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La résolution numérique de quelques équations intégrales et intégro-différentielles non linéaires par la méthode de collocation

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Thesis

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Numerical Solution of Some Classes of Nonlinear Integral and Integro-Differentiale Equations by Using Collocation Method

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DEDICATION

To my **parents**.

To my brothers and sisters.

To my children, my husband, and his family.

To everyone who encouraged me.

ABSTRACT

The main purpose of this thesis is to provide a direct, convergent and easy to implement numerical method to obtain the approximate solution for nonlinear Volterra integral equations and nonlinear Volterra integro-differential equations . Algorithms based on iterative collocation method is developed for the numerical solution of these kinds of equations. We also provide a rigorous error analysis. A theoretical proof is given and we present some numerical results which illustrate the performance of the methods.
Key Words: Nonlinear Volterra integral equations, Volterra integro-differential equations, Collocation method, Continuous collocation method, Iterative Method, Lagrange polynomials, Convergence analysis,Error estimation.

RÉSUMÉ

L'objectif essentiel de ce travail consiste à résoudre numériquement des équations intégrales de Volterra non linéaires et des équations integro-différentielles non linéaires de Volterra par la méthode de " collocation itérative" en utilisant les polynômes de Lagrange. Des exemples numériques sont présentés pour confirmer les estimations théoriques et illustrer la convergence de la méthode.

Mots-clés : Équations intégo-différentielle non linéaires de Volterra, Équations intégrales de Volterra, Méthode de collocation, Polynômes de Lagrange.

ملخص

قدمنا في هذه المذكرة طريقة مقترحة مع خوارزمية جديدة لحل معادلات فولتيرا التكاملية وكذا التكاملية–التفاضلية الغير الخطية. حيث يتم ايجاد الحل التقريبي لمعادلات فولتيرا التكاملية باستخدام طريقة التجميع التكرارية، بالاعتماد كثيرات حدود لاغرانج. كما من الممكن ملاحظة كفاءة الطريقة و سهولة الحسابات، فيها حيث تمت مقارنة نتائج هذه الطريقة مع نتائج طرق اخرى من خلال بعض الأمثلة التوضيحية لحل معادلات فولتيرا التكاملية وكذا التكاملية–التفاضلية الغير الخطية. وقد تم الحصول على نتائج جيدة.

الكلمات الكفتاحية

معادلات فولتيرا التكاملية، التكاملية–التفاضلية، الغير الخطية، طريقة التجميع التكرارية، كثيرات حدود لاغرانج

CONTENTS

INTRODUCTION

The theory and the applications of the integrals equations are important subject in applied mathematics. Integral equations or integro-differential equations describe many applications in science and engineering, also occur as reformulations of other mathematical problems. For examples the Volterra's population growth model, biological species living together, the heat transfer and the heat radiation are among many areas that are described by integral equations.

The first integral equation mentioned in the mathematical literature is due to Abel and can be found in almost any book on this subject (see, for instance, [20]). Abel found this equation in 1812, starting from a problem in mechanics. He gave a very elegant solution that was published in 1826.

Starting in 1896, Vito Volterra built up a theory of integral equations, viewing their solutions as a problem of finding the inverses of certain integral operators. In 1900, Ivar Fredholm made his famous contribution that led to a fascinating period in the development of mathematical analysis. Poincaré, Fréchet, Hilbert, Schmidt, Hardy and Riesz were involved in this new area of research. Volterra integral equations belong to its owner Vito Volterra, among the most popular types of integral equations. It arises in many varieties of mathematical, scientific, and engineering problems. One such problem is the solution of parabolic differential equations with initial boundary conditions [25].

The nonlinear Volterra integro-differential equation appeared after its establishment by Volterra. It appears in many physical applications such as glass-forming process, heat transfer, diffusion process in general, neutron diffusion and biological species coexisting together with increasing and decreasing rates of generating. More details about the sources where these equations arise can be found in physics, biology and books of engineering applications. In the following examples, we will briefly describe a class of Volterra integral and integro-differential equations. More applications and sources of Volterra integral equations can be found [12, 45, 67].

Example 01 : In this example, we will study the Volterra model for population growth of a species within a closed system [67]. The population model of Volterra is characterized by the nonlinear Volterra integro-differential equation

$$\frac{dP}{dt} = aP - bP^2 - cP \int_0^t P(x)dx, \quad P(0) = P_0$$

where $P = P(t)$ denotes the population at time t, a, b and c are constants and positive parameters $a > 0$ is the birth rate coffecient, $b > 0$ is the crowding coffecient, $c > 0$ is the toxicity coffecient and P_0 is the initial population. The coffecient c indicates the essential behavior of the population evolution before its level falls to zero in the long run.

There are many existing numerical methods for solving Volterra integro-differential equations, such as Legendre spectral collocation method [69], Runge-Kutta method [13], spectral method [70, 68], Bernstein's approximation method [47], Polynomial collocation method [14, 18, 16, 53, 64], Tau method [26, 34], Haar and Legendre wavelets method [43, 63], Taylor collocation method [41, 42].

The aim of this thesis is to apply a new direct iterative collocation method based on the use of Lagrange polynomials for nonlinear Volterra integrals equations and the nonlinear integro-differential equations. This method is based on the idea of approach-

ing the exact solution of a given integral equation using a suitable function, belonging to a chosen finite dimensional space. The approximate solution must satisfy the integral equation on a certain subset of the interval (called the set of collocation points). We consider as spaces of approximation, the real polynomial spline spaces. The main advantages of this direct iterative collocation method are:
(i) The approximate solution is given by using explicit formulas.
(ii) This method has a convergence order.
(iii) There is no algebraic system needed to be solved, which makes the proposed algorithm very effective and easy to implement.

Our thesis is organized as follows:
In the first chapter, we provide the fundamental notions, definitions and some necessary theorems will be needed for the following chapters, such as the classifications of integral and integro-differential equations, Leïbniz rule, the linearity and the homogeneity concepts of integral equations, the conversion process of an Initial Value Problem to Volterra integral and integro-differntaille equation and discrete inequalities.
The purpose of the first part, is to give a numerical method based on the use of Lagrange polynomials to construct a collocation solution in the two piecewise polynomials splines spaces $S_{m-1}^{(-1)}(\Pi_N)$ and $S_m^{(0)}(\Pi_N)$ for approximating the solution of nonlinear Volterra integral equations . We prove the convergence of the approximate solution to the exact solution. Numerical examples illustrate the theoretical results.
In the second part, we provide a new direct numerical method for first-order nonlinear Volterra integro-differential equations (VIDEs) in the space $S_m^{(0)}(\Pi_N)$ and $S_{m+1}^{(1)}(\Pi_N)$, the space of continuous polynomial spline functions . We developed an algorithm based on the use of Lagrange polynomials for the numerical solution . It is shown that this algorithm is convergent. Numerical results are presented to prove the effectiveness of the presented algorithm.
Finally, we summarize the contributions of this thesis and we suggest new avenues, improvements for future research and perspectives.

CHAPTER 1

PRELIMINARY AND AUXILIARY RESULTS

An integral equation is defined as an equation in which the unknown function $u(t)$ to be determined appear under the integral sign. The subject of integral equations is one of the most useful mathematical tools in both pure and applied mathematics. It has enormous applications in many physical problems. Many initial and boundary value problems associated with ordinary differential equation (ODE) and partial differential equation (PDE) can be transformed into problems of solving some approximate integral equations. The development of science has led to the formation of many physical laws, which, when restated in mathematical form, often appear as differential equations [67]. An integral equation is an equation in which the unknown function $u(t)$ appears under an integral sign. A standard integral equation in $u(t)$ is of the form:

$$u(t) = f(t) + \lambda \int_{g(t)}^{h(t)} K(t,s)u(s)ds,$$

where $g(t)$ and $h(t)$ are the limits of integration, λ is a constant parameter, and $k(t,s)$ is a function of two variables t and s called the kernel of the integral equation. The function $u(t)$ that will be determined appears under the integral sign, and it appears inside the integral sign and outside the integral sign as well. The functions $f(t)$ and $k(t,s)$ are given in advance. It is to be noted that the limits of integration $g(t)$ and $h(t)$ may be both variables, constants, or mixed.

An integro-differential equation is an equation in which the unknown function $u(t)$ appears under an integral sign and contains an ordinary derivative $u^{(n)}(t)$ as well. A standard integro-differential equation is of the form:

$$u^{(n)}(t) = f(t) + \lambda \int_{g(t)}^{h(t)} K(t,s)u(s)ds,$$

where $g(t)$, $h(t)$, $f(t)$, λ and the kernel $k(t,s)$ are as prescribed before. Integral equations and integro-differential equations will be classified into distinct types according to the limits of integration and the kernel $k(t,s)$. [67].

1.1 Classifications of integral and intergro-differential equations

The most integral and integro-differential equations fall under two main classes namely Fredholm and Volterra integral and integro-differential equations.

Fredholm integral and integro-differential equations

Fredholm integral equations:

Fredholm integral equations arise in many scientific applications. It was also shown that, this equation can be derived from boundary value problems. Erik Ivar Fredholm (1866-1927) is best remembered for his work on integral equations and spectral theory. Fredholm was a Swedish mathematician who established the theory of integral equations and his 1903 paper in Acta Mathematica played a major role in the establishment of operator theory (Wazwaz (2011)). The most standard form of Fredholm linear integral equations is given by the following form

$$v(t)u(t) = f(t) + \lambda \int_a^b K(t,s)u(s)ds, \quad a \leq t, \quad s \leq b, \tag{1.1}$$

where the limit of integration a and b are constants and the unknown function $u(t)$ appears under the integral sign. Where $k(t,s)$ is the kernel of the integral equation and λ is a parameter. The Eq. (1.1) is called linear because the unknown function $u(t)$ under the integral sign occurs linearly, i.e. the power of $u(t)$ is one.

The value of $v(t)$ will give the following kinds of Fredholm integral equations:

If $v(t) = 0$, then Eq. (1.1) yields

$$f(t) = \lambda \int_a^b K(t,s)y(s)ds, \quad a \leq t, \quad s \leq b,$$

which is called Fredholm integral equation of the first kind.
If the function $v(t) = 1$, then Eq. (1.1) becomes simply

$$u(t) = f(t) + \lambda \int_a^b K(t,s)u(s)ds, \quad a \leq t, \quad s \leq b,$$

and this equation is called Fredholm integral equation of second kind.
If $v(t) \neq 0$, then Eq.(1.1) becomes Fredholm integral equations of third kind. Fredholm integral equation is of the first kind if the unknown function $u(t)$ appears only under the integral sign.

Nonlinear Fredholm integral equations:

The nonlinear Fredholm integral equations of the second kind is given by the following form

$$u(t) = f(t) + \lambda \int_a^b K(t,s,u(s))ds, \quad a \leq t, \quad s \leq b,$$

where the unknown function $u(t)$ occurs inside and outside the integral sign, λ is a parameter, and a and b are constants. For this type of equations, the kernel k and the function $f(t)$ are given real-valued functions.

Nonlinear Fredholm-Hammerstein integral equations:

Nonlinear Fredholm-Hammerstein integral equations is given by the form,

$$u(t) = f(t) + \lambda \int_a^b K(t,s)F(s,u(s))ds, \quad a \leq t, \quad s \leq b,$$

Nonlinear Fredholm integro-differential equations:

The nonlinear Fredholm integro-differential equations is given by the following form,

$$u^n(t) = f(t) + \int_a^b K(t,s,u(s),u'(s),\ldots,u^{n-1}(s))ds, \quad u^k(a) = b_k, \quad 0 \leq k \leq n-1, \tag{1.2}$$

where $u^n(t) = \frac{d^n u}{dt^n}$ Because the resulted equation in (1.2) combines the differential operator and the integral operator, then it is necessary to define initial conditions $u(0),\ u^{'}(0),...,u^{n-1}(0)$ for the determination of the particular solution $u(t)$ of the equation (1.2). Any Fredholm integro-differential equation is characterized by the existence of one or more of the derivatives $u'(t),\ u^{''}(t),...$ outside the integral sign. The Fredholm integro-differential equations of the second kind appear in a variety of scientific applications such as the theory of signal processing and neural networks.

Nonlinear Fredholm-Hammerstein integro-differential equations:
The nonlinear Fredholm-Hammerstein integro-differential equations of the second kind is of the form,

$$u^n(t) = f(t) + \int_a^b K(t,s)F(s,u(s),u'(s),\ldots,u^{n-1}(s))ds,$$

Volterra integral and integro-differential equations

It is well known that linear and nonlinear Volterra integral equations arise in many scientific fields such as the population dynamics, spread of epidemics, and semi-conductor devices. Volterra started working on integral equations in 1884, but his serious study began in 1896. The name integral equation was given by du Bois-Reymond in 1888. However, the name Volterra integral equation was first coined by Lalesco in 1908 [67].
Volterra integral equations:
The standard form of linear Volterra integral equations, where the limits of integration are functions of t rather than constants, are of the form,

$$v(t)u(t) = f(t) + \lambda \int_a^t K(t,s)u(s)ds, \quad a \le t, \quad s \le b, \tag{1.3}$$

where the unknown function $u(t)$ under the integral sign occurs linearly as stated before. It is worth noting that (1.3) can be viewed as a special case of Fredholm integral equation when the kernel $k(t,s)$ vanishes for $s > t$, t is in the range of integration $[a,b]$.

As in Fredholm equations, Volterra integral equations fall under the following kinds, depending on the value of $v(t)$, namely:
First, when $v(t) = 0$, Eq. (1.3) becomes,

$$0 = f(t) + \lambda \int_a^t K(t,s)u(s)ds, \quad a \leq t, \quad s \leq b,$$

and in this case the integral equation is called Volterra integral equation of the first kind.
Secondly, when $v(t) = 1$, Eq. (1.3) becomes,

$$u(t) = f(t) + \lambda \int_a^t K(t,s)u(s)ds, \quad a \leq t, \quad s \leq b,$$

and in this case the integral equation is called Volterra integral equation of the second kind.
Thirdly, when $v(t) \neq 0$, Eq. (1.3) becomes Volterra integral equations of third kind.

Nonlinear Volterra integral equations:

The nonlinear Volterra integral equation of the second kind is represented by the form,

$$u(t) = f(t) + \lambda \int_a^t K(t,s,u(s))ds,$$

The nonlinear Volterra integral equation of the first kind is expressed in the form,

$$f(t) = \lambda \int_a^t K(t,s,u(s))ds,$$

Nonlinear Volterra-Hammerstein integral equations:

The nonlinear Volterra-Hammerstein integral equation of the second kind is represented by the form

$$u(t) = f(t) + \lambda \int_a^t K(t,s)F(s,u(s))ds,$$

Volterra Integro-differential equations:

Volterra, in the early 1900, studied the population growth, where new type of equations have been developed and was termed as integro-differential equations. In this type of equations, the unknown function $u(t)$ occurs in one side as an ordinary derivative, and appears on the other side under the integral sign. Several phenomena in physics and biology give rise to this type of integro-differential equations. Further, we point out that an integro-differential equation can be easily observed as an intermediate stage when we convert a differential equation to an integral equation in next section.
The Volterra integro-differential equation appeared after its establishment by Volterra. It then appeared in many physical applications such as glass forming process, nanohydrodynamics, heat transfer, diffusion process in general, neutron diffusion and biological species coexisting together with increasing and decreasing rates of generating, and wind ripple in the desert. More details about the sources where these equations arise can be found in physics, biology and engineering applications books (see, for example Brunner [12], Volterra [60]). To determine the exact solution for the integro-differential equation, the initial conditions should be given. The Volterra integro-differential equations can be converted to an integral equation by using Leibnitz rule .
Nonlinear Volterra integro-differential equations:
The nonlinear Volterra integro-differential equation of the second kind is in the form

$$u^{(n)}(t) = f(t) + \int_a^t K(t,s,u(s),u'(s),\dots,u^{n-1}(s))ds, \quad u^{(k)}(a) = b_k, \quad 0 \le k \le n-1,$$

and the standard form of the nonlinear Volterra integro-differential equation of the first kind is given by

$$\int_a^t K(t,s,u(s),u'(s),\dots,u^{n-1}(s))ds = f(t),$$

Nonlinear Volterra-Hammerstein integro-differential equations:
The nonlinear Volterra-Hammerstein integro-differential equation of the second kind is in the form

$$u^{(n)}(t) = f(t) + \int_a^t K(t,s)F(s,u(s),u'(s),\ldots,u^{n-1}(s))ds, \quad u^{(k)}(a) = b_k, \quad 0 \leq k \leq n-1,$$

Volterra-Fredholm integral and integro-differential equations

Volterra-Fredholm integral equations:
The Volterra-Fredholm integral equation, which is a combination of disjoint Volterra and Fredholm integrals, appears in one integral equation. The Volterra-Fredholm integral equations arise from the modelling of the spatiotemporal development of an epidemic, from boundary value problems and from many physical and chemical applications [67]. The standard form of the linear Volterra-Fredholm integral equation is in the form

$$u(t) = f(t) + \int_a^t K_1(t,s)u(s)ds + \int_a^b K_2(t,s)u(s)ds,$$

where $k_1(t,s)$ and $k_2(t,s)$ are the kernels of the equation.

Nonlinear Volterra-Fredholm integral equations:
The standard form of the Nonlinear Volterra-Fredholm integral equation is in the form

$$u(t) = f(t) + \int_a^t K_1(t,s,u(s))ds + \int_a^b K_2(t,s,u(s))ds,$$

Nonlinear Volterra-Fredholm-Hammerstein integral equations:
The standard form of the Nonlinear Volterra-Fredholm-Hammerstein integral equation

is in the form

$$u(t) = f(t) + \int_a^t K_1(t,s)F(s,u(s))ds + \int_a^b K_2(t,s)G(s,u(s))ds,$$

where $k_1(t,s)$ and $k_2(t,s)$ are the kernels of the equation.

Volterra-Fredholm integro-differential equations:

The Volterra-Fredholm integro-differential equation, which is a combination of disjoint Volterra and Fredholm integrals and differential operator, may appear in one integral equation. The Volterra-Fredholm integro-differential equations arise from many physical and chemical applications similar to the Volterra-Fredholm integral equations [5], [6], [61], [62]. The standard form of the Volterra-Fredholm integro-differential equation is in the form,

$$u^{(n)}(t) = f(t) + \int_a^t K_1(t,s,u(s),u'(s),\ldots,u^{n-1}(s))ds + \int_a^b K_2(t,s,u(s),u'(s),\ldots,u^{n-1}(s))ds.$$

Nonlinear Volterra-Fredholm-Hammerstein integro-differential equations:

$$u^{(n)}(t) = f(t) + \int_a^t K_1(t,s)F(t,s,u(s),u'(s),\ldots,u^{n-1}(s))ds + \int_a^b K_2(t,s,u(s),u'(s),\ldots,u^{n-1}(s))ds.$$

Singular integral equations

Volterra integral equations of the first kind,

$$f(t) = \lambda \int_{g(t)}^{h(t)} K(t,s)u(s)ds,$$

or of the second kind

$$u(t) = f(t) + \lambda \int_{g(t)}^{h(t)} K(t,s)u(s)ds,$$

are called singular if one of the limit of integration $g(t), h(t)$ is infinite or the kernel $k(t,s)$ becomes unbounded at one or more points in the interval of integration. We focus on concern on equation of the form:

$$u(t) = f(t) + \lambda \int_{0}^{t} \frac{1}{(t-s)^{\alpha}} u(s)ds, \quad 0 \leq \alpha \leq 1, \tag{1.4}$$

or of the second kind

$$f(t) = \lambda \int_{0}^{t} \frac{1}{(t-s)^{\alpha}} u(s)ds, \quad 0 \leq \alpha \leq 1, \tag{1.5}$$

The Eq. (1.4) and Eq.(1.5) are called generalized Abel's integral equation and weakly singular integral equations respectively.
On the other hand, the well known weakly singular Fredholm integral equations of the form,

$$u(t) = f(t) + \int_{0}^{1} k(t,s)u(s)ds, \quad 0 \leq \alpha \leq 1,$$

where the singularity of kernel may be stated in the forms $k(t,s) = \dfrac{1}{(t-s)^{\alpha}}$ or $k(t,s) = \dfrac{1}{(1-t)^{\alpha}}$.

Definition 1.1.1 ***(The homogeneity property)***
We set $f(t) = 0$ in Fredholm or Volterra integral and integro-differential equations as given in the above, the resulting equations is called a homogeneous integral and integro-differential equations, otherwise it is called nonhomogeneous or inhomogeneous integral and integro-differential equations.

Theorem 1.1.1 (Leibnits) *Let $f(x)$ be continuous $[a, b]$,so:*

$$\forall x \in [a, b], \int_0^x \int_0^{x_1} ... \int_0^{x_{n-1}} f(x_n) dx_n ... dx_1 = \frac{1}{(n-1)!} \int_a^x (x-t)^{n-1} f(t) dt.$$

1.2 Conversion of differential equations to integral equations

In general, the initial values problems (IVP) can be transformed to Volterra integral equations, and the boundary values problems (BVP) can be transformed to Fredholm integral equations and virse versa

IVP to Volterra integral equations:

In this section, we will study the technique that will convert an initial value problem (IVP) to an equivalent Volterra integral equation and Volterra integro-differential equation as well [67]. For simplicity reasons, we will apply this process to a second order initial value problem given by

$$u''(t) + p(t)u'(t) + q(t)u(t) = g(t), \tag{1.6}$$

$$u(0) = \alpha, u'(0) = \beta,$$

where α and β are constants. The functions $p(t)$ and $q(t)$ are analytic functions, and $g(t)$ is continuous through the interval of discussion. To achieve our goal we first set

$$u''(t) = v(t), \tag{1.7}$$

where $v(t)$ is a continuous function. Integrating both sides of (1.7) from 0 to t yields

$$u'(t) - u'(0) = \int_0^t v(s)ds,$$

or equivalently

$$u'(t) = \beta + \int_0^t v(s)ds, \tag{1.8}$$

Integrating both sides of (1.8) from 0 to t yields

$$u(t) - u(0) = \beta t + \int_0^t \int_0^s v(r)drds,$$

or equivalently

$$u(t) = \alpha + \beta t + \int_0^t (t-s)v(s)ds, \tag{1.9}$$

obtained upon using the formula that reduce double integral to a single integral that was discussed in the next section. Substituting (1.7), (1.8), and (1.9) into the initial value problem (1.6) yields the Volterra integral equation:

$$u(t) + p(t)\left[\beta + \int_0^t v(s)ds\right] + q(t)\left[\alpha + \beta t + \int_0^t (t-s)v(t)dt\right] = g(t).$$

The last equation can be written in the standard Volterra integral equation form:

$$v(t) = f(t) + \int_0^t k(t,s)v(s)ds, \tag{1.10}$$

where

$$k(t,s) = p(t) + q(t)(t-s),$$

and

$$f(t) = g(t) - [\beta p(t) + \alpha q(t) + \beta t q(t)] .$$

It is interesting to point out that by differentiating Volterra equation (1.10) with respect to t, using Leibnitz rule, we obtain an equivalent Volterra integro-differential equation in the form:

$$u'(t) + k(t,t) = f'(t) - \int_0^t \frac{\partial k(t,s)}{\partial t} u(s)ds, \quad u(0) = f(0),$$

The technique presented above to convert initial value problems to equivalent Volterra integral equations can be generalized by considering the general initial value problem:

$$u^{(n)}(t) + a_1 u^{n-1} + ... + a_{n-1} u^{'} + a_n u = g(t), \tag{1.11}$$

subject to the initial conditions

$$u(0) = c_0, u'(0) = c_1, u''(0) = c_2, ..., u^{n-1} = c_{n-1}.$$

Let $v(t)$ be a continuous function on the interval of discussion, and we consider the transformation:

$$u^{(n)}(t) = v(t). \tag{1.12}$$

Integrating both sides with respect to t gives

$$u^{(n-1)}(t) = c_{n-1} + \int_0^t v(t)dt.$$

Integrating again both sides with respect to t yields

$$\begin{aligned} u^{(n-2)}(t) &= c_{n-2} + c_{n-1} t + \int_0^t \int_0^t v(s) ds ds \\ &= c_{n-2} + c_{n-1} t + \int_0^t (t-s) v(s) ds, \end{aligned}$$

obtained by reducing the double integral to a single integral. Proceeding as before we find

$$u^{(n-3)}(t) = c_{n-3} + c_{n-2}t + \frac{1}{2}c_{n-1}t^2 + \int_0^t \int_0^t \int_0^t v(s)dsdsds$$

$$= c_{n-3} + c_{n-2}t + \frac{1}{2}c_{n-1}t^2 + \frac{1}{2}\int_0^t (t-s)^2 v(s)ds.$$

Continuing the integration process leads to

$$u(t) = \sum_{k=0}^{n-1} \frac{c_k}{k!}t^k + \frac{1}{(n-1)!}\int_0^t (t-s)^{n-1} v(s)ds. \tag{1.13}$$

Substituting (1.12)–(1.13) into (1.11) gives

$$u(t) = f(t) + \int_0^t k(t,s)v(s)ds, \tag{1.14}$$

where

$$k(t,s) = \sum_{k=1}^{n} \frac{a_n}{k-1!}(t-s)^k - 1,$$

and

$$f(t) = g(t) - \sum_{j=1}^{n} a_j \left(\sum_{k=1}^{j} \frac{c_n - k}{(j-k)!} t^j \right).$$

Notice that the Volterra integro-differential equation can be obtained by differentiating (1.14).

The following examples will highlight the process to convert initial value problem to an equivalent Volterra integral equation.

BVP to Fredholm integral equations:

In this section, we will convert a boundary value problem to an equivalent Fredholm integral equation. The method is similar to the method that was presented in the above section for converting Volterra equation to IVP, with the exception that boundary conditions will be used instead of initial values. In this case we will determine another initial condition that is not given in the problem. The technique requires more work if compared with the initial value problems when converted to Volterra integral equations. Without loss of generality, we will present two specific distinct boundary value problems (BVPs) to derive two distinct formulas that can be used for converting BVP to an equivalent Fredholm integral equation [67].

Type I: We first consider the following boundary value problem:

$$u''(t) + g(t)u(t) = h(t), \;\; 0 \leq t \leq 1, \tag{1.15}$$

with the boundary conditions:

$$u(0) = \alpha \quad and \quad u(1) = \beta,$$

we start as in the previous section and set

$$u''(t) = v(t), \tag{1.16}$$

integrating both sides of (1.16) from 0 to t we obtain

$$\int_0^t u''(s)ds = \int_0^t v(s)ds,$$

that gives

$$u'(t) = u'(0) + \int_0^t v(s)ds, \tag{1.17}$$

where the initial condition $u'(0)$ is not given in a boundary value problem. The condition $u'(0)$ will be determined later by using the boundary condition at $t = 1$. Integrating both sides of (1.17) from 0 to t gives

$$u(t) = u(0) + tu'(0) + \int_0^t \int_0^t v(s)dsds,$$

or equivalently

$$u(t) = \alpha + tu'(0) + \int_0^t (t-s)v(s)ds, \tag{1.18}$$

obtained upon using the condition $u(0) = \alpha$ and by reducing double integral to a single integral. To determine $u'(0)$, we substitute $t = 1$ into both sides of (1.15) and using the boundary condition at $u(1) = \beta$ we find

$$u(1) = \alpha + u'(0) + \int_0^1 (1-s)v(s)ds,$$

that gives

$$\beta = \alpha + u'(0) + \int_0^1 (1-s)v(s)ds.$$

This in turn gives

$$u'(0) = \beta - \alpha - \int_0^1 (1-s)v(s)ds. \tag{1.19}$$

Substituting (1.19) into (1.18) gives

$$u(t) = \alpha + (\beta - \alpha)t - \int_0^1 t(1-s)v(s)ds + \int_0^t (t-s)v(s)ds. \tag{1.20}$$

Substituting (1.16) and (1.20) into (1.15) yields

$$u(t) + \alpha g(t) + (\beta - \alpha) t g(t) - \int_0^1 t g(t)(1 - s) v(s) ds + \int_0^t g(t)(t - s) v(s) ds = h(t).$$

Hence, by using Chasles formula, we obtain

$$v(t) = h(t) - \alpha g(t) - (\beta - \alpha) t g(t) - \int_0^t g(t)(t - s) v(s) ds - t g(t) \left[\int_0^t (1 - s) v(s) ds + \int_t^1 (1 - s) v(s) ds \right],$$

that gives

$$v(t) = f(t) + \int_0^t s(1 - t) v(s) ds + \int_t^1 t(1 - s) g(t) v(s) ds, \tag{1.21}$$

that leads to the Fredholm integral equation:

$$v(t) = f(t) + \int_0^1 k(t, s) v(s) ds, \tag{1.22}$$

where

$$f(t) = h(t) - \alpha g(t) - (\beta - \alpha) t g(t),$$

and the kernel $k(t, s)$ is given by

$$k(t, s) = \begin{cases} s(1 - t) g(t), \ for\ 0 \leq s \leq t, \\ \\ s(1 - s) g(t), \ for\ t \leq s \leq 1. \end{cases}$$

An important conclusion can be made here. For the specific case where $u(0) = u(1) = 0$ which means that $\alpha = \beta = 0$, it is clear that $f(t) = h(t)$ in this case. This means that the resulting Fredholm equation in (1.22) is homogeneous or inhomogeneous if the boundary value problem in (1.15) is homogeneous or inhomogeneous respectively when $\alpha = \beta = 0$.

Type II: We next consider the following boundary value problem:
problem:

$$u''(t) + g(t)u(t) = h(t), \ \ 0 \le t \le 1, \tag{1.23}$$

with the boundary conditions:

$$u(0) = \alpha_1, \ \ u'(1) = \beta_1.$$

we again set

$$u''(t) = v(t), \tag{1.24}$$

integrating both sides of (1.21) from 0 to t we obtain

$$\int_0^t u''(s)ds = \int_0^t v(s)ds,$$

that gives

$$u'(t) = u'(0) + \int_0^t v(s)ds, \tag{1.25}$$

where the initial condition $u'(0)$ is not given in a boundary value problem. The condition $u'(0)$ will be derived later by $u'(1) = \beta_1$. Integrating both sides of (1.25) from 0 to t gives

$$u(t) = u(0) + tu'(0) + \int_0^t \int_0^t v(s)dsds,$$

or equivalently

$$u(t) = \alpha_1 + tu'(0) + \int_0^t (t - s)v(s)ds, \tag{1.26}$$

obtained upon using the condition $u(0) = \alpha_1$ and by reducing double integral to a single integral. To determine $u'(0)$, we first differentiate (1.26) with respect to t to get

$$u'(t) = u'(0) + \int_0^t v(s)ds, \tag{1.27}$$

where by substituting $t = 1$ into both sides of (1.27) and using the boundary condition at $u'(1) = \beta_1$ we find

$$u'(t) = \beta_1 + \int_0^t v(s)ds.$$

This in turn gives

$$u'(1) = u'(0) + \int_0^1 v(s)ds. \tag{1.28}$$

Using (1.28) into (1.26) gives

$$u'(0) = \beta_1 - \int_0^1 v(s)ds, \tag{1.29}$$

Substituting (1.24) and (1.29) into (1.23) yields

$$v(t) + \alpha_1 g(t) + \beta_1 t g(t) - \int_0^1 tg(s)v(s)ds + \int_0^t g(t)(t-s)v(s)ds = h(t).$$

Hence, by using Chasles formula, we obtain

$$v(t) = h(t) - (\alpha_1 + \beta_1 t)g(t) + tg(t)\left[\int_0^t v(s)ds + \int_t^1 v(s)ds\right] - g(t)\int_0^t (t-s)v(s)ds.$$

The last equation can be written as

$$v(t) = f(t) + \int_0^t sg(t))v(s)ds + \int_t^1 tg(t)v(s)ds,$$

that leads to the Fredholm integral equation:

$$u(t) = f(t) + \int_0^1 k(t,s)u(s)ds, \tag{1.30}$$

where

$$f(t) = h(t) - (\alpha_1 + \beta_1 t)g(t),$$

and the kernel $k(t,s)$ is given by

$$k(t,s) = \begin{cases} sg(t), \; for\; 0 \leq s \leq t, \\ \\ tg(t), \; for\; t \leq s \leq 1. \end{cases}$$

An important conclusion can be made here. For the specific case where $u(0) = u'(1) = 0$ which means that $\alpha_1 = \beta_1 = 0$, it is clear that $f(t) = h(t)$ in this case. This means that the resulting Fredholm equation in (1.30) is homogeneous or inhomogeneous if the boundary value problem in (1.23) is homogeneous or inhomogeneous respectively.

1.3 Conversion of Volterra integro-differential equations to Volterra integral equation

The following Volterra integro-differential equation

$$u^{(n)}(t) = f(t) + \lambda \int_0^t K(t,s)u(s)ds, \;\; u^{(k)}(0) = b_k, \;\; 0 \leq k \leq n-1, \tag{1.31}$$

can also be solved by converting it to an equivalent Volterra integral equation. It is obvious that the Volterra integro-differential equation (1.31) involves derivatives at the left side, and integral at the right side. To perform the conversion process, we need to integrate both sides n times to convert it to a standard Volterra integral equation. Firstly, Integration of derivatives: from calculus we observe the following:

$$\int_0^t u'(s)ds = u(t) - u(0),$$

$$\int_0^t \int_0^{t_1} u''(s)dsdt_1 = u(t) - tu'(0) - u(0),$$

$$\int_0^t \int_0^{t_1} \int_0^{t_2} u'''(s)dsdt_1dt_2 = u(t) - \frac{1}{2}t^2u''(0) - tu'(0) - u(0),$$

and so on for other derivatives.

Secondly, Reducing multiple integrals to a single integral as follows,

$$\int_0^x \int_0^{x_1} u(t)dtdx_1 = \int_0^x (x-t)u(t)dt,$$

$$\int_0^x \int_0^{x_1} (x-t)u(t)dtdx_1 = \frac{1}{2}\int_0^x (x-t)^2u(t)dt,$$

$$\int_0^x \int_0^{x_1} (x-t)^2u(t)dtdx_1 = \frac{1}{3}\int_0^x (x-t)^3u(t)dt$$

$$\int_0^x \int_0^{x_1} (x-t)^3u(t)dtdx_1 == \frac{1}{4}\int_0^x (x-t)^4u(t)dt,$$

and so on. This can be generalized in the form

$$\int_0^x \int_0^{x_1} ... \int_0^{x_{n-1}} (x-t)u(t)dtdx_{n-1}...dx_1 = \frac{1}{(n)!}\int_0^t (t-s)^n u(t)dt,$$

The conversion to an equivalent Volterra integral equation will be illustrated by studying the following examples.

Example 1.3.1 *Convert the following Volterra integro-differential equation to an Volterra* *integral equation:*

$$u'(x) = 1 + \int_0^x u(t)dt, \ \ u(0) = 0,$$

integrating both sides from 0 *to* x*, and using the aforementioned formulas we find*

$$u(x) - u(0) = x + \int_0^x \int_0^{x_1} u(t)dtdx_1,$$

using the initial condition gives the Volterra integral equation

$$u(x) = x + \int_0^x (x-t)u(t)dt,$$

1.4 Existence and uniqueness of the solution

Consider the nonlinear Volterra integro-differential equation (NVIDE)

$$y^n(x) = f(x) + \int_0^x K(x,t,y(t))ds, \ \ x \in [0,b], \tag{1.32}$$

with n initial conditions

$$u^{(k)}(0) = \alpha_k, \ \ 0 \le k \le n-1,$$

f and K are given smooth functions.

In this section, the existence and uniqueness of the solution for Eq. (1.32) are presented. First we give the following theorem from [45].

Theorem 1.4.1 *Consider the following nonlinear Volterra integral equations*

$$y(x) = f(x) + \int_{0}^{t} k(x,t,y(t))dt, \tag{1.33}$$

Assume that

(i) $f(x)$ *is continuous ,*

(ii) $k(x,t,y(t))$ *is a continuous function for* $0 \leq t \leq s \leq b$ *and* $-\infty \leq |\, y| \ \leq \infty$*,*

(iii) the kernel satisfies the Lipschitz condition

$$|k(x,t,y_1) - k(x,t,y_2)| \leq L|y_1 - y_2|. \tag{1.34}$$

wherer L is independent of t, t, y_1 *and* y_2*. Then the Eq.* (1.32) *has a unique continuous solution in* $0 \leq t \leq b$*.*

Now we consider some cases of the integro-differential equations and investigate existence and uniqueness of the solutions of them.

Corollary 1.4.1

$$y'(x) = f(x) + \int_{0}^{x} K(x,t,y(t))dt, \tag{1.35}$$

with initial condition $y(0) = \alpha$ *where* f *and* K *are continuous functions and* K *satisfies the Lipschitz condition*

$$|\, K(x,t,y_1) - K(x,t,y_2)| \ \leq L|\, y_1 - y_2| \,. \tag{1.36}$$

Then this problem has a unique continuous solution.

Proof. B, Section 1.3, Equation (1.35) transformed to the following Volterra integral equation

$$y(s) = \alpha + \int_0^x H(s, y(s))ds, \qquad (1.37)$$

where $H(s, y(s)) = f(s) + \int_0^s K(s, t, y(t))dt,$

which is in the form of Eq.(1.33), where obviously α and $H(s, y(s))$ are continuous. Therefore, for the existence and uniqueness of a continuous solution of the Eq.(1.35) it is sufficient to show that Eq. (1.37) satisfies the Lipschitz condition. To this end, we have

$$\begin{aligned} ||H(s, y_1(s)) - H(s, y_2(s))|| &= || \int_0^s (K(s, t, y_1(t)) - K(s, t, y_2(t)))dt|| \\ &\leq L_1||y_1 - y_2|| \int_0^s dt \\ &\leq L_1 b||y_1 - y_2||. \end{aligned}$$

So by Theorem (1.4.1), the Eq. (1.35) has a unique continuous solution. ■

Corollary 1.4.2

$$y'(x) + cy(x) = f(x) + \int_0^x K(x, t, y(t))dt, \qquad (1.38)$$

with initial condition $y(0) = \alpha$,the f and K are continuous (1.36) *then the equation* (1.38) *with given condition has a unique continuous solution.*

Proof. B, Section 1.3, Equation (1.38) transformed to the following Volterra integral equation

$$y(s) = \alpha + \int_0^x H(s, y(s)), \qquad (1.39)$$

where $H(s, y(s)) = f(s) + -cy(s) + \int_0^s K(s, t, y(t))dt$, similar to the previous corollary we only investigate the Lipschitz condition. To this end, we have

$$\begin{aligned} \|H(s, y_1(s)) - H(s, y_2(s))\| &= \|c[y_1(s) - y_2(s)] + \int_0^s (K(s, t, y_1(t)) - K(s, t, y_2(t)))dt\| \\ &\leq |c|\|y_1 - y_2\| + L_1\|y_1 - y_2\| \int_0^s dt \\ &\leq (c + bL_1)\|y_1 - y_2\|. \end{aligned}$$

Again, by Theorem (1.4.1), the Eq. (1.38) has a unique continuous solution. ■

Corollary 1.4.3

$$y''(x) + c_1 y(x) + c_2 y(x) = f(x) + \int_0^x K(x, t, y(t))dt, \tag{1.40}$$

with initial condition $y(0) = \alpha$, $y'(0) = \beta$ *,the* f *and* K *are continuous* (1.36) *Then the mentioned problem has a unique continuous solution.*

Proof. With the same manner, Volterra integro-differential equation(1.40) by converting it to the following Volterra integral equation

$$y(s) = \alpha + (\beta - c_1\alpha)z + \int_0^x H(s, y(s))dx.$$

where $H(s, y(s)) = -cy(s) + \int_0^x \left(f(s) - c_2 y(s) + \int_0^s K(s, t, y(t))dt \right) ds$, then we obtain

$$\|H(s, y_1(s)) - H(s, y_2(s))\|$$
$$= \|c_1[y_2(s) - y_1(s)] + \int_0^x \left(c_2(y_2(s) - y_{1(s)}) + \int_0^s (K(s,t,y_1(t)) - K(s,t,y_2(t))dt) \right) ds\|$$
$$\leq |c_1|\|y_1 - y_2\| + b|c_2|\|y_1 - y_2\| + L_1\|y_1 - y_2\| \int_0^x \int_0^s dtds$$
$$\leq (|c_1| + b|c_2| + b^2 L_1)\|y_1 - y_2\|.$$

Similar to previous cases, by Theorem (1.4.1), the Eq. (1.40) has a unique continuous solution. ■

The same conclusion can be drawn for the following Volterra integro-differential equation of order n

$$y^n(x) + \int_0^x K(x,t,y(t))ds = f(x), \quad x \in [0,b],$$

with conditions $y^i(0) = \alpha_i, \ \ i = 0,1,...,n-1$, and similar to the previous corollaries we can convert this problem to an equation of the form (1.32).

1.5 Piecewise polynomial spaces

Let:

$$I_h = \{t_n = t_n^{(N)} : 0 = t_0^{(N)} < t_1^{(N)} < ... < t_N^{(N)} = T\}$$

denote a mesh (or: grid) on the given interval $I = [0,T]$. Define the subintervals

$$\delta_n^{(N)} = \left[t_n^{(N)}, t_{n+1}^{(N)}\right]$$

Definition 1.5.1 *For a given mesh I_h the piecewise polynomial space $S_\mu^{(d)}(I_h)$ with $\mu \geq 0, -1 \leq d \leq \mu$, is given by*

$$S_\mu^{(d)}(I_h) = \{v \in C^d(I) : v|_{\sigma_n} \in \pi_\mu (0 \leq n \leq N-1)\}$$

Here , π_μ denotes the space of (real) polynomials of degree not exceeding μ .
It is readily verified that $S_\mu^{(d)}(I_h)$ is a (real) linear vector space whose dimension is given by

$$\dim S_\mu^{(d)}(I_h) = N(\mu - d) + d + 1$$

Remark 1.5.1 *The particular piecewise polynomial space $S_{m+d}^{(d)}(I_h)$ corresponding to $\mu = m + d$ with $m \geq 1$ and $d \geq -1$ will play a central role in the chapter 2 and 3.*
Since its dimension is

$$\dim S_{m+d}^{(d)}(I_h) = Nm + (d + 1), \tag{1.41}$$

it may be viewed as the 'natural' collocation space for the approximation of solutions to initial value problems for Volterra equations, the choice of the degree of regularity d will be governed by the number of prescribed initial conditions, while the term Nm suggests that m (distinct) collocation points are to be placed in each of the N subintervals σ_n . Thus, the natural choice of d in (1.41) is as follows:

- For Volterra integral equations (no initial condition) we choose $d = -1$; hence, the natural collocation space will be $S_{m-1}^{(-1)}(I_h)$. Its dimension is Nm.
- For first-order ODEs or Volterra integro-differential equations (one initial condition) we use $d = 0$, and the preferred collocation space is $S_m^{(0)}(I_h)$, with dimension equal to $Nm + 1$.
- For ODEs or VIDEs of ferst order with initial conditions the natural collocation space is $S_{m+1}^{(1)}(I_h)$, corresponding to the choice $d = 1$. The dimension of this space is $Nm + 2$.

1.6 Collocation method

A collocation method is based on the idea of approximating the exact solution of a given integral equation with a suitable function belonging to a chosen finite dimensional space

such that the approximated solution satisfies the integral equation on a certain subset of the interval on which the equation has to be solved (called the set of collocation points). In our thesis, we consider the polynomial spline space as the approximating space. In order to describe the relevant collocation method for given N, let Π_N be a uniform partition of a bounded interval $I = [0, T]$ with gride points $t_n = nh, n = 0, 1, ..., N$, where h is the stepsize. Define the subintervals $\delta_n = [t_n, t_{n+1}], n = 0, ..., N-1$.
So, the real polynomial spline spaces of degrees $m, m+k-1$, which will be used in chapters 2, 3, is defined as follows:

$$S_{m-1}^{(-1)}(I, \Pi_N) = \{u : u_n = u/_{\sigma_n} \in \pi_{m-1}, n = 0, .., N-1\}.$$

$$S_m^{(0)}(\Pi_N) = \{u \in C(\mathrm{I}, \mathbb{R}) : u_n = u/\sigma_n \in \pi_m, n = 0, ..., N-1, \}.$$

$$S_{m+1}^{(1)}(\Pi_N) = \{u \in C^1(\mathrm{I}, \mathbb{R}) : u_n = u/\sigma_n \in \pi_{m+1}, n = 0, ..., N-1\}.$$

1.7 Review of basic discrete Gronwall-type inequalities

In this section, we give general results of discrete Gronwall-type inequalities. We will need the following discrete Gronwall-type inequalities.

Lemma 1.7.1 *[12] Let $\{k_j\}_{j=0}^n$ be a given non-negative sequence and the sequence $\{\varepsilon_n\}$ satisfies $\varepsilon_0 \leq p_0$ and*

$$\varepsilon_n \leq p_0 + \sum_{i=0}^{n-1} k_i \varepsilon_i, \quad n \geq 1,$$

with $p_0 \geq 0$. Then ε_n can be bounded by

$$\varepsilon_n \leq p_0 \exp\left(\sum_{j=0}^{n-1} k_j\right), \quad n \geq 1.$$

Lemma 1.7.2 *[20] Assume that $(\alpha_n)_{n\geq 1}$ and $(q_n)_{n\geq 1}$ are given non-negative sequences and the sequence $(\varepsilon_n)_{n\geq 1}$ satisfies*

$\varepsilon_1 \le \beta$ and

$$\varepsilon_n \le \beta + \sum_{j=1}^{n-1} q_j + \sum_{j=1}^{n-1} \alpha_j \varepsilon_j, \ n \ge 2.$$

Then

$$\varepsilon_n \le \left(\beta + \sum_{j=1}^{n-1} q_j\right) \exp\left(\sum_{j=1}^{n-1} \alpha_j\right), \ n \ge 2.$$

Lemma 1.7.3 *[1] If $\{f_n\}_{n\ge0}$, $\{g_n\}_{n\ge0}$ and $\{\varepsilon_n\}_{n\ge0}$ are nonnegative sequences and*

$$\varepsilon_n \le f_n + \sum_{i=0}^{n-1} g_i \varepsilon_i, \quad n \ge 0.$$

Then,

$$\varepsilon_n \le f_n + \sum_{i=0}^{n-1} f_i g_i \exp\left(\sum_{k=0}^{n-1} g_k\right), \quad n \ge 0.$$

The following three lemmas will be used in this section.

Lemma 1.7.4 *[32] Assume that the sequence $\{\varepsilon_n\}_{n\ge0}$ of nonnegative numbers satisfies*

$$\varepsilon_n \le A\varepsilon_{n-1} + B \sum_{i=0}^{n-1} \varepsilon_i + K, \quad n \ge 0,$$

where A, B and K are nonnegative constants, then

$$\varepsilon_n \le \frac{\varepsilon_0}{R_2 - R_1}\left[(R_2 - 1)R_2^n + (1 - R_1)R_1^n\right] + \frac{K}{R_2 - R_1}\left[R_2^n - R_1^n\right],$$

where

$$\begin{aligned} R_1 = &\left(1 + A + B - \sqrt{(1 - A)^2 + B^2 + B + 2B}\right)/2, \\ R_2 = &\left(1 + A + B + \sqrt{(1 - A)^2 + B^2 + 2AB + 2B}\right)/2, \end{aligned} \tag{1.42}$$

therefore, $0 \le R_1 \le 1 \le R_2$.

Part I

Numerical solution of nonlinear Volterra integral equations

INTRODUCTION

In this part, we study a numerical method based on iterative and iterative continuous collocation method for the solution of nonlinear Volterra integral equations of the form,

$$x(t) = f(t) + \int_0^t K(t, s, x(s))ds, t \in I = [0, T], \quad (1.43)$$

where the functions f, K are sufficiently smooth.

The integral equations are often involved in various fields such as physics and biology (see, for example [12, 37, 46]), and they also occur as reformulations of other mathematical problems, such as ordinary differential equations and partial differential equations (see [37]).

There has been a growing interest in the numerical solution of Equation (1.43) (see, for example, [3, 4, 23, 24, 62, 29, 30, 36, 46, 49, 48, 54]) such as, Chebyshev approximation [3], Adomian's method [4, 46], Taylor polynomial approximations [62], homotopy perturbation method [29], the series expansion method [30], fixed point method [49], Haar wavelet method [48], rationalized Haar functions method [54]. Moreover, many collocation methods for approximating the solutions for Equation (1.43) have been developed recently (see, [12, 27, 52, 56, 71]) such as Lagrange spline collocation method

[12], cubic B-spline collocation method [27], quintic B-spline collocation method [52], Taylor collocation method [56], and sinc-collocation method for Volterra integral equations is used in [71]. The numerical solution of these equations has a high computational cost due to the nonlinearity and most of the collocation methods for nonlinear Volterra integral equations transform Equation (1.43) into a system of nonlinear algebraic equations.

iterative explicit solution to approximate the solution of nonlinear Volterra integral equation (1.43). The main advantages of the current collocation method are that it is direct and there is no algebraic system to be solved, which makes the proposed algorithm very effective, easy to implement and the calculation cost low.

This part is organized as follows: In chapter 1, we approximate the solution of (1.43) in the polynomial spline space of degree $m-1$ as follows:

$$S_{m-1}^{(-1)}(I,\Pi_N) = \{u : u_n = u|_{\sigma_n} \in \pi_{m-1}, n = 0,..,N-1\}.$$

In chapter 2, we approximate the solution of (1.43) in the polynomial spline space of degree m as follows

$$S_m^{(0)}(\Pi_N) = \{u \in C(\mathrm{I},\mathbb{R}) : u_n = u/\sigma_n \in \pi_m, n = 0,...,N-1\}.$$

CHAPTER 2

ITERATIVE COLLOCATION METHOD FOR SOLVING NONLINEAR VOLTERRA INTEGRAL EQUATIONS

2.1 Introduction

In this chapter, we consider the following nonlinear Volterra integral equations, (1.43) There are many collocation methods for approximating the solutions for Equation (1.43) have been developed recently (see, [12, 27, 52, 56, 71]) such as Lagrange spline collocation method [12], cubic B-spline collocation method [27], quintic B-spline collocation method [52], Taylor collocation method [56], and sinc-collocation method for Volterra integral equations is used in [71].
The numerical solution of these equations has a high computational cost due to the nonlinearity and most of the collocation methods for nonlinear Volterra integral equations transform Equation (1.43) into a system of nonlinear algebraic equations.
The remainder of the work is organized as follows. In section 2, we divide the interval $[0, T]$ into subintervals, and we approximate the solution of (1.43) in each interval by using iterative Lagrange polynomials. Global convergence is established in section 3. Numerical examples are provided in section 4. In the last section, we give a conclusion.

2.2 Description of the method

Let Π_N be a uniform partition of the interval $I = [0, T]$ defined by $t_n = nh, \quad n = 0, ..., N-1$, where the stepsize is given by $\dfrac{T}{N} = h$. Let the collocation parameters be $0 \leq c_1 < ...... < c_m \leq 1$ and the collocation points be $t_{n,j} = t_n + c_j h,\ j = 1, ..., m, n = 0, ..., N-1$.
Define the subintervals $\sigma_n = [t_n, t_{n+1}]$, and $\sigma_{N-1} = [t_{N-1}, t_N]$.
Moreover, denote by π_m the set of all real polynomials of degree not exceeding m.
We define the real polynomial spline space of degree m as follows:

$$S_{m-1}^{(-1)}(I, \Pi_N) = \{u : u_n = u|_{\sigma_n} \in \pi_{m-1}, n = 0, .., N-1\}.$$

This is the space of piecewise polynomials of degree at most $m-1$. Its dimension is Nm. It holds for any $x \in C^m([0,T])$ that

$$x(t_n + sh) = \sum_{j=1}^{m} L_j(s)x(t_{n,j}) + \epsilon_n(s), \;\; \epsilon_n(s) = h^m \frac{x^m(\zeta_n(s))}{m!} \prod_{j=1}^{m}(s - c_j), \tag{2.1}$$

where $s \in [0,1]$ and $L_j(v) = \prod_{l \neq j}^{m} \dfrac{v - c_l}{c_j - c_l}$ are the Lagrange polynomials associate with the parameters c_j, $j = 1, ..., m$.

Inserting (2.1) into(1.43), we obtain for each $j = 1, ..., m, , n = 0, ..., N-1$

$$\begin{aligned} x(t_{n,j}) = f(t_{n,j}) + h \sum_{p=0}^{n-1} \sum_{v=1}^{m} b_v K(t_{n,j}, t_{pv}, x(t_{pv})) + h \sum_{v=1}^{m} a_{j,v} K(t_{n,j}, t_{n,v}, x(t_{n,v})) \\ + o(h^m), \end{aligned} \tag{2.2}$$

such that $a_{j,v} = \int_0^{c_j} L_v(\eta) d\eta$ and $b_v = \int_0^1 L_v(\eta) d\eta$.

It holds for any $u \in S_{m-1}^{-1}(I, \Pi_N)$ that

$$u(t_n + sh) = \sum_{j=1}^{m} L_j(s) u(t_{n,j}), s \in [0,1]. \tag{2.3}$$

Now, we approximate the exact solution x by $u \in S_{m-1}^{-1}(I, \Pi_N)$ such that $u(t_{n,j})$ satisfy the following nonlinear system,

$$u(t_{n,j}) = f(t_{n,j}) + h \sum_{p=0}^{n-1} \sum_{v=1}^{m} b_v K(t_{n,j}, t_{pv}, u(t_{pv})) + h \sum_{v=1}^{m} a_{j,v} K(t_{n,j}, t_{n,v}, u(t_{n,v})), \tag{2.4}$$

for $j = 1, ..., m$, $n = 0, ..., N-1$. Since the above system is nonlinear, we will use an iterative collocation solution $u^q \in S_{m-1}^{-1}(I, \Pi_N), q \in \mathbb{N}$, to approximate the exact solution of (1.43) such that

$$u^q(t_n + sh) = \sum_{j=1}^{m} L_j(s) u^q(t_{n,j}), s \in [0,1], \tag{2.5}$$

where the coefficients $u^q(t_{n,j})$ are given by the following formula:

$$u^q(t_{n,j}) = f(t_{n,j}) + h\sum_{p=0}^{n-1}\sum_{v=1}^{m} b_v K(t_{n,j}, t_{pv}, u^q(t_{pv})) + h\sum_{v=1}^{m} a_{j,v} K(t_{n,j}, t_{n,v}, u^{q-1}(t_{n,v})), \tag{2.6}$$

such that the initial values $u^0(t_{n,j}) \in J$ (J is a bounded interval).
The above formula is explicit and the approximate solution u^q is given without needed to solve any algebraic system.
In the next section, we will prove the convergence of the approximate solution u^q to the exact solution x of (1.43), moreover, the order of convergence is m for all $q \geq m$.

2.3 Convergence analysis

In this section, we assume that the functions K satisfy the Lipschitz condition with respect to the third variable: there exist $L_1 \geq 0$ such that

$$|K(t,s,y_1) - K(t,s,y_2)| \leq L|y_1 - y_2|.$$

The following result gives the existence and the uniqueness of a solution for the nonlinear system (2.4).

Lemma 2.3.1 *For sufficiently small h, the nonlinear system* (2.4) *has a unique solution $u \in S^{-1}_{m-1}(I, \Pi_N)$. Moreover, the function u is bounded.*

Proof. Claim 1. The nonlinear system (2.4) has a unique solution in $S^{-1}_{m-1}(I, \Pi_N)$.
We will use the induction combined with the Banach fixed point theorem.

(i) On the interval $\sigma_0 = [t_0, t_1]$, the nonlinear system (2.4) becomes

$$u(t_{0,j}) = f(t_{0,j}) + h\sum_{v=1}^{m} a_{j,v} K(t_{0,j}, t_{0,v}, u(t_{0,v})), \, j = 1, ..., m.$$

We consider the operator Ψ defined by:

$$\Psi : \mathbb{R}^m \longrightarrow \mathbb{R}^m$$
$$x = (x_1, ..., x_m) \longmapsto \Psi(x) = (\Psi_1(x), ..., \Psi_m(x)),$$

such that for $j = 1, ..., m$, we have

$$\Psi_j(x) = f(t_{0,j}) + h \sum_{v=1}^{m} a_{j,v} K(t_{0,j}, t_{0,v}, x_v).$$

Hence, for all $x, y \in \mathbb{R}^m$, we have

$$\|\Psi(x) - \Psi(y)\| \leq hmaL\|x - y\|.$$

where $a = \max\{|a_{j,v}|, j = 1, ..., m, v = 1, ..., m\}$.
Since $hmaL < 1$ for sufficiently small h, then by Banach fixed point theorem, the nonlinear system (2.4) has a unique solution u on the interval σ_0.

(ii) Suppose that u exists and unique on the intervals $\sigma_i, i = 0, ..., n-1$ for $n \geq 1$ and we show that u exists and unique on the interval σ_n.
On the interval σ_n, the nonlinear system (2.4) becomes

$$u(t_{n,j}) = F(t_{n,j}) + h \sum_{v=1}^{m} a_{j,v} K(t_{n,j}, t_{n,v}, u(t_{n,v})), \ j = 1, ..., m \tag{2.7}$$

where, $F(t_{n,j}) = f(t_{n,j}) + h \sum\limits_{p=0}^{n-1} \sum\limits_{v=1}^{m} b_v K(t_{n,j}, t_{pv}, u(t_{pv}))$.
We consider the operator Ψ defined by:

$$\Psi : \mathbb{R}^m \longrightarrow \mathbb{R}^m$$
$$x = (x_1, ..., x_m) \longmapsto \Psi(x) = (\Psi_1(x), ..., \Psi_m(x)),$$

such that for $j = 1, ..., m$, we have

$$\Psi_j(x) = F(t_{n,j}) + h \sum_{v=1}^{m} a_{j,v} K(t_{n,j}, t_{n,v}, x_v).$$

Hence, for all $x, y \in \mathbb{R}^m$, we have

$$\|\Psi(x) - \Psi(y)\| \leq hmaL\|x - y\|$$

Since $hmaL < 1$ for sufficiently small h, then by Banach fixed point theorem, the nonlinear system (2.7) has a unique solution u on the interval σ_n.

Claim 2. The solution u is bounded.

We have, from (2.4), for $n = 0, ..., N-1$ and $j = 1, ..., m$,

$$\begin{aligned} \left|u(t_{n,j})\right| \leq & |f(t_{n,j})| + h \sum_{p=0}^{n-1} \sum_{v=1}^{m} |b_v||K(t_{n,j}, t_{pv}, 0))| + h \sum_{v=1}^{m} |a_{j,v}||K(t_{n,j}, t_{n,v}, 0))| \\ & + hL \sum_{p=0}^{n-1} \sum_{v=1}^{m} |b_v||u(t_{pv})| + hL \sum_{v=1}^{m} |a_{j,v}||u(t_{n,v})| \\ \leq & \|f\| + hbm\overline{K}T + Tma\overline{K} + hLb \sum_{p=0}^{n-1} \sum_{v=1}^{m} |u(t_{pv})| + hLa \sum_{v=1}^{m} |u(t_{n,v})|, \end{aligned} \tag{2.8}$$

where $\|f\| = \max\{|f(t)|, t \in I\}, b = \max\{|b_j|, j = 1, ..., m\}$,
$a = \max\{|a_{j,v}|, j = 1, ..., m, v = 1, ..., m\}$.
Now, we consider the sequence $y_n = \max\{u(t_{n,p}), p = 1, ..., m\}$ for $n = 0, ..., N-1$.
Then, from (2.8), y_n satisfies for $n = 0, ..., N-1$,

$$y_n \leq \underbrace{\|f\| + hbm\overline{K}T + Tma\overline{K}}_{\alpha} + hLbm \sum_{p=0}^{n-1} y_p + hLamy_n,$$

Hence, for $\bar{h} < \dfrac{1}{Lam}$, we have for all $h \in (0, \bar{h}]$

$$y_n \leq \frac{\alpha}{1-\bar{h}Lam} + \frac{hLbm}{1-\bar{h}Lam}\sum_{p=0}^{n-1} y_p.$$

We deduce, by Lemma (1.7.1), that for all $n = 0, ..., N-1$

$$y_n \leq \frac{\alpha}{1-\bar{h}Lam}\exp\left(\frac{TLbm}{1-\bar{h}Lam}\right).$$

Thus, by using (2.3), we deduce that u is bounded. ■

The following result gives the convergence of the approximate solution u to the exact solution x.

Theorem 2.3.1 *Let f, K be m times continuously differentiable on their respective domains. Then for sufficiently small h, the collocation solution u converges to the exact solution x, and the resulting error function $e := x - u$ satisfies:*

$$||e|| \leq Ch^m,$$

where C is a finite constant independent of h.

Proof. We have, from (2.4) and (2.2), for all $n = 0, ..., N-1$ and $j = 1, ..., m$

$$\begin{aligned} |e(t_{n,j})| \leq & hL\sum_{p=0}^{n-1}\sum_{v=1}^{m}|b_v|\left|e(t_{pv})\right| + hL\sum_{v=1}^{m}a_{j,v}\left|e(t_{n,v})\right| + \alpha h^m \\ \leq & hLb\sum_{p=0}^{n-1}\sum_{v=1}^{m}\left|e(t_{pv})\right| + hLa\sum_{v=1}^{m}\left|e(t_{n,v})\right| + \alpha h^m, \end{aligned} \tag{2.9}$$

where α is a positive number.

We consider the sequence $e_n = \max\{|e(t_{n,v})|, v = 1, ..., m\}$ for $n = 0, ..., N-1$.

Then, from (2.9), e_n satisfies for $n = 0, ..., N-1$,

$$e_n \leq hLbm \sum_{p=0}^{n-1} \sum_{v=1}^{m} e_p + hLame_n + \alpha h^m,$$

Hence, for $\overline{h} < \dfrac{1}{Lam}$, we have for all $h \in (0, \overline{h}]$

$$e_n \leq \frac{\alpha}{1 - \overline{h}Lam} h^m + \frac{hLbm}{1 - \overline{h}Lam} \sum_{p=0}^{n-1} e_p.$$

Then, by Lemma (1.7.1), for all $n = 0, ..., N-1$

$$e_n \leq \frac{\alpha}{1 - \overline{h}Lam} h^m \exp\left(\frac{TLbm}{1 - \overline{h}Lam}\right).$$

Therefore, by using (2.1) and (2.3), we obtain

$$\begin{aligned} \|e\| &\leq m\overline{L} \max\{e_n, n = 0, ..., N-1\} + \alpha h^m \\ &\leq m\overline{L} \frac{\alpha}{1 - \overline{h}Lam} \exp\left(\frac{TLbm}{1 - \overline{h}Lam}\right) h^m + \alpha h^m, \end{aligned}$$

where $\overline{L} = \max\{|L_j(s)|, j = 1, ..., m, s \in [0, 1]\}$.
Thus, the proof is completed by taking $C = m\overline{L} \frac{\alpha}{1 - \overline{h}Lam} \exp\left(\frac{TLbm}{1 - \overline{h}Lam}\right) + \alpha$. ■
The following result gives the convergence of the iterative solution u^q to the exact solution x.

Theorem 2.3.2 *Consider the iterative collocation solution* $u^q, q \geq 1$ *defined by* (2.5) *and* (2.6), *then for any initial condition* $u^0(t_{n,j}) \in J$, *the iterative collocation solution* $u^q, q \geq 1$ *converges to the exact solution* x. *Moreover, for sufficiently small* h, *the following error estimate holds*

$$\|u^q - x\| \leq d\beta^q h^q + Ch^m,$$

where d, β *and* C *are finite constants independent of* h.

Proof. We define the error e^q and ξ^q by $e^q(t) = u^q(t) - x(t)$ and $\xi^q = u^q(t) - u(t)$, where u is defined by lemma (2.3.1).
We have, from (2.4) and (2.6), for all $n = 0, ..., N-1$ and $j = 1, ..., m$

$$|\xi^q(t_{n,j})| \leq hLb \sum_{p=0}^{n-1} \sum_{v=1}^{m} \left|\xi^q(t_{pv})\right| + hLa \sum_{v=1}^{m} \left|\xi^{q-1}(t_{n,v})\right|.$$

Now, for each fixed $q \geq 1$, we consider the sequence $\xi_n^q = \max\{\left|\xi^q(t_{n,v})\right| v = 1....m\}$ for $n = 0, ..., N-1$, it follows that,

$$\xi_n^q \leq hLbm \sum_{p=0}^{n-1} \xi_p^q + hLam\xi_n^{q-1}.$$

Hence, by Lemma (1.7.3), for all $n = 0, ..., N-1$

$$\xi_n^q \leq hLam\xi_n^{q-1} + h^2L^2abm^2 \sum_{p=0}^{n-1} \xi_p^{q-1} \exp(TLam). \tag{2.10}$$

We consider the sequence $\eta^q = \max\{\xi_n^q, n = 0, ..., N-1\}$ for $q \geq 1$.
Then, from (2.10), η^q satisfies,

$$\begin{aligned}
\eta^q &\leq hLam\eta^{q-1} + ThL^2abm^2\eta^{q-1} \exp(TLam) \\
&= \underbrace{(Lam + TL^2abm^2 \exp(TLam))}_{\beta} h\eta^{q-1} \\
&\leq \beta^2h^2\eta^{q-2} \leq ... \leq \beta^q h^q \eta^0.
\end{aligned}$$

Since, $u^0(t_{n,j}) \in J$ (bounded interval) and u is bounded by lemma (2.3.1), then there exists $\delta > 0$ such that $\eta^0 < \delta$, which implies that, for all $q \geq 1$

$$\eta^q \leq \delta\beta^q h^q.$$

Therefore, by using (2.3) and (2.5), we obtain

$$\|\xi^q\| \leq m\overline{L}\eta^q \leq m\overline{L}\delta\beta^q h^q,$$

Hence, by theorem (2.3.1), we deduce that

$$\|e^q\| \leq \|\xi^q\| + \|u - x\| \leq d\beta^q h^q + Ch^m.$$

Thus, the proof is completed. ■

2.4 Numerical examples

To illustrate the theoretical results obtained in the previous section, we present the following examples with $T = 1$. All the exact solutions x are already known. In each example, we calculate the error between x and the iterative collocation solution u^m.
We compare our results by other methods in [66, 27, 36, 4, 52, 49].
The results in these examples confirm the theoretical results; moreover, the results obtained by the present method is very superior to that obtained by the methods in [66, 27, 36, 4, 52, 49].

Example 2.4.1 *([66]) We consider the following linear Volterra integral equation of second kind*

$$x(t) = (1 + \lambda t - \lambda t^2)e^t - \lambda t + \lambda \int_0^t tsx(s)ds, t \in [0, 1],$$

where $\lambda = \frac{1}{10}$ *and the exact solution is* $x(t) = e^t$.
The absolute errors for $N = 4$ *and* $m = q = 5$ *and* $m = q = 6$*at* $t = 0.25, 0.5, 0.75, 1$ *are compared with the absolute error of Iterative method [66] in Table 2.1 .*

Table 2.1: Comparison of the absolute errors of Example 2.4.1

t	Iterative method [66]		Present method	
	e_5	e_6	$m = 5$	$m = 6$
0.25	7.85 E −5	7.85 E −5	9.63 E −8	3.43 E −8
0.5	3.72 E −6	3.72 E −6	1.14 E −7	9.29 E −9
0.75	1.79 E −5	1.79 E −5	1.25 E −7	4.61 E −9
1.0	2.17 E −6	2.17 E −6	8.38 E −7	1.25 E −8

Example 2.4.2 *([27, 36]) Consider the following nonlinear Volterra integral equation*

$$x(t) = 1 + (\sin(t))^2 - \int_0^t 3\sin(t-s)(x(s))^2 ds, \; t \in [0,1],$$

where $u(x) = \cos(x)$ *is the exact solution. We used the new method and obtained the results shown in table 2.2.*
The absolute errors for $N = 10, 20$ *and* $m = 4$ *at* $t = 0, 0.1, ..., 1$ *are displayed in Table 2.2 . The numerical results of the present method are considerable accurate in comparison with the numerical results obtained by [27, 36].*

Table 2.2: Comparison of the absolute errors of Example 2.4.2

t	Method in [27]		Method in [36]		Our method	
	$N = 10$	$N = 20$	$N = 10$	$N = 20$	$N = 10$	$N = 20$
0.1	1.01 E −5	1.59 E −6	1.24 E −5	2.54 E −8	9.82 E −8	5.57 E −9
0.2	2.48 E −5	3.26 E −6	1.62 E −6	3.44 E −7	1.41 E −7	3.34 E −9
0.3	3.65 E −5	4.72 E −6	2.03 E −4	9.19 E −7	2.04 E −7	1.47 E −8
0.4	4.61 E −5	5.87 E −6	2.07 E −5	1.44 E −6	2.72 E −7	1.60 E −8
0.5	5.26 E −5	6.63 E −6	3.84 E −5	1.88 E −6	3.56 E −7	2.28 E −8
0.6	5.59 E −5	6.98 E −6	5.11 E −5	2.18 E −6	4.34 E −7	2.67 E −8
0.7	5.58 E −5	6.92 E −6	7.22 E −5	1.83 E −6	5.11 E −7	3.46 E −8
0.8	5.28 E −5	6.47 E −6	6.43 E −5	6.41 E −6	5.62 E −7	3.26 E −8
0.9	4.65 E −5	5.70 E −6	1.96 E −5	1.00 E −4	6.02 E −7	4.50 E −8
1	3.97 E −5	4.71 E −6	6.36 E −4	9.25 E −4	4.66 E −7	2.43 E −8

Example 2.4.3 *([4, 52, 49]) Consider the following linear Volterra integral equation with exact solution* $y(t) = 1 - \sinh(t)$*:*

$$x(t) = 1 - t - \frac{t^2}{2} \int_0^t (t-s)x(s)ds, \ t \in [0,1].$$

The absolute errors for $m = 4$ *and* $N = 20$ *at* $t = 0, 0.2, ..., 1$ *are displayed in Table 2.3. The numerical results obtained here are compared in Table 2.3 with the numerical results obtained by using the methods in [4, 52].*

It is seen from Table 2.3 that the results obtained by the present method is very superior to that obtained by the methods in [4, 52].

Table 2.3: Comparison of the absolute errors of Example 2. 4.3

t	Our method $N = 20$	Method in [4] $N = 20$	Method in [52] $N = 20$
0.0	1.00 E −9	0	1.98 E −14
0.1	4.80 E −10	5.63 E −6	1.21 E −7
0.2	1.85 E −9	2.20 E −5	2.35 E −7
0.3	2.05 E −9	4.82 E −5	3.54 E −7
0.4	1.69 E −9	8.33 E −5	4.77 E −7
0.5	2.90 E −9	1.26 E −4	6.05 E −7
0.6	4.05 E −9	1.77 E −4	7.39 E −7
0.7	4.56 E −9	2.34 E −4	8.80 E −7
0.8	5.41 E −9	2.97 E −4	1.03 E −6
0.9	6.89 E −9	3.65 E −4	1.19 E −6
1	1.55 E −8	4.38 E −4	1.36 E −6

Example 2.4.4 *([49]) We consider the following nonlinear Volterra integral equation*

$$x(t) = \frac{t}{e^{t^2}} + \int_0^t 2tse^{-x^2(s)}ds, t \in [0,1],$$

where the exact solution is $x(t) = t$.
The absolute errors for $N = 20$ *and* $m = q = 10$ *at* $t = 0, 0.2, ..., 1$ *are compared with the absolute error of the method in [49] in Table 2.4 .*

Table 2.4: Comparison of the absolute errors of Example 2. 4.4

t	Method in [66] $N = 20$	Our method $N = 20$
0	0	2.58 E −9
0.2	1.49 E −8	2.57 E −8
0.4	7.74 E −7	2.61 E −7
0.6	9.36 E −6	4.54 E −7
0.8	4.58 E −5	1.51 E −6
1	1.29 E −4	1.60 E −6

Example 2.4.5 *([49]) We consider the following nonlinear Volterra integral equation*

$$x(t) = t\cos(t) + \int_0^t t\sin(x(s))ds, t \in [0,1],$$

where the exact solution is $x(t) = t$.

The absolute errors for $N = 25$ *and* $m = q = 4$ *at* $t = 0.001, 0.2, 0.4, 0.6, 0.8, 1$ *are compared with the absolute error of the method in [49] in Table* 2.4.5 . *It is seen from Table 2.5 that the results obtained by the present method is very superior to that obtained by the method in [49].*

Table 2.5: Comparison of the absolute errors of Example 2. 4.5

t	Method in [49] $N = 25$	Our method $N = 25$
0.001	4.65 E −10	1.75 E −11
0.2	3.53 E −6	3.00 E −10
0.4	5.81 E −6	4.00 E −10
0.6	7.74 E −7	9.00 E −10
0.8	1.20 E −5	4.00 E −10
1	3.98 E −5	3.68 E −8

2.5 Conclusion

n this chapter, we have proposed a iterative collocation method based on the use of Lagrange polynomials to approximate the solution of the volterra integral equation (1.43) in the spline space $S_{m-1}^{(-1)}(I, \Pi_N)$ We have shown that the numerical solution is convergent. This method is easy to implement, and the coefficients of the approximation solution are determined by iterative formulas without the need to solve any system of algebraic equations. The numerical examples introduced have shown that the method is convergent with a good accuracy

CHAPTER 3

ITERATIVE CONTINUOUS COLLOCATION METHOD FOR SOLVING NONLINEAR VOLTERRA INTEGRAL EQUATIONS

3.1 Introduction

In this chapter, we consider the following nonlinear Volterra integral equations, (1.43) where the functions f, K are sufficiently smooth.

There are many collocation methods for approximating the solutions for Equation (1.43) have been developed recently (see,[27, 36, 52, 4, 49])

such as cubic B-spline collocation method [27],the Adomian decomposition method for nonlinear Volterra integral equations [36] , quintic B-spline collocation method [52], and Adomian decomposition method for linear Volterra integral equations is used in [4], Fixed point method for solving nonlinear Volterra-Hammerstein integral equation [49].
This chapter is concerned with the iterative continuous collocation method to obtain an approximate solution for Volterra integral, our method presents some advantages:

- It provides a global approximation of the solution
- Without needed to solve any algebraic system
- High order of convergence
- Provides an explicit numerical solution and easy to be implemented.

This chapter is organized as follows: In section 2, we divide the interval $[0, T]$ into subintervals, and we approximate the solution of (1.43) in each interval by using iterative Lagrange polynomials. Global convergence is established in section 3. Finally, we report our numerical results and demonstrate the efficiency and accuracy of the proposed numerical scheme by considering some numerical examples in section 4.

3.2 Description of the method

Let Π_N be a uniform partition of the interval $I = [0, T]$ defined by $t_n = nh, n = 0, \ldots, N-1$, where the stepsize is given by $\frac{T}{N} = h$. Let the collocation parameters be $0 \le c_1 < \ldots < c_m \le 1$ and the collocation points be $t_{n,j} = t_n + c_j h$, $j = 1, \ldots, m, n = 0, \ldots, N-1$. Define the subintervals $\sigma_n = [t_n, t_{n+1}]$.

Moreover, denote by π_m the set of all real polynomials of degree not exceeding m.

We define the real polynomial spline space of degree m as follows

$$S_m^{(0)}(\Pi_N) = \{u \in C(\mathrm{I}, \mathbb{R}) : u_n = u/\sigma_n \in \pi_m, n = 0, \ldots, N-1\}.$$

It holds for any $x \in C^{m+1}([0, T])$ that

$$x(t_n + sh) = L_0(s)x(t_n) + \sum_{j=1}^{m} L_j(s)x(t_{n,j}) + h^{m+1}\frac{x^{(m+1)}(\zeta_n(s))}{(m+1)!} s \prod_{j=1}^{m}(s - c_j), \tag{3.1}$$

where $s \in [0, 1]$, $L_0(v) = (-1)^m \prod_{l=1}^{m} \frac{v - c_l}{c_l}$ and $L_j(v) = \frac{v}{c_j} \prod_{l \neq j}^{m} \frac{v - c_l}{c_j - c_l}, j = 1, \ldots, m$ are the Lagrange polynomials associated with the parameters c_j, $j = 1, \ldots, m$.

Inserting (3.1) for the function $s \longmapsto K(t, s, x(s))ds$ into (1.43), we obtain for each $j = 1, \ldots, m, n = 0, \ldots, N-1$

$$\begin{aligned} x(t_{n,j}) = f(t_{n,j}) &+ h\sum_{p=0}^{n-1} b_0 K(t_{n,j}, t_p, x(t_p)) + h\sum_{p=0}^{n-1}\sum_{v=1}^{m} b_v K(t_{n,j}, t_{p,v}, x(t_{p,v})) \\ &+ ha_{j,0}K(t_{n,j}, t_n, x(t_n)) + h\sum_{v=1}^{m} a_{j,v}K(t_{n,j}, t_{n,v}, x(t_{n,v})) + o(h^{m+1}), \end{aligned} \tag{3.2}$$

such that $a_{j,v} = \int_0^{c_j} L_v(\eta)d\eta$ and $b_v = \int_0^1 L_v(\eta)d\eta, v = 0, \ldots, m$.

It holds for any $u \in S_m^0(I, \Pi_N)$ that

$$u_n(t_n + sh) = L_0(s)u_{n-1}(t_n) + \sum_{j=1}^{m} L_j(s)u_n(t_{n,j}), s \in [0, 1]. \tag{3.3}$$

Now, we approximate the exact solution x by $u \in S_m^0(I, \Pi_N)$ such that $u(t_{n,j})$ satisfies the following nonlinear system,

$$u_n(t_{n,j}) = f(t_{n,j}) + h\sum_{p=0}^{n-1} b_0 K(t_{n,j}, t_p, u_p(t_p)) + h\sum_{p=0}^{n-1}\sum_{v=1}^{m} b_v K(t_{n,j}, t_{p,v}, u_p(t_{p,v}))$$
$$+ h a_{j,0} K(t_{n,j}, t_n, u_{n-1}(t_n)) + h\sum_{v=1}^{m} a_{j,v} K(t_{n,j}, t_{n,v}, u_n(t_{n,v})), \qquad (3.4)$$

for $j = 1, \ldots, m$, $n = 0, \ldots, N-1$, where $u_{-1}(t_0) = x(0) = f(0)$.

Since the above system is nonlinear, we will use an iterative collocation solution $u^q \in S_m^0(I, \Pi_N), q \in \mathbb{N}$, to approximate the exact solution of (1.43) such that

$$u_n^q(t_n + sh) = L_0(s) u_{n-1}^q(t_n) + \sum_{j=1}^{m} L_j(s) u_n^q(t_{n,j}), s \in [0, 1], \qquad (3.5)$$

where the coefficients $u_n^q(t_{n,j})$ are given by the following formula:

$$u_n^q(t_{n,j}) = f(t_{n,j}) + h\sum_{p=0}^{n-1} b_0 K(t_{n,j}, t_p, u_p^q(t_p)) + h\sum_{p=0}^{n-1}\sum_{v=1}^{m} b_v K(t_{n,j}, t_{p,v}, u_p^q(t_{p,v}))$$
$$+ h a_{j,0} K(t_{n,j}, t_n, u_{n-1}^q(t_n)) + h\sum_{v=1}^{m} a_{j,v} K(t_{n,j}, t_{n,v}, u_n^{q-1}(t_{n,v})), \qquad (3.6)$$

such that $u_{-1}^q(t_0) = f(0)$ for all $q \in \mathbb{N}$ and the initial values $u^0(t_{n,j}) \in J$ (J is a bounded interval).

The above formula is explicit and the approximate solution u^q is obtained without solving any algebraic system. The complexity of the proposed algorithm can be measured in terms of how many times the function K must be evaluated at each collocation point.

From formula (3.5) it follows that the number of such evaluations is $O(mn)$ for each iteration. Since the optimal number of iterations is $q = m + 1$ (as it will be shown in the next section), we conclude that the total number of evaluations is $O(m^2 n)$, which makes this method competitive, in comparison with other methods where a nonlinear system of equations is solved by an iterative algorithm.

In the next section, we prove the convergence of the approximate solution u^q to the exact solution x of (1.43) is of order m for all $q \geq m$.

3.3 Convergence analysis

In this section, we assume that the function K satisfies the Lipschitz condition with respect to the third variable: there exist $L \geq 0$ such that

$$|K(t,s,y_1) - K(t,s,y_2)| \leq L|y_1 - y_2|.$$

The following result gives the existence and the uniqueness of a solution for the nonlinear system (3.4).

Lemma 3.3.1 *The nonlinear system* (3.4) *has a unique solution* $u \in S_m^0(I, \Pi_N)$ *for sufficiently small* h*.*

Proof. We will use the induction combined with the Banach fixed point theorem.

(i) On the interval $\sigma_0 = [t_0, t_1]$, the nonlinear system (3.4) becomes

$$u_0(t_{0,j}) = f(t_{0,j}) + ha_{j,0}K(t_{0,j}, t_0, f(0)) + h\sum_{v=1}^{m} a_{j,v}K(t_{0,j}, t_{0,v}, u_0(t_{0,v})), \, j = 1, \ldots, m.$$

We consider the operator Ψ defined by

$$\Psi : \mathbb{R}^m \longrightarrow \mathbb{R}^m$$
$$x = (x_1, \ldots, x_m) \longmapsto \Psi(x) = (\Psi_1(x), \ldots, \Psi_m(x)),$$

such that for $j = 1, \dots, m$, we have

$$\Psi_j(x) = f(t_{0,j}) + ha_{j,0}K(t_{0,j}, t_0, f(0)) + h\sum_{v=1}^{m} a_{j,v}K(t_{0,j}, t_{0,v}, x_v).$$

Hence, for all $x, y \in \mathbb{R}^m$, we have

$$\|\Psi(x) - \Psi(y)\| \leq hmaL\|x - y\|,$$

where $a = \max\{|a_{j,v}|, j = 1, \dots, m, v = 0, \dots, m\}$.

Since $hmaL < 1$ for sufficiently small h, then by the Banach fixed point theorem, the nonlinear system (3.4) has a unique solution u_0 on σ_0.

(ii) Suppose that u_i exists and unique on the intervals $\sigma_i, i = 0, \dots, n-1$ for $n \geq 1$, we show that u_n exists and is unique on the interval σ_n.

On the interval σ_n, the nonlinear system (3.4) becomes

$$u_n(t_{n,j}) = F(t_{n,j}) + h\sum_{v=1}^{m} a_{j,v}K(t_{n,j}, t_{n,v}, u_n(t_{n,v})), \tag{3.7}$$

where, $F(t_{n,j}) = f(t_{n,j}) + h\sum\limits_{p=0}^{n-1} b_0K(t_{n,j}, t_p, u_p(t_p)) +$
$h\sum\limits_{p=0}^{n-1}\sum\limits_{v=1}^{m} b_vK(t_{n,j}, t_{p,v}, u_p(t_{p,v})) + ha_{j,0}K(t_{n,j}, t_n, u_{n-1}(t_n))$.

We consider the operator Ψ defined by:

$$\Psi : \mathbb{R}^m \longrightarrow \mathbb{R}^m$$
$$x = (x_1, \dots, x_m) \longmapsto \Psi(x) = (\Psi_1(x), \dots, \Psi_m(x)),$$

such that for $j = 1, \dots, m$

$$\Psi_j(x) = F(t_{n,j}) + h\sum_{v=1}^{m} a_{j,v}K(t_{n,j}, t_{n,v}, x_v).$$

Hence, for all $x, y \in \mathbb{R}^m$

$$||\Psi(x) - \Psi(y)|| \leq hmaL||x - y||$$

Since $hmaL < 1$ for sufficiently small h, then by the Banach fixed point theorem, the nonlinear system (3.7) has a unique solution u_n on σ_n.

■ The following result gives the convergence of the approximate solution u to the exact solution x.

Theorem 3.3.1 *Let f, K be $m+1$ times continuously differentiable on their respective domains. If $-1 < R(\infty) = (-1)^m \prod_{l=1}^{m} \frac{1 - c_l}{c_l} < 1$, then, for sufficiently small h, the collocation solution u converges to the exact solution x, and the resulting error function $e := x - u$ satisfies:*

$$||e|| \leq Ch^{m+1},$$

where C is a finite constant independent of h.

Proof. Define the error e on σ_n by $e(t) = e_n(t) = x(t) - u_n(t)$ for all $n \in \{0, 1, \ldots, N-1\}$.

We have, from (3.4) and (3.2), for all $n = 0, \ldots, N-1$ and $j = 1, \ldots, m$

$$|e_n(t_{n,j})| \leq hbL \sum_{p=0}^{n-1} |e_p(t_p)| + hbL \sum_{p=0}^{n-1} \sum_{v=1}^{m} |e_p(t_{p,v})| + haL|e_{n-1}(t_n)| + haL \sum_{v=1}^{m} |e_n(t_{n,v})| + \alpha h^{m+1}, \quad (3.8)$$

where α is a positive number and $e_{-1}(t_0) = 0$.

We consider the sequence $\varepsilon_n = \sum_{v=1}^{m} |e_n(t_{n,v})|$ for $n = 0, \ldots, N-1$.

Then, from (3.8), ε_n satisfies for $n = 0, \dots, N-1$

$$\begin{aligned}\varepsilon_n &\leq hbLm \sum_{p=0}^{n-1} |e_p(t_p)| + hbLm \sum_{p=0}^{n-1} \varepsilon_p + haLm|e_{n-1}(t_n)| + haLm\varepsilon_n + \alpha m h^{m+1} \\ &\leq 2hbLm \sum_{p=0}^{n-1} \|e_p\| + hbLm \sum_{p=0}^{n-1} \varepsilon_p + haLm\varepsilon_n + \alpha m h^{m+1}.\end{aligned}$$

Hence, for $\bar{h} < \dfrac{1}{Lam}$ and $h \in (0, \bar{h}]$, we have

$$\varepsilon_n \leq \underbrace{\frac{2bLm}{1 - Lam\bar{h}}}_{\alpha_1} h \sum_{p=0}^{n-1} \|e_p\| + \underbrace{\frac{bLm}{1 - Lam\bar{h}}}_{\alpha_2} h \sum_{p=0}^{n-1} \varepsilon_p + \underbrace{\frac{\alpha m}{1 - Lam\bar{h}}}_{\alpha_3} h^{m+1}.$$

Then, by Lemma 1.7.2, for all $n = 0, \dots, N-1$

$$\varepsilon_n \leq \underbrace{\alpha_1 \exp(T\alpha_2)}_{\alpha_4} h \sum_{p=0}^{n-1} \|e_p\| + \underbrace{\alpha_3 \exp(T\alpha_2)}_{\alpha_5} h^{m+1}.$$

Therefore, by using (3.1) and (3.3), we obtain

$$\begin{aligned}\|e_n\| &\leq |R(\infty)|\|e_{n-1}\| + \rho\varepsilon_n + \beta h^{m+1} \\ &\leq |R(\infty)|\|e_{n-1}\| + \underbrace{\rho\alpha_4}_{\alpha_6} h \sum_{p=0}^{n-1} \|e_p\| + \underbrace{(\rho\alpha_5 + \beta)}_{\alpha_7} h^{m+1}.\end{aligned}$$

where $\rho = \max\{|L_j(t)|, t \in [0,1]; j = 1, \dots, m\}$.

Hence by Lemma 1.7.4, we obtain for all $n = 0, \dots, N-1$

$$\begin{aligned}\|e_n\| &\leq \frac{\|e_0\|}{R_2 - R_1}[(R_2 - 1)R_2^n + (1 - R_1)R_1^n] + \frac{\alpha_7 h^{m+1}}{R_2 - R_1}[R_2^n - R_1^n] \\ &\leq \frac{\|e_0\|}{R_2 - R_1}[(R_2 - 1)R_2^{\frac{T}{h}} + 1] + \frac{\alpha_7 h^{m+1}}{R_2 - R_1}[R_2^{\frac{T}{h}}] \\ &\leq \left(\frac{1}{R_2 - R_1}[(R_2 - 1)R_2^{\frac{T}{h}} + 1] + \frac{1}{R_2 - R_1}[R_2^{\frac{T}{h}}]\right)\alpha_7 h^{m+1},\end{aligned}$$

where R_1, R_2 are defined by (1.42) such that $A = |R(\infty)|, B = \alpha_6 h, K = \alpha_7 h^{m+1}$.

Since, $\lim\limits_{h\longrightarrow 0}(\frac{1}{R_2-R_1}[(R_2-1)R_2^{\frac{T}{h}}+1]+\frac{1}{R_2-R_1}[R_2^{\frac{T}{h}}]) = \frac{1}{1-|R(\infty)|}\exp(\frac{2T\alpha_6}{1-|R(\infty)|}) < +\infty$.

Then, there exists $\gamma > 0$ such that for all $h \in (0, \overline{h}]$.

$$\frac{1}{R_2-R_1}[(R_2-1)R_2^{\frac{T}{h}}+1]+\frac{1}{R_2-R_1}[R_2^{\frac{T}{h}}] \leq \gamma,$$

Thus, the proof is completed by taking $C = \alpha_7\gamma$. ■

The following result gives the convergence of the iterative solution u^q to the exact solution x.

Theorem 3.3.2 *Consider the iterative collocation solution* $u^q, q \geq 1$ *defined by* (3.5) *and* (3.6), *if* $-1 < R(\infty) = (-1)^m \prod\limits_{l=1}^{m} \frac{1-c_l}{c_l} < 1$, *then for any initial condition* $u^0(t_{n,j}) \in J$, *the iterative collocation solution* $u^q, q \geq 1$ *converges to the exact solution* x *for sufficiently small* h. *Moreover, the following error estimate holds*

$$||u^q - x|| \leq d\beta^q h^q + Ch^{m+1}$$

where d, β *and* C *are finite constants independent of* h.

Proof. We define the errors e^q and ξ^q by $e^q(t) = e_n^q(t) = u_n^q(t) - x(t)$ and

$\xi^q = \xi_n^q = u_n^q(t) - u_n(t)$ on $\sigma_n, n = 0, \ldots, N-1$, where u is defined by Lemma 3.3.1.

We have, from (3.4) and (3.6), for all $n = 0, \ldots, N-1$ and $j = 1, \ldots, m$

$$|\xi_n^q(t_{n,j})| \leq hbL\sum_{p=0}^{n-1}|\xi_p^q(t_p)| + hbL\sum_{p=0}^{n-1}\sum_{v=1}^{m}|\xi_p^q(t_{p,v})| + haL|\xi_{n-1}^q(t_n)|$$
$$+ haL\sum_{v=1}^{m}|\xi_n^{q-1}(t_{n,v})|.$$

Now, for each fixed $q \geq 1$, we consider the sequence

$\eta_n^q = \max\{|\xi_n^q(t_{n,v})|, v = 1, \ldots, m\}$ for $n = 0, \ldots, N-1$, it follows that

$$\begin{aligned}\eta_n^q &\le hbL\sum_{p=0}^{n-1}|\xi_p^q(t_p)| + hbLm\sum_{p=0}^{n-1}\eta_p^q + haL|\xi_{n-1}^q(t_n)| + haLm\eta_n^{q-1}\\ &\le 2hbL\sum_{p=0}^{n-1}||\xi_p^q|| + hbLm\sum_{p=0}^{n-1}\eta_p^q + haLm\eta_n^{q-1}.\end{aligned}$$

Hence, by Lemma 1.7.3, for all $n = 0, \ldots, N-1$

$$\begin{aligned}\eta_n^q \le 2hbL\sum_{p=0}^{n-1}||\xi_p^q|| + haLm\eta_n^{q-1} + \exp(TLbm)ab(hLm)^2\sum_{p=0}^{n-1}\eta_p^{q-1}\\ + 2\exp(TLbm)Tm(bL)^2h\sum_{p=0}^{n-1}||\xi_p^q||.\end{aligned} \tag{3.9}$$

We consider the sequence $\eta^q = \max\{\eta_n^q, n = 0, \ldots, N-1\}$ for $q \ge 1$.

Then, from (3.9), η^q satisfies

$$\eta_n^q \le \underbrace{2(bL + \exp(TLbm)Tm(bL)^2)}_{\alpha_1} h\sum_{p=0}^{n-1}||\xi_p^q|| + \alpha_2 h\eta^{q-1}.$$

where $\alpha_2 = (aLm + \exp(TLbm)abT(Lm)^2)$.

Therefore, by using (3.3) and (3.5), we obtain

$$\begin{aligned}||\xi_n^q|| &\le |R(\infty)|||\xi_{n-1}^q|| + \rho m\eta_n^q\\ &\le |R(\infty)|||\xi_{n-1}^q|| + \rho m\alpha_1 h\sum_{p=0}^{n-1}||\xi_p^q|| + \rho m\alpha_2 h\eta^{q-1}.\end{aligned}$$

Hence by Lemma 1.7.4, we obtain for all $n = 0, \dots, N-1$

$$\begin{aligned}
\|\xi_n^q\| &\leq \frac{\|\xi_0^q\|}{R_2 - R_1}[(R_2-1)R_2^n + (1-R_1)R_1^n] + \frac{\rho m \alpha_2 h \eta^{q-1}}{R_2 - R_1}[R_2^n - R_1^n] \\
&\leq \frac{\|\xi_0^q\|}{R_2 - R_1}[(R_2-1)R_2^{\frac{T}{h}} + 1] + \frac{\rho m \alpha_2 h \eta^{q-1}}{R_2 - R_1}[R_2^{\frac{T}{h}}] \\
&\leq \left(\frac{1}{R_2 - R_1}[(R_2-1)R_2^{\frac{T}{h}} + 1] + \frac{1}{R_2 - R_1}[R_2^{\frac{T}{h}}]\right)\rho m \alpha_2 h \eta^{q-1}, \qquad (3.10)
\end{aligned}$$

where R_1 and R_2 are defined by (1.42) such that $A = |R(\infty)|, B = \rho m \alpha_1 h$,

$K = \rho m \alpha_2 h \eta^{q-1}$.

Since, $\lim\limits_{h \longrightarrow 0}(\frac{1}{R_2-R_1}[(R_2-1)R_2^{\frac{T}{h}} + 1] + \frac{1}{R_2-R_1}[R_2^{\frac{T}{h}}]) = \frac{\exp(\frac{2T\rho m \alpha_1}{1-|R(\infty)|})}{1-|R(\infty)|} < +\infty$.

Then, there exists $\gamma > 0$ such that for all $h \in (0, \overline{h}]$

$$\frac{1}{R_2 - R_1}[(R_2-1)R_2^{\frac{T}{h}} + 1] + \frac{1}{R_2 - R_1}[R_2^{\frac{T}{h}}] \leq \gamma.$$

It follows, from (3.10), that for all $n = 0, \dots, N-1$

$$\|\xi_n^q\| \leq \gamma \rho m \alpha_2 h \eta^{q-1} \leq \gamma \rho m \alpha_2 h \|\xi^{q-1}\|,$$

which implies, for all $q \geq 1$, that

$$\|\xi^q\| \leq \gamma \rho m \alpha_2 h \|\xi^{q-1}\| \leq \dots \leq (\gamma \rho m \alpha_2)^q h^q \|\xi^0\|.$$

Since, $u_{-1}^0(t_0) = f(0), u^0(t_{n,j}) \in J$ (bounded interval), then by (3.3) the function

u^0 is bounded.

Hence there exists $d > 0$ such that $\|\xi^0\| = \|u^0 - u\| \leq \|u^0 - x\| + \|x - u\| < d$.

Which implies that, for all $q \geq 1$

$$\|\xi^q\| \leq d\underbrace{(\gamma \rho m \alpha_2)}_{\beta}{}^q h^q.$$

Hence, by theorem 3.3.1, we deduce that

$$\|e^q\| \leq \|\xi^q\| + \|u - x\| \leq d\beta^q h^q + Ch^{m+1}.$$

Thus, the proof is completed. ■

Remark 3.3.1 *From the error estimate in Theorem 3.3.2 it follows that the optimal number of iterations is $q = m + 1$. Actually, with $m + 1$ iterations the total error has the order of $O(h^{m+1})$, which will not be improved if more iterations are performed.*

3.4 Numerical examples

In order to test the applicability of the presented method, we consider the following examples with $T = 1$. These examples have been solved with various values of N, m and q. In each example, we calculate the error between x and the iterative collocation solution u^q.

The absolute errors at some particular points are given to compare our solutions with the solutions obtained by [4, 27, 36, 49, 52].

These results of these numerical examples are in agreement with the theory presented in Section 3 and they confirm the advantages of our method in comparison with those described in [4, 27, 36, 49, 52].

Example 3.4.1 *([27, 36]) Consider the following nonlinear Volterra integral equation*

$$x(t) = 1 + (\sin(t))^2 - \int_0^t 3\sin(t-s)(x(s))^2 ds, t \in [0, 1],$$

where $u(x) = \cos(x)$ is the exact solution.

The absolute errors for $N = 10, 20$ and $m = q = 3$ at $t = 0, 0.1, \ldots, 1$ are displayed in Table 3.1. We used the collocation parameters $c_i = \frac{i}{m+1} + \frac{1}{5}, i = 1, \ldots, m$ and $R(\infty) = -0.02$. The *numerical results obtained by the present method are considerably more accurate in comparison with the numerical results obtained in [27, 36].*

Table 3.1: Comparison of the absolute errors of Example 3.4.1

t	Method in [27]		Method in [36]		Our method	
	$N = 10$	$N = 20$	$N = 10$	$N = 20$	$N = 10$	$N = 20$
0.1	1.01E−5	1.59E−6	1.24E−5	2.54E−8	3.32E−8	7.92E−9
0.2	2.48E−5	3.26E−6	1.62E−6	3.44E−7	1.84E−9	5.15E−9
0.3	3.65E−5	4.72E−6	2.03E−4	9.19E−7	3.58E−8	3.87E−9
0.4	4.61E−5	5.87E−6	2.07E−5	1.44E−6	5.29E−8	8.00E−9
0.5	5.26E−5	6.63E−6	3.84E−5	1.88E−6	9.91E−8	8.90E−10
0.6	5.59E−5	6.98E−6	5.11E−5	2.18E−6	1.48E−7	5.90E−9
0.7	5.58E−5	6.92E−6	7.22E−5	1.83E−6	1.77E−7	9.71E−9
0.8	5.28E−5	6.47E−6	6.43E−5	6.41E−6	2.00E−7	3.34E−9
0.9	4.65E−5	5.70E−6	1.96E−5	1.00E−4	2.04E−7	2.07E−8
1	3.97E−5	4.71E−6	6.36E−4	9.25E−4	1.95E−7	5.13E−9

Example 3.4.2 *([4, 52]) Consider the following linear Volterra integral equation with exact solution* $x(t) = 1 - \sinh(t)$*:*

$$x(t) = 1 - t - \frac{t^2}{2} + \int_0^t (t-s)x(s)ds, t \in [0,1].$$

The absolute errors for $m = q = 3$ *and* $N = 20$ *at* $t = 0, 0.1, \ldots, 1$ *are displayed in Table 3.2. We used the collocation parameters* $c_i = \frac{i}{m+1} + \frac{1}{5}, i = 1, \ldots, m$ *and* $R(\infty) = -0.02$*. The numerical results obtained here are compared in Table 3.2 with the numerical results obtained by using the methods in [4, 52].*

It is seen from Table 3.2 that the results obtained by the present method are much more accurate than those obtained in [4, 52].

The absolute errors for $N = 5$ *and* $(q, m) \in \{(2,2), (3,2), (3,3), (3,5), (4,5)\}$ *at* $t = 0, 0.1, \ldots, 1$ *are presented in Table 3.3, we note that the absolute error reduces as q or m increases.*

We calculate the experimental order of convergence (EOC) at $t = 1$ *for* $N = 2^l, l = 1, 2, 3, 4, 5,$ $m = 1, 2, 3$ *and* $q = m + 1$ *in Table 3.4, the result confirm the theoretical result and suggest that that the order of convergence with* $q = m + 1$ *is* $m + 1$. As we have remarked (see Remark 3.3.1) this is the maximal convergence order that can be obtained with the present method.

Moreover, we calculate the run time to solve the approximate solution u for $N = 6, \ldots, 10, m = 7, \ldots, 10$ *and* $q = m + 1$*, the numerical results are solved by using Maple version* 16.

The computations were performed in a PC with a 2.16 *GHz processor, running with* 2.00 *Go RAM. As it could be expected, the computing time increases with m and N; however we cannot see a simple relationship between the computing time and the complexity of the algorithm, probably because this time depends on other factors than the number of evaluations of the function K. This table shows that accurate results can be obtained by our method in a small computer with a low computational cost.*

Table 3.2: Comparison of the absolute errors of Example 3.4.2

t	Our method		Method in [4]	Method in [52]
	$N = 10$	$N = 20$		
0.0	0	0	0	1.98E−14
0.1	1.30E−8	1.98E−9	5.38E−6	1.21E−7
0.2	3.35E−8	2.54E−9	2.20E−5	2.35E−7
0.3	3.14E−8	6.55E−9	4.82E−5	3.54E−7
0.4	5.98E−8	5.80E−9	8.33E−5	4.77E−7
0.5	6.94E−8	3.50E−9	1.26E−4	6.05E−7
0.6	8.01E−8	8.51E−10	1.77E−4	7.39E−7
0.7	1.00E−7	5.83E−9	2.34E−4	8.80E−7
0.8	1.15E−7	7.38E−9	2.97E−4	1.03E−6
0.9	1.37E−7	8.90E−9	3.65E−4	1.19E−6
1	1.62E−7	9.38E−9	4.38E−4	1.36E−6

Table 3.3: Absolute errors for Example 3.4.2

t	$q = 2$, $m = 2$	$q = 3$, $m = 2$	$q = 3$, $m = 3$	$q = 3$, $m = 5$	$q = 4$, $m = 5$
0	0.0	0.0	0.0	0.0	0.0
0.1	8.231E−6	7.282E−6	3.015E−7	7.451E−8	3.701E−8
0.2	8.563E−5	8.373E−5	4.115E−7	1.147E−6	8.474E−7
0.3	1.053E−5	7.583E−6	6.394E−7	5.824E−8	4.007E−8
0.4	1.027E−4	9.863E−5	8.478E−7	8.031E−7	4.328E−7
0.5	1.064E−5	5.410E−6	1.017E−6	2.316E−8	1.897E−8
0.6	1.143E−4	1.070E−4	1.324E−6	1.058E−7	4.785E−8
0.7	1.033E−5	2.283E−6	1.470E−6	1.309E−8	3.040E−8
0.8	1.297E−4	1.175E−4	1.909E−6	1.114E−7	7.258E−8
0.9	9.861E−6	1.815E−6	2.021E−6	8.470E−9	8.137E−10
1	1.514E−4	1.314E−4	2.620E−6	1.156E−7	4.245E−8

Example 3.4.3 *([49]) We consider the following nonlinear Volterra integral equation*

$$x(t) = \frac{t}{e^{t^2}} + \int_0^t 2tse^{-x^2(s)}ds, t \in [0,1],$$

where the exact solution is $x(t) = t$.

The absolute errors for $N = 20$ *and* $m = 3, q = 5$ *at* $t = 0, 0.2, \ldots, 1$ *are compared with the absolute error of the method in [49] in Table 3.5.*

Where the collocation parameters $c_i = \frac{i}{m+3} + \frac{1}{5}, i = 1, \ldots, m$ *and* $R(\infty) = -0.64$.

Table 3.4: EOC and the run-time/sec of Example 3.4.2

N	$m=1$	$m=2$	$m=3$
2			
4	2.04	2.91	4.00
8	2.05	2.96	4.01
16	2.04	2.91	4.00
32	2.04	2.91	4.00

EOC of Example3.4.2

N	$m=7$	$m=8$	$m=9$	$m=10$
6	3.9	5.6	9.9	14.8
7	5.8	9.9	17.9	31.1
8	8.9	16.7	24.3	56.6
9	13.3	32.7	43.9	118.4
10	17.4	35.9	126.3	232.3

run-time/sec of Example3.4.2

Table 3.5: Comparison of the absolute errors of Example 3. 3.4.3

t	Method in [49] $N=20$	Our method $N=20$
0	0	0
0.2	1.49E−8	8.9E−9
0.4	7.74E−7	2.69E−8
0.6	9.36E−6	8.90E−9
0.8	4.58E−5	3.39E−8
1	1.29E−4	2.49E−8

Example 3.4.4 *([49]) We consider the following nonlinear Volterra integral equation*

$$x(t) = t\cos(t) + \int_0^t t\sin(x(s))ds, t \in [0,1],$$

where the exact solution is $x(t) = t$.

The absolute errors for $N = 25$ *and* $m = q = 4$ *at* $t = 0.001, 0.2, 0.4, 0.6, 0.8, 1$ *are compared with the absolute error of the method in [49] in Table 3.6.*

Where the collocation parameters $c_i = \frac{i}{m+3} + \frac{1}{5}, i = 1, \ldots, m$ *and* $R(\infty) = 0.35$.

It is seen from Table 3.6 that the results obtained by the present method is very superior to that obtained by the method in [49].

Table 3.6: Comparison of the absolute errors of Example 3.4.4

t	Method in [49] $N = 25$	Our method $N = 25$
0.001	1.25E−12	1.75E−11
0.2	3.53E−6	6.30E−8
0.4	5.81E−6	5.40E−8
0.6	7.74E−7	9.60E−8
0.8	1.20E−5	6.00E−9
1	3.98E−5	7.20E−8

3.5 Conclusion

We have shown that the method yields an efficient and very accurate numerical method for the approximation of solution of the in the spline space In this chapter, we have used a iterative collocation method for the numerical solution of nonlinear volterra integral equation VIDEs (1.43) inthe spline space $S_m^{(0)}(I, \Pi_N)$. It is proved that the method is convergent with an experimental order of convergence $EOC = m + 1$. This method is easy to implement and the coefficients of the approximation solution are determined by using iterative formulas without the need to solve any system of algebraic equations. Numerical examples showing that the method is convergent with a good accuracy and the numerical results confirmed the theoretical estimates.

CONCLUSION

In this part, we have used an iterative collocation method based on the Lagrange polynomials for the numerical solution of Volterra integral equations (1.43) in the spline space $S_{m-1}^{(-1)}(\Pi_N)$ and Iterative Continuous Collocation Method for Solving Nonlinear Volterra Integral Equations (1.43) in the spline space $S_m^{(0)}(\Pi_N)$. The main advantages of this method that, is easy to implement, has high order of convergence and the coefficients of the approximation solution are determined by using iterative formulas without the need to solve any system of algebraic equations. Numerical examples showing that the method is convergent with a good accuracy and the comparison of the results obtained by the present method with the other methods reveals that the method is very effective and convenient.

Part II

Numerical solution of nonlinear Volterra integro-differential equation

INTRODUCTION

In this part, we investigate an continuous iterative colocation method for the following nonlinear Volterra integro-differential equation

$$\begin{aligned} x'(t) &= f(t) + Q(t, x(t)) + \int_0^t K(t, s, x(s), x'(s))ds, \quad t \in I = [0, T] \\ x(t_0) &= x_0, \end{aligned} \tag{3.11}$$

where the functions f, Q, K are sufficiently smooth.
There are several numerical methods for approximating the solution of equation (3.11). For example, spectral methods, implicit Runge-Kutta methods, Galerkin methods, collocation methods, and Legendre wavelets series, (cf, e.g. [38, 58, 13, 2, 39, 40], and references therein).
The purpose of this part is to solve equation (3.11) by the iterative collocation method. The main idea of the iterative collocation method is to obtain an explicit solution without needed to solve any algebraic system. Many authors used this method to solve integral equations. L. Hacia [31], used the Iterative-Collocation Method to solve integral equations of heat conduction problems, H. Brunner [15] applied the iterated collocation methods to approximate the solution for Volterra integral equations with delay arguments. In [9, 65] the variational iteration method is used to solve integral

and integro-differential equations.
The collocation method has been introduced for the first time in 1937 by Frazer et al [28]. After that, many reaschers used it to solve many phenomena in physics and engineering such as unsteady heat condition problems and a viscous fluid problem. Recently, the collocation method has been used over a wide range of problems (cf, e.g. [22, 33, 57, 59]). Collocation methods have many good properties such as high order of convergence, strong stability properties and flexibility in such a way that if we know some information of the exact solution, then it is possible to choose the collocation functions and the collocation points in order to rise the order of convergence, see for example [21] in the case of ordinary differential equations, [44, 12] in the case of Volterra integral equations, [17] in the case of Volterra integro-differential equations. and [41] in the case of Numerical solution of high-order linear Volterra integro-differential equations by using Taylor collocation method.
This part is concerned with the iterative collocation method to obtain an approximate solution for (3.11), our method presents some advantages:

- It provides a global approximation of the solution
- Without needed to solve any algebraic system
- High order of convergence
- Provides an explicit numerical solution and easy to be implemented.

The outlines of this part is as follows. In chapter 4, we approximate the solution of (3.11) in the space of continuous piecewise polynomials of degree m as follows

$$S_m^{(0)}(\Pi_N) = \{u \in C(\mathrm{I}, \mathbb{R}) : u_n = u/\sigma_n \in \pi_m, n = 0, ..., N-1\}.$$

and in chapter 5 we approximate the solution of (3.11) in the space of continuous piecewise polynomials of degree $m+1$ as follows

$$S_{m+1}^{(1)}(\Pi_N) = \{u \in C^1(\mathrm{I}, \mathbb{R}) : u_n = u/\sigma_n \in \pi$$

Each chapter are organized as follows.
In section 2, the spline polynomial has been used to approximate equation (3.11) based on the iterative collocation method, error analysis has been discussed in section 3, section 4 is devoted to present some numerical examples, in the last section, we give a conclusion.

CHAPTER 4

ITERATIVE CONTINUOUS COLLOCATION METHOD FOR SOLVING NONLINEAR VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS IN THE SPACE $S_M^{(0)}(\Pi_N)$

4.1 Introduction

In this chapter, we consider the following Volterra integro-differential equations,

$$\begin{aligned} x'(t) &= f(t) + Q(t, x(t)) + \int_0^t K(t, s, x(s), x'(s))ds, \quad t \in I = [0, T] \\ x(t_0) &= x_0, \end{aligned} \tag{4.1}$$

where the functions f, Q, K are sufficiently smooth. The existence and the uniqueness of the solution of (4.1) can be found, for example, in [12].
integro-differential equations find its applications in various fields of science and engineering. There are several numerical methods for approximating the solution of integro-differential equations are known and many different basic functions have been used. There are various methods to solve integro-differential equations such as Adomian decomposition method, successive substitutions, Laplace transformation method, Picard's method, etc (Wazwaz (2011)[67]). collocation theory is a relatively new and an emerging tool in applied mathematical research area. It has been applied in a wide range of engineering disciplines; particularly,and fast algorithms for easy implementation. collocation method have been applied for the numerical solution of different kinds of integral equations,

The aim of this chapter is to generalize the iterative continuous collocation method in [55] for construct an iterative continuous approximate solution for equation (4.1). In our method the approximate solution is explicit, direct and obtained by using simple iterative formulas.
In fact the applications of the iterative collocation method in the numerical analysis field possessing some of the well known advantages such as:

1. It is accurate,

2. It is possible to pick any point in the interval of integration and as well the approximate solutions and their derivatives will be applicable.

3. The method does not require discretization of the variables, and it is not affected

by computation Ind off errors and one is not faced with necessity of large computer memory and time.

The outlines of this chapter is as follows. In section 2, an iterative collocation method has been used to construct an approximate solution for (4.1) in the continuous spline polynomials space $S_m^{(0)}(\Pi_N)$, the convergence analysis has been given in section 3. Some numerical illustrations are provided in section 4.

4.2 Description of the method

Let Π_N be a uniform partition of the interval $I = [0, T]$ with grid points $t_n = nh, \quad n = 0, ..., N-1$, where the stepsize is given by $h = \dfrac{T}{N}$. Let the collocation parameters be $0 < c_1 < ...... < c_m < 1$ and the collocation points
$\Gamma_{N,m} = \{t_{n,j} = t_n + c_j h, j = 1, ..., m, n = 0, ..., N-1\}$.
Define the subintervals $\sigma_n = [t_n, t_{n+1}]$, and $\sigma_{N-1} = [t_{N-1}, t_N]$.
Moreover, denote by π_m the set of all real polynomials of degree not exceeding m.
We consider polynomial spline approximations $u(t)$ of the exact solution $x(t)$ in the spline space

$$S_m^{(0)}(\Pi_N) = \{u \in C(\mathrm{I}, \mathbb{R}) : u_n = u/\sigma_n \in \pi_m, n = 0, ..., N-1, \}.$$

This is the space of piecewise polynomials of degree at most m. Its dimension is $Nm+1$, the same as the number of collocation points.
We seek $u \in S_m^{(0)}(\Pi_N)$ satisfies the collocation equation

$$\begin{aligned} u'(t) &= f(t) + Q(t, u(t)) + \int_0^t K(t, s, u(s), u'(s))ds, \quad t \in \Gamma_{N,m}, \\ u(t_0) &= f(0). \end{aligned} \tag{4.2}$$

In what follows, we consider two equivalent reformulations of problem (4.2) by using the function $w(t) = u'(t) \in S_{m-1}^{(-1)}(\Pi_N)$. Since $w_n \in \pi_{m-1}$, it holds for $\mu \in (0, 1]$,

$$w_n(t_n + \mu h) = \sum_{v=1}^{m} L_v(s) w_n(t_{n,v}), \tag{4.3}$$

$$u_n(t_n + \mu h) = f(0) + h \sum_{p=0}^{n-1} \sum_{v=1}^{m} \left(\int_0^1 L_v(\tau) d\tau \right) w_p(t_{p,v}) + h \sum_{v=1}^{m} \left(\int_0^{\mu} L_v(\tau) d\tau \right) w_n(t_{n,v}), \tag{4.4}$$

where $L_j(\mu) = \prod_{l \neq j}^{m} \dfrac{\mu - c_l}{c_j - c_l}$ are the Lagrange polynomials associate with the parameters $c_j, j = 1, ..., m$. By using (4.4), the collocation equation (4.2) may be rewritten as the following nonlinear Volterra integro-differential equation with respect to w.

$$w_n(t) = f(t) + Q\left(t, f(0) + \int_0^t w(r)dr\right) + \int_0^t K\left(t, s, f(0) + \int_0^s w(r)dr, w(s)\right) ds$$

$$0 \le \tau \le s \le t = t_{n,j}, j = 1, ..., m.$$

Hence, for each $j = 1, ..., m, n = 0, ..., N-1$, $w_n(t_{n,j})$ satisfies the following nonlinear system,

$$\begin{aligned} w_n(t_{n,j}) =& f(t_{n,j}) + Q\left(t_{n,j}, f(0) + h \sum_{p=0}^{n-1} \int_0^1 w_p(t_p + \tau h) d\tau + h \int_0^{c_j} w_n(t_n + \tau h) d\tau\right) \\ &+ h \sum_{p=0}^{n-1} \int_0^1 K\left(t_{p,j}, t_p + \mu h, u_p(t_p + \mu h), w_p(t_p + \mu h)\right) d\mu \\ &+ h \int_0^{c_j} K\left(t_{n,j}, t_n + \mu h, u_n(t_n + \mu h), w_n(t_n + \mu h)\right) d\mu, \end{aligned} \tag{4.5}$$

Since the above system is nonlinear, we will use an iterative collocation solution $u^q \in S_m^0(I, \Pi_N), q \in \mathbb{N}$, to approximate the exact solution of (4.1) such that

$$w_n^q(t_n + \mu h) = \sum_{v=1}^{m} L_v(\mu) w_n^q(t_{n,v}), \tag{4.6}$$

and

$$u_n^q(t_n+\mu h) = f(0) + h\sum_{p=0}^{n-1}\sum_{v=0}^{m}\left(\int_0^1 L_v(\tau)d\tau\right)w_p^q(t_{p,v}) + h\sum_{v=0}^{m}\left(\int_0^{\mu} L_v(\tau)d\tau\right)w_n^q(t_{n,v}), \tag{4.7}$$

where the coefficients $w_n^q(t_{n,j})$ are given by the following formula:

$$\begin{aligned} w_n^q(t_{n,j}) =& f(t_{n,j}) + Q\left(t_{n,j}, f(0) + h\sum_{p=0}^{n-1}\int_0^1 w_p^q(t_p+\tau h)d\tau + h\int_0^{c_j} w_n^{q-1}(t_n+\tau h)d\tau\right)\\ &+ h\sum_{p=0}^{n-1}\int_0^1 K\left(t_{p,j}, t_p+\mu h, u_p^q(t_p+\mu h), w_p^q(t_p+\mu h)\right)d\mu\\ &+ h\int_0^{c_j} K\left(t_{n,j}, t_n+\mu h, H_n^{q-1}(t_n+\mu h), w_n^{q-1}(t_n+\mu h)\right)d\mu, \end{aligned} \tag{4.8}$$

where,

$$H_n^q(t_n+\mu h) = f(0) + h\sum_{p=0}^{n-1}\sum_{v=0}^{m}\left(\int_0^1 L_v(\tau)d\tau\right)w_p^q(t_{p,v}) + h\sum_{v=0}^{m}\left(\int_0^{\mu} L_v(\tau)d\tau\right)w_n^{q-1}(t_{n,v}).$$

Such that the initial values $w_n^0(t_{n,j})$ belong in a bounded interval J.

Remark 4.2.1 *The above formula is explicit and the approximate solution u^q is given without needed to solve any algebraic system.*

In the next section, we will prove the convergence of the approximate solution u^q to the exact solution x of (4.1), moreover, the order of convergence is m for all $q \geq m$.

4.3 Convergence analysis

In this section, we assume that the functions Q and K satisfy the following Lipschitz conditions: there exist $A_i \geq 0 \;\; i = 0, 1, 2$ such that

$$\begin{aligned} &|Q(t, x_1) - Q(t, x_2)| \leq A_0 |x_1 - x_2|, \\ &|K(t, s, x_1, y_1) - K(t, s, x_2, y_2)| \leq A_1 |x_1 - x_2| + A_2 |y_1 - y_2|. \end{aligned}$$

The following lemma will be used in this section. The following result gives the existence and the uniqueness of a solution for the nonlinear system (4.5).

Lemma 4.3.1 *For sufficiently small h, the nonlinear system* (4.5) *has a unique solution $u \in S_m^0(I, \Pi_N)$.*

Proof. We will use the induction combined with the Banach fixed point theorem.

(i) On the interval $\sigma_0 = [t_0, t_1]$, the nonlinear system (4.5) becomes

$$\begin{aligned} w_0(t_{0,j}) = & f(t_{0,j}) + Q\left(t_{0,j}, u_0 + h \sum_{v=1}^{m} \left(\int_0^{c_j} L_v(\tau) d\tau \right) w_0(t_{0,v}) \right) \\ & + h \int_0^{c_j} K\left(t_{0,j}, t_0 + \mu h, u_0 + h \sum_{v=1}^{m} \left(\int_0^{\mu} L_v(\tau) d\tau \right) w_0(t_{0,v}), \sum_{v=1}^{m} L_v(\mu) w_0(t_{0,v}) \right) d\mu. \end{aligned} \tag{4.9}$$

We consider the operator Ψ defined by:

$$\begin{aligned} \Psi : \mathbb{R}^m &\longrightarrow \mathbb{R}^m \\ x = (x_1, ..., x_m) &\longmapsto \Psi(x) = (\Psi_1(x), ..., \Psi_m(x)), \end{aligned}$$

such that for $j = 1, ..., m$, we have

$$\Psi_j(x) = f(t_{0,j}) + Q\left(t_{0,j}, u_0 + h\sum_{v=1}^{m}\left(\int_0^{c_j} L_v(\tau)d\tau\right)x_v\right)$$
$$+ h\int_0^{c_j} K\left(t_{0,j}, t_0 + \mu h, u_0 + h\sum_{v=1}^{m}\left(\int_0^{\mu} L_v(\tau)d\tau\right)x_v, \sum_{v=1}^{m} L_v(\mu)x_v\right)d\mu.$$

Hence, for all $x, y \in \mathbb{R}^m$, we have

$$\|\Psi(x) - \Psi(y)\|_\infty \leq hmb\,(A_0 + A_1 + A_2)\,\|x - y\|_\infty,$$

where $b = \max\{|L_v(\mu)|, \mu \in [0, 1], v = 1, ..., m\}$.
Since $hmb\,(A_0 + A_1 + A_2) < 1$ for sufficiently small h, then by Banach fixed point theorem, the nonlinear system (4.9) has a unique solution on the interval σ_0.

(ii) Suppose that u exists and unique on the intervals $\sigma_i, i = 0, ..., n-1$ for $n \geq 1$ and we show that u exists and unique on the interval σ_n.
On the interval σ_n, the nonlinear system (4.5) becomes

$$w_n(t_{n,j}) = F(t_{n,j}) + Q\left(t_{n,j}, G(t_{n,j}) + h\sum_{v=1}^{m}\left(\int_0^{c_j} L_v(\tau)d\tau\right)w_n(t_{n,v})\right)$$
$$+ h\int_0^{c_j} K\left(t_{n,j}, t_n + \mu h, R(t_{n,j}) + h\sum_{v=1}^{m}\left(\int_0^{\mu} L_v(\tau)d\tau\right)w_n(t_{n,v}), \sum_{v=1}^{m} L_v(\mu)w_n(t_{n,v})\right)d\mu,$$

where,

$$F(t_{n,j}) = f(t_{n,j}) + h\sum_{p=0}^{n-1}\int_0^1 K\left(t_{p,j}, t_p + \mu h, u_p(t_p + \mu h), w_p(t_p + \mu h)\right)d\mu.$$

$$G(t_{n,j}) = f(0) + h\sum_{p=0}^{n-1}\sum_{v=1}^{m}\left(\int_0^{1} L_v(\tau)d\tau\right)w_p(t_{p,v}).$$

We consider the operator Ψ defined by:

$$\Psi : \mathbb{R}^m \longrightarrow \mathbb{R}^m$$
$$x = (x_1, ..., x_m) \longmapsto \Psi(x) = (\Psi_1(x), ..., \Psi_m(x)),$$

such that for $j = 1, ..., m$, we have

$$\begin{aligned}\Psi_j(x) =& F(t_{n,j}) + Q\left(t_{n,j}, G(t_{n,j}) + h\sum_{v=1}^{m}\left(\int_0^{c_j} L_v(\tau)d\tau\right)x_v\right)\\ &+ h\int_0^{c_j} K\left(t_{n,j}, t_n + \mu h, G(t_{n,j}) + h\sum_{v=1}^{m}\left(\int_0^{\mu} L_v(\tau)d\tau\right)x_v, \sum_{v=1}^{m} L_v(\mu)x_v\right)d\mu,\end{aligned}$$

Hence, for all $x, y \in \mathbb{R}^m$, we have

$$\|\Psi(x) - \Psi(y)\|_\infty \leq hmb\,(A_0 + A_1 + A_2)\,\|x - y\|_\infty,$$

Since $hmb\,(A_0 + A_1 + A_2) < 1$ for sufficiently small h, then by Banach fixed point theorem, the nonlinear system (4.5) has a unique solution u on σ_n.

■ The following result gives the convergence of the approximate solution u to the exact solution x.

Theorem 4.3.1 *Let f, Q, K be m times continuously differentiable on their respective domains. Then for sufficiently small h, the collocation solution u converges to the exact solution x, and the resulting error function $e := x - u$ satisfies:*

$$\|e^v\|_{L^\infty(I)} \leq Ch^m,$$

for $v = 0, 1$, where C is a finite constant independent of h.

Proof. Let $y = x'$. It holds that

$$y_n(t_n + \mu h) = \sum_{j=1}^{m} L_j(\mu) y_n(t_{n,j}) + \epsilon_n(\mu),\ \epsilon_n(\mu) = h^m \frac{y^m(\zeta_n(\mu))}{m!} \prod_{j=1}^{m} (\mu - c_j). \tag{4.10}$$

Hence,

$$\begin{aligned} x_n(t_n + \mu h) = u_0 &+ h \sum_{p=0}^{n-1} \int_0^1 \left(\sum_{v=1}^{m} L_v(\tau) y_p(t_{p,v}) + h^m \frac{y^m(\zeta_p(\tau))}{m!} \prod_{j=1}^{m} (\tau - c_j) d\tau \right) \\ &+ h \int_0^{\mu} \left(\sum_{v=1}^{m} L_v(\tau) y_n(t_{n,v}) + h^m \frac{y^m(\zeta_n(\tau))}{m!} \prod_{j=1}^{m} (\tau - c_j) \right) d\tau, \end{aligned} \tag{4.11}$$

It follows that the errors $\xi = y - w$ and $e = x - u$ have the following representation

$$\xi_n(t_n + \mu h) = \sum_{j=1}^{m} L_j(\mu) \xi_n(t_{n,j}) + \epsilon_n(\mu),\ \epsilon_n(\mu) = h^m \frac{y^m(\zeta_n(\mu))}{m!} \prod_{j=1}^{m} (\mu - c_j), \tag{4.12}$$

$$\begin{aligned} e(t_n + \mu h) = &h \sum_{p=0}^{n-1} \int_0^1 \left(\sum_{v=1}^{m} L_v(\tau) \xi_p(t_{p,v}) + h^m \frac{y^m(\zeta_p(\tau))}{m!} \prod_{j=1}^{m} (\tau - c_j) \right) d\tau \\ &+ h \int_0^{\mu} \left(\sum_{v=1}^{m} L_v(\tau) \xi_n(t_{n,v}) + h^m \frac{y^m(\zeta_n(\tau))}{m!} \prod_{j=1}^{m} (\tau - c_j) \right) d\tau, \end{aligned} \tag{4.13}$$

where $\xi_n = \xi|_{\sigma_n}$ and $e_n = e|_{\sigma_n}$.

On the other hand, from (4.5), we have

$$\begin{aligned} |\xi_n(t_{n,j})| \le &hAb \sum_{p=0}^{n} \sum_{v=1}^{m} |\xi_p(t_{p,v})| \\ &+ hAb \sum_{p=0}^{n-1} \left(h \sum_{i=0}^{p-1} \sum_{v=1}^{m} |\xi_i(t_{i,v})| + h \sum_{v=1}^{m} |\xi_p(t_{p,v})| + \sum_{v=1}^{m} |\xi_p(t_{p,v})| \right) \\ &+ hAb \left(h \sum_{p=0}^{n-1} \sum_{v=1}^{m} |\xi_p(t_{p,v})| + h \sum_{v=1}^{m} |\xi_n(t_{n,v})| + \sum_{v=1}^{m} |\xi_n(t_{n,v})| \right) + \alpha h^m, \end{aligned} \tag{4.14}$$

where $A = \max\{A_i, i = 0, 1, 2\}$ and α is a positive number.

We consider the sequence $\xi_n = \max\{|\xi_n(t_{n,v})|$ for $n = 0, ..., N-1$.

Then, from (4.14), ξ_n satisfies for $n = 0, ..., N-1$,

$$
\begin{aligned}
\xi_n \leq & Ahbm \sum_{p=0}^{n} \xi_p + hAbm \sum_{p=0}^{n-1} \left(h \sum_{i=0}^{p-1} \xi_i + h\xi_p + \xi_p \right) \\
& + hAbm \left(h \sum_{p=0}^{n-1} \xi_p + h\xi_n + \xi_n \right) + \alpha h^m \\
\leq & h \underbrace{Abm(2+3T)}_{\alpha_1} \sum_{p=0}^{n-1} \xi_p + h \underbrace{Abm\,(T+2)}_{\alpha_2} \xi_n + \alpha h^m.
\end{aligned} \tag{4.15}
$$

Hence, for $\bar{h} < \dfrac{1}{\alpha_2}$, we have for all $h \in (0, \bar{h}]$

$$
\xi_n \leq \frac{\alpha}{1 - \bar{h}\alpha_2} h^m + \frac{\alpha_1}{1 - \bar{h}\alpha_2} h \sum_{p=0}^{n-1} \xi_p.
$$

Then, by Lemma 1.7.1, for all $n = 0, ..., N-1$

$$
\xi_n \leq \frac{\alpha}{1 - \bar{h}\alpha_2} h^m \exp\left(\frac{T\alpha_1}{1 - \bar{h}\alpha_2} \right).
$$

Therefore, by using (4.12), we obtain

$$
\begin{aligned}
||e|| &\leq mb \max\{\xi_n, n = 0, ..., N-1\} + \beta h^m \\
&\leq mb \frac{\alpha}{1 - \bar{h}\alpha_2} \exp\left(\frac{T\alpha_1}{1 - \bar{h}\alpha_2} \right) h^m + \beta h^m \\
&\leq \underbrace{\left(mb \frac{\alpha}{1 - \bar{h}\alpha_2} \exp\left(\frac{T\alpha_1}{1 - \bar{h}\alpha_2} \right) + \beta \right)}_{\alpha_3} h^m,
\end{aligned}
$$

where β is a positive number.

Therefore, by using (4.13), we obtain

$$\|e\| \leq hmb\sum_{p=0}^{n-1}\xi_p + hmb\xi_n + \gamma h^m \leq 2mbT\alpha_3 h^m + \gamma h^m,$$

where γ is a positive number,

Thus, the proof is completed by taking $C = \max(\alpha_3, 2mbT\alpha_3 + \gamma)$. ∎

The following result gives the convergence of the iterative solution u^q to the exacts solution x.

Theorem 4.3.2 *Consider the iterative collocation solution u^q defined by* (4.6), (4.7) *and* (4.8), *then for any initial conditions $(u')^0(t_{n,j}) = w^0(t_{n,j}) \in J$ (J is a bounded interval), the iterative collocation solution u^q converges to the exact solution x. Moreover, the following error estimates hold*

$$\|(u^q)^{(v)} - x^{(v)}\| \leq Ch^m + C'\beta^q h^q$$

for $v = 0, 1$, where C, C', β are finite constants independent of h and q.

Proof. We define the error $\xi^q, e^q, \varepsilon^q$ and ς^q by $\xi^q(t) = w^q(t) - y(t)$, $e^q(t) = u^q(t) - x(t)$, $\varepsilon^q = w^q(t) - w(t)$, $\varsigma^q = u^q(t) - u(t)$ where u is defined by lemma 4.3.1.

We have, from (4.5) and (4.8), for all $n = 0, ..., N-1$ and $j = 1, ..., m$

$$\begin{aligned}|\varepsilon_n^q(t_{n,j})| \leq & hAb\left(\sum_{p=0}^{n-1}\sum_{v=0}^{m}|\varepsilon_p^q(t_{p,v})| + \sum_{v=0}^{m}|\varepsilon_n^{q-1}(t_{n,v})|\right)\\ &+ hAb\sum_{p=0}^{n-1}\left(h\sum_{i=0}^{p-1}\sum_{v=0}^{m}|\varepsilon_i^q(t_{i,v})| + h\sum_{v=0}^{m}|\varepsilon_p^q(t_{p,v})| + \sum_{v=0}^{m}|\varepsilon_p^q(t_{p,v})|\right)\\ &+ hAb\left(h\sum_{p=0}^{n-1}\sum_{v=0}^{m}|\varepsilon_p^q(t_{p,v})| + h\sum_{v=0}^{m}|\varepsilon_n^{q-1}(t_{n,v})| + \sum_{v=0}^{m}|\varepsilon_n^{q-1}(t_{n,v})|\right),\end{aligned} \tag{4.16}$$

Now, for each fixed $q \geq 1$, we consider the sequence $\varepsilon_n^q = \max\{\left|\varepsilon_n^q(t_{n,v})\right| v = 1....m\}$. It follows, from (4.16), that for $n = 0, ..., N-1$

$$\begin{aligned}
\varepsilon_n^q &\leq hAbm\left(\sum_{p=0}^{n-1} \varepsilon_p^q + \varepsilon_n^{q-1}\right) + hAbm\sum_{p=0}^{n-1}\left(h\sum_{i=0}^{p-1} \varepsilon_i^q + h\varepsilon_p^q + \varepsilon_p^q\right) \\
&\qquad + hAbm\left(h\sum_{p=0}^{n-1} \varepsilon_p^q + h\varepsilon_n^{q-1} + \varepsilon_n^{q-1}\right) \qquad (4.17) \\
&\leq h\underbrace{Abm(2+3T)}_{\alpha_1}\sum_{p=0}^{n-1} \varepsilon_p^q + h\underbrace{Abm(2+T)}_{\alpha_2}\varepsilon_n^{q-1},
\end{aligned}$$

We consider the sequence $\eta^q = \max\{\varepsilon_n^q, n = 0, ..., N-1\}$ for $q \geq 1$.
Then, from (4.17), we obtain

$$\varepsilon_n^q \leq \alpha_1 h \sum_{p=0}^{n-1} \varepsilon_p^q + \alpha_2 h \eta^{q-1}. \qquad (4.18)$$

Hence, by Lemma 1.7.1, for all $n = 0, ..., N-1$

$$\eta^q \leq \underbrace{\alpha_2 \exp(\alpha_1 T)}_{\beta} h\eta^{q-1} \leq \beta^2 h^2 \eta^{q-2} \leq ... \leq \beta^q h^q \eta^0.$$

Since, $w^0(t_{n,j}) \in J$ (bounded interval) and w is bounded by Lemma 4.3.1, then there exists $\delta > 0$ such that $\eta^0 < \delta$, which implies that, for all $q \geq 1$

$$\eta^q \leq \delta\beta^q h^q.$$

Therefore, by using (4.3) and (4.6), we obtain

$$\|\varepsilon^q\| \leq mb\eta^q \leq \underbrace{mb\delta}_{d}\beta^q h^q,$$

Hence, by Theorem (4.3.1), we deduce that

$$\|\xi^q\| \leq \|\varepsilon^q\| + \|w - y\| \leq d\beta^q h^q + Ch^m.$$

On the other hand, from (4.4) and (4.7), we have

$$\|\varsigma^q\| \leq 2Tmb\|\varepsilon^q\| \leq 2Tmbd\beta^q h^q.$$

Finally, by using Theorem (4.3.1), we deduce that

$$\|e^q\| \leq \|\varsigma^q\| + \|u - x\| \leq \underbrace{2Tmbd}_{d'}\beta^q h^q + Ch^m.$$

Thus, the proof is completed by taking $C' = \max(d, d')$. ■

4.4 Numerical examples

In order to test the applicability of the presented method, we consider the following examples with $T = 1$. These examples have been solved with various values of N, m and $q = m$. We used the collocation parameters $c_i = \frac{i}{m+1}, i = 1, ..., m$. In each example, we calculate the error between x and the iterative collocation solution u^m.
The absolute errors at the particular points are given to compare our solutions with the solutions obtained by [61, 64].
The results in these examples confirm the theoretical results; moreover, the results obtained by the present method is very superior to that obtained by the methods in [61, 64].

Example 4.4.1 *Consider the nonlinear Volterra integro-differential equation given by*

$$x'(t) = f(t) + \int_0^t \cos(t + s + x(s) + x'(s)) + \frac{1}{1 + x^2(s)} ds, \ t \in [0, 1].$$

with f is chosen so that the exact solution is $x(t) = 2t + 5$. The absolute errors for $(N, m) \in \{(2, 3), (4, 3), (4, 4), (6, 4)\}$ at $t = 0, 0.1, ..., 1$ are presented in Table 4.1. From the Table 4.1, we note that the absolute error reduces as N or m increases.

Table 4.1: Absolute errors for Example 4.4.1

t	$N = 2$ $m = 3$	$N = 4$ $m = 3$	$N = 4$ $m = 4$	$N = 6$ $m = 4$
0	0.0	0.0	0.0	0.0
0.1	2.03 E −4	1.48 E −5	4.40 E −7	4.99 E −8
0.2	2.37 E −4	1.73 E −5	1.36 E −6	2.90 E −7
0.3	2.23 E −4	4.94 E −7	1.43 E −6	2.27 E −7
0.4	2.88 E −4	7.86 E −6	1.21 E −6	9.56 E −7
0.5	5.53 E −4	6.69 E −5	8.55 E −6	9.58 E −7
0.6	1.07 E −3	6.17 E −5	7.95 E −6	8.30 E −7
0.7	1.12 E −3	5.43 E −5	7.46 E −6	7.60 E −7
0.8	1.14 E −3	6.04 E −5	7.24 E −6	6.97 E −7
0.9	1.52 E −3	6.90 E −5	6.36 E −6	3.84 E −7
1	2.71 E −3	1.38 E −4	6.64 E −7	4.33 E −7

Example 4.4.2 *Consider the nonlinear Volterra integro-differential equation given by*

$$x'(t) = f(t) + \int_0^t \frac{\cos(t)}{1+t+(x'(s))^2} + \frac{t\sin(s)}{2+x^2(s)} ds, \ t \in [0,1].$$

with f is chosen so that the exact solution is $x(t) = 3\cos(t) + 1$. The absolute errors for $(N,m) \in \{(2,3),(4,3),(4,4),(6,4)\}$ at $t = 0, 0.1, ..., 1$ are presented in Table 4.2. From the Table 4.2, we note that the absolute error reduces as N or m increases.

Table 4.2: Absolute errors for Example 4.4.2

t	$N=2$ $m=3$	$N=4$ $m=3$	$N=4$ $m=4$	$N=6$ $m=4$
0	0.0	0.0	0.0	0.0
0.1	6.87 E −4	5.08 E −5	1.33 E −7	1.13 E −8
0.2	8.11 E −4	5.17 E −5	1.36 E −7	4.96 E −8
0.3	8.13 E −4	8.67 E −5	5.64 E −7	6.11 E −8
0.4	8.41 E −4	9.49 E −5	6.16 E −7	1.38 E −7
0.5	7.58 E −4	9.41 E −5	7.46 E −7	1.61 E −7
0.6	1.28 E −3	1.37 E −4	1.43 E −6	2.44 E −7
0.7	1.38 E −3	1.40 E −4	1.46 E −6	3.58 E −7
0.8	1.39 E −3	1.65 E −4	2.38 E −6	3.69 E −7
0.9	1.41 E −3	1.71 E −4	2.47 E −6	5.06 E −7
1	1.35 E −3	1.69 E −4	2.61 E −6	5.13 E −7

Example 4.4.3 *Consider the nonlinear Volterra integro-differential equation given by*

$$x'(t) = f(t) + \int_0^t (ts \arctan(s + x(s) + x'(s)) + \cos(t - s + x(s)))ds, \ t \in [0, 1].$$

with f *is chosen so that the exact solution is* $x(t) = 2t + 1$*. The absolute errors for* $(N, m) \in \{(2, 2), (2, 3), (4, 3), (4, 4)\}$ *at* $t = 0, 0.1, ..., 1$ *are presented in Table 4.2. From the Table 4.3, we note that the absolute error reduces as N or m increases.*

Table 4.3: Absolute errors for Example 4.4.3

t	$N = 2$ $m = 2$	$N = 2$ $m = 3$	$N = 4$ $m = 3$	$N = 4$ $m = 4$
0	0.0	0.0	0.0	0.0
0.1	1.26 E −4	5.86 E −7	1.40 E −8	3.03 E −9
0.2	2.47 E −4	1.58 E −6	5.66 E −8	8.11 E −9
0.3	3.61 E −4	3.42 E −6	1.00 E −7	1.49 E −9
0.4	4.70 E −4	6.53 E −6	1.13 E −7	6.14 E −9
0.5	5.73 E −4	1.13 E −5	1.47 E −7	2.00 E −9
0.6	5.66 E −4	1.13 E −5	1.43 E −7	2.16 E −9
0.7	5.63 E −4	1.11 E −5	1.41 E −7	1.01 E −8
0.8	5.66 E −4	1.06 E −5	1.31 E −7	4.96 E −10
0.9	5.73 E −4	1.01 E −5	1.26 E −7	1.76 E −10
1	5.84 E −4	9.46 E −6	1.22 E −7	2.00 E −9

Example 4.4.4 *([61, 64]) Consider the linear Volterra integro-differential equation given by*

$$x'(t) = 1 - \int_0^t x(s)ds, \; t \in [0, 1].$$

with the initial conditions $x(0) = 0$ *and the exact solution* $x(t) = \sin(t)$. *Here,* $f(t) = 1, g(t) = 0, K(t, s) = -1$.

The absolute errors for $N = 6, 10$ *and* $m = q = 5$ *at* $t = 0, 0.1, ..., 1$ *are displayed in Table 4.4. The numerical results of the absolute error function obtained by the present method are compared in Table 4.4 with the absolute error function of the Taylor method given in [61] and Bessel method [64] for an approximate polynomial solutions of degree* 5*.*

Table 4.4: Comparison of the absolute errors of Example 4. 4.4

t	Taylor method [61]	Bessel method [64]	Present method $N = 6$	Present method $N = 10$
0.0	0.0	0.0	0.0	0.0
0.1	2.00 E −11	2.49 E −7	1.58 E −9	1.26 E −10
0.2	2.50 E −9	4.02 E −7	5.45 E −10	9.50 E −11
0.3	4.33 E −8	3.00 E −7	2.71 E −9	2.61 E −10
0.4	3.24 E −7	2.05 E −7	8.90 E −10	1.08 E −10
0.5	1.54 E −6	2.83 E −7	5.60 E −9	2.04 E −10
0.6	5.52 E −6	3.75 E −7	7.20 E −9	4.96 E −12
0.7	1.62 E −5	1.65 E −7	1.19 E −9	4.62 E −10
0.8	4.12 E −5	1.81 E −7	1.36 E −10	5.00 E −10
0.9	9.38 E −5	1.18 E −6	1.27 E −9	1.72 E −10
1.0	1.95 E −4	9.66 E −6	7.85 E −9	9.92 E −10

4.5 Conclusion

In this chapter, we proposed the iterative collocation method for the numerical solution of integro-differential equations(4.1) in the spline space $S_m^{(0)}(\Pi_N)$. Our numerical results are compared with exact solutions and existing methods. Error analysis shows the accuracy and effectiveness of the proposed scheme Hence, the present method is approached through the illustrative examples which show the efficiency, validity and applicability.

CHAPTER 5

ITERATIVE COLLOCATION METHOD FOR NONLINEAR VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS IN THE SPACE $S_{M+1}^{(1)}(\Pi_N)$

5.1 Introduction

In this chapter, we consider the following Volterra integro-differential equations,

$$x'(t) = f(t) + \int_0^t K(t, s, x(s))ds, x(t_0) = x_0, t \in I = [0, T], \tag{5.1}$$

where the functions f, K are sufficiently smooth.
There are several numerical methods for approximating the solution of equation (5.1). For example, spectral methods, implicit Runge-Kutta methods, Galerkin methods, collocation methods, and Legendre wavelets series, (cf, e.g. [38, 58, 13, 2, 39, 40], and references therein).
The purpose of this paper is to solve equation (5.1) by the iterative collocation method. in the spline space $S^{(1)}_{m+1}(\Pi_N)$.
The outlines of this chapter is as follows. In section 2, the spline polynomial has been used to approximate equation (5.1) based on the iterative collocation method, error analysis has been discussed in section 3, section 4 reports some numerical examples , in the last section, we give a conclusion.

5.2 Description of the method

Let Π_N be a uniform partition of the interval $I = [0, T]$ defined by $t_n = nh, \quad n = 0, ..., N - 1$, where the stepsize is given by $\frac{T}{N} = h$. Let the collocation parameters be $0 \leq c_1 < ...... < c_m \leq 1$ and the collocation points be $t_{n,j} = t_n + c_j h$, $j = 1, ..., m, n = 0, ..., N - 1$. Define the subintervals $\sigma_n = [t_n, t_{n+1}]$, and $\sigma_{N-1} = [t_{N-1}, t_N]$. Moreover, denote by π_{m+1} the set of all real polynomials of degree not exceeding $m + 1$.
We define the real polynomial spline space of degree $m + 1$ as follows:

$$S^{(1)}_{m+1}(\Pi_N) = \{u \in C^1(\mathrm{I}, \mathbb{R}) : u_n = u/\sigma_n \in \pi_{m+1}, n = 0, ..., N - 1\}.$$

It holds for any $y \in C^{m+2}([0,T])$ that

$$x'(t_n + \mu h) = L_0(v)x'(t_n) + \sum_{v=1}^{m} L_v(\mu)x'(t_{n,v}) + h^{m+1}R^1_{m+1,n}(\mu). \tag{5.2}$$

The Peano remainder term and Peano kernel are given by

$$R^1_{m+1,n}(\mu) := \int_0^1 K_m(s,\tau)x^{(m+1)}(t_n + \tau)d\tau, \tag{5.3}$$

and

$$K_m(\mu,\tau) := \frac{1}{(m-1)!}\left\{(\mu-\tau)^{m-1}_+ - \sum_{v=1}^{m} L_v(\mu)(c_v - \tau)^{m-1}_+\right\}. \tag{5.4}$$

Integration of (5.2) leads to

$$x(t_n + \mu h) = x(t_n) + hB_0(\mu)x'(t_n) + h\sum_{v=1}^{m} B_v(\mu)x'(t_{n,v}) + h^{m+2}R_{m+1,n}(\mu), \tag{5.5}$$

where

$$R_{m+1,n}(\mu) := \int_0^{\mu} R^1_{m+1,n}(\tau)d\tau,$$

where $\mu \in [0,1]$, $B_0(\mu) = \int_o^{\mu} L_0(v)dv$ and $B_j(\mu) = \int_o^{\mu} L_j(v)dv$ with
$L_0(v) = (-1)^m \prod_{l=1}^{m} \frac{v - c_l}{c_l}$ and $L_j(v) = \frac{v}{c_j}\prod_{l\neq j}^{m} \frac{v - c_l}{c_j - c_l}$, $j = 1, ..., m$ are the Lagrange polynomials associate with the parameters c_j, $j = 1, ..., m$.

Inserting (5.5) for the function $s \longmapsto K(t,s,x(s))$ into (5.1), we obtain for each $j =$

$1, ..., m, , n = 0, ..., N-1$

$$\begin{aligned} x'(t_{n,j}) &= f(t_{n,j}) \\ &+ \sum_{p=0}^{n-1} \int_o^1 K\left(t_{n,j}, t_p + \mu h, x(t_p) + hB_0(\mu)x'(t_p) + h\sum_{v=1}^{m} B_v(\mu)x'(t_{p,v}) + h^{m+2}R_{m+1,p}(\mu)\right) d\mu \\ &+ h\int_o^{c_j} K\left(t_{n,j}, t_n + \mu h, x(t_n) + hB_0(\mu)x'(t_n) + h\sum_{v=1}^{m} B_v(\mu)x'(t_{n,v}) + h^{m+2}R_{m+1,n}(\mu)\right) d\mu. \end{aligned} \tag{5.6}$$

It holds for any $u \in S^1_{m+1}(I, \Pi_N)$ that

$$u'_n(t_n + \mu h) = L_0(\mu)u'_{n-1}(t_n) + \sum_{v=1}^{m} L_v(\mu)u'_n(t_{n,v}), \tag{5.7}$$

and

$$u_n(t_n + \mu h) = u_{n-1}(t_n) + hB_0(\mu)u'_{n-1}(t_n) + h\sum_{v=1}^{m} B_v(\mu)u'_n(t_{n,v}), \mu \in [0, 1]. \tag{5.8}$$

Now, we approximate x by $u \in S^1_{m+1}(I, \Pi_N)$ such that $u'(t_{n,j})$ satisfy the following non-linear system,

$$\begin{aligned} u'_n(t_{n,j}) &= f(t_{n,j}) \\ &+ \sum_{p=0}^{n-1} \int_o^1 K\left(t_{n,j}, t_p + \mu h, u_p(t_p) + hB_0(\mu)u'_p(t_p) + h\sum_{v=1}^{m} B_v(\mu)u'_p(t_{p,v})\right) d\mu \\ &+ h\int_o^{c_j} K\left(t_{n,j}, t_n + \mu h, u_{n-1}(t_n) + hB_0(\mu)u'_{n-1}(t_n) + h\sum_{v=1}^{m} B_v(\mu)u'_n(t_{n,v})\right) d\mu. \end{aligned} \tag{5.9}$$

for $n = 0, ..., N-1$, $j = 1, ..., m$ where $u'_{-1}(t_0) = x'(0) = f(0)$ and $u_{-1}(t_0) = x(0)$.
Since the above system is nonlinear, we will use an iterative collocation solution $u^q \in S^1_{m+1}(I, \Pi_N), q \in \mathbb{N}$, to approximate the solution of (5.1) such that

$$(u^q_n)'(t_n + \mu h) = L_0(\mu)(u^q_{n-1})'(t_n) + \sum_{v=1}^{m} L_v(\mu)(u^q_n)'(t_{n,v}), \mu \in [0, 1], \tag{5.10}$$

and

$$u^q_n(t_n+\mu h) = u^q_{n-1}(t_n) + hB_0(\mu)(u^q_{n-1})'(t_n) + h\sum_{v=1}^{m} B_v(\mu)(u^q_n)'(t_{n,v}),\ \mu\in[0,1], \tag{5.11}$$

where the coefficients $(u^q_n)'(t_{n,j})$ are given by the following formula:

$$\begin{aligned}(u^q_n)'(t_{n,j}) &= f(t_{n,j})\\ &+\sum_{p=0}^{n-1}\int_o^1 K\left(t_{n,j}, t_p+\mu h, u^q_p(t_p) + hB_0(v)(u^q_p)'(t_p) + h\sum_{v=1}^{m} B_v(v)(u^q_p)'(t_{p,v})\right)d\mu\\ &+ h\int_o^{c_j} K\left(t_{n,j}, t_n+\mu h, u^q_{n-1}(t_n) + hB_0(v)(u^q_{n-1})'(t_n) + h\sum_{v=1}^{m} B_v(v)(u^{q-1}_n)'(t_{n,v})\right)d\mu.\end{aligned} \tag{5.12}$$

Such that $(u^q_{-1})'(t_0) = f(0)$ and $u^q_{-1}(t_0) = x_0$ for all $q\in\mathbb{N}$ and the initial values $(u^0_n)'(t_{n,j})\in J$ (J is a bounded interval).
The above formula is explicit and the approximate solution u^q is given without needed to solve any algebraic system.
In the next section, we will prove the convergence of the approximate solution u^q to the exact solution x of (5.1), moreover, the order of convergence is m for all $q\geq m$.

5.3 Convergence analysis

In this section, we assume that the functions K satisfy the Lipschitz condition with respect to the third variable: there exists $\overline{K}\geq 0$ such that
$|K(t,s,x_1) - K(t,s,x_2| \leq \overline{K}|x_1 - x_2|$. The following three lemmas will be used in this section. The following result gives the existence and the uniqueness of a solution for the nonlinear system (5.8).

Lemma 5.3.1 *For sufficiently small h, the nonlinear system* (5.8) *defines a unique solution $u\in S^1_{m+1}(I,\Pi_N)$ which is given by* (5.9).

Proof. We will use the induction combined with the Banach fixed point theorem.

(i) On the interval $\sigma_0 = [t_0, t_1]$, the nonlinear system (5.9) becomes

$$u_n'(t_{0,j}) = f(t_{0,j}) + h\int_0^{c_j} K\left(t_{0,j}, t_0 + \mu h, x_0 + hf(0)B_0(\mu) + h\sum_{v=1}^{m} B_v(\mu)u_0'(t_{0,v})\right)d\mu.$$

We consider the operator Ψ defined by:

$$\Psi : \mathbb{R}^m \longrightarrow \mathbb{R}^m$$
$$x = (x_1, ..., x_m) \longmapsto \Psi(x) = (\Psi_1(x), ..., \Psi_m(x)),$$

such that for $j = 1, ..., m$, we have

$$\Psi_j(x) = f(t_{0,j}) + h\int_0^{c_j} K\left(t_{0,j}, t_0 + \mu h, x_0 + hf(0)B_0(\mu) + h\sum_{v=1}^{m} B_v(\mu)x_v\right)d\mu.$$

Hence, for all $x, y \in \mathbb{R}^m$, we have

$$\|\Psi(x) - \Psi(y)\|_\infty \leq h^2 mb\overline{K}\|x - y\|_\infty,$$

where $b = \max\{|B_v(\mu)|, \mu \in I, v = 0, ..., m\}$.
Since $h^2mb\overline{K} < 1$ for sufficiently small h, then by Banach fixed point theorem, the nonlinear system (5.9) has a unique solution $u'(t_{0,j}), j = 1, ..., m$.
Hence, the equation (5.8) defines a unique solution u_0 on σ_0.

(ii) Suppose that u_p exists and unique on the intervals $\sigma_p, p = 0, ..., n-1$ for $n \geq 1$ and we show that u_n exists and unique on the interval σ_n.
On the interval σ_n, the nonlinear system (5.9) becomes

$$u_n'(t_{n,j}) = F(t_{n,j}) + h\int_o^{c_j} K\left(t_{n,j}, t_n + \mu h, G(t_{n,j}) + h\sum_{v=1}^{m} B_v(\mu)u_n'(t_{n,v})\right)d\mu.$$

Where,

$$F(t_{n,j}) = f(t_{n,j}) + \sum_{p=0}^{n-1} \int_{o}^{1} K\left(t_{n,j}, t_p + \mu h, u_p(t_p) + hB_0(\mu)u'_p(t_p) + h\sum_{v=1}^{m} B_v(\mu)u'_p(t_{p,v})\right) d\mu.$$

$$G(t_{n,j}) = u_{n-1}(t_n) + hB_0(\mu)u'_{n-1}(t_n)$$

We consider the operator Ψ defined by:

$$\Psi : \mathbb{R}^m \longrightarrow \mathbb{R}^m$$
$$x = (x_1, ..., x_m) \longmapsto \Psi(x) = (\Psi_1(x), ..., \Psi_m(x)),$$

such that for $j = 1, ..., m$, we have

$$\Psi_j(x) = F(t_{n,j}) + h\int_{o}^{c_j} K\left(t_{n,j}, t_n + \mu h, G(t_{n,j}) + h\sum_{v=1}^{m} B_m(\mu)x_v\right) d\mu.$$

Hence, for all $x, y \in \mathbb{R}^m$, we have

$$||\Psi(x) - \Psi(y)||_\infty \leq h^2 mb\overline{K}||x - y||_\infty,$$

Since $h^2 mb\overline{K} < 1$ for sufficiently small h, then by Banach fixed point theorem, the nonlinear system (5.9) has a unique solution $u'(t_{n,j}), j = 1, ..., m$.
Hence, the equation (5.8) defines a unique solution u_n on σ_n.

■ The following result gives the convergence of the approximate solution u to the exact solution x.

Theorem 5.3.1 *Let f, K be $m+2$ times continuously differentiable on their respective domains. If $-1 < R(\infty) = (-1)^m \prod_{l=1}^{m} \frac{1-c_l}{c_l} < 1$, then, for sufficiently small h, the collocation solution u converges to the exact solution x, and the resulting errors functions $e^{(v)} := x^{(v)} - u^{(v)}$ for $v = 0, 1$ satisfies:*

$$||e^{(v)}||_{L^\infty(I)} \leq Ch^{m+1},$$

for $v = 0, 1$ *and* C *is a finite constant independent of* h.

Proof. We have, from (5.9) and (5.6), the error e' which is defined on σ_n, by $e'(t) = e'_n(t) = x'(t) - u'_n(t)$ for $n \in \{0, 1, ..., N-1\}$ satisfies for all $n = 0, ..., N-1$ and $j = 1, ..., m$

$$\begin{aligned}
|e'_n(t_{n,j})| \leq & h\overline{K} \sum_{p=0}^{n-1} |e_p(t_p)| + h^2 b\overline{K} \sum_{p=0}^{n-1} |e'_p(t_p)| + h^2 b\overline{K} \sum_{p=0}^{n-1} \sum_{v=1}^{m} |e'_p(t_{p,v})| \\
& + h\overline{K}|e_{n-1}(t_n)| + h^2 b\overline{K}|e'_{n-1}(t_n)| + h^2 b\overline{K} \sum_{v=1}^{m} |e'_n(t_{n,v})| + \alpha h^{m+2} \\
\leq & 2\overline{K}h \sum_{p=0}^{n-1} \|e_p\| + 2b\overline{K}h^2 \sum_{p=0}^{n-1} \|e'_p\| + b\overline{K}h^2 \sum_{p=0}^{n-1} \sum_{v=1}^{m} |e'_p(t_{p,v})| \\
& + b\overline{K}h^2 \sum_{v=1}^{m} |e'_n(t_{n,v})| + \alpha h^{m+2},
\end{aligned} \tag{5.13}$$

where α is a positive number.
Therefore, by using (5.5) and (5.8), we obtain for each $p = 0, ..., N-1$

$$\|e_p\| \leq bh \sum_{i=0}^{p-1} \|e'_i\| + bh \sum_{i=0}^{p-1} \sum_{v=1}^{m} |e'_i(t_{i,v})| + \alpha T h^{m+1}. \tag{5.14}$$

Inserting (5.14) into (5.13), we obtain for each $j = 1, ..., m, n = 0, ..., N-1$

$$\begin{aligned}
|e'_n(t_{n,j})| \leq 2\overline{K}b(T+h)h \sum_{p=0}^{n-1} \|e'_p\| + \overline{K}b(2T+h)h \sum_{p=0}^{n-1} \sum_{v=1}^{m} |e'_p(t_{p,v})| \\
+ b\overline{K}h^2 \sum_{v=1}^{m} |e'_n(t_{n,v})| + \overline{\alpha} h^{m+1}.
\end{aligned} \tag{5.15}$$

We consider the sequence $\varepsilon_n = \sum_{v=1}^{m} |e'_n(t_{n,v})|$ for $n = 0, ..., N-1$.
Then, from (5.15), ε_n satisfies for $n = 0, ..., N-1$,

$$\varepsilon_n \leq 2m\overline{K}b(T+h)h \sum_{p=0}^{n-1} \|e'_p\| + \overline{K}bm(2T+h)h \sum_{p=0}^{n-1} \varepsilon_p + h^2 mb\overline{K}\varepsilon_n + \overline{\alpha} m h^{m+1}. \tag{5.16}$$

Hence, for $\overline{h} < \dfrac{1}{\sqrt{mb\overline{K}}}$, we have for all $h \in (0, \overline{h}]$

$$\varepsilon_n \leq \underbrace{\frac{2m\overline{K}b(T+\overline{h})}{1-\overline{h}^2 mb\overline{K}}}_{\alpha_1} h \sum_{p=0}^{n-1} \|e'_p\| + \underbrace{\frac{\overline{K}bm(2T+\overline{h})}{1-\overline{h}^2 mb\overline{K}}}_{\alpha_2} h \sum_{p=0}^{n-1} \varepsilon_p + \underbrace{\frac{\overline{\alpha}m}{1-mb\overline{K}\overline{h}}}_{\alpha_3} h^{m+1}.$$

Then, by Lemma 1.7.3, for all $n = 0, ..., N-1$

$$\varepsilon_n \leq \underbrace{\alpha_1 \exp(T\alpha_2)}_{\alpha_4} h \sum_{p=0}^{n-1} \|e'_p\| + \underbrace{\alpha_3 \exp(T\alpha_2)}_{\alpha_5} h^{m+1}.$$

Therefore, by using (5.2) and (5.7), we obtain

$$\begin{aligned} \|e'_n\| &\leq |R(\infty)|\|e'_{n-1}\| + \rho\varepsilon_n + \beta h^{m+1} \\ &\leq |R(\infty)|\|e'_{n-1}\| + \underbrace{\rho\alpha_4}_{\alpha_6} h \sum_{p=0}^{n-1} \|e'_p\| + \underbrace{(\rho\alpha_5 + \beta)}_{\alpha_7} h^{m+1}, \end{aligned}$$

where $\rho = \max\{|L_j(t)|, t \in [0,1], j = 1, ..., m\}$ and $e'_{-1} = 0$.

Hence by Lemma 1.7.4, we obtain for all $n = 0, ..., N-1$

$$\begin{aligned} \|e'_n\| &\leq \frac{\|e'_0\|}{R_2 - R_1}[(R_2-1)R_2^n + (1-R_1)R_1^n] + \frac{\alpha_7 h^{m+1}}{R_2 - R_1}[R_2^n - R_1^n] \\ &\leq \frac{\|e'_0\|}{R_2 - R_1}[(R_2-1)R_2^{\frac{T}{h}} + 1] + \frac{\alpha_7 h^{m+1}}{R_2 - R_1}[R_2^{\frac{T}{h}}] \\ &\leq \left(\frac{1}{R_2 - R_1}[(R_2-1)R_2^{\frac{T}{h}} + 1] + \frac{1}{R_2 - R_1}[R_2^{\frac{T}{h}}]\right)\alpha_7 h^{m+1}, \end{aligned} \tag{5.17}$$

where R_1 and R_2 are defined by (1.42) such that $A = |R(\infty)|, B = \alpha_6 h, K = \alpha_7 h^{m+1}$.

Since, $\lim_{h\longrightarrow 0}\left(\frac{1}{R_2-R_1}[(R_2-1)R_2^{\frac{T}{h}} + 1] + \frac{1}{R_2-R_1}[R_2^{\frac{T}{h}}]\right) = \frac{1+\exp\left(\frac{T\alpha_6}{1-|R(\infty)|}\right)}{1-|R(\infty)|} < +\infty$.

Then, there exists $\gamma > 0$ such that for all $h \in (0, \overline{h}]$.

$$\frac{1}{R_2 - R_1}[(R_2-1)R_2^{\frac{T}{h}} + 1] + \frac{1}{R_2 - R_1}[R_2^{\frac{T}{h}}] \leq \gamma.$$

Thus, $\|e'(t)\| \leq \gamma\alpha_7 h^{m+1}$, which implies, by using (5.5) and (5.8), that

$$
\begin{aligned}
\|e_n\| &\leq \|e_{n-1}\| + hb\|e'_{n-1}\| + hbm\|e'_n\| + \alpha h^{m+2} \\
&\leq \|e_{n-1}\| + \underbrace{(b\gamma\alpha_7 + bm\gamma\alpha_7 + \alpha)}_{\lambda} h^{m+2} \\
&\leq \|e_{n-2}\| + 2\lambda h^{m+2} \leq ... \leq \|e_0\| + n\lambda h^{m+2} \qquad (5.18) \\
&\leq \underbrace{\|e_{-1}\|}_{=0} + \lambda h^{m+2} + T\lambda h^{m+1} \\
&\leq 2\lambda T h^{m+1}.
\end{aligned}
$$

Thus, the proof is completed by taking $C = \max(2\lambda T, \gamma\alpha_7)$. ■

The following result gives the convergence of the iterative solution u^q to the exact solution x.

Theorem 5.3.2 *Consider the iterative collocation solution $u^q, q \geq 1$ defined by* (5.11)*, if $-1 < R(\infty) = (-1)^m \prod_{l=1}^{m} \frac{1-c_l}{c_l} < 1$, then for any initial condition $(u')^0(t_{n,j}) \in J$, the iterative collocation solution $u^q, q \geq 1$ converges to the exact solution x for sufficiently small h. Moreover, the following errors estimates hold*

$$\|u^q - x\| \leq \gamma_1\beta^q h^{2q} + \gamma_2\beta^q h^{m+1+2q} + \beta_2 h^{m+1},$$

and

$$\|(u^q)' - x'\| \leq \beta_1 h^{2q} + \beta_2 h^{m+1+2q} + \beta_2 h^{m+1}.$$

where $\beta, \beta_1, \beta_2, \gamma_1, \gamma_2$ are finite constants independent of h and q.

Proof. We define the errors e^q and ξ^q by $e^q(t) = e^q_n(t) = u^q_n(t) - x(t)$ and $\xi^q = \xi^q_n = u^q_n(t) - u_n(t)$ on $\sigma_n, n = 0, ..., N-1$.

We have, from (5.9) and (5.12), for all $n = 0, ..., N-1$ and $j = 1, ..., m$

$$\begin{aligned}
|(\xi_n^q)'(t_{n,j})| &\leq h\overline{K}\sum_{p=0}^{n-1}\|\xi_p^q\| + h^2b\overline{K}\sum_{p=0}^{n-1}\|(\xi_p^q)'\| + h^2bm\overline{K}\sum_{p=0}^{n-1}\|(\xi_p^q)'\| + h\overline{K}\|\xi_{n-1}^q\| \\
&\qquad + h^2b\overline{K}\|(\xi_{n-1}^q)'\| + h^2b\overline{K}\sum_{v=1}^{m}|(\xi_n^{q-1})'(t_{n,v})| \\
&\leq 2h\overline{K}\sum_{p=0}^{n-1}\|\xi_p^q\| + 2h^2b\overline{K}\sum_{p=0}^{n-1}\|(\xi_p^q)'\| + h^2bm\overline{K}\sum_{p=0}^{n-1}\|(\xi_p^q)'\| \\
&\qquad + h^2b\overline{K}\sum_{v=1}^{m}|(\xi_n^{q-1})'(t_{n,v})|. \\
&\leq 2h\overline{K}\sum_{p=0}^{n-1}\|\xi_p^q\| + h^2b\overline{K}(2+m)\sum_{p=0}^{n-1}\|(\xi_p^q)'\| + h^2b\overline{K}m\|(\xi_n^{q-1})'\|.
\end{aligned} \tag{5.19}$$

We have, from (5.8) and (5.11), for all $n = 0, ..., N-1$ and $j = 1, ..., m$

$$\begin{aligned}
\|\xi_n^q\| &\leq \|\xi_{n-1}^q\| + hb\|(\xi_{n-1}^q)'\| + hbm\|(\xi_n^q)'\| \\
&\leq \|\xi_{n-1}^q\| + hbm\left(\|(\xi_{n-1}^q)'\| + \|(\xi_n^q)'\|\right) \\
&\leq \|\xi_{n-2}^q\| + hbm\left(\|(\xi_{n-2}^q)'\| + 2\|(\xi_{n-1}^q)'\| + \|(\xi_n^q)'\|\right) \\
&\leq \|\xi_{n-3}^q\| + hbm\left(\|(\xi_{n-3}^q)'\| + 2\|(\xi_{n-2}^q)'\| + 2\|(\xi_{n-1}^q)'\| + \|(\xi_n^q)'\|\right) \\
&\qquad \vdots \\
&\leq \|\xi_0^q\| + hbm\left(\|(\xi_0^q)'\| + 2\|(\xi_1^q)'\| + ... + 2\|(\xi_{n-1}^q)'\| + \|(\xi_n^q)'\|\right) \\
&\leq \|\xi_0^q\| + 2hbm\sum_{p=0}^{n}\|(\xi_i^q)'\| \\
&\leq \|\underbrace{\xi_{-1}^q}_{=0}\| + hb\|\underbrace{(\xi_{-1}^q)'}_{=0}\| + hbm\|(\xi_0^q)'\| + 2hbm\sum_{p=0}^{n}\|(\xi_i^q)'\| \\
&\leq 3hbm\sum_{i=0}^{n}\|(\xi_i^q)'\|.
\end{aligned} \tag{5.20}$$

Inserting (5.20) into (5.19), we obtain for each $j = 1, ..., m, n = 0, ..., N-1$

$$
\begin{aligned}
|(\xi_n^q)'(t_{n,j})| &\le 6hTbm\overline{K}\sum_{p=0}^{n-1}\|(\xi_p^q)'\| + h^2 b\overline{K}(2+m)\sum_{p=0}^{n-1}\|(\xi_p^q)'\| \\
&\quad + h^2 b\overline{K}m\|(\xi_n^{q-1})'(t_{n,v})\| \\
&\le h\underbrace{b\overline{K}(7Tm+2T)}_{c_1}\sum_{p=0}^{n-1}\|(\xi_p^q)'\| + h^2 b\overline{K}m\|(\xi_n^{q-1})'(t_{n,v})\|,
\end{aligned}
$$

which implies, by using (5.7) and (5.10), that

$$
\begin{aligned}
\|(\xi_n^q)'\| &\le |R(\infty)|\|(\xi_{n-1}^q)'\| + \rho m \max\{|(\xi_n^q)'(t_{n,j})|, j = 1, ..., m\} \\
&\le |R(\infty)|\|(\xi_{n-1}^q)'\| + \rho h c_1 \sum_{p=0}^{n-1}\|(\xi_p^q)'\| + \rho h^2 b\overline{K}m\|(\xi_n^{q-1})'\|.
\end{aligned}
$$

Now, let $\eta^q = \max\{\|(\xi_n^q)'\|, n = 0, ..., N-1\}$, it follows that, for all $n = 0, ..., N-1$

$$
\|(\xi_n^q)'\| \le |R(\infty)|\|(\xi_{n-1}^q)'\| + \rho h c_1 \sum_{p=0}^{n-1}\|(\xi_p^q)'\| + \rho h^2 b\overline{K}m\eta^{q-1}.
$$

Hence by Lemma 1.7.4, we obtain for all $n = 0, ..., N-1$

$$
\begin{aligned}
\|(\xi_n')^q\| &\le \frac{\|(\xi')_0^q\|}{R_2 - R_1}[(R_2-1)R_2^n + (1-R_1)R_1^n] + \frac{\rho h^2 b\overline{K}m\eta^{q-1}}{R_2 - R_1}[R_2^n - R_1^n] \\
&\le \frac{\|(\xi_0')^q\|}{R_2 - R_1}[(R_2-1)R_2^{\frac{T}{h}} + 1] + \frac{\rho h^2 b\overline{K}m\eta^{q-1}}{R_2 - R_1}R_2^{\frac{T}{h}} \\
&\le \left(\frac{1}{R_2 - R_1}[(R_2-1)R_2^{\frac{T}{h}} + 1] + \frac{1}{R_2 - R_1}R_2^{\frac{T}{h}}\right)\rho h^2 b\overline{K}m\eta^{q-1},
\end{aligned}
\tag{5.21}
$$

where R_1 and R_2 are defined by (1.42) such that $A = |R(\infty)|, B = \rho h c_1, K = \rho h^2 b\overline{K}m\eta^{q-1}$.
Since, $\lim\limits_{h\longrightarrow 0}\left(\frac{1}{R_2-R_1}[(R_2-1)R_2^{\frac{T}{h}} + 1] + \frac{1}{R_2-R_1}R_2^{\frac{T}{h}}\right) = \frac{1+\exp\left(\frac{T\rho c_1}{1-|R(\infty)|}\right)}{1-|R(\infty)|} < +\infty$.
Then, there exists $\gamma > 0$ such that for all $h \in (0, \overline{h}]$.

$$
\frac{1}{R_2 - R_1}[(R_2-1)R_2^{\frac{T}{h}} + 1] + \frac{1}{R_2 - R_1}[R_2^{\frac{T}{h}}] \le \gamma.
$$

It follows, from (5.21), that for all $n = 0, ..., N-1$.

$$\|(\xi'_n)^q\| \leq \gamma\rho h^2 b\overline{K}m\eta^{q-1},$$

which implies that, for all $q \geq 1$,

$$\eta^q \leq \gamma\rho h^2 b\overline{K}m\eta^{q-1} \leq ... \leq (\underbrace{\gamma\rho b\overline{K}m}_{\beta})^q h^{2q}\eta^0. \tag{5.22}$$

Since, $(u^0_n)'(t_{n,j}) \in J$ (bounded interval), then there exists $\alpha > 0$ such that $|(u^0_n)'(t_{n,j})| \leq \alpha$, which implies from (5.10) that, for all $n = 0, ..., N-1$

$$\begin{aligned}
\|(u^0_n)'\| &\leq |R(\infty)|\|(u^0_{n-1})'\| + \rho m\alpha \\
&\leq (|R(\infty)|)^2\|(u^0_{n-2})'\| + \rho m\alpha(1 + |R(\infty)|) \\
&\quad \vdots \\
&\leq (|R(\infty)|)^n\|(u^0_0)'\| + \rho m\alpha\left(1 + |R(\infty)| + ... + (|R(\infty)|)^{n-1}\right) \\
&\leq \|(u^0_0)'\| + \frac{\rho m\alpha}{1 - |R(\infty)|} \\
&\leq |R(\infty)||f(0)| + \rho m\alpha + \frac{\rho m\alpha}{1 - |R(\infty)|} = c_2.
\end{aligned}$$

Hence, there exist positive numbers c_3 and c_5 such that,

$$\|\eta^0\| = \|(u^0)' - u'\| \leq \|(u^0)' - x'\| + \|x' - u'\| \leq \beta_1 + \beta_2 h^{m+1}.$$

Then, from (5.22), we obtain for all $q \geq 1$

$$\|(\xi^q)'\| \leq \beta^q h^{2q}\left(\beta_1 + \beta_2 h^{m+1}\right). \tag{5.23}$$

By using Theorem (5.3.1), we deduce that

$$\|(e^q)'\| \leq \|(\xi^q)'\| + \|u' - x'\| \leq \beta_1\beta^q h^{2q} + \beta_2\beta^q h^{m+1+2q} + \beta_2 h^{m+1}.$$

On the other hand, from (5.11) and (5.8), we have by using (5.23)

$$\begin{aligned}
\|\xi_n^q\| &\leq \|\xi_{n-1}^q\| + hb\|(\xi_{n-1}^q)'\| + hbm\|(\xi_n^q)'\| \\
&\leq \|\xi_{n-1}^q\| + hb(m+1)\|(\xi^q)'\| \\
&\leq \|\xi_{n-2}^q\| + 2hb(m+1)\|(\xi^q)'\| \\
&\quad \vdots \\
&\leq \|\xi_0^q\| + nhb(m+1)\|(\xi^q)'\| \\
&\leq \underbrace{\|\xi_{-1}^q\|}_{=0} + hb(m+1)\|(\xi^q)'\| + Tb(m+1)\|(\xi^q)'\| \\
&\leq \underbrace{(\bar{h}+T)b(m+1)}_{\beta_3}\|(\xi^q)'\| \\
&\leq \beta_3\beta^q h^{2q}\left(\beta_1 + \beta_2 h^{m+1}\right).
\end{aligned}$$

By using Theorem (5.3.1), we deduce that

$$\|e^q\| \leq \|\xi^q\| + \|u' - x'\| \leq \underbrace{\beta_1\beta_3}_{\gamma_1}\beta^q h^{2q} + \underbrace{\beta_2\beta_3}_{\gamma_2}\beta^q h^{m+1+2q} + \beta_2 h^{m+1}.$$

Thus, the proof is completed.

■

The following result gives the convergence of the iterative solution u^q to the exact solution x.

Theorem 5.3.3 *Consider the iterative collocation solution* $u^q, q \geq 1$ *defined by* (5.11), *if* $-1 < R(\infty) = (-1)^m \prod_{l=1}^{m} \frac{1-c_l}{c_l} < 1$, *then for any initial condition* $(u')^0(t_{n,j}) \in J$, *the iterative collocation solution* $u^q, q \geq 1$ *converges to the exact solution* x *for sufficiently small* h. *Moreover, the following errors estimates hold*

$$\|u^q - x\| \leq \gamma_1\beta^q h^{2q} + \gamma_2\beta^q h^{m+1+2q} + \beta_2 h^{m+1},$$

and

$$\|(u^q)' - x'\| \leq \beta_1 h^{2q} + \beta_2 h^{m+1+2q} + \beta_2 h^{m+1}.$$

where $\beta, \beta_1, \beta_2, \gamma_1, \gamma_2$ *are finite constants independent of* h *and* q*.*

Proof. We define the errors e^q and ξ^q by $e^q(t) = e^q_n(t) = u^q_n(t) - x(t)$ and $\xi^q = \xi^q_n = u^q_n(t) - u_n(t)$ on $\sigma_n, n = 0, ..., N-1$.

We have, from (5.9) and (5.12), for all $n = 0, ..., N-1$ and $j = 1, ..., m$

$$\begin{aligned}
|(\xi^q_n)'(t_{n,j})| &\leq h\overline{K}\sum_{p=0}^{n-1}\|\xi^q_p\| + h^2 b\overline{K}\sum_{p=0}^{n-1}\|(\xi^q_p)'\| + h^2 bm\overline{K}\sum_{p=0}^{n-1}\|(\xi^q_p)'\| + h\overline{K}\|\xi^q_{n-1}\| \\
&\quad + h^2 b\overline{K}\|(\xi^q_{n-1})'\| + h^2 b\overline{K}\sum_{v=1}^{m}|(\xi^{q-1}_n)'(t_{n,v})| \\
&\leq 2h\overline{K}\sum_{p=0}^{n-1}\|\xi^q_p\| + 2h^2 b\overline{K}\sum_{p=0}^{n-1}\|(\xi^q_p)'\| + h^2 bm\overline{K}\sum_{p=0}^{n-1}\|(\xi^q_p)'\| \\
&\quad + h^2 b\overline{K}\sum_{v=1}^{m}|(\xi^{q-1}_n)'(t_{n,v})|. \\
&\leq 2h\overline{K}\sum_{p=0}^{n-1}\|\xi^q_p\| + h^2 b\overline{K}(2+m)\sum_{p=0}^{n-1}\|(\xi^q_p)'\| + h^2 b\overline{K}m\|(\xi^{q-1}_n)'\|.
\end{aligned} \tag{5.24}$$

We have, from (5.8) and (5.11), for all $n = 0, ..., N-1$ and $j = 1, ..., m$

$$\begin{aligned}
\|\xi_n^q\| &\le \|\xi_{n-1}^q\| + hb\|(\xi_{n-1}^q)'\| + hbm\|(\xi_n^q)'\| \\
&\le \|\xi_{n-1}^q\| + hbm\left(\|(\xi_{n-1}^q)'\| + \|(\xi_n^q)'\|\right) \\
&\le \|\xi_{n-2}^q\| + hbm\left(\|(\xi_{n-2}^q)'\| + 2\|(\xi_{n-1}^q)'\| + \|(\xi_n^q)'\|\right) \\
&\le \|\xi_{n-3}^q\| + hbm\left(\|(\xi_{n-3}^q)'\| + 2\|(\xi_{n-2}^q)'\| + 2\|(\xi_{n-1}^q)'\| + \|(\xi_n^q)'\|\right) \\
&\quad\vdots \\
&\le \|\xi_0^q\| + hbm\left(\|(\xi_0^q)'\| + 2\|(\xi_1^q)'\| + ... + 2\|(\xi_{n-1}^q)'\| + \|(\xi_n^q)'\|\right) \qquad (5.25)\\
&\le \|\xi_0^q\| + 2hbm\sum_{p=0}^{n}\|(\xi_i^q)'\| \\
&\le \|\underbrace{\xi_{-1}^q}_{=0}\| + hb\|\underbrace{(\xi_{-1}^q)'}_{=0}\| + hbm\|(\xi_0^q)'\| + 2hbm\sum_{p=0}^{n}\|(\xi_i^q)'\| \\
&\le 3hbm\sum_{i=0}^{n}\|(\xi_i^q)'\|.
\end{aligned}$$

Inserting (5.25) into (5.24), we obtain for each $j = 1, ..., m, n = 0, ..., N-1$

$$\begin{aligned}
|(\xi_n^q)'(t_{n,j})| &\le 6hTbm\overline{K}\sum_{p=0}^{n-1}\|(\xi_p^q)'\| + h^2 b\overline{K}(2+m)\sum_{p=0}^{n-1}\|(\xi_p^q)'\| \\
&\quad + h^2 b\overline{K}m\|(\xi_n^{q-1})'(t_{n,v})\| \\
&\le h\underbrace{b\overline{K}(7Tm+2T)}_{c_1}\sum_{p=0}^{n-1}\|(\xi_p^q)'\| + h^2 b\overline{K}m\|(\xi_n^{q-1})'(t_{n,v})\|,
\end{aligned}$$

which implies, by using (5.7) and (5.10), that

$$\begin{aligned}
\|(\xi_n^q)'\| &\le |R(\infty)|\|(\xi_{n-1}^q)'\| + \rho m \max\{|(\xi_n^q)'(t_{n,j})|, j = 1, ..., m\} \\
&\le |R(\infty)|\|(\xi_{n-1}^q)'\| + \rho h c_1\sum_{p=0}^{n-1}\|(\xi_p^q)'\| + \rho h^2 b\overline{K}m\|(\xi_n^{q-1})'\|.
\end{aligned}$$

Now, let $\eta^q = \max\{\|(\xi_n^q)'\|, n = 0, ..., N-1\}$, it follows that, for all $n = 0, ..., N-1$

$$\|(\xi_n^q)'\| \leq |R(\infty)|\|(\xi_{n-1}^q)'\| + \rho h c_1 \sum_{p=0}^{n-1} \|(\xi_p^q)'\| + \rho h^2 b\overline{K} m \eta^{q-1}.$$

Hence by Lemma 1.7.4, we obtain for all $n = 0, ..., N-1$

$$\begin{aligned}
\|(\xi_n')^q\| &\leq \frac{\|(\xi')_0^q\|}{R_2 - R_1}[(R_2-1)R_2^n + (1-R_1)R_1^n] + \frac{\rho h^2 b\overline{K} m \eta^{q-1}}{R_2 - R_1}[R_2^n - R_1^n] \\
&\leq \frac{\|(\xi_0')^q\|}{R_2 - R_1}[(R_2-1)R_2^{\frac{T}{h}} + 1] + \frac{\rho h^2 b\overline{K} m \eta^{q-1}}{R_2 - R_1} R_2^{\frac{T}{h}} \\
&\leq \left(\frac{1}{R_2 - R_1}[(R_2-1)R_2^{\frac{T}{h}} + 1] + \frac{1}{R_2 - R_1} R_2^{\frac{T}{h}}\right) \rho h^2 b\overline{K} m \eta^{q-1},
\end{aligned} \tag{5.26}$$

where R_1 and R_2 are defined by (1.42) such that $A = |R(\infty)|, B = \rho h c_1, K = \rho h^2 b\overline{K} m \eta^{q-1}$.
Since, $\lim_{h \longrightarrow 0} \left(\frac{1}{R_2 - R_1}[(R_2-1)R_2^{\frac{T}{h}} + 1] + \frac{1}{R_2 - R_1} R_2^{\frac{T}{h}}\right) = \frac{1+\exp\left(\frac{T\rho c_1}{1-|R(\infty)|}\right)}{1-|R(\infty)|} < +\infty$.
Then, there exists $\gamma > 0$ such that for all $h \in (0, \overline{h}]$.

$$\frac{1}{R_2 - R_1}[(R_2-1)R_2^{\frac{T}{h}} + 1] + \frac{1}{R_2 - R_1}[R_2^{\frac{T}{h}}] \leq \gamma.$$

It follows, from (5.26), that for all $n = 0, ..., N-1$.

$$\|(\xi_n')^q\| \leq \gamma \rho h^2 b\overline{K} m \eta^{q-1},$$

which implies that, for all $q \geq 1$,

$$\eta^q \leq \gamma \rho h^2 b\overline{K} m \eta^{q-1} \leq ... \leq (\underbrace{\gamma \rho b\overline{K} m}_{\beta})^q h^{2q} \eta^0. \tag{5.27}$$

Since, $(u_n^0)'(t_{n,j}) \in J$ (bounded interval), then there exists $\alpha > 0$ such that $|(u_n^0)'(t_{n,j})| \leq \alpha$, which implies from (5.10) that, for all $n = 0, ..., N-1$

$$\begin{aligned}
\|(u_n^0)'\| &\leq |R(\infty)|\|(u_{n-1}^0)'\| + \rho m\alpha \\
&\leq (|R(\infty)|)^2\|(u_{n-2}^0)'\| + \rho m\alpha(1 + |R(\infty)|) \\
&\quad \vdots \\
&\leq (|R(\infty)|)^n\|(u_0^0)'\| + \rho m\alpha \left(1 + |R(\infty)| + ... + (|R(\infty)|)^{n-1}\right) \\
&\leq \|(u_0^0)'\| + \frac{\rho m\alpha}{1 - |R(\infty)|} \\
&\leq |R(\infty)||f(0)| + \rho m\alpha + \frac{\rho m\alpha}{1 - |R(\infty)|} = c_2.
\end{aligned}$$

Hence, there exist positive numbers c_3 and c_5 such that,

$$\|\eta^0\| = \|(u^0)' - u'\| \leq \|(u^0)' - x'\| + \|x' - u'\| \leq \beta_1 + \beta_2 h^{m+1}.$$

Then, from (5.27), we obtain for all $q \geq 1$

$$\|(\xi^q)'\| \leq \beta^q h^{2q}\left(\beta_1 + \beta_2 h^{m+1}\right). \tag{5.28}$$

By using Theorem (5.3.1), we deduce that

$$\|(e^q)'\| \leq \|(\xi^q)'\| + \|u' - x'\| \leq \beta_1\beta^q h^{2q} + \beta_2\beta^q h^{m+1+2q} + \beta_2 h^{m+1}.$$

On the other hand, from (5.11) and (5.8), we have by using (5.28)

$$\begin{aligned}
\|\xi_n^q\| &\leq \|\xi_{n-1}^q\| + hb\|(\xi_{n-1}^q)'\| + hbm\|(\xi_n^q)'\| \\
&\leq \|\xi_{n-1}^q\| + hb(m+1)\|(\xi^q)'\| \\
&\leq \|\xi_{n-2}^q\| + 2hb(m+1)\|(\xi^q)'\| \\
&\quad \vdots \\
&\leq \|\xi_0^q\| + nhb(m+1)\|(\xi^q)'\| \\
&\leq \underbrace{\|\xi_{-1}^q\|}_{=0} + hb(m+1)\|(\xi^q)'\| + Tb(m+1)\|(\xi^q)'\| \\
&\leq \underbrace{(\bar{h}+T)b(m+1)}_{\beta_3}\|(\xi^q)'\| \\
&\leq \beta_3\beta^q h^{2q}\left(\beta_1 + \beta_2 h^{m+1}\right).
\end{aligned}$$

By using Theorem (5.3.1), we deduce that

$$\|e^q\| \leq \|\xi^q\| + \|u' - x'\| \leq \underbrace{\beta_1\beta_3}_{\gamma_1}\beta^q h^{2q} + \underbrace{\beta_2\beta_3}_{\gamma_2}\beta^q h^{m+1+2q} + \beta_2 h^{m+1}.$$

Thus, the proof is completed. ■

5.4 Numerical examples

In order to test the applicability of the presented method, we consider the following examples with $T = 1$. These examples have been solved with various values of N, m and $q = m$. In each example, we calculate the error between x and the iterative collocation solution u^m.

The absolute errors at the particular points are given to compare our solutions with the solutions obtained by [5, 6, 35, 61, 64].

The results in these examples confirm the theoretical results; moreover, the results obtained by the present method is very superior to that obtained by the methods in

[5, 6, 35, 61, 64].

Example 5.4.1 *([5, 6, 35]) Consider the following nonlinear Volterra integral equation*

$$x'(t) = 2\sin(t)\cos(t) + 3\int_0^t \cos(t-s)(x(s))^2 ds,\ t \in [0,1],$$

where $x(x) = \cos(x)$ *is the exact solution.*
The absolute errors for $N = m = q = 4$ *at* $t = 0, 0.1, ..., 1$ *are displayed in Table 5.1. We used the collocation parameters* $c_i = \frac{i}{m+1} + \frac{1}{4}, i = 1, ..., m$ *and* $R(\infty) = -\frac{11}{1989}$.
The numerical results of the present method are considerable accurate in comparison with the numerical results obtained by [5, 6, 35].

Table 5.1: Comparison of the absolute errors of Example 5.4.1

t	Method in [35] $N = 16$	Method in [6] $N = 32$	Method in [5] $N = 32$	Present method $N = 4$
0.0	0.0	0.0	− − −−	0.0
0.1	4.43 E −4	4.49 E −4	1.09 E −3	9.27 E −10
0.2	2.22 E −4	2.42 E −4	7.25 E −4	4.19 E −8
0.3	1.22 E −4	1.62 E −4	8.42 E −4	1.10 E −7
0.4	1.34 E −4	2.00 E −4	3.56 E −3	1.97 E −7
0.5	4.29 E −4	3.38 E −4	7.59 E −3	3.10 E −7
0.6	1.77 E −4	6.10 E −5	5.29 E −3	4.20 E −7
0.7	4.54 E −4	3.22 E −4	1.94 E −3	5.33 E −7
0.8	5.75 E −4	4.35 E −4	2.34 E −3	6.52 E −7
0.9	5.82 E −4	4.47 E −4	1.69 E −4	7.29 E −7
1.0	9.15 E −4	8.00 E −4	− − −−	7.84 E −7

Example 5.4.2 *([61, 64]) Consider the nonlinear Volterra integro-differential equation given by*

$$x'(t) = 1 - \int_0^t x(s)ds, \; t \in [0,1].$$

with the initial conditions $x(0) = 0$ *and the exact solution* $x(t) = sin(t)$ *. Here,* $f(t) = 1, g(t) = 0, K(t,s) = -1.$

The absolute errors for $N = 4, 8$ *and* $m = q = 4$ *at* $t = 0, 0.1, ..., 1$ *are displayed in Table 5.2. We used the collocation parameters* $c_i = \frac{i}{m+1} + \frac{1}{2}, i = 1, ..., m$ *and* $R(\infty) = \frac{1}{1001}$.

The numerical results of the absolute error function obtained by the present method are compared in Table 5.2 with the absolute error function of the Taylor method given in [61] and Bessel method [64] for an approximate polynomial solutions of degree 5.

Table 5.2: Comparison of the absolute errors of Example 5.4.2

t	Taylor method [61]	Bessel method [64]	Present method $N = 4$	Present method $N = 8$
0.0	0.0	0.0	0.0	0.0
0.1	2.00 E −11	2.49 E −7	7.46 E −9	9.38 E −12
0.2	2.50 E −9	4.02 E −7	7.42 E −9	2.74 E −9
0.3	4.33 E −8	3.00 E −7	1.13 E −8	5.27 E −9
0.4	3.24 E −7	2.05 E −7	2.28 E −8	9.43 E −9
0.5	1.54 E −6	2.83 E −7	2.40 E −8	8.80 E −9
0.6	5.52 E −6	3.75 E −7	4.87 E −8	8.80 E −9
0.7	1.62 E −5	1.65 E −7	5.26 E −8	5.90 E −9
0.8	4.12 E −5	1.81 E −7	6.71 E −8	6.14 E −9
0.9	9.38 E −5	1.18 E −6	9.05 E −8	6.90 E −9
1.0	1.95 E −4	9.66 E −6	9.04 E −8	7.34 E −9

Example 5.4.3 *Consider the nonlinear Volterra integro-differential equation given by*

$$x'(t) = f(t) + \int_0^t e^{s-t} x(s) ds, \ t \in [0, 1].$$

with f *is chosen so that the exact solution is* $x(t) = 2\cos(t) + 1$. *The absolute errors for* $(N, m) \in \{(5, 2), (5, 3), (6, 3), (10, 3)\}$ *at* $t = 0, 0.1, \ldots, 1$ *are presented in Table 5.3. We used the collocation parameters* $c_i = \frac{i}{m+1} + \frac{1}{3}, i = 1, \ldots, m$. *From the Table 5.3, we note that the absolute error reduces as N or m increases.*

Table 5.3: Absolute errors for Example 5.4.3

t	$N = 5$ $m = 2$	$N = 5$ $m = 3$	$N = 6$ $m = 3$	$N = 10$ $m = 3$
0	0.0	0.0	0.0	0.0
0.1	0.15 E −4	0.43 E −7	0.16 E −7	0.68 E −10
0.2	0.15 E −4	0.42 E −7	0.32 E −7	0.11 E −8
0.3	0.30 E −4	0.16 E −6	0.54 E −7	0.67 E −9
0.4	0.29 E −4	0.16 E −6	0.19 E −6	0.44 E −9
0.5	0.44 E −4	0.36 E −6	0.12 E −6	0.24 E −8
0.6	0.44 E −4	0.37 E −6	0.22 E −6	0.13 E −8
0.7	0.57 E −4	0.65 E −6	0.27 E −6	0.14 E −8
0.8	0.58 E −4	0.67 E −6	0.34 E −6	0.19 E −8
0.9	0.70 E −4	0.10 E −5	0.47 E −6	0.55 E −8
1	0.71 E −4	0.10 E −5	0.50 E −6	0.12 E −7

Example 5.4.4 *We Consider the linear Volterra integro-differential equation given by*

$$x'(t) = f(t) + \int_0^t scos(s+t)x(s)ds, \ t \in [0,1].$$

with f is chosen so that the exact solution is $x(t) = e^t + 2$.
The absolute errors e and e' for $m = q = 3$ $N \in \{4,5,10\}$ at $t = 0, 0.1, ..., 1$ are presented in Table 5.4. We used the collocation parameters $c_i = \frac{i}{m+1} + \frac{1}{4}, i = 1, ..., m$. From the Table 5.4, we note that the absolute error reduces as N or m increases.

Table 5.4: Absolute errors of Example 5.4.4

t	$N = 4$		$N = 10$	
	e	e'	e	e'
0.0	0.0	0.0	0.0	0.0
0.1	1.40 E −6	1.37 E −5	5.88 E −9	2.01 E −7
0.2	1.93 E −6	6.75 E −7	5.20 E −10	6.18 E −8
0.3	2.64 E −6	2.27 E −5	1.08 E −8	5.22 E −7
0.4	4.32 E −6	6.04 E −7	9.55 E −9	3.62 E −7
0.5	4.38 E −6	2.59 E −7	8.45 E −9	2.89 E −7
0.6	6.74 E −6	2.30 E −5	8.60 E −9	4.09 E −7
0.7	7.64 E −6	7.65 E −7	3.25 E −8	1.72 E −7
0.8	8.83 E −6	3.76 E −5	6.97 E −8	3.41 E −7
0.9	1.15 E −6	9.60 E −6	1.03 E −7	1.01 E −7
1.0	1.16 E −5	5.98 E −7	1.88 E −7	3.58 E −7

5.5 Conclusion

In this work we developed a method to find the solution of nonlinear Volterra integral equations. The presented method is based on the Lagrange polynomail formula and collocating by the real polynomial spline space of degree $m+1$ $S^{(1)}_{m+1}(\Pi_N)$ This method tested on four examples. Our method by the suggested method is compared with the methods in [5, 6, 35, 61, 64]. Our results verified the accurate nature of our approach

CONCLUSION

This part has considered the iterative collocation method approximation approach for solving nonlinear Volterra integro-differential equations equations. The method is easy to implement and has high order of convergence. The convergence of the presented algorithm is proved and an error estimate is established. Iterative collocation method can be extended to higher order integro- differential equations. Thus a possible area of future research is the application of the iterative collocation method method to higher dimensional problems. One could also investigate the application of iterative collocation method to singular PDE's

CONCLUSION AND PERSPECTIVE

In this thesis, we have developed a new numerical method by using iterative collocation method to approximate the solutions of nonlinear volterra integral and integro-differential equations. In the real polynomial spline space $S_{m-1}^{(-1)}(\Pi_N)$, $S_m^{(0)}(\Pi_N)$ and $S_{m+1}^{(1)}(\Pi_N)$. It is proved that the method is convergent and Error analysis shows the accuracy gives better. This method is easy to implement and the coefficients of the approximate solution are determined by iterative formulas without the need to solve any system of algebraic equations. Moreover, many numerical examples were introduced showing that the method is convergent with a good accuracy and the numerical results confirmed the theoretical estimates. The present scheme is very easy, accurate and effective.

Further researches on this kind of problems will be conducted by generalizing the current numerical method to approximate a high-order nonlinear Volterra integro-differential equations and high order nonlinear integro-differential equations with weakly singular kernels.

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