *and* $\alpha \in [\frac{2}{3}, 2]$ *then the equilibrium point* x^* *is unstable.*

3. *If* $D(P) < 0$, $a_1 > 0$, $a_2 > 0$, $a_3 > 0$, $a_4 > 0$, *then the equilibrium point* x^* *is locally asymptotically stable for all* $\alpha \in [0, \frac{1}{2}[$.

   Also, if $D(P) < 0$, $a_1 < 0$, $a_2 > 0$, $a_3 < 0$, $a_4 > 0$, *then the equilibrium point* x^* *is unstable for all* $\alpha \in [0, 2]$.

4. *If* $D(P) < 0, a_1 > 0, a_2 > 0, a_3 > 0, a_4 > 0$ *and* $a_2 = \frac{a_1 a_4}{a_3} + \frac{a_3}{a_1}$, *then the equilibrium point* x^* *is locally asymptotically stable, for all* $\alpha \in [0, 1[$ *and unstable for all* $\alpha \in]1, 2]$.

5. $a_4 > 0$ *is the necessary condition for the equilibrium point* x^* *to be locally asymptotically stable.*

Proof

1. We emphasis two cases:

   - For $\alpha \in [0, 1[$, assume that the conditions (3.3) are satisfied, then all real eigenvalues and all real parts of complex conjugate eigenvalues of Eq. (3.2) are negative, hence, these conditions (3.3) implies that all the eigenvalues of (3.2) lie in the left-half complex plane then $|\arg(\lambda_i)| > \frac{\pi}{2}$. Thus, $|\arg(\lambda_i)| > \frac{\pi}{2} > \alpha\frac{\pi}{2}$. Therefore x^* is locally asymptotically stable.

   - For $\alpha \in [1, 2]$, we have $\alpha\frac{\pi}{2} \geq \frac{\pi}{2}$. Assume that (2.5) is satisfied then $|\arg(\lambda_i)| > \alpha\frac{\pi}{2}$, implies that $|\arg(\lambda_i)| > \frac{\pi}{2}$. Therefore the asymptotic stability of x^* imply that the conditions (3.3) are satisfied.

2. Notice that if $D(P) > 0$ then there exists 4 distinct real roots r_1, r_2, r_3, r_4 or two pairs of complex eigenvalues $\lambda_{1,2} = \beta_1 \pm j\gamma_1$, and $\lambda_{3,4} = \beta_2 \pm j\gamma_2$.

In the case of real roots we have

$$\begin{aligned} a_1 &= -(r_1 + r_2 + r_3 + r_4), \\ a_2 &= r_1 r_2 + r_1 r_3 + r_1 r_4 + r_2 r_3 + r_2 r_4 + r_3 r_4, \\ a_3 &= -[r_1 r_2 r_3 + r_1 r_2 r_4 + r_1 r_3 r_4 + r_2 r_3 r_4], \\ a_4 &= r_1 r_2 r_3 r_4. \end{aligned}$$

Clearly, $a_2 < 0$ implies that at last two real roots have opposite signs. Hence the equilibrium point x^* is unstable. In the other case:

$$\begin{aligned} a_1 &= -2(\beta_1 + \beta_2), \\ a_2 &= \beta_1^2 + \gamma_1^2 + \beta_2^2 + \gamma_2^2 + 4\beta_1\beta_2, \\ a_3 &= -2[\beta_1(\beta_2^2 + \gamma_2^2) + \beta_2(\beta_1^2 + \gamma_1^2)], \\ a_4 &= (\beta_1^2 + \gamma_1^2)(\beta_2^2 + \gamma_2^2). \end{aligned}$$

We have $a_2 < 0$ imply that $\beta_2^2 \sec^2\theta + \beta_1^2 + \gamma_1^2 + 4\beta_1\beta_2 < 0$, where $\theta = |\arg \lambda_{3,4}|$. therefore $\beta_2^2 \sec^2\theta < -4\beta_1\beta_2$, which imply that $\beta_1\beta_2 < 0$ (i.e β_1 and β_2 are of opposite signs), Without loss of generality, suppose that $\beta_1 < 0,\ \beta_2 > 0$, then using the condition $a_1 > 0$, we get

$$\beta_2^2 \sec^2\theta < -4\beta_1\beta_2 < 4\beta_2^2.$$

This implies that $\theta < \pi/3$. Hence, the equilibrium point x^* is unstable for all $\alpha \in [\frac{2}{3}, 2]$.

3. If $D(P) < 0$ then there exists two real roots $\lambda_1 = r_1, \lambda_2 = r_2$, and one pair of complex eigenvalues $\lambda_{3,4} = \beta \pm j\gamma$. Then we have

$$\begin{aligned} a_1 &= -(r_1 + r_2 + 2\beta), \\ a_2 &= r_1 r_2 + \beta^2 + \gamma^2 + 2\beta(r_1 + r_2), \\ a_3 &= -2\beta r_1 r_2 - (r_1 + r_2)(\beta^2 + \gamma^2), \\ a_4 &= r_1 r_2(\beta^2 + \gamma^2). \end{aligned}$$

Assume that $a_1 > 0, a_2 > 0, a_3 > 0, a_4 > 0$ there are zero changes in sign of the coefficients of the characteristic polynomial $P(\lambda)$, then by Descartes' rule of signs, it follows that there is no positive real roots of $P(\lambda)$, this implies that $r_1 < 0$ and $r_2 < 0$. On the other hand $a_3 > 0$ implies that $2\beta r_1 r_2 + (r_1 + r_2)\beta^2 sec^2\theta < 0$.

We distinguish two cases:

1. If $\beta \leq 0$, then x^* is locally asymptotically stable for all $\alpha \in [0, 1[$, particularly for $\alpha \in [0, \frac{1}{2}[$.

2. If $\beta > 0$, then $-\frac{(r_1+r_2)}{2}\beta \sec^2\theta > r_1 r_2$ and $a_2 > 0$ implies that $r_1 r_2 > -2\beta(r_1 + r_2) - \beta^2 \sec^2\theta$, where $\theta = |\arg \lambda_{3,4}|$. Therefore, $-\frac{(r_1+r_2)}{2}\beta \sec^2\theta > -2\beta(r_1 + r_2) - \beta^2 \sec^2\theta$, then $-(r_1 + r_2)[\frac{\beta}{2}\sec^2\theta - 2\beta] > -\beta^2 \sec^2\theta$, it follow that $\beta^2 \sec^2\theta > -(r_1 + r_2)[-\frac{\beta}{2}\sec^2\theta + 2\beta]$, then $a_1 > 0$ implies that $-(r_1 + r_2) > 2\beta$, therefore $\beta^2 \sec^2\theta > 2\beta[-\frac{\beta}{2}\sec^2\theta + 2\beta]$. Thus, $\beta^2 \sec^2\theta > -\beta^2 \sec^2\theta + 4\beta^2$ which implies that $\sec^2\theta > 2$, therefore $\dfrac{\pi}{4} < \theta < \dfrac{\pi}{2}$. Then x^* is locally asymptotically stable for all $\alpha \in [0, \frac{1}{2}[$.

If the conditions $a_1 < 0, a_2 > 0, a_3 < 0, a_4 > 0$ are satisfied then there are zero changes in sign of the coefficients of the polynomial $P(-\lambda)$, then by Descartes' rule of signs, it follows that there is no positive real roots of $P(-\lambda)$, this mean that there is no negative real roots for the characteristic polynomial $P(\lambda)$, therefore $r_1 > 0$ and $r_2 > 0$. Thus, the equilibrium point x^* is unstable for all $\alpha \in [0, 2]$.

4. Notice that $D(P) < 0, a_1 > 0, a_2 > 0, a_3 > 0, a_4 > 0$ imply that there are two negative real eigenvalues, and the condition $a_2 = \frac{a_1 a_4}{a_3} + \frac{a_3}{a_1}$ implies that the two other eigenvalues are $\lambda_{3,4} = \pm i\sqrt{\frac{a_3}{a_1}}$, which lie on the imaginary axis (i.e $|\arg \lambda_{3,4}| = \frac{\pi}{2}$).
   Consequently, if $\alpha \in [0, 1[$, then all eigenvalues lie in the stable region, and if $\alpha \in]1, 2]$, then $\lambda_{3,4}$ lie on the unstable region.

5. The part (5) is proved in [60] for general n, which includes our current case.

■

Remark 3.3 • *Although the stability criteria given by inequality (2.5) with the fractional order α as the main variable, remain valid for both cases $\alpha \in [0, 1)$ and $\alpha \in (1, 2]$ the stability area in the parameter space does not remain the same as illustrated in Figure 3.1,* *where the stability region (green colour) for* $\alpha \in (1,2]$ *is restricted than the stability region for* $\alpha \in [0,1)$, *indicating a high requirement on the parameter to have stability for* $\alpha \in (1,2]$ *than for* $\alpha \in [0,1)$.

We have reported a common form of stability conditions on parameters for both cases $\alpha \in [0,1)$ *and* $\alpha \in (1,2]$, *for dimension* $n = 2$ *and* $n = 3$ *in proposition 3.2. and proposition3.4. respectively, but for* $n = 4$ *no common form where fond.*

- *Although the results presented in propositions 3.2-3.5 elaborate conditions on parameters for satisfying necessary and sufficient conditions for stability of equilibrium points, the proposed analysis is limited to restricted order of characteristic equation resulted from the Jacobian matrix.*

Remark 3.4 *The validity of Routh-Hurwitz conditions derived in [60], for fractional order differential systems, is limited to fractional order* $\alpha \in [0,1)$, *but the validity of conditions proposed in the present paper is demonstrated for fractional order* $\alpha \in [0,2)$. *Furthermore for the first time the stability diagram and phases portrait classification for fractional order planar differential systems in the* (τ, Δ) *plane are reported in the present paper.*

3.3 Applications to population dynamics

The interactions between population models either prey and predator species or epidemiological models can be predicted by simple mathematical models [64, 65, 66]. All population species posses the property of heredity which means the passing on of traits from parents to their offspring, either through asexual reproduction or sexual reproduction, the offspring cells or organisms acquire the genetic information of their parents. Through heredity. Variations between individuals can accumulate and cause species to evolve by natural selection. This property makes fractional differential equations may model more efficiently certain problems than ordinary differential equations. In this work we apply our theoretical results to three population fractional-order models. We consider some classical models existing in the literature, but modeled by a system of fractional differential equations. The first one is the fractional-order Holling-Tanner model [67], the second one is the fractional-order super-predator, predator and prey community model [68] and the last one is a Heroin epidemic model [69].

3.3.1 The fractional-order Holling-Tanner model

Example 3.1 *Let consider the fractional order Holling-Tanner model*

$$\begin{cases} D^{\alpha}x = r_1 x(1-\frac{x}{K}) - \frac{qxy}{m+x}, \\ D^{\alpha}y = r_2 y(1-\frac{y}{\gamma x}). \end{cases} \tag{3.16}$$

Where $\alpha \in [0,2)$, $x(t) \geq 0$ and $y(t) \geq 0$ are the density of prey and predator populations at time t respectively. The parameters r_1 and r_2 are the intrinsic growth rates, K represents the carrying capacity of the prey, q is the maximum number of prey that can be eaten per predator per unit of time, m is a saturation value, it corresponds to the number of prey necessary to achieve one half the maximum rate q, γ is a measure of the quality of the prey as a food for the predator. For example for $r_1 = 1,\ r_2 = 0.2,\ K = 7,\ q = \frac{6}{7},\ m = 1$ and $\gamma = 0.4$, the system (3.16) has two equilibrium points $E_0 = (7,0)$ and $E_1 = (5,2)$.

- *The characteristic polynomial of the Jacobian matrix evaluated at E_0 is given by*

$$P(\lambda) = \lambda^2 + 0.8\lambda - 0.2.$$

So $a_2 = -0.2 < 0$, then according to Proposition(3.2) E_0 is unstable for all $\alpha \in [0,2)$.

- *The characteristic polynomial of the Jacobian matrix evaluated at E_1 is given by*

$$P(\lambda) = \lambda^2 + \frac{71}{105}\lambda + \frac{16}{105}.$$

So $a_1 = \frac{71}{105}$ and $a_2 = \frac{16}{105} > 0$, according to Proposition (3.2) the critical value of α is

$$\alpha_c = \frac{2}{\pi} cos^{-1}(\frac{-a_1}{2\sqrt{a_2}}) \approx 1.6668.$$

Then E_1 is locally asymptotically stable for all $\alpha < \alpha_c$, Figure (3.2), illustrate these results. We observe that for $\alpha = 1.5$ and for $\alpha = 1.66$ the trajectory initiated near E_1 spiral toward E_1, which is locally asymptotically stable for all fractional order $\alpha < \alpha_c$, but for $\alpha = 1.67$

and $\alpha = 1.7$ the trajectories initiated near E_1 are repulsed by E_1 which is unstable for $\alpha > \alpha_c$. Particularly for α not too far from α_c the trajectories spiral toward an $S-$asymptotically periodic solution of (3.16) [70, 71], giving rise to a periodic behavior of the model.

3.3.2 The fractional-order super-predator, predator and prey community model

Example 3.2 *The fractional-order super-predator, predator and prey community model introduced in [68] by*

$$\begin{cases} D^\alpha x = x(\rho - \omega y), \\ D^\alpha y = y(-\mu + \beta x - \gamma z), \\ D^\alpha z = z(1-z) + \delta y z. \end{cases} \tag{3.17}$$

Where $\alpha \in [0,2)$, $x \geq 0$, $y \geq 0$ and $z \geq 0$ are the biomass densities of prey, predator and super-predator respectively. All parameters of the model are positive and constant values. The equilibrium point of (3.17) is $E^* = (x^*, y^*, z^*)$, such that:

$$x^* = \frac{\mu}{\beta} + \frac{\gamma}{\beta}(1 + \frac{\delta\rho}{\omega}),\ y^* = \frac{\rho}{\omega},\ z^* = 1 + \frac{\delta\rho}{\omega}.$$

The characteristic polynomial of the Jacobian matrix of (3.17) at E^* is

$$P(\lambda) = \lambda^3 + z^*\lambda^2 + (\gamma\delta z^* + \omega\beta x^*)y^*\lambda + \omega\beta x^* y^* z^*.$$

We have

$$\begin{cases} a_1 = z^* > 0, \\ a_2 = (\gamma\delta z^* + \omega\beta x^*)y^* > 0, \\ a_3 = \omega\beta x^* y^* z^* > 0. \end{cases}$$

If $D(P) > 0$, we have $a_1 a_2 > a_3$, by Proposition (3.3), we have the local asymptotic stability of E^* for all $\alpha \in [0, 2[$.

If $D(P) < 0$, then according to the Proposition (3.3), E^* is locally asymptotically stable for all $\alpha < \frac{2}{3}$ and it is unstable for all $\alpha > \frac{4}{3}$, as shown in Figure (3.3), where for $\alpha = 0.66 < \frac{2}{3}$ the trajectory starting near E^* is attracted by it indicating local asymptotic stability, but for $\alpha = 1.34 > \frac{4}{3}$ the trajectory starting near E^* is repulsed by it indicating its instability. two values

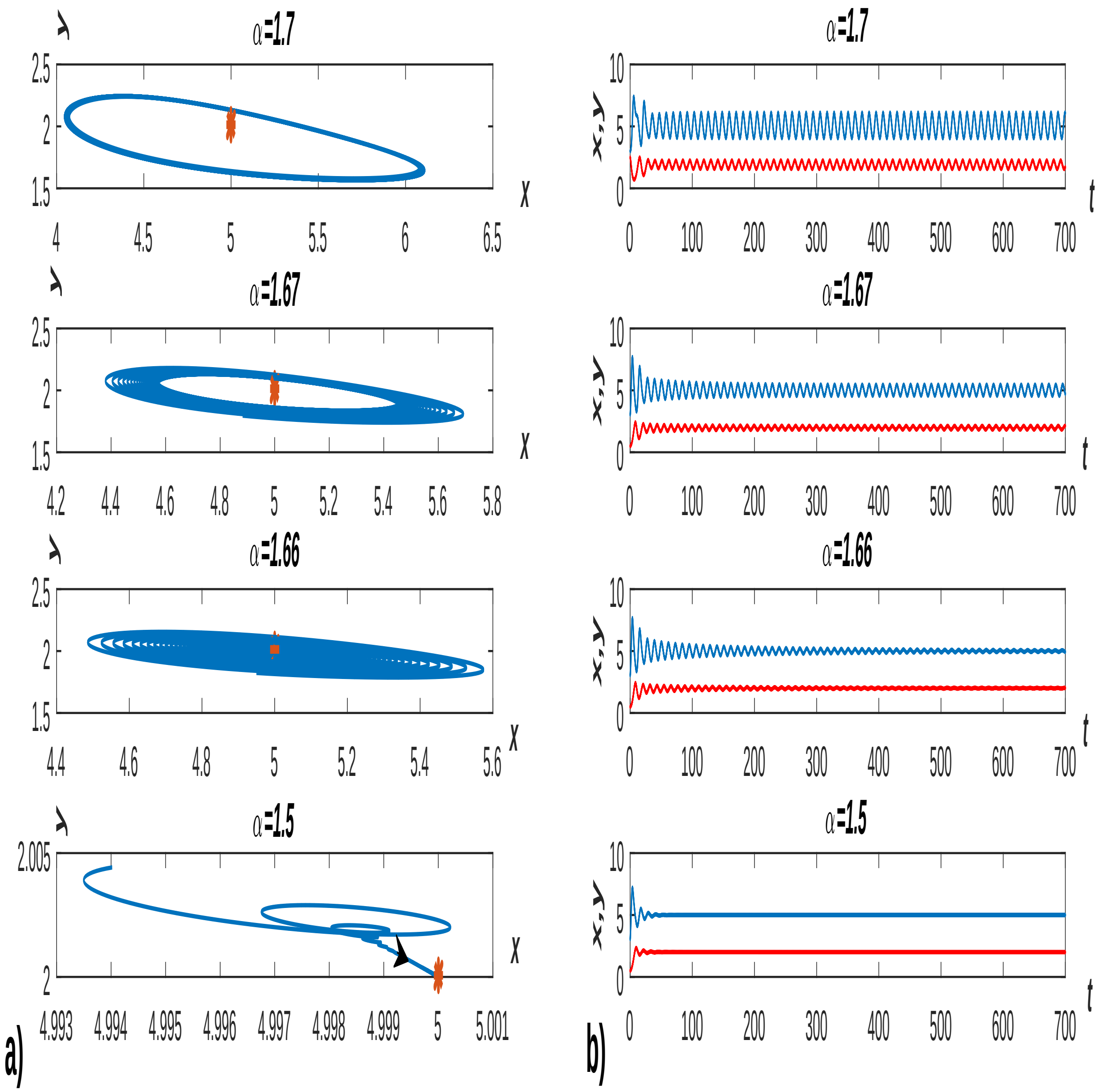


Figure 3.2: **a)** Phase portrait and **b)** Time evolutions of system (3.16) for some values of α with the parameter values $r_1 = 1,\ r_2 = 0.2,\ K = 7,\ q = \frac{6}{7},\ m = 1$ and $\gamma = 0.4$.

of the fractional order α.

For $\alpha \in [\frac{2}{3}, \frac{4}{3}]$, *we use the Proposition (3.4), for example for* $\omega = 1$, $\beta = 2$, $\mu = 1$, $\gamma = 1$, $\rho = 4$ *and* $\delta = 3$. *We have* $D(P) = -15109584 < 0$, $u = 7.2936$ *and* $v = -7.1142$.

The critical value of α *is*

$$\alpha_c = \pi - \frac{2}{\pi}|tan^{-1}(\frac{-3\sqrt{3}(u-v)}{3(u+v)+2a_1})| \approx 1.2169.$$

Thus the equilibrium point E^* *is locally asymptotically stable for all* $\alpha < \alpha_c$, *as illustrated in Figure (3.3), where for* $\alpha = 1.21 < \alpha_c$ *the trajectory starting in the vicinity of* E^* *is attracted by it which confirm that* E^* *is locally asymptotically stable, but for* $\alpha = 1.22 > \alpha_c$ *the trajectory starting in the vicinity of* E^* *is repulsed by it indicating its instability.*

3.3.3 The fractional-order Heroin epidemic model

Example 3.3 *Let consider the following fractional-order Heroin epidemic model of four sub-population [69], with susceptibles* x, *heroin users not in treatment* y, *heroin users undergoing treatment* z *and heroin users who have been successfully treated from heroin use* w:

$$\begin{cases} D^\alpha x = \Lambda - \beta y x - \mu x, \\ D^\alpha y = \beta y x + \rho y - (\mu + \delta_1 + \xi) y - \dfrac{\kappa y}{1 + \omega y}, \\ D^\alpha z = \dfrac{\kappa y}{1 + \omega y} - (\rho + \sigma + \delta_2 + \mu) z, \\ D^\alpha w = \sigma z + \xi y - \mu w. \end{cases} \tag{3.18}$$

Where $\alpha \in [0, 2)$, *and all parameters of the model are positive.*

The system (3.18) has two equilibrium points $E = (\frac{\Lambda}{\mu}, 0, 0, 0)$ *and* $E^* = (x^*, y^*, z^*, w^*)$, *such that*

$$\begin{cases} x^* = \dfrac{\Lambda}{\beta y^* + \mu}, \\ z^* = \dfrac{\kappa y^*}{Q_0 Q_2}, \\ w^* = \dfrac{\sigma \kappa y^* + \xi Q_0 Q_2 y^*}{\mu Q_0 Q_2}. \end{cases}$$

and y^* *is solution of the second order equation*

$$A y^{*2} + B y^* + C = 0,$$

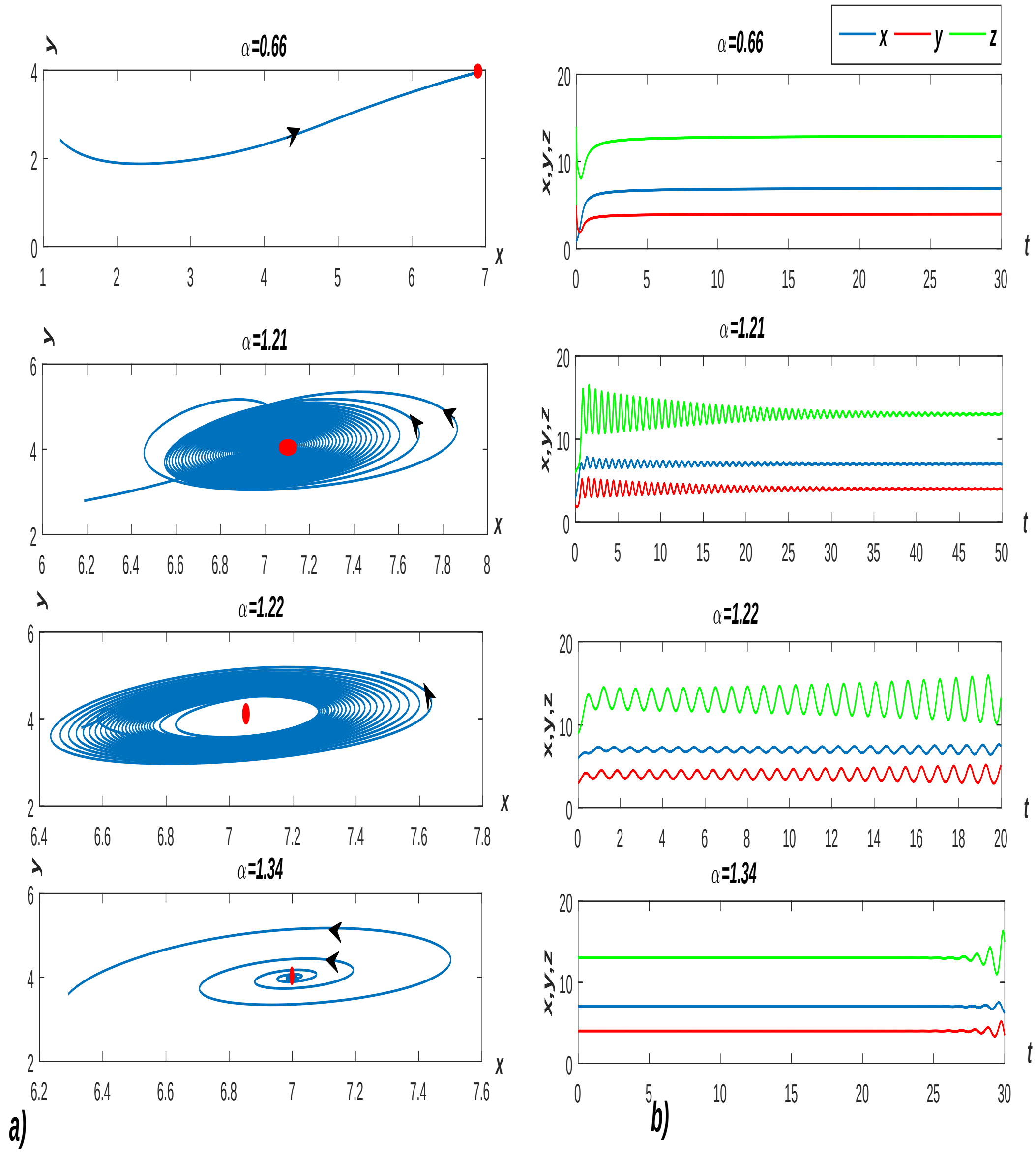


Figure 3.3: **a)** Phase portrait and **b)** Time evolutions of system (3.17) for some values of α, with the parameter values $\omega = 1$, $\beta = 2$, $\mu = 1$, $\gamma = 1$, $\rho = 4$ and $\delta = 3$

where

$$\begin{cases} A = \beta\omega Q_1 Q_2, \\ B = (\beta + \mu\omega)Q_1 Q_2 + \kappa\beta(Q_2 - \rho) - \beta\Lambda, \\ C = [\mu Q_1 Q_2 + \kappa\mu(Q_2 - \rho)](1 - R_0). \end{cases}$$

And

$$\begin{cases} Q_0 = 1 + \omega y^*, \\ Q_1 = \mu + \delta_1 + \xi, \\ Q_2 = \rho + \sigma + \delta_2 + \mu, \\ R_0 = \dfrac{\kappa\beta Q_2}{\mu Q_1 Q_2 + \kappa\mu(Q_2 - \rho)}. \end{cases}$$

The characteristic polynomial of the Jacobian matrix of (3.18) at E is

$$P_E(\lambda) = (\lambda + \mu)^2(\lambda^2 + p_1\lambda + p_2).$$

Where

$$\begin{cases} p_1 = Q_1 + Q_2 + \kappa - \frac{\beta\Lambda}{\mu}, \\ p_2 = Q_2(Q_1 + \kappa - \frac{\beta\Lambda}{\mu}) - \kappa\rho. \end{cases}$$

The characteristic polynomial of the Jacobian matrix of (3.18) at E^* is

$$P_{E^*}(\lambda) = \lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4.$$

Where

$$\begin{cases} a_1 = \mu + I_1 + I_2, \\ a_2 = I_1 I_2 + I_3 - I_4 + \mu(I_1 + I_2), \\ a_3 = I_1 I_3 - Q_2 I_4 + \mu(I_1 I_2 + I_3 - I_4), \\ a_4 = \mu(I_1 I_3 - Q_2 I_4), \end{cases}$$

and

$$\begin{cases} I_1 = \beta y^* + \mu, \\ I_2 = Q_1 + Q_2 + \frac{\kappa}{Q_0^2} - \beta x^*, \\ I_3 = Q_2(Q_1 + \frac{\kappa}{Q_0^2} - \beta x^*) - \frac{\kappa\rho}{Q_0^2}, \\ I_4 = \beta^2 x^* y^*. \end{cases}$$

For example we use the following parameter values $\Lambda = 4.434486182758694$, $\kappa = 0.5$, $\beta = 0.001185$, $\omega = 0.1654$, $\sigma = 20$, $\mu = 0.0099909$, $\rho = 0.001$, $\delta_1 = 0.001$, $\delta_2 = 0.002$, $\xi =$

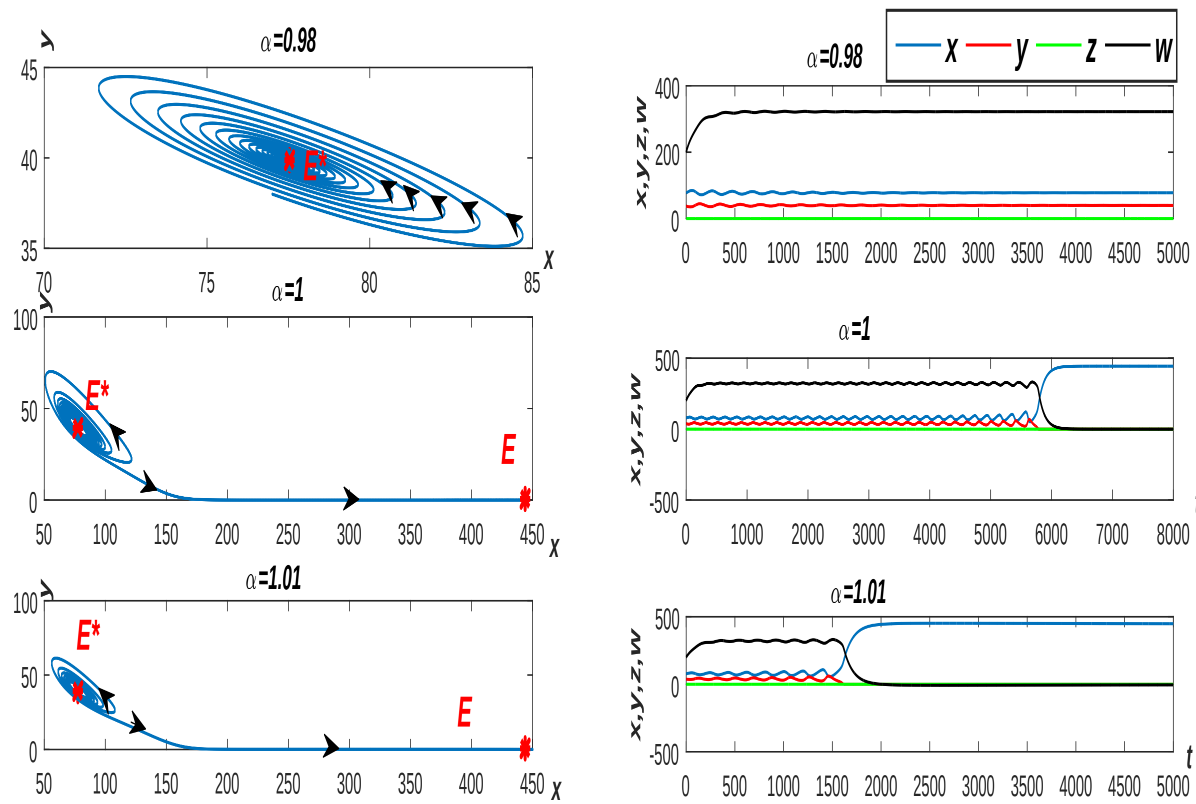


Figure 3.4: **a)** Phase portrait and **b)** Time evolutions of system (3.18) for some values of α, with the parameter values $\Lambda = 4.434486182758694$, $\kappa = 0.5$, $\beta = 0.001185$, $\omega = 0.1654$, $\sigma = 20$, $\mu = 0.0099909$, $\rho = 0.001$, $\delta_1 = 0.001$, $\delta_2 = 0.002$, $\xi = 0.014999324798155$.

0.014999324798155. *We have:*

$D(P_E) > 0$, $p_1 > 0$ *and* $p_2 > 0$, *it means that all the roots of* $P_E(\lambda) = 0$ *are real negative, then* E *is locally asymptotically stable for all* $\alpha \in [0, 2)$.

$D(P_{E^*}) < 0$, $a_i > 0$ *for all* $i = 1, 2, 3, 4$ *and* $a_2 = \frac{a_1 a_4}{a_3} - \frac{a_3}{a_1}$, *then according to proposition (3.5)* E^* *is locally asymptotically stable for all* $\alpha \in [0, 1[$ *and unstable for all* $\alpha \in]1, 2]$, *Figure (3.4) illustrates these results, where we observe that for* $\alpha \in [0, 1[$ *all trajectory initiated near* E^* *converge to it but for* $\alpha \in [1, 2]$ *all trajectory initiated near* E^* *are repulsed by it and attracted by* E *which is locally asymptotically stable for all* $\alpha \in [0, 2]$.

Remark 3.5 *Assume that a* $3-D$ *integer-order system displays a chaotic attractor and suppose* *that Ω is the set of equilibrium points surrounded by scrolls. A necessary condition for the corresponding fractional order system to exhibit a chaotic attractor similar to its integer order counterpart is instability of the equilibrium points in Ω. Otherwise, one of these equilibrium points becomes asymptotically stable and attracts the nearby trajectories [72, 73]. The proposed stability conditions are a powerful tool for determining regions of possible chaos (instability region) in the parameters space (including fractional order) where chaotic phenomenon can be developed. Different figures of the presented examples show variation of state evolution (from stationary to periodic and divergent) as value of fractional order α changes indicating possibility of chaos.*

Chapter 4

Periodic Solutions of Fractional Order Systems

The existence of periodicity properties in fractional-order derivatives are one of the main issues in qualitative theory of differential equations. this subject has attracted the attention of many mathematicians, including (Tavazoei, Haeri, Yasdani, Belmekki, Kaslik, Wang, Abdelouahab) in [24, 25, 26, 74, 75, 76, 27].

4.1 Fractional order derivatives of periodic functions and periodic solutions

4.1.1 Fractional order derivatives of periodic functions

The following theorems reveals a remarkable property for the fractional derivatives defined based on Caputo definition, Grünwald-Letnikov definition, Riemann-Liouville definition [24].

Theorem 4.1 *Suppose that $f(t)$ is a non constant periodic function with period T. If $f(t)$ is n-times differentiable, then the functions ${}_{a}^{C}D_{t}^{\alpha} f(t)$, where $0 < \alpha \notin N$ and n is the first integer greater than α, cannot be a periodic functions with period T.*

Theorem 4.2 *Suppose that $f(t)$ is $(n-1)$-times continuously differentiable and $f^{(n)}(t)$ is bounded. If $f(t)$ is a non-constant periodic function with period T, then the functions ${}_a^{GL}D_t^{\alpha} f(t)$ and ${}_a^{RL}D_t^{\alpha} f(t)$, where $0 < \alpha \notin N$ and n is the first integer greater than α, cannot be periodic functions with period T.*

The demonstrations of these Theorems ((4.1), (4.2)) can be found in [24].

Example 4.1 *Let $f(t) = \sin(t)$. We have*

$$\sin(t) = \sum_{p=0}^{\infty} (-1)^p \frac{t^{2p+1}}{(2p+1)!}.$$

So

$${}_a^{RL}D_t^{\alpha} \sin(t) = t^{1-\alpha} E_{2,2-\alpha}(-t^2).$$

Where $0 < \alpha < 1$ and $E_{\alpha,\beta}(t)$ is the generalized Mittag-Leffler function defined by

$$E_{\alpha,\beta}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\alpha k + \beta)}.$$

Numerical simulations show that $t^{1-\alpha}E_{2,2-\alpha}(-t^2)$ is not a periodic function where $0 < \alpha < 1$, even though for $\alpha = 1$ this function equals the periodic function $\cos(t)$.

4.1.2 Non existence of periodic solutions

From Theorems (4.1) and (4.2), we have the following corollary

Corollaire 4.1 *A differential equation of fractional-order in the form*

$${}_a^{\cdot}D_t^{\alpha} x(t) = f(x(t)), \tag{4.1}$$

where $0 < \alpha \notin \mathbb{N}$, cannot have any non-constant smooth periodic solution.

Proof Suppose $\tilde{x}(t)$ is a non-constant periodic solution with period T of (4.1), then

$$f(\tilde{x}(t)) = f(\tilde{x}(t+T)), \tag{4.2}$$

for all $t \geq 0$.

From (4.1) and (4.2), it is deduced that ${}_a D_t^{\alpha} \tilde{x}(t) = {}_a D_t^{\alpha} \tilde{x}(t+T)$ for all $t \geq 0$.

But according to Theorems (4.1) and (4.2), this equality is impossible, So $\tilde{x}(t)$ cannot be a non-constant periodic solution.

■

4.1.3 Existence of periodic solutions

The existence of periodic solutions is the basic fact in this subsection. in [26] the authors has been proved the existence of periodic solutions in fractional-order systems under some circumstances.

Theorem 4.3 *The fractional-order system*

$$\begin{cases} {}_a^c D_t^{\alpha} x = f(x) \\ x(a) = x_a \end{cases} ; \tag{4.3}$$

does not have any periodic solution unless the lower terminal of the derivative is $\pm\infty$ *(*$a = \pm\infty$*.)*

Proof Suppose $\phi(t, x_a)$ is a non-constant periodic solution with period T of (4.3), then

$$f(\phi(t, x_a)) = x_a + \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\phi(\tau, x_a)) d\tau, \tag{4.4}$$

and

$$f(\phi(t+T, x_a)) = x_a + \frac{1}{\Gamma(\alpha)} \int_a^{t+T} (t+T-\tau)^{\alpha-1} f(\phi(\tau, x_a)) d\tau, \tag{4.5}$$

$\phi(t, x_a)$ is a periodic solution and since the system (4.3) is autonomous it mains that

$$f(\phi(\tau+T, x_a)) = f(\phi(\tau, x_a)). \tag{4.6}$$

Putting $\tau^* = \tau - T$ and performing obvious substitutions of variables in (4.5), and by subtracting $\phi(t+T, x_a)$ from $\phi(t, x_a)$, we obtain

$$\phi(t, x_a) - \phi(t+T, x_a) = \frac{1}{\Gamma(\alpha)} \int_{a-T}^{a} (t-\tau)^{\alpha-1} f(\phi(\tau, x_a)) d\tau = 0, \tag{4.7}$$

While $f(\phi(\tau,x_a)) \neq 0$, it can be concluded that the limits of integral are equal (ie $a = a - T$) But $T \neq 0$, then $a = \pm\infty$.

■

4.2 Fractional order derivative with fixed memory length

4.2.1 The Grünwald-Letnikov fractional order derivative with fixed memory length

We first recall the Grünwald-Letnikov fractional-order derivative with fixed memory length introduced in [27].

Definition 4.1 *Let $\alpha \geq 0$, $L > 0$, n an integer such that $n-1 \leq \alpha < n$ and f an integrable function in the interval $[a-L, b]$. The operator ${}^{MG}_{\ L}D^{\alpha}_{t}$ defined by :*

$$ {}^{MG}_{\ L}D^{\alpha}_{t} f(t) = \lim_{h\to 0} \frac{1}{h^{\alpha}} \sum_{k=0}^{\frac{L}{h}} (-1)^{k} \frac{\Gamma(\alpha+1)}{k!\Gamma(\alpha-k+1)} f(t-kh), \quad t \in [a,b], \tag{4.8} $$

is called the Grünwald-Letnikov fractional derivative with fixed memory length.

The following proposition gives an evaluation of the limit in the definition of Grünwald-Letnikov fractional derivative with fixed memory length.

Proposition 4.1 *Under the assumptions of definition 4.1. if the function f is n-differentiable with $f^{(n)} \in L_1[a-L,b]$, then*

$$ {}^{MG}_{\ L}D^{\alpha}_{t} f(t) = \sum_{k=0}^{n-1} \frac{f^{(k)}(t-L)L^{k-\alpha}}{\Gamma(k-\alpha+1)} + \frac{1}{\Gamma(n-\alpha)} \int_{t-L}^{t} (t-\tau)^{n-\alpha-1} f^{(n)}(\tau)d\tau. \tag{4.9} $$

It have been demonstrated that this modified fractional-order derivative posses two useful properties: the first is the preservation of periodicity and the second one is the short memory, which reduces considerably the cost of numerical computations. Furthermore they have proven that contrary to fractional autonomous systems in term of classical fractional derivative, the fractional autonomous systems in term of the modified fractional derivative can generate exact periodic solutions.

In the following subsections we introduce a similar modification of the Caputo fractional-order derivative and the Riemann-Liouville fractional-order derivative.

4.2.2 The Caputo and the Riemann-Liouville fractional order derivatives with fixed memory length

Definition 4.2 *(The Caputo fractional derivative with fixed memory length) Let* $\alpha > 0$, $L > 0$, n *an integer such that* $n = \lceil \alpha \rceil$ *and* $f \in C^n[a-L, b]$. *We define the Caputo fractional derivative with fixed memory length by*

$$ {}_{L}^{MC}D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int\limits_{t-L}^{t} (t-\tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau. \tag{4.10}$$

Definition 4.3 *(The Riemann-Liouville fractional derivative with fixed memory length) Let* $\alpha \geq 0$, $L > 0$, n *an integer such that* $n-1 \leq \alpha < n$ *and* f *is a continuous function in* $[a-L, b]$, *we define the Riemann-Liouville fractional derivative with fixed memory length by:*

$$ {}_{L}^{MRL}D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_{t-L}^{t} (t-\tau)^{n-\alpha-1} f(\tau) d\tau, \tag{4.11}$$

Remark 4.1 *From (4.9) and (4.10) we get*

$$ {}_{L}^{MC}D_t^{\alpha} f(t) = {}_{L}^{MG}D_t^{\alpha} f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(t-L) L^{k-\alpha}}{\Gamma(k-\alpha+1)}. \tag{4.12}$$

Proposition 4.2 *Under the assumption that the function* $f(t)$ *is* n*-times continuously differentiable, we get*

$$ {}_{L}^{MRL}D_t^{\alpha} f(t) = {}_{L}^{MG}D_t^{\alpha} f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(t-L) L^{k-\alpha}}{\Gamma(k-\alpha+1)}. \tag{4.13}$$

Proof By differentiation and performing repeatedly integration by parts, we get

$$\begin{aligned}
{}^{MRL}_{\ L}D_t^{\alpha}f(t) &= \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_{t-L}^{t}(t-\tau)^{n-\alpha-1}f(\tau)d\tau,\\
&= -\frac{f^{(n-1)L^{n-\alpha-1}}(t-L)}{\Gamma(n-\alpha)} + \frac{1}{\Gamma(n-\alpha-1)}\frac{d^{n-1}}{dt^{n-1}}\int_{t-L}^{t}(t-\tau)^{n-\alpha-2}f(\tau)d\tau,\\
&\vdots\\
&= -\sum_{k=0}^{n-1}\frac{f^{(k)}(t-L)L^{k-\alpha}}{\Gamma(k-\alpha+1)} + \frac{1}{\Gamma(-\alpha)}\int_{t-L}^{t}(t-\tau)^{-\alpha-1}f(\tau)d\tau,
\end{aligned}$$

we pot $I = \frac{1}{\Gamma(-\alpha)}\int_{t-L}^{t}(t-\tau)^{-\alpha-1}f(\tau)d\tau$, performing a successive integration by part we obtain

$$\begin{aligned}
I &= \frac{f(t-L)L^{-\alpha}}{\Gamma(1-\alpha)} + \frac{1}{\Gamma(1-\alpha)}\int_{t-L}^{t}(t-\tau)^{-\alpha}f'(\tau)d\tau,\\
&= \frac{f(t-L)L^{-\alpha}}{\Gamma(1-\alpha)} + \frac{f'(t-L)L^{1-\alpha}}{\Gamma(2-\alpha)} + \frac{1}{\Gamma(2-\alpha)}\int_{t-L}^{t}(t-\tau)^{-\alpha+1}f^{(2)}(\tau)d\tau,\\
&\vdots\\
&= \sum_{k=0}^{n-1}\frac{f^{(k)}(t-L)L^{k-\alpha}}{\Gamma(k-\alpha+1)} + \frac{1}{\Gamma(n-\alpha)}\int_{t-L}^{t}(t-\tau)^{-\alpha+n-1}f^{(n)}(\tau)d\tau,\\
&= {}^{MG}_{\ L}D_t^{\alpha}f(t).
\end{aligned}$$

Thus

$${}^{MRL}_{\ L}D_t^{\alpha}f(t) = {}^{MG}_{\ L}D_t^{\alpha}f(t) - \sum_{k=0}^{n-1}\frac{f^{(k)}(t-L)L^{k-\alpha}}{\Gamma(k-\alpha+1)}.$$

■

Remark 4.2 *From (4.12) and (4.13) we have*

$${}^{MRL}_{\ L}D_t^{\alpha}f(t) = {}^{MC}_{\ L}D_t^{\alpha}f(t) = {}^{MG}_{\ L}D_t^{\alpha}f(t) - \sum_{k=0}^{n-1}\frac{f^{(k)}(t-L)L^{k-\alpha}}{\Gamma(k-\alpha+1)}. \tag{4.14}$$

In the following parts, we denote the operators of Caputo and Riemann-Liouville fractional derivative with fixed memory length by ${}^{M}_{L}D_t^{\alpha}$.

4.3 Fractional order derivative with fixed memory length of some functions

In this section, we give the fractional derivatives with fixed memory length of some basic functions periodic functions. The comparison between previous results of fractional order derivatives with fixed memory length and classical fractional-order derivatives are proposed in the second subsection

4.3.1 Fractional derivative of some elementary functions

In the following, we give the fractional derivative with fixed memory length of two elementary functions (the power function and the exponential function).

Fractional derivative of the power function

Let $f(t) = t^m$, $m \in N^*$, $\alpha > 0$, $L > 0$ and n is an integer such that $n-1 < \alpha < n$.

If $m < n$, then $f^{(n)}(t) = 0$, substituting in (4.10) yields ${}_{L}^{M}D_t^{\alpha}(t^m) = 0$.

If $m \geq n$ then by repeated integration by parts of the relation (4.10) we obtain

$$ {}_{L}^{M}D_t^{\alpha}(t^m) = \sum_{k=0}^{m-n} \frac{m! L^{-\alpha+n+k}(t-L)^{m-n-k}}{(m-n-k)!\Gamma(-\alpha+n+k+1)}. \tag{4.15}$$

Remark 4.3 *(Fractional derivative of a constant function)*
If f is a constant function (i.e.$f(t) = C$ for all $t \in [a-L, b]$, and C any constant including zero) then we have

$$ {}_{L}^{M}D_t^{\alpha} C = 0.$$

Fractional derivative of the exponential function

Let $f(t) = e^t = \sum\limits_{p=0}^{\infty} \frac{t^p}{p!}$, $\alpha > 0$, $L > 0$ and n is an integer such that $n-1 < \alpha < n$.

We have;

$$ {}_{L}^{M}D_t^{\alpha} e^t = {}_{L}^{M}D_t^{\alpha} \sum_{p=0}^{\infty} \frac{t^p}{p!} = \sum_{p=0}^{\infty} \frac{1}{p!} {}_{L}^{M}D_t^{\alpha} t^p.$$

From (4.15), we find that

$$\begin{aligned}
{}_{L}^{M}D_t^{\alpha}(e^t) &= \sum_{p=0}^{\infty}\sum_{k=0}^{p-n}\frac{L^{-\alpha+n+k}(t-L)^{p-n-k}}{(p-n-k)!\Gamma(-\alpha+n+1+k)},\\
&= \sum_{p=0}^{\infty}\sum_{k=0}^{p-n}\frac{L^{-\alpha+n+k}(t-L)^{p-n-k}}{(p-n-k)!\Gamma(k-\alpha+n+1)},\\
&= \sum_{p=0}^{\infty}\sum_{k=0}^{p}\frac{L^{-\alpha+n+k}(t-L)^{p-k}}{(p-k)!\Gamma(k-\alpha+n+1)},\\
&= \sum_{p=0}^{\infty}\frac{L^{-\alpha+n}(t-L)^{p}}{p!\Gamma(-\alpha+n+1)}+\sum_{p=0}^{\infty}\frac{L^{-\alpha+n+1}(t-L)^{p}}{p!\Gamma(-\alpha+n+2)}+\dots,\\
&= \left(\sum_{p=0}^{\infty}\frac{(t-L)^{p}}{p!\Gamma(-\alpha+n+1)}\right)\left(\sum_{k=0}^{\infty}\frac{L^{-\alpha+n+k}}{\Gamma(-\alpha+n+1+k)}\right),\\
&= e^{t-L}L^{-\alpha+n}\sum_{k=0}^{\infty}\frac{L^{k}}{\Gamma(-\alpha+n+1+k)},\\
&= e^{t-L}L^{n-\alpha}E_{1,n+1-\alpha}(L).
\end{aligned}$$

4.3.2 Fractional derivative of a periodic functions

The main result of this paper is stated in the following theorem.

Theorem 4.4 *Let $\alpha > 0$, $L > 0$ and n an integer such that $n-1 < \alpha < n$ and $f \in C^n[a-L, b]$. If f is a periodic function with period T. Then ${}_{L}^{M}D_t^{\alpha} f$ is a periodic function with the same period T.*

Proof If f is a periodic function with period T, then

$$\begin{aligned}
{}_{L}^{M}D_{t+T}^{\alpha}f(t+T) &= \frac{1}{\Gamma(n-\alpha)}\int_{t+T-L}^{t+T}(t+T-\tau)^{n-\alpha-1}f^{(n)}(\tau+T)d\tau,\\
&= \frac{1}{\Gamma(n-\alpha)}\int_{t-L}^{t}(t-s)^{n-\alpha-1}f^{(n)}(s+2T)ds,\\
&= \frac{1}{\Gamma(n-\alpha)}\int_{t-L}^{t}(t-s)^{n-\alpha-1}f^{(n)}(s)ds,\\
&= {}_{L}^{M}D_t^{\alpha}f(t).
\end{aligned}$$

So, ${}_{L}^{M}D_t^{\alpha} f$ is a periodic function with the same period T. ■

Fractional derivative of some fundamental periodic functions

We first recall that the functions ${}_{L}^{MG}D_t^{\alpha}\sin(t)$ and ${}_{L}^{MG}D_t^{\alpha}\cos(t)$ had been calculated in [27].

$$ {}_{L}^{MG}D_t^{\alpha}\sin(t) = L^{-\alpha}E_{2,1-\alpha}(-L^2)\sin(t-L) + L^{1-\alpha}E_{2,2-\alpha}(-L^2)\cos(t-L), $$

and

$$ {}_{L}^{MG}D_t^{\alpha}\cos(t) = L^{-\alpha}E_{2,1-\alpha}(-L^2)\cos(t-L) - L^{1-\alpha}E_{2,2-\alpha}(-L^2)\sin(t-L). $$

Example 4.2 *(Fractional derivative with fixed memory length of the sine function)*

We have

$$ {}_{L}^{M}D_t^{\alpha}f(t) = {}_{L}^{MG}D_t^{\alpha}f(t) - \sum_{k=0}^{n}\frac{f^{(k)}(t-L)L^{k-\alpha}}{\Gamma(k-\alpha+1)}. $$

Thus

$$ \begin{aligned} {}_{L}^{M}D_t^{\alpha}\sin(t) &= {}_{L}^{MG}D_t^{\alpha}\sin(t) - \sum_{k=0}^{n}\frac{\frac{d^k}{dt^k}(\sin(t-L))L^{k-\alpha}}{\Gamma(k-\alpha+1)}, \\ &= L^{-\alpha}\sin(t-L)E_{2,1-\alpha}(-L^2) + L^{1-\alpha}\cos(t-L)E_{2,2-\alpha}(-L^2) \\ &- L^{\alpha}\sum_{k=0}^{[\frac{n}{2}]}\frac{(-L^2)^k}{\Gamma(2k+1-\alpha)}\sin(t-L) - L^{1-\alpha}\sum_{k=0}^{[\frac{n-1}{2}]}\frac{(-L^2)^k}{\Gamma(2k+2-\alpha)}\cos(t-L), \\ &= L^{-\alpha}\sin(t-L)(E_{2,1-\alpha}(-L^2) - \sum_{k=0}^{[\frac{n}{2}]}\frac{(-L^2)^k}{\Gamma(2k+1-\alpha)}) \\ &+ L^{1-\alpha}\cos(t-L)(E_{2,2-\alpha}(-L^2) - \sum_{k=0}^{[\frac{n-1}{2}]}\frac{(-L^2)^k}{\Gamma(2k+2-\alpha)}), \\ &= a\sin(t-L) + b\cos(t-L). \end{aligned} \tag{4.16} $$

Where, $a = E_{2,1-\alpha}(-L^2) - \sum_{k=0}^{[\frac{n}{2}]}\frac{(-L^2)^k}{\Gamma(2k+1-\alpha)}$, $b = E_{2,2-\alpha}(-L^2) - \sum_{k=0}^{[\frac{n-1}{2}]}\frac{(-L^2)^k}{\Gamma(2k+2-\alpha)}$.
We observe that, ${}_{L}^{M}D_t^{\alpha}\sin(t)$ *is a periodic function with the period* $T = 2\pi$. *This analytical result is depicted in figure (4.1), for some value of* α *and* $L = 32.1$.

Example 4.3 *(Fractional derivative of cosine function)*

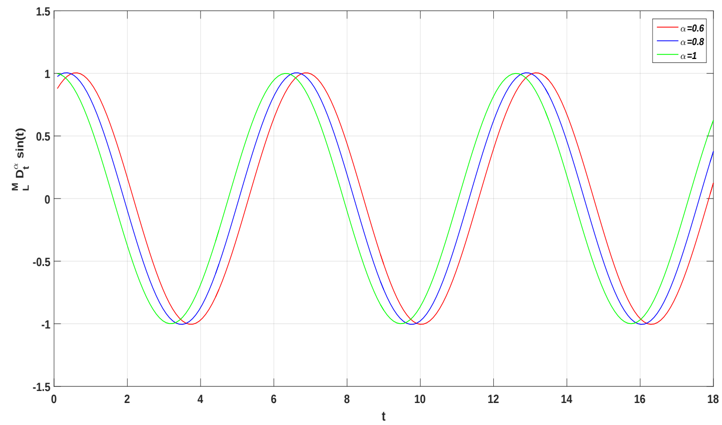


Figure 4.1: Fractional derivative of the Sine function for $L = 32.1$ and some value of α.

We have

$$\begin{aligned}
{}^{M}_{L}D^{\alpha}_{t}\cos(t) &= {}^{MG}_{\ \ L}D^{\alpha}_{t}\cos(t) - \sum_{k=0}^{n}\frac{\frac{d^k}{dt^k}(\cos(t-L))L^{k-\alpha}}{\Gamma(k-\alpha+1)}, \\
&= L^{-\alpha}\cos(t-L)E_{2,1-\alpha}(-L^2) - L^{1-\alpha}\sin(t-L)E_{2,2-\alpha}(-L^2) \\
&- L^{\alpha}\sum_{k=0}^{[\frac{n}{2}]}\frac{(-L^2)^k}{\Gamma(2k+1-\alpha)}\cos(t-L) + L^{1-\alpha}\sum_{k=0}^{[\frac{n-1}{2}]}\frac{(-L^2)^k}{\Gamma(2k+2-\alpha)}\sin(t-L), \\
&= L^{-\alpha}\cos(t-L)(E_{2,1-\alpha}(-L^2) - \sum_{k=0}^{[\frac{n}{2}]}\frac{(-L^2)^k}{\Gamma(2k+1-\alpha)}) \\
&- L^{1-\alpha}\sin(t-L)(E_{2,2-\alpha}(-L^2) - \sum_{k=0}^{[\frac{n-1}{2}]}\frac{(-L^2)^k}{\Gamma(2k+2-\alpha)}), \\
&= a\cos(t-L) - b\sin(t-L). \qquad (4.17)
\end{aligned}$$

Where, $a = E_{2,1-\alpha}(-L^2) - \sum_{k=0}^{[\frac{n}{2}]}\frac{(-L^2)^k}{\Gamma(2k+1-\alpha)}$, *and* $b = E_{2,2-\alpha}(-L^2) - \sum_{k=0}^{[\frac{n-1}{2}]}\frac{(-L^2)^k}{\Gamma(2k+2-\alpha)}$. *Clearly,* ${}^{M}_{L}D^{\alpha}_{t}\cos(t)$ *is a periodic function with the period* $T = 2\pi$.

4.4 An interpolation property

It is known that the operator of Grünwald-Letnikov fractional derivative with fixed memory length is an extension of the integer-order operator $\frac{d^n}{t^n}$, (see [27]).

The following proposition proves that the Caputo and Riemman-Lioville operators of fractional derivative with fixed memory length, verified this property for $\alpha \to n$, but not for $\alpha \to n-1$.

Proposition 4.3 *Let $L > 0$ and $0 \leq n-1 < \alpha < n$ such that n is an integer number, and let $f(t)$ has $(n+1)$ continuous bounded derivatives in $[a-L, b]$. Then , for all $t \in [a, b]$, we have*

$$\lim_{\alpha \to n} \ {}^{M}_{L}D^{\alpha}_{t} f(t) = f^{(n)}(t),$$

and

$$\lim_{\alpha \to n-1} \ {}^{M}_{L}D^{\alpha}_{t} f(t) = f^{(n-1)}(t) - f^{(n-1)}(t-L).$$

Proof We have

$$\begin{aligned}
\lim_{\alpha \to n} \ {}^{M}_{L}D^{\alpha}_{t} f(t) &= \lim_{\alpha \to n} \frac{1}{\Gamma(n-\alpha)} \int_{t-L}^{t} (t-\tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau, \\
&= \lim_{\alpha \to n} \frac{L^{n-\alpha} f^{(n)}(t-L)}{\Gamma(n-\alpha+1)} + \lim_{\alpha \to n} \frac{1}{\Gamma(n-\alpha+1)} \\
&\int_{t-L}^{t} (t-\tau)^{n-\alpha} f^{(n+1)}(\tau) d\tau, \\
&= f^{(n)}(t-L) + \int_{t-L}^{t} f^{(n+1)}(\tau) d\tau, \\
&= f^{(n)}(t).
\end{aligned}$$

For $\alpha \to n-1$, we have

$$\begin{aligned}
\lim_{\alpha \to n-1} \ {}^{MC}_{L}D^{\alpha}_{t} f(t) &= \lim_{\alpha \to n-1} \frac{1}{\Gamma(n-\alpha)} \int_{t-L}^{t} (t-\tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau, \\
&= \int_{t-L}^{t} f^{(n)}(\tau) d\tau, \\
&= f^{(n-1)}(t) - f^{(n-1)}(t-L).
\end{aligned}$$

■

Example 4.4 *Let $f(t) = e^t$, we have*

$$ {}_{L}^{M}D_t^{\alpha} e^t = e^{t-L} L^{n-\alpha} E_{1,n+1-\alpha}(L), $$

Thus,

$$ \lim_{\alpha \to n} {}_{L}^{M}D_t^{\alpha} e^t = e^{t-L} E_{1,1}(L) = e^t = f^{(n)}(t). $$

However

$$ \begin{aligned} \lim_{\alpha \to n-1} {}_{L}^{M}D_t^{\alpha} e^t &= e^{t-L} L E_{1,2}(L) = e^{t-L}(e^L - 1), \\ &= e^t - e^{t-L} = f^{(n)}(t) - f^{(n-1)}(t-L). \end{aligned} $$

Example 4.5 *Let $f(t) = t^m$, we have*

$$ {}_{L}^{M}D_t^{\alpha}(t^m) = \sum_{k=0}^{m-n} \frac{m! L^{-\alpha+n+k}(t-L)^{m-n-k}}{(m-n-k)!\Gamma(-\alpha+n+k+1)}. $$

Putting $N = m - n$ and $t - L = a$, then

$$ \begin{aligned} \lim_{\alpha \to n} {}_{L}^{M}D_t^{\alpha}(t^m) &= \sum_{k=0}^{N} \frac{m! L^k a^{N-k}}{(N-k)!k!}, \\ &= \frac{m!}{N!} \sum_{k=0}^{N} \frac{N! L^k a^{N-k}}{(N-k)!k!}, \\ &= \frac{m!}{N!}(a+L)^N = \frac{m!}{(m-n)!} t^{m-n}, \\ &= \frac{d^n}{dt} t^m = f^{(n)}(t). \end{aligned} $$

However

$$ \begin{aligned} \lim_{\alpha \to n-1} {}_{L}^{M}D_t^{\alpha}(t^m) &= \sum_{k=0}^{N} \frac{m! L^{k+1} a^{N-k}}{(N-k)!(k+1)!}, \\ &= \frac{m!}{(N+1)!} \sum_{k=0}^{N+1} \frac{(N+1)! L^k a^{N+1-k}}{(N+1-k)!k!} - \frac{m!}{(m-n+1)!}(t-L)^{m-n+1}, \\ &= \frac{m!}{(N+1)!} t^{N+1} - \frac{m!}{(m-(n-1))!}(t-L)^{m-(n-1)}, \\ &= \frac{m!}{(m-(n-1))!} t^{m-(n-1)} - \frac{m!}{(m-(n-1))!}(t-L)^{m-(n-1)}, \\ &= \frac{d^{n-1}}{dt} t^m - \frac{d^{n-1}}{dt}(t-L)^m = f^{(n-1)}(t) - f^{(n-1)}(t-L). \end{aligned} $$

Classical fractional derivative ${}^{C}_{a}D^{\alpha}_{t}$ or ${}^{RL}_{a}D^{\alpha}_{t}$	**Fractional derivative with fixed memory length** ${}^{M}_{L}D^{\alpha}_{t}$
${}^{C}_{a}D^{\alpha}_{t}f(t) = {}^{RL}_{a}D^{\alpha}_{t}f(t) - \sum_{k=0}^{m-1}\frac{f^{(k)}(a)(t-a)^{k-\alpha}}{\Gamma(k-\alpha+1)}$	${}^{MC}_{L}D^{\alpha}_{t}f(t) = {}^{MR}_{L}D^{\alpha}_{t}f(t)$
$\lim_{\alpha\to n} {}^{RL}_{a}D^{\alpha}_{t}f(t) = \lim_{\alpha\to n} {}^{C}_{a}D^{\alpha}_{t}f(t) = f^{(n)}(t)$	$\lim_{\alpha\to n} {}^{M}_{L}D^{\alpha}_{t}f(t) = f^{(n)}(t)$
$\lim_{\alpha\to n-1} {}^{RL}_{a}D^{\alpha}_{t}f(t) = f^{(n-1)}(t),$ $\lim_{\alpha\to n-1} {}^{C}_{a}D^{\alpha}_{t}f(t) = f^{(n-1)}(t) - f^{(n-1)}(a)$	$\lim_{\alpha\to n-1} {}^{M}_{L}D^{\alpha}_{t}f(t)$ $= f^{(n-1)}(t) - f^{(n-1)}(t-L)$
${}^{RL}_{0}D^{\alpha}_{t}(t^m) = {}^{C}_{0}D^{\alpha}_{t}(t^m) = \frac{\Gamma(m+1)}{\Gamma(m-\alpha+1)}t^{m-\alpha}$	${}^{M}_{L}D^{\alpha}_{t}(t^m) = \sum_{k=0}^{m-n}\frac{m!L^{-\alpha+n+k}(t-L)^{m-n-k}}{(m-n-k)!\Gamma(-\alpha+n+k+1)}$
${}^{RL}_{a}D^{\alpha}_{t}C = \frac{C}{\Gamma(1-\alpha)}(t-a)^{\alpha} \neq 0,$ ${}^{C}_{a}D^{\alpha}_{t}C = 0$	${}^{M}_{L}D^{\alpha}_{t}C = 0$
${}^{RL}_{a}D^{\alpha}_{t}\sin t = t^{1-\alpha}E_{2,2-\alpha}(-t^2)$	${}^{M}_{a}D^{\alpha}_{t}\sin t = a\sin(t-L) + b\cos(t-L).$

Table 4.1: Comparison between some results of classical fractional-order derivatives and fractional order derivatives with fixed memory length.

4.5 Comparison between some results of classical fractional-order derivatives and fractional order derivatives with fixed memory length

The previous results are summarized in the following table, for comparison purpose between classical fractional-order derivative and fractional-order derivative with fixed memory length.

4.6 Fractional-order autonomous systems with exact periodic solution

As previously mentioned any autonomous fractional-order system expressed in terms of classical fractional derivatives cannot have any exact periodic solutions [24, 25]. Now we present some examples (linear and non-linear) showing that fractional-order autonomous systems expressed in terms of fractional derivatives with fixed memory length can have exact periodic solutions.

4.6.1 Linear fractional-order autonomous system with exact periodic solution

Let consider the following linear fractional-order autonomous system

$$ {}_{2\pi}^{M}D_t^{\alpha} X(t) = AX(t), \tag{4.18} $$

where $X(t) \in R^2$ and $A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$, with $a = E_{2,1-\alpha}(-L^2) - \sum_{k=0}^{[\frac{n}{2}]} \frac{(-L^2)^k}{\Gamma(2k+1-\alpha)}$, $b = E_{2,2-\alpha}(-L^2) - \sum_{k=0}^{[\frac{n-1}{2}]} \frac{(-L^2)^k}{\Gamma(2k+2-\alpha)}$.

The vector function $X(t) = c \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix}$, $c \in R$ is an exact $2\pi-$periodic solution for the system (4.18).

Namely, we have ${}_{2\pi}^{M}D_t^{\alpha}X(t) = c\begin{pmatrix} {}_{2\pi}^{M}D_t^{\alpha}\cos(t) \\ {}_{2\pi}^{M}D_t^{\alpha}\sin(t) \end{pmatrix}$. Then, from (4.16) and (4.17) we obtain

$$
\begin{aligned}
{}_{2\pi}^{M}D_t^{\alpha}X(t) &= c\begin{pmatrix} a\cos(t-2\pi) - b\sin(t-2\pi) \\ a\sin(t-2\pi) + b\cos(t-2\pi) \end{pmatrix}, \\
&= c\begin{pmatrix} a & -b \\ b & a \end{pmatrix}\begin{pmatrix} \cos(t-2\pi) \\ \sin(t-2\pi) \end{pmatrix}, \\
&= cA\begin{pmatrix} \cos(t-2\pi) \\ \sin(t-2\pi) \end{pmatrix}, \\
&= AX(t).
\end{aligned}
$$

Thus, $X(t) = c\begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix}$ is an exact $2\pi-$periodic solution of (4.18).

4.6.2 The predator-prey model with Holling type II response function

All population species posses the property of heredity which means the passing on of traits from parents to their offspring, either through asexual reproduction or sexual reproduction, the offspring cells or organisms acquire the genetic information of their parents, through heredity. This property makes fractional differential systems may model more efficiently certain problems than ordinary differential ones. Motivated by this fact, we recall the fractional version of the Holling-Tanner model (3.16) [67] as follow

$$
\begin{cases} D^{\alpha}x = r_1x(1-\frac{x}{K}) - \frac{qxy}{m+x}, \\ D^{\alpha}y = r_2y(1-\frac{y}{\gamma x}). \end{cases} \tag{4.19}
$$

Where $D^{\cdot}$ denotes a standard fractional-order derivative operator and $\alpha \in [0,1]$ is the fractional order related to the hereditary property of the population (a value of α close to an integer number mean that the population has a weak hereditary property).

Since exact analytical resolution of this nonlinear system is unavailable, we resort to qualitative and numerical study, for this purpose the parameters are set to $r_1 = 1,\ r_2 = 0.2,\ K =$

25, $q = \frac{6}{7}$, $m = 1$ and $\gamma = 0.95$, the system (4.19) has two equilibrium points $E_0 = (25, 0)$ and $E_1 \approx (7.1429, 6.7857)$.

- The characteristic polynomial of the Jacobian matrix evaluated at E_0 is given by $P(\lambda) = \lambda^2 + 0.8\lambda - 0.2$. So $a_2 = -0.2 < 0$, then according to Proposition (3.2) E_0 is unstable for all $\alpha \in [0, 2)$.

- The characteristic polynomial of the Jacobian matrix evaluated at E_1 is given by $P(\lambda) = \lambda^2 - 0.1409\lambda + 0.0747$. So $a_1 \approx -0.1409$ and $a_2 \approx 0.0747 > 0$.

  Applying Hopf-Like Bifurcation theory [70, 22] and using Proposition (3.2), we obtain the Hopf-Like bifurcation value $\alpha^* = \frac{2}{\pi} cos^{-1}(\frac{-a_1}{2\sqrt{a_2}}) \approx 0.8341$, at which the fixed point E_1 losses its stability and a periodic motion ($S-$asymptotically periodic for the classical fractional derivative and exact periodic for fractional derivative with fixed memory length) appears.

To illustrate these results we solve numerically the system (4.19) by developing a Matlab code using a discretization technique based on the formula (4.14).
Choosing a value for α greater then α^*, for example $\alpha = 0.9$, then we compare between the solution of (4.19) in term of classical fractional operator and its solution in term of fractional operator with fixed memory length $L = 30$. The two trajectories are started from the same initial point $X_0 = (2.64, 4.88)$, predicted at the attracting limit cycle. The results are shown in Figure 4.2.
An $S-$asymptotically $T-$periodic solution with $T \approx 27.2$ is obtained for classical fractional operator as shown in Figure 4.2(a,b); and an exact $T-$periodic solution is obtained for the fractional derivative operator with fixed memory length as shown in Figure 4.2(c,d).

4.6.3 Fractional-order memristor-based circuit

A simplest memristor-based electrical circuit which posses a rich dynamical behavior (ranged from stationary and periodic behavior to chaotic behavior with a double scroll and four-scroll chaotic attractor) has been introduced in [21] and its fractional version has been also given and

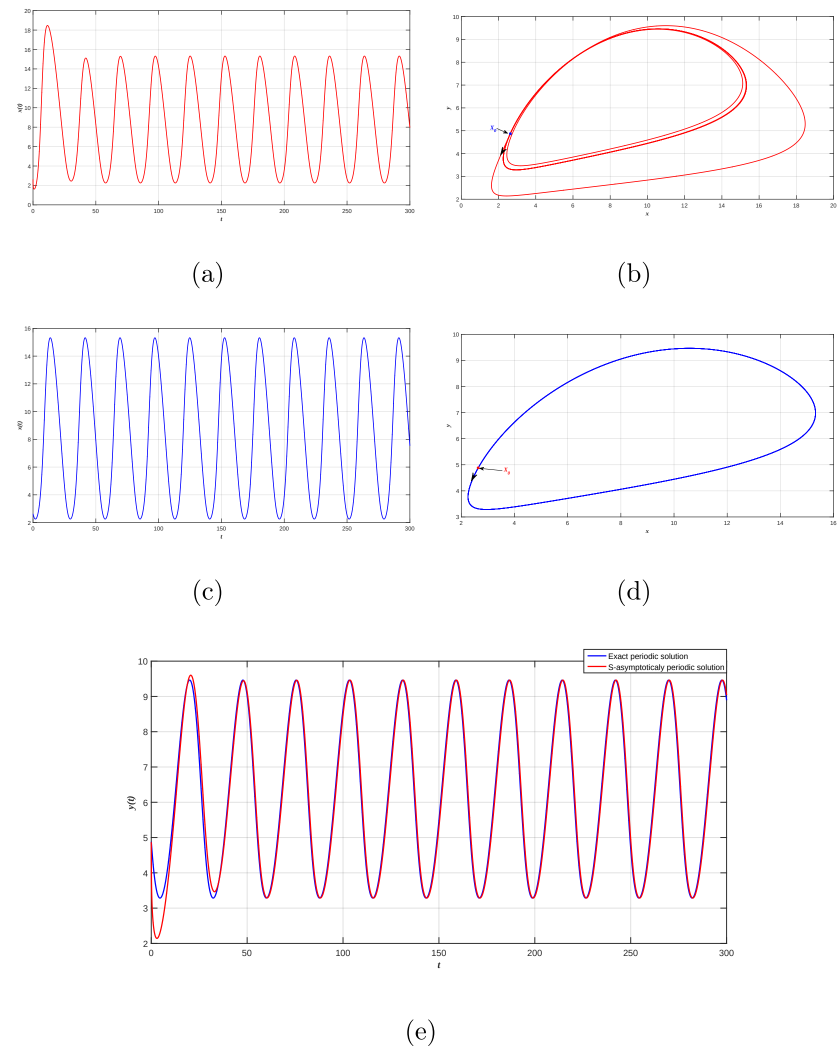


Figure 4.2: Time evolution and phase portrait of system (4.19) for $\alpha = 0.9$ (a,b) $S-$asymptotically $T-$periodic solution with $T \approx 27.2$ for classical fractional operator. (c,d) Exact $T-$periodic solution for the fractional derivative operator with fixed memory length. (e) Comparison between the two solutions.

analysed based on fractional-order linear capacitor and fractional-order inductor models proposed in [77, 78]. The dynamic of this fractional circuit is described by the mathematical model

$$\begin{cases} D^{\alpha_1}x &= y, \\ D^{\alpha_2}y &= -(x+(\gamma z^2-\beta)y)/3, \\ D^{\alpha_3}z &= -y-0.9z+y^2 z. \end{cases} \quad (4.20)$$

As in the previews example we adopt qualitative and numerical study. for this purpose the parameters are set to $\gamma = 0.1$, $\beta = 3.3$ and $\alpha_1 = \alpha_2 = \alpha_3 = \alpha$.
The model has only one fixed point $E = (0, 0, 0)$. Applying Hopf-Like Bifurcation criterion we obtain the Hopf-Like bifurcation value $\alpha^* = 0.1967$ [21], at which the fixed point E losses its stability and a periodic motion appears.
To illustrate these results we solve numerically the system (4.20). In order to localize the periodic interval we plot the bifurcation diagram of y versus the fractional order α in Figure 4.3; clearly we have a periodic motion for $\alpha \in]0.1967, 0.8216[$ which is agree with the obtained Hopf-Like Bifurcation value.
Now we choose a value for α in the periodic interval for example $\alpha = 0.5$, then we compare between the solution of (4.20) in term of classical fractional operator and its solution in term of fractional operator with fixed memory length $L = 80$. The two trajectories are started from the same initial point $X_0 = (1.39, 0.923, -3.62)$, predicted at the attracting limit cycle. The results are shown in Figure 4.4.
An $S-$asymptotically $T-$periodic solution with $T \approx 19.5$ is obtained for classical fractional operator as shown in Figure 4.4(a,b); and an exact $T-$periodic solution is obtained for the fractional derivative operator with fixed memory length as shown in Figure 4.4(c,d).

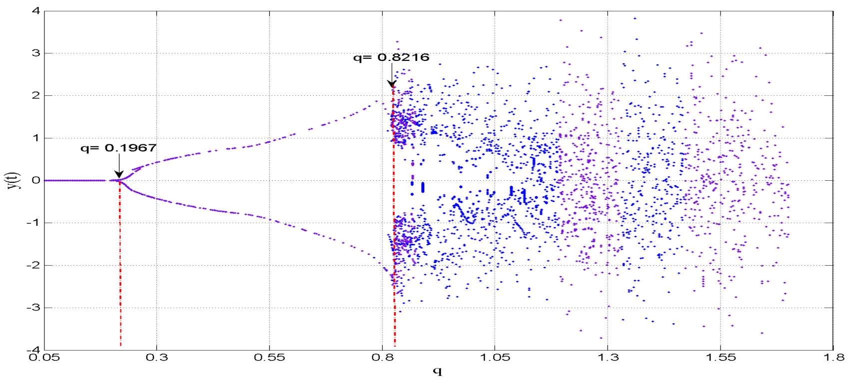
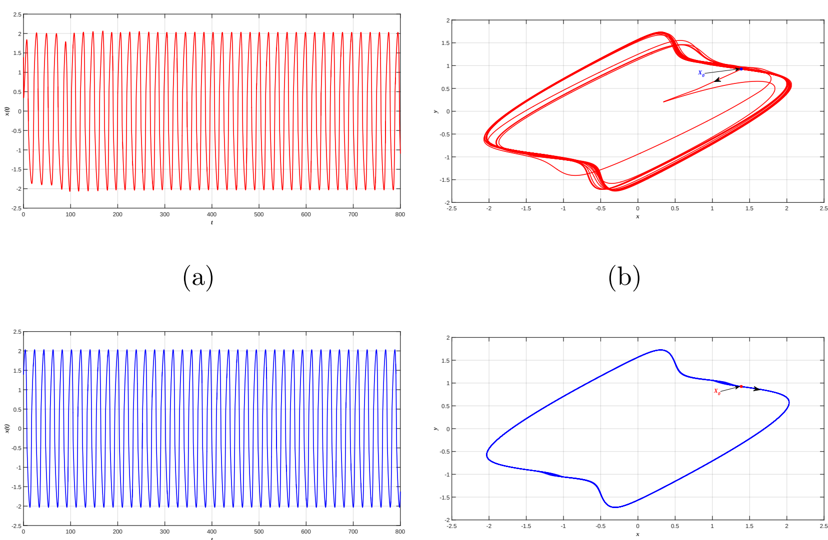


Figure 4.3: Bifurcation diagram of the state y versus the fractional-order α for $\beta = 3.3$.



(a) (b)

(c) (d)

Figure 4.4: Time evolution and phase portrait of system (4.20) for $\alpha = 0.5$ and $\beta = 3.3$. (a,b) $S-$asymptotically $T-$periodic solution with $T \approx 19.5$ for classical fractional operator. (c,d) Exact $T-$periodic solution for the fractional derivative operator with fixed memory length.

General Conclusion

In this thesis, we study the basic subjects in theory of fractional-order derivatives. We see that fractional-order derivatives are generalization of integer-order derivatives, moreover, we try to extend some optimal Routh-Hurwitz conditions to fractional systems of order $\alpha \in [0, 2)$ in order to facilitate and shorten the time to study stability of fractional-order systems, also we try to solve the problem of absence of periodic solutions in fractional-order systems. These two important topics are essential in the study of Bifurcations and Chaos in fractional-order systems. So, we have devoted the first part (chapter (1) and (2)) of this thesis to display the most famous definitions of fractional-order derivatives with the most important subjects such as stability, Bifurcations and Chaos theories. In the second part (chapter (3) and (4)) we have presented our works to the fractional calculus space. In the first paper we exposed our contributions in the stability theory of fractional-order system, we have derived some new optimal (non-improvable) *Routh-Hurwitz conditions* for fractional type models of orders between 0 and 2., i.e., some necessary and sufficient conditions guaranteeing that all zeros of the corresponding characteristic polynomial are located inside the Matignon stability sector. The effect of parameter α on the model dynamics has been highlighted. These results can be regarded as a generalization of the classical Routh-Hurwitz stability conditions. As application, the stability properties of some fractional-order mathematical models in population dynamics and epidemiology have been explored. Numerical simulations are provided to exemplify the theoretical findings. The second work deal with a modification of the Caputo and Rieman-Liouville fractional-order derivatives by fixing the memory length and varying the lower terminal of the derivative. It is shown that the modified fractional derivative operator preserves the periodicity. As a consequence periodic solutions can be expected in fractional-order systems expressed in term of the new operator. To confirm this assertion three examples have been investigated, one linear system for which an analytic expression of an exact periodic solution is given and two nonlinear systems for which exact periodic solutions are provided using qualitative and numerical methods.

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Abstract

In this thesis, we have presented some basic subjects that are needed to study the Bifurcations and Chaos in fractional-order systems. Namely, stability conditions and existence of periodic solutions. In order to facilitate the study of stability we have devoted the first work to extend the Routh-Hrwitz conditions to fractional systems of order $\alpha \in [0,2)$. In the second work, we extend the modification which consists of fixing the memory length and varying the lower terminal of fractional differential operators, this modification has enabled us to preserve the periodicity. to illustrate our theoretical results we have employed some numerical examples from some fractional-order systems.

Keyword: Fractional-order derivative, Bifurcations and Chaos, Stability, Routh-Hrwitz conditions, Fixed memory length, Periodic solution.

Résumé

Dans cette thèse, nous avons présenté quelques sujets de base qui sont nécessaires pour étudier la Bifurcations et le Chaos dans un système d'ordre fractionnaire. Notamment, les conditions de la stabilité et l'existence des solutions périodiques. Dans le but de faciliter l'étude de la stabilité nous avons consacré le premier travail pour prolonger les conditions de Routh-Hrwitz pour les systèmes fractionnaires d'ordre $\alpha \in [0,2)$ Dans le de le deuxième travail, nous étendons la modification qui consiste à fixer la longueur de la mémoire et à faire varier la borne inférieure des opérateurs de différentiation fractionnaire, cette modification nous a permis de conserver la périodicité. Pour illustrer nos résultats théoriques, nous avons utilisé quelques exemples numériques de certains systèmes d'ordre fractionnaire.

Mots clés : Dérivée d'ordre fractionnaire, Bifurcations et Chaos, Stabilité, Conditions de Routh-Hrwitz , Mémoire à longueur fixée, Solutions périodiques.

ملخص

في هذه الأطروحة ، قدمنا بعض المواد الأساسية اللازمة لدراسة التشعبات والفوضى في أنظمة ذات رتبة كسرية . وهي شروط الاستقرار ووجود الحلول الدورية، من أجل تسهيل دراسة الاستقرار، خصصنا أول عمل لتمديد شروط روث~هارويتز إلى أنظمة كسرية ذات الرتبة $\alpha \in [0,2)$. في العمل الثاني ، قمنا بتوسيع التعديل الذي يتكون من تثبيت طول الذاكرة وتغيير الطرف السفلي في عوامل التفاضل ذات الرتب الكسرية ، وقد مكننا هذا التعديل من الحفاظ على الدورية .لتوضيح نتائجنا النظرية ، استخدمنا بعض الأمثلة العددية من بعض الأنظمة ذات رتبة كسرية .

كلمات مفتاحية: مشتقات ذات رتبة كسرية، التشعبات والفوضى، اللإستقرار، شروط روث~هارويتز، ذاكرة بطول ثابت، الحلول الدورية.